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ABSTRACT

GENERAL PARALLEL-LEG MECHANISM: POSITION ANALYSIS

by
Hans Buus Gangwar

This thesis proposes to analytically express the positions of the legs (each leg has a length and two spin orientations) for a parallel leg mechanism in terms of a single variable. The only limitations imposed are the leg bases are all on the same reference plane and the tool orientation is normal to the mobile upper plate. This is done for a parallel leg mechanism having *any number* of legs. The variable chosen is the first leg's vertex position on the upper plate.

The state space is the space the configuration can occupy. By looking at a planar slice of the state space for a specified tool axis position and orientation on the upper plate, the different possible leg positions can be constructed. The three dimensional space can then be generated by overlaying a progressive set of planes.

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by
Hans Buus Gangwar

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GENERAL PARALLEL LEG MECHANISM: POSITION ANALYSIS

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To my dad for his unobtrusive support of all my decisions

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LIST OF LEG SYMBOLS

$\mathbf{i}$ leg subscript #
(when absent, the leg symbol is used as a general representation of all legs)

$\mathbf{r_i}$ constant ground radial distance of leg **i** from origin

$\mathbf{x_i}$ length of leg i $(\mathbf{x} \in [\mathbf{x_{min}}, \mathbf{x_{max}}])$

θ_i constant ground orientation of leg **i** about z_g-axis

η_i rotation of leg **i** about the y_B-axis at leg base

ψ_i rotation of leg **i** about the x'' axis so new coordinate x_L axis aligned with leg

LIST OF OTHER SYMBOLS

$\mathbf{B(r,\theta)}$	3x3 Dual Number matrix expressing first leg base coordinate $x_B y_B z_B$ placement in terms of ground (z_B in z_g direction)
$\mathbf{D}$	3x3 matrix of leg top positions along the x_G, y_G, z_G directions
$\mathbf{D^0}$	First column of matrix **D**
$\mathbf{D^1}$	Second column of matrix **D**
$\mathbf{D^2}$	Third column of matrix **D**
$\mathbf{d_i}$	1x3 column matrix representation of **D**
$\mathbf{d_{i,x}}$	x_G-component of $\mathbf{d_i}$
$\mathbf{d_{i,y}}$	y_G-component of $\mathbf{d_i}$
$\mathbf{d_{i,z}}$	z_G-component of $\mathbf{d_i}$
$\mathbf{L(\eta,\psi)}$	3x3 Dual Number matrix expressing second leg base coordinate $x_L y_L z_L$ placement (x_L in leg direction) in terms of $x_B y_B z_B$
$\mathbf{n}$	Tool axis orientation; assumed to be normal to the upper plate surface at the tool axis position **P**.
$\mathbf{n_x}$	x_G-component of **n**
$\mathbf{n_y}$	y_G-component of **n**
$\mathbf{n_z}$	z_G-component of **n**
$\mathbf{P}$	Position of tool axis on the upper plate with respect to the ground
$\mathbf{P_x}$	x_G-component of **P**
$\mathbf{P_y}$	y_G-component of **P**
$\mathbf{P_z}$	z_G-component of **P**

LIST OF OTHER SYMBOLS
(Continued)

$\mathbf{T}$	3x3 Dual number matrix expressing leg top in terms of ground; equal to $\mathbf{B}(r,\theta)\mathbf{L}(\eta,\psi)\mathbf{U}(x)$
$\mathbf{T}^0$	First column of dual number matrix **T**
$\mathbf{T}^1$	Second column of dual number matrix **T**
$\mathbf{Tp}$	3x3 matrix of primary (rotational) component of dual number matrix **T**
$\mathbf{Td}$	3x3 matrix of dual (translation multiplied by rotation) component of dual number matrix **T**
$\mathbf{U}(x)$	3x3 Dual Number matrix expressing leg top (or distal) coordinate in terms of $x_L y_L z_L$
$\mathbf{V}_i$	Fixed distance from $\mathbf{d}_i$ to **P**
γ_i	Fixed angle between $\lvert\mathbf{P}-\mathbf{d}_i\rvert$ and $\lvert\mathbf{P}-\mathbf{d}_1\rvert$

CHAPTER 1

INTRODUCTION

1.1 Advantages and Disadvantages of a Parallel-Leg Mechanism

There are three main advantages of a parallel-leg mechanism over a conventional serial link configuration: identical components, even distribution of weight, and absence of error buildup.

By the inherent nature of a parallel-leg configuration, the links and their ball joint connections do not vary from leg to leg. This component universality thereby reduces the costs of production and part replacement. A serial-link mechanism does not have this intrinsic repetition of parts, thus raising the maintenance and production costs.

Second, in a parallel-leg mechanism, the weight of the tool plate is distributed among the six legs. Thus, the legs do not need to be massive, nor do the servo-motors controlling the movement need to exert a large amount of torque. Whereas in a serial-link mechanism, the weight is *additive*, for the base link has to support all the other links in addition to the tool plate. As a consequence of better weight distribution, parallel-leg mechanisms exhibit better rigidity and higher acceleration.

Lastly, with the parallel-leg mechanism, the error in link orientation is isolated to the specific links (it is not felt between links), so the total error is not additive, and usually ends up being a little greater than the largest error [Behi, 1988]. However, with a serial arrangement, the error is *cumulative*.

Thus a parallel-leg mechanism is superior to a serial-link mechanism in performance, cost of manufacturing, and cost of maintenance.

The disadvantages are subtle. First, analysis of the movement for a serial mechanism is fundamentally simpler. Only forces and moments between links cumulate, and the effects of those are easily analyzed, especially with the effects being felt at the link connections. Thus the motion can be isolated along orthogonal axes. With a parallel mechanism, however, *link movement is coupled – when one link moves, the relative orientation of the others also changes.* Second, tool path generation in the operation of the mechanism is straightforward with a serial mechanism. Path generation is an everyday requirement for machinists with NC machines. Every different machine job needs a different program. Because the motion of a parallel-leg mechanism is complex, planning a trajectory is time intensive, difficult, and many times not optimal.

1.2 Background

Parallel-leg mechanism designs were first introduced back in the 1800's. It was not until 1947 that one was actually built by Gough. His Universal Tire Testing Rig was the first practical application of a parallel-leg mechanism. Gough's mechanism allowed the measuring of wear by pressing a rotating tire at different angles against a rough surface [Gough & Whitewall, 1956].

Parallel-leg mechanisms are popularly known as Stewart platforms, after Stewart's six degree of freedom flight simulator shown in **Figure 1** [Stewart, 1965]. However, it was not until 25 years after the Stewart platforms first appearance that theoretical research in robotics began to focus on parallel-leg mechanisms. The first analytical efforts were made by Behi, Griffs and Duffy, and Fichter.

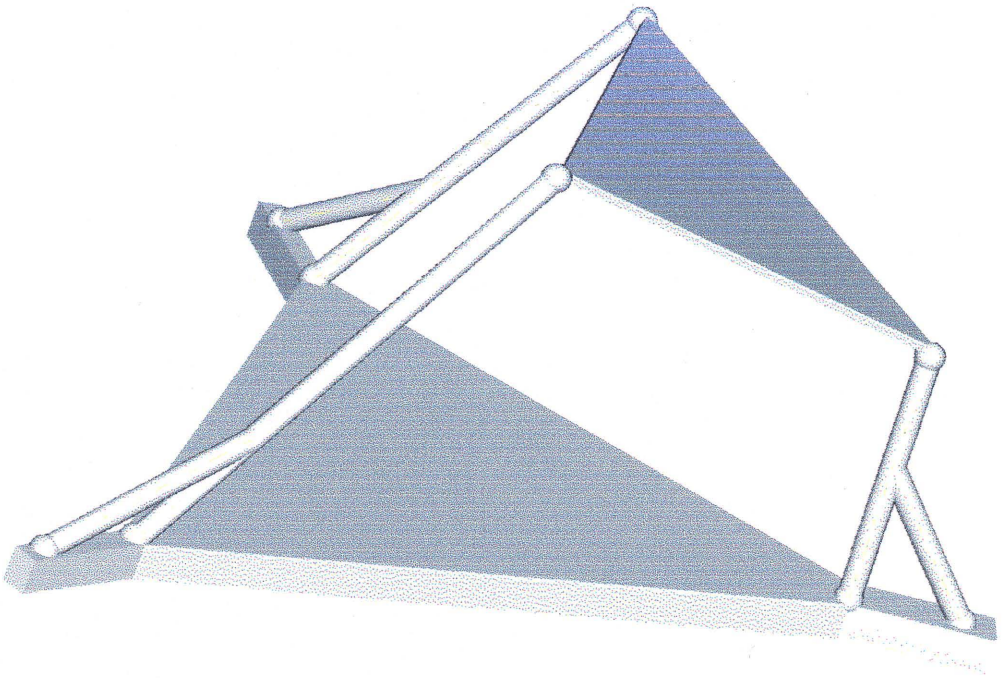


Figure 1 Stewart Platform

Fichter determined the leg lengths, velocities, and singular positions of the configuration for a Stewart's platform. His parameters were the position, velocity, and orientation of the upper plate [Fichter, 1986].

Behi [1988] also studied a Stewart's platform. He analyzed the platform for both reverse and forward displacements. In reverse displacement analysis, the orientation and center position of the upper plate are given, for which the leg lengths are solved. In forward displacement analysis, the leg lengths are given, for which the orientation and center position of the upper plate are solved. Then he looked as well at the workspace of the mechanism – the three dimensional space in which the mechanism operates.

Griffs and Duffy [1989] analyzed a modified Stewart platform with six legs attached at three separate ball joints at both the base and upper plates (shown in **Figure 2**). Eight reflected solution pairs for the leg lengths were arrived at, given the orientation and center position of the upper plate.

