ABSTRACT

THEORETICAL PREDICTION OF JOINT REACTION FORCE IN A DYNAMIC GENERAL 3-D PENDULUM TREE MODEL FOR HUMAN OR ANIMAL MOTION

by
Sucheta Goyal

Lagrangian dynamics and the method of superfluous coordinates are applied to predict dynamic joint reaction forces in an idealized flexible model of a branched 3-D pendulum tree system. The number of segments and joints on the tree are adjustable as is the branching tree pattern. The segments that comprise the tree are assumed to be one-dimensional rigid rods containing a discrete set of mass points that is both flexible in number and distribution on the tree. The idealized 3-D pendulum tree system is intended to provide a flexible theoretical framework to model and better understand the dynamics of human and animal movement as well as the forces associated with those movements. In particular, this work focuses on predicting the dynamic reaction forces that are produced in the simple idealized frictionless joints of the pendulum system during motion. The ability to predict dynamic joint reaction forces in this model system could prove helpful in assessing the potential effect of a posited movement technique in producing joint injury and/or pain. This thesis extends the findings of previous work on similar pendulum model systems in 2-D to model systems in 3-D.

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Om Ganeshay Namah!!!

I would like to dedicate this thesis to my grandmother the late Vimla Goyal; grandfather, Shyam Lal Mahaveer Parsad Goyal; father, Pawan Goyal; mother, Sumitra Goyal; brother, Kamal Goyal; and to all my friends and entire family. There is no doubt in my mind that without their continued support, blessings and encouragement I could not have achieved this.

My strong belief and conviction in Almighty GOD and
HIS grace has made my dreams come true.

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CHAPTER 1

INTRODUCTION AND MOTIVATION

The approach of modeling human and animal motion as a dynamic rigid pendulum system has a long history with too many examples to describe here. To understand aspects of human walking models as concise as a simple inverted pendulum to those containing many connecting and interacting segments for the lower and upper extremities have been used**.** “These models are useful in better understanding movement and in estimating clinical parameters that are significant but otherwise difficult to access. For example, joint reaction forces and dynamic stability indices are clinically important but difficult to assess” (Lacker, 1997).

In modeling any skilled movement task, it is important to ideally select the most direct representation that can best describe the motion. This choice can be difficult and requires scientific skill and experimentation perhaps, in part, for the same reason as it does to skillfully choose in real life which segments to move and what constraints to apply to produce an efficient motion technique. Complex interactions occur when connected segments move together that give rise to new dynamic forces that can *only* be ‘felt’ when the system is in *motion* and that have significant effects both upon stability and mechanical efficiency.

Since the segments dynamically engaged in performing a given complete motor task can and often will vary in time during the course of that task, the mechanical pendulum system used to model that motor task can change depending on the phase of the motion being considered. Because of this required model flexibility, it is desirable to be able to formulate the equations of motion (EOM) for a *general pendulum tree system* which can vary in the number of its segments, branching patterns and constraints applied to it. Having formulated the EOM for such a general pendulum system, it then becomes possible to solve for both the motions and the forces involved in producing those motions for any particular realization of that general pendulum system.

This thesis considers an idealized, frictionless, pendulum system consisting of S segments, J joints and P mass points distributed on it where S, J and P are freely adjustable. The connectivity of the segments is also adjustable so that different branching patterns for the pendulum system are possible. In this laboratory, Rajai and Lacker have developed equations for predicting the dynamic joint reaction forces for such a general pendulum system in 2D. (Rajai, 2007). The ability to predict these constraint forces provides a theoretical framework that could be helpful in assessing the potential effect of a posited movement technique in producing joint injury and/or pain. This thesis extends the findings of Rajai and Lacker for such a general pendulum system from 2D to 3D systems. In particular, it focuses on predicting the pivot reaction force (PRF) equation for a general 3D pendulum system. Predicting the PRF represents a significant step towards predicting the general joint reaction force (JRF) in such a system. The methods for determining the PRF are closely related to those for finding the general JRF. Predicting the general JRF equation concludes this thesis.

CHAPTER 2

METHODS

2.1 Introduction to Superfluous Coordinates For Finding Constraint Forces in Lagrangian Systems

Using Lagrangian dynamics to describe the motion of a mechanical system, a desired constraint can be expressed in the EOM by implicitly assuming that a given symbol such as the length, l, of a rigid rod is to be considered a non-dynamic constant. Alternatively, a constraint can be represented by imposing an additional equation into the system that explicitly expresses a definite relationship that a dynamic variable or several dynamic variables of the system are to satisfy.

For example, consider a single frictionless pendulum consisting of a rod without mass of constant length l and a point mass m at its end moving in 3-space where a constant vertically downward gravitational field of strength g, is present (see Figure 2.1).

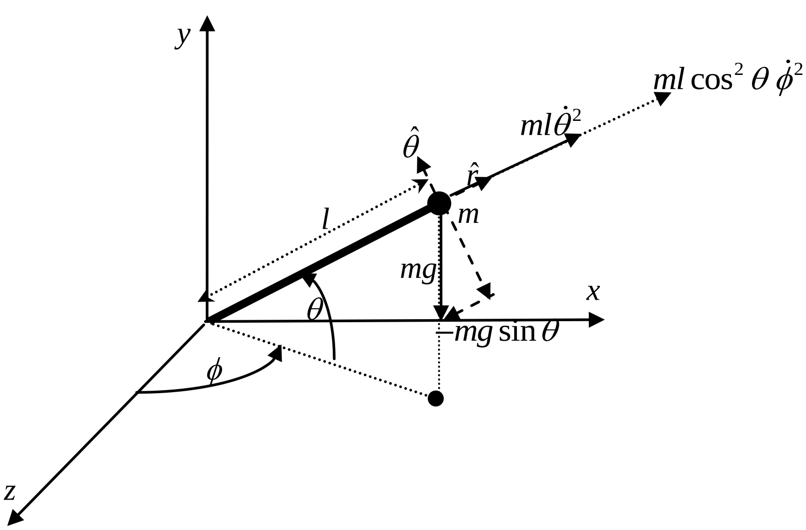

Figure 2.1 Single pendulum in 3-space.

Let θ be defined as the counterclockwise angle that the pendulum rod makes with its projection onto the x,z plane and let ϕ be defined as the counterclockwise angle that this projection makes with the z-axis (see Figure 2.1). In this case, there is a force of constraint that is often not explicit in the EOM but that must be present to keep *l* constant. To solve for the reactive force at the pivot joint, one needs to solve for the tension in the rod that is acting as the constraint force, keeping the rod at a constant length and preventing the mass from taking off into the air.

One method that can be used to solve for this dynamic constraint force is to introduce a superfluous coordinate variable into the system that in effect releases the *implicit* constraint on the system. This change will alter the equations of motion of the system in such a way that the forces that are required to maintain the constraint are revealed when the constraint is *explicitly* re-applied to the altered equations.

2.2 Application of Superfluous Coordinates and Lagrangian Dynamics to Determine the Pivot Reaction Force for a Single Pendulum in 3-D

In the case of the single pendulum described in the previous section (see Figure 2.1), instead of initially considering the distance of the mass point to the pivot as the constant *l* the method of superfluous coordinates releases this constraint and allows the length of the rod to be a new variable *r* which, in effect, treats the mass point of the pendulum as if it were a free particle. The Cartesian coordinates of the mass point (x, y, z) expressed in polar coordinates are $(r\cos\theta\sin\phi, r\sin\theta, r\cos\theta\cos\phi)$. The Potential Energy, *P*, of the mass point is given by,

$$P = mgy = mgr\sin\theta \tag{2.1}$$

Its Kinetic Energy, *K*, is

$$K = \frac{1}{2}mv^2 = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) = \frac{1}{2}m\left[\dot{r}^2 + r^2\left(\dot{\theta}^2 + \dot{\phi}^2\cos^2\theta\right)\right]. \tag{2.2}$$

The Lagrangian function, $L = K - P$, is therefore,

$$L = \frac{1}{2}m\left[\dot{r}^2 + r^2\left(\dot{\theta}^2 + \dot{\phi}^2\cos^2\theta\right)\right] - mgr\sin\theta. \tag{2.3}$$

The EOM for the system are found by applying

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\overline{x}}_i}\right) - \frac{\partial L}{\partial \overline{x}_i} = 0 \tag{2.4}$$

to each of the three dynamic variables of the system

$$\overline{x}_i = \begin{cases} r, & i = 1 \\ \theta, & i = 2 \\ \phi, & i = 3 \end{cases}. \tag{2.5}$$

The resulting EOM for the system is

$$mr^2\ddot{\theta} = -2mr\dot{r}\dot{\theta} - \frac{1}{2}mr^2\sin 2\theta\,\dot{\phi}^2 - mgr\cos\theta \qquad (\hat{\theta}\text{ direction}) \qquad (2.6)$$

$$mr^2\cos^2\theta\,\ddot{\phi} = -2mr\cos^2\theta\,\dot{r}\dot{\phi} + mr^2\sin 2\theta\,\dot{\theta}\dot{\phi} \qquad (\hat{\phi}\text{ direction}) \qquad (2.7)$$

$$m\ddot{r} = mr\dot{\theta}^2 + mr\cos^2\theta\,\dot{\phi}^2 - mg\sin\theta \qquad (\hat{r}\text{ direction}) \qquad (2.8)$$

The Lagrangian in Equation (2.3) has the same form for the system without the superfluous coordinate but in that case *l* replaces *r* and *l* is implicitly assumed to be a constant so the equation in the $\hat{r}$ direction is not part of the EOM for the system without superfluous coordinate *r*.

