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ABSTRACT

NONLINEAR LONG-WAVE INTERFACIAL STABILITY OF TWO-LAYER GAS-LIQUID FLOW

by

Tetyana Segin

The flow of two immiscible viscous fluids in a thin inclined channel is considered, in either a cocurrent or countercurrent régime. Following the air-water case, which is found in a variety of engineering systems, we allow the upper fluid to be either compressible or incompressible. The disparity of the length scales and the density and viscosity ratios of the two fluids is exploited through a lubrication approximation of the conservation of mass and the Navier-Stokes equations. As a result of this long-wave theory, a coupled nonlinear system of partial differential equations is obtained that describes the evolution of the interfacial thickness and the leading-order pressure. This system includes the effects of viscosity stratification, inertia, shear, and capillarity, and reduces to the single-phase falling film Benney equation for sufficiently thin liquid films and constant gas density.

The case of two incompressible fluids is investigated first. Since the experimental conditions for this effective system are unclear, we consider several ways to drive the flow: either by fixing the volumetric flow rate of the gas phase or by fixing the total pressure drop over a downstream length of the channel, or by fixing liquid flow rate and gas pressure drop. The forcing with prescribed pressure drop results in a single evolution equation whose dynamics depends nonlocally on the interfacial shape. From weakly nonlinear analysis in this case, we obtain the modified Kuramoto-Sivashinsky equation with an additional integral term, influencing the speed of propagation but not the shape of the interfacial wave. For the strongly nonlinear case, admissible criteria for Lax shocks, undercompressive shocks and rarefaction waves are investigated. Through a numerical verification we find that
these criteria do not depend significantly on the inertial effects within the more dense layer. The choice of the local/nonlocal boundary conditions appears to play a role in the transient growth of undercompressive shocks, and may relate to the phenomena observed near the onset of flooding.

We then perform a linear stability analysis when the gas phase is compressible. The base-state profile for the density is spatially dependent when a pressure drop over the length of the channel is prescribed. The case when zero pressure drop is prescribed is amenable to a normal-mode analysis. When the liquid film thickness is sufficiently thin, the stability matches that of the single-phase falling film case with the exception that the compressible quiescent gas is stabilizing. When the liquid film thickness is sufficiently thick, the density mode within the thin gas layer is destabilizing. In the general case, over a finite domain, a general stability diagram of film thickness and pressure drop is found. For sufficiently large countercurrent pressure drops, the interfacial mode becomes unstable, with the location of the largest deformation found near the liquid inlet.
NONLINEAR LONG-WAVE INTERFACIAL STABILITY OF TWO-LAYER GAS-LIQUID FLOW

by

Tetyana Segin

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SEGIN, T. M., AND KONDIC, L., AND TILLEY, B. S.
To my family
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CHAPTER 1

INTRODUCTION

Liquid films are encountered in many physical situations, including applications in cooling systems, coating processes and biological applications. Other examples of their practical application include condensate flow in gas wells (Duenckel [23]), oil and gas flow through subsea tiebacks (Moritis [52]), as well as on-chip cooling of micro-electromechanical (MEMS) devices (Pettigrew et al. [57], Trebotich et al. [69], Kirshberg et al. [41]). Two-phase gas-liquid flows are also important in a number of space operations including the design and operation of spacecraft environmental systems, storage and transfer of cryogenic fluids and safety and performance issues related to space nuclear power systems (Dukler et al. [24], Bousman et al. [14]). Experimentally, Furukawa and Fukano [31] observed the strong dependence of liquid viscosity on flow pattern transitions of upward air-liquid two-phase systems. Knowledge about the physical properties of fluids and their effects on flow characteristics is important to understand the fundamental nature of two-phase flow.

Recently, attention has been paid to the prediction of flow patterns in inclined pipes (Soleimani and Hanratty [64]) and microchannels (capillaries) (Triplett et al. [70], [71]). The knowledge about capillary two-phase characteristics, including flow patterns and two-phase pressure drop, is necessary in design and operation of systems that include gas-liquid flows. In capillaries, surface tension is dominant and renders the flow characteristics independent of channel orientation with respect to gravity. With respect to flow patterns, for example, due to the dominance of surface tension, stratified flow is essentially absent, and slug and churn flow patterns occur over extensive range of parameters.
For single-phase liquid films flowing down an inclined channel, the long-wave linear stability theory was first developed by Yih and Benjamin [3]. At small angles of inclination, Floryan et al. [28] studied a falling single-phase film and found that growth rates of disturbances could be reduced by increasing the surface tension or decreasing the angle of inclination. They also reported that the critical Reynolds number ($Re$) of the shear mode varied non-monotonically with the inclined angle or surface tension parameter. Smith [63] used the model of a thin liquid film with a deformable top surface flowing down a rigid inclined plane to discuss the mechanisms of instability. These include the initiating mechanisms (shear and/or velocity induced) that drive the dominant motion in a perturbed film and the growth mechanism (due to inertial stress) that produce the unstable motion at the interface. Joo et al. [34] found that waves can steepen and increase in height to a point where long-wave assumptions cease to be valid. When the disturbance wave number is sufficiently small, a numerical solution of the evolution equation shows that the wave grows initially at the exponential rate of linear theory, but later grows super-exponentially. Peaks grow much faster than troughs deepen. The front of the peaks steepen toward the vertical, showing the incipient breaking, and the second trough grows behind the peak. Kalliadasis et al. [36] showed that the solitary wave solutions obtained by Joo et al. [34] for a thin layer of a viscous fluid flowing down a uniformly heated planar wall were unrealistic (with branch multiplicity and limit points) above $O(1)$ Reynolds number. To avoid this problem, the integral-boundary-layer (IBL) approximation of the Navier-Stokes/energy equations and associated free-surface boundary conditions was developed. This method (IBL) predicts the existence of solitary waves for all $Re$. Kalliadasis et al. [36] found that in the region of small thickness where the Marangoni forces dominated inertia forces, the IBL system reduces to a single equation for the film thickness that contained one parameter. When this parameter tends to zero, both the solitary wave speed and the maximum amplitude tend to infinity. Another aspect
of free-surface thin-film flows is the stability of flow over topography (step-down). Kalliadasis and Homsy [37] found that the free-surface develops a ridge just before the entrance to the step. The ridge is stable for a wide range of the pertinent parameters. An energy analysis indicates that the strong stability of this capillary ridge is governed by rearrangement of fluid in the flow direction owing to the net pressure gradient induced by the topography at small wave numbers and by surface tension at high wave numbers. Chang [17] used a phenomenological model to investigate the interfacial behavior of two-phase flow under the assumption of passive upper phase, allowing turbulent shear stress on the interface. He derived a dispersive modification of the Kuramoto-Sivashinsky (KS) equation for moderate flow rates, which is valid for long waves. Chang [18], Oron et al. [55] reviewed the modeling of the single phase wave evolution problem, presenting the mechanism of wave evolution on the interface, and discussed the transitions between various wave solutions. In addition, Oron et al. [55] discussed the dynamics of films with spatial dependence of the base-state solutions. This spatial dependence occurs in the dynamics of free liquid films, bounded films with interfacial viscosity, and dynamics of surfactant in free and bounded films.

The KS equation appeared first in the paper of Benney [4]. This equation was obtained independently for two-phase plane Poiseuille flow by Shiang [62], and Hooper and Grimshaw [33] for a horizontal channel. The results of Hooper and Grimshaw [33] were corrected by Charru and Fabre [19] who suggested symmetry checks to verify the evolution equation. In this case, if the ratio of densities and viscosities is equal to one and the surface tension coefficient equals zero, the interface does not affect stability since it becomes only a streamline in the flow. The KS equation is extensively discussed in the papers of Kevrekidis et al. [40], and Chen and Chang [20]. This equation is capable of generating solutions in the form of irregularly fluctuating quasi-periodic waves. It also provides a mechanism to explain the saturation of an instability in which the energy in long-wave instabilities is
transferred to short wave modes, which are then damped by surface tension (Joseph and Renardy [35]). Frenkel et al. [30] investigated the Poiseuille core-annular flow of two fluids having the same properties to understand the influence of surface tension on the interfacial behavior. The dynamics of the interface was determined by the KS equation with both stabilizing and destabilizing terms related to surface tension and a nonlinear term related to the base flow. Their combination leads to growth and subsequent saturation of initial disturbances. Frenkel et al. [30] identified the second derivative of the interface with respect to the axial distance as responsible for the induction of capillary instability. As the deformation of the interface increases, nonlinear convection induces an asymmetry and steepening of the wave. The fourth derivative of the interface with respect to the axial distance arising from the curvature operator then becomes important and prevents breakup. In the case of two-phase core-annular flows, the KS equation was modified by Papageorgiou et al. [56] for a thin film at a pipe wall. Two length scales were used: a radial scale (the film thickness) and the axial scale (the pipe radius). The problem involves one natural small parameter, $\epsilon$, defined as the ratio of the two length scales, and the solution is sought as an asymptotic expansion in $\epsilon$. The densities of the fluids are equal and the disturbances are assumed to be axisymmetric. Due to different viscosities of the fluids, the interfacial evolution equation contains an additional integral term. For $Re = O(\epsilon)$ and small surface tension, this term corresponds to a purely dispersive effect. However, for $Re = O(1)$ and at a high enough surface tension, it accounts for both dispersive and dissipative effects. Numerical studies of Papageorgiou et al. [56] show that in both cases this new integral term is responsible for a tendency to organize the typically chaotic solutions of the KS equation into doubly periodic traveling waves.

Two-phase cocurrent (both phases flow in the same direction) down-flow of air and water was extensively studied by Kouris and Tsamopoulos [44] in a host of
geometries ranging from model arrangements to single vertical constricted tubes. The ratio of viscosity of the fluid in the annulus to that in the core of the tube, $\mu$, was taken to be greater or equal to one. They found that the difference in viscosity of the two fluids induces an interfacial velocity, which is directly responsible for the transition from chaotic interactions obtained at viscosity ratio close to one, to well organized wave trains obtained for larger values of $\mu$. The viscosity ratio does not, however, affect the saturated wave shape and amplitude. The increase of surface tension causes a proportional increase in the amplitude of the resulting wave. Additionally, the increase of surface tension dramatically reduces the wave speed of the saturated wave. For larger values of $\mu$, the initial condition only slightly affects the number of crests which compose the saturated wave. It does not affect their type, i.e., whether they are chaotic or organized. For viscosity ratio ($O(10^{-3})$) Kouris and Tsamopoulos [43] found that stationary solutions were steady and the most unstable eigenvalue remained real. Generally, steady core-annular flow in a tube of sinusoidally varying cross-section was more susceptible to instability than in straight tube; in addition, for similar ranges of parameters, it might be generated by different mechanisms. Decreasing the thickness of the annular fluid or the density of the core fluid stabilized the flow. For stability reasons, the viscosity ratio had to remain strictly below one and it had an optimum value that maximized the range of allowed $Re$. For the case of two-layer flow the initiating mechanism for the long-wave instability is richer than one for the single-layer (due to viscosity and/or density stratification). Linear stability of two superposed layers of fluids was first studied by Yih [72] for the plane Couette-Poiseuille flow in a horizontal channel. Using a long-wave assumption, he showed that the interface was susceptible to instability due to viscosity stratification.

The phenomenon of generation of water waves by wind has been studied extensively both in the context of deep water and thin films. Miles [48], [49], [50] discussed physical mechanisms that may be responsible for the energy transfer from
wind to these waves. Miles analyzed the air-water stability problem in the context of deep water, while Craik [21] performed this analysis for thin films. Miles discovered that interfacial waves for which gravity provided the restoring force can be driven unstable via a resonant interaction with the ambient wind. A key role in Miles's asymptotic theory is played by the critical (resonant) layer, in which the wave speed is equal to the mean wind speed. The velocity profile is characterized by a viscous sublayer adjacent to the water surface, where the velocity profile is linear and a turbulent region above this layer with negative curvature of the velocity profile. If the critical layer is located outside the viscous sublayer, this negative curvature induces a destabilizing positive and constant Reynolds stress at all heights up to the critical one, provided the wave length is long enough and that the wind speed is not too large. If the critical layer is located inside the viscous sublayer where the velocity profile is identically zero, this mechanism cannot play a role in wave generation (Boomkamp and Miesen [13]). In Craik's paper [21], the hydrodynamic stability of thin liquid films (typically thinner than $10^{-3}$ m) was studied both experimentally and theoretically for relatively small $Re$ (typically smaller than 10). The experiments showed the presence of two types of waves: “fast” waves, having wave speed larger than the surface velocity of the liquid film, and “slow” waves, having wave velocities somewhat smaller than the surface velocity. The “fast” waves were only present at high enough $Re$ of the film. The “slow” waves were present only at lower $Re$. Later, Miersen and Boersma [47] removed the free surface approximation from the Miles problem [51] in order to analyze the dynamic effect of the gas. This effect appeared to be very large due to the fact that the imaginary part of the wave speed was very small. A second mode of instability was found which had a phase velocity larger than the maximum liquid velocity and corresponded to capillary-gravity waves.

Tilley et al. [67] investigated the influence of the channel thickness and the mean interfacial height on the stability of two-layer superposed fluid flow, and identified the
mechanisms for linear stability in the long-wave limit where the flow rate of each layer is prescribed. They considered only spatially periodic boundary conditions. Gravity-driven flow of two incompressible immiscible viscous fluids on a periodic domain was studied numerically as a fully nonlinear free-boundary problem by Zhang et al. [73]. It was found that fingers were formed as unstable waves flowing downstream. In addition, it was observed that increasing the viscosity of the upper fluid, and decreasing the angle of inclination, made the flow more stable. Their conclusion is similar to the results in Floryan et al. [28]. The attempt to classify the instabilities in parallel two-phase flow is made by Boomkamp and Miesen [13]. The criteria for dividing the mechanisms of instabilities into classes is based on how energy is being transformed from the primary to the disturbed flow. These mechanisms find their origin in one of the following properties of the flow system: density stratification and orientation (Rayleigh-Taylor and instability induced by tangential disturbances, i.e., viscosity and/or gravity instabilities), velocity profile curvature (Miles-instability), viscosity stratification, shear effects or a combination of the last two.

The problem of two-fluid flow which can flow cocurrently or countercurrently (fluids flow in opposite directions, Figure 1.1) is partially motivated by the phenomenon called flooding, found in countercurrent flows. This term is used to describe various aspects of the transition from countercurrent to cocurrent flow adverse to gravity, as the pressure gradient is increased. Two examples of phenomena which are considered as flooding are: 1) large interfacial deformations of the liquid that prevents the flow of gas, and 2) transitions during which the liquid flow rate is reduced or inverted. Flooding has been investigated extensively both phenomenologically (Chang [17], Fowler and Lisseter [29]) and experimentally (see Bankoff and Lee [2] and the references therein, Mouza et al. [53]) but the criteria for onset of flooding is still an open question. We note that during the transition from countercurrent to cocurrent upstream flow, a whole range of waves is observed from
possibly chaotic small-amplitude ones to large-amplitude waves that impede the flow of the upper fluid. Countercurrent flow returns only after the pressure gradient is decreased below the flooding point.

![Diagram of cocurrent and countercurrent flow in channel](image)

**Figure 1.1** Cocurrent and countercurrent flow in channel. Domain of interest in the model to follow is shown in box.

Next, we turn our attention to a different system consisting of a single phase driven by an interfacial shear stress induced by a thermal gradient (Marangoni effect) with counteracting gravitational force. These films arise in thin coating flows and are of great technical and scientific interest. Practical examples include spin coating, thin films coating the lungs and microfluidic devices. Until recently, it was believed that for such systems only compressive shocks (in which characteristics enter the shock from each side) occur. However, recently it has been discovered that the interfacial dynamics of these films include the development of undercompressive shocks. There is also a growing amount of theoretical work indicating that undercompressive shocks are observed in other physical systems. Kluwick *et al.* [42] find these waves in a modified Korteweg-de-Vries-Burgers equation which describes the evolution of weakly nonlinear concentration waves in suspensions of particles in fluids. According to Marchesin
and Plohr [46] undercompressive shocks in water alternating gas recovery have a
great practical potential for improving the efficiency of this process. Scientifically,
the advancing front of a driven film is an important example of shock formation in
systems described by the scalar conservation law.

The first indication of the more complicated interfacial dynamics with the
presence of undercompressive shocks comes from the comparison of two series of
experiments. The first experimental investigation of this particular thin-film behavior
was done by Ludviksson and Lightfoot [45]. Using squalane oil spreading on a
silver substrate with surface stresses on the order of 0.2 dyn/cm² or less produces
stable spreading films with a straight-edged moving front. Using interferometry,
they reconstructed the film thickness profiles and showed that the film thickness
decreases monotonically toward the substrate with no evidence of a capillary rim at
the advancing front. In contrast, Cazabat et al. [16] report a well-formed fingering
instability (Fig. 1.2) at the leading edge of a climbing film which develops within
minutes of applying a vertical temperature gradient. In these experiments, a silicone
oil film subject to stresses of approximately 0.5 dyn/cm² and higher was made to coat
a silicon wafer. This “contradiction” between results of experiments of Ludviksson
and Lightfoot [45] and Cazabat et al. [16] inspired a number of new analytical and
experimental studies (Carles and Cazabat [15], Bertozzi et al. [8], Bertozzi et al. [9]).
Bertozzi et al. [8] observe experimentally the formation of single very pronounced
overshoot or “bump” on the leading edge of the shock (Fig. 1.3). This structure is
often referred to as a capillary ridge in experiments. The ridge continues to broaden
as it advances up the plate. Despite the large capillary ridge, the contact line remains
stable. Bertozzi et al. [8] emphasize that undercompressive shock plays a key role
in preventing the contact line from fingering (i.e., breaking up into rivulets). These
solutions are known to arise in equations with nonconvex fluxes and combined diffusive
and dispersive effects (Bertozzi et al. [8], [10]). Schneemilch and Cazabat [61] showed
experimentally that a novel undercompressive shock (for which characteristics enter
the shock on one side and leave on the other) theory developed by Bertozzi and
coworkers for infinite films and substrates can be applied to real systems of finite
dimensions. Their experiments stimulated new theory for these problems resulting
in discovery of reverse undercompressive shock (Sur et al. [66], Munch [54]). The
reverse undercompressive shock forms the trailing edge of a double shock wave in a
thin liquid film that moves up the wafer similarly to a solitary wave. The velocity
of the wave and the thickness of the enclosed film can be varied by changing the
temperature gradient or the inclination angle, and the total amount of fluid can be
changed by modifying the initial conditions. This shock is reverse because it involves
a thicker film advancing upward from a thinner film, which is the reverse of the type
found by Schneemilch and Cazabat [60].

Figure 1.2 Fingering instability for film thickness of 0.6 μm, surface tension
gradient 0.18 Pa and angle of inclination $\alpha = \pi/2$. Equal-thickness interference
fringes are used to reconstruct thickness profiles. From Bertozzi et al. [8].

Single-layer driven thin films are examples of an open system, in which
the pressure at the interface is determined by a constant ambient pressure and
Figure 1.3  Experimental profile (+) in the dimensionless units at $t = 215$ with upstream film height 0.21. This profile shows large pronounced bump (undercompressive wave) in Bertozzi et al. [8].

capillarity. The air plays a passive role in interfacial evolution. In two-layer systems, however, pressure acts to ensure that the masses of both fluids are conserved, and plays an active role in the interfacial dynamics. Hence, global constraints on the pressure or on the flow parameters themselves play a significant role in the interfacial evolution. Almost all models related to the two-layer problems were investigated under the assumption of constant-flux boundary conditions. A more realistic boundary condition is to fix the pressure drop over the length of the channel. If the assumption of a passive upper layer is relaxed and two-fluid flow considered, one can expect that the difference between the constant flow rate and constant pressure drop régimes in gas phase will be more pronounced. We note that in experiments and applications, it is very difficult to maintain a constant flow rate, especially when the upper fluid is a gas. Therefore, there is a strong motivation for considering the constant liquid flow rate and constant gas pressure drop régimes. In this work, we compare the interfacial profile dynamics obtained using conditions of (i) a fixed gas
volumetric flow rate; (ii) fixed overall pressure drop; (iii) a fixed liquid volumetric
flow rate (or equivalently a fixed liquid Reynolds number) and a fixed gas pressure
drop. We find that the linear stability of a flat interfacial solution remains the same
in all cases. But unlike the traveling-wave dynamics found in (i) and (ii), in (iii) the
mean interfacial height varies in time as the spatial interfacial deflections evolve.

Most of the existing work on two fluid flows neglects compressibility of both
fluids, even if one of them is a gas. However, compressibility cannot be neglected
in problems having to do with the flow of gases through narrow channels, where an
appreciable pressure difference may exist between the inlet and outlet regions (Faber
[27]). In our problem, we allow for significant differences in the pressure jumps of
the gas flow, which accounts for the appropriate gas velocity changes. Since we
are interested in understanding interfacial dynamics in the full range of gas velocity
changes, the compressibility is an important physical effect to be investigated.

There are several papers devoted to weakly compressible flow (Mach number
being sufficiently small and the initial data are almost incompressible). In this case,
Hagstrom and Lorenz [32] found that the solution remains smooth for all times
and, to the leading order, it consists of the corresponding incompressible flow plus
a highly oscillatory part describing sound waves. Alexakis et al. [1] motivated
by an astrophysical problem (where mixing across material interfaces driven by
shear flows may significantly affect the dynamical evolution) showed in their linear
stability analysis that compressibility decreases the growth rate. They reached the
conclusion that the deviation from the incompressible case is not very large, even
for relatively strong (but still subsonic) winds. In addition, Rusak and Lee [59]
investigated the influence of the compressibility on the appearance of instabilities
and transition (breakdown) phenomena in a compressible inviscid axisymmetric and
rotating columnar flow of perfect gas in a finite-length straight circular pipe. Rusak
and Lee [59] reported the stabilizing effect of compressibility on vortex flows.
This work is organized as follows. In Chapter 2, we employ a long wave asymptotic analysis to derive a coupled nonlinear system of equations. We discuss the assumptions under which our model is valid, and apply the air-water limit to simplify the system of highly nonlinear evolution and pressure gradient/density equations. In the limit of air being incompressible, this system reduces to one highly nonlinear evolution equation. Here, we concentrate on a two fluid system driven by (i) constant gas flow rate, (ii) constant gas pressure drop, (iii) constant liquid flow rate and gas pressure drop, and gravity.

Chapter 3 describes linear stability and weakly nonlinear analysis for the flow of two incompressible fluids. From a weakly nonlinear analysis, the modified KS equation is derived. This equation contains the additional integral term for constant pressure drop formulation. This term represents the nonlocal dependence on the interfacial height. Thus, we investigate the behavior of a modified KS equation near bifurcation points in order to better understand the influence of this additional term. We show that it modifies the propagation speed of the interfacial waves, while not changing the shape of interfacial profiles. Linear stability results are used in the next chapter to verify the implemented numerical scheme.