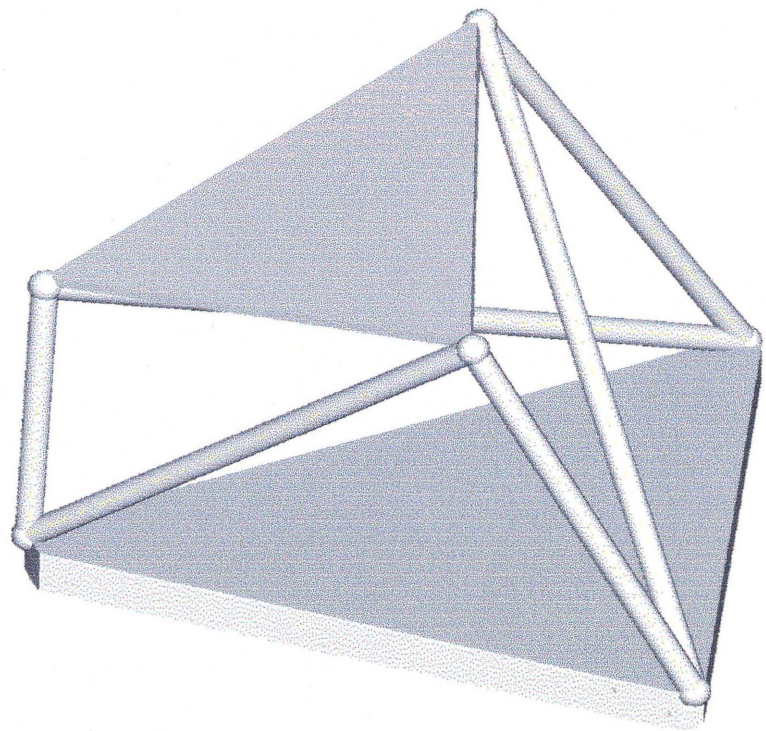


Figure 2 Griffs and Duffy's Modified Stewart Platform

The most intensive study was done by Innocenti and Parenti-Castelli [1990], whom analyzed the mechanism shown in **Figure 3.** They solved for the orientation and mobile upper plate position in terms of the leg parameters. Their approach was first closure of the loop equation for the legs, then construction of the orientation and center position vectors for the upper plate by means of the three top vertices. Using complex numbers, their result was a 16^{th} order polynomial in terms of one variable.

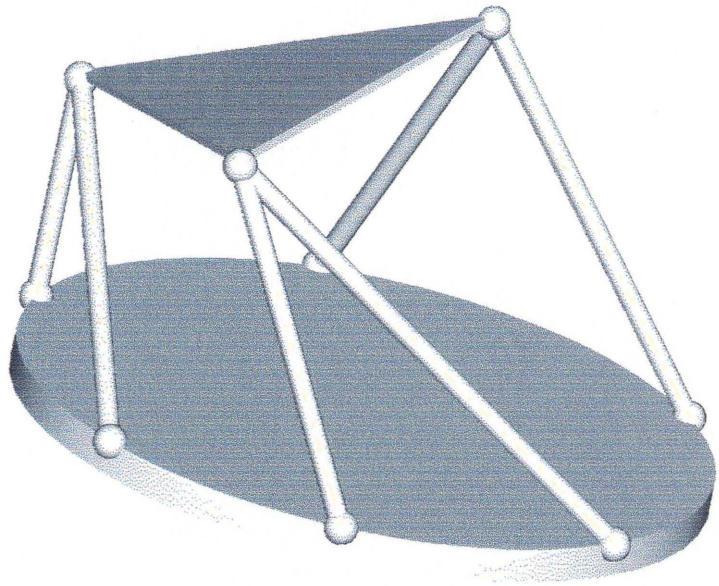


Figure 3 Innocenti and Parenti-Castelli's Parallel-Leg Mechanism

1.3 Motivation and Objective

In machining, it is important to be able to move a parallel leg mechanism through a prescribed path. Since most tools are symmetric, the mobile upper plate can be rotated about the tool axis without changing the tool orientation. Thus a single degree of freedom exists for the upper plate, a rotation about the tool axis. Having an equation expressing the leg variables in terms of a single point on the upper plate (in addition to the tool axis position and orientation) would enable the generation of the leg positions for different rotations of the upper plate. From inputting the different points on the tracer line of the tool axis position, this process can be repeated to generate a set of such solutions. In addition the tool orientation need not be held constant over tracer path.

The objective is then to determine all possible configurations for a specified tool position and orientation.

1.4 Design Parameters

Each leg is given an axial length **x**, an angular orientation η about the y_B-axis, and a second angular orientation ψ about the z_L axis as shown in **Figure 4**. There are six legs, so there are 6 * 3 variables. Given the tool axis orientation and position, a point in the upper plate can lie on a circle of solutions. Therefore the objective of this thesis is to express the other variables in terms of the first leg's upper vertex position on the mobile plate, $\mathbf{d_1}$. This is done in order to provide the equations to map out the solution space over the range of the single independent variable.

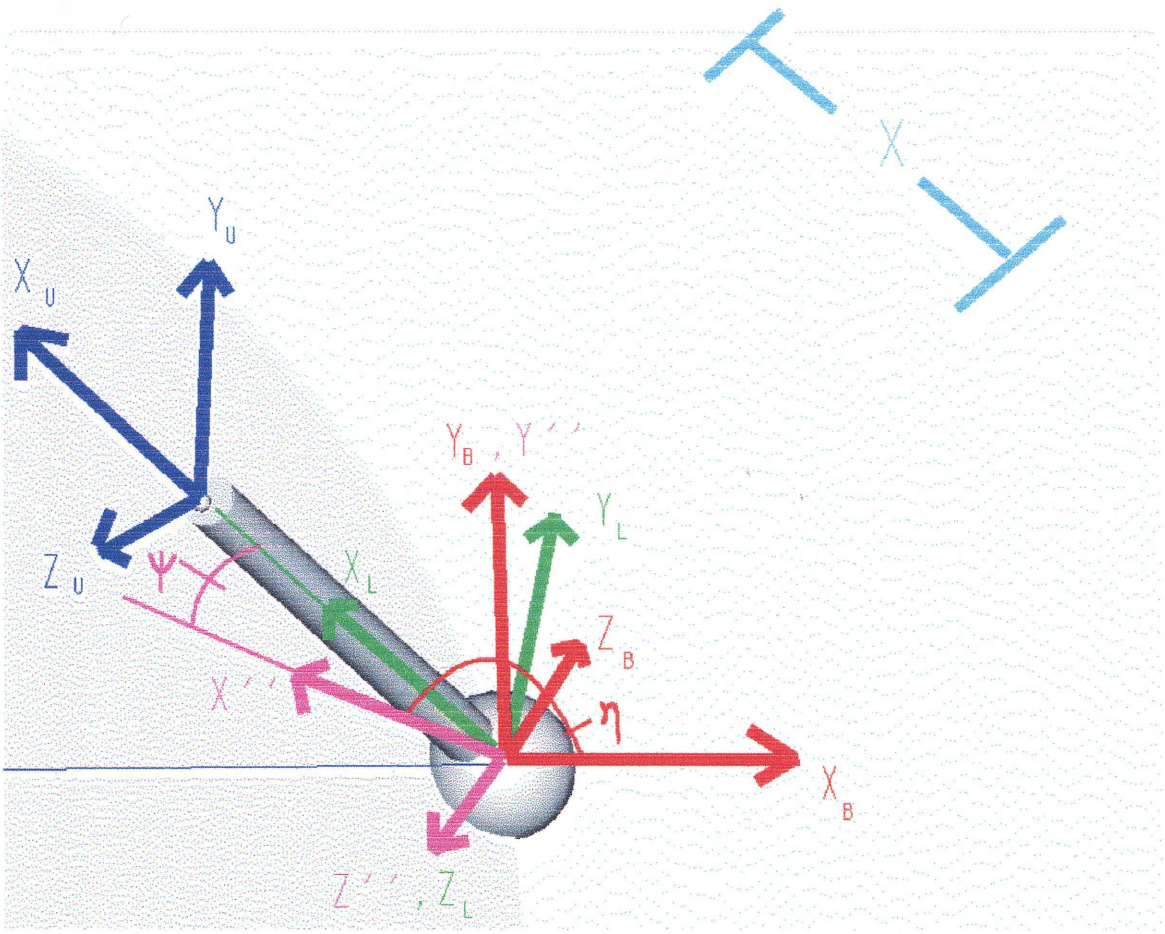


Figure 4 Leg Coordinates

The tool axis orientation **n** is assumed to be normal to the upper plate surface at the tool axis position **P.** The distance between **P** and leg vertex $\mathbf{d_i}$ is $\mathbf{V_i}$ (see **Figure 5**)