Equation (2.8) reveals the forces acting on the free mass point in the $\hat{r}$ direction that extend or contract its distance from the pivot point (origin). Therefore, a reactive constraint force, F_C must be applied that is equal and opposite to these forces to prevent the constraint condition *r=l* from being violated. Explicitly imposing the constraint condition $r = l$ (a constant) on Equation (2.8) which implies that $\dot{r} = \ddot{r} = 0$ and adding F_C, the reactive tension in the rigid rod, required to maintain the constraint yields the following expression for F_C:

$$m\ddot{r} = 0 = ml\dot{\theta}^2 + ml\cos^2\theta\,\dot{\phi}^2 - mg\sin\theta + F_C \qquad (2.9)$$

or

$$F_C = -ml\dot{\theta}^2 - ml\cos^2\theta\,\dot{\phi}^2 + mg\sin\theta \qquad (\hat{r}\text{ direction}) \qquad (2.10)$$

Equation (2.10) shows that the dynamic constraint force, F_C must oppose three dynamic forces, $mg\sin\theta$ comes from the contribution of the gravitational force on the mass and $ml\dot{\theta}^2$ and $ml\cos^2\theta\,\dot{\phi}^2$ are dynamic contributions due to inertial (centrifugal) forces. Equation (2.10) does not represent a solution for the dynamic constraint force, F_C until the solution $\theta(t),\phi(t)$ for the dynamical system is found. This solution can be obtained by solving the closed system that results from explicitly substituting the constraint condition $r=l$ (a constant), $\dot{r}=0$, $\bar{r}=0$ into Equations (2.6) - (2.7) for the free mass giving

$$ml^2\,\ddot{\theta} = -\frac{1}{2}ml^2\sin 2\theta\;\dot{\phi}^2 - mgl\cos\theta \qquad (\hat{\theta}\text{ direction}) \qquad (2.11)$$

$$ml^2\cos^2\theta\;\ddot{\phi} = ml^2\sin 2\theta\;\dot{\theta}\dot{\phi} \qquad (\hat{\phi}\text{ direction}) \qquad (2.12)$$

This system of equations is the same as would be obtained by applying Lagrange's equations (Equation (2.4)) to the Lagrangian function (Equation (2.3)) with constant l replacing the dynamic r. Note that the inertial (Coriolis) force terms $-2mr\dot{r}\dot{\theta}$ and $-2mr\cos^2\theta\,\dot{r}\dot{\phi}$ in Equations (2.6) - (2.7) disappear. The idealized rigid rod does not yield to extension or contraction forces and bending forces do not exist in the rod because of the idealized frictionless pivot joint. It should also be noted that Lagrange's EOM are not always force equations per Equation (2.8) is a true force equation, but Equations (2.6)- (2.7) do not have units of force but rather units of torque or angular force. This arises because the coordinates chosen do not always have units of length. In this case, only the third coordinate *r* has units of length while θ and ϕ are in radians. For these reasons, the terms for the EOM that are derived using Lagrange's method are often called generalized forces and the dynamic variables chosen are referred to as generalized coordinates.

Finally, the unit vectors of the coordinate system $(\hat{r}, \hat{\theta}, \hat{\phi})$ in which the EOM are expressed represent those of a system that moves with the rod (see Figure (2.1)). This coordinate system is both natural for a pendulum system and simplifies the form of the EOM for that system but because the dynamic coordinate system is rotary it represents a non-inertial (accelerating) frame of reference in which inertial (fictitious) force terms arise such as the centrifugal and Coriolis terms in Equations (2.6)-(2.8). While these forces are "fictitious" in terms of their origin, they do generate "real' forces (tension) in the rods or segments that react to them. For a complex dynamic mechanical system, the dynamic value of the reactive forces that maintain a constraint may not be easy to deduce or evaluate. The application of superfluous coordinates in this context will prove to be invaluable when later applied to finding the joint reaction forces for a complex pendulum tree system.

CHAPTER 3

3-D DYNAMIC PENDULUM TREE

This work applies the method of superfluous coordinates to a generalized, frictionless 3D pendulum tree system that will now be described. The pendulum system consists of *S* segments with J joints and *P* mass points distributed on it. A 2-D example of such a branched tree system consisting of *S*= *5* segments, *J*=*4* joints and *P*= *6* mass points is shown in Figure 3.1 below:

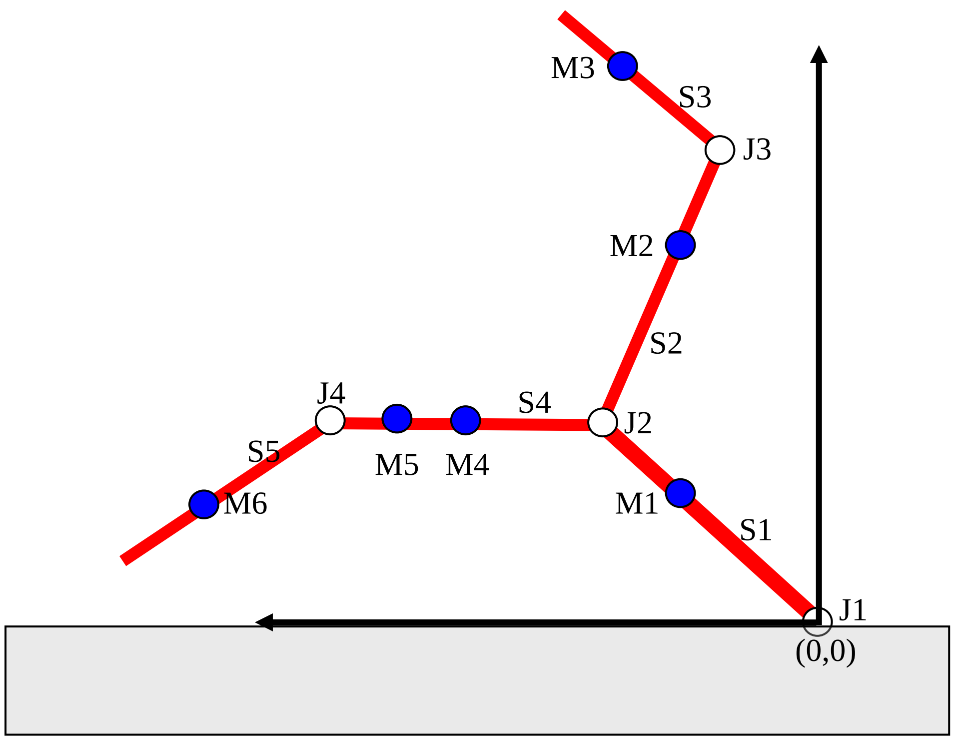

Figure 3.1 A 2-D pendulum tree with six point masses and five segments and four joints.

The global origin of the system is located where the pendulum tree attaches to the ceiling, wall or floor. This joint (J1 in Figure 3.1) will be called the root pivot point and its attached segment will be referred to as the root segment of the system (S1 in Figure 3.1). All other segments of the tree will be connected or descended from this root segment. Each segment of the tree belongs to a generation of the tree depending upon how far removed it is from the root segment. If it is directly connected to the root, it is a generation 1 or child of the root. If it is directly connected to a child of the root, then it is a generation 2 or grandchild of the root and so on.

Each segment of the tree is given a number. A convenient labeling system is to give the segments with numbers in order of their relation to the root in such a way that each child of a segment has a number higher than any of its parent's sibling segments (aunts or uncles). Every branch point of the tree will represent a joint of the pendulum system and all joints except the origin of the system will have at least one proximal and one distal joint segment attached to it. The origin pivot joint will have only the root segment attached to it. No closed loops are allowed in the tree.

All joints and mass points on the tree are also given numbers. A convenient labeling system is to number each joint so that every joint more distally related to the root joint will have a higher joint number than a more proximally related joint. Similarly, the mass points (the fruit) on the tree can be numbered so that no mass point on a given segment has a lower number than another mass point that is on a segment that is more central to the given segment, and so that all mass points that are on the same segment are numbered in order of their distance from the most proximal joint of that segment.

Let the length of the j^{th} segment be denoted by L_j and let the distance of the i^{th} mass point from its most proximal joint be denoted by d_i. Define a matrix R called the relation matrix whose entries consist of 0's and the lengths d_i and L_j. There are S columns in R and P rows. The i^{th} row refers to the i^{th} mass point and the j^{th} column to j^{th} segment. If the j^{th} segment has the i^{th} mass point on it then the entry $R_{i,j} = d_i$. If the j^{th} segment is a parent, grandparent, great-grandparent, or any fore-parent of the segment that the i^{th} mass point is on then $R_{i,j} = l_i$. For all other entries, $R_{i,j} = 0$. The relation matrix for Figure 3.1 is therefore, given by:

