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## ABSTRACT

### **A Study of Droplet Burning in the Nearly Adiabatic Limit**

by  
**Juan C. Gómez**

We consider a small drop of liquid fuel that burns in an oxidizing gaseous environment and translates slowly (relative to flow 'at infinity') under the action of gravity. Practical applications include the burning of liquid fuels as sprays in domestic and industrial oil-fired burners, diesel engines, and liquid-propellant rocket motors. More relevant to the simple physical set-up of the present study are well-characterized laboratory experiments on the burning of a single, isolated fuel drop.

The drop burns in a nearly spherical, diffusion flame, flame sheet regime. We consider a specific example, or limit, referred to as 'nearly adiabatic burning', in which the temperature of the gas mixture at the flame sheet is close to the ambient temperature at infinity. Temperature gradients everywhere outside the flame sheet are therefore small. The problem is solved by perturbation methods, primarily, with a distinguished limit between the inverse nondimensional activation energy  $\epsilon$  and the translational Reynolds number  $Re$ . We include time dependence far from the drop as a quasisteady effect, and this influences the near field region of the drop via matching between near field and far field.

Evaluation of quantities such as the 'drag' force exerted by the fluid on the drop, the flame sheet shape, and the speed of translation necessitates numerical solution of a higher order problem in the perturbation scheme. Results predicted for the behavior of a heptane fuel drop will be presented.

**A STUDY OF DROPLET BURNING  
IN THE NEARLY ADIABATIC LIMIT**

by  
**Juan C. Gómez**

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**Department of Mathematical Sciences, NJIT  
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### A Study of Droplet Burning in the Nearly Adiabatic Limit

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To our friend, Carlos Urdaneta,  
his optimism will be inspiration and guide  
always

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# CHAPTER 1

## INTRODUCTION

We consider a small drop of liquid fuel that burns in an oxidizing gaseous environment. Practical applications include the burning of liquid fuels as sprays in domestic and industrial oil-fired burners, diesel engines, and liquid-propellant rocket motors. More relevant to the simple physical set-up of the present study are well-characterized laboratory experiments on the burning of a single, isolated fuel drop.

The drop is small, of the order of  $10 - 100\mu m$  in diameter, and is assumed to remain spherical under the action of surface tension. The ambient gas at infinity is taken to be at rest and of uniform, constant composition, temperature, and pressure. Internal motion within the drop is neglected, since the ratio of shear viscosities of the gaseous environment and liquid fuel, which enters the condition for continuity of tangential stress at the drop interface, is small. The drop falls under the action of gravity, but its small size and weight imply that its translational speed relative to the ambient gas at infinity is also small.

In the absence of gravity the translational speed is zero. For a single component fuel that undergoes a one-step, highly activated chemical reaction the drop burns with a spatially narrow, spherical flame sheet located at some non-negative stand-off distance from the drop surface. The chemical reaction within the flame sheet is assumed to be highly exothermic, so that heat generated at the flame sheet is conducted back to the drop surface and vaporizes the liquid fuel. Since the density of the gaseous vaporized fuel is far less than the density of the liquid drop, gaseous fuel is forced outwards towards the flame sheet. Simultaneously, oxidizing gas migrates inwards from infinity towards the flame sheet under the action of diffusion through the gas mixture. This simple structure for the combustion of a fuel drop is referred to as a spherical diffusion flame, and under suitable conditions it can be self-sustaining.

The distinction between a 'diffusion flame' of this type and a 'premixed flame' is that for the diffusion flame the sources of fuel and oxidizer are separated spatially, so that diffusion is always an essential part of mixing the reactants on the molecular level as is required for reaction to occur. For the premixed flame, gaseous fuel and oxidizer are mixed first and react subsequently, so that diffusion, although it often plays an important part in the process, is not essential. For the diffusion flame, gas velocities must be very small relative to the speed of sound, whereas for the premixed flame this is often but not necessarily true.

The radial symmetry of the spherical diffusion flame is broken if the drop falls under gravity. However, for the small drop at slow translational speed, the departure from spherical symmetry is small. Here we apply perturbation methods to determine how weak translation can influence features such as the flame shape, the rate of burning, and the transfer of energy from the drop to its surroundings.

We consider a specific example, or limit, of the spherical diffusion flame which is referred to as 'nearly adiabatic burning'. Here, the temperature of the gas mixture at the flame sheet is close to the ambient temperature at infinity, so that temperature gradients everywhere outside the flame sheet are small. One benefit of considering this limit is that the structure of the spatially narrow flame sheet can be resolved by completely analytical, as opposed to numerical, means. A further benefit is the fact that the response relations for the basic, steady state solution found in the nearly adiabatic limit encompass the full range of burning conditions with respect to the Damkohler number, from the limit of 'frozen flow' at low Damkohler number to the 'Burke-Schumann' limit of near-infinite Damkohler number, by use of a single set of scalings.

Whether or not the nearly adiabatic burning limit applies in practice depends on conditions at infinity. Under circumstances where the ambient temperature is close to room temperature the limit clearly does not apply, but for circumstances

where a spray is burnt in an enclosure the enclosure walls can be allowed to reach temperatures close to the reaction temperature by radiation. Also, in the central regions of a dilute spray that is not enclosed the ambient temperature experienced by a single drop may be maintained near the reaction temperature by the presence of neighboring drops, while other drop-to-drop interactions are weak.

The literature on droplet burning is extensive; background and review material can be found in the texts [1] to [3]. The nearly adiabatic burning limit is discussed in [3], and a thorough study of diffusion flame structure is given in [4]. Two groups of studies that consider the effects of weak translation on an otherwise spherical diffusion flame by perturbation methods are those of Fendell and co-workers [5-6], which date to the sixties, and Ayyaswamy and co-workers [7-8], from the eighties and nineties.

Both groups of authors consider the related problem of condensation and evaporation of a translating drop in an inert environment, see [9] and [10-13]. These studies are particularly relevant to the warm-up or pre-ignition stage which the drop undergoes prior to initiation of reaction at the diffusion flame, and draw on earlier work on slow viscous flow past a sphere [14-15], where thermal expansion due to temperature gradients and heat transfer effects causes the flow, although slow, to be compressible. These, in turn, can usefully be considered as a development of the classical problem of slow viscous flow past a sphere in the isothermal and incompressible limit. This last problem (the origin of the 'Whitehead paradox') was resolved by the introduction of separate but matched near field and far field regions during early development and formalism of the method of matched asymptotic expansions for low Reynolds number flow in the late fifties [16-17].

Here we consider the Lewis numbers of fuel and oxidizer, that is the ratios of thermal diffusivity to species diffusivity in the gas mixture, to be arbitrary. We also consider the inverse nondimensional activation energy  $\epsilon$  to be small but non-

zero. This enables us to consider the influence that finite rate chemistry, with its implication of a small but non-zero width of the flame sheet, has on the overall process. A distinguished limit between  $\epsilon$  and the translational Reynolds number  $Re$  is introduced in the perturbation scheme. Both assumptions were relaxed in the existing studies [5-8], so that with  $\epsilon$  set to zero chemical reaction occurs at a discontinuity in the derivatives of the flow field. Use of Lewis numbers set to unity in [5-8] allows the reaction rate to be eliminated from the governing equations by introduction of two separate combinations of the temperature and species mass fractions as dependent variables. The location of the flame sheet discontinuity in the flow field is then determined by taking the Damkohler number to be effectively infinite, so that the reaction discontinuity is considered in the Burke-Schumann limit.

When we set the ratio  $\epsilon/Re$  to zero and set the Lewis numbers of fuel and oxidizer to unity in the results of our study, we recover the same set of jump conditions in the derivatives of the flow field across the flame sheet as were found in [5-8]. However, we are not able to fully compare some other results of our study with this earlier work: during the course of its lifetime there is a slow decrease in radius of the drop and a slow decrease in its weight. This implies that the translational speed of the drop relative to the flow at infinity varies with time, and induces time-dependent effects in the far field flow region. Here, we include time dependence in the far field as a quasisteady effect, which influences the near field region via matching between the near field and far field. In addition, decrease in the drop mass also influences its rate of change of momentum, which enters the overall force balance of the drop.

A second point of comparison concerns the Burke-Schumann limit. There, since the oxidizer mass fraction at the flame sheet is zero, the ratio of flame sheet radius to drop radius is constant. This implies that the mass flux of gas leaving the drop surface is directly proportional to the drop radius, and overall conservation of mass between the gas and liquid phases implies that the square of the drop radius decreases

linearly in time. This is referred to as the d-squared law of droplet burning, and although it is a long-established result of fundamental theory, experimental studies indicate that it is not always accurately followed in practice. Here, use of the nearly adiabatic burning limit with its ready inclusion of a Damkohler number that is large but finite leads to time dependence of the ratio of flame sheet to drop radius, and the possibility of departure from the d-squared law. We include numerical results predicted by the model which are based on parameter values for n-heptane fuel.