Chapter 4 describes the numerical method that we use. We proceed with an analytical and numerical discussion of the dynamics of the interface and find that the waves formed are very similar to the ones that are found in the one fluid flow discussed above. This is particularly interesting since the governing equations of the two fluid flows are much more complex (including nonlocality in the case of constant pressure drop), compared to the one fluid case. Here, the traveling-wave solutions are investigated and, we identify the regions of their existence (necessary, but not sufficient) in the cases of flows driven by constant gas volumetric flow rate (with and without inertial effects) and discuss the case of constant gas pressure drop. We verify numerically our analytical results and find that the structure of the interfacial profile
for a given set of system parameters (ratio of densities, dynamical viscosities etc.) involves the Lax (classical) shock, double shock (Lax and undercompressive) and the combination of the rarefaction wave with undercompressive shock. In addition, in the system driven by fixed liquid flow rate and gas pressure drop, the interfacial dynamics results in unsteady solutions.

In Chapter 5, we discuss the linear stability theory for weakly compressible gas. We show that under certain assumptions on the gas density and thickness of liquid layer, the two-fluid flow exhibits the same dynamics as a single-phase film flow. We analyze the influence of compressibility on the stability properties of two-fluid flow for very thin liquid films. For the liquid films of general thickness from the boundary condition for the density equation (prescribed values of the density at the ends of the channel are different) the basic density state depends on the downstream coordinate. However, this transverse dependence does not occur if the fixed densities are the same. In this case, we proceed with the normal mode analysis on a periodic domain to investigate the behavior of both interfacial and density modes. Next, we turn our attention to the domain of fixed length and investigate the dynamics of countercurrent two-fluid flow.

In the appendices, we present details of the derivation of inertial terms and differences in the derivation of a full system of equations when the incompressible assumption for the upper fluid is used.
CHAPTER 2

FORMULATION AND GOVERNING EQUATIONS

We approach the problem of two-fluid flow in a closed channel by first formulating the governing equations, and then discussing the assumptions under which our system can be asymptotically expanded. This procedure leads to the reduced system of evolution and pressure gradient/density equations for flows with compressible gas. Additionally, we simplify the reduced system under the assumption of incompressible gas. This approximation will be relaxed later in Chapter 5.

2.1 General Formulation

Consider the flow of a viscous compressible fluid over an incompressible one in a channel of height $d$ and length $L$ (Figure 2.1), where $n$ is the unit normal to the interface pointing from phase 1 into phase 2, $t$ is the unit tangent vector at the interface. The equations that govern this system are continuity and Navier-Stokes

![Figure 2.1](image)

Figure 2.1  Configuration of two-layer flow in an inclined channel.

(asterisks denote dimensional variables):
\[ \nabla \cdot \mathbf{u}^{*(1)} = 0, \]
\[ \rho_i^{*} \left( \frac{\partial \mathbf{u}^{*(1)}}{\partial t^{*}} + \mathbf{u}^{*(1)} \cdot \nabla \mathbf{u}^{*(1)} \right) = -\nabla p^{*(1)} + \rho_i^{*} \mathbf{g} + \mu_i^{*} \nabla^2 \mathbf{u}^{*(1)}, \]
\[ \rho_g^{*} \left( \frac{\partial \mathbf{u}^{*(2)}}{\partial t^{*}} + \mathbf{u}^{*(2)} \cdot \nabla \mathbf{u}^{*(2)} \right) = -\nabla p^{*(2)} + \rho_g^{*} \mathbf{g} + \mu_2^{*} \left( \nabla^2 \mathbf{u}^{*(2)} + \frac{1}{3} \nabla (\nabla \cdot \mathbf{u}^{*(2)}) \right). \]

The superscripts \(^{(2)}\) and \(^{(1)}\) on the dependent velocity and pressure variables correspond to the compressible (upper) fluid and incompressible (lower) fluid, respectively, with corresponding densities of upper fluid \(\rho_g^{*}\) (\(g\) for gas) and lower fluid \(\rho_l^{*}\) (\(l\) for liquid), dynamical viscosities \(\mu_i^{*}\) and pressures \(p^{*(i)}\), \(g\) is the gravitational acceleration. The velocities \(\mathbf{u}^{*(i)} = (u^{*(i)}, w^{*(i)})\) satisfy the boundary conditions on the channel walls: \(\mathbf{u}^{*(1)} = 0\) on \(z^{*} = 0\) and \(\mathbf{u}^{*(2)} = 0\) on \(z^{*} = d\), as well as the balance of normal stress, balance of tangential stress, continuity of normal and tangential components of velocity and kinematic condition at \(z^{*} = h^{*}(x^{*}, t^{*})\):

\[ [\mathbf{n} \cdot \mathbf{T}^{*} \cdot \mathbf{n}] = \sigma^{*} \kappa^{*}, \quad (2.1) \]
\[ [\mathbf{t} \cdot \mathbf{T}^{*} \cdot \mathbf{n}] = 0, \quad (2.2) \]
\[ [\mathbf{u}^{*} \cdot \mathbf{n}] = 0, \quad (2.3) \]
\[ [\mathbf{u}^{*} \cdot \mathbf{t}] = 0, \quad (2.4) \]
\[ h_i^{*} + u^{*} h_x^{*} - w^{*} = 0, \quad (2.5) \]

where the jump \([f]\) of the quantity \(f\) across the interface is denoted by \([f] = f^{(2)} - f^{(1)}\); \(\mathbf{T}^{*}\) denotes the stress tensor, \(\sigma^{*}\) is the surface tension between the two fluids, and \(\kappa^{*}\) is twice the mean curvature of the interface, given by

\[ \kappa^{*} = -h_x^{*} \left(1 + h_x^{*2}\right)^{-3/2}. \]
We scale lengths by $d$, time by $\nu_1^* / dg$, and densities by $\rho_1^*$, and velocities by $d^2g / \nu_1^*$, where $\nu_1^* = \mu_1^* / \rho_1^*$ is the kinematic viscosity of the lower fluid and $g$ is the acceleration of gravity. The pressure scale is then $\rho_1^* dg$. Thus, we obtain

$$\nabla \cdot \mathbf{u}^{(1)} = 0,$$

$$G \left( \frac{\partial \mathbf{u}^{(1)}}{\partial t} + \mathbf{u}^{(1)} \cdot \nabla \mathbf{u}^{(1)} \right) = -\nabla p^{(1)} + \hat{g} + \nabla^2 \mathbf{u}^{(1)},$$

$$\tilde{\rho}_t + \nabla \cdot (\tilde{\rho} \mathbf{u}^{(2)}) = 0,$$

$$G \tilde{\rho} \left( \frac{\partial \mathbf{u}^{(2)}}{\partial t} + \mathbf{u}^{(2)} \cdot \nabla \mathbf{u}^{(2)} \right) = -\nabla p^{(2)} + \tilde{\rho} \hat{g} + \tilde{\mu} \left( \nabla^2 \mathbf{u}^{(2)} + \frac{1}{3} \nabla (\nabla \cdot \mathbf{u}^{(2)}) \right).$$

where $\tilde{\rho} = \rho_2^* / \rho_1^*$ is the density ratio, $\tilde{\mu} = \mu_2^* / \mu_1^*$ is the viscosity ratio, $G = gd^3 / \nu_1^{*2}$ is the ratio of buoyancy to viscous effects of the incompressible fluid, $\hat{g}$ is the unit vector in the direction of gravity and velocities are $\mathbf{u}^{(i)} = (u^{(i)}, w^{(i)})$.

The boundary conditions on the channel walls become: $\mathbf{u}^{(1)} = 0$ on $z=0$ and $\mathbf{u}^{(2)} = 0$ on $z = 1$ (Figure 2.1). The conditions (2.1)-(2.5) at $z = h(x, t)$ become

$$[\mathbf{n} \cdot \mathbf{T} \cdot \mathbf{n}] = \sigma \kappa,$$

$$[\mathbf{t} \cdot \mathbf{T} \cdot \mathbf{n}] = 0,$$

$$[\mathbf{u} \cdot \mathbf{n}] = 0,$$

$$[\mathbf{u} \cdot \mathbf{t}] = 0,$$

$$h_t + uh_x - w = 0,$$

where

$$\sigma = \frac{\sigma^*}{\rho_1 d^2 g}.$$

Since we are interested in gas-liquid systems, the ratios of density and dynamical viscosities of upper fluid to lower fluid are small. Following the case of air-water (under standard conditions: $\rho_2^* / \rho_1^* = 8 \cdot 10^{-4}, \mu_2^* / \mu_1^* = 2 \cdot 10^{-2}$), we assume that $\tilde{\rho} = \rho_2^* / \rho_1^*$ is of the order $\epsilon^2$ and $\tilde{\mu} = \mu_2^* / \mu_1^*$ is of the order $\epsilon$, where $\epsilon = d/L$ is the
aspect ratio of channel thickness to the channel length, i.e., \( \bar{\rho} = \varepsilon^2 \rho, \bar{\mu} = \varepsilon \mu, \) and \( \rho, \mu \) are \( O(1) \). Although limited in its range of applicability, this assumption allows us to capture the dominant physical effects in this system, while simplifying the analysis considerably. We assume that changes of the flow occur on a much longer spatial scale than of the channel thickness (Tilley et al. [68]). Therefore, we use scaled variables \( \xi = \varepsilon x \) and \( \zeta = z \). From balancing terms in the kinematic boundary condition (2.14) we then requires rescaling of time as \( \tau = \varepsilon t \).

Our system of equations (2.6)-(2.9) in the long-wave limit can be written in coordinate form as

\[
\begin{align}
\varepsilon u^{(1)}_\xi + u^{(1)}_\zeta &= 0, \\
Ge \left[ u^{(1)}_\tau + u^{(1)} u^{(1)}_\xi + \frac{1}{\varepsilon} w^{(1)} u^{(1)}_\zeta \right] &= -\varepsilon p^{(1)}_\xi + \dot{g} + \varepsilon^2 \left( u^{(1)}_{\xi \xi} + \frac{1}{\varepsilon^2} w^{(1)}_{\zeta \zeta} \right), \\
Ge \left[ w^{(1)}_\tau + u^{(1)} w^{(1)}_\xi + \frac{1}{\varepsilon} w^{(1)} u^{(1)}_\zeta \right] &= -p^{(1)}_\zeta + \dot{g} + \varepsilon^2 \left( w^{(1)}_{\xi \xi} + \frac{1}{\varepsilon^2} w^{(1)}_{\zeta \zeta} \right), \\
\varepsilon \rho_\tau + \varepsilon \left( \rho u^{(2)} \right)_\xi + \left( \rho w^{(2)} \right)_\zeta &= 0,
\end{align}
\]

\( \varepsilon = \varepsilon x \) and \( \zeta = z \). From balancing terms in the kinematic boundary condition (2.14) we then requires rescaling of time as \( \tau = \varepsilon t \).

Our system of equations (2.6)-(2.9) in the long-wave limit can be written in coordinate form as

\[
\begin{align}
G \varepsilon^2 \rho \left[ \varepsilon u^{(2)}_\xi + \varepsilon u^{(2)} u^{(2)}_\xi + w^{(2)} u^{(2)}_\zeta \right] \\
= -\varepsilon p^{(2)}_\xi + \varepsilon^2 \rho \dot{g} + \varepsilon^3 \mu \left( \frac{4}{3} w^{(2)}_{\xi \xi} + \frac{1}{\varepsilon^2} w^{(2)}_{\zeta \zeta} + \frac{11}{\varepsilon} w^{(2)}_\zeta \right), \\
G \varepsilon^2 \rho \left[ \varepsilon w^{(2)}_\tau + \varepsilon u^{(2)} w^{(2)}_\xi + w^{(2)} w^{(2)}_\zeta \right] \\
= -p^{(2)}_\zeta + \varepsilon^2 \rho \dot{g} + \varepsilon^3 \mu \left( \frac{4}{\varepsilon^2} w^{(2)}_{\zeta \zeta} + w^{(2)}_\xi + \frac{1}{\varepsilon^3} w^{(2)}_\xi \right).
\end{align}
\]

We assume a perturbation expansion for \( u \) in \( \varepsilon \):

\[
u^{(1)}(\xi, \zeta, \tau) = u_0^{(1)}(\xi, \zeta, \tau) + \varepsilon u_1^{(1)}(\xi, \zeta, \tau) + \ldots,
\]

and then from the continuity equation (2.15), it follows that

\[
w^{(1)}(\xi, \zeta, \tau) = \varepsilon \left\{ w_0^{(1)}(\xi, \zeta, \tau) + \varepsilon w_1^{(1)}(\xi, \zeta, \tau) + \ldots \right\}.
\]
Since the system is driven by a pressure gradient and possibly gravity as described in (2.16), (2.19), we assume, based on the pressure scale, that the downstream pressure gradients are of order one:

\[ p^{(i)}(\xi, \zeta, \tau) = \frac{1}{\epsilon} \left\{ p_0^{(i)}(\xi, \zeta, \tau) + \epsilon p_1^{(i)}(\xi, \zeta, \tau) + \ldots \right\}, \quad i = 1, 2. \]

Before specifying the expansion of the velocity in the gas phase, we want to balance the inertial terms and viscous terms in (2.19). Thus, we require the leading-order terms in the tangential gas velocity equation to satisfy

\[ -p_\xi^{(2)} + \mu u_\zeta^{(2)} = 0. \]

From here, we can allow gas velocities to be large

\[ u^{(2)} \sim O\left(\frac{1}{\epsilon}\right) \]

while inertial effects remain higher order due to the small density ratio \( \rho \). Therefore, the leading order tangential gas velocity is \( O\left(\frac{1}{\epsilon}\right) \).

The gas velocities in the second fluid have an asymptotic expansion of the form:

\[ u^{(2)}(\xi, \zeta, \tau) = \frac{1}{\epsilon} \left\{ u_0^{(2)}(\xi, \zeta, \tau) + \epsilon u_1^{(2)}(\xi, \zeta, \tau) + \ldots \right\}. \]

From the continuity equation (2.18):

\[ w^{(2)}(\xi, \zeta, \tau) = w_0^{(2)}(\xi, \zeta, \tau) + \epsilon w_1^{(2)}(\xi, \zeta, \tau) + \ldots. \]

In addition, we assume large surface tension and define the unit-order parameter \( S = \epsilon^2 \sigma \).

Note that a similar expansion was performed by Tilley et al. [68] without the assumption of a particular scaling of the viscosities and the densities. Therefore, their formulation is much more complex than the one derived here.
We obtain the sequence of linear problems from (2.6)-(2.14)

\[
O \left( \epsilon^{-1} \right): \\
\begin{align*}
\rho \nabla_\xi &= 0, \\
\rho \nabla_\xi &= \rho \nabla_\xi \quad (\zeta = h(\xi, \tau)), \\
u &= 0 \quad (\zeta = h(\xi, \tau)),
\end{align*}
\]

\[
O(1): \\
\begin{align*}
- \rho \nabla_\xi + \mu u \nabla_\xi &= 0, \\
- \rho \nabla_\xi + \sin \beta + u \nabla_\xi &= 0, \\
p \nabla_\xi + \cos \beta &= 0, \\
\rho \nabla_\xi &= 0, \\
u \nabla_\xi &= 0, \\
(\rho u \nabla_\xi) \nabla_\xi + (\rho u \nabla_\xi) \nabla_\xi &= 0, \\
u \nabla_\xi - \rho \nabla_\xi &= 0 \quad (\zeta = h(\xi, \tau)), \\
\mu u \nabla_\xi - u \nabla_\xi &= 0 \quad (\zeta = h(\xi, \tau)), \\
p \nabla_\xi - p \nabla_\xi &= -Sh \xi \xi \quad (\zeta = h(\xi, \tau)), \\
w \nabla_\xi - h \xi u &= 0 \quad (\zeta = h(\xi, \tau)), \\
u &= 0 \quad (\zeta = 0), \\
u &= 0 \quad (\zeta = 1), \\
u &= 0 \quad (\zeta = 0), \\
u &= 0 \quad (\zeta = 1).
\end{align*}
\]

From equations \( O(\epsilon^{-1}) \), (2.21) and (2.22), we find that

\[
p \nabla_\xi = \rho \nabla_\xi = \rho \nabla_\xi = p \nabla_\xi,
\]

The \( O(1) \) \( \zeta \)-momentum equations, (2.26) and (2.27), with condition (2.32) on the interface yield,

\[
p \nabla_\xi - (\cos \beta) \nabla_\xi + P \nabla_\xi,
\]
and

\[ p_1^{(2)}(\xi, \tau) = -h(\xi, \tau) \cos \beta + P_1(\xi, \tau) + \bar{S} h_{\xi \xi}. \]

In what follows, we refer to \( P_1(\xi, \tau) \) as the pressure correction.

The \( O(1) \) \( \xi \)-momentum equations, (2.24), (2.25), continuity equations (2.28), (2.29) with boundary conditions (2.34) - (2.37), and continuity of the tangential component of the velocity and shear stress at interface (2.23), (2.31) yield

\[ u_0^{(1)} = \frac{p_0 \xi - \sin \beta}{2} \zeta^2 + \frac{2h \sin \beta - (h + 1)p_0 \xi}{2} \zeta, \]

\[ w_0^{(1)} = -\frac{p_0 \xi}{6} \zeta^3 + \left\{ -h \xi \sin \beta + \frac{p_0 \xi}{2} (h + 1) + p_0 \xi \frac{h}{2} \right\} \frac{\zeta^2}{2}, \]

\[ u_0^{(2)} = \frac{p_0 \xi}{2\mu} (\zeta - 1)^2 + \frac{p_0 \xi (1 - h)}{2\mu} (\zeta - 1), \]

\[ w_0^{(2)} = -\frac{(\rho p_0 \xi) \xi}{6\rho \mu} (\zeta - 1)^3 + \frac{(\rho p_0 \xi [h - 1]) \xi}{4\rho \mu} (\zeta - 1)^2. \]

At \( O(\epsilon) \), the \( \xi \)-momentum and continuity equations (2.6)-(2.9) yield

\[ G \left[ u_0^{(1)} + u_0^{(1)} u_0^{(1)} + u_0^{(1)} w_0^{(1)} \right] = -p_1^{(1)} + u_1^{(1)}, \]

\[ G \rho \left( u_0^{(2)} u_0^{(2)} + u_0^{(2)} w_0^{(2)} \right) = -p_1^{(2)} + \mu u_1^{(2)}, \]

\[ u_1^{(1)} + w_1^{(1)} = 0, \]

\[ \rho \tau + (\rho u_1^{(2)})_{\xi} + (\rho w_1^{(2)})_{\zeta} = 0, \]

\[ u_1^{(1)} = 0 \quad (\zeta = 0), \]

\[ u_1^{(2)} = 0 \quad (\zeta = 1), \]

\[ w_1^{(1)} = 0 \quad (\zeta = 0), \]

\[ w_1^{(2)} = 0 \quad (\zeta = 1), \]

\[ \mu u_1^{(2)} - u_1^{(1)} = 0 \quad (\zeta = h(\xi, \tau)), \]

\[ w_1^{(2)} - h_\xi u_1^{(2)} - w_0^{(1)} + h_\xi u_0^{(1)} = 0 \quad (\zeta = h(\xi, \tau)). \]
Using equations (2.42)-(2.45) and the boundary and interfacial conditions (2.46)-(2.51), similar to the analysis at the previous order, we arrive at the solution for $u_1^{(i)}$

\[
\begin{align*}
    u_1^{(1)} &= P_1 \zeta^2 + \phi \zeta + F^{(1)}(\xi, \zeta, \tau), \\
    u_1^{(2)} &= \frac{P_1 + \alpha}{2\mu}(\zeta - 1)^2 + \psi(\zeta - 1) + F^{(2)}(\xi, \zeta, \tau),
\end{align*}
\[(2.52)\]

where

\[
\begin{align*}
    \phi &= \mu F^{(2)}_\zeta(\xi, h, \tau) - F^{(1)}(\xi, h, \tau) - \frac{\mu}{h - 1} F^{(2)}_\zeta(\xi, h, \tau) \\
    &+ \frac{\mu h}{2(h - 1)} \left\{ h \sin \beta - p_0 \xi \right\} + \alpha \frac{h - 1}{2} - P_1 \xi \frac{h + 1}{2}, \\
    \psi &= \frac{F^{(2)}(\xi, h, \tau)}{1 - h} + \frac{(P_1 + \alpha)(1 - h)}{2\mu} + \frac{h(h \sin \beta - p_0 \xi)}{2(h - 1)}, \\
    \alpha &= \bar{S}h_{\xi \xi \xi} - h_{\xi} \cos \beta.
\end{align*}
\[(2.54)\]

We refer to Appendix A for the definition of the inertial terms $F^{(i)}$. The continuity of the normal velocity (2.12) then gives the equation for the pressure gradient

\[
\begin{align*}
    \left[ \frac{\rho_0 \xi (h - 1)^3}{12\mu} \right]_\xi + \epsilon \left( \frac{h^2(h + 3)}{12} \rho_0 \xi \rho_0 - \frac{h^2 h_{\xi}}{1 - h} \rho_0 \xi + \frac{h^2 h_{\xi}}{2} \rho G \sin \beta \right) + \epsilon \int_h^1 \rho \zeta d\xi \\
    + \epsilon \int_h^1 \left( \frac{\alpha + P_1 \xi}{2\mu} \rho(\zeta - 1)^2 + \rho \psi(\zeta - 1) + \rho F^{(2)}(\xi, \zeta, \tau) \right)_\xi d\zeta = 0.
\end{align*}
\[(2.57)\]

The interface shape is found using the kinematic boundary condition (2.14):

\[
    h_\tau + h_{\xi} u_0^{(1)} - u_0^{(1)} + \epsilon \{ h_{\xi} u_1^{(1)} - w_1^{(1)} \} = 0.
\]

After substituting the appropriate terms for the velocity expansion $u_0^{(1)}, w_0^{(1)}, u_1^{(1)}, w_1^{(1)}$ in the equation above, it simplifies to:

\[
    h_\tau + \tilde{A}_1 + \epsilon \left( S \frac{h^3 h_{\xi \xi \xi}}{3} - \frac{h_{\xi}^3}{3} h_{\xi} \cos \beta + \bar{A}_2 + GI \right)_\xi = 0.
\]
\[(2.58)\]
where

\[ \bar{A}_1 = h^2 h_{\xi} \sin \beta - \left[ \frac{h^2(h + 3)}{12} \bar{p}^{(2)}_{\xi} \right]_{\xi}, \]

\[ \bar{A}_2 = \frac{\mu h^3 p_{0\xi}}{4(1 - h)} - \frac{\mu h^4}{4(1 - h)} \sin \beta, \]

\[ I = \frac{h^4(7h + 25)}{240} p_{0\xi} + \frac{13}{480} h^2(1 - h)^{\frac{1}{2}} \frac{\rho}{\mu^2 p_{0\xi}} \left\{ -h_{\xi} p_{0\xi} + \frac{\rho_{\xi}}{\rho} p_{0\xi}(1 - h) \right\} 
- \frac{h^5(10h^2 + 7h + 77)}{3360(1 - h)} h_{\xi} p_{0\xi}^2 + \frac{2}{15} h^6 h_{\xi} \sin^2 \beta - \frac{h^5(29h^2 + 161h - 378)}{20160} \frac{\rho_{\xi}}{\rho} p_{0\xi}^2 
+ \sin \beta \left[ \frac{h^6(109h + 147)}{10080} \frac{\rho_{\xi}}{\rho} p_{0\xi} + \frac{h^5(41h^2 - 49h - 56)}{840(1 - h)} p_{0\xi} h_{\xi} \right]. \]

This general reduced system of two equations (2.57), (2.58) is closed using the equation of state of the upper fluid and the appropriate boundary conditions.