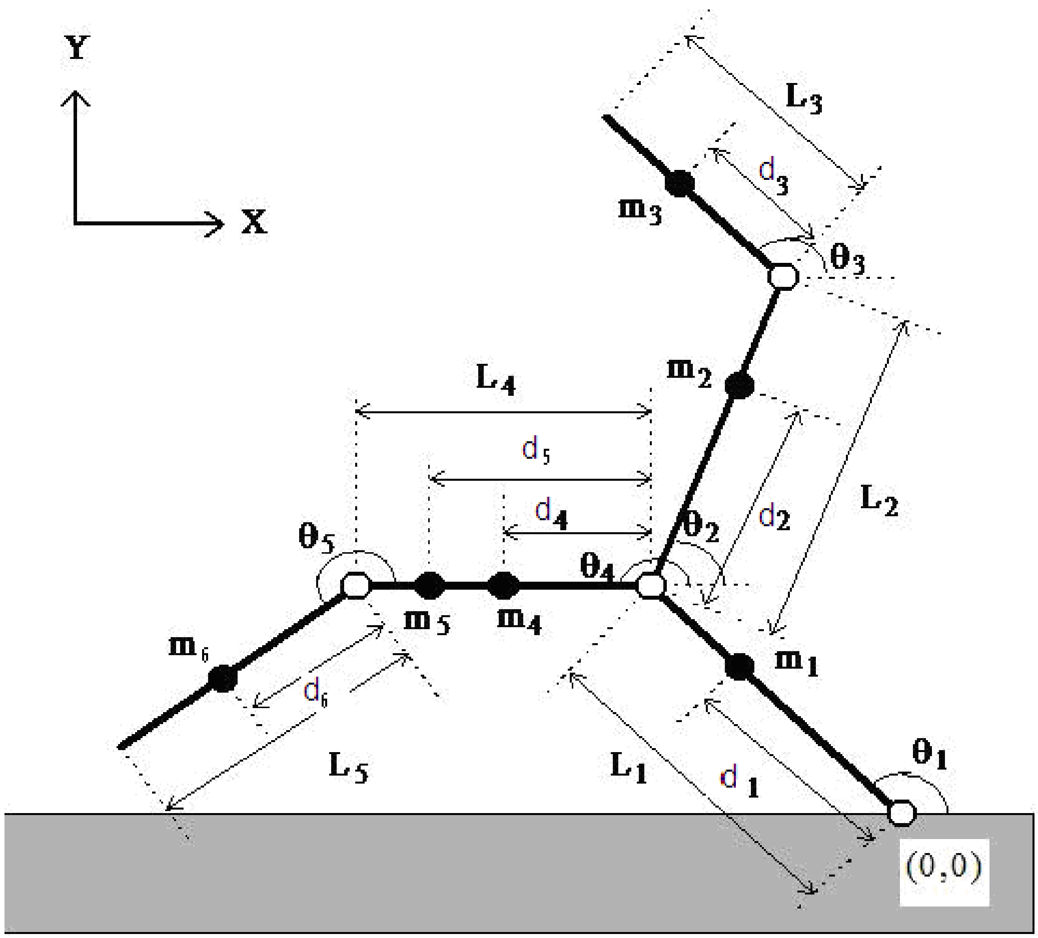

Figure 3.2 The same dynamic pendulum tree as in Figure 3.1 but with segment lengths and mass point distances labeled from their proximal joints. The dynamic angle for each segment is also shown in this figure.

$$\mathbf{R} = \begin{bmatrix} d_1 & 0 & 0 & 0 & 0 \\ L_1 & d_2 & 0 & 0 & 0 \\ L_1 & L_2 & d_3 & 0 & 0 \\ L_1 & 0 & 0 & d_4 & 0 \\ L_1 & 0 & 0 & d_5 & 0 \\ L_1 & 0 & 0 & L_4 & d_6 \end{bmatrix} \tag{3.1}$$

In general, the relation matrix R is a rectangular $P \times S$ matrix whose components R_{ij} are lengths that summarize the connectivity of the mass points according to their position on the segment that they lie on and that segment's relative ancestry from the root segment of the tree:

$$R_{ij} \equiv \begin{cases} d_i & \text{if } i^{th} \text{ mass point is on (or belongs to) the } j^{th} \text{ segment,} \\ L_j & \text{if the } j^{th} \text{ segment is a forefather segment of the segment} \\ & \text{that contains the } i^{th} \text{ mass point,} \\ 0 & \text{otherwise.} \end{cases} \tag{3.2}$$

The dynamic segment angles for the example pendulum tree $\theta_i(t),\ i = 1, \cdots, S = 5$ are also shown in Figure 3.2. These dynamic segment angles are defined in the following way. In addition to the global coordinate system whose origin is located where the root segment attaches to the lab frame (ceiling, wall or floor), local coordinate systems can be defined that are parallel to the global system but whose origins are translated to each ancestral joint of the system.

For a 2-D pendulum system, the dynamic position of the i$^{\text{th}}$ segment can be uniquely defined by the polar coordinate angle $\theta_i(t)$ that the segment makes with the positive x-axis of the local coordinate system whose origin (vertex) lies on the joint at the segment's proximal end (counterclockwise positive). For a 2-D pendulum system, there is only one polar coordinate needed to define the angle that the segment makes with its proximal attached joint.

For a 3-D pendulum system two polar coordinates $\theta_i(t), \phi_i(t)$ are required to uniquely define the dynamic position of the i$^{\text{th}}$ segment. The angle $\theta_i(t)$ is the dynamic angle (counterclockwise positive) that the i$^{\text{th}}$ segment makes with its projection onto the *(x,z)*-coordinate plane of the local coordinate system whose origin (vertex) lies on the joint at the segment's proximal end. The dynamic angle $\phi_i(t)$ is the angle formed by the above mentioned projection and the z-axis of the local coordinate system described above (see Figure 3.3).

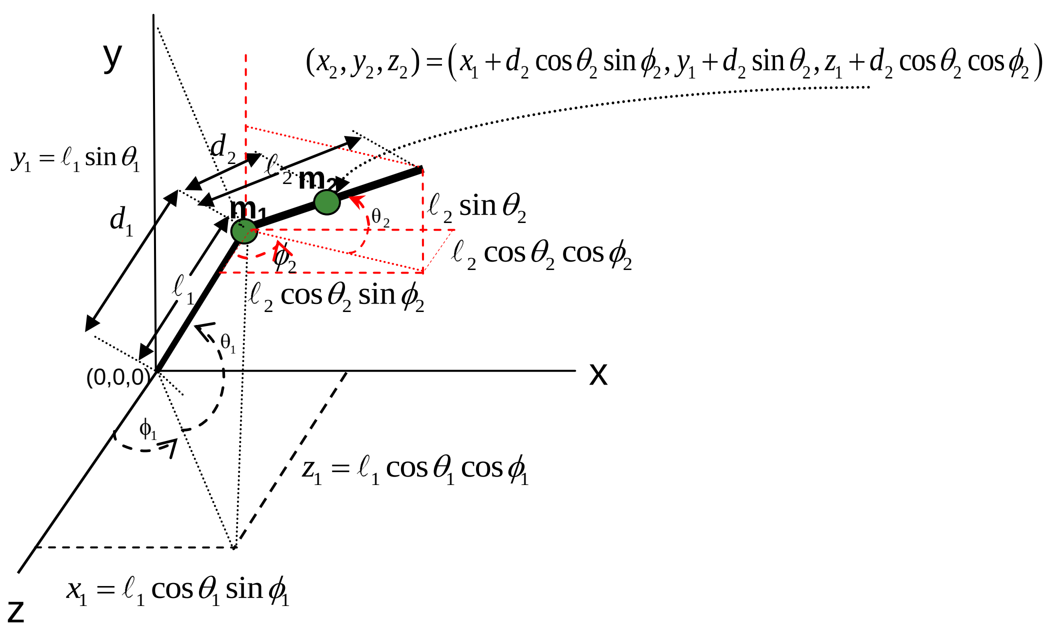

Figure 3.3 Local coordinate system is shown in red for the 2$^{\text{nd}}$ joint in this double pendulum example.

The generalized dynamic configuration, *X,* for the 3D pendulum tree is defined as the vector $X = (\theta, \phi)^T$ where $\theta = (\theta_1, \cdots, \theta_S)^T$ and $\phi = (\phi_1, \cdots, \phi_S)^T$. For a pendulum with *S* segments $X(t)$ will have dimension *2S*. The dynamic position of the i^{th} mass point on the tree in the global coordinate system in Cartesian coordinates is (x_i, y_i, z_i) where,

$$x_i = \sum_{j=1}^{S} R_{ij} \cos\theta_j \sin\phi_j \ , \quad y_i = \sum_{j=1}^{S} R_{ij} \sin\theta_j , \quad z_i = \sum_{j=1}^{S} R_{ij} \cos\theta_j \cos\phi_j \qquad (3.3)$$

and R is the relation matrix described above. Since this matrix only relates the connectivity of the segments of the pendulum system it remains unchanged for the same pendulum system restricted to planar 2-D motion or for that system dynamically allowed to move in 3-D space.

CHAPTER 4

3-D DYNAMIC PENDULUM TREE WITH SUPERFLUOUS COORDINATE

To determine the joint reaction force at the root pivot (origin) of the 3-D pendulum system described in the previous section, the distance d_1 from the origin to the first mass point on root segment will now be considered to be an additional *dynamic variable* of the system. The new dynamic generalized position vector is $\overline{x}(t) = \left(d_1(t), X(t)\right) = \left(d_1(t), \theta(t), \phi(t)\right)$. In effect adding this superfluous variable will allow the tree to "fly-away" from its pivot point. The constraint that d_1 is to be held constant by the reaction force (tension) in the root segment will be applied explicitly after the inertial (motion dependent) and gravitational forces are determined from the new equation of motion that must be added to the pendulum system with superfluous d_1.

The new equation is obtained by applying Lagrange's method in the radial direction, $\hat{r}_1$, of the root segment to obtain,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{d}_1}\right) - \frac{\partial L}{\partial d_1} = 0 \qquad (\hat{r}_1 \ \textit{direction}) \tag{4.1}$$

The pivot reaction force (PRF) must act equal and opposite to the inertial and gravitational force components in this direction to prevent the system from 'flying off' in the radial direction, $\hat{r}_1$. The method is quite analogous to the method described earlier in Section 2.2 to obtain the joint reaction force for a single 3-D pendulum.

4.1 Lagrangian for Pendulum Tree with Superfluous Coordinate

The Lagrangian for the system with superfluous coordinate is obtained from the kinetic and potential energies

$$L\left(\bar{x},\dot{\bar{x}}\right)=K\left(\bar{x},\dot{\bar{x}}\right)-P\left(\bar{x}\right), \tag{4.2}$$

where

$$\bar{x}^T(t)=\left(d_1(t),X(t)\right)=\left(d_1(t),\theta(t),\phi(t)\right),\quad \left(\theta\right)_i=\theta_i(t),\quad \left(\phi\right)_i=\phi_i(t)\quad i=1,\cdots,S\ \ .$$

Expressions for these energies will now be developed for the general pendulum tree system with superfluous coordinate d_1 .