## CHAPTER 2

### FORMULATION

**Governing equations.** We analyze the combustion of a single component, spherical fuel drop which moves slowly relative to a uniform stream at infinity. To simplify the governing equations we neglect the Soret and Dufour effects, pressure gradient diffusion, and radiation effects. We also assume that no body forces (such as gravity) act on the gas mixture. We assume that chemical changes taking place in the gas mixture are governed by a single, one-step, irreversible reaction of the form



where  $Y_i$  and  $\nu_i$ , for  $i = F, O, P$  are the mass fractions and stoichiometric coefficients of the fuel, oxidizer, and products, respectively. The gas mixture includes an inert species I as well, and the mass fractions sum to unity by definition, i.e.,  $\sum_{i=1}^4 Y_i = 1$ .

The reaction (2.1) has a single, well-defined rate of production  $w$ . Using the Arrhenius law, we may write this as

$$w = f(\rho, T, Y_F, Y_O) \exp\left(-\frac{E}{R^o T}\right) \quad (2.2)$$

where  $E$  is the activation energy of the reaction,  $R^o$  is the universal gas constant, and  $T$  is the absolute temperature of the mixture. An example of the function  $f$  based on the law of mass action, which we shall use below, is  $f(\rho, T, Y_F, Y_O) = B\rho_\infty Y_F^{\nu_F} Y_O^{\nu_O}$ , where  $\rho_\infty$  is the ambient density of the mixture at infinity, and  $B$  is a constant which is equal to the collision frequency for binary encounters in the mixture under ambient conditions. Additional algebraic dependence of  $f$  on  $\rho$  and  $T$  can be included but will not be of concern here.

For a mixture of ideal gases with constant and equal specific heats  $c_p$ , the equations expressing conservation of mass, momentum, reactant mass fractions, and

energy in dimensional variables are

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \quad (2.3)$$

$$\rho \left( \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) \mathbf{v} = -\nabla p + \nabla \cdot \sigma \quad (2.4)$$

$$\rho \left( \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) Y_i = \nabla \cdot (\rho D_i \nabla Y_i) - \nu_i W_i w \quad i = F, O \quad (2.5)$$

$$\rho c_p \left( \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) T + \rho \left( \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) \frac{\mathbf{v} \cdot \mathbf{v}}{2} = Qw + \nabla \cdot (\lambda \nabla T) + \nabla \cdot (\mathbf{v} \cdot \sigma) \quad (2.6)$$

respectively. Here

$$\sigma_{ij} = \mu \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) - \frac{2}{3} \hat{\lambda} (\nabla \cdot \mathbf{v}) \delta_{ij} \quad (2.7)$$

is the deviatoric part of the viscous stress tensor,  $\mu$  is the molecular or shear viscosity and  $\hat{\lambda} = \mu - (3\hat{\kappa}/2)$  where  $\hat{\kappa}$  is the bulk viscosity. We shall retain the bulk viscosity, although for a mixture of ideal gases the Stokes relation applies, that is, the deviatoric stress is traceless so that  $\hat{\kappa}$  is zero and thus  $\hat{\lambda} = \mu$ . This condition also holds under all but the most extreme conditions for real gases. The dimensional heat release  $Q$  is defined as

$$Q = \nu_F W_F h_F^0 + \nu_O W_O h_O^0 - \nu_P W_P h_P^0 > 0 \quad (2.8)$$

where  $h_i^0$  are the specific enthalpies for each species.

The vorticity equation is derived directly from the equations for conservation of mass and momentum, and in the case of constant molecular and bulk viscosities is

$$\frac{D}{Dt} \left( \frac{\vec{\omega}}{\rho} \right) - \left( \frac{\vec{\omega}}{\rho} \cdot \nabla \right) \mathbf{v} = \nabla \left( \frac{1}{\rho} \right) \wedge \frac{D\mathbf{v}}{Dt} + \frac{\mu}{\rho^2} \nabla^2 \vec{\omega}, \quad (2.9)$$

where  $\frac{D}{Dt} = \left( \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right)$  is the advective time derivative following a material particle, and the vorticity is  $\vec{\omega} = \nabla \wedge \mathbf{v}$ .

For a mixture of  $N$  perfect gases in local thermodynamic equilibrium the total pressure is

$$p = \rho R^0 T \sum_{i=1}^N \frac{Y_i}{W_i} \quad (2.10)$$

and from the definition of the specific heats for each species

$$\frac{R^0}{W_i} = c_{pi} - c_{vi} = c_{pi} \left( \frac{\gamma_i - 1}{\gamma_i} \right). \quad (2.11)$$

If the specific heats are the same for all species then  $c_{pi} = c_p$  and  $c_{vi} = c_v$ , so that the quantity in (2.11) is  $R = R^0/W$  and the equation of state simplifies to

$$p = R\rho T. \quad (2.12)$$

If further each gas is polytropic, so that its specific heats are independent of temperature, then  $R$  is constant. We shall use this as a suitable model for the equation of state.

**Boundary conditions.** The liquid to gas interface of the drop surface is at  $\mathbf{x} = \mathbf{a}_d$ . We assume that the drop remains spherical under the action of surface tension, which is a good approximation for the small drop sizes of most applications under most conditions.

The origin of coordinates is taken to be at the drop center, relative to which each gas phase species has velocity  $\mathbf{v}_i$ . This is related to the mass-averaged mixture velocity  $\mathbf{v} = \sum_{i=1}^N Y_i \mathbf{v}_i$  and the diffusion velocity  $\mathbf{V}_i$  by  $\mathbf{v}_i = \mathbf{v} + \mathbf{V}_i$ . In the formulation of the governing equations above we have assumed that diffusion of each species within the mixture is governed by Fick's law with a diffusivity  $D_i$ , that is,

$$\mathbf{V}_i = -\frac{D_i}{Y_i} \nabla Y_i. \quad (2.13)$$

The definition of a mass fraction is that the  $i$ th species has density  $\rho_i = \rho Y_i$ , and therefore

$$\begin{aligned} \text{mass flux of species } i \text{ relative to the origin} &= \rho Y_i \mathbf{v}_i \\ &= \rho Y_i \mathbf{v} - \rho D_i \nabla Y_i. \end{aligned} \quad (2.14)$$

Relative to the origin, the drop interface has velocity or regression rate  $\frac{d\mathbf{a}_d}{dt}$ , so that

$$\begin{aligned} \text{mass flux of species } i \text{ relative to the interface} &= \rho Y_i (\mathbf{v}_i - \frac{d\mathbf{a}_d}{dt}) \\ &= \rho Y_i (\mathbf{v} - \frac{d\mathbf{a}_d}{dt}) - \rho D_i \nabla Y_i. \end{aligned} \quad (2.15)$$

Summing over all species, since the definition of mass averaged mixture velocity implies that  $\sum_{i=1}^N Y_i \mathbf{V}_i = 0$ , we find that

$$\text{mixture mass flux relative to the interface} = \rho (\mathbf{v} - \frac{d\mathbf{a}_d}{dt}). \quad (2.16)$$

The drop is assumed to consist of pure fuel throughout, so that other species are not absorbed by it. Relative to the interface the normal mass flux of the fuel is therefore equal to the normal mass flux of the mixture, and this condition can be written as

$$\rho D_F \nabla Y_F \cdot \mathbf{n} = -\rho (1 - Y_F) (\mathbf{v} - \frac{d\mathbf{a}_d}{dt}) \cdot \mathbf{n}. \quad (2.17)$$

where  $\mathbf{n} = \mathbf{e}_r$  is the outward unit normal from the drop surface. Similarly, zero normal mass flux of oxidizer relative to the interface implies that

$$\rho D_O \nabla Y_O \cdot \mathbf{n} = \rho Y_O (\mathbf{v} - \frac{d\mathbf{a}_d}{dt}) \cdot \mathbf{n}. \quad (2.18)$$

In the absence of radiative effects, evaporation of the liquid fuel occurs by conduction of heat to the drop surface alone, so that the conductive heat flux is equal to the rate at which energy is supplied as latent heat in evaporating the fuel. When this condition is written in terms of the mixture mass flux from (2.17) it becomes

$$\lambda \nabla T \cdot \mathbf{n} = \rho L (\mathbf{v} - \frac{d\mathbf{a}_d}{dt}) \cdot \mathbf{n}, \quad (2.19)$$

where  $L$  is the latent heat of vaporization.

Vaporization occurs when the fuel's vapor pressure is equal to the pressure of the adjacent atmosphere, and this is expressed by the Clausius-Clapeyron equation. However, in common with many studies on droplet burning, we replace this by

assuming that the fuel vaporizes at a well-defined boiling temperature  $T_s$ , which is known for a given fuel at a given pressure. Thus

$$T = T_s \quad (2.20)$$

at the interface. For a translating drop, radial symmetry of the temperature and velocity fields is broken, and there is a small but non-zero component of the tangential velocity of the gas mixture at the interface. Continuity of tangential stress implies that a circulation of liquid velocity is set up within the drop, and this modifies the drag force it experiences. We neglect this effect, as is common in many studies, and assume that the tangential velocity component vanishes at the interface, that is,

$$\mathbf{v} \wedge \mathbf{n} = \mathbf{0} \quad (2.21)$$

which is justified in the limit when the viscosity of the liquid is much greater than the viscosity of the gas.