We consider the isothermal ideal gas equation of state, therefore ignoring the thermal effects

\[ p^{(2)} = K^* \rho_g^* \]

We nondimensionalize this equation by applying the scalings from page 17

\[ p^{(2)} = K \bar{\rho} \quad (2.59) \]

where \( K = K^*/d\rho. \)

Since \( \bar{\rho} = \epsilon^2 \rho \) with \( \rho \) being \( O(1) \) we require \( K = \epsilon^{-3} D \), \( D \) is \( O(1) \). Thus, we close the system (2.57), (2.58) with the equation relating pressure and density

\[ p^{(2)} = D \rho. \]

Our next objective is to understand the interfacial dynamics through the linear stability analysis of the obtained system of equations.
2.2 Reduction to Incompressible Flow

System (2.57), (2.58) reduces to the incompressible case in the limit $\rho_\xi \to 0, \rho_r \to 0$. We refer to Appendix B for the details of derivation. We obtain a system of equations consisting of the coupled nonlinear evolution equation and of the pressure gradient equation

$$h_r + A_1(h) h_\xi + \epsilon \left[ -\frac{h^3}{3} (\cos \beta) h_\xi + S \frac{h^3}{3} h_{\xi \xi \xi} + A_2(h) + GI(h) \right]_{\xi} = 0, \quad (2.60)$$

$$\left[ p_0 \xi (1 - h)^3 \right]_{\xi} = 0, \quad (2.61)$$

where

$$A_1(h) = h^2 \sin \beta - \frac{h(h+1)}{2(1-h)} p_0 \xi,$$

$$A_2(h) = \frac{\mu h^3 p_0 \xi}{4(1-h)} + \frac{\mu h^4}{4(h-1)} \sin \beta$$

$$+ \frac{\mu h^2 (h+3)}{(1-h)^3} \left[ \gamma - \frac{h^2 (h+3)}{12} \sin \beta - \frac{h(h-1)}{4} p_0 \xi \right]$$

$$I(h) = R_1 + R_2 + R_3,$$

with

$$R_1 = \frac{h^4 (7h + 25)}{240} p_{0 \xi \xi} + \frac{-41 h^2 + 49h + 56 h^5 h_\xi p_0 \xi}{840(h-1)} \sin \beta,$$

$$R_2 = \frac{h^5 (10h^2 + 7h + 77)}{3360(h-1)} h_\xi p_0^2 \xi + \frac{2}{15} h^6 h_\xi \sin^2 \beta,$$

$$R_3 = -\frac{13}{480} h^2 (h-1)^4 \frac{h_\xi p_0^2 \xi}{\mu^2} + \frac{17}{3360} \frac{\rho}{\mu^2} h^2 (h+3)(1-h)^3 h_\xi p_0^2 \xi,$$

and

$$\gamma = \int_0^\xi \frac{h(h+1) h_\xi p_0 \xi}{2(1-h)} d\xi.$$

The third, fourth and sixth terms in the evolution equation describe hydrostatic pressure effects, surface tension and inertial effects. The second term in the evolution equation corresponds to advection effects and consists of two sub-terms: the first
one describes wave propagation and steepening and the second one the influence of pressure gradient. Integrating (2.61) with respect to $\xi$ gives

$$p_0 \xi = \frac{\Phi(\tau)}{(1 - h)^3}.$$  \hfill (2.62)

where $\Phi(\tau)$ depends on the conditions which are imposed on the flow.

Next, we discuss three types of conditions for the pressure gradient equation (2.61). We show later that these conditions lead to different interfacial dynamics.

**Case I:** Constant gas volumetric flow rate.

One possibility is to prescribe the gas volumetric flow rate $q$:

$$q = \int_h^1 u^{(2)} d\zeta.$$  

After substituting the appropriate expression for tangential gas velocity (2.40), we obtain

$$q = -\frac{(1 - h)^3}{12\mu} p_0 \xi, \quad \Phi_1(\tau) = -12\mu q.$$  \hfill (2.63)

Thus, if $q$ is held fixed, the pressure gradient is determined by the interfacial deflection $h(\xi, \tau)$.

**Case II:** Constant gas pressure drop.

A second possibility is to define pressure drop $\Delta P = p_0(0) - p_0(1)$ and then

$$\Phi_2(\tau) = -\frac{\Delta P}{\int_0^1 \frac{d\xi}{(1 - h)^3}}.$$  \hfill (2.64)

In this case, the pressure gradient is determined by the global effect of the interfacial height. Condition (2.64) provides the solution for the pressure gradient equation (2.61). The prescribed boundary conditions for $h(\xi, \tau)$ then fully determine the problem.

Case I and Case II are related to each other through the nonlocal dependence on the interfacial height.
Case III: Constant liquid volumetric flow rate and constant gas pressure drop.

A third possibility of formulating this problem is to prescribe the liquid flow rate \( Q_l \) and pressure drop \( \Delta P \) over the length of the channel. These are more physical conditions because in the experiments (see Bankoff and Lee [2], Chang [17]) the liquid flow rate and pressure drop are known. We see below that \( Q_l \) and \( \Delta P \) can not be prescribed independently. We define liquid flow rate \( Q_l \) by

\[
Q_l = \int_0^l u_0^{(1)} d\zeta.
\]

After substitution of the corresponding expression for tangential velocity of the lower fluid (2.38), we obtain

\[
Q_l = -\frac{p_0 \xi (h + 3)}{12} + \frac{h^3}{3} \sin \beta. \tag{2.66}
\]

To obtain the pressure drop \( \Delta P \), we integrate (2.66) to obtain

\[
\Delta P = 4 \int_0^1 \frac{3Q_l - h^3 \sin \beta}{h + 3} d\xi. \tag{2.67}
\]

Equation (2.67) is the required relation which \( Q_l \) and \( \Delta P \) need to satisfy.

We note that the pressure gradient equation (2.61) is a first-order differential equation in \( p_0 \xi \). The evolution equation (2.60) is then formulated in terms of pressure gradient. Thus, fixing the pressure drop (2.67), we impose an additional restriction on our flow. To satisfy (2.66) and (2.67) for a given \( Q_l \) and \( \Delta P \), and to avoid illposedness (number of unknowns is less than the number of equations), we allow the mean (average) interfacial height (the averaged integral of the interfacial height \( h(\xi, \tau) \)) to vary in time.
We consider all three sets of the conditions. Therefore, we require either the gas volumetric flow rate and the interfacial heights at the inlet and outlet of channel fixed and \( p_0 \) found explicitly in terms of the interfacial shape or pressure drop fixed or both \( Q_i \) and \( \Delta P \) are prescribed. We discuss the differences between these scenarios further in Chapters 3 and 4.
CHAPTER 3

LINEAR STABILITY AND WEAKLY NONLINEAR ANALYSIS FOR INCOMPRESSIBLE FLOW

First, we perform the linear and weakly nonlinear analysis of the system of equations (2.60) and (2.61). Our linear stability analysis is the special case of the approach in Tilley et al. [68], since we assume the limit such that the ratio of densities is of order $O(\epsilon^2)$ and the ratio of dynamic viscosities is $O(\epsilon)$, as specified on page 18. The linear stability theory and weakly nonlinear analysis allow us to understand basic features of the flow and instability development. In addition, these analyses are also beneficial in helping us formulate a numerical method for the simulation of the full evolution equation for incompressible flow.

3.1 Linear Stability Analysis

To explore the linear stability of long waves, we perform a linear stability analysis of the evolution equation (2.60) coupled with pressure gradient equation (2.61). With this in mind we introduce the basic pressure gradient $p_{0\xi}^{(0)}$ corresponding to the pressure gradient at the basic interfacial height $h_0$. The form of $p_{0\xi}^{(0)}$ following from the pressure gradient equation (2.61) will be discussed below.

In order to determine the response of the system to small perturbations, we perturb the basic state as $h = h_0 + \delta \tilde{h}$, where $\delta \ll \epsilon$ and $h_0$ is a constant. Next, we linearize the evolution equation in the perturbation parameter $\delta$, and introduce normal modes $\tilde{h} = e^{ik(\xi - \epsilon t)}$ with $k$ being a wave number, to arrive to the following characteristic equation:

$$- ikc + ikA_1(h_0, p_{0\xi}^{(0)}) + \epsilon \left[ ikA_2(h_0, p_{0\xi}^{(0)}) + S\frac{h_0^3}{3} k^4 - k^2 \left\{ -\frac{h_0^3}{3} \cos \beta + GI(h_0, p_{0\xi}^{(0)}) \right\} \right] = 0.$$
This equation yields the following expressions for the phase speed $c_r$ and growth rate $kc_i$ (with $\text{Im}(kc) > 0$ for growing waves):

$$c_r = A_1(h_0, P_0^{(0)} + \epsilon A_2(h_0, p_0^{(0)}),$$

$$kc_i = \epsilon k^2 \left[ P(h_0) + GI(h_0, p_0^{(0)} - SC(h_0)k^2 \right].$$

Here

$$A_1(h_0, p_0^{(0)}) = h_0^2 \sin \beta + \frac{h_0(h_0 + 1) p_0^{(0)}}{2(h_0 - 1)},$$

$$A_2(h_0, p_0^{(0)}) = \frac{\mu \sin \beta h_0^3}{4(1 - h_0)^4} \left[ 18h_0^3 - 45h_0^2 + 36h_0 - 25 \right]$$

$$- \frac{h_0^3(h_0^2 - 74h_0 + 57)}{4(1 - h_0)^4} \mu p_0^{(0)},$$

$$P(h_0) = -C(h_0) \cos \beta,$$

$$C(h_0) = \frac{h_0^3}{3},$$

$$I(h_0, p_0^{(0)}) = p_0^{(0)^2} D_1(h_0, \mu, \rho) + p_0^{(0)} D_2(h_0, \mu, \rho) + D_3(h_0, \mu, \rho),$$

$$D_1(h_0, \mu, \rho) = \frac{h_0^5(5h_0^3 + 72h_0^2 + 371h_0 + 224)}{1680(1 - h_0)^2} + \frac{\rho h_0^2(1 - h_0)^3(27h_0 - 10)}{180},$$

$$D_2(h_0, \mu, \rho) = -\frac{h_0^5(65h_0^3 + 623h_0 + 112)}{1680(1 - h_0)} \sin \beta,$$

$$D_3(h_0, \mu, \rho) = \frac{2}{15} h_0^5 \sin^2 \beta.$$
and hydrostatic effects are stabilizing, the system is stable to long wave perturbations. If hydrostatic effects are stabilizing and inertial effects are destabilizing then under sufficient shear (defined by the imposed pressure gradient at basic interfacial state in (3.4)), the system becomes unstable (Figure 3.1b). The shape of neutral stability curves in Fig. 3.1b is a sketch. The minimum of the corresponding neutral stability branches occurs at some $k$ that depends on the value of the imposed pressure gradient.

![Figure 3.1](image)

**Figure 3.1** Possible long-wave neutral stability curves for 2-layer flow. Unstable regions are denoted by a “U” and stable regions are denoted by as “S”.

Our conclusions about the influences of the surface tension and hydrostatic pressure effects hold true for all three cases of the boundary conditions for pressure gradient/density equation (2.61). Next, we discuss the inertial terms (3.4) for each case of the boundary conditions. Note that the characteristic equation (3.2) is quadratic in the pressure gradient at the basic state. To understand the dependence of inertial terms on $q$, $\Delta P$ or $Q_l$, we expand the pressure drop, gas volumetric flow rate and liquid flow rate in power series of $\epsilon$ for different families of pressure gradient boundary conditions. These expansions lead to the following forms of the inertial term (3.4):
1. Constant gas volumetric flow rate:

\[
I(h_0, q_0) = q_0^2 \tilde{D}_1(h_0, \mu, \rho) - q_0 \tilde{D}_2(h_0, \mu, \rho) + D_3(h_0, \mu, \rho),
\]

where

\[
\tilde{D}_1(h_0, \mu, \rho) = D_1(h_0, \mu, \rho) \left( \frac{12\mu}{(1 - h_0)^3} \right)^2,
\]

\[
\tilde{D}_2(h_0, \mu, \rho) = D_2(h_0, \mu, \rho) \frac{12\mu}{(1 - h_0)^3}.
\]

2. Constant pressure drop:

\[
I(h_0, (\Delta P)_0) = (\Delta P)_0^2 D_1(h_0, \mu, \rho) - (\Delta P)_0 D_2(h_0, \mu, \rho) + D_3(h_0, \mu, \rho).
\]

3. Constant liquid flow rate and constant gas pressure drop:

\[
I(h_0, Q_{10}) = Q_{10}^2 \tilde{D}_1(h_0, \mu, \rho) - Q_{10} \tilde{D}_2(h_0, \mu, \rho) + \tilde{D}_3(h_0, \mu, \rho),
\]

where

\[
\tilde{D}_1(h_0, \mu, \rho) = D_1(h_0, \mu, \rho),
\]

\[
\tilde{D}_2(h_0, \mu, \rho) = \frac{12}{h_0 + 3} \left( D_2(h_0, \mu, \rho) + 2D_1(h_0, \mu, \rho) \frac{4h_0^3 \sin \beta}{3 + h_0} \right),
\]

\[
\tilde{D}_3(h_0, \mu, \rho) = D_1(h_0, \mu, \rho) \left( \frac{4h_0^3 \sin \beta}{3 + h_0} \right)^2 + D_2(h_0, \mu, \rho) \frac{4h_0^3 \sin \beta}{3 + h_0} + D_3(h_0, \mu, \rho).
\]

We observe that the dynamics caused by the inertial terms are similar in all three sets of boundary conditions (inertia is quadratic in the applied force). The appearance of the different coefficients in this quadratic dependence of the inertial terms on the driving force does not change qualitatively the behavior of the system.

In conclusion, the linear stability analysis predicts similar interfacial dynamics for all three families of boundary conditions for pressure gradient: constant \( q \), constant
\( \Delta P \), and constant \( Q_l \) and constant \( \Delta P \). We are particularly interested in the constant pressure drop case since it is relevant for the compressible flow (being the natural boundary condition in that case). Next, we proceed with a weakly nonlinear analysis of our system (2.60), (2.61) under different boundary conditions for the pressure gradient equation.

### 3.2 Weakly Nonlinear Analysis for Fixed Pressure Drop

To determine how the nonlinear dynamics change as the applied pressure drop is modified, we perform a weakly nonlinear analysis of the system of the pressure gradient (2.61) and evolution equations (2.60). Owing to the complexity of the evolution equation, we consider a small-amplitude analysis in a neighborhood slightly beyond the linear neutral stability boundary. Thus, we introduce a “slow” time scale \( \tau_1 = \epsilon \tau \) keeping the “fast” time scale \( \tau \) in our system (analog of multiscale analysis), and expand the gas flow rate and the interface position about the basic state:

\[
q(\tau, \tau_1) = q_0 + \epsilon q_1(\tau, \tau_1), \quad h(\xi, \tau, \tau_1) = h_0 + \epsilon H(\xi, \tau, \tau_1). \quad (3.5)
\]

The equation, describing the relation between \( q \) and \( \Delta P \)

\[
\frac{\Delta P}{12 \mu} = q \int_0^1 \frac{d\xi}{(1 - h)^3}
\]

can be written in the form

\[
\frac{\Delta P}{12 \mu} = q_0 \int_0^1 \frac{d\xi}{(1 - h_0)^3} + \epsilon \left[ q_1 \int_0^1 \frac{d\xi}{(1 - h_0)^3} + 3q_0 \int_0^1 \frac{H d\xi}{(1 - h_0)^4} \right] + O(\epsilon^2).
\]

By equating the corresponding powers of \( \epsilon \) we obtain:

\[
q_0 = \frac{\Delta P (1 - h_0)^3}{12 \mu}, \quad q_1 = -\frac{3q_0}{1 - h_0} \int_0^1 H d\xi. \quad (3.6)
\]
We substitute the expansions for $q$ and $h$ (3.5), (3.6) into the evolution equation (2.60) and obtain
\[
\epsilon \left[ H_{\tau} + \left\{ h_0^2 \sin \beta - \frac{6\mu q_0 h_0(1 + h_0)}{(1 - h_0)^4} \right\} H_{\xi} \right] + \epsilon^2 \left[ H_{\tau \tau} + \alpha_2 H_{\xi} \right] \int_0^1 H \, d\xi + \alpha_1 H H_{\xi} + \lambda H_{\xi \xi} + \gamma H_{\xi \xi \xi \xi} + \theta H_{\xi} = 0, \tag{3.7}
\]
where
\[
\alpha_1 = 2h_0 \sin \beta - \frac{6\mu q_0(2h_0^2 + 5h_0 + 1)}{(1 - h_0)^5},
\]
\[
\alpha_2 = \frac{18\mu q_0 h_0(1 + h_0)}{(1 - h_0)^5},
\]
\[
\gamma = \frac{S}{3} h_0^3,
\]
\[
\lambda = G \left[ \frac{\mu q_0}{70(1 - h_0)^4} \frac{h_0^5(41h_0^2 - 49h_0 - 56)}{35(1 - h_0)^3} \sin \beta + \frac{3\mu^2 q_0^2 h_0^5(10h_0^2 + 7h_0 + 77)}{70(1 - h_0)^7} \right] + G \left[ \frac{2}{15} h_0^6 \sin^2 \beta + \frac{6h_0^2 h_0(27h_0 - 10)}{35(1 - h_0)^3} \rho q_0^2 \right] - \frac{h_0^3}{3} \cos \beta,
\]
\[
\theta = \frac{\mu h_0^3(3h_0 - 4)}{4(1 - h_0)^2} \sin \beta + \frac{3\mu^2 q_0 h_0^2(h_0 + 3)}{(1 - h_0)^5} + \mu h_0^2(3 + h_0) \left\{ -\frac{h_0(h_0 + 2)}{4} \sin \beta + \frac{3\mu q_0(1 + h_0)^2}{(1 - h_0)^4} \right\} = \frac{\mu h_0^3(2h_0^2 - h_0 + 5)}{2} \sin \beta + \frac{6\mu^2 q_0 h_0^2(h_0 + 3)(h_0^2 + 1)}{(1 - h_0)^7}.
\]

At $O(\epsilon)$, we obtain the advection equation. We can put ourselves in a moving frame, the speed of which is defined in equation (3.7) by
\[
s_{mf} = h_0^2 \sin \beta - \frac{6\mu q_0 h_0(1 + h_0)}{(1 - h_0)^4}.
\]

If we let the speed of the moving frame $s_{mf}$ to vanish, we can investigate interfacial dynamics at "slower" time scale.

At $O(\epsilon^2)$, we obtain an equation of KS type with an additional integral term:
\[
H_{\tau \tau} + \alpha_2 H_{\xi} \int_0^1 H \, d\xi + \alpha_1 H H_{\xi} + \lambda H_{\xi \xi} + \gamma H_{\xi \xi \xi \xi} + \theta H_{\xi} = 0. \tag{3.8}
\]
This additional integral term appears in the equation due to the constant pressure drop condition and represents the nonlocal dependence on the interfacial height. Moreover, since this integral is a constant, we expect that it effectively only changes the speed of the deviation wave.

In order to understand fully the influence of this additional term and, to relate it to the previous results of Tilley et al. [68], we consider the periodic domain $[0, 2\pi]$ with periodic boundary conditions for $H$. We apply the Galilean transformation (since $\xi$ is $2\pi$-periodic, this transformation is consistent with periodic boundary conditions)

$$x = \xi - \theta t,$$

where $\theta$ defines the speed of the moving frame at "slower" time scale. Introducing the new time scale $t = \gamma t_1$, we obtain KS equation with additional integral term

$$H_t + \alpha' HH_x + \lambda' H_{xx} + H_{xxxx} + \alpha_2' H_x \int_0^{2\pi} H d\xi = 0$$

(3.9)

where $\lambda' = \lambda/\gamma$ is the ratio of inertial and hydrostatic effects to capillarity; $\alpha' = \alpha_1/\gamma$ and $\alpha_2' = \alpha_2/\gamma$ (for simplicity we drop primes below). We note that for $\alpha_2=0$ equation (3.9) reduces to KS equation.

The KS equation corresponding to the constant gas volumetric flow rate regime has the following invariance property: if $(u(y, t), \lambda, \alpha)$ forms a solution set, then so does $(ku(ky, k^4t), k^2\lambda, k^2\alpha)$. The modified KS equation corresponding to the constant pressure drop regime has the following modified invariance property: if $(u(y, t), \lambda, \alpha, \alpha_2)$ forms a solution set, then so does $(ku(ky, k^4t), k^2\lambda, k^2\alpha, k^2\alpha_2)$.

Since for zero-mean solutions the integral term vanishes, we consider only non-zero-mean solutions. In order to compare with previous works of the KS equation, we let $\alpha = \lambda$ be a bifurcation parameter since the KS equation can then be rescaled to depend only on a single parameter. We use the initial condition:

$$h(x) = 0.1 + 0.0002 \sin x.$$
In Tilley et al. [68], it is shown that bifurcation points of the KS equation occur on the basic state \( u = 0 \) for \( \lambda = n^2, n = 1, 2, \ldots \) (Fig. 3.2). Similarly to the problem for a thin film at the pipe wall (Papageorgiou et al. [56]) where the KS equation contains the integral term, this term changes the interfacial profile from chaotic to doubly periodic waves. We also expect to observe the changes in the interfacial dynamics. In order to see these changes, we investigate the interfacial behavior near the first two bifurcation points. As an example, we use \( \lambda = 1.001 \) and \( \lambda = 4.05 \).

![Bifurcation diagram](image)

**Figure 3.2** Bifurcation diagram for steady-state, spatially periodic solutions of the KS equation: stable solution branches (solid line); unstable solution branches (dashed); bifurcations to traveling-wave solutions (diamonds). From Tilley et al. [68].

**Case \( \lambda = 1.001 \)**

Figure 3.3 shows profiles of steady waves for values of \( \alpha_2 \). This figure shows that \( \alpha_2 \) modifies only the speed of the wave, but not its shape. Based on the fact that shape is preserved, we conjecture that stability properties are unchanged as well. This type of behavior is also confirmed by asymptotic analysis (Appendix C).

To understand the influence of the additional integral term in equation (3.8), we next analyze the speed and maximum value of the interfacial height for different
values of $\alpha_2$. Since we expect that the integral term modifies the propagation speed, we consider the coefficients in front of the first derivative of $H$. The sum of these coefficients, $\alpha_2 \int_0^{2\pi} H d\xi + \alpha H_{\text{max}}$ (with $H_{\text{max}}$ being the maximum height of the interfacial deviation) is defined as the "analytically" obtained speed. The "numerical" speed is calculated as the time derivative of the position of the maximum of the wave. We obtain that this speed is constant and, in particular, equal to $\approx 1.3$ for $\alpha_2 = 2$. This constant speed confirms the steadiness of the traveling wave. Figure 3.4 shows that the numerical and analytical speeds overlap for smaller values of $\alpha_2$ (approximately in the range from -6 to 6), with small differences for larger $|\alpha_2|$.