POTENTIAL ENERGY

The potential energy, P, is

$$P=\sum_{i=1}^{P}m_i\ g\ y_i=g\sum_{i=1}^{P}m_iy_i=g\sum_{i=1}^{P}m_i\sum_{j=1}^{S}R_{ij}\sin\theta_j\ =\ g\sum_{j=1}^{S}\left(\sum_{i=1}^{P}m_iR_{ij}\right)\sin\theta_j\ , \tag{4.3}$$

Defining the vector $\underline{M}^T$ of mass points of the system as

$$\underline{M}^T\equiv(m_1,m_2,\cdots m_P)\,, \tag{4.4}$$

then the sum $\sum_{i=1}^{P} m_i R_{ij}$ in Equation (4.3) can be written as the j^{th} component of the vector matrix product $\underline{M}^T R$ where R is the relation matrix previously defined by Equation (3.2).

$$\sum_{i=1}^{P} m_i R_{ij} = \left(\underline{M}^T R\right)_j . \tag{4.5}$$

Therefore, the potential energy can be written in the form

$$P = g\sum_{j=1}^{S}\left(\underline{M}^T R\right)_j \sin\theta_j . \tag{4.6}$$

Although this form for P is the same as for the *3-D* pendulum system without superfluous coordinate, d_1, there is a significant difference. In the pendulum system without superfluous coordinate R is a non-dynamic (time independent) matrix but since R has terms that depend upon the new variable d_1, R is *time dependent* for the pendulum system with superfluous coordinate d_1. This dynamic property of R has significant consequences in the calculations that will follow and the dependence of R on d_1 will now be described more precisely.

Even though the superfluous variable d_1 allows the tree to “fly-away” from its pivot point, this affects only the root segment length L_1. All other segment lengths remain intact and are not changed by making d_1 a variable. Since the root segment of the tree is a forefather to all other branches every length in the first column of R will change with the superfluous coordinate d_1 but all lengths in the other columns of R will be independent of d_1 (see Example in Figure 3.2 with explicit R given by Equation (3.2)). More precisely the dependence of R on the superfluous coordinate d_1 is given by,

$$R_{ij} = \begin{cases} d_1, \quad i=1; \quad j=1 \\ L_{1,} = d_1 + Const, \quad j=1; \quad i=2,\cdots,P \\ L_j \quad (d_1 \text{ independent}), \quad j=2,\cdots,P; \quad i=1,\cdots,P \\ 0, \quad \text{otherwise} \end{cases}, \tag{4.7}$$

The $Const$ in $L_{1,} = d_1 + Const$ represents the remaining fixed length from the variable position of the first mass point to the end of the root segment.

KINETIC ENERGY

The Cartesian velocity of the i^{th} mass point $v_i = \left(\dot{x}_i, \dot{y}_i, \dot{z}_i\right)$ is obtained by differentiating Equation (3.3) with respect to time resulting in

$$\dot{x}_i = \dot{d}_1 \cos\theta_1 \sin\phi_1 + \left(\dot{x}_i\right)_o \tag{4.8}$$

$$\dot{y}_i = \dot{d}_1 \sin\theta_1 + \left(\dot{y}_i\right)_o \tag{4.9}$$

$$\dot{z}_i = \dot{d}_1 \cos\theta_1 \cos\phi_1 + \left(\dot{z}_i\right)_o, \tag{4.10}$$

where

$$\left(\dot{x}_i\right)_o = -\sum_{j=1}^{S} R_{ij} \sin\theta_j \sin\phi_j \,\dot{\theta}_j + \sum_{j=1}^{S} R_{ij} \cos\theta_j \cos\phi_j \,\dot{\phi}_j \tag{4.11}$$

$$\left(\dot{y}_i\right)_o = \sum_{j=1}^{S} R_{ij} \cos\theta_j \,\dot{\theta}_j \tag{4.12}$$

$$\left(\dot{z}_i\right)_o = -\sum_{j=1}^{S} R_{ij} \sin\theta_j \cos\phi_j \,\dot{\theta}_j - \sum_{j=1}^{S} R_{ij} \cos\theta_j \sin\phi_j \,\dot{\phi}_j \,, \tag{4.13}$$

are the velocity components for the system *without* superfluous coordinate .

The velocity squared of the i^{th} mass point is $v_i^2 = \dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2$. Substituting the above Equations (4.8)-(4.10) and simplifying the expression using trigonometric identities results in

$$v_i^2 = (v_i^2)_o + \dot{d}_1^{\,2} + \dot{d}_1 \sum_{j=1}^{S} R_{ij} \begin{bmatrix} \left(\left(\sin C_{1j} + \sin A_{1j} \right) + \frac{1}{2} \sum_{k=1}^{4} \sin\psi_{k_{1j}} \right) \dot{\theta}_j \\ + \frac{1}{2} \sum_{k=1}^{4} \left(-1\right)^k \sin\psi_{k_{1j}} \dot{\phi}_j \end{bmatrix}, \tag{4.14}$$

where

$$\psi_{1_{ij}} = A_{ij} - B_{ij} \,, \quad \psi_{2ij} = A_{ij} + B_{ij} \,, \quad \psi_{3ij} = -C_{ij} - B_{ij} \,, \quad \psi_{4ij} = -C_{ij} + B_{ij} \,, \tag{4.15}$$

and

$$A_{ij} = \theta_i - \theta_j, \quad B_{ij} = \phi_i - \phi_j, \quad C_{ij} = \theta_i + \theta_j, \quad i = 1: \quad j = 1, \cdots, S. \qquad (4.16)$$

Equation (4.14) shows the explicit dependence of v_i^2 on $\dot{d}_1$. The term $(v_i^2)_o$ represents the velocity squared of the i^{th} mass point for the original pendulum system without superfluous coordinate. The kinetic energy of the i^{th} mass point, $K_i = \frac{1}{2} m_i v_i^2$ and the total kinetic energy $K = \sum_{i=1}^{P} K_i$. Substituting Equation (4.14) into the expression for K yields

$$K = K_o + K_n \qquad (4.17)$$

$$K_o = \frac{1}{2} \sum_{i=1}^{P} m_i (v_i^2)_o, \qquad (4.18)$$

$$K_n = \frac{1}{2} m_T \dot{d}_1^2 + \dot{d}_1 \left(u^T \dot{\theta} + q^T \dot{\phi} \right) \qquad (4.19)$$

where

$K_o = \frac{1}{2} \sum_{i=1}^{P} m_i (v_i^2)_o$ is the total kinetic energy of the original system without superfluous coordinate d_1. The two vectors $\vec{u}$ and $\vec{q}$ are defined by,

$$u_j \equiv \left(\underline{M}^T R \right)_j \left[\frac{1}{2} \left(\sin C_{1j} + \sin A_{1j} \right) + \frac{1}{4} \sum_{k=1}^{4} \sin \psi_{k_{1j}} \right], \qquad (4.20)$$

$$q_j \equiv \frac{1}{4}\left(\underline{M}^T R\right)_j \sum_{k=1}^{4}\left(-1\right)^k \sin\psi_{k_{1j}}, \quad j = 1,\ldots,S \tag{4.21}$$

$m_T \equiv \sum_{i=1}^{P} m_i$ is the total mass of the system and the vectors $\underline{M}^T$ and $\underline{M}^T R$ are defined as in Equations (4.4)-(4.5).

The kinetic energy, K_o, of the branched 3D-pendulum tree system without superfluous coordinate can be expressed in terms of the generalized coordinates (Lacker, Unpublished).

$$K_0\left(X,V\right) = \frac{1}{2}V^T M\left(X\right)V, \tag{4.22}$$

where

$$\begin{gathered} X^T \equiv (\theta,\phi), \quad V^T \equiv (\dot{\theta},\dot{\phi}), \quad \left(\theta\right)_j \equiv \theta_j, \\ \left(\dot{\theta}\right)_j \equiv \dot{\theta}_j, \quad \left(\phi\right)_j \equiv \phi_j, \quad \left(\dot{\phi}\right)_j \equiv \dot{\phi}_j, \quad j = 1,\cdots,S \end{gathered} \tag{4.23}$$

$$\begin{gathered} M\left(X\right) = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}, \quad M_{IJ\,i\,j}, \\ I = 1,2;\ J = 1,2; \quad i = 1,\cdots,S;\ j = 1,\cdots,S \end{gathered} \tag{4.24}$$

$$M_{11_{i\,j}} = \bar{C}_{i\,j}\mu_{11_{i\,j}} = M_{11_{j\,i}}\ (M_{11} \text{ is } \textit{symmetric}) \tag{4.25}$$

$$M_{12_{i\,j}} = \bar{C}_{i\,j}\mu_{12_{i\,j}} = M_{21_{j\,i}} \tag{4.26}$$

$$M_{21_{ij}} = \bar{C}_{ij}\mu_{21_{ij}} \tag{4.27}$$

$$M_{22_{ij}} = \bar{C}_{ij}\mu_{22_{ij}} = M_{22_{ji}} (M_{22} \text{ is } \textit{symmetric}) \tag{4.28}$$

$$\bar{C} = R^T M_D R \quad (\textit{symmetric}) \tag{4.29}$$

$$(M)_{kl} = (M)_{lk}, l = 1,\cdots,2S; k = 1,\cdots,2S \quad (M \text{ is } \textit{symmetric}) \tag{4.30}$$

$$(M_D)_{nq} = \begin{cases} m_i, & n = q \\ 0, & n \neq q \end{cases}, n,q = 1,\cdots,P \; (\textit{diagonal}). \tag{4.31}$$

.