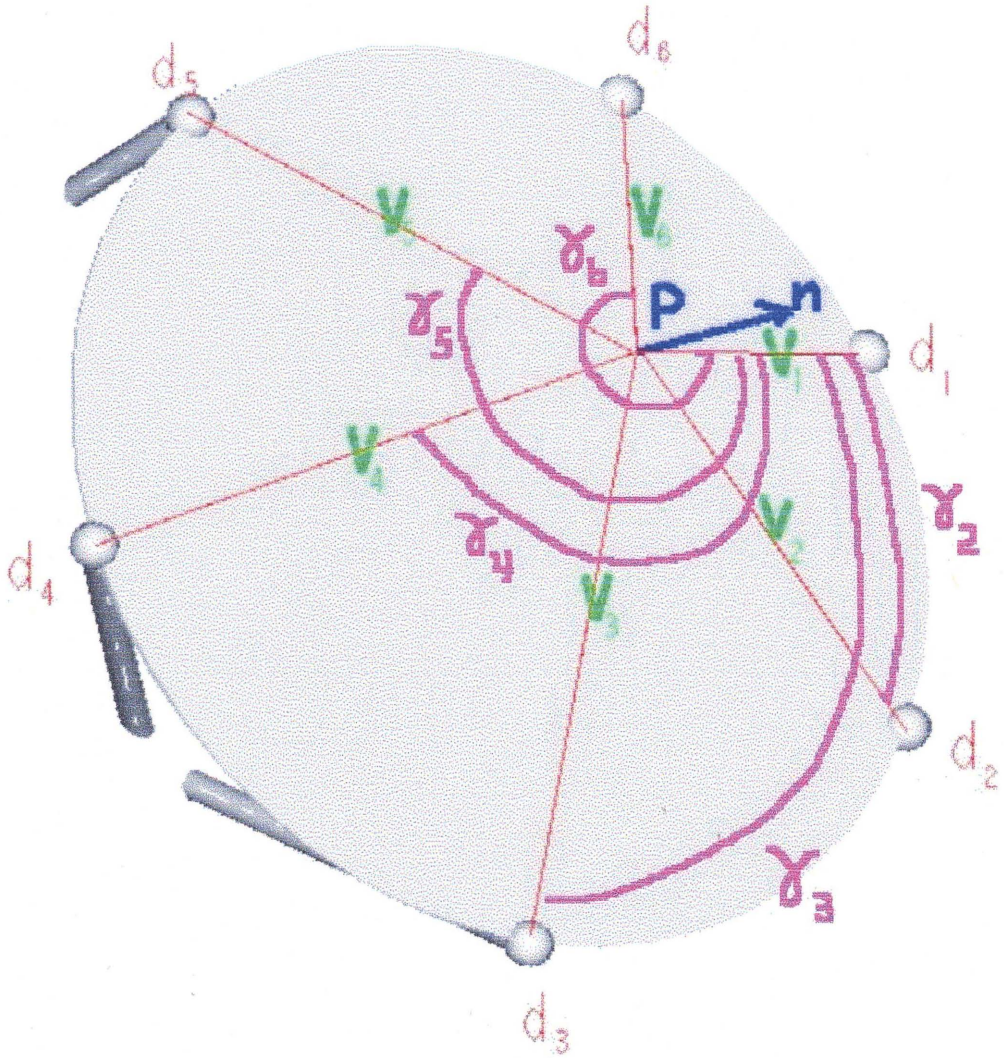


Figure 5 Upper Plate Parameters

The last parameters needed are the base positions of each leg. The assumption made is that each leg ***i*** is on a circle of radius $\mathbf{r_i}$ at an angle θ_i (**Figure 6** illustrates the base coordinates for the case of six legs).

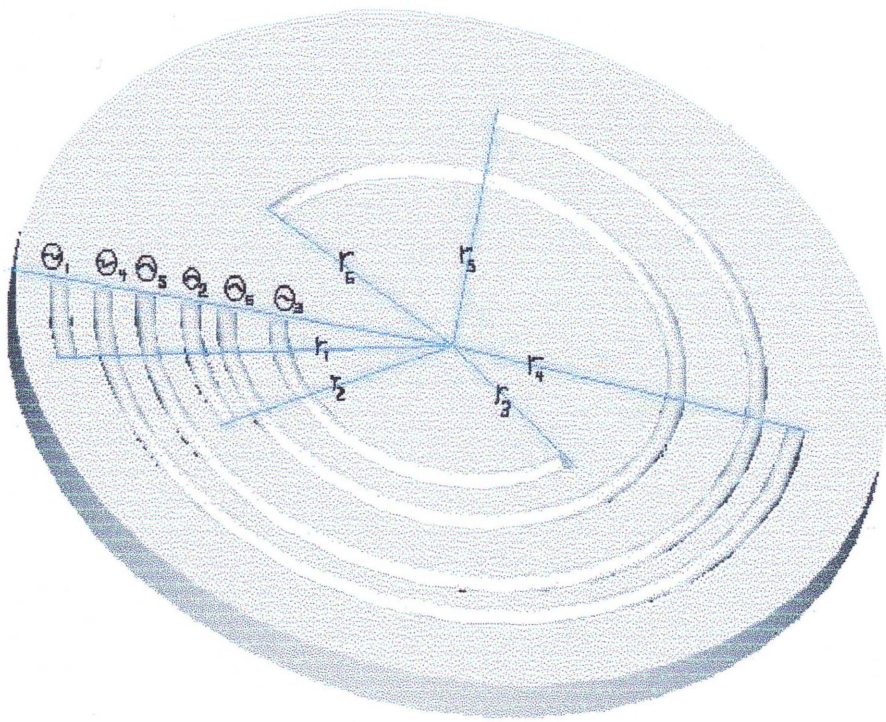


Figure 6 Base Coordinates for Six Legs

CHAPTER 2

POSITION ANALYSIS

2.1 Approach

A parallel leg mechanism's position depends on the tool axis position **P** and orientation **n** on the upper plate. The main quantity of interest controlling the position of the upper plate is the legs' lengths. Since the legs are attached to the upper plate at the vertices, in order to express the legs' lengths in terms of **P** and **n**, first the equation describing the position of the vertices is needed. To accomplish this, a general equation will be developed in Section 2.2 for any arbitrary leg. This equation will be called the *Single Leg Equation.* Once the *Single Leg Equation* is known, the leg variables (length **x** and angular orientations η and ψ) can be expressed in terms of the vertex distance **d**. This will be done in Section 2.3. The equations derived in Sections 2.2 and 2.3 are for an arbitrary leg and corresponding vertex. Since the vertices are not independent, but related by the geometry of the upper plate and by specifying the tool axis position and orientation, there is only a single degree of freedom (once one vertex is specified, the plate is fully constrained). As such, the other vertices after the first can be expressed in terms of the first. All that is needed is the distance $\mathbf{V_i}$ between vertex $\mathbf{d_i}$ and tool axis position **P**, and the angle γ_i between the axis position and the vertices $\mathbf{d_i}$ and $\mathbf{d_1}$. Section 2.4 derives the equation relating the vertices. From this set of equations, all leg lengths are expressible as a function of the first vertex $\mathbf{d_1}$.

2.2 Single Leg Equation

The purpose of this section is to express the distance from a vertex to the ground in terms of the leg variables. This expression is called the *Single Leg Equation*, and can be used to relate any vertex position in terms of its leg's variables **x**, η, and ψ.

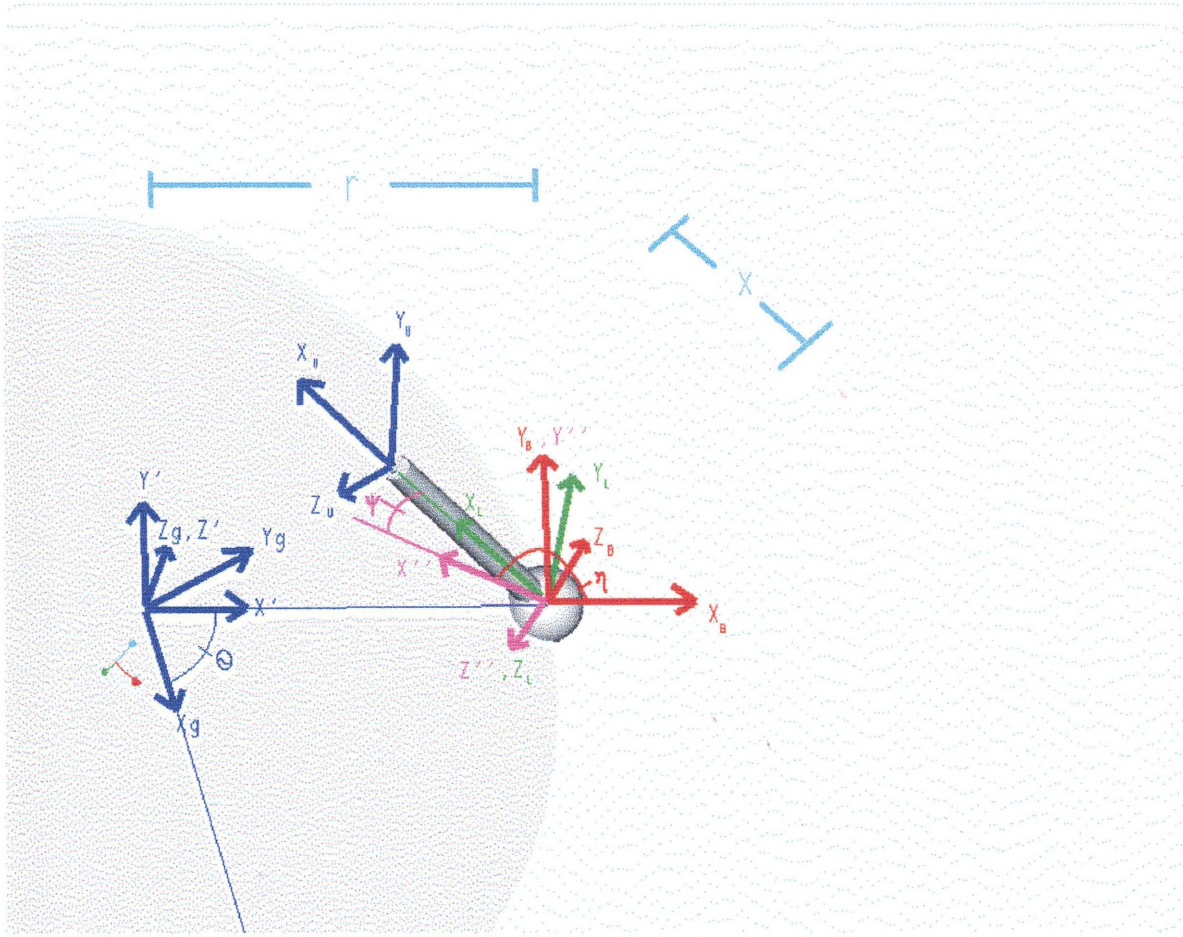


Figure 7 Coordinate Frames

A leg is broken into the three different coordinate frames shown in **Figure 7**: its reference base frame {**B**} on the base plate, its leg direction frame {**L**} whose origin coincides with {**B**} but whose x-axis points in the direction of the leg, and its upper connection frame {**U**} which lies on the connection of the leg to the upper plate. By formulating dual number transformations from the ground {**G**} to {**B**}, from {**B**} to {**L**},

and from {**L**} to {**U**}, the vertex **d** becomes simply the transformation **T** from the ground {**G**} to the upper connection (vertex) {**U**}. The appendix describes the dual number transforms in detail.