Mass conservation between the gas and liquid phases implies that the rate at which the drop loses mass is equal to the mass flux of the mixture leaving its surface, and the last quantity is referred to as the (mass) burning rate. If the liquid density is  $\rho_l$  then for a spherical drop with radius  $a_d = |\mathbf{a}_d|$ , we have

$$-\frac{d}{dt} \left( \rho_l \frac{4\pi a_d^3}{3} \right) = \int_{S_{a_d}} \rho (\mathbf{v} - \frac{d\mathbf{a}_d}{dt}) \cdot d\mathbf{S}, \quad (2.22)$$

where  $S_{a_d}$  denotes the drop surface. The force balance on the drop is given by equating the sum of its weight, corrected for buoyancy, plus the total drag force  $\vec{\mathcal{F}}$  it experiences to the time rate of change of momentum, ie,

$$(\rho_l - \rho_{|\mathbf{x}|=a_d}) \frac{4\pi a_d^3}{3} g(-\hat{\mathbf{k}}) + \vec{\mathcal{F}} = \frac{d}{dt} \left( \rho_l \frac{4\pi a_d^3}{3} U_\infty \right) (-\hat{\mathbf{k}}), \quad (2.23)$$

where  $\hat{\mathbf{k}}$  is a unit vector in the upward vertical direction.

At infinity all variables, such as the temperature and gas velocity, approach their ambient values. Hence, as  $|\mathbf{x}| \rightarrow \infty$ ,

$$(T, Y_F, Y_O, \mathbf{v}, \rho, p) \rightarrow (T_\infty, 0, Y_{O_\infty}, U_\infty \hat{\mathbf{k}}, \rho_\infty, p_\infty). \quad (2.24)$$

**Transport of mass, momentum and energy.** The mass transferred from the liquid to the gas phase at the drop surface is given by

$$\mathcal{M} = \int_{S_{a_d}} \rho \left( \mathbf{v} - \frac{d\mathbf{a}_d}{dt} \right) \cdot \mathbf{n} dS, \quad (2.25)$$

where, as in the boundary conditions at the drop surface, the normal component of the gas velocity relative to the interface appears. Since mass is conserved in the region outside the drop, this is also equal to the mass transferred to infinity in the free stream.

The momentum transfer at the drop surface is related to the drag force. The total drag force exerted by the fluid on the drop is given by integrating the momentum flux over its surface, where the contribution due to mass transfer from the liquid to gas phase is corrected by the drop regression rate  $\frac{da_d}{dt}$ . The total drag force exerted by the gas on the drop is thus

$$\vec{\mathcal{F}} = - \int_{S_{a_d}} \rho \mathbf{v} \left( \mathbf{v} - \frac{d\mathbf{a}_d}{dt} \right) \cdot \mathbf{n} + p \mathbf{n} - \sigma \cdot \mathbf{n} dS. \quad (2.26)$$

When contributions to the energy balance due to the kinetic energy density of the gas and the heat flux due to Joule heating are neglected, the rate of transfer of thermal energy is given by integrating the thermal energy flux over the drop surface or over a sphere at infinity. The difference between these two quantities is due to the transfer of chemical energy to thermal energy within the flow field as the reaction proceeds. The rate of transfer of thermal energy to the drop is given by

$$\mathcal{E}_a = - \int_{S_{a_d}} \left( c_p \rho \left( \mathbf{v} - \frac{d\mathbf{a}_d}{dt} \right) T - \lambda \nabla T \right) \cdot \mathbf{n} dS. \quad (2.27)$$

The rate of transfer of thermal energy from the drop to the flow at infinity is given by

$$\mathcal{E}_\infty = \int_{S_\infty} (c_p \rho \mathbf{v} T - \lambda \nabla T) \cdot \mathbf{e}_r dS \quad (2.28)$$

where  $S_\infty$  is the sphere at infinity.

**Non-dimensionalization.** We introduce the following nondimensional variables as indicated by a \* superscript

$$\mathbf{x} = a_0 \mathbf{x}^*, \quad t = t_0 t^*, \quad T = T_\infty T^*, \quad (2.29)$$

$$\mathbf{v} = U_{\infty 0} \mathbf{v}^*, \quad \rho = \rho_\infty \rho^*, \quad p = p_\infty p^*, \quad \sigma = \frac{\rho_\infty \nu_\infty U_\infty}{a_0} \sigma^*, \quad (2.30)$$

with nondimensional transport coefficients and heat release

$$D_F = D_{F_\infty} D_F^*, \quad D_O = D_{O_\infty} D_O^*, \quad Q = c_p T_\infty q^*,$$

$$\lambda = \lambda_\infty \lambda^*, \quad \nu = \nu_\infty \nu^*, \quad \hat{\lambda} = \rho_\infty \nu_\infty \hat{\lambda}^*. \quad (2.31)$$

An infinity subscript denotes a reference value in the ambient atmosphere as  $|\mathbf{x}| \rightarrow \infty$ ,  $a_0$  is the initial radius of the drop,  $U_{\infty 0}$  is the initial speed of the free stream relative to the drop, and  $t_0$  is a characteristic time for unsteady flow phenomena. Since both the radius of the drop and, in general, its speed relative to the free stream vary with time, we set

$$a_d = a_0 a \quad \text{and} \quad U_\infty = U_{\infty 0} v_\infty \quad (2.32)$$

where  $a$  and  $v_\infty$  are the dimensionless, time-dependent drop radius and free stream speed, respectively, and which therefore have initial values  $a = 1$  and  $v_\infty = 1$ . We also introduce the thermal diffusivity  $\kappa_\infty = \lambda_\infty / \rho_\infty c_p$ .

The following nondimensional parameter groupings appear:

$$\begin{aligned} \bar{\omega} &= \frac{a_0}{U_{\infty 0} t_0}, \quad \text{Da} = \frac{a_0}{U_{\infty 0} t_f}, \quad \text{Ma} = \frac{U_{\infty 0}}{\sqrt{p_\infty / \rho_\infty}}, \quad \text{Re} = \frac{U_{\infty 0} a_0}{\nu_\infty}, \quad \text{Fr} = \frac{U_{\infty 0}^2}{a_0 g}, \\ \text{Pr} &= \frac{\nu_\infty}{\kappa_\infty}, \quad \text{Le}_F = \frac{\kappa_\infty}{D_{F_\infty}}, \quad \text{Le}_O = \frac{\kappa_\infty}{D_{O_\infty}}, \quad l = \frac{L}{c_p T_\infty}. \end{aligned} \quad (2.33)$$

These, respectively, are the Strouhal number, Damkohler number, Mach number, Reynolds number, Froude number, Prandtl number, the Lewis numbers of the fuel

and oxidizer, and a dimensionless coefficient representing the latent heat of vaporization.

The reaction rate term  $w$  of (2.2) has the dimensions of density over time, and the nondimensionalization for this time is taken with respect to the typical time  $t_f$  between binary molecular encounters in the gas mixture under conditions of ambient temperature and pressure at infinity, so that  $1/t_f$  is the ambient collision frequency. This is given by

$$\frac{1}{t_f} = B_\infty \frac{a_\infty^2}{\nu_\infty} \quad (2.34)$$

where  $B_\infty$  is a known dimensionless constant which is dependent on the properties of the mixture and is of order unity,  $a_\infty$  is the ambient, equilibrium or isothermal sound speed which is given by  $a_\infty^2 = RT_\infty$ , and  $\nu_\infty$  is the ambient kinematic viscosity which is typical of the value of the diffusivities associated with molecular transport effects. We therefore nondimensionalize (2.2) as

$$w = \frac{\rho_\infty}{t_f} w^*. \quad (2.35)$$

Corresponding to the Arrhenius form of  $w$  given by the right hand side of (2.2) the form for the dimensionless reaction rate  $w^*$  is

$$w^* = f^*(\rho^*, T^*, Y_F, Y_O) \exp\left(-\frac{1}{\epsilon T^*}\right), \quad (2.36)$$

where  $\epsilon$  is the inverse nondimensional activation energy,  $\epsilon = (E/R^0 T_\infty)^{-1} \ll 1$ , and  $f^*(\cdot)$  is a nondimensional function such that  $f^*(\cdot) = f(\cdot)t_f/\rho_\infty$ . With the choice of  $f$  given below (2.2),  $B = 1/t_f$  and we have

$$f^* = Y_F^{\nu_F} Y_O^{\nu_O} \quad (2.37)$$

We can also define a chemical time  $t_{ch}$  which is a local property of the mixture and is related to  $t_f$  by

$$\frac{1}{t_{ch}} = \frac{w^*}{t_f}$$

so that  $w = \rho_\infty/t_{ch}$ . In our example then

$$\frac{1}{t_{ch}} = \frac{f^* e^{-\frac{1}{\epsilon T}}}{t_f}.$$

The quantity  $1/t_{ch}$  is the local rate of production of species. Our definition of the Damkohler number is that it is the ratio of the flow time of the translational motion  $a_0/U_{\infty 0}$  to the ambient collision time  $t_f$ .