**Case $\lambda = 4.05$**

In paper of Tilley et al. [68], it is shown that the basic state $u = 0$ loses its stability for $\lambda > 1$ in sense that from the pitchfork bifurcation at $\lambda = 1$ (Fig. 3.2), a steady-state branch emerges. Following this branch, a symmetry-breaking bifurcation ($\lambda = 3.25$, labeled as SB1 in Fig. 3.2) occurs, from which a branch representing a pair of asymmetric traveling waves emerges.
For $\lambda = 4.05$, KS has two stable solutions (Tilley et al. [68]) in the form of two asymmetric waves traveling in the opposite directions with the same speed. Depending on the initial condition, the solution can be attracted to either one of those stable waves and stay there for a long time. Figure 3.5 shows that in our case (with integral term) there are two stable asymmetric wave profiles corresponding to different values of $\alpha_2$. The change of sign of $\alpha_2$ corresponds to flipping the wave profiles.

After a long time, the wave corresponding to case $\alpha_2 = 2$ reaches its stable profile since its maximum height reaches the constant value of 4.9. The wave corresponding to $\alpha_2 = -2$ also reaches steady state with a maximum absolute height of 4.77. The wave propagates in the corresponding direction (defined by the sign of $\alpha_2$) with constant speed (Fig. 3.6). The speed is calculated in the same manner as in the previous case $\lambda = 1.001$.

From weakly nonlinear analysis for a periodic domain near two bifurcation points, we find that the final interfacial states are unaffected (except for their resulting phase speed) by the additional integral term.
So far, we have considered the periodic domain. In the case of a fixed domain we consider the local disturbance of the interface to confirm our conclusion above. Figure 3.7) presents the interfacial profiles at time $t=0.4$ corresponding to flows driven by either the prescribed pressure drop or constant gas volumetric flow rate. We observe that at the given time the profile corresponding to the constant pressure drop moves further than the one corresponding to constant gas volumetric flow rate, while the shape is preserved.

Thus, the integral term appearing in the KS equation due to the constant pressure drop regime only modifies the propagation velocity but it does not modify the stability properties. This conjecture about unchanged stability properties has been reached through combination of analytical and numerical methods. We use a similar numerical approach to solve the full problem. Therefore, the computations performed here also serve as a check of our numerical method which we use to solve the full problem of the evolution and pressure gradient equations.

![Figure 3.5](image_url)  

**Figure 3.5** Wave profile for different values of $\alpha_2$ for positive mean at time $t=49$. 
Figure 3.6  Speed profile for two values $\alpha_2=-2, 2$ for positive mean of interfacial height.

Figure 3.7  Interfacial height profiles versus $x$ for constant $\Delta P > 0$ ($\alpha_2 = 900$), and for constant $q$ (obtained from the solution of equation (3.8)). Common parameters are $\lambda=-0.04$, $\gamma=0.004$, $\alpha_1=4800$ and $\theta=0$. The length of domain is $2\pi$, $x$ is the downstream coordinate.
CHAPTER 4

UNDERCOMPRESSIVE SHOCKS

We proceed with the discussion of the numerical simulation of the full problem (2.60), (2.61) for three cases of prescribed driving forces: (i) constant gas volumetric flow rate $q$ (Case I), (ii) constant gas pressure drop $\Delta P$ (Case II), and (iii) fixed liquid flow rate $Q_l$ and gas pressure drop $\Delta P$ (Case III). We use the results from our linear stability and weakly nonlinear analyses to verify our numerical approach. Next, we turn our attention to the contact line problems, i.e., problems with initial profile with smoothed step function (the interfacial height at the left end of the channel is bigger than the height at the right end of the channel). These problems allow developing shocks in the interfacial profile.

We also need to address the boundary condition at the fluid front. It is well known that adherence to the no-slip boundary condition at a moving contact line (defined as the position where the liquid meets the solid substrate) produces a singularity in the stress in situations involving the spreading of a liquid on a solid (Dussan and Davis [26]). One of the proposed approaches to alleviate this problem assumes the existence of a precursor film beyond the nominal contact line. This precursor film can be purposely coated onto a substrate or it can develop naturally as a result of evaporation/condensation processes or surface diffusion occurring at the advancing front of the spreading film. The thickness of this precoating film can therefore assume macroscopic to microscopic dimensions. In either case its presence removes the singularity at the contact line.

Problems allowing development of shocks on the interface are worth pursuing since the large-amplitude counterpropagating waves, being observed in the flooding phenomenon, may actually be related to the developing undercompressive shocks.
4.1 Numerical Scheme

Before outlining the details of the numerical simulations, we discuss the imposed boundary conditions. At the inlet of the channel, we assume the liquid film thickness as a given height $h$ for the flow driven by prescribed gas volumetric flow rate (Case I) or pressure drop (Case II). We allow interfacial height to adjust for flows driven by fixed liquid flow rate and pressure drop (Case III).

We choose the simplest boundary condition at the outlet of the channel, a precursor film of thickness $h_+$ (Bertozzi and Brenner [6], Trojan and Kataoka [38], [39]). At the inlet of the channel, we prescribe the height of the liquid film to be equal to $h_-$ for cases of fixed $q$ or $\Delta P$. We close the system of differential equations assuming either the interface remains flat at the boundaries of the channel: i) $h_x(x = 0) = 0$ and $h_x(x = L) = 0$ for fixed $q$ (Case I) or $\Delta P$ (Case II); ii) $h_x(x = 0) = h_{xxx}(x = 0)$ and $h_x(x = L) = h_{xxx}(x = L) = 0$ for fixed $Q_l$ and $\Delta P$ (Case III).

Next, we draw attention to the numerical simulation and related issues. We use a finite difference method with equally spaced grid points to solve the highly nonlinear evolution equation (2.60) with parameters $\beta$, $G$, $S$, $\mu$, $\rho$. The time discretization is performed by a $\Theta$-scheme

$$\frac{h_k^{n+1} - h_k^n}{\Delta t^n} + \theta f_k^{n+1} + (1 - \theta)f_k^n = 0, \quad k = 1, ..., N. \quad (4.1)$$

where $0 \leq \theta \leq 1$, $n$ stands for the time level $t^n$ and $k$ corresponds to the grid point $x_k$ and $N$ to the number of grid points. Here, $\theta = 0$ gives the forward Euler explicit scheme, $\theta = 1$ gives the backward Euler implicit scheme, and $\theta = 1/2$ (which we use) yields the Crank-Nicholson scheme.

Equations (4.1) form a nonlinear algebraic system of $N$ equations, which is solved using iterative Newton-Kantorovich's method (for the description of method, we refer to Diez and Kondic [22]). The idea is to linearize the nonlinear algebraic system of equations around the guess of the interfacial height $h_k^n$ and obtain the
system of linear equations for the correction $\delta h_k$ for the guess $h_k^*$. The obtained system of linear equations leads to a penta-diagonal matrix of coefficients which is solved using the biconjugate gradient method, one-dimensional in space (see Press et al. [58]). As the guess for the solution $h_k^*$ we use the solution at the previous time level. If the maximum value of the correction $\delta h_k$ is greater than the tolerance (typically, $10^{-10}$), then we choose a new initial guess as $h_k^* + \delta h_k$, $k = 1, \ldots, N$. This procedure is performed iteratively until the maximum value of the correction is less than the tolerance.

To verify the implemented numerical method, we compare the numerical simulations with the predictions of linear stability theory. With this purpose in mind, we perturb the flat interface $h_0 = 0.5$ with a small sinusoidal wave (Fig. 4.1). We compute the phase speed of the wave in the vertical channel for zero pressure gradient (Table 4.1). Phase speed is calculated as the difference approximation to the velocity at times 0.818 and 0.409. For a simpler description, we define the numerical error calculated by

$$\text{Error} = \left| \frac{c_{\text{num}} - c_{\text{lin.stab.}}}{c_{\text{lin.stab.}}} \right| \cdot 100\%$$

where $c_{\text{num}}$ denotes the phase speed obtained numerically and $c_{\text{lin.stab.}}$ is from our linear stability theory (3.1). From Table 4.1, we conclude that with more points in the numerical grid, the phase speed calculated numerically converges to the phase speed predicted by the linear stability analysis. Similar results were obtained for growth rate $k_c$, i.e., $k_c$ calculated numerically is equal to $-2.01 \cdot 10^{-6}$, while from the linear stability analysis, we obtain value $-1.25 \cdot 10^{-6}$.

Next, we investigate the counter propagating large-amplitude traveling waves that can lead to the undercompressive shocks.
Figure 4.1 Interfacial dynamics for constant gas volumetric flow rate case (Case I) for $h_0=0.5$, $q=0$, $\beta = \pi/2$, $S = 3$, $\rho = \mu = 1$, $G = 0$, $\epsilon = 0.001$. Solid line shows the initial profile with disturbance wave.

Table 4.1 Comparison of numerical and linear stability phase speed.

<table>
<thead>
<tr>
<th>Number of points ($N$)</th>
<th>$c_{num}$</th>
<th>$c_{rin.stab.}$</th>
<th>Error, %</th>
</tr>
</thead>
<tbody>
<tr>
<td>50</td>
<td>0.2521</td>
<td>0.2500</td>
<td>0.8</td>
</tr>
<tr>
<td>100</td>
<td>0.2513</td>
<td>0.2500</td>
<td>0.52</td>
</tr>
<tr>
<td>200</td>
<td>0.2510</td>
<td>0.2500</td>
<td>0.4</td>
</tr>
<tr>
<td>400</td>
<td>0.2499</td>
<td>0.2500</td>
<td>0.04</td>
</tr>
</tbody>
</table>
4.2 Admissible Solutions

In this section, we examine traveling-wave solutions for a set of fluid and flow parameters. Next, we derive \textit{a priori} bounds for the values of $h(x, t)$ in the system with and without inertial effects with the particular external driving force (constant gas volumetric flow rate (Case I) or constant pressure drop (Case II) and gravity.

Before concentrating on our problem, we briefly outline the main conclusions of the related works by Bertozzi \textit{et al.} [9], which analyzed the model resulting from a single phase flow driven by a thermal gradient induced shear stress in the opposite sense of gravity

$$h_t + (h^2 - h^3)_x = -\epsilon(h^3 h_{xxx})_x,$$

where $h$ is the fluid thickness and $\epsilon$ is a small parameter. The traveling wave solutions connect an upstream height $h_\infty$ to a downstream height $b < h_\infty$, and steepen to shock wave solutions of (4.2) as $\epsilon \to 0$. If the speed $s$ of a resulting traveling wave satisfies $\lambda(b) < s < \lambda(h_\infty)$ with $\lambda(h) = 2h - 3h^2$ then the shock is called a compressive or Lax shock. Shocks violating this condition are called undercompressive. It is found that if $h_\infty - b$ is small, then the shock is compressive. If $h_\infty$ increases, there are multiple traveling waves approximating a Lax shock. In this range of $h_\infty$, the stable solution of (4.2) is found, which is composed of two waves traveling with different speeds. The slower wave corresponds to a Lax shock joining $h_\infty$ to a height $h_{uc} > h_\infty$, where $h_{uc}$ (height of the undercompressive shock) is independent of $h_\infty$, while the faster wave corresponds to an undercompressive shock from $h_{uc}$ to $b$. Experimentally the observed transition is due to a fundamental change in structure of the front, from a classical Lax shock (for negligible gravity), which is linearly unstable to perturbations, to a structure (for non-negligible gravity), that includes an "undercompressive shock", which is linearly stable to perturbations at the leading front (Troian and Kataoka [38], [39]). The undercompressive structure also manifests itself in larger bumps that
continue to broaden. The corresponding value of \( h_{uc} \) is found through numerical analysis and values of \( b \) and \( h_{\infty} \) are prescribed based on the nonconvex flux function behavior.

To explore traveling-wave solutions in our problem we extend the domain to the entire plane in \( \xi \)-direction and then equation (2.60) for constant gas volumetric flow rate takes the form

\[
h_{\tau} + f_\xi + \epsilon \left[ \frac{S}{3} h^3 h\xi\xi - \frac{h^3}{3} h\xi \cos \beta + G\bar{I} \right] = 0, \tag{4.3}
\]

where

\[
f(h) = \frac{h^3}{3} \sin \beta + \frac{6h^2 - 3h + 1}{(1-h)^3} \mu q + \epsilon \left[ -\frac{12\mu^2 q h^3}{(1-h)^5} - \frac{\mu \sin \beta h^4 (h^2 + 3)}{3(1-h)^3} + \frac{\mu h^2 (h+3)}{(1-h)^3} \gamma \right], \tag{4.4}
\]

\[
\bar{I} = g(h) h\xi, \tag{4.5}
\]

with

\[
g(h) = \frac{h^5(65h^2 + 665h + 112)}{140(1-h)^4} \mu q \sin \beta + \frac{3\mu^2 q^2}{35} \frac{h^5(5h^3 + 72h^2 + 392h + 245)}{(1-h)^8}
+ \frac{6pq^2}{35} \frac{h^2(27h - 10)}{(1-h)^3} + \frac{2}{15} h^6 \sin^2 \beta, \tag{4.6}
\]

\[
\gamma = \mu q \left[ -\frac{4}{(1-h)^3} + \frac{9}{(1-h)^2} - \frac{6}{1-h} \right]
+ \mu q \left[ -\frac{4}{(1-h_-)^3} + \frac{9}{(1-h_-)^2} + \frac{6}{1-h_-} \right]. \tag{4.7}
\]

Next, we formulate necessary conditions for existence of traveling-wave solutions, discuss the interfacial dynamics for constant gas volumetric flow rate, and the influence of the liquid inertia.

Note that the capillary shock profile is a traveling-wave solution \( h(\xi - st) \) of the PDE (4.3) where \( s \) is the shock speed and satisfies the Rankine-Hugoniot condition

\[
s = \frac{f(h_-) - f(h_+)}{h_- - h_+}.
\]
We derive bounds on the maximum height of any traveling wave solution and also bounds on the admissible far field states \((h_\pm)\) for such a solution. A traveling wave \(h(\xi - st)\) connecting the state \(h_-\) to the state \(h_+\) satisfies

\[-sh_x + f_x = -\alpha(h^3h_{xxx})_x + \theta(h^3h_x)_x - \epsilon G\bar{I}_x,\]  

(4.8)

where

\[\alpha = \frac{\epsilon S}{3},\]

\[\theta = \frac{\epsilon \cos \beta}{3}.

We integrate equation (4.8) once to obtain

\[-sh + f = -\alpha h^3h_{xx} + \theta h^3h_x - \epsilon G\bar{I} + Q,\]  

(4.9)

where

\[Q = -sh_+ + f(h_+).

Our approach of finding the necessary conditions for the existence of the traveling-wave solution follows the approach of Bertozzi et al. [9] to (4.2). Basically, it consists of two parts: 1) formulate a Lyapunov function for equation (4.9); 2) use this Lyapunov function to find \textit{a priori} pointwise upper and lower bounds of traveling waves. For the rigorous proof of the existence of traveling waves with undercompressive shocks we refer to paper of Bertozzi and Shearer [12] which can be generalized to our problem. In presentation here, we concentrate only on those aspects that are different from the Marangoni thin film problem.

The Lyapunov function for equation (4.9) is of the form

\[L(h) = \alpha h_{xx}h_x + R(h).\]
Note that without inertia present in the system, the Lyapunov function is increasing. Here, we first concentrate our attention on the simpler problem without inertial effects, and discuss the effects of fluid inertia in Section 4.3.2.

In order to find lower and upper bounds for traveling-wave solutions, we look for the critical points of the Lyapunov function:

\[-sh + f - Q = 0. \quad (4.12)\]
By construction $h_+$ and $h_-$ are zeroes of $R'(h)$. We are interested to find traveling-wave solutions connecting $h_+$ and $h_-$. After we substitute the expressions for $s$ and $Q$ into this equation, it can be rewritten as

$$[f(h_-) - f(h)][h_- - h_+] = [f(h_-) - f(h_+)][h_- - h].$$

From here one can see that $h_+$ and $h_-$ are the roots. We can expect more roots as the equation is of the 8th order (see the definition of $f(h)$ in equation (4.4)).

Firstly, let us consider the case when $R'(h)$ has exactly two zeroes at $h_-$ and $h_+$ and other roots come in pairs (either real or complex). Figure 4.2 shows the corresponding structure of $R(h)$. From the graph of $R(h)$ and the fact that $R(h_-) < R(h) < R(h_+)$ we conclude that there exist values $h_* < h_+$ and $h_{**} > h_-$ such that

$$R(h_*) = R(h_-),$$

$$R(h_{**}) = R(h_+).$$

Note that all bounded solutions connecting $h_-$ and $h_+$ lie between $h_*$ and $h_{**}$. The chord connecting $h_+$ and $h_-$ does not cross the flux function graph defined by equation (4.4), similarly to the convex flux case. The interfacial dynamics in this case is characterized by the development of Lax shock solutions (see Fig. 4.8 for numerical solution).

Next, Fig. 4.3 shows the structure of $R(h)$ with three extrema ($h_+, h_-, h_3$). This case is more complex but the extrema of the traveling-wave solution still satisfy

$$h_* \leq h \leq h_{**},$$

where

$$h_* = \min \{ h | R(h) \geq R(h_2) \},$$

$$h_{**} = \max \{ h | R(h) \geq R(h_2) \}.$$
In this case, the chord connecting $h_+$ and $h_-$ crosses the graph of the flux function at a third point. This is a sign of a nonconvex function. Taking into account the regularization in our equation (surface tension term) of the fourth order, we can expect the appearance of undercompressive shocks in the interfacial dynamics. This expectation is verified by numerical simulation (see Fig. 4.13).

Thus, we have found necessary condition for the existence of the traveling-wave solutions in our system. More detailed discussion of the sufficient conditions for the existence of traveling-wave solutions can be found in Bertozzi and Shearer [12]. Next, we determine which values of $h_-$ and $h_+$ allow us to obtain these solutions.

### 4.3 Entropy-entropy Flux Pairs and Constraints on Admissible Capillary Shock Profiles

In order to determine the values of $h_+$ and $h_-$ that lead to the formation of weak solutions of equation (4.3), we proceed with the entropy-entropy flux pair concept to arrive at entropy inequality. We discuss the derivation of the entropy inequality.
Figure 4.3  \( R(h) \) versus \( h \) for using system parameters and notation as in Fig. 4.2, \( h_- = 0.6 \). Here, the function \( R(h) \) has three extrema. In this case, we expect the developing of undercompressive shocks at the leading front of the interfacial profile (see Fig. 4.13).

under different prescribed boundary conditions: constant \( q \) (Case I) and constant \( \Delta P \) (Case II). Since qualitatively the system of equations describing the flow driven by constant \( Q_l \) and constant \( \Delta P \) (Case III) is similar to the one corresponding to Case I, we only discuss Case I in full details. In addition, we analyze the influence of the inertial terms on the boundaries of the region of admissible values \( h_+ \) and \( h_- \).

4.3.1 Constant Gas Volumetric Flow Rate without Inertial Effects

Any scalar conservation law of the form

\[
h_t + (f(h))_x = 0
\]

(4.13)

can be rewritten as

\[
\bar{G}(h)_t + F(h)_x = 0,
\]
where $\bar{G}, F$ are called an entropy-entropy flux pair: $\bar{G}$ is convex and $F$ is related to $\bar{G}$ by compatibility with the conservation law (4.13)

$$F'(h) = \bar{G}'(h)f'(h).$$

This notation of an entropy-entropy flux pair is used since the function $\bar{G}$ for equation

$$h_t + (f(h)h_{xxx})_x = 0$$

possesses the property $\int L \bar{G}(h(x, t))dx \geq 0$ for all $t$ (Bertozzi and Pugh [11]). This inequality is similar to the balance equation of the second law of thermodynamics expressed as $S > 0$ where $S$ is the entropy. In addition, for the case $f(h) = h$, the entropy $\int \bar{G}(h)$ is of the form $\int h \log h$ (Bertozzi [5]) which corresponds to the entropy of ideal gas.

We use this concept to derive a priori bounds for an admissible capillary shock. In order to do this, we note that equation (4.3) can be reduced to an ODE (4.8). This equation (4.8) is coupled with the function $\bar{G}$ after appropriate integration results in the entropy inequality. Here, we use the important restriction on $\bar{G}$ being convex (the second derivative positive). From there we proceed to find which $h_-$ and $h_+$ admit a traveling-wave solution of equation (4.3).

Let us apply the entropy-entropy flux pair concept to equation (4.3) to find traveling-wave solutions that connect the left state $h_-$ to the right state $h_+ < h_-$. Let us assume that $h_+ > 0$. If we have such a solution, $h(\xi - st)$, it must satisfy equation (4.8). Now we consider a function of $h$, $\bar{G}(h)$, that satisfies $\bar{G}''(h) > 0$ on some range of $h$. Multiplying equation (4.8) by $\bar{G}'(h)$, and integrating by parts from $-\infty$ to $\infty$ gives

$$-s [\bar{G}(h)] + [F] = \int \bar{G}'r(h)dx,$$

(4.14)

where

$$r(h) = \epsilon \left[ -\frac{S}{3} \left\{ h^3 h_{xxx} \right\}_x + \frac{\cos \beta}{3} \left\{ h^3 h_x \right\}_x - GI_x \right].$$
The first term in integral (4.14)

\[ \int -\epsilon \frac{S}{3} \left\{ h^3 h_{xxx} \right\} x \tilde{G}' dx = \epsilon \frac{S}{3} \int h^3 h_{xxx} h_x \tilde{G}'' dx \]

is non-positive for \( \tilde{G}''(h) = h^{p-3} \) provided that \( p \) satisfies the inequality \(-\frac{1}{2} < p \leq 1\). This result is obtained following the analysis of Bertozzi et al. [9], [7] for the weak solutions to the fourth order degenerate diffusion equation. Case \( p = 1 \) corresponds to the entropy function \( \tilde{G}(h) = -\ln(h) \).

The second term in integral (4.14), describing the hydrostatic pressure effects, is negative for \( 0 < h < 1 \)

\[ \epsilon \int \frac{\cos \frac{\beta}{3}}{3} \left\{ h^3 h_x \right\} x \tilde{G}' dx = -\epsilon \frac{\cos \frac{\beta}{3}}{3} \int \tilde{G}'' h^3 h^2_x dx. \quad (4.15) \]

Thus, from (4.14) it follows

\[ -s \left[ \tilde{G}(h) \right] + [F] \leq 0. \quad (4.16) \]

The jump in the entropy can be written as

\[ [\tilde{G}] = -\ln(h_+) + \ln(h_-), \]

and the jump in the flux is

\[ [F] = F(h_+) - F(h_-), \]

where

\[
F(h) = -\frac{h^2}{2} - 6\mu q \sin \beta \left[ \frac{2}{3(1-h)^3} - \frac{1}{2(1-h)^2} \right] - \epsilon \left[ 6\mu^2 q \sin \beta \left\{ -\frac{8}{3(1-h)^6} + \frac{34}{5(1-h)^5} - \frac{7}{(1-h)^4} + \frac{10 + 2C_1}{3(1-h)^3} - \frac{1 + C_1}{(1-h)^2} \right\} \right] - \epsilon \mu \left\{ \frac{(1-h)^2}{2} - 3(1-h) + 3 \ln(1-h) - \frac{3}{(1-h)} + \frac{4}{(1-h)^2} - \frac{4}{3(1-h)^3} \right\},
\]
Figures 4.4 and 4.5 show the left- and right-hand sides of the inequality equation (4.17). For prescribed parameters of the system, and for given $h_+$ and $h_+$, there are maximum and minimum values of $h_-$ for which traveling waves exist. If we fix $\epsilon$, $\mu$, $\beta$, and $h_+$ and increase $q$ from -0.01 (Fig. 4.4) to -0.001 (Fig. 4.5) then the region of admissible $h_-$ increases from zero to the range $h_+ = 0.01 \leq h_- \leq 0.635$. The changes in the range of admissible $h_-$ can also be caused by the variation of other parameters (e.g., $p$, $q$, $\beta$), as it can be seen from equation (4.17).