$B_i(r,\theta)$ expresses the coordinates of the reference base of leg ***i*** in terms of the ground. Two coordinate transforms are needed. First rotate about z_g by θ to arrive at the new x'- and y'- axes. Then translate along the new x'-axis by r to arrive at the x_B- and y_B- axes. Since the purpose is to express a coordinate point (x_B,y_B,z_B) as (x_g,y_g,z_g), the order of matrix operation for $B_i(r,\theta)$ is backwards.

$B = Z(\theta,0)X(0,r).$

$$B(r,\theta) := \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\varepsilon \cdot r \\ 0 & \varepsilon \cdot r & 1 \end{bmatrix}$$

Now each leg ***i*** is presumed to be connected at $\{B_i\}$ by a ball joint. Looking at the leg ***i***, a second coordinate system can be established at $\{B_i\}$, where the x_L-axis points in the leg direction. Since we are only looking at the joint right now to see the directional orientation of the leg, the translation r = 0. To arrive at the new coordinate system, first rotate $\{B_i\}$ about the y_B-axis by η_i until the leg is in the x''y'' (or x''y_B) plane, then rotate about the z'' axis by ψ_i to align x_L with the leg. So undoing this operation to arrive at the coordinate transformation,

$L = Y(\eta,0)Z(\psi,0)$

$$L(\eta,\psi) := \begin{bmatrix} \cos(\eta) & 0 & \sin(\eta) \\ 0 & 1 & 0 \\ -\sin(\eta) & 0 & \cos(\eta) \end{bmatrix} \cdot \begin{bmatrix} \cos(\psi) & -\sin(\psi) & 0 \\ \sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Next, each leg has a length x_i, so to move from the coordinate system at {**L**} to the coordinate system at {**U**} (upper part or distal end of the leg), a simple translation occurs along the x_L-axis.

$$U(x) := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\varepsilon \cdot x \\ 0 & \varepsilon \cdot x & 1 \end{bmatrix}$$

Combining the equations, the position of the top of each leg can be expressed in terms of the ground coordinate. **T = B * L * U**

$$T = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\varepsilon \cdot r \\ 0 & \varepsilon \cdot r & 1 \end{bmatrix} \cdot \begin{bmatrix} \cos(\eta) & 0 & \sin(\eta) \\ 0 & 1 & 0 \\ -\sin(\eta) & 0 & \cos(\eta) \end{bmatrix} \cdot \begin{bmatrix} \cos(\psi) & -\sin(\psi) & 0 \\ \sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\varepsilon \\ 0 & \varepsilon \cdot x & 1 \end{bmatrix}$$

Evaluating the matrix symbolically

$$T(r,\theta,\eta,\psi,x)^{<0>} \rightarrow \begin{bmatrix} \cos(\theta)\cdot\cos(\eta)\cdot\cos(\psi) - \sin(\theta)\cdot\sin(\psi) - \sin(\theta)\cdot\varepsilon\cdot r\cdot\sin(\eta)\cdot\cos(\psi) \\ \sin(\theta)\cdot\cos(\eta)\cdot\cos(\psi) + \cos(\theta)\cdot\sin(\psi) + \cos(\theta)\cdot\varepsilon\cdot r\cdot\sin(\eta)\cdot\cos(\psi) \\ \varepsilon\cdot r\cdot\sin(\psi) - \sin(\eta)\cdot\cos(\psi) \end{bmatrix}$$

$$T(r,\theta,\eta,\psi,x)^{<1>} \rightarrow \begin{bmatrix} -\cos(\theta)\cdot\cos(\eta)\cdot\sin(\psi) - \sin(\theta)\cdot\cos(\psi) + \sin(\theta)\cdot\varepsilon\cdot r\cdot\sin(\eta)\cdot\sin(\psi) + \varepsilon\cdot x\cdot\cos(\theta)\cdot\sin(\eta) + \varepsilon^2\cdot x\cdot\sin(\theta)\cdot r\cdot\cos(\eta) \\ -\sin(\theta)\cdot\cos(\eta)\cdot\sin(\psi) + \cos(\theta)\cdot\cos(\psi) - \cos(\theta)\cdot\varepsilon\cdot r\cdot\sin(\eta)\cdot\sin(\psi) + \varepsilon\cdot x\cdot\sin(\theta)\cdot\sin(\eta) - \varepsilon^2\cdot x\cdot\cos(\theta)\cdot r\cdot\cos(\eta) \\ \varepsilon\cdot r\cdot\cos(\psi) + \sin(\eta)\cdot\sin(\psi) + \cos(\eta)\cdot\varepsilon\cdot x \end{bmatrix}$$

$$T(r,\theta,\eta,\psi,x)^{<2>} \rightarrow \begin{bmatrix} \varepsilon\cdot x\cdot\cos(\theta)\cdot\cos(\eta)\cdot\sin(\psi) + \varepsilon\cdot x\cdot\sin(\theta)\cdot\cos(\psi) - \varepsilon^2\cdot x\cdot\sin(\theta)\cdot r\cdot\sin(\eta)\cdot\sin(\psi) + \cos(\theta)\cdot\sin(\eta) + \sin(\theta)\cdot\varepsilon\cdot r\cdot\cos(\eta) \\ \varepsilon\cdot x\cdot\sin(\theta)\cdot\cos(\eta)\cdot\sin(\psi) - \varepsilon\cdot x\cdot\cos(\theta)\cdot\cos(\psi) + \varepsilon^2\cdot x\cdot\cos(\theta)\cdot r\cdot\sin(\eta)\cdot\sin(\psi) + \sin(\theta)\cdot\sin(\eta) - \cos(\theta)\cdot\varepsilon\cdot r\cdot\cos(\eta) \\ -\varepsilon^2\cdot x\cdot r\cdot\cos(\psi) - \varepsilon\cdot x\cdot\sin(\eta)\cdot\sin(\psi) + \cos(\eta) \end{bmatrix}$$

Using the property $\varepsilon^2 = 0$, the result is

$$T(r,\theta,\eta,\psi,x)^{<0>} \rightarrow \begin{bmatrix} \cos(\theta)\cdot\cos(\eta)\cdot\cos(\psi) - \sin(\theta)\cdot\sin(\psi) - \sin(\theta)\cdot\varepsilon\cdot r\cdot\sin(\eta)\cdot\cos(\psi) \\ \sin(\theta)\cdot\cos(\eta)\cdot\cos(\psi) + \cos(\theta)\cdot\sin(\psi) + \cos(\theta)\cdot\varepsilon\cdot r\cdot\sin(\eta)\cdot\cos(\psi) \\ \varepsilon\cdot r\cdot\sin(\psi) - \sin(\eta)\cdot\cos(\psi) \end{bmatrix}$$

$$T(r,\theta,\eta,\psi,x)^{<1>} = \begin{bmatrix} -\cos(\theta)\cdot\cos(\eta)\cdot\sin(\psi) - \sin(\theta)\cdot\cos(\psi) + \varepsilon\cdot r\cdot\sin(\theta)\cdot\sin(\eta)\cdot\sin(\psi) + \varepsilon\cdot x\cdot\cos(\theta)\cdot\sin(\eta) \\ -\sin(\theta)\cdot\cos(\eta)\cdot\sin(\psi) + \cos(\theta)\cdot\cos(\psi) - \varepsilon\cdot r\cdot\cos(\theta)\cdot\sin(\eta)\cdot\sin(\psi) + \varepsilon\cdot x\cdot\sin(\theta)\cdot\sin(\eta) \\ \varepsilon\cdot r\cdot\cos(\psi) + \sin(\eta)\cdot\sin(\psi) + \varepsilon\cdot x\cdot\cos(\eta) \end{bmatrix}$$

$$T(r,\theta,\eta,\psi,x)^{<2>} = \begin{bmatrix} \varepsilon\cdot x\cdot\cos(\theta)\cdot\cos(\eta)\cdot\sin(\psi) + \varepsilon\cdot x\cdot\sin(\theta)\cdot\cos(\psi) + \cos(\theta)\cdot\sin(\eta) + \varepsilon\cdot r\cdot\sin(\theta)\cdot\cos(\eta) \\ \varepsilon\cdot x\cdot\sin(\theta)\cdot\cos(\eta)\cdot\sin(\psi) - \varepsilon\cdot x\cdot\cos(\theta)\cdot\cos(\psi) + \sin(\theta)\cdot\sin(\eta) - \varepsilon\cdot r\cdot\cos(\theta)\cdot\cos(\eta) \\ -\varepsilon\cdot x\cdot\sin(\eta)\cdot\sin(\psi) + \cos(\eta) \end{bmatrix}$$

Separating the primary component $\mathbf{T_p}$ and dual component $\mathbf{T_d}$

$$Tp(r,\theta,\eta,\psi,x) := \begin{bmatrix} \cos(\theta)\cdot\cos(\eta)\cdot\cos(\psi) - \sin(\theta)\cdot\sin(\psi) & -\cos(\theta)\cdot\sin(\psi)\cdot\cos(\eta) - \sin(\theta)\cdot\cos(\psi) & \cos(\theta)\cdot\sin(\eta) \\ \sin(\theta)\cdot\cos(\eta)\cdot\cos(\psi) + \cos(\theta)\cdot\sin(\psi) & -\sin(\theta)\cdot\sin(\psi)\cdot\cos(\eta) + \cos(\theta)\cdot\cos(\psi) & \sin(\theta)\cdot\sin(\eta) \\ -\sin(\eta)\cdot\cos(\psi) & \sin(\eta)\cdot\sin(\psi) & \cos(\eta) \end{bmatrix}$$

$$Td(r,\theta,\eta,\psi,x) := \begin{bmatrix} -r\cdot\sin(\theta)\cdot\sin(\eta)\cdot\cos(\psi) & x\cdot\cos(\theta)\cdot\sin(\eta) + r\cdot\sin(\theta)\cdot\sin(\eta)\cdot\sin(\psi) & x\cdot(\cos(\theta)\cdot\cos(\eta)\cdot\sin(\psi) + \sin(\theta)\cdot\cos(\psi)) + r\cdot\sin(\theta)\cdot\cos(\eta) \\ r\cdot\cos(\theta)\cdot\sin(\eta)\cdot\cos(\psi) & x\cdot\sin(\theta)\cdot\sin(\eta) - r\cdot\cos(\theta)\cdot\sin(\eta)\cdot\sin(\psi) & x\cdot(\sin(\theta)\cdot\cos(\eta)\cdot\sin(\psi) - \cos(\theta)\cdot\cos(\psi)) - r\cdot\cos(\theta)\cdot\cos(\eta) \\ r\cdot\sin(\psi) & x\cdot\cos(\eta) + r\cdot\cos(\psi) & -x\cdot\sin(\eta)\cdot\sin(\psi) \end{bmatrix}$$

Now, to deal with the coupling of leg ends, we can look at the origins, which have to coincide. Letting $\{\mathbf{U_i}\}$ = the reference coordinates for the upper (distal) end of leg **i**, the matrix **T** can be decomposed as follows.