The rate of mass, momentum, and energy transfer are made dimensionless with respect to reference values by putting

$$\begin{aligned} \mathcal{M} &= 4\pi\rho_\infty\kappa_\infty a_0 M, & \vec{\mathcal{F}} &= 6\pi\rho_\infty\nu_\infty U_{\infty 0} a_0 \mathbf{F}, \\ \mathcal{E}_a &= 4\pi\rho_\infty\kappa_\infty a_0 c_p T_\infty E_a, & \mathcal{E}_\infty &= 4\pi\rho_\infty\kappa_\infty a_0 c_p T_\infty E_\infty, \end{aligned} \quad (2.38)$$

where  $M, \mathbf{F}, E_a$ , and  $E_\infty$  are dimensionless. The reference value for the drag force is the (leading order, i.e., Stokes) drag for isothermal, viscous flow past a sphere at low Reynolds number. Thus

$$\begin{aligned} M &= \frac{\text{Pe}}{4\pi} \int_{S_a} \rho(\mathbf{v} - \bar{w} \frac{d\mathbf{a}}{dt}) \cdot \mathbf{n} dS, \\ \mathbf{F} &= \frac{-1}{6\pi} \int_{S_a} \text{Re} \rho \mathbf{v}(\mathbf{v} - \bar{w} \frac{d\mathbf{a}}{dt}) \cdot \mathbf{n} + \frac{\text{Re}}{\text{Ma}^2} p \mathbf{n} - \boldsymbol{\sigma} \cdot \mathbf{n} dS, \\ E_a &= \frac{-1}{4\pi} \int_{S_a} \left( \text{Pe} \rho(\mathbf{v} - \bar{w} \frac{d\mathbf{a}}{dt}) T - \lambda \nabla T \right) \cdot \mathbf{n} dS, \\ E_\infty &= \frac{1}{4\pi} \int_{S_\infty} (\text{Pe} \rho \mathbf{v} T - \lambda \nabla T) \cdot \mathbf{e}_r dS, \end{aligned} \quad (2.39)$$

where  $\text{Pe} = \text{Re Pr}$  is the Peclet number.

The governing equations in the form given above do not admit exact solutions, but our aim is to construct approximate solutions in a limit relevant to values of the nondimensional parameters (2.33) that are typical of applications under many circumstances. The translational Reynolds number  $\text{Re}$  is typically small, since for much of its lifetime the drop moves at small velocity relative to the ambient free stream. The reaction is assumed to be highly activated, so that the inverse nondimensional activation energy  $\epsilon$  is also small. Since the flow near the drop is dominated

by a diffusion flame, the gas speed there is small relative to the ambient sound speed, and hence the Mach number  $Ma$  is small. In this case we find that in the equation for conservation of energy (2.6), the kinetic energy density  $(\partial_t + \mathbf{v} \cdot \nabla)(\mathbf{v} \cdot \mathbf{v}/2)$  is always small and of order  $O(Ma^2)$  relative to the thermal energy density. The term is therefore neglected in the analysis that follows. Similarly, since the Prandtl number is typically of order  $O(1)$ , the viscous effect of Joule heating in the energy equation  $\nabla \cdot (\mathbf{v} \cdot \sigma)$  is always small and it too is neglected.

Dropping the asterisk  $*$  on nondimensional quantities, we obtain the following system of nondimensional conservation equations

$$\bar{\omega} \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \quad (2.40)$$

$$\rho \left( \bar{\omega} \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) \mathbf{v} = -\frac{1}{Ma^2} \nabla p + \frac{1}{Re} \nabla \cdot \sigma \quad (2.41)$$

$$\rho \left( \bar{\omega} \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) Y_i = \frac{1}{RePrLe_i} \nabla \cdot (\rho D_i \nabla Y_i) - \nu_i W_i Da f \exp\left(-\frac{1}{\epsilon T}\right) \quad i = F, O \quad (2.42)$$

$$\rho \left( \bar{\omega} \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) T = \frac{1}{RePr} \nabla \cdot (\lambda \nabla T) + q Da f \exp\left(-\frac{1}{\epsilon T}\right) \quad (2.43)$$

$$p = \rho T. \quad (2.44)$$

The nondimensional stress tensor  $\sigma$  of (2.41) is given by formally the same expression as (2.7) but now all variables are dimensionless and the Stokes relation for zero bulk viscosity implies that (with the asterisk dropped)  $\hat{\lambda} = 1$ . The vorticity equation, for constant molecular and bulk viscosities, is

$$\left( \bar{\omega} \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) \left( \frac{\vec{\omega}}{\rho} \right) - \left( \frac{\vec{\omega}}{\rho} \cdot \nabla \right) \mathbf{v} = \nabla \frac{1}{\rho} \wedge \left( \bar{\omega} \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) \mathbf{v} + \frac{1}{Re} \frac{1}{\rho^2} \nabla^2 \vec{\omega}. \quad (2.45)$$

The nondimensional boundary conditions at the droplet surface  $\mathbf{x} = \mathbf{a}$  are given by

$$\rho D_F \nabla Y_F \cdot \mathbf{n} = -RePrLe_F \rho (1 - Y_F) \left( \mathbf{v} - \bar{\omega} \frac{d\mathbf{a}}{dt} \right) \cdot \mathbf{n} \quad (2.46)$$

$$\rho D_O \nabla Y_O \cdot \mathbf{n} = \text{RePrLe}_O \rho Y_O (\mathbf{v} - \bar{\omega} \frac{d\mathbf{a}}{dt}) \cdot \mathbf{n} \quad (2.47)$$

$$\lambda \nabla T \cdot \mathbf{n} = \text{RePr}l \rho (\mathbf{v} - \bar{\omega} \frac{d\mathbf{a}}{dt}) \cdot \mathbf{n} \quad (2.48)$$

$$T = T_s \quad (2.49)$$

$$\mathbf{v} \wedge \mathbf{n} = \mathbf{0}. \quad (2.50)$$

At infinity, the nondimensional boundary conditions are

$$(T, Y_F, Y_O, \mathbf{v}, \rho, p) \rightarrow (1, 0, Y_{O\infty}, v_\infty \hat{\mathbf{k}}, 1, 1) \quad (2.51)$$

where  $\hat{\mathbf{k}} = \cos \theta \mathbf{e}_r - \sin \theta \mathbf{e}_\theta$  in spherical polar coordinates, so that  $\hat{\mathbf{k}}$  and  $\theta = 0$  have the same, i.e., upward vertical, direction and sense.

In dimensionless terms, the relation (2.22) for conservation of mass between the liquid and gas phases at the drop interface becomes

$$-\alpha_l a^2 \frac{da}{dt} = M \quad \text{where} \quad \alpha_l = \bar{\omega} \text{Pe} \frac{\rho_l}{\rho_\infty} \quad (2.52)$$

and  $M$  is defined in equation (2.40). Similarly, if the higher order effect of correction of the drop weight for buoyancy is omitted and the dimensionless drag is written  $\mathbf{F} = F \hat{\mathbf{k}}$ , the relation (2.23) for the force balance on the drop becomes

$$\beta_g a^3 - \frac{9}{2} \text{Pr} F = \alpha_l \frac{d}{dt} (a^3 v_\infty) \quad \text{where} \quad \beta_g = \frac{\text{Pe}}{\text{Fr}} \frac{\rho_l}{\rho_\infty}. \quad (2.53)$$

In the analysis that follows we consider the limit in which the inverse nondimensional activation energy  $\epsilon$  and the translational Reynolds number  $\text{Re}$  are small and of the same order, so that effects of finite rate chemistry and translation relative to the free stream are comparable. Thus

$$\text{Re} \sim \epsilon. \quad (2.54)$$

The drop radius  $a$  decreases with time as fuel evaporates from its surface to maintain the reaction, and this is a source of unsteady effects. The relations (2.52) and (2.53) for the mass and momentum balance of the drop both imply a time scale where the Strouhal number  $\bar{\omega}$ , the density ratio  $\rho_\infty/\rho_l$ , and the Peclet number  $Pe$  are such that the grouping  $\alpha_l$  is of order one as  $Re$  and  $\epsilon$  tend to zero. Since the Prandtl number is also of order one this can be written

$$\frac{\rho_\infty}{\rho_l} \sim \bar{\omega} Re. \quad (2.55)$$

We shall assume, however, that this is the only source of unsteady effects. That is, *all other* terms containing time derivatives in the governing equations (2.40) to (2.44) and boundary conditions (2.46) to (2.51) are negligible. This is consistent with a specific choice of scaling relations or asymptotic limit: the Strouhal number  $\bar{\omega}$  is zero under completely steady circumstances and not zero but small under conditions of evolution on a long time scale. If we consider the order of the Strouhal number (relative to  $Re$  and  $\epsilon$ ) to increase from zero, so as to include unsteady effects as a perturbation of a steady idealization, then time derivatives in the governing equations and boundary conditions would first appear in the Oseen approximation of the far field (Chapter 5). Consideration of the size of these terms shows that they can be neglected, with the far field thus quasisteady and time derivatives retained only in equations (2.52) and (2.53), provided

$$\bar{\omega} = o(Re). \quad (2.56)$$

Inclusion of gravity via the drop weight in the drop momentum balance (2.53) implies that the grouping  $\beta_g$  defined there must be of order one. Dividing the definition of  $\beta_g$  by the definition of  $\alpha_l$  in (2.52), we find that the Froude number  $Fr$  is therefore such that

$$Fr \sim \frac{1}{\bar{\omega}}. \quad (2.57)$$

Although we do not present the necessary details, it can be seen by including a gravitational body force in the gas phase momentum balance (i.e., by including a term  $\rho g \hat{\mathbf{k}}$  on the right hand side of equation (2.4)) that (i) in the limit of nearly adiabatic burning which we consider here, for which temperature and density variations outside the reaction sheet are everywhere small, and (ii) to the order of calculation given, it is consistent to neglect gravity in the gas phase momentum balance while retaining it in the drop momentum balance.

