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# ABSTRACT <br> PROPAGATION AND SCATTERING OF COLLIMATED BEAM WAVE IN VEGETATION USING SCALAR TRANSPORT THEORY 

by<br>Michael Yu-Chi Wu

The scalar time-dependent equation of radiative transfer is used to develop a theory of pulse beamwave propagation and scattering in a medium characterized by many random discrete scatterers which scatter energy strongly in the forward scattering direction. Applications include the scattering of highly collimated millimeter waves in vegetation and optical beams in the atmosphere. The specific problem analyzed is that of a periodic sequence of Gaussian shaped pulses normally incident from free space onto the planar boundary surface of a random medium half-space, such as a forest, that possesses a power scatter (phase) function consisting of a strong, narrow forward lobe superimposed over an isotropic background. After splitting the specific intensity into the reduced incident and diffuse intensities, the solution of the transport equation expressed in cylindrical coordinates in the random medium half-space is obtained by expanding the angular dependence of both the scatter function and the diffuse intensity in terms of Associate Legendre polynomials, by using a Fourier series/Hankel transform to obtain the equation of transfer for each spatial frequency, and by employing the weighted residual method to satisfy the boundary condition that the forward traveling diffuse intensity be zero at the interface. Data generated from the solution will be compared to results obtained from a computationally intensive second method of solution, which follows the procedure used by Chang and Ishimaru to study the propagation and scattering of monochromatic beam waves in random media. In this second method, the timedependent scalar transport equation is solved using a Fourier Series/Hankel transform along with the two-dimensional Gauss quadrature formula and an eigenvalue eigenvector technique. Numerical results are given for received power at different penetration depths, different beam sizes and different scatter directions.

# PROPAGATION AND SCATTERING OF COLLIMATED BEAM WAVE IN VEGETATION USING SCALAR TRANSPORT THEORY 

## by <br> Michael Yu-Chi Wu

> A Thesis
> Submitted to the Faculty of New Jersey Institute of Technology
> in Partial Fulfillment of the Requirements for the Degree of Master of Science in Electrical Engineering

Department of Electrical and Computer Engineering Department


## APPROVAL PAGE

PROPAGATION AND SCATTERING OF COLLIMATED BEAM WAVE IN VEGETATION USING SCALAR TRANSPORT THEORY

## Michael Yu-Chi Wu

Dr. Gerald M. Whitman, Thesis Advisor Professor of Electrical Engineering, NJIT

Dr. Ed ip Niver
Associate of Professor Electrical Engineering, NJIT

Dr. Marek Sósnowski
Date

Associate Professor of Electrical Engineering, NJIT

## BIOGRAPHICAL SKETCH

Author:Michael Yu-Chi WuDegree:Date:Master of ScienceMay 2001
Undergraduate and Graduate Education:

- Master of Science in Electrical Engineering, New Jersey Institute of Technology, Newark, NJ, 2001
- Master of Science in Computer Science, New Jersey Institute of Technology, Newark, NJ, 2000
- Bachelor of Science in Electrical Engineering, New Jersey Institute of Technology, Newark, NJ, 1998
Major: Electrical Engineering

To my beloved family.

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## CHAPTER 1

## INTRODUCTION

For line-of-sight communication, cellular communication in particular, current interest centers on radio-link performance, and how it is affected by wave attenuation, fading and co-channel interference. When vegetation, such as a forest, lies along the path of a radio-link, the radio performance will be affected by strong multiscattering effects. This needs to be understood and therefore warrants investigation.

There are two methods that are usually used to study multiscattering effect, namely, analytical theory and transport theory[1]. Analytical theory is a very rigorous mathematical approach based on Maxwell's Equations. It is very complex and obtaining solutions often requires introducing strong simplications which limit the applicable parameter ranges. In contrast, radiative transfer theory deals with the transfer of energy through the multscattering medium. In this theory, the basic equation that is solved is the equation of radiative transfer or tranpsort theory. The radiative transfer theory developed heuristically from the conservation of energy principle in radiation space. The transport equation is equivalent to Boltzmann's equation found in the kinetic theory of gases and in neutron transport theory and is less rigorously than the analytical theory. However, transport theory has been very successfully applied in the study of many radiation problems, such as, optical propagation through the atmosphere, remote sensing and radiation from stars.

In previous work, continuous wave (CW) millimeter wave and plane wave pulse propagation in vegetation were studied using the scalar transport theory [2-6]. In these studies, interest focused on the determination of the range and directional dependency of the received power as well as on pulse broadening and distortion. The scalar transport equation is capable of specifying the total energy density of radiation in two orthogonal polarizations, but not polarization or depolarization effects (see [4] for experimental justification of their neglect in these studies). In
the earlier developed theory of a plane wave incident upon the forest half-space, it was shown that the range dependence in the forest (treated as a random medium) is not be simply an exponential decrease at constant attenuation rate. What actually occurs for the received power is a high attenuation rate at short distances into the medium that evolves into a much lower attenuation rate at large distances. The theory explains this in terms of the interaction between the coherent and incoherent field components. The coherent component, dominating at short distances, is highly attenuated (by absorption and scattering ) while the incoherent component, which is generated by the scattering of the coherent component, does not loose power by further (multiple) scattering - it scatters into itself - and thus dominates at large distances into the forest, decreasing at a much reduced attenuation rate. In the transition region between the high and low attenuation regimes significant beam broadening and pulse broadening occurs.

In this study, the scalar time-dependent equation of radiative transfer is used to develop a theory for the propagation and scattering of pulsed beam waves of finite cross-section in a medium that is characterized by many random discrete scatterers (vegetation). Such a medium scatters energy strongly in the forward scattering direction. Applications include the scattering of millimeterwaves in vegetation and the scattering of optical beams in the atmosphere. Strong forward scattering occurs at millimeter and optical frequencies since all scatter objects in a forest or in the atmosphere are large compared to wavelength. Again of interest are the range and directional dependency of received power, pulse broadening and distortion, in addition to the effect of a finite beamwidth when the incident field is not a plane wave. This case differs basically from the plane wave case in that scattering out of the beam occurs (while in the plane wave case any multiscattered wave trains will always remain within the infinitely wide beam); this is likely to have a significant effect on range dependence, as well as on beam broadening and pulse broadening.

## CHAPTER 2

FORMULATION AND SOLUTION

### 2.1 Introduction

The forest is modeled as a statistically homogeneous half-space of randomly distributed particles, which scatter and absorb electromagnetic energy. A periodic sequence of Gaussian pulses is taken to be normally incident from free space onto the planar boundary of a forest. The incident pulse train is assumed to be a collimated beam-wave; see Figure 2.1.1.


Figure 2.1.1 Collimated beam wave pulse train normally incident onto a forest half-space
Chang and Ishimaru [7] used scalar transport theory to study the scattering of a monochromatic collimated beam-wave in a random medium. Their approach, however, is computational intensive, and does not provide numerical data for off-axis beam scattering. In the method presented here - which also involves using the scalar transport equation in cylindrical coordinates - a more analytical development is achieved, which permits numerical data to be obtained for off-axis beam scattering in the forest.

### 2.2 Incident Gaussian Beam Pulses

A collimated beam wave pulse train is assumed to be normally incident from the air region ( $z \leq 0$ ) to the random scattering medium (i.e., the forest), which occupies the half-space region $z \geq 0$. At $z=0$ and $\rho=0$ (see Figure A. 1 in Appendix for the geometry), the magnitude of the instantaneous Poynting vector of the incident beam wave is given by $S(z=0, \rho=0, t)=2 S_{p} f(0, t) \cos ^{2}\left(\omega_{c} t\right)$, where $\mathrm{S}_{\mathrm{p}}$ is the incident Poynting vector time-averaged with respect to the carrier frequency $\omega_{c}$. Being a positive even function of time that is periodic with period $T \gg T_{c}=\frac{2 \pi}{\omega_{c}}, f(0, t)$ is normalized such that

$$
\begin{equation*}
\frac{1}{T} \int_{-T / 2}^{T / 2} f(0, t) d t=1 \tag{2.2.1}
\end{equation*}
$$

For Gaussian incident pulses, $\mathrm{f}(0, \mathrm{t})$ is taken to be

$$
\begin{align*}
& f(0, t)=\frac{\alpha_{0}}{\sqrt{\pi}} e^{-\left(\alpha_{o} t / T\right)^{2}} \quad, \quad-\frac{T}{2} \leq t \leq \frac{T}{2}  \tag{2.2.2}\\
& \alpha_{0} \equiv \text { const } .
\end{align*}
$$

Since the incident beam wave pulses are even, this periodic function of time can be represented by an even Fourier series at $z=0$ :

$$
\begin{equation*}
f(0, t)=\frac{b_{o}}{2}+\sum_{v=1}^{\infty} b_{v} \cos (v \omega t)=\operatorname{Re}\left\{\sum_{v=0}^{\infty} f_{\nu} e^{i v \omega t}\right\}, \tag{2.2.3}
\end{equation*}
$$

where

$$
\begin{equation*}
\omega=\frac{2 \pi}{T} \quad, \quad b_{v}=\frac{2}{T} \int_{-T / 2}^{T / 2} f(0, t) \cos (v \omega t) d t \tag{2.2.4}
\end{equation*}
$$

Hence, for the Gaussian beam wave pulse train,

$$
f_{v}=\frac{\varepsilon_{v} b_{v}}{2}=\varepsilon_{v} e^{-\left(\pi v / \alpha_{o}\right)^{2}} \quad, \quad \varepsilon_{v}=\left\{\begin{array}{ll}
1, & v=0  \tag{2.2.5}\\
2, & v \neq 0
\end{array}\right\}
$$

$\alpha_{o}$ has to be chosen properly to ensure that the Gaussian function in (2.2.2) approaches zero as $t \rightarrow \pm T / 2$ allowing the limit of the integration in (2.2.4) to be replaced by to $\pm \infty$.

The specific intensity (power per unit area and per unit solid angle) of the incident beam wave pulse train travels through air at the speed of light " c " in the positive z direction and is given by

$$
\begin{equation*}
I_{p}=S_{p} e^{-(\rho / w)^{2}} f(t-z / c) \frac{\delta(\theta)}{2 \pi \sin (\theta)} \tag{2.2.6}
\end{equation*}
$$

with

$$
\begin{equation*}
f(t-z / c)=\operatorname{Re}\left\{\sum_{v=0}^{\infty} f_{v} e^{i v \omega(t-z / c)}\right\} . \tag{2.2.7}
\end{equation*}
$$

In (2.2.6), $\delta(\theta)$ is the Dirac delta function, and $\theta$ is defined as the scatter angle measured positive from the positive direction of the $z$-axis (see Figure A.1.).

### 2.3 Phase Function

The random scatter medium is characterized by an absorption cross-section per unit volume ( $\sigma_{a}$ ), the scattering cross-section per unit volume ( $\sigma_{s}$ ) and a power scatter or phase function $p\left(\hat{s}, \hat{s}^{\prime}\right)$. The phase function depends on both the incident power unit vector direction ( $\hat{s}^{\prime}$ ) and the scatter power unit vector direction ( $\hat{s}$ ).