$\mathbf{T}= [R]\{\mathbf{U_i}\} + \varepsilon[DR]\{\mathbf{U_i}\}$.

The dual component **Td** of matrix **T** represents the total translation [D] multiplied by the total rotation [R] (which makes sense by considering the matrix operation in dual numbers... [D1] ε X [D2] ε = 0; so only the {[R1][D2] + [R2][D1]}ε dual term remains; or in three terms, {[R3][R2][D1] + [R3][D2][R1] + [D3][R2][R1]}ε).

So $[D] = [DR]\,[R]^T$.

(Note: $[R]^{-1} = [R]^T$ because [R] is obtained from the rotation matrices X, Y, Z whose inverses are their transposes; since [R] is composed directly from these matrices, its inverse is its transpose – decompose then take its inverse step by step to see). (Fischer 1999, p. 83)

$$D(r,\theta,\eta,\psi,x) := Td(r,\theta,\eta,\psi,x)\cdot Tp(r,\theta,\eta,\psi,x)^{T}$$

$$D(r,\theta,\eta,\psi,x)^{\langle 0\rangle} \rightarrow \begin{bmatrix} -r\cdot\sin(\theta)\cdot\sin(\eta)\cdot\cos(\psi)^2\cdot\cos(\theta)\cdot\cos(\eta) - r\cdot\sin(\theta)\cdot\sin(\eta)\cdot\sin(\psi)^2\cdot\cos(\theta)\cdot\cos(\eta) + \cos(\theta)\cdot\sin(\eta)\cdot r\cdot\sin(\theta)\cdot\cos(\eta) \\ r\cdot\cos(\theta)^2\cdot\sin(\eta)\cdot\cos(\psi)^2\cdot\cos(\eta) - \sin(\theta)^2\cdot\sin(\eta)\cdot x\cdot\cos(\psi) + r\cdot\cos(\theta)^2\cdot\sin(\eta)\cdot\sin(\psi)^2\cdot\cos(\eta) - x\cdot\cos(\theta)^2\cdot\sin(\eta)\cdot\cos(\psi) - \cos(\theta)^2\cdot\sin(\eta)\cdot r\cdot\cos(\eta) \\ -r\cdot\sin(\psi)^2\cdot\sin(\theta) - x\cdot\cos(\theta)\cdot\cos(\eta)^2\cdot\sin(\psi) - \cos(\eta)\cdot x\cdot\sin(\theta)\cdot\cos(\psi) - r\cdot\cos(\psi)^2\cdot\sin(\theta) - \sin(\eta)^2\cdot\sin(\psi)\cdot x\cdot\cos(\theta) \end{bmatrix}$$

$$D(r,\theta,\eta,\psi,x)^{\langle 1\rangle} \rightarrow \begin{bmatrix} -r\cdot\sin(\theta)^2\cdot\sin(\eta)\cdot\cos(\psi)^2\cdot\cos(\eta) + x\cdot\cos(\theta)^2\cdot\sin(\eta)\cdot\cos(\psi) - r\cdot\sin(\theta)^2\cdot\sin(\eta)\cdot\sin(\psi)^2\cdot\cos(\eta) + \sin(\theta)^2\cdot\sin(\eta)\cdot x\cdot\cos(\psi) + \sin(\theta)^2\cdot\sin(\eta)\cdot r\cdot\cos(\eta) \\ r\cdot\sin(\theta)\cdot\sin(\eta)\cdot\cos(\psi)^2\cdot\cos(\theta)\cdot\cos(\eta) + r\cdot\sin(\theta)\cdot\sin(\eta)\cdot\sin(\psi)^2\cdot\cos(\theta)\cdot\cos(\eta) - \cos(\theta)\cdot\sin(\eta)\cdot r\cdot\sin(\theta)\cdot\cos(\eta) \\ r\cdot\sin(\psi)^2\cdot\cos(\theta) - x\cdot\sin(\theta)\cdot\cos(\eta)^2\cdot\sin(\psi) + \cos(\eta)\cdot x\cdot\cos(\theta)\cdot\cos(\psi) + r\cdot\cos(\psi)^2\cdot\cos(\theta) - \sin(\eta)^2\cdot\sin(\psi)\cdot x\cdot\sin(\theta) \end{bmatrix}$$

$$D(r,\theta,\eta,\psi,x)^{\langle 2\rangle} \rightarrow \begin{bmatrix} r\cdot\sin(\theta)\cdot\sin(\eta)^2\cdot\cos(\psi)^2 + \sin(\eta)^2\cdot\sin(\psi)\cdot x\cdot\cos(\theta) + \sin(\eta)^2\cdot\sin(\psi)^2\cdot r\cdot\sin(\theta) + x\cdot\cos(\theta)\cdot\cos(\eta)^2\cdot\sin(\psi) + \cos(\eta)\cdot x\cdot\sin(\theta)\cdot\cos(\psi) + r\cdot\sin(\theta)\cdot\cos(\eta)^2 \\ -r\cdot\cos(\theta)\cdot\sin(\eta)^2\cdot\cos(\psi)^2 + \sin(\eta)^2\cdot\sin(\psi)\cdot x\cdot\sin(\theta) - \sin(\eta)^2\cdot\sin(\psi)^2\cdot r\cdot\cos(\theta) + x\cdot\sin(\theta)\cdot\cos(\eta)^2\cdot\sin(\psi) - \cos(\eta)\cdot x\cdot\cos(\theta)\cdot\cos(\psi) - r\cdot\cos(\theta)\cdot\cos(\eta)^2 \\ 0 \end{bmatrix}$$

where [D] is the matrix result of the cross of **d** by the unit vectors of frame {**U**}

$$D\cdot e \equiv \begin{bmatrix} d_x \\ d_y \\ d_z \end{bmatrix} \times \begin{bmatrix} e_{u,x} \\ e_{u,y} \\ e_{u,z} \end{bmatrix} \equiv \begin{bmatrix} 0 & -d_z & d_y \\ d_z & 0 & -d_x \\ -d_y & d_x & 0 \end{bmatrix} \cdot \begin{bmatrix} e_{u,x} \\ e_{u,y} \\ e_{u,z} \end{bmatrix}$$

As a check, the 1-1 element should be zero

$$-r\cdot\sin(\theta)\cdot\sin(\eta)\cdot\cos(\theta)\cdot\cos(\eta)\cdot\left(\cos(\psi)^2 + \sin(\psi)^2\right) + \cos(\theta)\cdot\sin(\eta)\cdot r\cdot\sin(\theta)\cdot\cos(\eta) \equiv 0$$

As a check, the 2-2 element should be zero

$$r\cdot\sin(\theta)\cdot\sin(\eta)\cdot\cos(\psi)^2\cdot\cos(\theta)\cdot\cos(\eta) + r\cdot\sin(\theta)\cdot\sin(\eta)\cdot\sin(\psi)^2\cdot\cos(\theta)\cdot\cos(\eta) - \cos(\theta)\cdot\sin(\eta)\cdot r\cdot\sin(\theta)\cdot\cos(\eta)$$

$$r\cdot\sin(\theta)\cdot\sin(\eta)\cdot\cos(\theta)\cdot\cos(\eta) - \cos(\theta)\cdot\sin(\eta)\cdot r\cdot\sin(\theta)\cdot\cos(\eta) \equiv 0$$

Now, the 2-1 element = $\mathbf{d_z}$

$$r\cdot\cos(\theta)^2\cdot\sin(\eta)\cdot\cos(\eta)\cdot\left(\cos(\psi)^2 + \sin(\psi)^2\right) - x\cdot\left(\cos(\theta)^2 + \sin(\theta)^2\right)\cdot\sin(\eta)\cdot\cos(\psi) - \cos(\theta)^2\cdot\sin(\eta)\cdot r\cdot\cos(\eta)$$

$$-x\cdot\sin(\eta)\cdot\cos(\psi) \equiv d_z$$

And, the 3-1 element = $\mathbf{d_x}$

$$\left[r\cdot\left(\cos(\psi)^2 + \sin(\psi)^2\right)\cdot\cos(\theta) - x\cdot\sin(\theta)\cdot\left(\cos(\eta)^2 + \sin(\eta)^2\right)\cdot\sin(\psi)\right] + \cos(\eta)\cdot x\cdot\cos(\theta)\cdot\cos(\psi)$$

$$r\cdot\cos(\theta) - x\cdot\sin(\theta)\cdot\sin(\psi) + x\cdot\cos(\theta)\cdot\cos(\eta)\cdot\cos(\psi) \equiv d_x$$

Lastly, the 1-3 element = $\mathbf{d_y}$

$$r\cdot\sin(\theta)\cdot\sin(\eta)^2\cdot\left(\cos(\psi)^2+\sin(\psi)^2\right)+\left(\sin(\eta)^2+\cos(\eta)^2\right)\cdot\sin(\psi)\cdot x\cdot\cos(\theta)+\cos(\eta)\cdot x\cdot\sin(\theta)\cdot\cos(\psi)+r\cdot\sin(\theta)\cdot\cos(\eta)^2$$

$$r\cdot\sin(\theta)\cdot\left(\sin(\eta)^2+\cos(\eta)^2\right)+x\cdot\cos(\theta)\cdot\sin(\psi)+x\cdot\sin(\theta)\cdot\cos(\eta)\cdot\cos(\psi)$$

$$r\cdot\sin(\theta)+x\cdot\cos(\theta)\cdot\sin(\psi)+x\cdot\sin(\theta)\cdot\cos(\eta)\cdot\cos(\psi)=d_y$$

or combining the results for $\mathbf{d_x}$, $\mathbf{d_y}$, and $\mathbf{d_z}$.

$$d(r,\theta,\eta,\psi,x):=\begin{bmatrix} r\cdot\cos(\theta)-x\cdot\sin(\theta)\cdot\sin(\psi)+x\cdot\cos(\theta)\cdot\cos(\eta)\cdot\cos(\psi) \\ r\cdot\sin(\theta)+x\cdot\cos(\theta)\cdot\sin(\psi)+x\cdot\sin(\theta)\cdot\cos(\eta)\cdot\cos(\psi) \\ -x\cdot\sin(\eta)\cdot\cos(\psi) \end{bmatrix}$$

Single Leg Equation

This equation is applicable to any leg, whatever the top constraint is.