To summarize the scaling relations of the last three paragraphs, we consider the limit in which  $Re$  and  $\epsilon$  tend to zero with

$$Re \sim \epsilon, \quad \bar{\omega} \sim \frac{1}{Fr} = o(Re), \quad \text{and} \quad \frac{\rho_\infty}{\rho_l} \sim \bar{\omega} Re = o(Re^2). \quad (2.58)$$

Although temperature changes across the flow field can be sufficiently large for significant variations in the molecular transport coefficients to occur, we set the nondimensional diffusion coefficients  $\rho D_O$ ,  $\rho D_F$ ,  $\lambda$ , and  $\mu$  equal to one, that is, equal to their ambient values at infinity.

### CHAPTER 3

#### THE NEAR FIELD AT LEADING (ZEROTH) ORDER

The mixture velocity in the near field region is nearly spherically symmetric, i.e. nearly radial, and is of a different order of magnitude to the slow translational velocity far from the drop. Near the drop the flow field is dominated by convection and diffusion of vaporizing fuel which migrates outwards from the drop towards a diffusion flame and oxidizer which migrates inwards by diffusion from the ambient free stream at infinity. The near field velocity scale is therefore based on the speed of thermal diffusion, termed the diffusional thermal velocity,  $\kappa_\infty/a_0$ , and its ratio with the translational speed  $U_{\infty 0}$  is the Peclet number  $Pe = RePr = U_{\infty 0}a_0/\kappa_\infty \ll 1$ .

Temperature variations are typically large in this region, and it includes a thin reaction layer or inner region of intense chemical activity where the gaseous fuel and oxidizer meet and react at the reaction temperature  $T = T_*$ . The narrowness of the reaction layer relative to other length scales such as the initial drop radius is implied by the smallness of the inverse nondimensional activation energy  $\epsilon$ . If  $T_s$  is the temperature at the surface of the drop, which is equal to the fuel's vaporization temperature, then necessarily  $T_s < T_*$  so that heat is transferred from the reaction layer to the drop surface, providing the latent heat of vaporization for the fuel. The reaction layer is referred to as a diffusion flame, and separates the source of each reactant species, irrespective of its particular regime or mode of combustion as determined by the Damköhler number and other parameters.

Within the near field but away from the reaction layer and on either side of it, there are two outer regions where the flow is effectively free of reaction, since either the gas is in a state of chemical equilibrium (one of the reactant species being absent) or the temperature is too low to maintain chemical activity (termed 'frozen flow'). Analysis of the reaction layer is given in the next chapter, but we note here that since effects of chemical reaction are confined to this narrow region it acts as a

source of heat and a sink of reactants. Thus, at leading order all dependent variables are continuous across the reaction layer, while the temperature and species mass fractions have a jump in their first normal derivatives. A schematic description of the flow field is given in Figure 1.

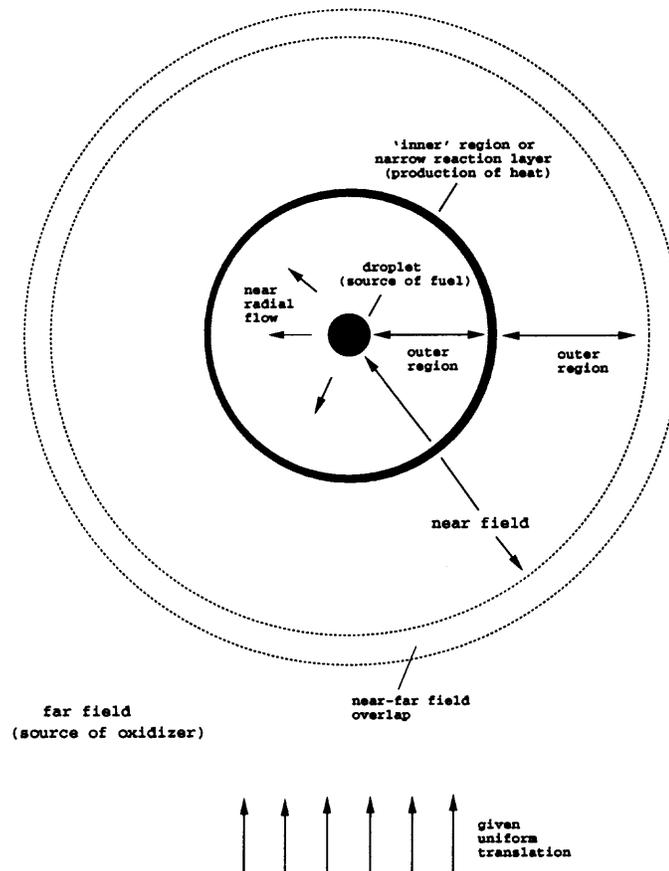


Figure 1. Schematic representation of the drop, flame, and flow field

The dimensionless mass burning rate, or drop mass flux, is defined in (2.40) as

$$M = \frac{\text{RePr}}{4\pi} \int_{S_a} \rho(\mathbf{v} - \bar{\omega} \frac{d\mathbf{a}}{dt}) \cdot \mathbf{n} dS. \quad (3.1)$$

Here, the velocity scale is that used for the original nondimensionalization, i.e.  $U_{\infty 0}$ . In order to capture the physics of the near field region and retain convection, diffusion, and reaction effects at leading order we rescale the velocity field  $\mathbf{v}(r, \theta)$  to the diffusional thermal velocity by setting

$$\mathbf{v}(r, \theta) = \frac{1}{\text{RePr}} \tilde{\mathbf{v}}(r, \theta), \quad (3.2)$$

The vorticity and viscous stress are also rescaled by putting

$$\vec{\omega} = \frac{1}{\text{RePr}} \tilde{\vec{\omega}} \quad \text{and} \quad \sigma_{ij}(r, \theta) = \frac{1}{\text{RePr}} \tilde{\sigma}_{ij}(r, \theta) \quad (3.3)$$

so that  $\vec{\omega} = \nabla \wedge \tilde{\mathbf{v}}$  and  $\tilde{\sigma}_{ij}(r, \theta)$  is given by

$$\tilde{\sigma}_{ij} = \left( \frac{\partial \tilde{v}_i}{\partial x_j} + \frac{\partial \tilde{v}_j}{\partial x_i} \right) - \frac{2\hat{\lambda}}{3} (\nabla \cdot \tilde{\mathbf{v}}) \delta_{ij}. \quad (3.4)$$

The rescaling for the near field velocity then gives

$$M = \frac{1}{4\pi} \int_{S_a} \rho(\tilde{\mathbf{v}} - \bar{\omega} \text{Pe} \frac{d\mathbf{a}}{dt}) \cdot \mathbf{n} dS, \quad (3.5)$$

where, since  $\bar{\omega} \text{Pe}$  is sufficiently small, the term in the regression rate can be neglected to the order of calculation given.

When the rescalings (3.2) and (3.3) are substituted in the governing equations (2.40) to (2.43) for steady flow with constant molecular transport coefficients, the governing equations in the near field are

$$\nabla \cdot (\rho \tilde{\mathbf{v}}) = 0 \quad (3.6)$$

$$\rho \tilde{\mathbf{v}} \cdot \nabla T = \nabla^2 T + q \text{DaRePr} f(\cdot) \exp\left(\frac{-1}{\epsilon T}\right) \quad (3.7)$$

$$\rho \tilde{\mathbf{v}} \cdot \nabla Y_i = \frac{1}{\text{Le}_i} \nabla^2 Y_i - \nu_i W_i \text{DaRePr} f(\cdot) \exp\left(\frac{-1}{\epsilon T}\right), \quad i = \text{F, O} \quad (3.8)$$

$$\rho \tilde{\mathbf{v}} \cdot \nabla \tilde{\mathbf{v}} = - \left( \frac{\text{RePr}}{\text{Ma}} \right)^2 \nabla p + \text{Pr} \nabla \cdot \tilde{\sigma}. \quad (3.9)$$

$$\rho T = p. \quad (3.10)$$

The vorticity equation is now

$$(\tilde{\mathbf{v}} \cdot \nabla) \left( \frac{\tilde{\omega}}{\rho} \right) - \left( \frac{\tilde{\omega}}{\rho} \cdot \nabla \right) \tilde{\mathbf{v}} = \nabla \left( \frac{1}{\rho} \right) \wedge (\tilde{\mathbf{v}} \cdot \nabla) \tilde{\mathbf{v}} + \frac{\text{Pr}}{\rho^2} \nabla^2 \tilde{\omega}. \quad (3.11)$$

If the gas mixture velocity is written in terms of its components as  $\tilde{\mathbf{v}} = u\mathbf{e}_r + v\mathbf{e}_\theta$ , then since the drop is assumed to remain spherical, the boundary conditions on the drop surface  $r = a$  are

$$T = T_s \quad (3.12)$$

$$\frac{\partial T}{\partial r} = l\rho u \quad (3.13)$$

$$\frac{\partial Y_F}{\partial r} = -\text{Le}_F \rho u (1 - Y_F) \quad (3.14)$$

$$\frac{\partial Y_O}{\partial r} = \text{Le}_O \rho u Y_O \quad (3.15)$$

$$v = 0. \quad (3.16)$$

Boundary conditions at infinity are implied by asymptotic matching as  $r \rightarrow \infty$  with a separate far field solution.