The μ submatrices are

$$\begin{aligned}\mu_{11_{ij}} = \frac{1}{2}\left[\cos(\theta_i + \theta_j) + \cos(\theta_i - \theta_j)\right] \\ + \frac{1}{4}\left(\cos\psi_{1_{ij}} + \cos\psi_{2_{ij}} - \cos\psi_{3_{ij}} - \cos\psi_{4_{ij}}\right)\end{aligned} \; (\textit{symmetric}) \tag{4.32}$$

$$\mu_{12_{ij}} = \frac{1}{4}\left[\sum_{k=1}^{4}(-1)^k \cos\psi_{k_{ij}}\right] = \mu_{21_{ji}} \tag{4.33}$$

$$\mu_{22_{ij}} = \frac{1}{4}\left[\sum_{k=1}^{4}\cos\psi_{k_{ij}}\right] (\textit{symmetric}), \tag{4.34}$$

where ψ_{kij} are defined as in Equation (4.15).

In summary, the Lagrangian for the system with superfluous coordinate d_1 is

$$L\left(\bar{x},\dot{\bar{x}}\right)=K-P=K_o+\frac{1}{2}m_T\dot{d}_1^{\,2}+\dot{d}_1\left(u^T\dot{\theta}+q^T\dot{\phi}\right)-g\sum_{j=1}^{S}\left(\underline{M}^T R\right)_j\ \sin\theta_j \qquad (4.35)$$

As was the case for *P,* the form for K_o is the same as for the *3-D* pendulum system without superfluous coordinate but with the superfluous coordinate, d_1, R becomes time dependent.

4.2 Lagrange's Equation of Motion Applied to the Superfluous Variable

As explained in the section introducing this chapter and also in Section 2.2, using the method of superfluous coordinates to determine the reaction force at the root pivot requires that Lagrange's equation of motion be applied to the superfluous component, Equation (4.1), where the Lagrangian *L* is that given by Equation (4.35) above. The derivation will proceed by calculating each component of Equation (4.1) separately.

CALCULATION OF $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{d}_1}\right)$

Taking the partial derivative of *L =K-P*, with respect to $\dot{d}_1$ gives

$$\frac{\partial L}{\partial \dot{d}_1}=\frac{\partial K}{\partial \dot{d}_1}-\frac{\partial P}{\partial \dot{d}_1}=\left(\frac{\partial K_o}{\partial \dot{d}_1}+\frac{\partial K_n}{\partial \dot{d}_1}\right)-\frac{\partial P}{\partial \dot{d}_1}, \qquad (4.36)$$

where P, K_{o} and K_n are defined as in Equations (4.3),(4.18) and (4.19). Note that in Equation (4.6) the potential energy does not depend upon the speed $\dot{d}_1$, therefore $\dfrac{\partial P}{\partial \dot{d}_1} = 0$.

Furthermore although the kinetic energy, K_o, depends on d_1 it is also independent of $\dot{d}_1$, thus, Equation (4.36) can be expressed as,

$$\left(\frac{\partial L}{\partial \dot{d}_1}\right) = \frac{\partial K_n}{\partial \dot{d}_1} = m_T \dot{d}_1 + u^T \dot{\theta} + q^T \dot{\phi} \tag{4.37}$$

Taking the derivative of Equation (4.37) with respect to time, gives

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{d}_1}\right) = m_T \ddot{d}_1 + u^T \ddot{\theta} + q^T \ddot{\phi} + \dot{u}^T \dot{\theta} + \dot{q}^T \dot{\phi} \tag{4.38}$$

Taking the time derivative of Equation (4.20) gives

$$\begin{aligned} \dot{u}_i = \left(\frac{d}{dt}u\right)_i &= \left(\underline{M}^T R\right)_i \left[\frac{1}{2}\left(\dot{C}_{1i} \cos C_{1i} + \dot{A}_{1i} \cos A_{1i}\right) + \frac{1}{4}\sum_{k=1}^{4} \dot{\psi}_{k_{1i}} \cos \psi_{k_{1i}}\right] \\ &+ \left[\frac{1}{2}\left(\sin C_{1i} + \sin A_{1i}\right) + \frac{1}{4}\sum_{k=1}^{4} \psi_{k_{1i}}\right] \frac{d}{dt}\left(\underline{M}^T R\right)_i, \quad i = 1, \ldots, S \end{aligned} \tag{4.39}$$

Since,

$$\frac{d}{dt}\left(\underline{M}^{T}R\right)_{i} = \frac{\partial}{\partial d_{1}}\left(\underline{M}^{T}R_{i}\right)\dot{d}_{1}, \tag{4.40}$$

and,

$$\frac{\partial}{\partial d_{1}}\left(\underline{M}^{T}R\right)_{i} = \frac{\partial}{\partial d_{1}}\left(\sum_{j=1}^{p} m_{j}R_{ji}\right) = \left(\sum_{j=1}^{p} m_{j}\frac{\partial R_{ji}}{\partial d_{1}}\right) \quad . \tag{4.41}$$

As described in Section 4.1 and Equation (4.7) only the first column of R has d_1 in each of its terms so

$$\frac{\partial R_{ji}}{\partial d_{1}} = \begin{cases} 1, & i = 1 \\ 0, & i \neq 1 \end{cases} \quad , \tag{4.42}$$

$$\frac{\partial}{\partial d_{1}}\left(\underline{M}^{T}R\right)_{i} = \sum_{j=1}^{p} m_{j}\frac{\partial R_{ji}}{\partial d_{1}} = \begin{cases} \sum_{j=1}^{p} m_{j} = m_{T}, & i = 1 \\ 0, & i = 2,\cdots,S \end{cases} \tag{4.43}$$

but when i=*1*, $\frac{1}{2}\left(\sin C_{1i} + \sin A_{1i}\right) + \frac{1}{4}\sum_{k=1}^{4}\psi_{k_{1i}} = 0$ which implies that

$$\left[\frac{1}{2}\left(\sin C_{1i} + \sin A_{1i}\right) + \frac{1}{4}\sum_{k=1}^{4}\psi_{k_{1i}}\right]\frac{d}{dt}\left(\underline{M}^{T}R\right)_{i} = 0 \quad \forall\, i\,. \tag{4.44}$$

Therefore, the vector $\dot{u}$ simplifies to

$$\dot{u}_i = \left(\underline{M}^T R\right)_i \left[\frac{1}{2}\left(\dot{C}_{1i} \cos C_{1i} + \dot{A}_{1i} \cos A_{1i}\right) + \frac{1}{4}\sum_{k=1}^{4} \dot{\psi}_{k_{1i}} \cos\psi_{k_{1i}}\right], \quad . \qquad (4.45)$$
$$i = 1, \dots, S$$

A similar argument that was used above for $\dot{u}$ can be applied to the vector $\dot{q}$ giving

$$\dot{q}_i = \frac{1}{4}\left(\underline{M}^T R\right)_i \sum_{k=1}^{4} (-1)^k \dot{\psi}_{k_{1i}} \cos\psi_{k_{1i}}, \quad i = 1, \dots, S\,. \qquad (4.46)$$

Summarizing the results for the first term of Equation (4.1),

$$
\begin{aligned}
&\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{d}_1}\right) = m_T \ddot{d}_1 + u^T \ddot{\theta} + q^T \ddot{\phi} + \dot{u}^T \dot{\theta} + \dot{q}^T \dot{\phi}, \\
&\text{where the vectors } u, \dot{u}, q, \dot{q} \text{ are} \\
&u_i = \left(\underline{M}^T R\right)_i \left[\frac{1}{2}\left(\sin C_{1i} + \sin A_{1i}\right) + \frac{1}{4}\sum_{k=1}^{4} \sin \psi_{k_{1i}}\right], \\
&\dot{u}_i = \left(\underline{M}^T R\right)_i \left[\frac{1}{2}\left(\dot{C}_{1i} \cos C_{1i} + \dot{A}_{1i} \cos A_{1i}\right) + \frac{1}{4}\sum_{k=1}^{4} \dot{\psi}_{k_{1i}} \cos \psi_{k_{1i}}\right], \\
&q_i = \frac{1}{4}\left(\underline{M}^T R\right)_i \sum_{k=1}^{4} (-1)^k \sin \psi_{k_{1i}}, \\
&\dot{q}_i = \frac{1}{4}\left(\underline{M}^T R\right)_i \sum_{k=1}^{4} (-1)^k \dot{\psi}_{k_{1i}} \cos \psi_{k_{1i}}, \quad i = 1, \ldots, S \\
&\text{and} \\
&\psi_{k_{1i}} = \begin{cases} \theta_1 - \theta_i - \phi_1 + \phi_i, & k = 1 \\ \theta_1 - \theta_i + \phi_1 - \phi_i, & k = 2 \\ -\theta_1 - \theta_i - \phi_1 + \phi_i, & k = 3 \\ -\theta_1 - \theta_i + \phi_1 - \phi_i, & k = 4 \end{cases}, \quad
\dot{\psi}_{k_{1i}} = \begin{cases} \dot{\theta}_1 - \dot{\theta}_i - \dot{\phi}_1 + \dot{\phi}_i, & k = 1 \\ \dot{\theta}_1 - \dot{\theta}_i + \dot{\phi}_1 - \dot{\phi}_i, & k = 2 \\ -\dot{\theta}_1 - \dot{\theta}_i - \dot{\phi}_1 + \dot{\phi}_i, & k = 3 \\ -\dot{\theta}_1 - \dot{\theta}_i + \dot{\phi}_1 - \dot{\phi}_i, & k = 4 \end{cases}. \\
&A_{1i} = \theta_1 - \theta_i, \quad B_{1i} = \phi_1 - \phi_i, \quad C_{1i} = \theta_1 + \theta_i \\
&\dot{A}_{1i} = \dot{\theta}_1 - \dot{\theta}_i, \quad \dot{B}_{1i} = \dot{\phi}_1 - \dot{\phi}_i, \quad \dot{C}_{1i} = \dot{\theta}_1 + \dot{\theta}_i
\end{aligned}
\quad (4.47)
$$

The vectors $\underline{M}^T$ and $\underline{M}^T R$ are defined as in Equations (4.4), (4.5) and m_T is the total mass of the system.