2.3 Leg Variable Dependence

The *Single Leg Equation* can solved for **x**, η, & ψ in terms of $\mathbf{d_x}$, $\mathbf{d_y}$, and $\mathbf{d_z}$. Now, the subscript **i** is introduced, for clarity of differentiation between legs.

$$\begin{bmatrix} d_{i,x} \\ d_{i,y} \\ d_{i,z} \end{bmatrix}=\begin{bmatrix} r_i\cdot\cos(\theta_i)-x_i\cdot\sin(\theta_i)\cdot\sin(\psi_i)+x_i\cdot\cos(\theta_i)\cdot\cos(\eta_i)\cdot\cos(\psi_i) \\ r_i\cdot\sin(\theta_i)+x_i\cdot\cos(\theta_i)\cdot\sin(\psi_i)+x_i\cdot\sin(\theta_i)\cdot\cos(\eta_i)\cdot\cos(\psi_i) \\ -x_i\cdot\sin(\eta_i)\cdot\cos(\psi_i) \end{bmatrix}$$

Single Leg Equation

$$\begin{bmatrix} -x_i\cdot\sin(\theta_i)\cdot\sin(\psi_i)+x_i\cdot\cos(\theta_i)\cdot\cos(\eta_i)\cdot\cos(\psi_i) \\ x_i\cdot\cos(\theta_i)\cdot\sin(\psi_i)+x_i\cdot\sin(\theta_i)\cdot\cos(\eta_i)\cdot\cos(\psi_i) \\ -x_i\cdot\sin(\eta_i)\cdot\cos(\psi_i) \end{bmatrix}=\begin{bmatrix} d_{i,x}-r_i\cdot\cos(\theta_i) \\ d_{i,y}-r_i\cdot\sin(\theta_i) \\ d_{i,z} \end{bmatrix}$$

Multiplying the first equation by $\cos(\theta_i)$ and the second equation by $\sin(\theta_i)$.

$$\begin{bmatrix} -x_i\cdot\sin(\theta_i)\cdot\cos(\theta_i)\cdot\sin(\psi_i)+x_i\cdot\cos(\theta_i)^2\cdot\cos(\eta_i)\cdot\cos(\psi_i) \\ x_i\cdot\sin(\theta_i)\cdot\cos(\theta_i)\cdot\sin(\psi_i)+x_i\cdot\sin(\theta_i)^2\cdot\cos(\eta_i)\cdot\cos(\psi_i) \\ -x_i\cdot\sin(\eta_i)\cdot\cos(\psi_i) \end{bmatrix}=\begin{bmatrix} \cos(\theta_i)\cdot\left(d_{i,x}-r_i\cdot\cos(\theta_i)\right) \\ \sin(\theta_i)\cdot\left(d_{i,y}-r_i\cdot\sin(\theta_i)\right) \\ d_{i,z} \end{bmatrix}$$

Then adding the first and second equation to form another

$$\begin{bmatrix} x_i \cdot \cos(\eta_i) \cdot \cos(\psi_i) \\ -x_i \cdot \sin(\eta_i) \cdot \cos(\psi_i) \end{bmatrix} = \begin{bmatrix} \left[\cos(\theta_i) \cdot (d_{i,x} - r_i \cdot \cos(\theta_i))\right] + \sin(\theta_i) \cdot (d_{i,y} - r_i \cdot \sin(\theta_i)) \\ d_{i,z} \end{bmatrix} \quad \text{Eq. 1}$$

Another similar transform can be performed by multiplying the first equation by $\sin(\theta_i)$ and the second by $\cos(\theta_i)$.

$$\begin{bmatrix} -x_i \cdot \sin(\theta_i)^2 \cdot \sin(\psi_i) + x_i \cdot \sin(\theta_i) \cdot \cos(\theta_i) \cdot \cos(\eta_i) \cdot \cos(\psi_i) \\ x_i \cdot \cos(\theta_i)^2 \cdot \sin(\psi_i) + x_i \cdot \sin(\theta_i) \cdot \cos(\theta_i) \cdot \cos(\eta_i) \cdot \cos(\psi_i) \\ -x_i \cdot \sin(\eta_i) \cdot \cos(\psi_i) \end{bmatrix} = \begin{bmatrix} \sin(\theta_i) \cdot (d_{i,x} - r_i \cdot \cos(\theta_i)) \\ \cos(\theta_{i_e}) \cdot (d_{i,y} - r_i \cdot \sin(\theta_i)) \\ d_{i,z} \end{bmatrix}$$

Then subtracting the first from the second

$$\begin{bmatrix} x_i \cdot \sin(\psi_i) \\ -x_i \cdot \sin(\eta_i) \cdot \cos(\psi_i) \end{bmatrix} = \begin{bmatrix} \cos(\theta_i) \cdot (d_{i,y} - r_i \cdot \sin(\theta_i)) - \sin(\theta_i) \cdot (d_{i,x} - r_i \cdot \cos(\theta_i)) \\ d_{i,z} \end{bmatrix} \quad \text{Eq. 2}$$

Combining Equations 1 and 2

$$\begin{bmatrix} x_i \cdot \sin(\psi_i) \\ x_i \cdot \cos(\eta_i) \cdot \cos(\psi_i) \\ -x_i \cdot \sin(\eta_i) \cdot \cos(\psi_i) \end{bmatrix} = \begin{bmatrix} \cos(\theta_i) \cdot (d_{i,y} - r_i \cdot \sin(\theta_i)) - \sin(\theta_i) \cdot (d_{i,x} - r_i \cdot \cos(\theta_i)) \\ \cos(\theta_i) \cdot (d_{i,x} - r_i \cdot \cos(\theta_i)) + \sin(\theta_i) \cdot (d_{i,y} - r_i \cdot \sin(\theta_i)) \\ d_{i,z} \end{bmatrix} \quad \text{Eq. 3}$$

Taking the square of each row and adding the lines.

$$(x_i)^2 = (d_{i,y} - r_i \cdot \sin(\theta_i))^2 + (d_{i,x} - r_i \cdot \cos(\theta_i))^2 + (d_{i,z})^2$$

$$x_i = \sqrt{(d_{i,x})^2 + (d_{i,y})^2 + (d_{i,z})^2 + (r_i)^2 - 2 \cdot r_i \cdot (d_{i,y} \cdot \sin(\theta_i) - d_{i,x} \cdot \cos(\theta_i))}$$

(The leg length can not be negative, so only the positive square root is taken.)

From the first row of Eq. 3

$$\sin(\psi_i) = \frac{\cos(\theta_i)\cdot\left(d_{i,y} - r_i\cdot\sin(\theta_i)\right) - \sin(\theta_i)\cdot\left(d_{i,x} - r_i\cdot\cos(\theta_i)\right)}{x_i}$$

$$\sin(\psi_i) = \frac{\cos(\theta_i)\cdot d_{i,y} - \sin(\theta_i)\cdot d_{i,x}}{x_i}$$

$$\cos(\psi_i) = \frac{\sqrt{(x_i)^2 - \left(\cos(\theta_i)\cdot d_{i,y} - \sin(\theta_i)\cdot d_{i,x}\right)^2}}{x_i}$$

The square root can be either +/-

Since $d_{i,z}$ in the third row of Eq. 3 can not be negative, either $\sin(\eta_i)$ is negative or $\cos(\psi_i)$ is negative.

$$\left(-x_i\cdot\sin(\eta_i)\right)\cos(\psi_i) = d_{i,z} \qquad x_i\cdot\cos(\eta_i)\cos(\psi_i) = \cos(\theta_i)\cdot\left(d_{i,x} - r_i\cdot\cos(\theta_i)\right) + \sin(\theta_i)\cdot d_{i,y} - r_i\cdot\sin(\theta_i$$

$$\sin(\eta_i) = \frac{-d_{i,z}}{x_i\cdot\cos(\psi_i)}$$

$$\cos(\eta_i) = \frac{d_{i,x}\cdot\cos(\theta_i) + d_{i,y}\cdot\sin(\theta_i) - r_i}{x_i\cdot\cos(\psi_i)}$$

So there are two solutions based off the sign of $\cos(\psi_i)$.