The dependent variables  $T$ ,  $Y_i (i = F, O)$ ,  $\tilde{\mathbf{v}}$ ,  $\rho$ , and  $p$  have only radial dependence at leading order in a parameter expansion for small  $\epsilon$  and  $\text{Re}$ , so that dependence on the meridional angle  $\theta$  occurs as a higher order effect. Thus, the flow field and flame sheet are spherically symmetric at leading order. In the near field and away from the reaction layer we introduce the following expansions with

respect to the parameter  $\epsilon \ll 1$ .

$$T = T_0(r) + \epsilon T_1(r, \theta) + \dots \quad (3.17)$$

$$Y_i = Y_{i0}(r) + \epsilon Y_{i1}(r, \theta) + \dots \quad i = F, O \quad (3.18)$$

$$\tilde{\mathbf{v}} = u\mathbf{e}_r + v\mathbf{e}_\theta = u_0(r)\mathbf{e}_r + \epsilon(u_1(r, \theta)\mathbf{e}_r + v_1(r, \theta)\mathbf{e}_\theta) + \dots \quad (3.19)$$

$$\rho = \rho_0(r) + \epsilon \rho_1(r, \theta) + \dots \quad (3.20)$$

$$p = 1 + \left(\frac{\text{Ma}}{\text{RePr}}\right)^2 (p_0 + \epsilon p_1(r, \theta) + \dots). \quad (3.21)$$

At leading order, the continuity equation (3.6) implies that  $\frac{d}{dr}(r^2 \rho_0 u_0) = 0$ .

Integrating this equation and applying an expansion

$$M = M_0 + \epsilon M_1 + \dots \quad (3.22)$$

for the drop mass flux (3.5) we obtain  $r^2 \rho_0 u_0 = M_0$ , which holds throughout the near field since  $\rho_0$  and  $u_0$  are continuous across the reaction layer. Using this result in the remaining equations at leading order we find the leading order equations

$$r^2 \rho_0 u_0 = M_0 \quad (3.23)$$

$$M_0 \frac{dT_0}{dr} - \frac{d}{dr} \left( r^2 \frac{dT_0}{dr} \right) = 0 \quad (3.24)$$

$$M_0 \frac{dY_{i0}}{dr} - \frac{1}{\text{Le}_i} \frac{d}{dr} \left( r^2 \frac{dY_{i0}}{dr} \right) = 0 \quad (3.25)$$

$$\frac{dp_0}{dr} = -\frac{M_0}{r^2} \frac{du_0}{dr} + 2\text{Pr} \left(1 - \frac{\hat{\lambda}}{3}\right) \frac{d}{dr} \left( \frac{1}{r^2} \frac{d}{dr} (r^2 u_0) \right) \quad (3.26)$$

$$\rho_0 T_0 = 1. \quad (3.27)$$

At this order the boundary conditions on  $r = a_0$ , expressed in terms of  $M_0$ , are

$$T_0 = T_{s0} \quad (3.28)$$

$$\frac{dT_0}{dr} = \frac{lM_0}{a^2} \quad (3.29)$$

$$\frac{dY_{F0}}{dr} = -\frac{M_0 \text{Le}_F}{a^2} (1 - Y_{F0}) \quad (3.30)$$

$$\frac{dY_{O0}}{dr} = \frac{M_0 \text{Le}_O}{a^2} Y_{O0} \quad (3.31)$$

Matching conditions with the far field solution imply that the dependent variables approach their ambient free stream values, that is,

$$(T_0, Y_{F_0}, Y_{O_0}, u_0, \rho_0, p_0) \rightarrow (1, 0, Y_{O_\infty}, 0, 1, 0) \quad (3.32)$$

as  $r \rightarrow \infty$ .

‘Nearly adiabatic burning’ is the term used to describe the case where the transfer of thermal energy between the reaction layer and the ambient far field is small, of order  $O(\epsilon)$ . This is the special limiting case we consider here, and it is such that the flame temperature  $T = T_*$  is close to the ambient far field temperature,  $T = 1$ , with  $T_* - 1 = O(\epsilon)$ . In this case, we see from analysis of the reaction layer that the flow field must be in a state of chemical equilibrium in the region outside the reaction layer and in a state of frozen flow inside the reaction layer. These conditions correspond to the ‘premixed combustion regime’ of diffusion flames for all values of the Damkohler number and other parameters, with no leakage of fuel through reaction layer. The temperature and mass fractions of oxidizer and fuel at a well-defined point within the reaction layer can be introduced as unknown parameters with expansions of the form

$$T_* = 1 + \epsilon T_{*1} + \dots \quad (3.33)$$

$$Y_{F*} = \epsilon Y_{F*1} + \dots \quad (3.34)$$

$$Y_{O*} = Y_{O*0} + \epsilon Y_{O*1} + \dots \quad (3.35)$$

$$\text{at } r = r_* = r_{*0} + \epsilon r_{*1}(\theta) + \dots$$

Continuity of the leading order solutions for the temperature and species mass fractions across the reaction layer therefore implies that

$$T_0(r_{*0}^\pm) = 1, \quad Y_{F_0}(r_{*0}^\pm) = 0, \quad Y_{O_0}(r_{*0}^\pm) = Y_{O*0}, \quad (3.36)$$

as  $r \rightarrow r_{*0}$  in both left and right hand limits.

We note that in the limit of nearly adiabatic burning the system simplifies on the domain  $r_* < r < \infty$ , since  $T_0(r) = 1$  there. We now consider solution of the leading order system (3.24) to (3.32) with the continuity conditions and parameterization (3.36).

For  $a_0 < r < r_*$ . The equations (3.24) and (3.25) for  $T_0$  and  $Y_{i0}$ , together with the boundary conditions (3.28) to (3.31) at  $r = a$  and the conditions of (3.36) that  $Y_{F_0}(r_{*0}) = 0$  and  $Y_{O_0}(r_{*0}) = Y_{O_{*0}}$ , give

$$T_0 = (T_{s0} - l) + le^{M_0(\frac{1}{a} - \frac{1}{r})} \quad (3.37)$$

$$Y_{F_0} = 1 - e^{-Le_F M_0(\frac{1}{r} - \frac{1}{r_{*0}})} \quad (3.38)$$

$$Y_{O_0} = Y_{O_{*0}} e^{-Le_O M_0(\frac{1}{r} - \frac{1}{r_{*0}})}. \quad (3.39)$$

The condition of (3.36), that  $T_0(r_{*0}) = 1$ , then implies that the leading order mass flux  $M_0$  is given in terms of  $r_{*0}$  and other parameters (i.e.,  $a$ ,  $T_{s0}$ , and  $l$ ) by

$$M_0 = \frac{\ln\left(1 + \frac{1 - T_{s0}}{l}\right)}{\frac{1}{a} - \frac{1}{r_{*0}}}. \quad (3.40)$$

The mixture density  $\rho_0$  and velocity  $u_0$  can now be found from the equation of state (3.27) and equation of continuity (3.23) and written in terms of  $T_0$  as

$$\rho_0 = \frac{1}{T_0} \quad u_0 = \frac{M_0}{r^2} T_0, \quad (3.41)$$

while the pressure  $p_0$  is given by

$$p_0 = \frac{-M_0^2(T_{s0} - l)}{2r^4} - \frac{le^{M_0(\frac{1}{a} - \frac{1}{r})}}{M_0^2} \left( \left\{1 - 2\text{Pr}\left(1 - \frac{\hat{\lambda}}{3}\right)\right\} \left(\frac{M_0}{r}\right)^4 + 2\left(\frac{M_0}{r}\right)^3 + 6\left(\frac{M_0}{r}\right)^2 + 12\left(\frac{M_0}{r}\right) + 12 \right) + c_{p0} \quad (3.42)$$

where  $c_{p0}$  is a constant.