A forest scatters energy symmetrically about the direction of the incident radiation because the scattering surfaces in a forest essentially have random orientation. As a result, the scattering depends only on the angle $\gamma$ between $\hat{s}^{\prime}$ and $\hat{s}$, where $\gamma=\cos ^{-1}\left(\hat{s}^{\prime} \cdot \hat{s}\right)$ and therefore the phase can be written as

$$
\begin{equation*}
p\left(\hat{s}, \hat{s}^{\prime}\right)=p\left(\hat{s} \cdot \hat{s}^{\prime}\right)=p(\cos \gamma) . \tag{2.3.1}
\end{equation*}
$$

Since all scatter objects in a forest are large compare to the wavelength at millimeter-wave and optical frequencies, a forest scatters energy strongly in the forward direction but weakly in all other directions. For that reason, the scatter function can be assumed to be characterized by a
strong narrow lobe superimposed over an isotropic background. This type of scatter function can analytically be expressed as a Gaussian function added to a homogeneous term, i.e.

$$
\begin{equation*}
p(\gamma)=\alpha q(\gamma)+(1-\alpha) \quad, \quad q(\gamma)=\left(\frac{2}{\Delta \gamma_{s}}\right)^{2} e^{-\left(\gamma / \Delta \gamma_{s}\right)^{2}}, \Delta \gamma_{s} \ll \pi \tag{2.3.2}
\end{equation*}
$$

which is normalized such that

$$
\begin{equation*}
\iint_{4 \pi} p(\gamma) d \Omega=4 \pi \tag{2.3.3}
\end{equation*}
$$

$d \Omega$ is the differential solid angle about the scatter angle $\hat{s} . \Delta \gamma_{s}$ denotes the width of the forward lobe in the scatter pattern. $\alpha$ is the ratio of the forward scattered power to the total scattered power.

### 2.4 The Scalar Time-Dependent Transport Equation in Cylindrical Coordinates

### 2.4.1 Scalar Transport or Radiative Transfer Equation

In transport theory, the specific intensity " $I$ " of the field in a random medium is governed by the radiative transfer equation (transport equation). In the normalized cylindrical coordinate system ( $\left.\rho^{\prime}, \psi^{\prime}, z^{\prime}\right)$ for symmetric scattering about the direction of the incident radiation, the scalar transport equation takes the form [10]:

$$
\begin{equation*}
\frac{\partial}{\partial t^{\prime}} I\left(\bar{r}, t^{\prime}, \hat{s}\right)+\hat{s} \cdot \nabla^{\prime} I\left(\bar{r}, t^{\prime}, \hat{s}\right)=-I\left(\bar{r}, t^{\prime}, \hat{s}\right)+\frac{W_{o}}{4 \pi} \iint_{4 \pi} p\left(\hat{s} \cdot \hat{s}^{\prime}\right) I\left(\bar{r}, t, \hat{s}^{\prime}\right) d \Omega^{\prime} \tag{2.4.1}
\end{equation*}
$$

where

$$
\begin{align*}
& \hat{s} \cdot \nabla^{\prime} \equiv \cos (\theta) \frac{\partial}{\partial z}+\sin (\theta) \cos (\psi) \frac{\partial}{\partial \rho}-\frac{1}{\rho} \sin (\theta) \sin (\psi) \frac{\partial}{\partial \psi}  \tag{2.4.1a}\\
& \hat{s} \cdot \hat{s}^{\prime}=\cos \gamma=\mu \mu^{\prime}+\sqrt{1-\mu^{2}} \sqrt{1-\mu^{\prime 2}} \cos \left(\psi-\psi^{\prime}\right)  \tag{2.4.1b}\\
& d \Omega^{\prime}=\sin \theta^{\prime} d \theta^{\prime} d \psi^{\prime}, \psi^{\prime}=\phi_{x}-\phi^{\prime}, \mu^{\prime}=\cos \theta^{\prime} \tag{2.4.1c}
\end{align*}
$$

The normalized space and time variable in (2.4.1) are given by $\rho^{\prime}=\sigma_{t} \rho, z^{\prime}=\sigma_{t} z, t^{\prime}=\sigma_{t} t$, and $W_{0}=\sigma_{s} / \sigma_{t}, \sigma_{t}=\sigma_{s}+\sigma_{a}$. The parameter $W_{0}$ is called the albedo and the parameters $\sigma_{t}, \sigma_{s}$ and $\sigma_{a}$ are the extinction, the scatter and the absorption cross-sections per unit volume, respectively. See Figure A.1.

To obtain a unique solution to (2.4.1) with I assumed to be time periodic, requires satisfying two boundary conditions that take the form

$$
\begin{array}{lll}
I_{r i}=I_{p} & , I_{d}=0 & \text { at } \quad z^{\prime}=0, \quad 0 \leq \theta \leq \frac{\pi}{2}  \tag{2.4.2}\\
I_{r i} \rightarrow 0 & & \text { at } \quad z^{\prime} \rightarrow \infty
\end{array}
$$

### 2.4.2 Intensities

As is customary, the specific intesntiy is separated into two components, namely, the reduced incident intensity $I_{r i}$ and the diffuse intensity $I_{d}$ by letting

$$
\begin{equation*}
I=I_{r i}+I_{d} . \tag{2.4.3}
\end{equation*}
$$

Substituting (2.4.3) into (2.4.1) and (2.4.2) yield the defining equations for $I_{r i}$ and $I_{d}$, which take the forms

$$
\begin{align*}
& \frac{\partial}{\partial t^{\prime}} I_{r i}+\hat{s} \cdot \nabla I_{r i} \frac{\partial}{\partial z^{\prime}} I_{r i}+I_{r i}=0  \tag{2.4.4}\\
& \frac{\partial}{\partial t^{\prime}} I_{d}+\hat{s} \cdot \nabla I_{d}=-I_{d}+\frac{W_{o}}{4 \pi} \iint_{4 \pi} p\left(\hat{s} \cdot \hat{s}^{\prime}\right)\left[I_{r i}+I_{d}\right] d \Omega^{\prime}, \tag{2.4.5}
\end{align*}
$$

where $\hat{s} \cdot \nabla$ is defined in (2.4.1b) with boundary conditions

$$
\begin{array}{ll}
I_{r i}=I_{p} \quad, \quad I_{d}=0 & \text { at } \quad z^{\prime}=0, \quad 0 \leq \theta \leq \frac{\pi}{2} .  \tag{2.4.6}\\
I_{r i} \rightarrow 0 & \text { at } \quad z^{\prime} \rightarrow \infty
\end{array}
$$

### 2.4.3 The Reduced Incident Intensity

To solve (2.4.4) and (2.4.5), Fourier series representations are introduced for the time dependence of the intensities:

$$
\begin{equation*}
I_{j}\left(\rho^{\prime}, z^{\prime}, t^{\prime}, \theta, \psi\right)=\operatorname{Re}\left\{\sum_{v=0}^{\infty} I_{j, v}\left(\rho^{\prime}, z^{\prime}, \theta, \psi\right) e^{i \omega \omega^{\prime} t^{\prime}}\right\}, j=p, r i, d \tag{2.4.7}
\end{equation*}
$$

where $T^{\prime}=\sigma_{t} c T$ and $\omega^{\prime}=2 \pi / T^{\prime}$. Note that because (2.4.1) is linear, the angular frequency $\omega^{\prime}$ in (2.4.7) is the normalized version of the frequency $\omega$ in (2.2.3). Note also that although the specific intensity (power quantity) is always positive, the individual Fourier constituents $I_{j \nu}$ may be negative; hence, they cannot physically represent power. Substituting (2.4.7) into (2.4.4) yields for $z^{\prime}>0$,

$$
\begin{equation*}
i v \omega^{\prime} I_{r i, v}+\hat{s} \cdot \nabla I_{r i, v}+I_{r i, v}=0 \tag{2.4.8a}
\end{equation*}
$$

subject to the boundary conditions

$$
\begin{align*}
& I_{r i, v}=I_{p v} \quad \text { at } \quad z^{\prime}=0 \quad, \quad 0 \leq \theta \leq \frac{\pi}{2},  \tag{2.4.8b}\\
& I_{r i, v} \rightarrow 0 \quad \text { as } \quad z^{\prime} \rightarrow \infty
\end{align*}
$$

where $I_{p}$ is the specific intensity of the incident beam wave pulse train. Solution to (2.4.8a) and (2.4.8b) gives

$$
\begin{equation*}
I_{r i, v}=S_{p} e^{-\left(\rho^{\prime} / w^{\prime}\right)^{2}} f\left(t^{\prime}-z^{\prime}\right) \frac{\delta(\theta)}{2 \pi \sin (\theta)} \tag{2.4.9}
\end{equation*}
$$

where $f\left(t^{\prime}-z^{\prime}\right)$ is defined in (2.2.7) expressed in normalized variables. $\delta(\theta)$ is the Dirac delta function, $w^{\prime}=\sigma_{t} w$ is the normalized beamwidth, $t^{\prime}$ is the normalized time and $S_{p}$ is the magnitude of the time-averaged Poynting vector. Variables $\rho^{\prime}$ and $\theta$, are depicted in Figure A.1.

### 2.4.4 The Diffuse Intensity

Table 2.4.1 Ranges of the original independent variables and the transform and/or discretized variables that are used in the calculations.

| Independent <br> Variables | Ranges | Transform/Discretized <br> Variables | Ranges |
| ---: | :---: | :---: | :--- |
| $\rho^{\prime}$ | $\rho^{\prime} \geq 0$ |  | $k^{\prime}$ |
| $z^{\prime}$ | $0 \leq k^{\prime} \leq k_{\max }$ |  |  |
| $t^{\prime} \geq 0$ |  |  |  |
| $\theta$ | $0 \leq t^{\prime}<\infty$ |  | $l=0,1,2, \ldots, v_{\max }$ |
| $\psi$ | $0 \leq \theta \leq \pi$ |  | $l=m, m+1, \ldots, N$ |
| $\psi$ | $0 \leq \psi \leq 2 \pi$ |  | $m=0,1,2, \ldots, N$ |

2.4.4.1 Transport Equation for $I_{d v}$ To solve for the diffuse intensity in (2.4.5), the Fourier series representation is introduced again via (2.4.7) to obtain

$$
\begin{align*}
& i v \omega^{\prime} I_{d v}+\hat{s} \cdot \nabla I_{d v}= \\
& \quad-I_{d v}+\frac{W_{o}}{4 \pi} \iint_{4 \pi} p\left(\hat{s} \cdot \hat{s}^{\prime}\right)\left[I_{r i, v}+I_{d v}\right] \sin \theta^{\prime} d \theta^{\prime} d \psi^{\prime} \tag{2.4.10}
\end{align*}
$$

with boundary conditions

$$
\begin{array}{llll}
I_{r i, v}=I_{p v} & , \quad I_{d v}=0 \quad \text { at } \quad z^{\prime}=0 \quad, \quad 0 \leq \theta \leq \frac{\pi}{2},  \tag{2.4.10a}\\
I_{r i, v} \rightarrow 0 \quad, \quad I_{d v} \rightarrow 0 \quad \text { as } \quad z^{\prime} \rightarrow \infty
\end{array}
$$

where $v=0,1,2, \cdots$ Substitution of the reduced incident intensity from (2.4.9) into (2.4.10a) gives for $z^{\prime}>0$,

$$
\begin{align*}
& i v \omega^{\prime} I_{d v}+\hat{s} \cdot \nabla I_{d v} \\
& \quad=-I_{d v}+\frac{W_{o}}{4 \pi} \iint_{4 \pi}\left[p(\cos (\gamma)) I_{d v}\right] \sin \theta^{\prime} d \theta^{\prime} d \psi^{\prime}+\frac{W_{o}}{4 \pi} S_{p} f_{v} e^{-z^{\prime}} p(\theta) e^{-\left(\rho^{\prime} / w^{\prime}\right)^{2}} \tag{2.4.11}
\end{align*}
$$