2.4 Vertex Dependence

The position of a vertex depends on both the tool orientation **n** and position **P** on the upper plate and internal geometry. The tool orientation of the upper plate describes the plane in which the plate lies. It is also the normal vector to that plane. The vector curl or cross product of two lines in the upper plate's plane will determine the normal vector. Since the normal vector is normalized ($\mathbf{n_x}^2 + \mathbf{n_y}^2 + \mathbf{n_z}^2 = 1$), the lines used in the cross product need to be normalized as well (divided by the line length). The lines from **P** to

$\mathbf{d_1}$ and $\mathbf{P}$ to $\mathbf{d_i}$ are used, whose line lengths are $\mathbf{V_1}$ and $\mathbf{V_i}$ respectively. (Note: this is done in order to develop equations relating the vertex $\mathbf{d_i}$ in terms of $\mathbf{d_1}$.)

$$\frac{|P - d_1|}{V_1} \times \frac{|P - d_i|}{V_i} = \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} e_{u,x} & e_{u,y} & e_{u,z} \\ P_x - d_{1,x} & P_y - d_{1,y} & P_z - d_{1,z} \\ P_x - d_{i,x} & P_y - d_{i,y} & P_z - d_{i,z} \end{bmatrix} = \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix} \cdot V_1 \cdot V_i$$

Which can be rewritten as

$$V_1 \cdot V_i \cdot n_x = (P_y - d_{1,y}) \cdot (P_z - d_{i,z}) - (P_z - d_{1,z}) \cdot (P_y - d_{i,y})$$

$$V_1 \cdot V_i \cdot n_y = (P_z - d_{1,z}) \cdot (P_x - d_{i,x}) - (P_x - d_{1,x}) \cdot (P_z - d_{i,z})$$

$$V_1 \cdot V_i \cdot n_z = (P_x - d_{1,x}) \cdot (P_y - d_{i,y}) - (P_y - d_{1,y}) \cdot (P_x - d_{i,x})$$

or expressing in terms of $\mathbf{d_i}$.

$$(P_z - d_{1,z}) \cdot d_{i,y} - (P_y - d_{1,y}) \cdot d_{i,z} = (P_z - d_{1,z}) \cdot P_y - (P_y - d_{1,y}) \cdot P_z + V_1 \cdot V_i \cdot n_x$$

$$(P_z - d_{1,z}) \cdot d_{i,x} - (P_x - d_{1,x}) \cdot d_{i,z} = (P_z - d_{1,z}) \cdot P_x - (P_x - d_{1,x}) \cdot P_z - V_1 \cdot V_i \cdot n_y \qquad \text{Eq. 4}$$

$$-(P_y - d_{1,y}) \cdot d_{i,x} + (P_x - d_{1,x}) \cdot d_{i,y} = (P_x - d_{1,x}) \cdot P_y - (P_y - d_{1,y}) \cdot P_x - V_1 \cdot V_i \cdot n_z \qquad \text{Eq. 5}$$

Since the upper plate's geometry is fixed, the angle γ_i between the lines from $\mathbf{P}$ to $\mathbf{d_1}$ and $\mathbf{P}$ to $\mathbf{d_i}$ does not change.

$$\frac{|P - d_1|}{V_1} \cdot \frac{|P - d_i|}{V_i} = \cos(\gamma_i)$$

or

$$(P_x - d_{1,x}) \cdot (P_x - d_{i,x}) + (P_y - d_{1,y}) \cdot (P_y - d_{i,y}) + (P_z - d_{1,z}) \cdot (P_z - d_{i,z}) = V_1 \cdot V_i \cdot \cos(\gamma_i)$$

which can be rewritten as

$$\left(P_x - d_{1,x}\right)\cdot d_{i,x} + \left(P_y - d_{1,y}\right)\cdot d_{i,y} + \left(P_z - d_{1,z}\right)\cdot d_{i,z} = P^2 - \left(P_x\cdot d_{1,x} + P_y\cdot d_{1,y} + P_z\cdot d_{1,z}\right) - V_1\cdot V_i\cdot\cos\left(\gamma_i\right)$$

This equation is used with Eq. 4 and Eq. 5 to find an expression for $\mathbf{d_i}$:

$$-\left(P_y - d_{1,y}\right)\cdot d_{i,x} + \left(P_x - d_{1,x}\right)\cdot d_{i,y} = \left(P_x - d_{1,x}\right)\cdot P_y - \left(P_y - d_{1,y}\right)\cdot P_x - V_1\cdot V_i\cdot n_z$$

$$\left(P_z - d_{1,z}\right)\cdot d_{i,y} - \left(P_y - d_{1,y}\right)\cdot d_{i,z} = \left(P_z - d_{1,z}\right)\cdot P_y - \left(P_y - d_{1,y}\right)\cdot P_z + V_1\cdot V_i\cdot n_x$$

$$\left(P_x - d_{1,x}\right)\cdot d_{i,x} + \left(P_y - d_{1,y}\right)\cdot d_{i,y} + \left(P_z - d_{1,z}\right)\cdot d_{i,z} = P^2 - \left(P_x\cdot d_{1,x} + P_y\cdot d_{1,y} + P_z\cdot d_{1,z}\right) - V_1\cdot V_i\cdot\cos\left(\gamma_i\right)$$

Expressing in matrix notation

$$\begin{bmatrix} -\left(P_y - d_{1,y}\right) & \left(P_x - d_{1,x}\right) & 0 \\ 0 & \left(P_z - d_{1,z}\right) & -\left(P_y - d_{1,y}\right) \\ \left(P_x - d_{1,x}\right) & \left(P_y - d_{1,y}\right) & \left(P_z - d_{1,z}\right) \end{bmatrix} \cdot \begin{bmatrix} d_{i,x} \\ d_{i,y} \\ d_{i,z} \end{bmatrix} = \begin{bmatrix} \left(P_x - d_{1,x}\right)\cdot P_y - \left(P_y - d_{1,y}\right)\cdot P_x - V_1\cdot V_i\cdot n_z \\ \left(P_z - d_{1,z}\right)\cdot P_y - \left(P_y - d_{1,y}\right)\cdot P_z + V_1\cdot V_i\cdot n_x \\ P^2 - \left(P_x\cdot d_{1,x} + P_y\cdot d_{1,y} + P_z\cdot d_{1,z}\right) - V_1\cdot V_i\cdot\cos\left(\gamma_i\right) \end{bmatrix}$$

or moving the coefficients of $\mathbf{d_i}$ to the right side

$$\begin{bmatrix} d_{i,x} \\ d_{i,y} \\ d_{i,z} \end{bmatrix} = \begin{bmatrix} -\left(P_y - d_{1,y}\right) & \left(P_x - d_{1,x}\right) & 0 \\ 0 & \left(P_z - d_{1,z}\right) & -\left(P_y - d_{1,y}\right) \\ \left(P_x - d_{1,x}\right) & \left(P_y - d_{1,y}\right) & \left(P_z - d_{1,z}\right) \end{bmatrix}^{-1} \cdot \begin{bmatrix} \left(P_x - d_{1,x}\right)\cdot P_y - \left(P_y - d_{1,y}\right)\cdot P_x - V_1\cdot V_i\cdot n_z \\ \left(P_x - d_{1,x}\right)\cdot P_y - \left(P_y - d_{1,y}\right)\cdot P_x - V_1\cdot V_i\cdot n_z \\ P^2 - \left(P_x\cdot d_{1,x} + P_y\cdot d_{1,y} + P_z\cdot d_{1,z}\right) - V_1\cdot V_i\cdot\cos\left(\gamma_i\right) \end{bmatrix}$$

Which can be expressed as

$$\begin{bmatrix} d_{i,x} \\ d_{i,y} \\ d_{i,z} \end{bmatrix} = \begin{bmatrix} \dfrac{\left(d_{1,y} - P_y\right)^2 + \left(d_{1,z} - P_z\right)^2}{-C_1} & \dfrac{\left(d_{1,x} - P_x\right)\cdot\left(d_{1,z} - P_z\right)}{C_0} & \dfrac{\left(d_{1,x} - P_x\right)}{C_1} \\ \dfrac{\left(d_{1,x} - P_x\right)}{C_1} & \dfrac{\left(d_{1,z} - P_z\right)}{C_1} & \dfrac{\left(d_{1,y} - P_y\right)}{C_1} \\ \dfrac{\left(d_{1,x} - P_x\right)\cdot\left(d_{1,z} - P_z\right)}{C_0} & \dfrac{\left(d_{1,x} - P_x\right)^2 + \left(d_{1,y} - P_y\right)^2}{C_0} & \dfrac{\left(d_{1,z} - P_z\right)}{C_1} \end{bmatrix} \cdot \begin{bmatrix} d_{1,y}\cdot P_x - d_{1,x}\cdot P_y - V_1\cdot V_i\cdot n_z \\ d_{1,y}\cdot P_z - d_{1,z}\cdot P_y + V_1\cdot V_i\cdot n_x \\ P^2 - \left(P_x\cdot d_{1,x} + P_y\cdot d_{1,y} + P_z\cdot d_{1,z}\right) - V_1\cdot V_i\cdot\cos\left(\gamma_i\right) \end{bmatrix}$$

where

$$C_1 = -\left[\left(d_{1,x} - P_x\right)^2 + \left(d_{1,y} - P_y\right)^2 + \left(d_{1,z} - P_z\right)^2\right] \qquad C_0 = -\left(d_{1,y} - P_y\right)\cdot\left[\left(d_{1,x} - P_x\right)^2 + \left(d_{1,y} - P_y\right)^2 + \left(d_{1,z} - P_z\right)^2\right]$$

CHAPTER 3

CONCLUSION

It is important to be able to move a parallel-leg mechanism through a prescribed path. Since most tools are symmetric, the mobile upper plate of a parallel leg mechanism can be rotated about its tool axis without changing the tool orientation. Thus a single degree of freedom exists for the upper plate, a rotation about the tool axis. Having an equation expressing the leg variables in terms of a single point on the upper plate (in addition to the tool axis position and orientation) would enable the generation of the leg positions for different rotations of the upper plate. The point chosen is the vertex $\mathbf{d_1}$, where the first leg connects onto the upper plate.

Thus, all leg variables $\mathbf{x_i}$, η_i, & ψ_i have been expressed first as functions of $\mathbf{d_i}$. Then the vertices have been expressed as functions of the single variable, $\mathbf{d_1}$, and the two parameters, tool axis orientation **n** and tool axis position **P**.