CALCULATION OF $\frac{\partial L}{\partial d_1}$

Since,

$$\frac{\partial L}{\partial d_1} = \frac{\partial K}{\partial d_1} - \frac{\partial P}{\partial d_1}, \qquad (4.48)$$

each term will be calculated separately, starting with $\frac{\partial P}{\partial d_1}$.

Taking the partial derivative of Equation (4.6) yields

$$\frac{\partial P}{\partial d_1} = g\sum_{j=1}^{S} \frac{\partial \left(\underline{M}^T R \right)_j}{\partial d_1} \sin\theta_j \ . \qquad (4.49)$$

Substituting Equation (4.5) into the above expression and exchanging the order of summation produces

$$\frac{\partial P}{\partial d_1} = g\sum_{k=1}^{P} m_k \sum_{j=1}^{S} \frac{\partial R_{kj}}{\partial d_1} \ \sin\theta_j \ . \qquad (4.50)$$

Using Equation (4.42) gives the final form

$$\boxed{\frac{\partial P}{\partial d_1} = m_T g \sin \theta_1}. \tag{4.51}$$

Physically Equation (4.51) represents that component of the total gravitational force (weight) of the system that is in the radial direction of the root segment $\hat{r}_1$.

The second term of Equation (4.48) $\frac{\partial K}{\partial d_1}$ will now be considered. Taking the partial derivative of Equation (4.17) with respect to the superfluous coordinate yields

$$\frac{\partial K}{\partial d_1} = \frac{\partial K_o}{\partial d_1} + \frac{\partial K_n}{\partial d_1} = \frac{\partial K_o}{\partial d_1} + \dot{d}_1 \left(\left(\frac{\partial u}{\partial d_1} \right)^T \dot{\theta} + \left(\frac{\partial q}{\partial d_1} \right)^T \dot{\phi} \right) \tag{4.52}$$

Taking the partial derivative of Equations (4.20) and (4.21) with respect to the superfluous coordinate produces

$$\frac{\partial u_i}{\partial d_1} = \frac{\partial \left(\underline{M}^T R \right)_i}{\partial d_1} \left[\frac{1}{2} \left(\sin C_{1i} + \sin A_{1i} \right) + \frac{1}{4} \sum_{k=1}^{4} \sin \psi_{k_{1i}} \right], \tag{4.53}$$

$$\frac{\partial q_i}{\partial d_1} = \frac{\partial \left(\underline{M}^T R \right)_i}{\partial d_1} \left[\frac{1}{4} \sum_{k=1}^{4} (-1)^k \sin \psi_{k_{1i}} \right], \quad i = 1, \ldots, S \tag{4.54}$$

It has been shown in Equation (4.43) that all components of $\frac{\partial}{\partial d_1} \left(\underline{M}^T R \right)_i = 0$ except for the first, $i = 1$, component but when $i = 1$, $\sum_{k=1}^{4} (-1)^k \sin \psi_{k_{1i}} = 0$ and

$\frac{1}{2}\left(\sin C_{1i} + \sin A_{1i}\right) + \frac{1}{4}\sum_{k=1}^{4}\sin\psi_{k_{1i}} = 0$. Therefore, the vectors $\frac{\partial u}{\partial d_1}$ and $\frac{\partial q}{\partial d_1}$ are both equal to $\vec{0}$ and Equation (4.52) above reduces to

$$\frac{\partial K}{\partial d_1} = \frac{\partial K_o}{\partial d_1}\,. \tag{4.55}$$

Taking the partial derivative of Equation (4.22) with respect to the superfluous coordinate yields

$$\frac{\partial K}{\partial d_1} = \frac{\partial K_o}{\partial d_1} = \frac{1}{2}V^T\,\frac{\partial M}{\partial d_1}\,V = \frac{1}{2}(\dot{\theta},\dot{\phi})\begin{pmatrix} \frac{\partial M_{11}}{\partial d_1} & \frac{\partial M_{12}}{\partial d_1} \\ \frac{\partial M_{21}}{\partial d_1} & \frac{\partial M_{22}}{\partial d_1} \end{pmatrix}\begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} \tag{4.56}$$

Expanding the quadratic form gives

$$\frac{\partial K}{\partial d_1} = \frac{1}{2}\left(\dot{\theta}^T\,\frac{\partial M_{11}}{\partial d_1}\,\dot{\theta} + \dot{\theta}^T\,\frac{\partial M_{12}}{\partial d_1}\,\dot{\phi} + \dot{\phi}^T\,\frac{\partial M_{21}}{\partial d_1}\,\dot{\theta} + \dot{\phi}^T\,\frac{\partial M_{22}}{\partial d_1}\,\dot{\phi}\right). \tag{4.57}$$

The dependency of each submatrix $\frac{\partial M_{IJ}}{\partial d_1}$, $I = 1,2;\ J = 1,2;$ on the superfluous variable d_1 occurs through

$$\frac{\partial \bar{C}_{i\,j}}{\partial d_1} = \frac{\partial}{\partial d_1}\left(R^T M_D R\right) = \sum_{k=1}^{P} m_k \frac{\partial\left(R_{ki} R_{kj}\right)}{\partial d_1} = \sum_{k=1}^{P} m_k R_{ki} \frac{\partial R_{kj}}{\partial d_1} + \sum_{k=1}^{P} m_k R_{kj} \frac{\partial R_{ki}}{\partial d_1} \quad . \qquad (4.58)$$

$$i = 1, \cdots, S; \; j = 1, \cdots, S$$

Applying Equation (4.58) above to the first term in Equation (4.57) gives

$$\dot{\theta}^T \frac{\partial M_{11}}{\partial d_1} \dot{\theta} = \Psi_1 + \Psi_2, \qquad (4.59)$$

where,

$$\Psi_1 \equiv \sum_{i=1}^{S} \dot{\theta}_i \sum_{j=1}^{S} \dot{\theta}_j \mu_{11ij} \sum_{k=1}^{P} m_k R_{ki} \frac{\partial R_{kj}}{\partial d_1} \qquad (4.60)$$

$$\Psi_2 \equiv \sum_{i=1}^{S} \dot{\theta}_i \sum_{j=1}^{S} \dot{\theta}_j \mu_{11ij} \sum_{k=1}^{P} m_k R_{kj} \frac{\partial R_{ki}}{\partial d_1} \quad . \qquad (4.61)$$

Using Equation (4.42) in the form,

$$\frac{\partial R_{kj}}{\partial d_1} = \begin{cases} 1, & j = 1; \quad k = 1, \cdots, P \\ 0, & j \neq 1; \quad k = 1, \cdots, P \end{cases} . \qquad (4.62)$$

and summing Equation (4.60) over j yields,

$$\Psi_1 = \dot{\theta}_1 \sum_{i=1}^{S} \dot{\theta}_i \mu_{11_{i1}} \sum_{k=1}^{P} m_k R_{ki} \ . \tag{4.63}$$

Substituting Equation (4.5) into the equation above gives

$$\Psi_1 = \dot{\theta}_1 \sum_{i=1}^{S} \dot{\theta}_i \left(\underline{M}^T R \right)_i \mu_{11_{i1}} \tag{4.64}$$

Summing Equation (4.61) over i and applying Equation (4.62) in the form

$$\frac{\partial R_{ki}}{\partial d_1} = \begin{cases} 1, & i = 1; \quad k = 1, \cdots, P \\ 0, & i \neq 1; \quad k = 1, \cdots, P \end{cases} \text{ yields,} \tag{4.65}$$

$$\Psi_2 = \dot{\theta}_1 \sum_{i=1}^{S} \dot{\theta}_i \left(\underline{M}^T R \right)_i \mu_{11_{1i}} \ . \tag{4.66}$$

Since μ_{11} is symmetric $\mu_{11_{i1}} = \mu_{11_{1i}}$ (see Equation (4.32)), the first term of the quadratic form in Equation (4.57) evaluates to

$$\dot{\theta}^T \frac{\partial M_{11}}{\partial d_1} \dot{\theta} = \Psi_1 + \Psi_2 = 2 \dot{\theta}_1 \sum_{i=1}^{S} \dot{\theta}_i \left(\underline{M}^T R \right)_i \mu_{11_{i1}} \ . \tag{4.67}$$

Using the same procedure to evaluate the remaining terms of Equation (4.57) gives the following result

$$\boxed{\begin{aligned}\frac{\partial K}{\partial d_1} &= \dot{\theta}_1 \sum_{i=1}^{S} \dot{\theta}_i \left(\underline{M}^T R\right)_i \mu_{11_{i1}} + \dot{\theta}_1 \sum_{i=1}^{S} \dot{\phi}_i \left(\underline{M}^T R\right)_i \mu_{12_{1i}} \\ &+ \dot{\phi}_1 \sum_{i=1}^{S} \dot{\theta}_i \left(\underline{M}^T R\right)_i \mu_{12_{i1}} + \dot{\phi}_1 \sum_{i=1}^{S} \dot{\phi}_i \left(\underline{M}^T R\right)_i \mu_{22_{i1}}\end{aligned}}, \qquad (4.68)$$

where the submatrices are defined as Equations (4.32) – (4.34) and where ψ_{kij} are defined as in Equations (4.15) – (4.16) with $j=1$.