### 2.4.4.2 Fourier-Series/Hankel-Transforms and Associated Legendre Functions In order

 to solve (2.4.12), $I_{d v}$ is represented in terms of a Fourier series in $\psi$ and a Hankel transform in $\rho$, i.e.$$
\begin{align*}
& I_{d v}\left(\rho^{\prime}, z^{\prime}, \theta, \psi\right)= \\
& \quad=\sum_{m=0}^{\infty} \int_{k=0}^{\infty} A_{m}^{v}\left(k^{\prime} ; z^{\prime}, \theta\right)\left[J_{m}\left(k^{\prime} \rho^{\prime}\right) \cos (m \psi)\right] k^{\prime} d k^{\prime} \tag{2.4.12}
\end{align*}
$$

Accordingly, the representation in (2.4.12) for $I_{d v}$ is an expansion in terms of basis functions, $\cos (m \psi) J_{m}\left(k^{\prime} \rho^{\prime}\right)$, which are complete and obey well known orthogonality properties. In addition, the $\theta$-dependent expansion coefficient is expanded in terms of Associate Legendre functions

$$
\begin{equation*}
A_{m}^{v}\left(k^{\prime} ; z^{\prime}, \theta\right)=\sum_{l=m}^{\infty} A_{m, l}^{v}\left(k^{\prime} ; z^{\prime}\right) P_{l}^{m}(\cos (\theta))=\sum_{l=m}^{\infty}(2 l+1) \bar{A}_{m, l}^{v}\left(k^{\prime} ; z^{\prime}\right) P_{l}^{m}(\cos (\theta)) \tag{2.4.13}
\end{equation*}
$$

Consequently,

$$
\begin{equation*}
I_{d v}^{F}(\rho, z, \theta, \psi)=\sum_{m=0}^{\infty} \int_{k=0}^{\infty} \sum_{l=m}^{\infty}(2 l+1) \bar{A}_{m, l}^{v}\left(k^{\prime} ; z^{\prime}\right) P_{l}^{m}(\cos (\theta)) J_{m}\left(k^{\prime} \rho^{\prime}\right) \cos (m \psi) k^{\prime} d k^{\prime} \tag{2.4.14}
\end{equation*}
$$

2.4.4.3 Phase Function as a Series Expansion in Legendre Polynomials Conveniently, the phase function is represented as a series expansion in terms of Legendre polynomials $P_{l}$ as follows

$$
\begin{equation*}
p(\gamma)=\sum_{l=0}^{\infty}(2 l+1) g_{l} P_{l}(\cos (\gamma)), \tag{2.4.15}
\end{equation*}
$$

where because of the orthogonality of Legendre polynomials, the expansion coefficients $g_{l}$ are found to be

$$
\begin{equation*}
g_{l}=\frac{2 \alpha}{\Delta \gamma_{s}^{2}} \int_{\gamma=0}^{\pi} e^{-\left(\gamma / \Delta \gamma_{s}\right)^{2}} P_{l}(\cos \gamma) \sin \gamma d \gamma+(1-\alpha) \delta_{0 l} \tag{2.4.16}
\end{equation*}
$$

$$
\delta_{0 l}=\left\{\begin{array}{ll}
1, & \text { for } l=0  \tag{2.4.16a}\\
0, & \text { for } l \neq 0
\end{array}\right\}
$$

Appropriate for the theoretical development here, the Legendre polynomials are expressed in terms of Assiciated Legendre functions via the well-known relationship:

$$
\begin{align*}
P_{l}(\cos (\gamma)) & =P_{l}(\mu) P_{l}\left(\mu^{\prime}\right)+2 \sum_{n=1}^{l} \frac{(l-n)!}{(l+n)!} P_{l}^{n}(\mu) P_{l}^{n}\left(\mu^{\prime}\right) \cos \left(n\left(\psi-\psi^{\prime}\right)\right)  \tag{2.4.17}\\
& =\sum_{n=0}^{l} \frac{2(l-n)!}{\varepsilon_{n}(l+n)!} P_{l}^{n}(\mu) P_{l}^{n}\left(\mu^{\prime}\right) \cos \left(n\left(\psi-\psi^{\prime}\right)\right)
\end{align*}
$$

with $\mu$ defined in (2.4.1c)

$$
\varepsilon_{n}=\left\{\begin{array}{l}
2, n=0  \tag{2.4.17a}\\
1, n=1,2, \ldots
\end{array}\right.
$$

Substituting (2.4.17) into (2.4.15) gives

$$
\begin{equation*}
p(\cos \gamma)=\sum_{l=0}^{\infty} \sum_{n=0}^{l}(2 l+1) \frac{2(l-n)!}{\varepsilon_{n}(l+n)!} g_{l} P_{l}^{n}(\mu) P_{l}^{n}\left(\mu^{\prime}\right) \cos n\left(\psi-\psi^{\prime}\right) \tag{2.4.18}
\end{equation*}
$$

2.4.4.4 System of Linear Equations Substitution of $p(\cos (\gamma))$ and $I_{d v}\left(\rho^{\prime}, z^{\prime}, \theta, \psi\right)$ from (2.4.18) and (2.4.16), respectively, into (2.4.11) yields the inhomogeneous system of linear firstorder differential equations

$$
\begin{align*}
& (l-m)\left[\frac{\partial}{\partial z^{\prime}} \bar{A}_{m, l-1}^{v}-i v \omega^{\prime} \bar{A}_{m, l-1}^{v}\right]+(l+m+1)\left[\frac{\partial}{\partial z^{\prime}} \bar{A}_{m, l+1}^{v}-i v \omega^{\prime} \bar{A}_{m, l+1}^{v}\right] \\
& +(2 l+1)\left[1-W_{0} g_{l}+i v \omega^{\prime}\right] \bar{A}_{m, l}^{v} \\
& +\frac{k^{\prime}}{2}\left[(l-m-1)(l-m) \bar{A}_{m+1, l-1}^{v}-(l+m+1)(l+m+2) \bar{A}_{m+1, l+1}^{v}\right]+  \tag{2.4.19}\\
& \frac{k^{\prime}}{2} \varepsilon_{m-1}\left[\bar{A}_{m-1, l-1}^{v}-\bar{A}_{m-1, l+1}^{v}\right] \\
& =g_{l}(2 l+1) \bar{Q}^{v}\left(k^{\prime}, w^{\prime}\right) \delta_{m 0} e^{-z^{\prime}}
\end{align*}
$$

where upon truncation

$$
m=0,1,2, \cdots, N, l=m, \quad m+1, \cdots, N
$$

and

$$
\begin{equation*}
\bar{Q}^{v}\left(k^{\prime}, w^{\prime}\right)=\frac{W_{0}}{4 \pi} S_{p} f_{v}\left(\frac{\left(w^{\prime}\right)^{2}}{2} e^{-\left(k^{\prime} w^{\prime} / 2\right)^{2}}\right) \tag{2.4.19a}
\end{equation*}
$$

Convenient normalizations are introduced into (2.4.12) and (2.4.19):

$$
\begin{align*}
& \bar{A}_{m, l}^{v}\left(k^{\prime} ; z^{\prime}\right)=b_{m, l}^{v}\left(k^{\prime} ; z^{\prime}\right) \frac{Q^{v}\left(k^{\prime}, w^{\prime}\right)}{U_{l}^{m} V^{m}},  \tag{2.4.20a}\\
& Q^{v}\left(k^{\prime}, w^{\prime}\right)=\sqrt{\pi} \frac{W_{0}}{4 \pi} S_{p} f_{v} w^{\prime 2} e^{-\left(k^{\prime} w^{\prime} / 2\right)^{2}},  \tag{2.4.20b}\\
& U_{l}^{m}=\sqrt{\frac{2}{2 l+1} \frac{(l+m)!}{(l-m)!}}  \tag{2.4.20c}\\
& V^{m}=\sqrt{\pi \varepsilon_{m}}, \text { and } \tag{2.4.20d}
\end{align*}
$$

$$
\varepsilon_{m}=\left\{\begin{array}{ccc}
2, & m=0  \tag{2.4.20e}\\
1, & m=1,2, \ldots
\end{array}\right.
$$

Thus for (2.4.15) is rewritten as

$$
\begin{align*}
& I_{d v}(\rho, z, \theta, \psi)= \\
& \quad \sum_{m=0}^{\infty} \int_{k=0}^{\infty} \sum_{l=m}^{\infty}(2 l+1) Q^{\nu}\left(k^{\prime}, w^{\prime}\right) b_{m, l}^{v}\left(k^{\prime} ; z^{\prime}\right) \frac{P_{l}^{m}(\cos (\theta))}{U_{l}^{m}} \frac{\cos (m \psi)}{V^{m}} J_{m}\left(k^{\prime} \rho\right) k^{\prime} d k^{\prime} \tag{2.4.21}
\end{align*}
$$

and simplifies (2.4.19) to produce

$$
\begin{align*}
& \alpha_{1}\left[\frac{\partial}{\partial z^{\prime}} b_{m, l-1}^{v}-i v \omega^{\prime} b_{m, l-1}^{v}\right]+\alpha_{2}\left[\frac{\partial}{\partial z^{\prime}} b_{m, l+1}^{v}-i v \omega^{\prime} b_{m, l+1}^{v}\right]+ \\
& \alpha_{3}\left[1-W_{0} g_{l}+i v \omega^{\prime}\right] b_{m, l}^{v}+  \tag{2.4.22}\\
& \frac{k^{\prime}}{2} \bar{\varepsilon}_{m}\left[\alpha_{4} b_{m+1, l-1}^{v}-\alpha_{5} b_{m+1, l+1}^{v}\right]+\frac{k^{\prime}}{2} \delta_{m}\left[\alpha_{6} b_{m-1, l-1}-\alpha_{7} b_{m-1, l+1}^{v}\right]=g_{l} \delta_{m 0} e^{-z^{\prime}}
\end{align*}
$$

with

$$
\begin{align*}
& \bar{\varepsilon}_{m}= \begin{cases}\sqrt{2} & , \quad m=0 \\
1, & m=1,2,3, \ldots, N-1, N\end{cases}  \tag{2.4.22a}\\
& \delta_{m}=\left\{\begin{array}{lll}
0 & , & m=0 \\
\sqrt{2} & , & m=1 \\
1 & , & m=2,3, \ldots, N-1, N
\end{array}\right.  \tag{2.4.22b}\\
& \delta_{m 0}=\left\{\begin{array}{lll}
1 & , & m=0 \\
0 & , & m \neq 0
\end{array},\right.  \tag{2.4.22c}\\
& \alpha_{1} \equiv \sqrt{\frac{(l-m)(l+m)(2 l-1)}{(2 l+1)^{2}}}  \tag{2.4.22d}\\
& \alpha_{2} \equiv \sqrt{\frac{(l-m+1)(l+m+1)(2 l+3)}{(2 l+1)^{2}}} \tag{2.4.22e}
\end{align*}
$$

$$
\begin{align*}
& \alpha_{3} \equiv \sqrt{2 l+1}  \tag{2.4.22f}\\
& \alpha_{4} \equiv \sqrt{\frac{(l-m-1)(l-m)(2 l-1)}{(2 l+1)^{2}}}  \tag{2.4.22g}\\
& \alpha_{5} \equiv \sqrt{\frac{(l+m+1)(l+m+2)(2 l+3)}{(2 l+1)^{2}}}  \tag{2.4.22h}\\
& \alpha_{6} \equiv \sqrt{\frac{(l+m)(l+m-1)(2 l-1)}{(2 l+1)^{2}}}  \tag{2.4.22i}\\
& \alpha_{7} \equiv \sqrt{\frac{(l-m+2)(l-m+1)(2 l+3)}{(2 l+1)^{2}}} \tag{2.4.22j}
\end{align*}
$$

2.4.4.5 Homogeneous Solution The solution of (2.4.22) requires determination of both the homogeneous and particular solutions. For the former, the right hand side of (2.4.22) is set equal to zero an the homogeneous solution is taken to be of the form

$$
\begin{equation*}
b_{m, l}^{\nu, h}\left(z^{\prime}, k^{\prime}\right)=G_{m, l}^{v}\left(k^{\prime}\right) e^{-z^{\prime} / \lambda} \tag{2.4.23}
\end{equation*}
$$