For the leg **i**, the solution is expressible in terms of $\mathbf{d_i}$.

$$x_i = \sqrt{(d_{i,x})^2 + (d_{i,y})^2 + (d_{i,z})^2 + (r_i)^2 - 2 \cdot r_i \cdot (d_{i,y} \cdot \sin(\theta_i) - d_{i,x} \cdot \cos(\theta_i))}$$

$$\sin(\psi_i) = \frac{\cos(\theta_i) \cdot d_{i,y} - \sin(\theta_i) \cdot d_{i,x}}{x_i} \qquad \cos(\psi_i) = \frac{\sqrt{(x_i)^2 - (\cos(\theta_i) \cdot d_{i,y} - \sin(\theta_i) \cdot d_{i,x})^2}}{x_i}$$

The square root can be either +/-

$$\sin(\eta_i) = \frac{-d_{i,z}}{x_i \cdot \cos(\psi_i)} \qquad \cos(\eta_i) = \frac{d_{i,x} \cdot \cos(\theta_i) + d_{i,y} \cdot \sin(\theta_i) - r_i}{x_i \cdot \cos(\psi_i)}$$

However taking η_i and ψ_i into account together, the Single Leg Equation still needs to be satisfied. As such, $\mathbf{d_z}$ can not be negative, so either $\sin(\eta_i)$ or $\cos(\psi_i)$ is negative.

These equations combined with

$$\begin{bmatrix} d_{i,x} \\ d_{i,y} \\ d_{i,z} \end{bmatrix} = \begin{bmatrix} \frac{(d_{1,y}-P_y)^2+(d_{1,z}-P_z)^2}{-C_1} & \frac{(d_{1,x}-P_x)\cdot(d_{1,z}-P_z)}{C_0} & \frac{(d_{1,x}-P_x)}{C_1} \\ \frac{(d_{1,x}-P_x)}{C_1} & \frac{(d_{1,z}-P_z)}{C_1} & \frac{(d_{1,y}-P_y)}{C_1} \\ \frac{(d_{1,x}-P_x)\cdot(d_{1,z}-P_z)}{C_0} & \frac{(d_{1,x}-P_x)^2+(d_{1,y}-P_y)^2}{C_0} & \frac{(d_{1,z}-P_z)}{C_1} \end{bmatrix} \cdot \begin{bmatrix} d_{1,y}\cdot P_x - d_{1,x}\cdot P_y - V_1\cdot V_i\cdot n_z \\ d_{1,y}\cdot P_z - d_{1,z}\cdot P_y + V_1\cdot V_i\cdot n_x \\ P^2 - (P_x\cdot d_{1,x} + P_y\cdot d_{1,y} + P_z\cdot d_{1,z}) - V_1\cdot V_i\cdot\cos(\gamma_i) \end{bmatrix}$$

where

$$C_1 = -\left[(d_{1,x}-P_x)^2+(d_{1,y}-P_y)^2+(d_{1,z}-P_z)^2\right] \qquad C_0 = -(d_{1,y}-P_y)\cdot\left[(d_{1,x}-P_x)^2+(d_{1,y}-P_y)^2+(d_{1,z}-P_z)^2\right]$$

expresses all the legs in terms of $\mathbf{d_1}$.

From inputting the different points on the tracer line of the tool axis position **P**, this process can be repeated to generate a set of such solutions. In addition, when constructing the tracer path, the tool orientation need not be held constant.

Another approach would be to map the state space numerically by constructing a series of different tool axis positions' **P** and orientations' **n**. Each different value of **P** and **n** corresponds to a different plane the mobile upper plate can lie in. The three dimensional space can then be generated by overlaying a progressive set of planes and removing the unattainable configurations.

This analysis can be extended to consider velocity by simply taking the derivatives with respect to time of $\mathbf{x_i}$, η_i, & ψ_i where only the variables $\mathbf{d_1}$, **n**, and **P** have time derivatives. The second derivative could also be taken to find the accelerations of the leg variables. Acceleration and velocity are important in design of mechanisms because the control systems need to produce them.

Also, instead of constraining the leg bases to lie on the same ground plane, the Single Leg Equation can be modified to incorporate an elevation **z** in the $\mathbf{z_g}$- direction at base point **B**.

Using

$$B(r,\theta,z) := \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\varepsilon \cdot r \\ 0 & \varepsilon \cdot r & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -\varepsilon \cdot z & 0 \\ \varepsilon \cdot z & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

It can be shown that the Modified Single Leg Equation would be:

$$d(r,\theta,z,\eta,\psi,x) := \begin{bmatrix} r\cdot\cos(\theta) - x\cdot(\sin(\theta)\cdot\sin(\psi) - \cos(\theta)\cdot\cos(\eta)\cdot\cos(\psi)) \\ r\cdot\sin(\theta) + z\cdot(\cos(\theta) - \sin(\theta))\cdot(\sin(\eta)\cdot\cos(\psi)\cdot\sin(\psi)) + x\cdot(\sin(\psi)\cdot\cos(\theta) + \cos(\eta)\cdot\cos(\psi)\cdot\sin(\theta)) \\ z - x\cdot\sin(\eta)\cdot\cos(\psi) \end{bmatrix}$$

From this equation the leg parameters could be expressed as functions of the variables $\mathbf{d_1}$, **n**, and **P**.

A last modification that could also be done is not to constrain the tool axis orientation to be normal to the mobile upper plate. Then **n** would not be the tool axis orientation, and the upper plate would be able to eccentrically rotate about the tool axis instead.

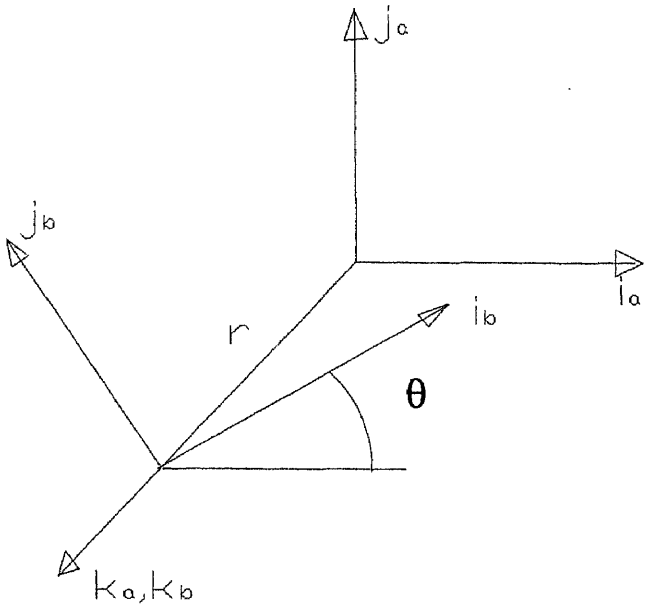
The reason why an elevation and eccentricity were not addressed was because in parallel leg mechanisms the leg bases do primarily lie on the same base plane and the tool is also primarily normal to the mobile upper plate.

APPENDIX

DUAL NUMBER REPRESENTATION

Dual numbers are used to determine the leg vertex (top) position in terms of the ground. The following is an overview of dual numbers. Refer to Fischer's Dual-Number Methods in Kinematics, Statics and Dynamics for an in-depth derivation.

A dual number is comprised of a primary (real) component and a dual (ε) component, where $\varepsilon^2 = 0$. This is a useful representation of simultaneous rotation and translation. The rotation is done in the primary plane and the translation is done in the ε plane. (The reason why translation is done in the ε plane is because from one coordinate transform to another, you never multiply separate translations, but multiply instead the translation by the next rotation; thus in dual numbers, a rotation times a translation ends up in the ε plane, pure rotations stay in the primary plane, and any multiplied translations disappear.) Now a dual angle is the rotation (vector curl) about a coordinate axis and a translation along that same coordinate axis (the translation does not effect the plane of rotation, but simply elevates the rotated plane). The breakdown of coordinate transformation is the same as if done with a pure rotation, i.e. a 3X3 matrix M, whose det(M) =1 in all cases. However, to represent the $\cos(\theta+r\varepsilon)$, a Taylor series expansion about the primary component is done. (Note, the expansion of $f(a+\varepsilon a_0) = f(a) + \varepsilon\, a_0$ $[df(a) / da]$; all other terms disappear because $\varepsilon^2 = 0$.) On the next page are the dual angle rotations and translations about the X, Y, & Z axes respectively.



$$M=\begin{bmatrix} i_b & j_b & k_b \end{bmatrix}$$

$$M=\begin{bmatrix} \cos(\theta + r\varepsilon) & -\sin(\theta + r\varepsilon) & 0 \\ \sin(\theta + r\varepsilon) & \cos(\theta + r\varepsilon) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

So M transforms coordinates in **b** into **a**. However to do this, it is important to remember $(\theta+r\varepsilon)$ is measured in **a**, not **b**.

To decompose a dual vector, it is important to realize that $N = No + (r \times No)\varepsilon$, where No is the angular orientation: projections along i, j, & k axes.

Now the dual vectors for transformations about the X, Y, & Z axes are

$$c(\theta,r) := \cos(\theta) - \varepsilon \cdot r \cdot \sin(\theta) \qquad s(\theta,r) := \sin(\theta) + \varepsilon \cdot r \cdot \cos(\theta)$$

$$X(\sigma,r) := \begin{bmatrix} 1 & 0 & 0 \\ 0 & c(\sigma,r) & -s(\sigma,r) \\ 0 & s(\sigma,r) & c(\sigma,r) \end{bmatrix} \quad Y(\eta,r) := \begin{bmatrix} c(\eta,r) & 0 & s(\eta,r) \\ 0 & 1 & 0 \\ -s(\eta,r) & 0 & c(\eta,r) \end{bmatrix} \quad Z(\theta,r) := \begin{bmatrix} c(\theta,r) & -s(\theta,r) & 0 \\ s(\theta,r) & c(\theta,r) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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