Figure 4.6 shows the regions of admissible values of $h_-$ for different values of $q$. With the increase of $q$, the corresponding region of admissible $h_-$ decreases. Thus, the size of the region where we guarantee the traveling-wave solutions depends on the value of gas volumetric flow rate. The fact that, in particular, for large absolute
values of $q$, this range is not very wide explains the absence of experimental evidence of undercompressive shocks in countercurrent flows. Our results predict that these solutions should be easier to observe in the system characterized by smaller absolute value of gas volumetric flow rate.

4.3.2 Constant Gas Volumetric Flow Rate Including Inertial Effects

The inertial effects manifest themselves in terms denoted by $\tilde{I}$ in equation (4.3). To see whether inertial effects can change the interfacial dynamics significantly, we go back to the integration of (4.14) with the additional term

$$-\epsilon \int G \tilde{l}_x G' dx = \epsilon G \int \tilde{I} \tilde{G}'' h_x dx = \epsilon G \int g \tilde{G}'' h_x^2 dx. \tag{4.18}$$

In order to be consistent with a similar analysis performed for the first two terms, we require the nonpositivity of the right-hand side of equation (4.14), i.e., we discuss the competing effects of hydrostatic pressure and inertia. Thus, we arrive at the
Figure 4.6 Necessary but not sufficient conditions for admissible solutions to exist for the values of $h_-$ inside the regions bounded by the curves, while outside the region we can not guarantee the traveling-wave solution: $q = -0.001$ (solid curve), $q = -0.005$ (dotted curve) and $q = -0.007$ (dashed curve) ($G = 0$).

\[
\frac{h^3 \cos \beta}{3} \geq G \left[ \frac{\mu q \sin \beta h^5 (65h^2 + 665h + 112)}{140(1 - h)^4} + \frac{2}{15} h^6 \sin^2 \beta \right] \\
+ G \left[ \frac{3}{35} (\mu q)^2 h^5 (5h^3 + 72h^2 + 392h + 245)}{(1 - h)^8} \right] \\
+ G \left[ \frac{6}{35} \rho q^2 h^2 (27h - 10)}{(1 - h)^3} \right].
\] (4.19)

Figure 4.7 shows the region of admissible values of $h_-$ (bounded by solid line) following from this condition. We see that the admissible region decreases significantly compared to the region bounded by the dashed line (the case without inertial effects). However, we note that this restriction may be too strong, since we do not take into account the influence of surface tension.

To estimate the relevance of inertial effects if surface tension is included, we note that the inequality, equation (4.19), has five parameters: $\beta$, $G$, $\rho$, $\mu$, $q$. For simplicity, we prescribe values of $\rho$, $\mu$ equal to 1. Also, we fix $\beta = \pi/2$ (vertical
Figure 4.7 Admissible values of $h_-$ for $q = -0.001$ with inertia (solid) and without inertia (dashed). The theory guarantees admissible solutions in the region bounded by the solid curves but it does not preclude their existence in the rest of the dashed region.

Next, we balance inertial terms which are of order $q^2 h^2$, and surface tension which is of order $Sh^3$ in equation (4.3). Taking into account this balance, using the magnitude of constant gas volumetric flow rate $q \approx -0.001$ (corresponds to countercurrent flow), and surface tension $S = 3$, we conjecture from equation (4.3) and inequality (4.19) that inertial effects are expected to be negligible for countercurrent flows in vertical channels. This estimate is supported by the numerical results presented in section 4.4.2. The situations when inertial terms become relevant and important is discussed in more details in Section 4.4.2.

### 4.3.3 Flow Driven by Constant Pressure Drop

Next, we consider the flow driven by constant pressure drop (Case II). Here, instead of equation (4.3), we obtain the following equation

$$h_\tau + A_1(h, \Delta P)h_\xi + \epsilon [SC(h)h_{\xi\xi\xi} + A_2(h, \Delta P) + GI(h, \Delta P) + P(h)h_\xi]_\xi = 0,$$
The integral term depends on the unknown interfacial height \( h \). Therefore, we cannot perform the analysis similar to the case of constant gas volumetric flow rate. Instead, we resort to numerical simulations presented in the next section.

### 4.4 Numerical Simulation

In this section, we first consider the interfacial dynamics of the flow driven by the constant gas volumetric flow rate. Next, we analyze the influence of the inertia on the solution profile. Finally, we discuss the interfacial dynamics in the flows driven

where

\[
A_1(h, \Delta P) = h^2 \sin \beta - \frac{h(h+1)\Delta P}{2(1-h)^4 \int_0^1 \frac{d\xi}{(1-h)^3}},
\]

\[
C(h) = \frac{h^3}{3},
\]

\[
P(h) = -\frac{h^3}{3} \cos \beta,
\]

\[
A_2(h, \Delta P) = -\frac{\mu h^3 \Delta P}{4(1-h)^4 \int_0^1 \frac{d\xi}{(1-h)^3}} - \frac{\mu h^4}{4(1-h)} \sin \beta
+ \frac{\mu h^5(h+3)}{(1-h)^3} \left[ \gamma - \frac{h^2(h+3)}{12} \sin \beta - \frac{h \Delta P}{4(1-h)^2 \int_0^1 \frac{d\xi}{(1-h)^3}} \right],
\]

\[
I(h, p_0, \xi) = R_1 + R_2 + R_3,
\]

with

\[
R_1 = \frac{h^4(7h + 25)}{240} p_0 \xi + \frac{-4h^2 + 49h + 56}{840(1-h)^4 \int_0^1 \frac{d\xi}{(1-h)^3}} h^5 h \xi \Delta P \sin \beta,
\]

\[
R_2 = -\frac{h^5(10h^2 + 7h + 77)}{3360(1-h)^7} h_\xi \left[ \frac{\Delta P}{\int_0^1 \frac{d\xi}{(1-h)^3}} \right]^2 + \frac{2}{15} h^6 h_\xi \sin^2 \beta,
\]

\[
R_3 = \frac{h^2(27h - 10)}{840(1-h)^3} \frac{h \xi \rho}{\mu^2} \left[ \frac{\Delta P}{\int_0^1 \frac{d\xi}{(1-h)^3}} \right]^2,
\]

where

\[
\gamma = -\frac{\Delta P}{\int_0^1 \frac{d\xi}{(1-h)^3}} \int_0^\xi \frac{h(h+1)h_\xi \Delta P}{2(1-h)^4 d\xi}.
\]

The integral term depends on the unknown interfacial height \( h \). Therefore, we cannot perform the analysis similar to the case of constant gas volumetric flow rate. Instead, we resort to numerical simulations presented in the next section.
by a constant pressure drop and constant liquid flow rate and constant gas pressure drop.

4.4.1 Constant Gas Volumetric Flow Rate without Inertial Effects

The numerical simulations are performed in a traveling reference frame moving with the speed

$$s(h_-, h_+) = \frac{f(h_-) - f(h_+)}{h_- - h_+}.$$  \hspace{1cm} (4.20)

All cases discussed below use $G = 0$, $q = -0.001$, $h_+ = 0.1$, $\rho = \mu = 1$, $S = 3$, $\beta = \pi/2$; however, the same qualitative dynamics emerge for all $h_+$ that are in the admissible ($q$-dependent) range presented in Fig. 4.6. In each case, we consider a range of left states $h_-$ for which the dynamics of the PDE (4.3) has certain observed characteristic behavior. We note that the values $h_1$, $h_2$ and $h_{UC}$ discussed below all depend upon the particular choice of system parameters and the right state $h_+$.

The cases considered below are analogous to those investigated by Bertozzi et al. [9] for (4.2).

**Case 1: $h_+ < h_- < h_1$: Unique weak Lax shock.**

Given $h_+$, there is a value $h_1$ such that for all $h_+ < h_- < h_1$, the solution of the PDE (4.3) is always observed to evolve to a unique capillary shock profile connecting the states $h_-$ and $h_+$. We consider two initial profiles

$$h_0(x) = [\tanh(-x + 20) + 1] \frac{h_- - h_+}{2} + h_+$$  \hspace{1cm} (4.21)

and

$$h_0(x) = \begin{cases} 
\frac{h_+ - h_-}{2} \tanh(x - 18) + \frac{h_+ + h_-}{2} & \text{if } x < 21 \\
-\frac{h_+ - h_-}{2} \tanh(x - 24) + \frac{h_+ + h_-}{2} & \text{if } x > 21 
\end{cases}$$  \hspace{1cm} (4.22)
connecting the states $h_-$ and $h_+$. Figures 4.8 and 4.9 show that the solutions, computed in the reference frame moving with speed $s$ (4.20), settle down to the unique steady traveling wave solution, which is independent of the initial condition.

**Case 2:** $h_1 < h_- < h_2$: *Multiple Lax shock.*

For $h_-$ in the range $h_1 < h_- < h_2$, there are multiple capillary shock profiles connecting the same two left ($h_-$) and right ($h_+$) states. Figures 4.10 and 4.11 show several solutions connecting $h_+$ and $h_- = 0.431$. Depending on whether the initial conditions (4.21) or (4.22) are used, we obtain either the single Lax shock or double shock structure, evolving into the undercompressive shock (leading shock) and Lax shock (trailing shock). The trailing shock moves with a slower speed than the leading shock. Due to the difference in speeds of propagation the capillary ridge broadens. This slower front speed is the characteristic feature of the undercompressive shock.

Figure 4.12 shows the dependence of the speed of propagation of Lax (dash-dot) and undercompressive (solid line) shocks on $h_-$. We see that the difference in the
Figure 4.10 Interfacial profile resulting in steady Lax shock (in moving frame) for $h_\gamma = 0.431$ and other parameters as in Fig. 4.8 using initial condition given by equation (4.21).
Figure 4.11 Double shock wave profile at time $t=1800$ for $h_-=0.431$ and other parameters as in Fig. 4.8, using the initial condition given by equation (4.22). Note that this double shock structure is no longer steady. It moves to the left (in the direction of the arrows) with capillary ridge broadening in width.

The propagation speeds of Lax and undercompressive shocks is much more pronounced for larger values of $h_-$. Note that the extrapolated value $h_-$, corresponding to the intersection point of the two speed lines in Fig. 4.12, is the same value ($h_1$) which was found numerically as the limiting height for the Case 1 (the critical height below which we observe the unique Lax shock).

**Case 3:** $h_2 < h_- < h_{UC}$: Undercompressive double shock structure.

For $h_- in this range, there are no capillary shock profiles joining $h_-$ and $h_+$. All initial conditions converge to a solution with the same double shock structure. Figure 4.13 illustrates this behavior for the case $h_- = 0.433$. The solution at the later time is characterized by the presence of two shocks. The leading shock is the undercompressive shock while the trailing shock is a classical Lax shock. We note that these are not steady solutions in the moving frame, since the capillary ridge grows in width with time.

**Case 4:** $h_- > h_{UC}$: Rarefaction-undercompressive shock.
**Figure 4.12** Dependence of Lax and undercompressive shocks speeds on the height in front of the shock $h_-$ for constant gas volumetric flow rate $q = -0.001$ and other parameters as in Fig. 4.8.

**Figure 4.13** Undercompressive double shock structure for $t = 300$, $h_- = 0.5$ and other parameters as in Fig. 4.8, with the initial condition given by equation (4.21). The direction of propagation is indicated by the arrows.
For \( h_\geq h_{UC} \) we obtain a two-wave structure in which the slower wave is a rarefaction wave solution of our PDE. These simulations are performed in a larger domain in order to analyze the dynamics for longer times. Furthermore, these simulations are performed in the laboratory frame. Here, we use \( h_- = 0.68 \) and the initial condition

\[
h_0(x) = [\tanh(-x + 925) + 1] \frac{h_- - h_+}{2} + h_+
\]

(we shift the initial condition (4.21) in order to follow the dynamics for longer times).

Figure 4.14 shows a combination of rarefaction wave and undercompressive shock. Since the undercompressive shock moves with a speed \( s(h_{UC}, b) \), which is greater than the speed \( f'(h_{UC}) \) of the right-hand side (leading edge) of the rarefaction wave, the undercompressive shock separates from the rarefaction wave to produce a separated rarefaction shock profile (Bertozzi et al. [9]).

If we increase \( h_- \) even further, we are near the boundary of the region which guarantees the existence of traveling-wave solutions (see Fig. 4.6 - solid line). Figure 4.15 shows the solution at time \( t = 100 \); this solution still involves the combination of rarefaction wave and undercompressive shock. However, for this \( h_- \), we also observe formation of a ‘step’ which is not present for smaller \( h_- \). Since this step-like profile is not present in the results for a single fluid flow (Bertozzi et al. [9]), we conjecture that this structure is the result of the influence of the top of the channel on the flow dynamics. We also note that Fig. 4.15 is the first case considered so far, where the liquid flow rate is reversed. We now discuss this new feature in more details.

Figure 4.16 shows the dependence of the liquid flow rate \( (Q_l) \) on \( h_- \) for different gas volumetric flow rates \( (g) \). If we fix the constant gas volumetric flow rate to be rather small \((-10^{-4})\), then we see the countercurrent liquid flow which persists for all admissible values of \( h_- \). If we increase the absolute value of gas volumetric flow rate to \( g = -10^{-3} \), then for some \( h_- \), the direction of the liquid flow changes from
countercurrent to cocurrent upstream flow. In other words, for \( q \) in some range, their exists a critical value of \( h_\epsilon \), such that if \( h_- > h_\epsilon \), there is flow reversal. We note that in all explored instances, this reversal coincides with the formation of the 'step' profile as in Fig. 4.15. For even larger absolute values of constant gas volumetric flow rate (e.g., \( q = -10^{-2} \)) we see this reversal for all admissible values of \( h_- \). We conjecture that this liquid flow reversal is possibly related to the onset of flooding (Dukler and Smith [25]).

Figure 4.17 summarizes all our numerical simulations presented so far. In this figure, for particular values of \( q = -0.001, S = 3, \epsilon = 0.01, \mu = \rho = 1, \beta = \pi/2 \), we plot the regions where the four cases considered above appear. Note that numerically found values of \( h_1, h_2 \) and \( h_{UC} \) for case \( h_+ = 0.1 \) are different from the corresponding values of \( h_1, h_2 \) and \( h_{UC} \) for \( h_+ = 0.2 \) and \( h_+ = 0.3 \). We observe that the increase of value \( h_+ \) leads to a decrease of the size of regions where we obtain unique Lax shock, multiple Lax shocks, double shock structure, or combination of rarefaction wave and undercompressive shock.
Figure 4.15  Interfacial profile for $h_\text{=} = 0.8$ and other parameters as in Fig. 4.8 at time 100. The dynamics of the solution indicates the influence of the top of the channel on the interfacial profile resulting in the development of a "step".

Figure 4.16  The dependence of the liquid flow rate, $Q_l$ on $h_\text{=}$ for fixed gas volumetric flow rate. Note the change from countercurrent flow for $q = -10^{-4}$ to cocurrent upstream flow for $q = -10^{-2}$.
Figure 4.17  Regions for admissible traveling shock solutions for $q = -0.001$ and other parameters as in Fig. 4.8. The circles denote $h_1$, diamonds $h_2$, squares corresponds to $h_1$ for constant pressure drop case (see section 4.4.3), and light right triangles denote the combination of rarefaction wave and undercompressive wave.

4.4.2 Constant Gas Volumetric Flow Rate Including Inertial Effects

We now consider the role of the inertia of the denser layer on the interfacial dynamics. Figure 4.18 shows the interfacial profiles in the countercurrent flows with and without inertial effects. In this case of countercurrent flow, the presence of fluid inertia neither changes the interfacial profiles, nor the speed of propagation. However, in the case of the cocurrent downstream flow, we observe different behavior. Figure 4.19 shows the interfacial profiles for this case, with large positive gas volumetric flow rate ($q = 1$) when inertial effects are included ($G = 4$, solid line), or not ($G = 0$, dashed line). The initial profile is shown as dash-dot line. We see that the position of the classical Lax shock at the time $t = 20$ is significantly different (compared to almost overlapped profiles in Fig. 4.18) in these two cases, although the shape of the shock profiles remains the same. This suggests that the influence of fluid inertia is more pronounced in the cocurrent downstream flow. We have also verified numerically that the fluid
4.4.3 Flow Driven by Constant Pressure Drop

Next, we analyze the case of constant pressure drop (Case II). Thus, we allow both the gas and liquid volumetric flow rates to change in time (see section 2.2). Of course, the incompressibility condition requires that the total flow rate remains constant.

Figures 4.20 and 4.21 show the formation of the unique Lax shock solutions and double shock wave structures evolving into undercompressive shock, respectively. These interfacial profiles are very similar to the ones we obtained for constant gas volumetric flow rate case for similar values of $h_{-}$, see Figs. 4.9 and 4.11.

Following the approach we applied to the flow characterized by constant gas volumetric flow rate, we analyze the parameter space $(h_{+}, h_{-})$ for the constant pressure drop case. We find the admissible values of $h_{-}$ for particular values of $h_{+}$ and $\Delta P$, corresponding initially to $q = -0.001$. From Fig. 4.17, we deduce that inertia does not change the boundaries of the shock traveling wave regions, shown in Fig. 4.17.
**Figure 4.19** Interfacial profiles for constant gas volumetric flow rate case without inertia ($G = 0$, dashed line) and with inertia ($G = 4$, solid line) using $q = 1.0$, $h_\_ = 0.25$ and other parameters as in Fig. 4.8. Note the change of sign of $q$ compared to the ones used in other figures.

**Figure 4.20** Interfacial dynamics of traveling unique Lax shock wave solution for constant pressure drop case ($\Delta P$ is chosen so it corresponds to $q = -0.001$ at time $t = 0$) using $h_\_ = 0.25$ and other parameters as in Fig. 4.8.
the regions, where the interfacial profile changes from the Lax shock to multiple Lax shocks involving the undercompressive shocks, are very close for these two cases.

Figure 4.22 compares interfacial profiles for countercurrent constant pressure flow in two cases: with and without inertia. Clearly, inertial effects do not change the interfacial dynamics in this case. Thus, we conclude that inertial effects are negligible in the countercurrent case for both types of external driving force: constant gas volumetric flow rate (Case I) and constant pressure drop (Case II). We have verified that with inertia present in the system, we still obtain the solutions characterized by a unique Lax shock, or a double shock structure, similar to the case of constant flux flow.

Thus, one could assume that the dynamics of constant gas volumetric flow rate and constant pressure drop are similar. However, there is one significant difference, already visible in Fig. 4.21: the height of the capillary ridge.

Figure 4.23 compares this height for the two considered cases. While in the case of the constant volumetric flow rate, height of the capillary ridge is constant in time,
for constant pressure drop, the height of the ridge grows in time. Therefore, it appears that the constant pressure condition may lead to a behavior which qualitatively resembles that of the onset of flooding. This does not occur for equivalent constant flux configuration. The observation that the choice of boundary conditions (or, more physically, of the mechanism driving the flow) is important in determining the onset of flooding could be of significant relevance for future work on this problem.

Next, we turn our attention to the flow driven by prescribed liquid flow rate and pressure drop (Case III).

4.4.4 Flow Driven by Constant Liquid Flow Rate and Constant Pressure Drop with Inertial Effects

We proceed to examine four regions obtained for the case of constant gas volumetric flow rate. We investigate whether different imposed boundary conditions for the pressure and interfacial height (constant $Q_l$ and constant $\Delta P$), which are more realistic in practice, lead to a different interfacial dynamics.
Figure 4.23 Height of the capillary ridge versus time for constant pressure drop (solid) and constant gas volumetric flow rate (dashed) cases using $h_- = 0.5$ and other parameters as in Fig. 4.8. The corresponding interfacial profile for the constant pressure drop case is shown in Fig. 4.22 and for the constant gas volumetric flow rate in Fig. 4.18. After initial transients, we see steady increase of the height in the case of the flow driven by constant pressure drop.

To generalize the problem, we include inertial effects into our system in order to explore whether these effects are important with these flow conditions. All cases discussed below use $G = 1$, $Q_t = 0.01$, $h_+ = 0.1$, $\rho = \mu = 1$, $S = 3$, $\beta = \pi/2$. Fixing the liquid flow rate along with the initial condition determines the applied pressure drop, $\Delta P$, which we keep fixed in each case from now on.

**Case 1: Weak Lax shock.**

Figure 4.24 shows the initial interfacial profile (solid line) evolving into the Lax shock (dashed line). Contrary to Fig. 4.8 where the profile reaches its steady state, we observe that the Lax shock is no longer steady. Since this may not be clear from Fig. 4.24, we plot the phase speed $s$ given by (4.20) in Fig. 4.25a. This figure shows that $s$ does not saturate to a constant value. Moreover, the average (mean) interfacial height (calculated as as integral of $h(x,t)$ over the length of the channel) increases
in time (Figure 4.25b). Both graphs therefore show that Lax shock does not settle down to an unique steady profile.

Thus, in this case, we observe a classical Lax shock which is unsteady.

**Case 2:** \( h_1 < h_- < h_2 \): *Multiple Lax shock.*

Next, we proceed with simulations for \( h_- \) corresponding to multiple capillary shock profiles connecting the same two left \( (h_-) \) and right \( (h_+) \) states. Figure 4.26 shows several solutions connecting \( h_+ = 0.1 \) and \( h_- = 0.431 \). Depending on whether the initial conditions (4.21) or (4.22) are used, we obtain either the single Lax shock or a double shock structure, evolving into the undercompressive shock (leading shock) and Lax shock (trailing shock). Similar to Case 1, the Lax shock profile is no longer steady, as shown in Fig. 4.27a. However, the double shock structure reaches its steady state (Fig. 4.27b). Here, the maximum height of the interface reaches the constant value, equal to 0.67. Based on the last figure, we can conjecture that our system of two fluids reaches its equilibrium.

**Case 3:** \( h_2 < h_- < h_{UC} \): *Undercompressive double shock structure.*
Figure 4.25  Speed profile (a) and mean interfacial height (b) for a Lax shock with the parameters as in Fig. 4.24.

Figure 4.26  Interfacial profiles for $h_- = 0.431$ and other parameters as in Fig. 4.24. (a) Weak Lax shock (in moving frame), using the initial condition (dashed line) given by (4.21). (b) Double shock wave profiles using the initial condition (dashed line) given by (4.22). Note that this double shock structure is no longer steady in the sense that it moves to the left with capillary ridge broadening before it approaches its steady state (solid line) at time $t = 3500$. 
Figure 4.27  (a) Change of the average interfacial height (defined on page 72) for the parameters from Fig. 4.26a. (b) Maximum interfacial height for parameters from Fig. 4.26b.