Generalized eigenvalue equation

$$
\begin{equation*}
\left[\mathbf{A}_{\mathbf{0}}\right] \mathbf{G}=\lambda\left[\mathbf{C}_{\mathbf{0}}\right] \mathbf{G} . \tag{2.4.24}
\end{equation*}
$$

The matrices $\left[\mathbf{A}_{\mathbf{0}}\right]$ and $\left[\mathrm{C}_{\mathbf{0}}\right]$ are given Appendix B . The eigenvalues $(\lambda)$ and eigenvectors ( $\mathbf{G}$ ) of (2.4.24) are determined by using the QZ method algorithm in the MATLAB library. The homogeneous system of linear equations corresponding to (2.4.24) takes the form

$$
\begin{align*}
& \alpha_{1} G_{m, l-1}^{v}+\alpha_{2} G_{m, l+1}^{v}= \\
& \lambda\left\{-c_{v}\left[\alpha_{1} G_{m, l-1}^{v}+\alpha_{2} G_{m, l+1}^{v}\right]+b_{v l}\left[\alpha_{3} G_{m, l}^{v}\right]+\right. \\
& \frac{k^{\prime}}{2} \bar{\varepsilon}_{m}\left[\alpha_{4} G_{m+1, l-1}^{v}-\alpha_{5} G_{m+1, l+1}^{v}\right]+  \tag{2.4.25}\\
& \left.\frac{k^{\prime}}{2} \delta_{m}\left[\alpha_{6} G_{m-1, l-1}^{v}-\alpha_{7} G_{m-1, l+1}^{v}\right]\right\}
\end{align*}
$$

where

$$
\bar{\varepsilon}_{m}= \begin{cases}\sqrt{2} & , \quad m=0  \tag{2.4.25a}\\ 1 & , \quad m=1,2,3, \ldots, N-1, N^{\prime}\end{cases}
$$

$$
\begin{align*}
& \delta_{m}=\left\{\begin{array}{lll}
0 & , & m=0 \\
\sqrt{2} & , & m=1 \\
1 & , & m=2,3,4, \ldots, N-1, N
\end{array},\right.  \tag{2.4.25b}\\
& b_{\nu l}=1-W_{0} g_{l}+i v \omega^{\prime},  \tag{2.4.25c}\\
& c_{v}=i v \omega^{\prime}, \tag{2.4.25d}
\end{align*}
$$

and $\alpha_{n}(n=1,2,3,4,5,6,7)$ are given in (2.4.22a) to (2.4.22j)
2.4.4.6 Particular Solution The particular solution to (2.4.22) is obtained by assuming

$$
\begin{equation*}
b_{m, l}^{v p}\left(z^{\prime}, k^{\prime}\right)=F_{m, l}^{v}\left(k^{\prime}\right) e^{-z^{\prime}}, \tag{2.4.26}
\end{equation*}
$$

which when substituted into (2.4.22) gives in matrix form

$$
\begin{equation*}
\left[\mathbf{B}_{0}\right] \mathbf{F}=\mathbf{g} \tag{2.4.27}
\end{equation*}
$$

The matrix $\left[\mathbf{B}_{0}\right]$ in (2.4.27) is shown in Appendix $B$. The system of linear equations corresponding to (2.4.27) and taken the form

$$
\begin{align*}
& \alpha_{1}\left[-F_{m, l-1}^{v p} a_{v}\right]+\alpha_{2}\left[-F_{m, l+1}^{v p} a_{v}\right]+\alpha_{3} b_{v l} F_{m, l}^{v p}+ \\
& \frac{k^{\prime}}{2} \bar{\varepsilon}_{m}\left[\alpha_{4} F_{m+1, l-1}^{v p}-\alpha_{5} F_{m+1, l+1}^{v p}\right]+\frac{k^{\prime}}{2} \delta_{m}\left[\alpha_{6} F_{m-1, l-1}^{v}-\alpha_{7} F_{m-1, l+1}^{v}\right],  \tag{2.4.28}\\
& =g_{l} \delta_{m 0} e^{-z^{\prime}}
\end{align*}
$$

where

$$
\begin{align*}
& \delta_{m 0}=\left\{\begin{array}{lll}
1 & , & m=0 \\
0 & , & m \neq 0^{\prime}
\end{array}\right.  \tag{2.4.28a}\\
& b_{v l}=1-W_{0} g_{l}+i v \omega^{\prime}, \text { and }  \tag{2.4.28b}\\
& a_{v}=1+i v \omega^{\prime} . \tag{2.4.28c}
\end{align*}
$$

The LU factorization with iterative refinement found in the LAPACK is used to find the particular solution. The solution vector $\mathbf{F}$ is shown in Appendix $B$ and $\mathbf{g}$ is given below:
$\mathbf{g}=\left(\begin{array}{c}-g_{0} \\ -g_{1} \\ -g_{2} \\ -g_{3} \\ \vdots \\ -g_{N-1} \\ -g_{N} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0\end{array}\right)$.
The general solution to (2.4.22) is the superposition of the particular solution and the $N_{i}=(N+1)^{2} / 4$ allowable homogeneous solutions, which obey the condition that $\operatorname{Re}\{1 / \lambda\}>0$ to ensure that solutions decay as $z^{\prime} \rightarrow \infty$. The general solution is

$$
\begin{align*}
b_{m, l}^{v} & =b_{m, l}^{v p}\left(k^{\prime}, z^{\prime}\right)+\sum_{i=1}^{N_{i}} \alpha_{i} b_{m, l}^{v, i}\left(k^{\prime}, z^{\prime}\right)  \tag{2.4.30}\\
& =F_{m, l}^{v p}\left(k^{\prime}, z^{\prime}\right) e^{-z^{\prime}}+\sum_{i=1}^{N_{i}} \alpha_{i} G_{m, l}^{v, i}\left(k^{\prime}, z^{\prime}\right) e^{-z^{\prime} / \lambda_{i}}
\end{align*}
$$

The diffuse intensity as expressed in (2.4.21) becomes with the proper truncation

$$
\begin{align*}
I_{d v}\left(\rho^{\prime}, z^{\prime}, \theta, \psi\right)= & \int_{k^{\prime}=0}^{k_{\max }^{\prime}} \sum_{m=0}^{N} \sum_{l=m}^{N}(2 l+1) Q^{v}\left(k^{\prime}, w^{\prime}\right)\left[F_{m, l}^{v} e^{-z^{\prime}}+\sum_{i=0}^{N_{i}-1} \alpha_{i} G_{m, l}^{v, i} e^{-z^{\prime} / \sigma_{i}}\right] \cdot  \tag{2.4.31}\\
& {\left[\frac{P_{l}^{m}(\cos (\theta))}{U_{l}^{m}}\right]\left[\frac{\cos (m \psi)}{V^{m}}\right] J_{m}\left(k^{\prime} \rho^{\prime}\right) k^{\prime} d k^{\prime} }
\end{align*}
$$

2.4.4.7 Boundary Condition for the Determination of Constants $\alpha_{i}$ From (2.4.10a), $I_{d v}$ satisfies the boundary condition

$$
\begin{equation*}
I_{d v}\left(\rho^{\prime}, z^{\prime}, \theta, \psi\right)=0 \quad \text { for } z^{\prime}=0, \quad 0 \leq \theta \leq \frac{\pi}{2}, \quad 0 \leq \rho^{\prime}<\infty, 0 \leq \psi \leq 2 \pi \tag{2.4.32}
\end{equation*}
$$

It can then be shown using the orthogonality and completeness properties of Bessel functions that

$$
\begin{align*}
& I_{d v}\left(k^{\prime} ; z^{\prime}=0, \theta, \psi\right) \equiv \\
& \qquad \sum_{m=0}^{N} \sum_{l=m}^{N}(2 l+1) Q^{v}\left(k^{\prime}, w^{\prime}\right) b_{m, l}^{v}\left(k^{\prime} ; z^{\prime}\right) \frac{P_{l}^{m}(\cos (\theta))}{U_{l}^{m}} \frac{\cos (m \psi)}{V^{m}}=0  \tag{2.4.33}\\
& \quad z^{\prime}=0, \quad 0 \leq \theta \leq \frac{\pi}{2}, \quad 0 \leq k^{\prime}<\infty, \quad 0 \leq \psi \leq 2 \pi
\end{align*}
$$

The above boundary condition is satisfied by using the normalized spherical harmonic functions as testing functions in the weighted residual method [12]. Hence,

$$
\begin{equation*}
\int_{\psi=0}^{2 \pi} \int_{\theta=0}^{\pi / 2} I_{d v}\left(k^{\prime} ; z^{\prime}=0, \theta, \psi\right)\left[\frac{P_{l}^{m}(\cos (\theta))}{U_{l}^{m}}\right]\left[\frac{\cos (m \psi)}{V^{m}}\right] \sin (\theta) d \theta d \psi=0 \tag{2.4.34}
\end{equation*}
$$

for values of $(l-m)$ which are odd.
Substituting (2.4.33) into (2.4.34) yields

$$
\begin{align*}
& Q^{\nu}\left(k^{\prime}, w^{\prime}\right) \sum_{m^{\prime}=0}^{N} \sum_{l^{\prime}=m^{\prime}}^{N}\left(2 l^{\prime}+1\right)\left[F_{m^{\prime}, l^{\prime}}^{v}+\sum_{i=0}^{N_{i}-1} \alpha_{i} G_{m^{\prime}, l^{\prime}}^{v, i}\right] . \\
& \underbrace{\int_{\psi=0}^{2 \pi}\left[\frac{\cos \left(m^{\prime} \psi\right)}{V^{m^{\prime}}}\right]\left[\frac{\cos (m \psi)}{V^{m}}\right] d \psi}_{\equiv \delta_{m m^{\prime}}=\left\{\begin{array}{l}
1, m=m^{\prime} \\
0, m \neq m^{\prime}
\end{array}\right.} \underbrace{\int_{\theta=0}^{\pi / 2}\left[\frac{P_{l^{\prime}}^{m^{\prime}}(\cos (\theta))}{U_{l^{\prime}}^{m^{\prime}}}\right]\left[\frac{P_{l}^{m}(\cos (\theta))}{U_{l}^{m}}\right] \sin (\theta) d \theta}_{=I_{l, l^{\prime}}^{\prime \prime}}=0 \tag{2.4.35}
\end{align*}
$$

(2.4.35) shows the values for two uncoupled integrations, which yield

$$
\begin{equation*}
\sum_{i=0}^{N_{i}-1} \alpha_{i}\left(k^{\prime}\right) \underbrace{\left[\sum_{l^{\prime}=m}^{N}\left(2 l^{\prime}+1\right) G_{m, l^{\prime}}^{v, i}\left(k^{\prime}\right) \hat{I}_{l, l^{\prime}}^{m}\right]}_{\equiv S_{m, l}^{v, i}\left(k^{\prime}\right)}=\underbrace{\left[-\sum_{l^{\prime}=m}^{N}\left(2 l^{\prime}+1\right) F_{m, l}^{v}\left(k^{\prime}\right) \hat{I}_{l, l^{\prime}}^{m}\right]}_{\equiv T_{m, l}^{v}\left(k^{\prime}\right)} \tag{2.4.36}
\end{equation*}
$$