For  $r_{*0} < r < \infty$ . The energy equation (3.7) with the boundary and continuity conditions (3.32) and (3.36) implies that  $T_0(r) = 1$  in the limit of nearly adiabatic burning. The system therefore simplifies in this region, and the solution is given by

$$T_0 = 1, \quad Y_{F_0} = 0, \quad (3.43)$$

$$Y_{O_0} = \frac{1}{1 - e^{-\frac{Le_O M_0}{r_{*0}}}} \left( Y_{O_{*0}} - Y_{O_\infty} e^{-\frac{Le_O M_0}{r_{*0}}} + (Y_{O_\infty} - Y_{O_{*0}}) e^{-\frac{Le_O M_0}{r}} \right), \quad (3.44)$$

and

$$\rho_0 = 1, \quad u_0 = \frac{M_0}{r^2}, \quad p_0 = -\frac{M_0^2}{2r^4} + c_{p0+}. \quad (3.45)$$

The constant  $c_{p0+}$  will later be shown by matching to be zero. It is also useful to anticipate the expression for  $Y_{O_{*0}}$  in terms of  $M_0$  and  $r_{*0}$  which is found later at (4.17), where (3.44) is simplified to become

$$Y_{O_0} = -\frac{\nu_o W_O}{\nu_F W_F} + \left( Y_{O_\infty} + \frac{\nu_o W_O}{\nu_F W_F} \right) e^{-\frac{Le_O M_0}{r}}. \quad (3.46)$$

Since the velocity field is axisymmetric (i.e.,  $\mathbf{v}$  lies in the meridional plane spanned by  $\mathbf{e}_r$  and  $\mathbf{e}_\theta$  and is independent of the azimuthal angle  $\phi$ ) the vorticity vector lies in the azimuthal direction. We therefore set

$$\vec{\omega} \equiv \nabla \wedge \vec{\mathbf{v}} = \tilde{\omega} \mathbf{e}_\phi, \quad (3.47)$$

where

$$\tilde{\omega} = \frac{1}{r} \left( \frac{\partial(rv)}{\partial r} - \frac{\partial u}{\partial \theta} \right). \quad (3.48)$$

The departure of the near field velocity from spherical symmetry occurs at order  $O(\epsilon)$ , so that the vorticity is at most of order  $O(\epsilon)$ , with expansion

$$\tilde{\omega} = \epsilon \omega_1 + \epsilon^2 \omega_2 + \dots, \quad \text{where} \quad \omega_i = \frac{1}{r} \left( \frac{\partial(rv_i)}{\partial r} - \frac{\partial u_i}{\partial \theta} \right) \quad i = 1, 2, \dots \quad (3.49)$$

In the next chapter, analysis of the reaction layer provides jump conditions to be satisfied by the first derivatives of the leading order near field solution together with jump conditions which are to be satisfied by the solution at the next order. Consideration of the vorticity expedites this analysis.

## CHAPTER 4

### THE REACTION LAYER

We denote values of variables evaluated at a well-defined point within the reaction layer by an asterisk (\*) subscript. The point can be described as the center of the reaction layer and is denoted by  $r = r_*$ , which depends on  $\theta$ ,  $\epsilon$  and other parameters, and has an expansion  $r_* = r_{*0} + \epsilon r_{*1}(\theta) + \dots$ . In the previous chapter we stated and used the fact that the solutions for the temperature and species mass fractions in the near field outer regions are continuous across the reaction layer at leading order; we also stated that across the layer these leading order solutions have a jump in their first derivative, while all components of the solution at the next order,  $O(\epsilon)$ , have a finite jump discontinuity across the layer. These results are derived in the course of the analysis below.

To analyze the structure of the reaction layer, we introduce a new local radial coordinate  $\xi$  as independent variable, together with local dependent variables, defined by

$$\begin{aligned}
 r &= r_* + \epsilon \xi = r_{*0} + \epsilon(r_{*1} + \xi) + \dots \\
 T &= 1 - \epsilon \theta_1 - \epsilon^2 \theta_2 + \dots \\
 Y_F &= \epsilon y_{F1} + \epsilon^2 y_{F2} + \dots \\
 Y_O &= Y_{O_{*0}} + \epsilon y_{O1} + \epsilon^2 y_{O2} + \dots \\
 \tilde{\mathbf{v}} &= u_{*0} \mathbf{e}_r + \epsilon(u_1 \mathbf{e}_r + v_1 \mathbf{e}_\theta) + \dots \\
 \rho &= 1 + \epsilon \rho_1 + \dots \\
 p &= 1 + \left( \frac{\text{Ma}}{\text{RePr}} \right)^2 (p_0 + \epsilon p_1 + \dots).
 \end{aligned} \tag{4.1}$$

The vorticity therefore has local expansion

$$\omega = \epsilon \omega_1 + \epsilon^2 \omega_2 + \dots \tag{4.2}$$

Where ambiguity may occur because symbols  $u_1, v_1, \rho_1, p_i$ , and  $\omega_i$  are used to refer to quantities in both the reaction layer and in the near field outer regions the argument  $\xi$  or  $r$  will be included to distinguish between the two.

**Leading order analysis.** We make the change of independent variable from  $r$  to  $\xi$  and substitute the local expansions into the near field equations (3.6) to (3.10) of Chapter 3 using the form (2.37) for dependence of the rate term, i.e.,  $f(\cdot) = Y_F^{\nu_F} Y_O^{\nu_O}$ , and introduce a scaled Damkohler number  $\bar{D}a$  defined by

$$\bar{D}a = e^{-\frac{1}{\epsilon}} \epsilon^{\nu_F+1} Da Re Pr, \quad (4.3)$$

which will be found to be of order  $O(1)$ .

The energy equation (3.7) and equation for the species mass fractions (3.8) at leading order imply that

$$\frac{\partial^2 \theta_1}{\partial \xi^2} = q \bar{D}a Y_{O_*0}^{\nu_O} y_{F_1}^{\nu_F} e^{-\theta_1} \quad (4.4)$$

$$\frac{\partial^2 y_{i_1}}{\partial \xi^2} = (Le_i \nu_i W_i) \bar{D}a Y_{O_*0}^{\nu_O} y_{F_1}^{\nu_F} e^{-\theta_1} \quad i = F, O. \quad (4.5)$$

Asymptotic matching with the near field outer solution to either side of the reaction layer requires that  $\theta_1$  and  $y_{F_1}$  have the behavior

$$\theta_1 = \begin{cases} -T'_0(r_{*0}^-) \xi - (r_{*1} T'_0(r_{*0}^-) + T_1(r_{*0}^-, \theta)) + o(1) & \xi \rightarrow -\infty \\ -T_1(r_{*0}^+, \theta) + o(1) & \xi \rightarrow \infty \end{cases} \quad (4.6)$$

$$y_{F_1} = \begin{cases} Y_{F_0}'(r_{*0}^-) \xi + r_{*1} Y_{F_0}'(r_{*0}^-) + Y_{F_1}(r_{*0}^-, \theta) + o(1) & \xi \rightarrow -\infty \\ o(1) & \xi \rightarrow \infty \end{cases} \quad (4.7)$$

where a prime ( $\prime$ ) is used to denote a partial derivative with respect to  $r$ . In writing these 'matching conditions' we have used the fact that in the leading order outer solution for  $r > r_{*0}$ ,  $T_0(r) = 1$ , following the assumption of nearly adiabatic burning, and  $Y_F = 0$  to all algebraic orders, so that there is no leakage of fuel through the

reaction layer. The second of these results is seen (to the order of calculation given) to be a necessary condition for existence of a solution in the nearly adiabatic limit. Since there is no general simplification of the near field outer solution for  $Y_O$ , we have

$$y_{O_1} = Y_{O_0}'(r_{*0}^\pm)\xi + r_{*1}Y_{O_0}'(r_{*0}^\pm) + Y_{O_1}(r_{*0}^\pm, \theta) + o(1) \quad \text{as } \xi \rightarrow \pm\infty. \quad (4.8)$$

We note that asymptotic matching between the reaction layer and near field shows the continuity of  $T_0$  and  $Y_{i_0}$  at  $r = r_{*0}$ , that is, per (3.36),  $T_0(r_{*0}^\pm) = 1$ ,  $Y_{F_0}(r_{*0}^\pm) = 0$ , and  $Y_{O_0}(r_{*0}^\pm) = Y_{O_{*0}}$ .