For $h_-$ in this range, there are no capillary shock profiles joining $h_-$ and $h_+$. All initial conditions converge to a solution with the same double shock structure. Figure 4.28a illustrates this behavior for the case $h_-=0.5$. The solution at the later time is characterized by the presence of two shocks. We observe here that the double shock wave structure exhibits unsteady growth, in contrast to the results shown in Fig. 4.13a. Figure 4.28b presents growth of the maximum of the interfacial height as the thickness of the precursor film decreases (corresponding to the fact that the mean interfacial height decreases in time).

**Case 4: $h_- > h_{UC}$: Rarefaction-undercompressive shock.**

For $h_- > h_{UC}$, we obtain a two-wave structure in which the slower wave is a rarefaction wave solution and the other is undercompressive shock. Here, we observe some fascinating dynamics. We use $h_-=0.68$ and the initial condition

$$h_0(x) = [\tanh(-x + 925) + 1] \frac{h_- - h_+}{2} + h_+.$$
Figure 4.28 (a) Interfacial profile with undercompressive shock for \( h_- = 0.5 \) and other parameters as in Fig. 4.24. Dotted line shows the initial condition, dashed and dashed-dot lines show undercompressive shock. We use the initial condition given by (4.21). (b) Change of the maximum height of undercompressive shock in time.

Note that for this case, simulations are performed in the fixed frame in order to capture two-wave structure.

Figure 4.29 shows a combination of rarefaction wave and undercompressive shock. By comparison with Fig. 4.13b, we see some features in Fig. 4.29a which were not present before. Here, the rarefaction wave not only separates the undercompressive double shock structure but it also divides classical and undercompressive shocks within the undercompressive wave. Further, the obtained solution is no longer steady in the sense that interfacial height grows at the inlet of the channel. This can be easily seen in Fig. 4.29a from the comparison of the initial profile to the one at time 600. Also, Figure 4.29b shows the maximum of the interfacial height which is an increasing function of time. Thus, the height of interfacial profile exhibits the unsteady growth.

If we increase the inlet height \( h_- \) even further, we reach the region where we cannot analytically guarantee the existence of traveling-wave solutions. However,
Fig. 4.30 shows that we still obtain the combination of two waves: rarefaction and undercompressive. In addition, in Fig. 4.31a we see that the interfacial profile still exhibits unsteady growth, similarly to the results shown in Fig. 4.29b. We conclude that in the case with prescribed liquid flow rate and gas pressure drop (Case III), it is more likely that interfacial profile eventually reaches the top of the channel (example of flooding phenomenon) compared to the flow driven by constant gas volumetric flow rate (Case I).

Another significant difference between these two cases is that the overall average interfacial height is no longer a constant. Figure 4.31b shows the gradual increase of the mean height for \( h_- = 0.8 \). This is another indication that the interfacial profile grows to the degree that the liquid film impedes the gas flow.

Thus, in all four cases discussed above, we observe unsteady growth of the interfacial height. Additionally, the average interfacial height as well as the maximum of the interfacial profile grow in time. We note that the inertial effects do not modify these conclusions: very similar results are obtained in their absence (\( G = 0 \) in Figs. 4.24-4.31).
Figure 4.29  (a) Interfacial profiles for \( h_- = 0.68 \) and other parameters as in Fig. 4.24. Dashed line is initial condition and solid line shows the solution at time \( t=600 \). (b) Maximum of interfacial height.

Figure 4.30  Interfacial profile at \( t = 0 \) (dashed line) and \( t = 55 \) (solid line) for \( h_- = 0.8 \) and other parameters as in Fig. 4.24.
Figure 4.31 Change of the maximum of interfacial height (a) and average interfacial height (b) in time for parameters as in Fig. 4.29. Both graphs show that the height of interfacial profile and the mean height grow in time.
 CHAPTER 5

LONG-WAVE LINEAR STABILITY THEORY FOR WEAKLY COMPRESSIBLE GAS

So far we have investigated the flow of two incompressible fluids through linear stability analysis, weakly nonlinear analysis and numerical simulation of the reduced system of the evolution and leading-order pressure gradient equations. Now, we relax the assumption of the upper fluid being incompressible. In this case, as in Chapter 2, the Navier-Stokes equations with corresponding boundary and interfacial conditions are reduced to the system of pressure gradient/density and evolution equations (we neglect $O(\varepsilon^2)$ terms)

\[ -\left[ \frac{\rho \varepsilon^{(2)}(1-h)^3}{12\mu} \right]_\xi + \varepsilon \left( \rho \left(1 - h - \frac{h(h+2)}{4} \right) \rho \varepsilon \sin \beta + \frac{h^2(1-h)}{4} \rho \varepsilon \sin \beta \right) \]

\[ + \varepsilon \left( \frac{h^2(h+3)}{12} \rho \varepsilon p - \frac{(1+h)^2}{4(1-h)} \rho h \varepsilon p + GT \right) = 0, \]

\[ h_r + A_1 + \varepsilon \left( S \frac{h^3}{3} h_{\varepsilon \varepsilon \varepsilon} - S \frac{h^3}{3} h_{\varepsilon \cos \beta} + A_2 + GI \right)_\xi = 0, \]  

(5.1)

(5.2)

where

\[ A_1 = h^2 h \varepsilon \sin \beta - \left[ \frac{h^2(h+3)}{12} \varepsilon^{(2)} \right]_\xi, \]

\[ A_2 = \frac{\mu h^3 \varepsilon}{4(1-h)} - \frac{\mu h^4}{4(1-h)} \sin \beta, \]

\[ I = \frac{h^4(7h+25)}{240} \varepsilon^r + \frac{13}{480} h^2(1-h) \frac{P}{\mu^2} \varepsilon \left\{ -h \varepsilon^r + \frac{\varepsilon^r}{\rho} \varepsilon(1-h) \right\} \]

\[ - \frac{h^5(10h^2+7h+77)}{3360(1-h)} h \varepsilon^2 + \frac{2}{15} \frac{h^6 h \varepsilon \sin^2 \beta}{\rho} - \frac{h^5(29h^2+161h-378) \varepsilon \rho}{\rho^2} \varepsilon^2 \]

\[ + \sin \beta \left[ \frac{h^6(109h+147)}{10080} \frac{\varepsilon^r}{\rho} + \frac{h^5(41h^2-49h-56)}{840(1-h)} \varepsilon^r \varepsilon h \varepsilon \right], \]

\[ T = -\frac{17}{3360 \mu^3} \left[ \rho \varepsilon^{(2)}(1-h)^6 h - \rho \varepsilon \rho \varepsilon^2(1-h)^7 \right]_\xi, \]
This reduced system (5.1), (5.2) is closed using the relation

\[ p^{(2)} = D \rho. \]

which essentially assumes isothermal conditions for an ideal gas. The ideal gas equation of state \((pV = nRT)\), taking into account scaling for pressure (page 17), gives (see also page 23)

\[ K^* \equiv \epsilon^3 D dg = \frac{nRT}{m}, \]

We note that \(m/n\) (molar density) is equal to approximately 0.030 kg for air, \(R\) is the gas constant (8.3143 \(J/°K/mole\)), acceleration \(g\) is \(\approx 10 m/s^2\). We choose temperature \(T = 300 K\), and the small parameter \(\epsilon = 10^{-2}\). Then the value of \(D\), that we are interested in, can be expressed in terms of the dimensional height of the channel \(d\). Since the particular scale for surface tension \(S\) and gravitational factor \(G\) is based on \(d\), we obtain, for example, for \(d \approx 10^{-3} m\), \(D \approx 8\). Therefore, we require parameter \(D = O(1)\) in our problem.

The appropriate boundary conditions for interfacial height and density fully determine the problem. In the evolution equation (5.2), we require that the first and third derivatives vanish at both ends of the channel. This choice of boundary conditions is motivated by the fact that it is consistent with sinusoidal perturbations that will be used in linear stability analysis. In the density equation (5.1), we prescribe \(\rho(0, t) = \rho_1\) and \(\rho(1, t) = \rho_2\), therefore also prescribing a constant pressure drop \((\Delta P = D[\rho_2 - \rho_1])\).

From the leading order pressure gradient/density equation (5.1), we find that

\[ (\rho_0^2)_{\xi} = \frac{2C(\tau)}{(h - 1)^3} \]

for the density basic state, where \(C(\tau)\) is a function of time resulting from integration. Given the boundary conditions for \(\rho\), we can solve this equation explicitly
We now proceed to perform linear stability analysis on the system (5.1), (5.2) using (5.3), (5.4), (5.5).

\[ \rho_0^2(\xi) = \tilde{Y} \rho_2^2 + (1 - \tilde{Y}) \rho_1^2, \]  

(5.3)

where

\[ \tilde{Y} = \frac{Y(\xi)}{Y(1)}, \quad Y(\xi) = \int_0^\xi \frac{1}{(h - 1)^3} dv. \]

Inertial terms in both equations (5.1), (5.2) can be simplified after taking appropriate derivatives to

\begin{align*}
I_\xi = & \frac{Dh^3(7h + 20)}{48} \rho_\xi + \frac{Dh^4(7h + 25)}{240} \rho_\xi \rho_T \\
& - \frac{13D^2h^2(1 - h)^4}{480 \mu^2} [\rho_\xi^3 h_\xi + 2 \rho \rho_\xi \rho_T h_\xi + \rho \rho_\xi^2 h_\xi] \\
& - \frac{h^5(10h^2 + 7h + 77)}{3360(1 - h)} [2D^2 \rho \rho_\xi h_\xi + D^2 \rho_\xi^2 h_\xi] + \frac{2}{15} h^6 \rho_\xi \sin^2 \beta \\
& - \frac{D^2}{20160} h^4(203h^2 + 966h - 1890) h_\xi \rho_\xi^2 - \frac{D^2 h^5(29h^2 + 161h - 378)}{20160} \left[ \frac{3 \rho_\xi^2 \rho_\xi}{\rho} - \rho_\xi^2 \right] \\
& + \frac{13D^2 h(1 - h)^4 (2 - 7h)}{480 \mu^2} \rho_\xi^2 h_\xi + \frac{13D^2 h^2(1 - h)^5}{480 \mu^2} \left[ \rho_\xi^3 + 2 \rho \rho_\xi \rho_\xi \right] \\
& + \sin \beta \left[ \frac{D h^5(41h^2 - 49h - 56)}{840(1 - h)} \left\{ \rho_\xi \rho_\xi h_\xi + \rho_\xi h_\xi \right\} \right] \\
& + \sin \beta \left[ \frac{D h^6 (109h + 147)}{10080} \left\{ 2 \rho \rho_\xi h_\xi \rho_\xi h_\xi + \rho_\xi^2 \right\} + \frac{D}{1440} h^5 (109h + 126) h_\xi \rho_\xi^2 \right], 
\end{align*}

(5.4)

\begin{align*}
T = & - \frac{51}{1120 \mu^3} \rho_\xi^2 (1 - h)^5 h_\xi - \frac{17}{1680 \mu^3} \rho_\xi^2 \rho_\xi (1 - h)^5 h_\xi \\
& - \frac{17}{3360 \mu^3} \rho_\xi^2 (1 - h)^5 h_\xi + \frac{17}{3360 \mu^3} \rho_\xi^4 (1 - h)^6 + \frac{17}{1120 \mu^3} \rho_\xi^2 \rho_\xi (1 - h)^6. 
\end{align*}

(5.5)

We now proceed to perform linear stability analysis on the system (5.1), (5.2) using (5.3), (5.4), (5.5).

If we assume that the basic interfacial height is a constant, from equation (5.3) it follows that the basic density state depends on the downstream coordinate. Due to this spatial dependence of the basic density state, we cannot use the linear stability analysis based on the normal mode expansion. However, normal mode analysis can
be carried out in the special case when $\rho_2 = \rho_1$. The normal mode analysis for this special case is presented in Section 5.2.1. First, we consider a different limit of a single phase flow and explore under which conditions the dynamics of two-fluid flow mimics the single phase flow dynamics. For simulations, we typically use the following parameters: $\mu=1$, $\rho_1=1$, $\beta = \pi/2$, $S=3$, $\epsilon = 0.01$.

### 5.1 Linear Stability for Thin Liquid Films ($h_0 \ll d$)

Single-phase film flow has been investigated for more than a half of a century. The analysis of this flow was started first by Yih and Benjamin [3], followed by Joo, Davis and Bankoff [34]. A lot of attention was paid to linear stability analysis in order to obtain unique criteria for the instability development. Significant results in form of inequalities involving system parameters were obtained. Here, we compare our results of two-fluid problem to the results obtained for single-phase flow.

To compare the stability of the two-fluid and of a single-fluid flow and to compare to the previous works on the single-fluid flow, we rescale our system of equations (5.1), (5.2) using a small parameter $h_0 = h_0^*/L$ describing the ratio of fluid thickness to the channel thickness; we assume $h_0 = h_0(\epsilon)$. Then, in the limit $h_0 \to 0$ and assuming a passive gas, we expect to observe the interfacial dynamics in the two-fluid system similar to the single-phase flow. In order to perform this comparison we, first, write equations (5.1) and (5.2) in dimensional form

\[
- \frac{\mu_1^*}{\mu_2^*} \left[ \frac{dL^* \rho_2^* p_{2x}^* (1 - h^*/d)^3}{12} \right]_{x^*} + \left( \frac{\nu_1^*}{g \rho_1^* \rho_2^*} (1 - h^*/d) + \frac{h^*(h^*/d + 2)}{4} \frac{\rho_2^*}{\rho_1^*} h_{x^*}^* \sin \beta \right) \\
+ \frac{h^*(h^*/d)}{4} \frac{\rho_2^*}{\rho_1^*} \sin \beta + \frac{L h^*}{12dg} (h^*/d + 3) \frac{\rho_2^*}{\rho_1^*} p_{2x}^* = 0, \\
\frac{\nu_1^*}{g} h_{t^*}^* + A_1 + \left( \frac{\sigma^* h^*}{3} h_{x^*}^* \cdot \left( \frac{h^3}{3} h_{x^*}^* \cos \beta + A_2 + \vec{f} \right) \right)_{x^*} = 0,
\]

where

\[
A_1 = h^2 h_{x^*}^* \sin \beta - \frac{L}{\rho_1^* g} \left[ \frac{h^2(h^*/d) + 3}{12} p_{2x}^* \right]_{x^*}.
\]
\[ A_2 = \frac{\mu_2^*}{d \mu_1^*} \frac{h^*}{4(1 - h^*/d)} \left( \frac{L}{\rho_1^* g} p_{z^*}^* - h^* \sin \beta \right), \]

\[ \bar{I} = \frac{g}{\nu_1^*} \left[ \frac{h^*}{240} \left( \frac{7h^*/d + 25}{\rho_1^* g^2 p_{z^*}^*} + \frac{2}{15} h^* \delta_{z^*}^* \sin^2 \beta \right) \right]. \]

Next, we scale all our variables similar to Chapter 2 with one difference: as a length scale, we now use the basic height \( h_0^* \)

\[
\begin{align*}
\tau &= t^* g h_0^{*2} / \nu_1^* L, \\
h &= h^* / h_0^*, \\
\xi &= x^* / L, \\
p &= p^* / \rho_1^* h_0^* g, \\
S &= (\delta_{h_0}^*) (\sigma^*/\rho_1^* h_0^{*2} g), \\
\mu &= L \mu_2^*/h_0^* \mu_1^*, \\
\rho &= L^2 \rho_2^*/h_0^{*2} \rho_1^*, \\
G &= g h_0^* / \nu_1^{*2}, \\
\epsilon &= d / L.
\end{align*}
\]

We then arrive to the nondimensional system of equations

\[ - \epsilon \left[ \frac{pp_{\xi}}{12 \mu} (1 - \frac{\bar{h}_0}{\epsilon} h)^3 \right] + \bar{h}_0 \rho_{\tau} \left( 1 - \frac{\bar{h}_0}{\epsilon} h \right) = 0, \quad (5.7) \]

\[
\begin{align*}
\bar{h}_\tau + \left[ h^2 h_\xi \sin \beta - \left( \frac{h^2}{12} p_\xi (h \bar{h}_0 / \epsilon + 3) \right) \right] + G \left[ \frac{h^4}{240} (7\bar{h}_0 h + 25) p_{\xi\tau} \right] \\
+ \bar{h}_0 \left[ \frac{S}{3} h^3 h_{\xi\xi} - \frac{h^3}{3} h_\xi \cos \beta + \frac{2}{15} Gh^6 \delta_{z^*}^* \sin^2 \beta \right] &= 0. \quad (5.8)
\end{align*}
\]

This coupled system describes the interaction between the gas and liquid phases, assuming that \( 0 \ll \bar{h}_0 \ll \epsilon \ll 1. \)
If we assume \( \rho_\xi \to 0 \) and \( \rho_r \to 0 \), then equation (5.7) is automatically satisfied, and from equation (5.8), we obtain

\[
h_r + h^2 h_\xi \sin \beta + \tilde{h}_0 \left[ \frac{S}{3} h^3 h_\xi \xi - \frac{h^3}{3} h_\xi \cos \beta + \frac{2}{15} G h^6 h_\xi \sin^2 \beta \right]_\xi = 0. \tag{5.9}
\]

The same equation results from Joo et al. [34], where time scale and surface tension coefficient are defined as \( \tau/G \) and \( SG \) respectively.

From equation (5.9), after a normal mode expansion

\[
h(\xi, \tau) = 1 + \delta e^{ik\xi + \omega \tau},
\]
we arrive at the dispersion relation

\[
\omega + ik \sin \beta + \tilde{h}_0 k^2 \left[ \frac{S}{3} k^2 + \frac{1}{3} \cos \beta - \frac{2}{15} G \sin^2 \beta \right] = 0. \tag{5.10}
\]

When surface tension is neglected, this relation yields the instability condition

\[
G \sin \beta > \frac{5}{2} \cot \beta. \tag{5.11}
\]

This same criteria was also obtained by Benjamin [3], Yih [72] and Joo et al. [34]. The fact that we obtain the same criteria shows that our model, in the limit when \( \rho_\xi \to 0, \rho_r \to 0, \) and \( \tilde{h}_0 \to 0 \) reduces to the single-phase thin film flow.

If surface tension is present, equation (5.10) shows that there is a cutoff wave number \( k_c \) above which the flow is stable. The value of \( k_c \) is found by setting the real part \( \omega_r = 0 \). The maximum growth rate then occurs at \( k = k_c/\sqrt{2} \).

Now we analyze the value of the cutoff wave number in the two-fluid system, concentrating, in particular, on the influence of compressibility. To better understand this problem, we compare the dynamics of two-phase flow driven by different prescribed pressure drops. Here we use \( \epsilon = 0.05 \) to see more clearly the changes in stability. Figure 5.1 shows the interfacial height mode growth rates corresponding to
$D$ changing from 0 to 10 (for $D=0$ we solve equation (5.10) to find growth rate). Part b) shows a closer view of the region $0 < k < 0.7$.

This figure shows that growth rates of the interfacial mode are very similar for small values of $D < D_{c,1} \approx 10^{-3}$ and also for $D > D_{c,2} \approx 1$.

![Figure 5.1](image.png)

**Figure 5.1** Growth rates of interfacial height for two fluid flow driven by gravity and pressure gradient (different $D$) vs wave number for $\varepsilon = 0.05$, $\tilde{h}_0 = 0.01$, $S = 0.5$, $G = 5$ for thin liquid films. For small and large values of $D$ the interfacial growth rates are almost the same. However, for moderately small values of $D$ (approximately $10^{-3} < D < 1$) the growth rates are qualitatively different. Part b is a closer view of the interfacial mode behavior for $0 < k < 0.7$.

However, we see different behavior for $D_{c,1} < D < D_{c,2}$. Here, in contrast to the incompressible case ($D=0$), and other values of $D$, we find stability of long wavelengths (see Fig. 5.1). The size of this stable region decreases with increase of $D$, and it vanishes for $D > D_{c,2}$. Therefore, for $D_{c,1} < D < D_{c,2}$ there exist two non-zero wave numbers where stability changes: $k_{c,1}$ (from stable to unstable) and $k_{c,2}$ (from unstable to stable). We discuss $D$-dependence of these wave numbers next.

Figure 5.2 shows how $k_{c,1}$ and $k_{c,2}$ change with $D$. For $D < D_{c,1}$, long wavelengths are unstable, as already noted in Fig. 5.1. Then, a stability island forms as $D$ increases beyond $D_{c,1}$. The size of this island decreases as $D$ grows. The
upper branch ($k_{c,2}$ in Fig. 5.2) shows that, except for very small $k$'s, the critical wave number does not depend strongly on $D$. However, the value of $k_{c,2}$ is smaller than the value $k^*_c=2$ found for incompressible flow ($D=0$).

![Figure 5.2](image)

**Figure 5.2** Dependence of the wave number $k_c$ where stability changes on $D$ for parameters as in Fig. 5.1 for very thin liquid films. $k^*_c$ corresponds to the single phase limit. “S” corresponds to the regions with negative growth rates and “U” denotes regions with positive growth rates.

To conclude, we find that compressibility of the gas may lead to stabilization of the long wavelengths for a certain range of $D$'s in the case of thin liquid films. Now, we proceed with analysis of the stability for general fluid thickness.

## 5.2 Linear Stability Theory for General Liquid Film Thickness

To pursue the linear stability analysis for general liquid film thickness, we return to our system of equations (5.1) - (5.2). To verify the approach used here, we consider first simplified problem where we assume the values of the density are equal at both ends of the channel ($\rho_1 = \rho_2$). This case is also amenable to a normal mode analysis which we consider next.
5.2.1 Normal Modes Analysis \((\rho_1 = \rho_2)\)

In order to apply normal modes analysis we use periodic boundary conditions, and introduce normal modes

\[
\begin{align*}
  h(\xi, \tau) &= h_0 + \delta_0 e^{ik(\xi - ct)}, \\
  \rho_0(\xi, \tau) &= \bar{\rho}_0 + \Delta_0 e^{ik(\xi - ct)}.
\end{align*}
\]

(the bar on \(\bar{\rho}_0\) is dropped below for simplicity). After substitution of these expansions into equations (5.1), (5.2), we arrive at the following system of characteristic equations

\[
\begin{align*}
  -iku\Delta_0 + ika_2\delta_0 + ika_4\Delta_0 + (ik)^2 a_5 \Delta_0 &= 0, \\
  -iku\delta_0 - (ik)^3 c\alpha \Delta_0 + b_1 (ik)^2 \Delta_0 + ikb_3 \delta_0 + (ik)^2 b_4 \delta_0 + b_5 (ik)^4 \delta_0 &= 0,
\end{align*}
\]

where \(k\) is a wave number, and

\[
\begin{align*}
  a_2 &= \frac{h_0(h_0 + 2)}{4(1 - h_0)} \rho_0 \sin \beta, \\
  a_4 &= \frac{h_0^2}{4} \sin \beta, \\
  a_5 &= -\frac{D(1 - h_0)^2}{12\epsilon\mu} \rho_0, \\
  b_1 &= -\frac{D}{12} h_0^2(h_0 + 3) + \epsilon \frac{\mu D h_0^3}{4(1 - h_0)}, \\
  b_3 &= h_0^2 \sin \beta - \frac{\epsilon \mu}{4} \sin \beta \left( h_0^3(4 - 3h_0) \right) (1 - h_0)^2, \\
  b_4 &= -\frac{h_0^3}{3} \cos \beta + \frac{2\epsilon G}{15} h_0^6 \sin^2 \beta, \\
  b_5 &= \frac{S h_0^3}{3}, \\
  \alpha &= \frac{D}{240} h_0^3(7h_0 + 25).
\end{align*}
\]

Thus, we obtain a system of two coupled equations (5.12), (5.13), which we solve to find \(iku\); the growth rate is given by the imaginary part of \(ku\).