Equation (2.4.36) defines a second linear system of equations as

$$
\begin{equation*}
\sum_{i=0}^{N_{i}-1} \alpha_{i}\left(k^{\prime}\right) S_{m, l}^{\nu, i}=T_{m, l}^{v}\left(k^{\prime}\right) \tag{2.4.37a}
\end{equation*}
$$

where

$$
\begin{align*}
& S_{m, l}^{v, i}\left(k^{\prime}\right)=\sum_{l^{\prime}=m}^{N}\left(2 l^{\prime}+1\right) G_{m, l^{\prime}}^{v, i}\left(k^{\prime}\right) \hat{I}_{l, l^{\prime}}^{m}, \text { and }  \tag{2.4.37b}\\
& T_{m, l}^{v}\left(k^{\prime}\right)=-\sum_{l^{\prime}=m}^{N}\left(2 l^{\prime}+1\right) F_{m, l^{\prime}}^{v}\left(k^{\prime}\right) \hat{I}_{l, l^{\prime}}^{m}, \tag{2.4.37c}
\end{align*}
$$

and $\hat{I}_{l, l^{\prime}}^{m}$ is defined in (2.4.35) to be

$$
\begin{equation*}
\hat{I}_{l, l^{\prime}}^{m}=\frac{I_{l, l^{\prime}}^{m}}{U_{l^{\prime}}^{m} U_{l}^{m}}=\int_{\theta=0}^{\pi / 2}\left[\frac{P_{l^{\prime}}^{m}(\cos (\theta))}{U_{l^{\prime}}^{m}}\right]\left[\frac{P_{l}^{m}(\cos (\theta))}{U_{l}^{m}}\right] \sin (\theta) d \theta, \tag{2.4.38}
\end{equation*}
$$

The evaluation of $\hat{I}_{l, l^{\prime}}^{m}$ is displayed in Table 2.4.3; to summarize,

$$
\begin{gather*}
\hat{I}_{l, l^{\prime}}^{m}=\left\{\begin{array}{l}
\hat{J}_{l, l^{\prime}}^{m}, \text { for } m \geq 0 \text { such that } l-m=o d d, l-l^{\prime}=\text { even } \\
\frac{1}{2}, \text { for } m \geq 0 \text { such that } l-m=o d d, l-l^{\prime}=0
\end{array}\right.  \tag{2.4.39}\\
\hat{J}_{l, l^{\prime}}^{m}=\frac{1}{2}(-1)^{\left(l^{\prime}+l\right) / 2+m+1 / 2} \frac{\sqrt{(2 l+1)\left(2 l^{\prime}+1\right)\left((l+m)^{2}-m^{2}\right)}}{\left(l^{\prime}-l\right)\left(l^{\prime}+l+1\right)} .  \tag{2.4.39a}\\
R\left(l^{\prime}-m\right) R\left(l^{\prime}+m\right) R(l+m+1) R(l-m+1)
\end{gather*}
$$

with

$$
\begin{equation*}
R(\alpha) \equiv \frac{\sqrt{\alpha!}}{2^{\alpha / 2}\left(\frac{\alpha}{2}\right)!} \tag{2.4.39b}
\end{equation*}
$$

Table 2.4.2 Boundary Condition Coefficient $\hat{I}_{l, l^{\prime}}^{m}$

| $m \bmod 2$ | $l \bmod 2$ | $l^{\prime} \bmod 2$ | $\hat{I}_{l, l^{\prime}}^{m}$ | $m \bmod 2$ | $l \bmod 2$ | $l^{\prime} \bmod 2$ | $\hat{I}_{l, l^{\prime}}^{m}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 |
| 0 | $l=l^{\prime}$ | 0 | 0 | 1 | $l=l^{\prime}$ | 0 | 1/2 |
| 0 | 0 | 1 | 0 | 1 | 0 | 1 | $\hat{J}_{l, l^{\prime}}^{m}$ |
| 0 | 1 | 0 | $\hat{J}_{l, l^{\prime}}^{m}$ | 1. | 1 | 0 | 0 |
| 0 | 1 | 1 | 0 | 1 | 1 | 1 | 0 |
| 0 | $l=l^{\prime}$ | 1 | $1 / 2$ | 1 | $l=l^{\prime}$ | 1 | 0 |

### 2.5 Received Power

The power received by a highly directive, narrow beamwidth antenna located in the forest is derived in Appendix C given by

$$
\begin{equation*}
P^{\prime}\left(z^{\prime}, \rho^{\prime}, t^{\prime}, \theta_{M}, \psi_{M}\right)=\operatorname{Re}\left\{\sum_{v=0}^{v_{\text {max }}} P_{v}^{\prime}\left(z^{\prime}, \rho^{\prime}, \theta_{M}, \psi_{M}\right) e^{i v \omega^{\prime}\left(t^{\prime}-z^{\prime}\right)}\right\}, \tag{2.4.40}
\end{equation*}
$$

where

$$
\begin{align*}
& P_{\nu}^{\prime}\left(z^{\prime}, \rho^{\prime}, \theta_{M}, \psi_{M}\right)=P_{r i, v}^{\prime}\left(z^{\prime}, \rho^{\prime}, \theta_{M}, \psi_{M}\right)+P_{d, v}^{\prime}\left(z^{\prime}, \rho^{\prime}, \theta_{M}, \psi_{M}\right)  \tag{2.4.40a}\\
& P_{r i, v}^{\prime}\left(z^{\prime}, \rho^{\prime}, \theta_{M}, \psi_{M}\right)=\frac{D\left(\theta_{M}\right)}{D(0)} e^{-\left(\rho^{\prime} / w^{\prime}\right)^{2}} f_{v} e^{-z^{\prime}}  \tag{2.4.40b}\\
& P_{d, v}^{\prime}=\frac{4 \pi}{S_{p} D(0)} I_{d \nu}\left(z^{\prime}, \rho^{\prime}, \theta_{M}, \psi_{M}\right) \tag{2.4.40c}
\end{align*}
$$

and

$$
\begin{align*}
& D\left(\gamma_{R}\right)=\left(\frac{2}{\Delta \gamma_{M}}\right)^{2} e^{-\left(\gamma_{R} / \Delta \gamma_{M}\right)^{2}}, \quad \Delta \gamma_{M} \ll \pi  \tag{2.4.40d}\\
& D(0)=\left(\frac{2}{\Delta \gamma_{M}}\right)^{2} \tag{2.4.40e}
\end{align*}
$$

$v_{\max }$ is the truncated value to ensure that the summation converges.

## CHAPTER 3

## DATA INACCURACIES

### 3.1 Introduction

Numerical solutions by their very nature are approximate solutions. Therefore, attention must be paid to sources of inaccuracies. As several data are examined, characteristic graphs are provide to indicate which graphs are sufficiently accurate and which ones need to be improved. For instance, because the Quadrature method in [11] used a completely different approach than in the current method - the Legendre method (Pn-method), the plots of the data that were generated by these two distinct methods are compared to validate the results.

In general, errors can be minimized by increasing the value for $N$ at the expense of the computational time. In Appendix B, the size of the matrix for solving the system of linear equation and generalized eigenvalue problem is

$$
\begin{equation*}
M^{2}=\left(\frac{(N+1)(N+2)}{2}\right)^{2} \tag{3.1.1}
\end{equation*}
$$

It appears from (3.1.1) that the Matrix is in the order of $N^{4}$. The computational complexity for the solving linear system of equationsplus the generalized eigenvalue problem is in the order of $n^{3}$ [13], where $n$ is the size of the square matrix. When $N$ is increased slightly, the time to simulate a certain system will take longer than the time to complete the exact same system when $N$ is taken slightly smaller. Moreover, because all of the matrices are stored into the memory of a computer during the simulations, the computer will require vaster memory storage and faster memory transfer speed to accelerate these simulations.

The following global parameters are used in all of the simulations in this thesis since they were utilized in [2] and [11]:

$$
W_{0}=0.75
$$

$a=0.8$
$\alpha_{0}=4 \sqrt{5}$
$\Delta \gamma_{s}=0.3$
$\Delta \gamma_{M}=0.012$
$T^{\prime}=2$
The following parameters vary among different simulations: $N, w^{\prime}, \rho^{\prime}, z^{\prime}, \theta, \psi^{\prime}$.

### 3.2 Scatter or Phase Function



Figure 3.2.1 Scatter or phase function $p(\cos \gamma)$ simplified by (2.3.2) and by (2.4.17) and truncated at $\mathrm{N}=31$.


Figure 3.2.2 Error analysis of the phase function. n is the number of terms for the truncation of (2.4.18)

Figure 3.2.1 shows the polar plot of both the exact value (2.3.2) and the truncated series expansion (2.4.17) of the phase function. Needless to say, the figure shows no difference between the exact phase function and the approximated phase function. Figure 3.2.2 shows the semilog plot of the error $\varepsilon_{p}$ between the exact and the approximated phase function, i.e.
$\varepsilon_{p}=\left|\frac{p_{\text {exact }}-p_{\text {approximated }}}{p_{\text {exact }}}\right|$.
Figure 3.2.2 shows that the error between the exact and the approximated phase function is small when taking a large number of terms for the Legendre polynomial expansion.

### 3.3 Convergence in the Received Diffuse Power

Several graphs are provided to allow one to observe the convergence in the numerical data that is provided for the received diffuse power. By changing several parameters, particularly the value of $N$, and for different observation point, the convergence for received power curves is seen to improve.


Figure 3.3.1 Normalized received diffuse power versus time for different values of N for the plane wave case $w^{\prime}=\infty$ at $\rho^{\prime}=0, z^{\prime}=1, \theta=0^{\circ} ; \psi$ is undefined when $\rho^{\prime}=0$.

When the incident beam is a plane wave, all of the received diffuse power curves in the crest region overlap with each for different values of $N$. When $N$ is taken as 23 , the trough region for the received diffuse power is seen to be the most accurate. When $N$ is 47 , the received diffuse power in the trough region is observed to be the most improved. Therefore, by increasing N , better convergence is obtained.


Figure 3.3.2 Normalized received diffuse power versus normalized time for $\mathrm{N}=23,27,39$ at $w^{\prime}=10, \rho^{\prime}=0, z^{\prime}=1, \theta=0^{\circ}$.

Like the plane wave case, the collimated beam wave with the beamwidth of ten shows a problematic convergence in the trough region. Even when $N$ is thirty-nine, the curve in the trough region does not converge nicely. It should be noted the values of power in the trough region are very small which indicates that numerical inaccuracy becomes evident in this region unless much larger values of N are taken to improve the series representation of the solution.


Figure 3.3.3 Normalized received diffuse power versus normalized time for different values of $\theta$ at $\mathrm{N}=31, \mathrm{w}^{\prime}=\infty$ (plane wave), $\rho^{\prime}=0, z^{\prime}=1$.

Received power in the trough region seems to converge for better values near $\theta=9^{\circ}$ but appears to require larger value of N as $\theta$ approaches zero.


Figure 3.3.4 Normalized received diffuse power versus normalized time at $N=39, w^{\prime}=10$, $\rho^{\prime}=0, z^{\prime}=1, \theta=0^{\circ}$ and $4.83^{\circ}$.

Similar to the plane wave case in Figure 3.3.3, Figure 3.3 .4 shows that as $\theta$ approaches zero, inaccuracies appear in the triangle region. As shown in Figure 3.3.1, it is expected that if N were to be increasing, better results whould be obtained in the trough region. Since very low power coccurs in the trough region and increasing $N$ necessitates a considerable increase in computational time, it was not warranted generating data for large $w^{\prime}$. As will be seen shortly, small beam widths, say $w^{\prime}=1$ does not yield such inaccuracies in trough regions.


Figure 3.3.5 Normalized received diffuse power versus normalized time for $w^{\prime}=1$ and 10 and for $w^{\prime}=\infty$ (plane wave) at $\rho^{\prime}=0, z^{\prime}=1, \theta=0^{\circ}$.