CHAPTER 5

PIVOT REACTION FORCE FOR THE 3-D PENDULUM TREE SYSTEM

Now that both terms in Lagrange's equation of motion for the superfluous component (Equation (4.1)) are evaluated, it can be used to determine the dynamic equation for the pivot reaction force of the 3-D pendulum tree. Substitution of Equations (4.47), (4.48) and (4.51) into Equation (4.1) gives

$$m_T \ddot{d}_1 + u^T \ddot{\theta} + q^T \ddot{\phi} = \frac{\partial K}{\partial d_1} - \left(\dot{u}^T \dot{\theta} + \dot{q}^T \dot{\phi} \right) - m_T g \sin \theta_1 . \tag{5.1}$$

Note that both $\dot{u}^T \dot{\theta}$ and $\dot{q}^T \dot{\phi}$ in Equation (5.1) are each composed of quadratic terms in the velocities $\dot{\theta}_i, \dot{\phi}_i$ (see summary Equation (4.47)). Terms with the same quadratic velocity components can be collected and compared to similar terms in Equation (4.68) for $\frac{\partial K}{\partial d_1}$. The result produces considerable cancellation and simplification so that the RHS of Equation (5.1) can be written in the form

$$\frac{\partial K}{\partial d_1} - \left(\dot{u}^T \dot{\theta} + \dot{q}^T \dot{\phi} \right) - m_T g \sin \theta_1 = w_1^T \dot{\theta}^2 + w_2^T \dot{\theta} \dot{\phi} + w_3^T \dot{\phi}^2 - m_T g \sin \theta_1 \tag{5.2}$$

where the vectors $\vec{w}_1, \vec{w}_2, \vec{w}_3$ are defined by

$$w_{1_i} = \frac{1}{2}\left(\underline{M}^T R\right)_i \left[\cos\left(\theta_1 - \theta_i\right) - \cos\left(\theta_1 + \theta_i\right) + \frac{1}{2}\left(\sum_{k=1}^{4} \cos\psi_{k_{i1}}\right)\right] \tag{5.3}$$

$$w_{2_i} = \frac{1}{2}\left(\underline{M}^T R\right)_i \left[\sum_{k=1}^{4} (-1)^k \cos\psi_{k_{i1}}\right] \tag{5.4}$$

$$w_{3_i} = \frac{1}{4}\left(\underline{M}^T R\right)_i \left[\sum_{k=1}^{4} \cos\psi_{k_{i1}}\right] \qquad i = 1, \ldots, S\,. \tag{5.5}$$

Lagrange's EOM for the superfluous component (Equation (4.1)) in the radial direction of the root segment $\hat{r}_1$ is therefore

$$\boxed{m_T \ddot{d}_1 + u^T \ddot{\theta} + q^T \ddot{\phi} = -m_T g \sin\theta_1 + w_1^{\,T} \dot{\theta}^2 + w_2^{\,T} \dot{\theta}\dot{\phi} + w_3^{\,T} \dot{\phi}^2} \tag{5.6}$$

5.1 Physical Interpretation of the Pivot Reaction Force Equation for the 3-D Pendulum Tree System

To interpret Equation (5.6) physically it is helpful to write it in the following form

$$m_T \ddot{d}_1 = -m_T g \sin\theta_1 + w_1^{\,T} \dot{\theta}^2 + w_2^{\,T} \dot{\theta}\dot{\phi} + w_3^{\,T} \dot{\phi}^2 - u^T \ddot{\theta} - q^T \ddot{\phi}\,. \tag{5.7}$$

The RHS of Equation (5.7) reveals the force terms that are acting on the pendulum system in the $\hat{r}_1$ direction to extend or contract the distance d_1 from the pivot.

Besides the first term which represents the conservative gravitational force component, all of the remaining force terms are motion dependent. When the pendulum system is at rest, these inertial forces disappear leaving only the gravitational force component to change d_1. The first three inertial force terms are quadratic in angular velocity (centrifugal) while the last two depend on angular acceleration.

To maintain the constant length of root segment in the pendulum tree a reactive constraint force, F_C must be applied that is equal and opposite to the net force that is acting to change the distance d_1. This reactive constraint force is the tension in the idealized rod that represents the root segment. Explicitly imposing the constraint condition that d_1 is constant implies that both $\dot{d}_1$ $\dot{r} = \ddot{r} = 0$ and $\ddot{d}_1$ are both 0. Adding the pivot reaction force term F_C that maintains the constraint to Equation (5.7) gives the equation for the pivot reaction force (PRF),

$$-m_T g \sin\theta_1 + w_1^T \dot{\theta}^2 + w_2^T \dot{\theta}\dot{\phi} + w_3^T \dot{\phi}^2 - u^T \ddot{\theta} - q^T \ddot{\phi} + F_c = m_T \ddot{d}_1 = 0 \qquad (5.8)$$

or

$$F_c = m_T g \sin\theta_1 - w_1^T \dot{\theta}^2 - w_2^T \dot{\theta}\dot{\phi} - w_3^T \dot{\phi}^2 + u^T \ddot{\theta} + q^T \ddot{\phi}\,. \qquad (5.9)$$

Since Equation (5.9) represents the final form for the PRF it will be written now with all terms previously defined collected here for future reference:

$$F_c = m_T g \sin\theta_1 - w_1^T \dot{\theta}^2 - w_2^T \dot{\theta}\dot{\phi} - w_3^T \dot{\phi}^2 + u^T \ddot{\theta} + q^T \ddot{\phi}$$

where the vectors $w_1, w_2, \mathrm{w}_3, u, q, \dot{\theta}^2, \dot{\theta}\dot{\phi}, \ \dot{\phi}^2, \ddot{\theta}, \ddot{\phi}$ are defined by

$$w_{1_i} = \frac{1}{2}\left(\underline{M}^T R\right)_i \left[\cos\left(\theta_1 - \theta_i\right) - \cos\left(\theta_1 + \theta_i\right) + \frac{1}{2}\left(\sum_{k=1}^{4} \cos\psi_{k_{i1}}\right)\right]$$

$$w_{2_i} = \frac{1}{2}\left(\underline{M}^T R\right)_i \left[\sum_{k=1}^{4} (-1)^k \cos\psi_{k_{i1}}\right]$$

$$w_{3_i} = \frac{1}{4}\left(\underline{M}^T R\right)_i \left[\sum_{k=1}^{4} \cos\psi_{k_{i1}}\right]$$

$$u_i = \left(\underline{M}^T R\right)_i \left[\frac{1}{2}\left(\sin\left(\theta_1 - \theta_i\right) + \sin\left(\theta_1 + \theta_i\right)\right) + \frac{1}{4}\sum_{k=1}^{4} \sin\psi_{k_{1i}}\right]$$

$$q_i = \frac{1}{4}\left(\underline{M}^T R\right)_i \sum_{k=1}^{4} (-1)^k \sin\psi_{k_{1i}}$$

$$\left(\dot{\theta}^2\right)_i = \dot{\theta}_i^{\,2}, \left(\dot{\theta}\dot{\phi}\right)_i = \dot{\theta}_i \dot{\phi}_i, \left(\dot{\theta}^2\right)_i = \dot{\phi}_i^{\,2}, \left(\ddot{\theta}\right)_i = \ddot{\theta}_i, \left(\ddot{\phi}\right)_i = \ddot{\phi}_i$$

$$\psi_{1_{ij}} = A_{ij} - B_{ij}, \psi_{2_{ij}} = A_{ij} + B_{ij}, \psi_{3_{ij}} = -C_{ij} - B_{ij}, \psi_{4_{ij}} = -C_{ij} + B_{ij}$$

$$A_{ij} = \theta_i - \theta_j, \quad B_{ij} = \phi_i - \phi_j, \quad C_{ij} = \theta_i + \theta_j, \quad i = 1, \cdots, S: \quad j = 1, \cdots, S$$

$$\underline{M}^T = (m_1, m_2, \cdots m_P)$$

$$R_{ij} = \begin{cases} d_i & \text{if } i^{th} \text{ mass point is on (or belongs to) the } j^{th} \text{ segment,} \\ L_j & \text{if the } j^{th} \text{ segment is a forefather segment of the segment} \\ & \text{that contains the } i^{th} \text{ mass point,} \\ 0 & \text{otherwise.} \end{cases} \tag{5.10}$$

5.2 Consistency Checks of the Pivot Reaction Force Equation for the 3-D Pendulum Tree System

Consider the special case when the number of segments S in the pendulum tree system is reduced to one and where there is only one mass point of mass m at the end of the segment whose length is l Equation (5.10) should then simplify to PRF for a single 3-D pendulum that was derived in Section 2.2, (Equation (2.10) where $m_T = m_1 = m, l_1 = l, \theta_1 = \theta, \phi_1 = \phi$. In this special case $\underline{M} = m, R = l, \underline{M}^T R = ml$. In this case the vectors u, q, w_2 evaluate to 0 and $w_1 = ml, w_3 = ml\cos^2\theta$. Therefore, Equation (5.10) reduces to

$$F_c = mg\sin\theta - ml\dot{\theta}^2 - ml\left(\cos^2\theta\right)\dot{\phi}^2 \ , \tag{5.11}$$

which is the same as Equation (2.10) in Section 2.2. If the pendulum is hanging down at rest in its stable equilibrium position $\theta = -\frac{\pi}{2}$ then, $F_c = -mg$. Since the radial direction of the root segment $\hat{r}_1$ is pointing down in this configuration therefore F_c is upward, equal and opposite to the mass's weight. If the single pendulum motion is constrained to a 2-D motion in the x,y plane, then $\dot{\phi} = 0$ and Equation (5.10) reduces to

$$F_c = mg\sin\theta - ml\dot{\theta}^2 \tag{5.12}$$

Now the force of constraint is dynamic and must oppose both the gravitational component and the outward centrifugal force. The gravitational component can be either outward (positive) or inward (negative) relative to $\hat{r}_1(t)$ depending on the pendulum's dynamic angular position $\theta(t)$. If the motion is restricted to be in the x,z plane, then $\theta = \dot{\theta} = 0$ and $F_c = -ml\dot{\phi}^2$ acts only to oppose the outward centrifugal force. If the motion is uniform circular motion, then $\dot{\phi}$ and F_c are constant and F_c has the correct magnitude and direction.