First, we consider \(D = 0\) case. Here, the system (5.12), (5.13) decouples. This corresponds to the situation when the gas is passive (does not influence the interfacial
height dynamics), so only the interfacial mode is relevant. In this case, the system of equations (5.12), (5.13) simplifies to

\[- ikc\delta_0 + ikb_3\delta_0 + (ik)^2b_4\delta_0 + b_5(ik)^4\delta_0 = 0, \tag{5.14} \]
\[- ikc\Delta_0 + ik\alpha_2\delta_0 + ik\alpha_4\Delta_0 = 0. \tag{5.15} \]

From equation (5.14), we obtain the same growth rate for interfacial height as for the incompressible case (3.2), reproduced here for convenience

\[k\alpha_i = \epsilon k^2 \left[ -\frac{S}{3} h_0^3 k^2 - \frac{h_0^3}{3} \cos \beta + G \frac{2}{15} h_0^6 \sin^2 \beta \right], \tag{5.16} \]

The phase speed of the interfacial mode is defined by the advection term \(b_3\). In both cases (compressible and incompressible), there is a finite region of unstable wave numbers corresponding to the positive growth rate.

From equation (5.15), after division by \(ik\), we obtain the characteristic equation. Since \(a_2\) (advection due to gravity) and \(a_4\) (advection due to compressibility) in this characteristic equation are real coefficients, we conclude that the growth rate for the density mode (defined by the imaginary part of \(c\)) is not affected by the deviation from the interfacial profile. However, the density phase speed depends not only on the system parameters but also on the basic interfacial height and the deviation from it.

When \(D \neq 0\), the density and interfacial modes are coupled. If we assume that the interface is flat, the density mode is stable, as can be verified from equation (5.12). In this case, the density growth rate is given by

\[k\alpha_i = -\frac{D(1 - h_0)^2}{12\epsilon\mu} \rho_o k^2. \tag{5.17} \]

Clearly, we always obtain a stable density mode for a flat interface.

When the interfacial deviations are non-zero, we proceed with the general case defined by equations (5.12) - (5.13). We solve these equations numerically and plot...
in Fig. 5.3 interfacial height growth rates for cases of an incompressible ($D=0$) and compressible ($D \neq 0$) gas for $h_0=0.5$. The growth rate for the incompressible case is calculated using equation (5.16). For this $h_0$, we find that the growth rates for the interfacial modes are similar for $D=0$ and $D > 0$, although there is some $D$ dependence. The density mode, which is always stable, is not plotted for clarity, since it decays much faster than the interfacial height mode.

![Figure 5.3](image)

**Figure 5.3** Growth rate curves for interfacial mode vs wave number for parameters $\rho_2 = 1$, $h_0 = 0.5$, $G = 1$, and different values of $D$.

Figure 5.4 shows that when we increase the basic interfacial height further, the density mode (dashed-dot line) becomes unstable for a finite range of wave numbers $k < k_c$ ($k_c$ is not shown in the figure), while the interfacial height mode is stable. This $k_c$ is determined by the balance of stabilizing surface tension and hydrostatic pressure effects and destabilizing inertial effects. The growth rate of the interfacial height mode in this case (dashed line) is smaller compared to the incompressible case (solid line).

In order to analyze the dependence of $k_c$ on $D$, we plot in Fig. 5.5 the density mode for four different values of $D$. For larger $D$'s, we find larger range
of unstable wave numbers characterized by larger growth rates. Thus, the presence
of a compressible gas destabilizes the flow for channels with a high liquid to gas ratio
(note that for $h_0=0.5$, density mode is always stable). Additional simulations have
shown that this destabilizing effect is slightly influenced by liquid inertia. However,
the effect of liquid inertia is not as significant as it has been expected based on the
single-film flow results.

Figure 5.4 Density and interfacial growth rate curves vs wave number for
parameters $\rho_2 = 1$, $h_0 = 0.85$, $G = 1$ and $D=1$.

Next, we proceed with examining the influence of compressibility on a two-fluid
flow with $\rho_2 \neq \rho_1$ in a channel of a fixed length.

5.2.2 Linear Stability for a General Case ($\rho_2 > \rho_1$)
If we relax the assumption of equal values of densities at the ends of the channel,
we need to approach the stability problem more generally. In particular, periodic
boundary conditions are not appropriate. In order to test our approach to this more
general problem, we first apply it to $\rho_1 = \rho_2$ case. Therefore, we consider an interval
of length $2\pi$ and perturb the interfacial height and density

$$ h(\xi, \tau) = h_0 + \delta(\xi)e^{\alpha \tau}, \quad (5.18) $$
Figure 5.5 Growth rate curves for density mode vs wave number for parameters $\rho_2 = 1$, $h_0 = 0.85$, $G = 1$ and different values of $D$.

$$\rho_0(\xi, \tau) = \bar{\rho}_0(\xi) + \Delta(\xi)e^{\sigma \tau},$$  \hfill (5.19)

where $\delta(\xi), \Delta(\xi) \ll \epsilon$.

The basic state can be found from equation (5.3) using $h(\xi, \tau) = h_0$

$$\rho_0^2(\xi) = \xi \rho_2^2 + (1 - \xi) \rho_1^2.$$  \hfill (5.20)

After substitution of expansion (5.18), (5.19) into the linearized system of equations (5.1), (5.2) with equation (5.20) we arrive at the system of characteristic equations

$$\sigma \delta + \sigma \alpha \Delta_{\xi \xi} + \bar{b}_6 \Delta + \bar{b}_7 \Delta_{\xi} + \bar{b}_1 \Delta_{\xi \xi} + \bar{b}_2 \delta + \bar{b}_3 \delta_{\xi} + \bar{b}_4 \delta_{\xi \xi} + b_5 \delta_{\xi \xi \xi} = 0, \hfill (5.21)$$

$$\sigma \Delta + \bar{a}_4 \delta + \bar{a}_2 \delta_{\xi} + \bar{a}_6 \delta_{\xi \xi} + \bar{a}_3 \Delta + \bar{a}_4 \Delta_{\xi} + \bar{a}_5 \Delta_{\xi \xi} \Delta_{\xi} = 0, \hfill (5.22)$$

where

$$\bar{b}_6 = \epsilon G \theta_4,$$

$$\bar{b}_7 = \epsilon G \theta_5,$$
\[ \bar{b}_1 = b_1 + \epsilon G \theta_6, \]
\[ \bar{b}_2 = b_2 + \epsilon G \theta_1, \]
\[ \bar{b}_3 = b_3 + \epsilon G \theta_2, \]
\[ \bar{b}_4 = b_4 + \epsilon G \theta_3, \]
\[ \bar{a}_1 = a_1 + G \gamma_3, \]
\[ \bar{a}_2 = a_2 + G \gamma_1, \]
\[ \bar{a}_3 = a_3 + G \gamma_4, \]
\[ \bar{a}_4 = a_4 + G \gamma_5, \]
\[ \bar{a}_5 = a_5 + G \gamma_6, \]
\[ \bar{a}_6 = G \gamma_2, \]
\[ \alpha = \frac{D}{240} h_0^4 (7 h_0 + 25), \]

with

\[ a_1 = \frac{D - h_0^2 + 2 h_0 + 3}{12 (1 - h_0)^2} \rho_0 \xi + \frac{D (1 - h_0)}{6 \epsilon \mu} \left[ \rho_0^2 + \rho_0 \rho_0 \xi \right], \]
\[ a_2 = -\frac{D}{4} \left( \frac{1 + h_0}{1 - h_0} \right)^2 \rho_0 \rho_0 \xi + \frac{h_0 (h_0 + 2)}{4 (1 - h_0)} \rho_0 \sin \beta + \frac{D}{4 \epsilon \mu} \rho_0 \rho_0 \xi (1 - h_0), \]
\[ a_3 = -\frac{D}{12 \epsilon \mu} (1 - h_0)^2 \rho_0 \xi, \]
\[ a_4 = \frac{h_0^2}{4} \sin \beta + \frac{D h_0^2 (h_0 + 3)}{6 (1 - h_0)} \rho_0 \xi - \frac{D (1 - h_0)^2}{6 \epsilon \mu} \rho_0 \xi, \]
\[ a_5 = -\frac{D (1 - h_0)^2}{12 \epsilon \mu} \rho_0, \]
\[ b_1 = -\frac{D}{12} h_0^2 (h_0 + 3) + \epsilon \frac{\mu D h_0^3}{4 (1 - h_0)}, \]
\[ b_2 = \frac{D}{4} h_0 (h_0 + 2) \rho_0 \xi + \frac{\epsilon \mu D}{4} \left[ \frac{h_0^3 \rho_0 \xi}{(1 - h_0)^2} + \frac{3 h_0^2}{1 - h_0} \rho_0 \xi \right], \]
\[ b_3 = h_0^2 \sin \beta - \frac{D}{4} h_0 (h_0 + 2) \rho_0 \xi - \frac{\epsilon \mu D}{4} \frac{h_0^3 (4 - 3 h_0)}{(1 - h_0)^2} + \frac{\epsilon \mu D h_0^3 (3 - 2 h_0)}{4 (1 - h_0)^2} \rho_0 \xi, \]
\[ b_4 = -\frac{\epsilon h_0^3}{3} \cos \beta, \]
\[ b_5 = \frac{\epsilon S}{3} h_0^3, \]
\[
\theta_1 = - \frac{D^2}{20160} \left[ 3 \frac{\rho_{0\xi}^2 \rho_{0\xi\xi}}{\rho_0} - \frac{\rho_{0\xi}^4}{\rho_0^2} \right] h_0^4(203h_0^2 + 966h_0 - 1890) \\
+ \frac{13}{240} \frac{D^2}{\mu^2} \rho_0 \rho_{0\xi} \rho_{0\xi\xi} h_0(2 - 7h_0)(1 - h_0)^4 \\
+ \frac{D}{10080} \sin \beta h_0^5(763h_0 + 882) \left[ \frac{2\rho_{0\xi} \rho_{0\xi\xi}}{\rho_0} - \frac{\rho_{0\xi}^3}{\rho_0^2} \right],
\]
\[
\theta_2 = - \frac{D^2}{1680} \frac{h_0^5(10h_0^2 + 7h_0 + 77)}{1 - h_0} \rho_{0\xi} \rho_{0\xi\xi} - \frac{D^2}{20160} \frac{\rho_{0\xi}^3}{\rho_0} h_0^4(203h_0^2 + 966h_0 - 1890) \\
- \frac{13}{480} \frac{D^2}{\mu^2} h_0^5(1 - h_0)^4 \left[ \frac{\rho_{0\xi}^3}{\rho_0} + 2\rho_0 \rho_{0\xi} \rho_{0\xi\xi} \right] + \frac{13}{480} \frac{D^2}{\mu^2} h_0(1 - h_0)^4(2 - 7h_0) \rho_0 \rho_{0\xi}^2 \\
+ \sin \beta \left[ \frac{D}{840} \frac{h_0^5(41h_0^2 - 49h_0 - 56)}{1 - h_0} \rho_{0\xi} \rho_{0\xi\xi} + \frac{D}{1440} \frac{\rho_{0\xi}^3}{\rho_0} h_0^5(109h_0 + 126) \right],
\]
\[
\theta_3 = - \frac{D^2}{3360} \frac{h_0^5(10h_0^2 + 7h_0 + 77)}{1 - h_0} \rho_{0\xi}^2 + \frac{2}{15} h_0^6 \sin^2 \beta - \frac{13}{480} \frac{D^2}{\mu^2} \rho_0 \rho_{0\xi} \rho_{0\xi\xi}^2 h_0^2(1 - h_0)^4 \\
+ \frac{D}{840} \sin \beta \frac{h_0^5(41h_0^2 - 49h_0 - 56)}{1 - h_0} \rho_{0\xi} \rho_{0\xi\xi},
\]
\[
\theta_4 = - \frac{D^2}{20160} \left[ \frac{2\rho_{0\xi}^4}{\rho_0^2} - \frac{3\rho_{0\xi}^2 \rho_{0\xi\xi}}{\rho_0^2} \right] h_0^5(29h_0^2 + 161h_0 - 378) \\
+ \frac{13}{240} \frac{D^2}{\mu^2} \rho_0 \rho_{0\xi} \rho_{0\xi\xi} h_0^5(1 - h_0)^5 + \frac{D}{5040} \sin \beta h_0^6(109h_0 + 147) \left[ \frac{\rho_{0\xi}^3}{\rho_0} - \frac{\rho_{0\xi} \rho_{0\xi\xi}}{\rho_0^2} \right],
\]
\[
\theta_5 = - \frac{D^2}{10080} \left[ 3 \frac{\rho_{0\xi} \rho_{0\xi\xi}}{\rho_0} - \frac{2\rho_{0\xi}^3}{\rho_0^2} \right] h_0^5(29h_0^2 + 161h_0 - 378) \\
+ \frac{13}{240} \frac{D^2}{\mu^2} \rho_0 \rho_{0\xi} \rho_{0\xi\xi} h_0^5(1 - h_0)^5 + \frac{D}{10080} \sin \beta h_0^6(109h_0 + 147) \left[ \frac{2\rho_{0\xi} \rho_{0\xi\xi}}{\rho_0} - \frac{3\rho_{0\xi}^2}{\rho_0^2} \right],
\]
\[
\theta_6 = - \frac{D^2}{6720} \frac{\rho_{0\xi}^2 h_0^5(29h_0^2 + 161h_0 - 378)}{\rho_0} + \frac{13}{240} \frac{D^2}{\mu^2} \rho_0 \rho_{0\xi} \rho_{0\xi\xi} h_0^2(1 - h_0)^5 \\
+ \frac{D}{5040} \sin \beta h_0^6(109h_0 + 147) \frac{\rho_{0\xi}}{\rho_0},
\]
\[
\gamma_1 = - \frac{51}{1120\mu^3} \rho_0 \rho_{0\xi}^3(1 - h_0)^5 - \frac{17}{1680\mu^3} \rho_0^2 \rho_{0\xi} \rho_{0\xi\xi}(1 - h_0)^5,
\]
\[
\gamma_2 = - \frac{3360\mu^3}{17} \rho_0^2 \rho_{0\xi}^2(1 - h_0)^5,
\]
\[
\gamma_3 = \frac{17}{560\mu^3} \rho_{0\xi}^4(1 - h_0)^5 - \frac{51}{560\mu^3} \rho_0 \rho_{0\xi} \rho_{0\xi\xi}(1 - h_0)^5,
\]
\[
\gamma_4 = \frac{17}{1120\mu^3} \rho_{0\xi} \rho_{0\xi\xi}(1 - h_0)^6,
\]
\[
\gamma_5 = \frac{17}{840\mu^3} \rho_{0\xi}^3(1 - h_0)^6 + \frac{17}{560\mu^3} \rho_0 \rho_{0\xi} \rho_{0\xi\xi}(1 - h_0)^6,
\]
We solve equations (5.21) - (5.22) numerically using a finite difference method. Applying a finite difference approximation to this system gives the equation in matrix form

\[ A \cdot x + \sigma B \cdot x = 0, \]

where \( A \) is a matrix of coefficients resulting from finite difference approximation, \( B \) is a matrix formed from the coefficients in front of \( \sigma \), and vector \( x \) is a column of unknowns \( (\delta_1, \ldots, \delta_N, \Delta_1, \ldots, \Delta_N) \).

We perform the power method (Strang [65]) on the matrix \( A \) to find the growth rate for a given wave number. We use a grid with \( N = 64 \) grid points to approximate growth rate for the wave number \( k=1 \) and \( N = 128 \) for \( k=2 \).

The power method is based on an iterative scheme with the initial guess \( y_0 \)

\[
\begin{align*}
    u_1 &= Ay_0, & y_1 &= u_1/||u_1||, \\
    & \vdots \\
    u_{l+1} &= Ay_l, & y_{l+1} &= u_{l+1}/||u_{l+1}||.
\end{align*}
\]

After each iteration we normalize the solution \( u_{l+1} \). The ratio of the corresponding elements of the vectors \( u \) and \( y \) of two consecutive iterations approximates the corresponding eigenvalue. In the limit as \( l \) approaches infinity, we obtain an estimate of the given eigenvalue of matrix \( A \)

\[ \lambda \approx \frac{u_{l+1}^{(j)}}{y_{l+1}^{(j)}}, \]

where \( y^{(j)} \) is the \( j \)-th non-zero element of the vector \( y \). Here, matrix \( A \) has eleven diagonals (i.e., in equation (5.21) five diagonals come from the fourth order derivative
of the interfacial height and other lower-order derivatives, and three diagonals from density derivatives) and is sparse, so the power method is easy to perform.

Table 5.1 shows that there is a good agreement between growth rates at low wave numbers. This may not be true at higher wave numbers since the conditional number of matrix goes to infinity. The reason for large conditional number of matrix is that the matrix has two zero eigenvalues which correspond to the translational invariance of the density and interfacial height equations, respectively.

The corresponding eigenfunctions reproduce the deviations from the interfacial height and density profiles. As an example, Fig. 5.6 shows these deviations for the wave number \( k = 1 \). The deviations are symmetric, confirming periodicity of eigenfunctions.

Table 5.1  Linear stability results for power method and normal modes analysis.

<table>
<thead>
<tr>
<th>Wave number</th>
<th>Power method</th>
<th>Normal modes (( kc_1 ))</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-2.081660</td>
<td>-2.081644</td>
</tr>
<tr>
<td>2</td>
<td>-8.3266</td>
<td>-8.3316</td>
</tr>
</tbody>
</table>

Our system of equations (5.1) - (5.2) was originally scaled on the interval of unit length, so now we turn our attention to finite domain of length 1.

First, we simplify our problem by neglecting inertial effects, i.e. we put \( G = 0 \). The system of characteristic equations (5.21) -(5.22) reduces to

\[
\sigma \delta + b_1 \Delta_{\xi\xi} + b_2 \delta + b_3 \delta_{\xi} + b_4 \delta_{\xi\xi} + b_5 \delta_{\xi\xi\xi\xi} = 0, \tag{5.23}
\]

\[
\sigma \Delta + a_1 \delta + a_2 \delta_{\xi} + a_3 \Delta + a_4 \Delta_{\xi} + a_5 \Delta_{\xi\xi} = 0, \tag{5.24}
\]

where \( a_i, b_i \) are defined on page 92.
To verify the convergence of our numerical scheme (finite difference approximation of system (5.23)-(5.24)) we plot logarithm (base 10) of the difference of growth rates of the most unstable mode vs. the number of grid points $N$ (Fig. 5.7). Negative slope of the line shows convergence as $N$ is increased.

Next, we investigate the influence of the interfacial basic height $E_0$ and of $A_2$ on the stability properties of the two-fluid flow. Figure 5.8 presents the dependence of the maximum growth rate on parameter $D$ for few different cases. First, we fix $A_2 = 2$ and observe that the increase of the basic interfacial height changes the dynamics of the flow drastically with increase of $D$: from stable flow ($h_0=0.1$) to unstable ($h_0=0.5$). Here, we expect that the observed behavior of the maximum growth rate results from the competition between advection (due to the significant value of the induced driving force) and smoothing surface tension. At the cutoff value of $D (=3.7$ for $A_2=2$) these two mechanisms balance each other. When we fix $h_0=0.5$ (Fig. 5.8) and change $A_2$ from 2 (solid line) to 5 (dashed line), we observe that instabilities
Figure 5.7 Logarithm (base 10) of the differences of maximum growth rate vs number of grid points $N$ for parameters $\rho_2 = 2$, $h_0 = 0.5$, $G = 0$, $D = 1$.

develop at smaller value of $D$ for larger $\rho_2$. Thus, the increase of both $h_0$ and $\rho_2$ destabilizes the flow even in the absence of liquid inertia.

We proceed with more complicated problem involving the inertial effects. Note that in single phase liquid films the liquid inertia is the prime source of instability (Yih [72], Benjamin [3]). The convergence of our numerical scheme for this problem is verified by tracking the most unstable mode (corresponding to the maximum real part of eigenvalue) for different number of points $N$ similarly as before. Figure 5.9 shows that the method converges as $N$ is increased. Figure 5.10 presents the real parts of the corresponding eigenfunction for the most unstable mode, showing that the difference is very small for $N=100$ and $N=180$.

From numerical solution of the system of equations (5.21) - (5.22) for the interfacial basic height $h_0=0.5$ when we fix $\rho_2 = 2$ (equivalent to pressure drop $\Delta P=1$) we obtain that both interfacial and density modes are stable since the maximum growth rate is negative. However, if we increase $h_0$ to 0.85, we find unstable waves. Similarly, if we increase $\rho_2$ (keeping $\rho_1$ and $h_0$ fixed), we observe instability. Therefore,
Figure 5.8  Dependence of the maximum growth rate on $D$ for $G=0$. There exists cutoff $D$ above which instability develops.

Figure 5.9  Logarithm (base 10) of the differences of the maximum growth rate vs number of grid points $N$ for parameters $G = 1$, $\rho_2 = 1.5$ (countercurrent flow), $h_0 = 0.2$, $D = 1$. 
both increase of density differential (i.e., pressure drop) and of $h_0$ lead to instability. Next, we analyze this result in more detail.

Figure 5.11 shows the maximum growth rates vs. $h_0$ for four different combinations of $\rho_2$ and $D$. Part b) of this figure shows in more detail the region $0.6 < h_0 < 0.8$, which we discuss in more detail later. The main features of the results shown in Fig. 5.11 are as follows: (i) For sufficiently small $h_0$, the flow is stable; (ii) This region of stable $h_0$'s is decreased as $\rho_2$ and/or $D$ are increased; (iii) The growth rates are more sensitive to an increase of $D$ compared to an increase of $\rho_2$.

Sharp corner in the plot of growth rate for case $\rho_2 = 5$ and $D = 5$ (Fig. 5.11) corresponds to the crossing eigenvalues, meaning that the most unstable mode changes as $h_0$ is increased through the corner. In order to confirm this, we look closer at the maximum growth rates near interfacial height 0.23 for $\rho_2 = 5$ and $D = 5$. Figure 5.12 shows the spatial eigenfunction profiles (corresponding to the maximum growth rate/eigenvalue) for interfacial and density modes, for three different heights in the
Figure 5.11  The growth rate of the most dangerous mode vs interfacial height for parameters $G = 1$. Sharp corner in the growth rate for $\rho_2 = 5$, $D = 5$ corresponds to the crossing eigenvalues. Figure b) is a closer view of the “corners” for all cases for $0.6 < h_0 < 0.8$.

vicinity of $h_0=0.23$. The change in the profile of the eigenfunction suggests that we start to follow different eigenvalue at height 0.23, i.e. we switch between eigenvalues.

For the case $\rho_2 = 5$, $D=1$, for which stability changes close to $h_0 \approx 0.7$, we plot in Fig. 5.13 the maximum growth rate (maximum real part of the eigenvalue) vs frequency (corresponding imaginary part). The eigenvalue path crosses the neutral stability line in three points. This fact shows that the flow is stable for basic interfacial heights less than 0.485 and also in the range $0.669 < h_0 < 0.69$.