Regardless of different values of N used, it appears as shown in Figure 3.3.5 that as the width ( $w$ ') of the incident beam wave becomes smaller, better convergence is obtained in the trough region.

### 3.4 Other Justifications



Figure 3.4.1 Comparisons between Quadrature method (Q) [11] and Polynomial method (Pn) of normalized received power versus normalized time for $z^{\prime}=1,3,5,10$ with $w^{\prime}=1$ and $\rho^{\prime}=0, \theta=4.83^{\circ}$.

The curves that are generated by the Quadrature method (lighter and thinner lines) lie very close to the curves that are generated by the polynomial method (darker and thicker lines), which are developed in this thesis.


Figure 3.4.2 Comparison between Quadrature method (Q) [11] and Polynomial method (Pn) of the normalized received power versus normalized time for $w^{\prime}=0.5,1,3,5, \infty$ and $\rho^{\prime}=0, z^{\prime}=1, \theta=4.83^{\circ}$.

The two set of curves in Figure 3.4.2 display simlar shapes and characteristics. Values nearly match each other over the crest region and possess the same shapes in the trough region. The Quadrature (Q) method yields consistently lower values for received power in the trough region and in the vicinity of the crest maximum. Since the plane wave result was obtained using the method presented in [2] and agrees with the Legendre polynomial (Pn) method, it is reasonable to assume that the Pn-method is more accurate than the Q-method, which is also an expected result since the Q -method is highly numerical as compared to the Pn-method. Hence, curves that are generated by the Pn-method are substatiate.

### 3.5 Discussion

$v_{\max }$ must be at least 10 in order for the received diffuse power curves to be accurate. In all of the graphs, $v_{\max }$ is chosen to lie between ten and fifteen, in which the choice of the value of $v_{\max }$ to be used is determined by a minimum error criteria. The normalized reduced incident power for different value of $v_{\max }$ is plotted in Figure 3.5.1. Figure 3.5 .1 shows that $v_{\max } \geq 12$ is needed for sufficiently accurate results to be obtained.


Figure 3.5.1 Normalized reduced incident power versus normalized time for $\rho^{\prime}=0, z^{\prime}=1, \theta=0^{\circ}$, for different values of $v_{\text {max }}$.

The $k_{\max }^{\prime}$ value is required in the truncation of the Hankel transform in (2.4.12) so that more than $99 \%$ of the integrand is included. To ensure that this be the case, $k_{\max }^{\prime}$ is selected to be $\frac{5}{w^{\prime}}$, where $w^{\prime}$ is the normalized beam width.

The Gaussian Quadrature method was used to perform the two integration needed in the simulation. The $\mathrm{k}^{\prime}$-integration (see 2.4.12) is approximated by thity-two terms whilst there are ninety-six terms for the integration over $\gamma$, which is needed to determine the expansion coeffcient $g_{l}$ in (2.4.16).

The solutions to the linear system of equations (2.4.27) or (2.4.28) for finding the particular solution and (2.4.37) for solving the boundary condition at $z^{\prime}=0$ were obtained by using LU factorization, linear equation solver, and iterative refinement packages provided by the optimized LAPACK. The eigenvalue solutions were obtained by using the Matlab library that is based on the QZ method in EISPACK. When tested, both of these procedures gave absolute errors ranging from $10^{-10}$ to $10^{-16}$. This accuracy is unable to be improve because the computer handles all variables using double precision, which means a precision that is accurate up to about sixteen digits.

The phase function is normalized such that

$$
\begin{equation*}
\iint_{4 \pi} p(\gamma) d \Omega=4 \pi \tag{3.5.1}
\end{equation*}
$$

This dictates that $g_{0}$ equals 1 . However, $g_{0}$ does not equal one when determined numerically from (2.4.16) which gives $g_{0}=0.9881$ when $\Delta \gamma_{s}=0.3, \alpha=0.8$. To ensure that the $g_{0}$ is unity, the phase function $p(\gamma)$ is redefined as $p_{\text {norm }}(\gamma)=p(\gamma) / g_{0}$ which guarantees that

$$
\begin{equation*}
\iint_{4 \pi} p_{\text {norm }}(\gamma) d \Omega=4 \pi \tag{3.5.2}
\end{equation*}
$$

## CHAPTER 4

## NUMERICAL RESULTS

### 4.1 Boundary Conditions



Figure 4.1.1 Diffuse Intensity versus $\theta$ for $\rho^{\prime}=0, z^{\prime}=0, w^{\prime}=10$.
The diffuse intensity is plotted in Figure 4.1.1 to indicate that the boundary condition

$$
\begin{equation*}
I_{d}=0 \quad \text { at } \quad z^{\prime}=0 \quad, \quad 0 \leq \theta \leq \frac{\pi}{2} \tag{4.1.1}
\end{equation*}
$$

is satisfied. As seen in the graph, when $\theta$ lies between $0^{\circ}$ and $90^{\circ}$, the diffuse intensity is nearly zero. As N increases, the diffuse intensity becomes even smaller and closer to zero at $z^{\prime}=0$ over the range $0 \leq \theta \leq \pi / 2$.


Figure 4.1.2 Diffuse intensity versus $\theta$ for beamwidths $w^{\prime}=0.5,1,2,3,5,7,10, \infty$ when $v=0$, $N=31, z^{\prime}=0, \rho^{\prime}=0$.

Figure 4.1.2 shows that for different beamwidths $\left(w^{\prime}\right)$, the boundary confition (4.1.1) is very well satisfied for the case $v=0$. For $v \neq 0$,(4.1.1) was also shown to be well satisfied; see Figure 4.1.3. As seen in Figure 4.1.2, numerical inaccuracies produce the negligibly values for $I_{d 0}$, which physically ought to be positive real.


Figure 4.1.3 Magnitude of diffuse intensity $I_{d 1}$ versus $\theta$ for $w^{\prime}=1,2$ and $v=1, \rho^{\prime}=0, z^{\prime}=0$.
Observe that the values of $\left|I_{d 1}\right|$ are significantly smaller than $\left|I_{d 0}\right|$ as indicated in Figure 4.1.2 using the same parameters. From Figure 4.1.3, the boundary condition (4.1.1) is well satisfied for $v=1$. Since $v \neq 0$, the diffuse intensity is complex, so its magnitude is plotted as distinct from the $v=0$ case which is purely real.

### 4.2 Power Attenuations



Figure 4.2.1 Time independent diffuse intensity $(v=0)$ versus $z^{\prime}$ for $w^{\prime}=1, \rho^{\prime}=0, \theta=0$.
The diffuse intensity in Figure 4.2.1 attenuates after reaching its maximum value in the vicinity of $z^{\prime}=1$. Notice that the diffuse intensity first increases, reaches a maximum level and then decreases as the beam wave penetrates deeper into the forest.


Figure 4.2.2 Normalized received power versus normalized time for $\rho^{\prime}$ varying from 0 to 5 and $N=31, w^{\prime}=1, z^{\prime}=1, \theta=0^{\circ}, \psi=0^{\circ}$.

Observe that the pulse in Figure 4.2.2 is strongest on the beam axis ( $\rho^{\prime}=0$ ) and as expected gets weaker the further the receiver is from $\rho^{\prime}=0$.


Figure 4.2.3 Normalized received power versus normalized time for $z^{\prime}=0.5,0.7,1,1.5,2,3$ and $\mathrm{N}=27, \mathrm{w}^{\prime}=1, \theta=0, \psi=0$, (a) $\rho^{\prime}=0$, (b) $\rho^{\prime}=1$, (c) $\rho^{\prime}=2$, (d) $\rho^{\prime}=3$.

As $\rho^{\prime}$ increases, the curves in Figure 4.2.3(a)-(d) shift downwards. This means that the received power $[\mathrm{dB}]$ is more attenuated as one moves further away from the $z$-axis.


Figure 4.2.4 Normalized received power versus normalized time for $\rho^{\prime}=0,1,2,3$ and $N=27$, $w^{\prime}=1, z^{\prime}=3, \theta=0, \psi=0^{\circ}$.

Observe in Figure 4.2.4 that when the point of observation moves away from the $z^{\prime}$-axis, the power tends to decrease more. Distortion in the trough region occurs due to small numerical inaccuracy at the very low power levels.


Figure 4.2.5 Normalized received power versus normalized time $t^{\prime}$ for $z^{\prime}=1,3,5,10$ and for $w^{\prime}=1.0, N=27, \rho^{\prime}=0, \theta=4.83^{\circ}, \psi=0^{\circ}$.

The graph in Figure 4.2.5 indicates that as the observation point moves away from the $z^{\prime}=0$ boundary between the forest and the air, the power, which for $\theta \neq 0$ is the diffuse power, attenuates and distorts due to pulse spreading.


Figure 4.2.6 Received power versus normalized time when (a) $w^{\prime}=2$, (b) $w^{\prime}=3$, (c) $w^{\prime}=5$, (d) $w^{\prime}=7$ for $z^{\prime}=1, \rho^{\prime}=0,1,5,10, \theta=4.83^{\circ}, \psi=0^{\circ}$.

In Figure 4.2.6, the curves of received power that were below -70 dB were not included because of inaccuracies.

### 4.3 Angular Spread



Figure 4.3.1 Normalized received diffuse power versus normalized time for beam wave with beamwidths $w^{\prime}=0.5,1,2,3,5,7$, and for a plane wave. $N=31$ for the plane wave case and $N=27$ for collimated beam waves; $\rho=0^{\circ}, z^{\prime}=1, \theta=5^{\circ}$.

Figure 4.3.1 shows that in the crest region all curves lie close together. The only difference occurs in the trough region; these more pulse distortion occurs as the beam width $\mathrm{w}^{\prime}$ gets smaller. Note that the $w^{\prime}=7$ case lies extremely close to the plane wave result, which shows the correct behavior of the solution as the beamwidth approaches large values.


Figure 4.3.2 Normalized received diffuse power versus normalized time for beam waves with beamwidth $w^{\prime}=0.5,1,2,3,5,7$, and for a plane wave. $N=31$ for the plane wave case and $N=27$ for collimated beam waves; $\rho=0^{\circ}, z^{\prime}=1, \theta=62^{\circ}$.

Figure depicts various plots of normalized received power for $\theta=62^{\circ}$. It can be seen that the received power is extremely low and exhibits considerable distortion. The smaller widths possess the lowest received power in the trough region. The power received for the beam wave with $w^{\prime}=7$ resembles the plane wave case as noted previously.


Figure 4.3.3 Normalized received diffuse power versus normalized time for beam waves with beamwidth $w^{\prime}=0.5,1,2,3,5,7$, and for a plane wave. $N=31$ for the plane wave case and $N=27$ for collimated beam waves; $\rho=0^{\circ}, z^{\prime}=1$, (a) $\theta=87^{\circ}$, (b) $\theta=118^{\circ}$, (a) $\theta=150^{\circ}$, (a) $\theta=175^{\circ}$.

Figure 4.3.3 shows that the received powers lose their distinctive pulse shape for $90^{\circ}<\theta<180^{\circ}$. In this range, the time dependence of $\mathrm{P}^{\prime}$ remain fairly constant but with magnitudes that continue to decrease as $\theta$ approaches $180^{\circ}$. These results are indications of the fact that for $\theta>0^{\circ}$, the antenna receives the diffuse (incoherent) intensity only, and in the range of backscatter directions $90^{\circ}<\theta<180^{\circ}$, the antenna faces the unbounded region $\left(z^{\prime} \rightarrow \infty\right)$, in which considerable multiscattering occurs.

### 4.4 Pulse Broadening Effect



Figure 4.4.1 Normalized received power versus normalized time for different values of $\theta_{M}$ and for (a) $z^{\prime}=1$, (b) $z^{\prime}=3,(c) z^{\prime}=5$, and (d) $z^{\prime}=10$ with $w^{\prime}=1, N=27, \rho^{\prime}=0$.