A more complicated special case is to consider a general 2-D pendulum tree with S segments but with those segments restricted to move in the x,y plane only, that is, for the case when $\phi_i = \dot{\phi}_i = \ddot{\phi}_i = 0 \; \forall \; i = 1, \cdots, S$. For this special case, w_2 and q are both 0 as are the terms $w_3{}^T \dot{\phi}^2$ and $q^T \ddot{\phi}$. Equation (5.10) simplifies to

$$F_c = m_T g \sin\theta_1 - w_1{}^T \dot{\theta}^2 + u^T \ddot{\theta} \quad , \tag{5.13}$$

where the vectors w_1 and u also simplify to

$$w_{1_i} = \left(\underline{M}^T R\right)_i \cos\left(\theta_1 - \theta_i\right), \tag{5.14}$$

$$u_i = \frac{1}{4}\left(\underline{M}^T R\right)_i \sin\left(\theta_1 - \theta_i\right). \tag{5.15}$$

This is the same result found in the page 30, Equation (4.20), (Rajai, 2007) for the general 2-D pendulum tree.

5.3 Dynamic Solution of the Pivot Reaction Force for the 3-D Pendulum Tree System

As the previous section illustrates, Equation (5.10) for the PRF does not yet represent an explicit dynamic solution for $F_c(t)$ until the dynamic functions $\theta_i(t), \phi_i(t)$ are specified for each of the $i = 1, \cdots, S$ segments of the general 3-D pendulum tree. The dynamic functions $\theta_i(t), \phi_i(t)$ can be theoretically predicted by solving Lagrange's Differential

EOM for the 3-D pendulum tree system without the superfluous coordinate. The system of differential equations can be written in the form

$$M\ddot{X} = F_{Grav} + F_{Inertial}, \tag{5.16}$$

where M is the generalized mass matrix used in Equation (4.22) to express the system kinetic energy K , $\ddot{X}^T = \left(\ddot{\Theta}, \ddot{\Phi}\right)$ is the generalized acceleration vector with 2S components F_{Grav} is the generalized gravitational force (torque)

$$F_{iGrav} = \begin{cases} -g\left(m^T R\right)_i \cos\theta_i & i = 1,\cdots,S \\ 0, & i = S+1,\cdots,2S \end{cases} \tag{5.17}$$

and $F_{Inertial} = \begin{pmatrix} F1_{Inertial_i}, & i = 1,\cdots,S \\ F2_{Inertial_i}, & i = 1,\cdots,S \end{pmatrix}$ represents a vector of generalized inertial (centrifugal) force terms quadratic in the angular velocity components $\dot{\theta}_i(t), \dot{\phi}_i(t)$ (Lacker, unpublished).

Equation (5.16) can be solved numerically by standard methods used to solve systems of ordinary differential equations with suitable initial value or boundary conditions (see, for example, Numerical Recipes (Rajai, 2007)).

CHAPTER 6

GENERAL JOINT REACTION FORCE (JRF) FOR THE 3-D PENDULUM TREE

Consider the J^{th} joint of the pendulum tree system. If it is not a branch point of the system, then all segments distal to it and descended from it (children, grandchildren…) form a sub-tree of the system. In general, this sub-tree will have $\underline{S}$ segments and $\underline{P}$ mass points distributed on it. The immediate distal segment whose proximal end attaches to the J^{th} joint forms the root segment of this sub-tree. The distance from the J^{th} joint to the first mass point on this sub-tree root segment can be treated as a superfluous coordinate in the same way that d_1 was treated for the root pivot joint of the whole tree.

In effect, the J^{th} joint has become the *root pivot joint for this sub-tree of the J^{th} joint*. Associated with this sub-tree of the J^{th} joint is its own relation matrix of segment connectivity $\underline{R}^{(J)}$ with its own superfluous coordinate $d_1^{[J]}$ that allows the sub-tree to "fly away" from the J^{th} joint. The $\underline{R}^{(J)}$ matrix is simply a particular submatrix of the general relation matrix of the whole tree R. The superscript J of $\underline{R}^{[J]}$ and $d_1^{[J]}$ does not here represent power multiplication but rather is simply a way to denote which joint in the tree is to be associated with this particular submatrix and its corresponding superfluous coordinate. The brackets around the superscript will be used to distinguish it from the usual meaning of multiplication to the J^{th} power. The submatrix $\underline{R}^{[J]}$ is in general a rectangular matrix with $\underline{P}$ rows and $\underline{S}$ columns.

The submatrix $\underline{R}^{[J]}$ is in every way analogous to R except that it refers only to the sub-tree associated with the J^{th} joint. In particular, only the first column of $\underline{R}^{[J]}$ will have

$d_1^{[J]}$ dependence. Because all other entries of $\underline{R}^{[J]}$ are independent of the superfluous coordinate $d_1^{[J]}$, the derivative

$$\frac{\partial \underline{R}_{ki}^{[J]}}{\partial d_1^{[J]}} = \begin{cases} 1, & i = 1; \quad k = 1, \cdots, \underline{P} \\ 0, & i \neq 1; \quad k = 1, \cdots, \underline{P} \end{cases}, i = 1, \cdots, \underline{S} \ , \tag{6.1}$$

which is analogous to Equation (2.1) for the whole tree. All the steps that were used to obtain the PRF for the whole tree will proceed to yield the same reaction force equation for the tension in the root segment of the sub-tree associated with the J^{th} joint with $\underline{R}^{[J]}$ replacing R in Equation (5.10) and with the dynamic angle variables renamed to refer to the corresponding segments of the sub-tree associated with the J^{th} joint. The reactive tension in the root segment of this sub-tree of the J^{th} joint will be called PRF_J. In a simple non-branching joint, it is the dynamic tension in the connecting segment just distal to the J^{th} joint and is given by Equation (5.10) with suitable replacement for R and renaming of the angle variables as just described.

However, there is a significant difference between the pivot root of the whole tree and that of the J^{th} joint 'root' of the sub-tree. The pivot root of the whole tree is fixed to the ceiling or wall of the lab frame and does not move with respect to it. The J^{th} joint is not fixed but rather is a moving part of the dynamic system. It will move due to the reactive tension in the connecting segment that is just proximal to it. *The joint reaction force (JRF) on a simple non-branching joint of the system will be the vector sum of the two reactive tensions in the connecting segments just distal and proximal to it.*

The dynamic tension force in any given segment at each of its ends, is equal and opposite. Therefore, the reactive tension exerted on the J^{th} joint from its proximal segment must be equal and opposite to the reactive tension exerted on the joint just proximal to it. Denoting this joint as the $(J\text{-}1)^{th}$ joint, then this joint has its own associated *(J-1)*sub-tree and the root segment of this *(J-1)*sub-tree is the proximal segment to the J^{th} joint. The dynamic tension associated with its sub-tree root is PRF_(J-1) and is given by Equation (5.10) with $\underline{R}^{[J-1]}$ replacing R and the dynamic angles properly interpreted for the *(J-1)*sub-tree.

The joint reaction force (JRF) on a simple non-branching joint of the system will be the vector sum of the two reactive tensions in the connecting segments just distal and proximal to it. Therefore, if the J^{th} joint is non-branching then JRF_J for this joint is given by

$$JRF_J = \mathrm{PRF\_}(\mathrm{J}) - \mathrm{PRF\_}(\mathrm{J}-1) \quad . \tag{6.2}$$

What if the J^{th} joint is not simple but rather a branch point of the pendulum tree? A simple example occurs in the joint labeled J2 in the specific tree illustrated in Figure 3.1. This joint has two distal segments S2 and S4 connected to it and therefore two distal sub-trees associated with it. In this case the (J-1) proximal joint is J1 the root of the whole tree. Labeling the PRF of the two distal sub-trees of J2 as $\mathrm{PRF}_1\mathrm{\_}(\mathrm{J=2})$ and $\mathrm{PRF}_2\mathrm{\_}(\mathrm{J=2})$ The JRF for J2 in this case will be

$$JRF_{J=2} = \mathrm{PRF}_1\mathrm{\_}(2) + \mathrm{PRF}_2\mathrm{\_}(2) - \mathrm{PRF} \tag{6.3}$$

In general, because there are no closed loops allowed in the tree structure system that is considered in this thesis, any given joint may have more than one directly connecting distal tree associated with it but will have only one directly connecting (J-1) proximal joint. If the J^{th} joint has K distal branches, then

$$JRF_J = \sum_{k=1}^{K} \mathrm{PRF}_{\mathrm{k}} \_ (J) - \mathrm{PRF}(J-1) \quad . \tag{6.4}$$

CHAPTER 7

CONCLUSION

In this thesis, the idealized 3-D pendulum tree system is intended to provide a flexible theoretical framework to model and better understand the dynamics of human and animal movement as well as the forces associated with those movements. The 3-D pendulum model developed here is particularly applicable to ballistic movements where muscles act primarily to impulsively initiate a movement phase such as the swing phase of human walking (toe-off to heel-strike). In such ballistic movements, impulsive muscular force generates the initial velocity of the system while the bulk of the motion phase is completed efficiently with relatively little further muscular effort by gravity and momentum transfer between body segments. Many skillful motor tasks probably have significant phases that can be modeled effectively as ballistic movement.

This thesis applies Lagrangian dynamics and the method of superfluous coordinates to determine constraint forces on the 3-D pendulum tree model system during ballistic movement. In particular, the dynamic reaction forces that are acting on the pivot joint and other joints in the general tree model system are considered. While consistency checks have been applied to the pivot joint, further consistency checks on the general joint reaction force equation developed in the thesis are required in future work.

The modeling approach developed in this thesis can be used to help estimate clinical parameters that are significant but otherwise difficult to access. It is hoped that in the future the ability to predict dynamic joint reaction forces in this dynamic general 3-D pendulum tree model will be used to help assess in a given individual the potential effect of proposed new movement techniques in producing joint injury and/or pain.

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