Figure 5.14 presents the wave forms of the interfacial and density modes at value of $h_0$ corresponding to the neutral stability growth rates for the same parameters as in Fig. 5.11. We note that basic interfacial heights corresponding to neutral stability states decreases when we increase $D$ and/or $\rho_2$. The wave forms of the interfacial and density modes change their shape with the increase of $\rho_2$ and $D$.

Next, we turn our attention to parametric plots for fixed parameter $D$. Figure 5.15 shows stable and unstable regions in $h_0$-$\rho_2$ plane for $D=1$ (part a) and $D=10$.
Figure 5.12 Dependence of the form of the eigenfunction on the basic interfacial height for parameters as in Fig. 5.11, $D=5$, $\rho_2=5$. The change in the shape of the eigenfunction corresponds to crossing eigenvalues and explains the sharp corner in the maximum growth rate plot shown in Fig. 5.11.

(part b). In Fig. 5.15a we see a peak in the plot of the neutral stability curve. This peak results from the second stable region which we observed in Fig. 5.13. In contrast to Fig. 5.12 we find that in this case there is no switch between the eigenvalues, i.e. we still follow the same eigenvalue.

The comparison of Fig. 5.15a and Fig. 5.15b shows that there is a significant decrease of the size of stable region for larger $D$ and $\rho_2$. In addition, the width of the peak that we obtained in Fig. 5.15a decreases significantly compared to the peak in Fig. 5.15b. We observe that at significantly high values of $\rho_2$ the changes in the cutoff values of the interfacial height are more gradual compared to smaller values of $\rho_2$.

In order to show that both $D$ and $\rho_2$ influence stability properties of the flow, we vary these two quantities but keep $\Delta P=$constant. For example, for the combination $D=1$ and $\rho_2=7$ flow is stable for $h_0 < 0.32$. For $D = 10$ and $\rho_2=1.6$, the corresponding regions of stable flow are $h_0 < 0.44$ and $0.68 < h_0 < 0.72$. So, we find that both $\rho_2$ and $D$ have significant influence on the stability of the flow.
Figure 5.13 Graph of the growth rate (real part of the maximum eigenvalue) vs frequency (imaginary part of the maximum eigenvalue) for parameters $\rho_2=5$, $D=1$.

To conclude this chapter, we summarize our findings. Two-phase flow under the appropriate scaling and assumptions reduces to a single-phase film flow. For very thin liquid films in two-fluid system we obtain that there exists the region of small wave numbers where flow is stable, in contrast to single-phase flow. This result holds for $D$'s approximately between $0.001 < D < 1$. For $D$'s outside of this range the dynamics of two-phase flow is very similar to single-film flow.

For two-fluid flows where thickness of the liquid film is allowed to be comparable with channel height we perform normal modes analysis for the case $\rho_1 = \rho_2$. In this case of moderate interfacial basic heights we obtain that both density and interfacial modes are stable. The increase of $h_0$, however, causes the destabilization of the density mode. In the case $\rho_2 > \rho_1$, we perform general analysis. Here, we find that an increase of $\rho_2$ and/or $D$ destabilizes the flow.
Figure 5.14 Wave forms for interfacial \((Re(\delta(\xi)))\) and density modes \((Re(\Delta(\xi)))\) at neutral stability state for parameters as in Fig. 5.11.
Figure 5.15 Neutral stability curves for $\rho_2$ vs $h_0$. Stable regions are located below the cutoff values of $h_0$ and flow is unstable for larger values of $h_0$. 
CHAPTER 6

CONCLUSIONS

We have investigated the nonlinear evolution of the interface between two immiscible fluids in an inclined channel. The lower fluid is significantly denser than the compressible upper fluid. Motivated by air-water systems, through a lubrication approximation, we derive a system of nonlinear evolution equation that governs the motion of the interface between the two fluids and the gas pressure. The lubrication approach includes the inertial effects of the liquid layer and the Reynolds stress terms in the gas. For incompressible flow the system reduces to a single evolution equation; however, in the case of compressible flow, this reduction is not possible. In incompressible case, we consider three different forcing scenarios. The first, where the gas volumetric flow rate is fixed, and the third, where the liquid volumetric flow rate and gas pressure drop are prescribed, result in a standard lubrication approximation, and the interfacial dynamics depends only on local variations of the interfacial shape. However, the third scenario results in an additional constraint on the flow. The second, where the pressure difference over the length of the channel is fixed, results in a nonlocal dependence of the interfacial dynamics on the interfacial shape.

Continuing under the incompressibility assumption, we extend the linear stability analysis from the usual constant flow rate (gas or liquid) boundary condition to the more realistic and relevant case of constant pressure drop, with the goal of understanding better this situation which is appropriate for the case of (compressible) gas flows. The weakly nonlinear analysis for an incompressible gas flow is then performed for all types of boundary conditions. We reproduce the Kuramoto-Sivashinsky (KS) equation that governs the dynamics. In the case of constant pressure drop, we derive a modified KS equation, with an additional integral term representing the nonlocal dependence on the interfacial height. We show that
the integral term modifies the propagation speed of the deviation wave, but it does not influence its shape.

Next, we perform fully nonlinear simulations of the evolution equations for all three types of boundary conditions, and find excellent agreement for the growth rate and phase speed with linear stability theory in the range of its validity. In all three cases, we find that the undercompressive shock paradigm found in Marangoni-driven fluid layers is applicable in the counter-current flow régime. Small differences in the upstream and downstream interfacial height results in the formation of Lax shocks. Larger differences in the upstream and downstream heights result in bistability among multiple Lax shocks. Increasing this differential still further results in the formation of undercompressive shocks and finally the formation of rarefaction waves. In the case of the flow driven by a constant pressure drop and a given liquid flow rate, we observe the unsteady growth of the interfacial profiles in all four regions. This is especially interesting regarding undercompressive shock height, $h_{UC}$. In this régime, it is more likely that the advection and inertial effects (comparable to surface-tension effects) may result in capturing the final state of the transient related to a flooding scenario – $h_{UC}$ actually reaches the upper channel wall. Numerical simulations confirm analytical results for necessary conditions for admissible solutions.

If we relax the assumption of incompressibility of the gas, then the ideal gas equation of state closes our system of evolution and pressure gradient/density equations. We use linear stability theory to understand the influence of compressibility. For a thin liquid film ($h_0 \ll d$), and keeping for a moment $\rho$ constant, we find that a two-fluid system mimics the single-phase flow. However, if the gas is allowed to compress, we find a stabilizing effect for moderately small values of dimensionless parameter $D$ (resulting from the equation of state).

One of the significant differences of linear stability theory between compressible and incompressible cases is that the basic state solution for the gas density in the
compressible case depends on the downstream coordinate. However, this basic-state solution is constant if we prescribe the same values of density at both ends of the channel. Using a normal-mode expansion in this régime, we find that both interfacial and density modes are stable for moderate basic interfacial heights. But the increase of the basic interfacial height causes the destabilization of flow due to compressibility of the gas layer. In the case of countercurrent flow (value of density $\rho_1$ on the left (upper) end of the channel is smaller than the value $\rho_2$ on the right (lower) end), a general approach in linear stability analysis is required. Here, an increase of $D$ and/or $\rho_2$ causes the destabilization of flow for smaller basic interfacial heights.

There are several overall conclusions of this work. Firstly, at the onset of a rarefaction wave in the case of constant gas volumetric flow rate for incompressible flow, we find that the liquid volumetric flow rate decreases as the wave develops, eventually reversing. This is qualitatively similar to the description of flow reversal, where a prescribed liquid volumetric flow rate cannot be sustained in a countercurrent régime. In addition, the final state appears to result in a hanging film: a state seen in an experiment after the flooding transition has taken place. Secondly, in the case of the flow driven by a constant pressure drop, we observe the transient growth of the undercompressive shock height $h_{UC}$. Interestingly, this transient may be related to a flooding scenario where the size of the hump actually reaches the upper channel wall. Thirdly, the primary modes of instability for the compressible gas case in a finite-length channel suggest that the region of instability would be seen experimentally first near the liquid inlet. Such a phenomenon is seen in experiment slightly before the flooding phenomenon begins.

These initial results from a lubrication model suggest that a more detailed study of these equations in the future, perhaps in slightly different limits, would yield better insight in the dominant physical mechanisms to the flooding phenomenon. Firstly, the evolution of the rarefaction wave to the hanging film state, in the case presented,
requires that the upstream liquid film thickness be sufficiently large. Clearly the presence of the upper channel wall is playing a significant role, but it is not clear if the physical mechanism leading to this phenomenon is due to differences in the gas pressure or due to the interfacial shear stress. A study in which these effects could be better isolated will help in determining the appropriate mechanism. Similarly, the mechanisms leading to the transient growth of the undercompressive shocks need to be investigated to see if the secondary roles of gas and liquid inertia may mitigate this transient effect. Finally, the nonlinear coupling of the gas density with the interfacial deflection, given that the initial unstable disturbance is found near the liquid inlet, may prove to be the most fruitful study of understanding the underlying physical mechanisms that lead to the flooding phenomenon in channels. Once the mechanisms are well understood in this Cartesian frame, extending the results to three-dimensional disturbances and in cylindrical geometries, both of which are found in engineering heat-transfer applications, will lead to criteria to better design these individual applications.
APPENDIX A

DERIVATION OF THE INERTIAL TERMS $F^{(i)}(\xi, \zeta, \tau)$ FOR COMPRESSIBLE CASE

To find the inertial term $F^{(2)}(\xi, \zeta, \tau)$ in (2.52) and (2.53), we need to solve the problem:

$$F^{(2)}_{\zeta \zeta}(\xi, \zeta, \tau) = G \frac{\rho}{\mu} \left( u_0^{(2)} p_0^{(2)} + u_{0\zeta}^{(2)} w_0^{(2)} \right), \quad (A.1)$$

$$F^{(2)}(\xi, 1, \tau) = 0, \quad (A.2)$$

$$F^{(2)}_{\zeta}(\xi, 1, \tau) = 0. \quad (A.3)$$

After integrating (A.1) and, using the boundary conditions (A.2), (A.3), we obtain:

$$F^{(2)}(\xi, \zeta, \tau) = a_1^{(2)} (\zeta - 1)^6 + a_2^{(2)} (\zeta - 1)^5 + a_3^{(2)} (\zeta - 1)^4,$$

$$a_1^{(2)} = G \frac{\rho p_0}{120 \mu^3} \left( \frac{h_{\xi p_0}}{1 - h} - \frac{\rho_{\xi}}{\rho} p_0 \right),$$

$$a_2^{(2)} = G \frac{\rho p_0}{80 \mu^3} (1 - h) \left( \frac{h_{\xi p_0}}{1 - h} - \frac{\rho_{\xi}}{\rho} p_0 \right),$$

$$a_3^{(2)} = G \frac{\rho p_0}{48 \mu^3} (1 - h)^2 \left( \frac{h_{\xi p_0}}{1 - h} - \frac{\rho_{\xi}}{\rho} p_0 \right).$$

Similarly for $F^{(1)}(\xi, \zeta, \tau)$ we have:

$$F^{(1)}_{\zeta \zeta}(\xi, \zeta, \tau) = G \left[ u_{0r}^{(1)} + u_0^{(1)} u_{0\zeta}^{(1)} + u_{0\zeta}^{(1)} w_0^{(1)} \right], \quad (A.4)$$

with the following boundary conditions:

$$F^{(1)}(\xi, 0, \tau) = 0, \quad (A.5)$$

$$F^{(1)}_{\zeta}(\xi, 0, \tau) = 0. \quad (A.6)$$

After integrating (A.4) and applying the boundary conditions (A.5), (A.6):

$$F^{(1)}(\xi, \zeta, \tau) = a_1^{(1)} \zeta^6 + a_2^{(1)} \zeta^5 + a_3^{(1)} \zeta^4 + a_4^{(1)} \zeta^3, \quad (A.7)$$
where

\[ a_1^{(1)} = G \frac{p_0 \xi}{360} \left( p_0 \xi - \sin \beta \right), \]

\[ a_2^{(1)} = G \frac{p_0 \xi}{60} \left\{ h \sin \beta - \frac{p_0 \xi}{2} (h + 1) \right\}, \]

\[ a_3^{(1)} = G \frac{1}{24} \left\{ p_{0 \xi} + \left[ h \sin \beta - \frac{p_0 \xi}{2} (h + 1) \right] \left[ h \xi \sin \beta - \frac{p_0 \xi}{2} (h + 1) - \frac{p_0 \xi h \xi}{2} \right] \right\}, \]

\[ a_4^{(1)} = G \frac{2h \xi \sin \beta - p_{0 \xi} (h + 1) - h \xi p_0 \xi}{12}. \]
APPENDIX B

DERIVATION OF THE $O(\epsilon)$ PART OF THE EVOLUTION EQUATION FOR INCOMPRESSIBLE CASE

We start from the $O(1)$ $\xi$-momentum equations (2.24) and (2.25), with boundary conditions (2.34) and (2.35). We apply the continuity of shear stress and of the normal and tangential components of velocity on the interface, which results in the velocity profiles given by (2.38), (2.40), (2.41) and

$$w_0^{(1)} = \frac{p_0 h_\xi}{2(h-1)} \zeta^3 - \left( \frac{(h+2)h_\xi}{h-1} p_0 + \sin \beta h_\xi \right) \frac{\zeta^2}{2}.$$

Comparing velocity profiles for compressible and incompressible case, we see that to this order there is an influence of compressibility in the normal component of velocity of the liquid (bottom fluid) since the leading order pressure gradient equation (2.57) in the compressible flow evolves the density dependence which is absent in the corresponding equation (2.61) of the incompressible flow.

At $O(\epsilon)$, the $\xi$-momentum and continuity equations yield

$$G \left( u_0^{(1)} + u_0^{(1)} u_{0\zeta} + u_0^{(1)} w_0^{(1)} \right) = -p_1^{(1)} + u_{1\zeta}^{(1)},$$
$$G p \left( u_0^{(2)} u_{0\zeta} + u_0^{(2)} w_0^{(2)} \right) = -p_1^{(2)} + \mu u_{1\zeta}^{(2)},$$

$$(B.1) \quad u_{1\xi}^{(1)} + w_{1\zeta}^{(1)} = 0,$$
$$u_{1\xi}^{(2)} + w_{1\zeta}^{(2)} = 0,$$

and

$$u_1^{(1)} = 0 \quad (\zeta = 0),$$
$$u_1^{(2)} = 0 \quad (\zeta = 1),$$

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At order (f), we can solve explicitly equation (B.3) (i) using the fact that pressure in the gas phase has no z-dependance, (ii) assuming that the correction to pressure $P_i$ is eliminated from the evolution equation, leaving only the leading order pressure gradient term. To solve the reduced equation for pressure we need to specify how we force our system.

Using equations (B.1) similar to the analysis in the previous order, we use the boundary and interfacial conditions (B.2) to arrive at the solution for $u_i^{(i)}$

\[
\begin{align*}
    w_1^{(1)} &= 0 \quad (\zeta = 0), \\
    w_1^{(2)} &= 0 \quad (\zeta = 1), \\
    \mu u_{i\zeta}^{(2)} - u_{i\zeta}^{(1)} &= 0 \quad (\zeta = h(\xi, \tau)), \\
    w_1^{(2)} - h\xi u_1^{(2)} - w_0^{(1)} + h\xi u_0^{(1)} &= 0 \quad (\zeta = h(\xi, \tau)).
\end{align*}
\]

where $\phi, \psi, \alpha$ are defined by (2.54), (2.55), (2.56), respectively. We refer to Appendix A for the definition of inertial terms $F^{(i)}(\xi, \zeta, \tau)$ under the assumption of incompressibility. The continuity of the normal velocity (2.12) then gives the equation for the pressure gradient

\[
\left[ p_0\xi(h - 1)^3 + \epsilon \left\{ (h - 1)^3(P_{i\xi} + \alpha) + 12\mu\theta \right\} \right]_{\xi} = \frac{6\mu(h + 1)}{(1 - h)} h\xi p_0\xi,
\]

where

\[
\theta = \frac{h^2(h + 3)}{12} \sin \beta + \frac{h(h - 1)}{4} p_0\xi - G \frac{17}{3360} \frac{\rho}{\mu^3} h\xi p_0^2\xi.
\]

At order ($\epsilon$), we can solve explicitly equation (B.3) (i) using the fact that pressure in the gas phase has no z-dependance, (ii) assuming that the correction to pressure $P_i$ obeys the zero pressure drop ($P_i(\xi = 1) - P_i(\xi = 0) = 0$). We eliminate $P_i$ from the evolution equation, leaving only the leading order pressure gradient term. To solve the reduced equation for pressure we need to specify how we force our system.
To find the local behavior (in time) of the modified KS equation (valid to $O(\varepsilon^3)$)

$$u_t + u_{xxxx} + \lambda u_{xx} + \alpha uu_x + \alpha_2 \left( \int_0^{2\pi} u dx \right) u_x = 0,$$  \hspace{1cm} (C.1)

local to the bifurcation points $k = 1, 2, \ldots$, along the basic state, we expand

$$\lambda = k^2 + \delta \varepsilon^2,$$

$$\tau = \varepsilon^2 t,$$

$$u(x, t, \varepsilon) \sim \varepsilon u_1(x, t) + \varepsilon^2 u_2(x, t) + \ldots.$$  

This particular expansion is motivated by the fact that the equation (C.1) is valid on a "slow" time scale $\varepsilon t$ and we consider even slower time scale $\tau = \varepsilon^2 t$. Since the basic state for velocity field is zero, then we start expansion for velocity at order $\varepsilon$. We substitute the expansions in the modified KS equation (C.1). At $O(\varepsilon)$ we define operator $L$ to be equal to:

$$Lu_1 = u_{1xxxx} + k^2 u_{1xx} = 0.$$  

Solution of this equation is

$$u_1(x, \tau) = A_1(\tau)e^{ikx} + c.c. + B_1(\tau),$$

where c.c. denotes the complex conjugate.

At $O(\varepsilon^2)$, we obtain

$$Lu_2 = -\alpha u_1 u_{1x} - \alpha_2 u_{1xx} \int_0^{2\pi} u_1 dx.$$  

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Note that

\[ \int_{0}^{2\pi} u_1 dx = 2\pi B_1(\tau). \]

We then obtain

\[ Lu_2 = -\alpha [ikA_1(\tau)B_1(\tau)e^{ikx} + ikA_1^2(\tau)e^{2ikx} + c.c.] - 2\alpha_2 \pi B_1(\tau)[ikA_1(\tau)e^{ikx} + c.c]. \tag{C.2} \]

We obtain the solvability condition in the form

\[ \left[ikA_1(\tau)e^{ikx} + c.c.\right] B_1(\tau) [\alpha + 2\pi \alpha_2] = 0.\]

If \( \alpha \neq -2\pi \alpha_2 \) then we require \( B_1(\tau) = 0 \), otherwise \( B_1(\tau) \) is an unknown function.

Thus, equation (C.2) simplifies to

\[ Lu_2 = -\alpha ikA_1^2(\tau)e^{2ikx} + c.c. \]

and the solution is given by

\[ u_2 = A_2(\tau)e^{ikx} - \frac{i\alpha A_1^2(\tau)}{12k^3}e^{2ikx} + c.c. + B_2(\tau). \]

At \( O(\varepsilon^3) \) we obtain

\[ Lu_3 = -u_1\varepsilon u_{1xx} - \alpha (u_1u_2)_x - \alpha_2 u_{2x} \int_{0}^{2\pi} u_1 dx - \alpha_2 u_{1x} \int_{0}^{2\pi} u_2 dx. \]

Note that

\[ \int_{0}^{2\pi} u_1 dx = \begin{cases} 2\pi B_1(\tau) & \alpha = -2\pi \alpha_2, \\ 0 & \alpha \neq -2\pi \alpha_2, \end{cases} \]

and

\[ \int_{0}^{2\pi} u_2 dx = 2\pi B_2(\tau). \]
It follows that
\[
Lu_3 = \left[ -\dot{A}_1 + \delta k^2 A_1 - \frac{\alpha^2}{12k^2} |A_1|^2 A_1 - \alpha B_2 i k A_1 - 2\pi i k \alpha_2 A_1 B_2 \right] e^{ikz} + \text{c.c.} \\
+ \dot{B}_1 - \alpha B_1(\tau) u_{2\tau} - 2\pi \alpha_2 u_{2\tau} B_1(\tau).
\]
(C.3)

Since we look for nonlinear effects, then for \( \alpha = -2\pi \alpha_2 \), we require that \( B_1(\tau) = 0 \) and the last three terms in the equation (C.3) will vanish. The solvability condition for the equation (C.3) then requires a Landau equation for \( A_1 \):
\[
\dot{A}_1 = \delta k^2 A_1 - \frac{\alpha^2}{12k^2} |A_1|^2 A_1 - i k \alpha B_2 A_1 - 2\pi i k \alpha_2 A_1 B_2.
\]

To find \( B_2(\tau) \), we look at \( O(\epsilon^4) \):
\[
Lu_4 = -u_{2\tau} - \delta u_{2\tau} = -\dot{B}_2 - \alpha_2 2 pi B_2 u_{2\tau} - \ldots,
\]
and by the argument used for nonlinear effects, we put \( B_2 = 0 \) and solve
\[
\dot{A}_1 = \delta k^2 A_1 - \frac{\alpha^2}{12k^2} |A_1|^2 A_1.
\]

Let us seek the solutions in form
\[
A_1(\tau) = \rho(\tau) e^{i\theta(\tau)}.
\]

Then we have to solve the system of equations
\[
\begin{aligned}
\dot{\rho} &= \delta k^2 \rho - \frac{\alpha^2}{6k^2} \rho^3, \\
\rho \dot{\theta} &= 0.
\end{aligned}
\]

Let \( z = 1/\rho^2 \). Then the first equation can be rewritten as
\[
-\frac{z}{2z} = \delta k^2 - \frac{\alpha^2}{12k^2 z},
\]
and the solution is given by
\[
z = z_0 e^{-2\delta k^2 \tau} + \frac{\alpha^2}{12\delta k^4}.
\]
Going back to the original variable

\[ \rho = \pm \frac{k^2 \sqrt{12\delta}}{\sqrt{\alpha^2 + 12\delta k^4 z_0 e^{-2\delta k^2 \tau}}} \]

as \( \dot{\theta} = 0 \) then \( \theta = \theta_0 \), and the solution of the modified KS equation is

\[ u \sim \epsilon \left( \pm \frac{k^2 \sqrt{12\delta}}{\sqrt{\alpha^2 + 12\delta k^4 z_0 e^{-2\delta k^2 \tau}}} e^{i(kx + \theta_0)} + c.c. \right) + O(\epsilon^2), \]

which for long times approaches a constant value. As \( \alpha \to 0 \), the stabilizing effect of \( uu_x \) diminishes and allows the unbounded growth of \( \rho(\tau) \).

Thus, for long time the solution looks like

\[ u \sim (\lambda - k^2)^{1/2}, \]

and the phase speed is

\[ c \sim \alpha_2 \int_0^{2\pi} u \, dx. \]
REFERENCES