Figure 4.4 .1 show what happens to the received power when changing the antenna received angle $\theta_{M}=0^{\circ}$. Observe that the crest of the received power decreases as $\theta_{M}$ becomes larger. These graphs also indicate the effect of beam broadening and pulse broadening as the beam penetrates the forest.

## CHAPTER 5

## CONCLUSIONS

The theory of beam wave pulse propagation and scattering in vegetation was presented. This theory was based on the solution to the scalar radiative transfer equation in vegetation. The vegetation was modeled as a statistically homogeneous half-space of randomly distributed particles, which scatter and absorb electromagnetic energy and are large compare to the wavelength. The results obtained show how vegetation attenuates, broadens, and distorts a beam wave pulse train both on and off the beam-axis.

## APPENDIX A

## DIAGRAMS

Diagrams for the problem configuration of the vegetation and spy plots of the sparse matrices are provided in this appendix.


Figure A. 1 Basic geometry for an incident beam wave pulse train enters the forest halfspace. The "sphere" represents a scatter point in the forest. The tiled cylinder represents a received antenna, which is shown in Figure A.2.


Figure A. 2 The coordinate geometry of the received antenna, which is depicted as a tilted cylinder and has a main beam direction $\left(\theta_{M}, \psi_{M}\right)$; note $\theta=\theta_{M}$.


Figure $\mathbf{A} .3$ Spy plot of sparse matrix $\left[\mathbf{A}_{\mathbf{0}}\right]$. nz=number of nonzeros; $N=27$


Figure $\mathbf{A . 4}$ Spy plot of sparse matrices $\left[\mathbf{B}_{0}\right]$ and $\left[\mathbf{C}_{0}\right]$. nz=number of nonzeros; $\mathrm{N}=27$. Notice that the pseudo-banded off-diagonal spies indicate that $k^{\prime}$ is not 0 .

## APPENDIX B

## COMPUTATIONAL METHODOLOGIES

## B. 1 The Method for Indexing the Matrix

The particular and homogeneous solutions can be written in matrix notation as

$$
\begin{equation*}
\left[\mathbf{B}_{0}\right] \mathbf{F}=\mathbf{g} \tag{B.1}
\end{equation*}
$$

$$
\begin{equation*}
\left[\mathbf{A}_{\mathbf{0}}\right] \mathbf{G}=\lambda\left[\mathbf{C}_{\mathbf{0}}\right] \mathbf{G} \tag{B.2}
\end{equation*}
$$

The elements of the vector solutions are written as

$$
\begin{align*}
& F_{n(m, l)}^{v}  \tag{B.3}\\
& G_{n(m, l)}^{v, i} \tag{B.4}
\end{align*}
$$

with

$$
\begin{align*}
& n(m, l) \equiv \frac{2 m \bar{N}-m^{2}-m}{2}+l  \tag{B.4a}\\
& \bar{N} \equiv N+1  \tag{B.4b}\\
& m=0,1,2, \ldots, N  \tag{B.4c}\\
& l=m, m+1, m+2, \ldots, N
\end{align*}
$$

Table B. 1 The method of indexing $m, l$ for $N=7$, where the values in the cell inside the table are represented as the values of $l$.

| $m=0$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | $m=0$ |  |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| $m=1$ |  | 1 | 2 | 3 | 4 | 5 | 6 | 7 | $m=1$ |  |
| $m=2$ |  |  | 2 | 3 | 4 | 5 | 6 | 7 | $m=2$ |  |
| $m=3$ |  |  |  |  | 3 | 4 | 5 | 6 | 7 | $m=3$ |
| $m=4$ |  |  |  |  |  | 4 | 5 | 6 | 7 | $m=4$ |
| $m=5$ |  |  |  |  |  | 5 | 6 | 7 | $m=5$ |  |
| $m=6$ |  |  |  |  |  |  | 6 | 7 | $m=6$ |  |
| $m=7$ |  |  |  |  |  |  |  | 7 |  |  |
| $m=7$ |  |  |  |  |  |  |  |  |  |  |

$\bar{N}$ is the total number ( $m$ ) used. $n(m, l)$ in (B.4a) is the index of the matrix that is shown in Table B.1. The following is a proof of (B.6a). As one can see from Table B.1, the index
has a triangular distribution. We consider this triangle distribution to be summed from 0 to $m-1$ plus offset $(l-m)$. As a result, $n(m, l)$ becomes

$$
\begin{align*}
& n(m, l)=\sum_{i=0}^{m-1}(\bar{N}-i)+(l-m)  \tag{B.5a}\\
& \sum_{i=0}^{m-1}(\bar{N}-i)=m \bar{N}-\sum_{i=0}^{m} i=m \bar{N}-(0+1+2+3+\cdots+(m-2)+(m-1))=m \bar{N}-\frac{(m-1) m}{2} \tag{B.5b}
\end{align*}
$$

Substituting (B.5b) into (B.5a) gives(B.4a):

$$
\begin{equation*}
n(m, l)=m \bar{N}-\frac{(m-1) m}{2}+(l-m)=\frac{2 m \bar{N}-m^{2}-m}{2}+l \tag{B.6}
\end{equation*}
$$

For simplicity, we rewrite the expression in (B.3) and (B.4) as follows:

$$
\begin{align*}
& F_{m, l}^{v}  \tag{B.7}\\
& G_{m, l}^{v, i} \tag{B.8}
\end{align*}
$$

In general, any elements of a vector that has a subscript of $m, l$ represents one dimensional rather than the usual two dimensional matrix element. Thus, $F_{m, l}^{v}$ in (B.7) has one dimension, namely, $(m, l) . \quad v$ in the superscript means that $F_{m, l}^{v}$ depends on integer $v$. For equation (B.8), $G_{m, l}^{v, i}$ has two dimensions, $(m, l)$ and $i$ and depends on the integer $v . i$ in $G_{m, l}^{v, i}$ is the index for the eigenvalue that is associated with an eigenvector. For example, $F_{m, l}^{v}$ can be expressed as follows

$$
\mathbf{F}=\left(\begin{array}{c}
F_{0,0}^{v} \\
F_{0,1}^{v} \\
\vdots \\
F_{0, N}^{v} \\
F_{1,1}^{v} \\
F_{1,2}^{v} \\
\vdots \\
F_{1, N}^{v} \\
F_{2,2}^{v} \\
F_{2,3}^{v} \\
\vdots \\
F_{2, N}^{v} \\
\vdots \\
F_{N-1, N-1}^{v} \\
F_{N-1, N}^{v} \\
F_{N, N}^{v}
\end{array}\right)
$$

where

$$
\begin{align*}
& F_{m, l \equiv}^{U} \frac{2 m N-m^{2}-m}{2}+l \tag{B.9b}
\end{align*}
$$

## B. 2 Matrix Representation

The homogeneous system of equations in (2.4.25) is rewritten here as

$$
\begin{align*}
& \alpha_{1} G_{m, l-1}^{v}+\alpha_{2} G_{m, l+1}^{v}= \\
& \lambda\left\{-c_{v}\left[\alpha_{1} G_{m, l-1}^{v}+\alpha_{2} G_{m, l+1}^{v}\right]+b_{v l}\left[\alpha_{3} G_{m, l}^{v}\right]+\right. \\
& \frac{k^{\prime}}{2} \bar{\varepsilon}_{m}\left[\alpha_{4} G_{m+1, l-1}^{v}-\alpha_{5} G_{m+1, l+1}^{v}\right]+  \tag{B.10}\\
& \left.\frac{k^{\prime}}{2} \delta_{m}\left[\alpha_{6} G_{m-1, l-1}^{v}-\alpha_{7} G_{m-1, l+1}^{v}\right]\right\}
\end{align*}
$$

The particular system of equations in (2.4.28) is rewritten here as

$$
\begin{align*}
& \alpha_{1}\left[-F_{m, l-1}^{v p} a_{v}\right]+\alpha_{2}\left[-F_{m, l+1}^{v p} a_{\nu}\right]+\alpha_{3} b_{v l} F_{m, l}^{v p}+ \\
& \frac{k^{\prime}}{2} \bar{\varepsilon}_{m}\left[\alpha_{4} F_{m+1, l-1}^{v p}-\alpha_{5} F_{m+1, l+1}^{v p}\right]+\frac{k^{\prime}}{2} \delta_{m}\left[\alpha_{6} F_{m-1, l-1}^{v}-\alpha_{7} F_{m-1, l+1}^{v}\right]  \tag{B.11}\\
& =g_{l} \delta_{m 0} e^{-z^{\prime}}
\end{align*}
$$

The remaining variables in (B.10) and (B.11) are found in equations (2.4.22a~j), (2.4.25a~d), and $(2.4 .28 \mathrm{a} \sim \mathrm{c})$. Table B. 2 denotes the values for the square matrices $\left[\mathbf{A}_{0}\right],\left[\mathbf{B}_{0}\right],\left[\mathbf{C}_{0}\right]$ as used in (B.1) and (B.2) from (B.10) and (B.11), respectively.

The size of the square matrices $\left[\mathbf{A}_{0}\right],\left[\mathbf{B}_{0}\right],\left[\mathbf{C}_{0}\right]$ are calculated by

$$
\begin{equation*}
M=\sum_{i=\bar{N}}^{1} i=\bar{N}+(\bar{N}-1)+\cdots+2+1=\frac{\bar{N}(\bar{N}+1)}{2} \tag{B.12}
\end{equation*}
$$

The proof of (B.12) follows from the triangle in Table B.1. The spy graphs or the plot of nonzero sparse matrices for $\left[\mathbf{A}_{0}\right],\left[\mathbf{B}_{0}\right]$, and $\left[\mathbf{C}_{0}\right]$ are shown in Figure A. 3 and Figure A. 4 in Appendix $B$.

Table B. 2 Two-dimensional matrices represented by $\left[\mathbf{B}_{0}\right] \mathbf{F}=\mathbf{g}$ and $\left[\mathbf{A}_{\mathbf{0}}\right] \mathbf{G}=\lambda\left[\mathbf{C}_{\mathbf{0}}\right] \mathbf{G}$.
$n(m, l) \equiv \frac{2 m \bar{N}-m^{2}-m}{2}+l$, and $\bar{N}=N+1$ are the same as (B.6a) and (B.6b), respectively.

| $\left[\mathbf{A}_{\mathbf{0}}\right]_{n(m, l), n(y, x)}$ |  |  |  | $\left[\mathbf{C}_{\mathbf{0}}\right]_{n(m, l), n(y, x)}$ |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $y \Downarrow / x \Rightarrow$ | $l-1$ | $l$ | $l+1$ | $y \Downarrow / x \Rightarrow$ | $l-1$ | $l$ | $l+1$ |
| $m-1$ | 0 | 0 | 0 | $m-1$ | $\frac{k^{\prime}}{2} \delta_{m} \alpha_{6}$ | 0 | $-\frac{k^{\prime}}{2} \delta_{m} \alpha_{7}$ |
| $m$ | $\alpha_{1}$ | 0 | $\alpha_{2}$ | $m$ | $-c_{v} \alpha_{1}$ | $b_{v l} \alpha_{3}$ | $-c_{v} \alpha_{2}$ |
| $m+1$ | 0 | 0 | 0 | $m+1$ | ${ }_{\frac{k^{\prime}}{2}} \bar{\varepsilon}_{m} \alpha_{4}$ | 0 | $-\frac{k^{\prime}}{2} \bar{\varepsilon}_{m} \alpha_{5}$ |


| $\left[\mathbf{B}_{0}\right]_{n(m, l), n(y, x)}$ |  |  |  |
| :---: | :---: | :---: | :---: |
| $y \Downarrow / x \Rightarrow$ | $l-1$ | $l$ | $l+1$ |
| $m-1$ | $\frac{k^{\prime}}{2} \delta_{m} \alpha_{6}$ | 0 | $-\frac{k^{\prime}}{2} \delta_{m} \alpha_{7}$ |
| $m$ | $-a_{v} \alpha_{1}$ | $b_{v i} \alpha_{3}$ | $-a_{v} \alpha_{2}$ |
| $m+1$ | $\frac{k^{\prime}}{2} \bar{\varepsilon}_{m} \alpha_{4}$ | 0 | $-\frac{k^{\prime}}{2} \bar{\varepsilon}_{m} \alpha_{S}$ |

## APPENDIX C

## POWER RECEIVED BY HIGHLY DIRECTIVE ANTENNA

In this study of beam wave pulses normally incident on a semi-infinite medium, the power is assumed to be received by a highly directive antenna placed in the forest. The power calculations were introduced in Section 2.5 and are repeated here for convenience [2].

Assume that a highly directive, lossless antenna of narrow beamwidth and narrow bandwidth is located inside the forest. This receiving antenna is characterized by an effective aperture $A\left(\gamma_{R}\right)$, where $\gamma_{R}$ is the angle included between the direction of observation $\left(\theta_{R}, \psi_{R}\right)$ and the pointing direction of the antenna axis, i.e., the main beam direction $\left(\theta_{M}, \psi_{M}\right)$; see Appendix A. Hence,

$$
\begin{equation*}
\cos \gamma_{R}=\cos \theta_{R} \cos \theta_{M}+\sin \theta_{R} \sin \theta_{M} \cos \left(\psi_{R}-\psi_{M}\right) . \tag{C.1}
\end{equation*}
$$

In transport theory powers add. Hence, the instantaneous power received by the antenna is the sum of the intensity contributions coming from all directions multiplied by the effective aperture of the antenna, i.e.,

$$
\begin{equation*}
P_{R}^{t o t}\left(z^{\prime}, \rho^{\prime}, t^{\prime}, \theta_{M}, \psi_{M}\right)=\operatorname{Re}\left\{\sum_{\nu=0}^{\infty} P_{R v}^{t o t}\left(z^{\prime}, \rho^{\prime}, \theta_{M}, \psi_{M}\right) e^{i v \omega^{\prime}\left(t^{\prime}-z^{\prime}\right)}\right\} \tag{C.2}
\end{equation*}
$$

where

$$
\begin{equation*}
P_{R \nu}^{t o t}\left(z^{\prime}, \rho^{\prime}, \theta_{M}, \psi_{M}\right)=\iint_{4 \pi} A_{e}\left(\gamma_{R}\right) I_{v}^{t o t}\left(z^{\prime}, \rho^{\prime}, \theta_{R}, \psi_{R}\right) \sin \theta_{R} d \theta_{R} d \psi_{R} \tag{C.3}
\end{equation*}
$$

and

$$
\begin{equation*}
I_{v}^{t o t}=I_{d v}+I_{r i, v} . \tag{C.4}
\end{equation*}
$$

Note that $\theta=\theta_{R}$ and $\psi=\psi_{R}$.
For millimeter waves, the carrier frequency is very large and, therefore, the bandwidth of the received signal is narrow. For such a small bandwidth, the effective aperture and gain of the receiving antenna can be taken to be independent of frequency and to be related by the general expression

$$
\begin{equation*}
A_{e}\left(\gamma_{R}\right)=\frac{\lambda_{o}^{2}}{4 \pi} D\left(\gamma_{R}\right) \tag{C.5}
\end{equation*}
$$

where $\lambda_{o}$ is the free space wavelength and $D\left(\gamma_{R}\right)$ is the directive gain of the antenna at the carrier frequency.

For analytical convenience, the directive gain is assumed to be Gaussian with a narrow beamwidth $\Delta \gamma_{M}$ and no sidelobes, i.e.,

$$
\begin{equation*}
D\left(\gamma_{R}\right)=\left(\frac{2}{\Delta \gamma_{M}}\right)^{2} e^{-\left(\gamma_{R} / \Delta \gamma_{M}\right)^{2}} \quad, \quad \Delta \gamma_{M} \ll \pi \tag{C.6}
\end{equation*}
$$

which is normalized such that

$$
\begin{equation*}
\iint_{4 \pi} D\left(\gamma_{R}\right) \sin \theta_{R} d \theta_{R} d \psi_{R}=4 \pi . \tag{C.7}
\end{equation*}
$$

Using the normalized directive gain $D\left(\gamma_{R}\right)$ in (C.6) and the total intensity expressed as in (C.4), the total received instantaneous power is obtained as the sum of diffuse power $P_{R, d}$ and reduced incident power $P_{R, r i}$. The received diffuse power is obtained as follows

$$
\begin{equation*}
P_{R, d}\left(z^{\prime}, \rho^{\prime}, t^{\prime}, \theta_{M}, \psi_{M}\right)=\operatorname{Re}\left\{\sum_{v=0}^{\infty} P_{R, d v}\left(z^{\prime}, \rho^{\prime}, \theta_{M}, \psi_{M}\right) e^{i v \omega^{\prime}\left(t^{\prime}-z^{\prime}\right)}\right\} \tag{C.8}
\end{equation*}
$$

where

$$
\begin{align*}
P_{R, d v}\left(z^{\prime}, \rho^{\prime}, \theta_{M}, \psi_{M}\right) & =\iint_{4 \pi} A_{e}\left(\gamma_{R}\right) I_{d v}\left(z^{\prime}, \rho^{\prime}, \theta_{R}, \psi_{R}\right) \sin \theta_{R} d \theta_{R} d \psi_{R} \\
& =\frac{\lambda_{o}^{2}}{4 \pi} \iint_{4 \pi} D\left(\gamma_{R}\right) I_{d v}\left(z^{\prime}, \rho^{\prime}, \theta_{R}, \phi_{R}\right) \sin \theta_{R} d \theta_{R} d \psi_{R}  \tag{C.9}\\
& \cong \frac{\lambda_{o}^{2}}{4 \pi} I_{d v}\left(z^{\prime}, \rho^{\prime}, \theta_{M}, \psi_{M}\right) \iint_{4 \pi} D\left(\gamma_{R}\right) \sin \theta_{R} d \theta_{R} d \psi_{R} \\
& =\lambda_{o}^{2} I_{d v}\left(z^{\prime}, \rho^{\prime}, \theta_{M}, \psi_{M}\right) .
\end{align*}
$$

Similarly, the received reduced incident power is obtained as follows

$$
\begin{equation*}
P_{R, r i}\left(z^{\prime}, \rho^{\prime}, t^{\prime}, \theta_{M}, \phi_{M}\right)=\operatorname{Re}\left\{\sum_{v=0}^{\infty} P_{R, r i, v}\left(z^{\prime}, \rho^{\prime}, \theta_{M}, \phi_{M}\right) e^{i v \omega^{\prime}\left(t^{\prime}-z^{\prime}\right)}\right\}, \tag{C.10}
\end{equation*}
$$

where

$$
\begin{align*}
P_{R, r i, v}\left(z^{\prime}, \rho^{\prime}, \theta_{M}, \phi_{M}\right) & =\iint_{4 \pi} A_{e}\left(\gamma_{R}\right) I_{r i, v}\left(z^{\prime}, \rho^{\prime}, \theta_{R}, \phi_{R}\right) \sin \theta_{R} d \theta_{R} d \phi_{R} \\
& \approx \frac{\lambda_{o}^{2}}{4 \pi} S_{p} f_{v} e^{-\left(\rho^{\prime} / w^{\prime}\right)^{2}} e^{-z^{\prime}} \int_{0}^{2 \pi} \int_{0}^{\pi} D\left(\gamma_{R}\right) \frac{\delta\left(\theta_{R}\right)}{2 \pi \sin \theta_{R}} \sin \theta_{R} d \theta_{R} d \psi_{R}  \tag{C.11}\\
& =\frac{\lambda_{o}^{2}}{4 \pi} S_{p} f_{v} e^{-\left(\rho^{\prime} / w^{\prime}\right)^{2}} e^{-z^{\prime}} D\left(\theta_{M}\right)
\end{align*}
$$

The instantaneous received power is normalized to the received time-averaged power at $z^{\prime}=0$, $\rho^{\prime}=0, \theta_{M}=0$ and $\psi_{M}=0$, which is given by

$$
\begin{equation*}
\left\langle P_{R}\left(0,0, t^{\prime}, 0,0\right)\right\rangle \equiv \frac{1}{T^{\prime}} \int_{-T^{\prime} / 2}^{T^{\prime} / 2} P_{R}\left(0,0, t^{\prime}, 0,0\right) d t^{\prime}=\frac{\lambda_{o}^{2}}{4 \pi} D(0) S_{p} \tag{C.12}
\end{equation*}
$$

Thus, the normalized total instantaneous power is the sum of the reduced incident and the diffuse normalized received powers, namely,

$$
\begin{equation*}
P^{\prime}\left(z^{\prime}, \rho^{\prime}, t^{\prime}, \theta_{M}, \psi_{M}\right)=\frac{P_{R}\left(z^{\prime}, \rho^{\prime}, t^{\prime}, \theta_{M}, \psi_{M}\right)}{\left\langle P_{R}\left(0,0, t^{\prime}, 0,0\right)\right\rangle}=P_{r i}^{\prime}+P_{d}^{\prime} \tag{C.13}
\end{equation*}
$$

Using the expressions in (C.9) and (C.11), the total normalized instantaneous received power takes the form:

$$
\begin{equation*}
P^{\prime}\left(z^{\prime}, \rho^{\prime}, t^{\prime}, \theta_{M}, \psi_{M}\right)=\operatorname{Re}\left\{\sum_{\nu=0}^{\infty} P_{\nu}^{\prime}\left(z^{\prime}, \rho^{\prime}, \theta_{M}, \psi_{M}\right) e^{i v \omega^{\prime}\left(t^{\prime}-z^{\prime}\right)}\right\} \tag{C.14}
\end{equation*}
$$

where

$$
\begin{equation*}
P_{v}^{\prime}\left(z^{\prime}, \rho^{\prime}, \theta_{M}, \psi_{M}\right)=\frac{D\left(\theta_{M}\right)}{D(0)} e^{-\left(\rho^{\prime} / w^{\prime}\right)^{2}} f_{v} e^{-z^{\prime}}+\frac{4 \pi}{S_{p} D(0)} I_{d v}\left(z^{\prime}, \rho^{\prime}, \theta_{M}, \psi_{M}\right) \tag{C.15}
\end{equation*}
$$

The first term in (C.15) combined with (C.14) yields

$$
\begin{align*}
P_{r i}^{\prime} & =\sum_{v=0}^{\infty} P_{v, r i}^{\prime} \\
& =\frac{D\left(\theta_{M}\right)}{D(0)} e^{-\left(\rho^{\prime} / w^{\prime}\right)^{2}} e^{-z^{\prime}} \operatorname{Re} \sum_{v=0}^{\infty} f_{v} e^{i v \omega^{\prime}\left(t^{\prime}-z^{\prime}\right)}  \tag{C.16}\\
& =\frac{D\left(\theta_{M}\right)}{D(0)} e^{-\left(\rho^{\prime} / w^{\prime}\right)^{2}-\tau} f\left(t^{\prime}-z^{\prime}\right)
\end{align*}
$$

Note that at $z^{\prime}=0, \rho^{\prime}=0$ and for $\theta_{M}=0$,

$$
\begin{equation*}
P_{r i}=f\left(0, t^{\prime}\right) . \tag{C.17}
\end{equation*}
$$

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