**Local Shvab-Zeldovich variables.** We eliminate the reaction rate between equations (4.4) and (4.5) to find that the local Shvab-Zeldovich variables

$\theta_1 - \frac{q}{Le_i\nu_i W_i} y_{i_1}$ , ( $i = F, O$ ) satisfy

$$\frac{d^2}{d\xi^2} \left( \theta_1 - \frac{q}{Le_i\nu_i W_i} y_{i_1} \right) = 0 \quad i = F, O. \quad (4.9)$$

Hence

$$\theta_1 - \frac{q}{Le_i\nu_i W_i} = A_{i_1} + B_{i_1}\xi \quad i = F, O, \quad (4.10)$$

where  $A_{i_1}$  and  $B_{i_1}$  are constants, i.e. independent of  $\xi$ . Expressions for  $A_{i_1}$  and  $B_{i_1}$  are given by applying the asymptotic matching conditions (4.6) to (4.8) as  $\xi \rightarrow \infty$  and, separately, as  $\xi \rightarrow -\infty$ . Equating the two expressions that are found for each of  $A_{i_1}$  and  $B_{i_1}$ , we find the first of our jump conditions. These relate the jump in the derivatives of the leading order near field outer solutions for  $T_0$  and  $Y_{i_0}$  across the reaction layer, and a jump in the solution at the next order, namely,

$$\left[ \frac{dT_0}{dr} \right]_{r_{*0}} + \frac{q}{Le_i\nu_i W_i} \left[ \frac{dY_{i_0}}{dr} \right]_{r_{*0}} = 0 \quad (4.11)$$

and

$$[T_1]_{r_{*0}} + \frac{q}{Le_i\nu_i W_i} [Y_{i_1}]_{r_{*0}} = 0. \quad (4.12)$$

The square bracket notation is used to denote the difference between right and left hand limits, i.e.,  $[f]_{r_{*0}} \equiv \lim_{r \rightarrow r_{*0}^+} f - \lim_{r \rightarrow r_{*0}^-} f$ .

The specific form of the leading order solution for the near field was given in Chapter 3, and is now used to determine relations between the parameters that are implied by (4.11). The derivatives in the right-hand limit  $r \rightarrow r_{*0}^+$  are

$$\frac{dT_0}{dr}|_{r_{*0}^+} = 0, \quad \frac{dY_{F_0}}{dr}|_{r_{*0}^+} = 0, \quad \frac{Y_{O_0}}{dr}|_{r_{*0}^+} = \frac{Le_O M_0}{r_{*0}^2} \left( \frac{Y_{O_\infty} - Y_{O_{*0}}}{e^{\frac{Le_O M_0}{r_{*0}}} - 1} \right), \quad (4.13)$$

and in the left-hand limit  $r \rightarrow r_{*0}^-$

$$\frac{dT_0}{dr}|_{r_{*0}^-} = \frac{M_0}{r_{*0}^2} (1 - T_{s_0} + l), \quad \frac{dY_{F_0}}{dr}|_{r_{*0}^-} = -\frac{Le_F M_0}{r_{*0}^2}, \quad \frac{dY_{O_0}}{dr}|_{r_{*0}^-} = \frac{Le_O M_0 Y_{O_{*0}}}{r_{*0}^2}. \quad (4.14)$$

Substituting these values in (4.11) with  $i=F$  we obtain

$$1 - T_{s_0} + l = \frac{q}{\nu_F W_F} \quad (4.15)$$

and with  $i=O$ , using this last relation, we obtain

$$Y_{O_{*0}} = -\frac{\nu_O W_O}{\nu_F W_F} + \left( Y_{O_\infty} + \frac{\nu_O W_O}{\nu_F W_F} \right) e^{-\frac{Le_O M_0}{r_{*0}}}. \quad (4.16)$$

These two relations can be used to simplify some of the expressions for the leading order near field solution, and in particular we shall find it useful to rewrite (3.44) for the oxidizer mass fraction  $Y_{O_0}(r)$  outside the reaction layer as

$$Y_{O_0} = -\frac{\nu_O W_O}{\nu_F W_F} + \left( Y_{O_\infty} + \frac{\nu_O W_O}{\nu_F W_F} \right) e^{-\frac{Le_O M_0}{r}}, \quad r_{*0} < r < \infty. \quad (4.17)$$

Further discussion of (4.15) and (4.16) appears in the section below on the leading order response relations.

The local Shvab-Zeldovich variables allow the mass fractions  $y_{i_1}$  within the reaction layer to be written in terms of  $\theta_1$ . From evaluation of the constants  $A_{i_1}$  and  $B_{i_1}$  as  $\xi \rightarrow \infty$  we have

$$y_{F_1} = \frac{Le_F \nu_F W_F}{q} (\theta_1 + T_1(r_{*0}^+, \theta)) \quad (4.18)$$

$$y_{O_1} = \frac{Le_O \nu_O W_O}{q} (\theta_1 + T_1(r_{*0}^+, \theta)) + Y_{O_0}'(r_{*0}^+) \xi + r_{*1} Y_{O_0}'(r_{*0}^+) + Y_{O_0}(r_{*0}^+, \theta) \quad (4.19)$$

**The structure equation for the temperature.** Equation (4.18) can now be used to eliminate  $y_{F_1}$  from the energy equation (4.4), which on defining

$$\Lambda = q\bar{D}aY_{O_{*0}}\nu_o \left( \frac{Le_F\nu_F W_F}{q} \right)^{\nu_F} \quad (4.20)$$

gives the following two point boundary value problem to be satisfied by  $\theta_1$ :

$$\frac{d^2\theta_1}{d\xi^2} = \Lambda(\theta_1 + T_1(r_{*0}^+, \theta))^{\nu_F} e^{-\theta_1} \quad \xi \in (-\infty, \infty) \quad (4.21)$$

subject to the boundary conditions (4.6), which we repeat here for convenience,

$$\theta_1 = \begin{cases} -T_0'(r_{*0}^-)\xi - (r_{*1}T_0'(r_{*0}^-) + T_1(r_{*0}^-, \theta)) + o(1) & \xi \rightarrow -\infty \\ -T_1(r_{*0}^+, \theta) + o(1) & \xi \rightarrow \infty \end{cases} \quad (4.22)$$

Fuel leakage across the reaction layer, as represented by a non-zero fuel mass fraction at order  $O(\epsilon)$  in the outer region for  $r > r_{*0}$ , would change the right hand side of equation (4.21) by adding a term  $Y_{F_1}(r_{*0}^+, \theta) \neq 0$  to the factor  $\theta_1 + T_1(r_{*0}^+, \theta)$  on the right hand side. However, the boundary condition (4.22) as  $\xi \rightarrow \infty$  would remain unchanged, and a simple phase plane argument shows that the boundary value problem would then have no solution. This confirms the absence of fuel leakage at this order.

Equation (4.21) can be integrated once, and on applying the boundary condition as  $\xi \rightarrow \infty$  gives the first integral

$$\left( \frac{d\theta_1}{d\xi} \right)^2 = 2\Lambda e^{T_1(r_{*0}^+, \theta)} \gamma(\nu_F + 1, \theta_1 + T_1(r_{*0}^+, \theta)) \quad (4.23)$$

where  $\gamma(a, z) = \int_0^z s^{a-1} e^{-s} ds$  is the incomplete gamma function. On also applying the boundary condition as  $\xi \rightarrow -\infty$  the value of  $\Lambda$  is determined by

$$\left( \frac{dT_0}{dr} \Big|_{r_{*0}^-} \right)^2 = 2\Lambda e^{T_1(r_{*0}^+, \theta)} \Gamma(\nu_F + 1), \quad (4.24)$$

where  $\Gamma(a) = \gamma(a, \infty)$  is the gamma function. This last result shows that:

(i) The parameter  $\Lambda$ , which is an eigenvalue of the two point boundary value problem

(4.21) and (4.22), is of order  $O(1)$  since all other quantities in (4.24) are of order  $O(1)$ . Hence,  $\bar{D}a$  as defined by (4.3) is also of order  $O(1)$ .

(ii)  $T_1(r_{*0}^+, \theta)$ , which is the correction to the near field temperature immediately outside the reaction layer, is independent of the meridional angle  $\theta$ , since all other quantities in (4.24) are independent of  $\theta$ . We therefore omit the second argument and write  $T_1(r_{*0}^+, \theta)$  as  $T_1(r_{*0}^+)$ .

When (4.14) and (4.15) are used to evaluate  $T_0'(r_{*0}^-)$  from the outer solution for  $r < r_{*0}$ , we find that (4.24) becomes

$$\frac{M_0^2}{r_{*0}^4} = 2 \left( \frac{\nu_F W_F}{q} \right)^2 \Lambda e^{T_1(r_{*0}^+)} \Gamma(\nu_F + 1). \quad (4.25)$$

**The leading order response relations.** We now collate the dimensionless equations that provide the leading order response relations or solution branches for the diffusion flame. Since the translational flow in the far field is slow and the departure of the near field solution from spherical symmetry is a higher order effect, these are the response relations for the spherically symmetric problem, which are well-known and in the nearly adiabatic limit have been given in [3].

The location  $r = r_{*0}$  of the reaction layer and the drop mass flux  $M_0$  are the principal unknowns to be determined from the response relations. However, it is convenient to consider the response relations in parametric form with  $r_{*0}$  as parameter. From equation (3.40) and (4.15), the drop mass flux or burning rate is given in terms of  $r_{*0}$  by

$$M_0 = \frac{\ln \left( \frac{q}{\nu_F W_F} \right)}{\frac{1}{a_0} - \frac{1}{r_{*0}}}, \quad (4.26)$$

so that  $M_0$  is a monotone decreasing function of  $r_{*0}$ . The oxidizer mass fraction at the layer,  $Y_{O,*0}$ , is given in (4.16) by

$$Y_{O,*0} = -\frac{\nu_O W_O}{\nu_F W_F} + \left( Y_{O,\infty} + \frac{\nu_O W_O}{\nu_F W_F} \right) e^{-\frac{Le_O M_0}{r_{*0}}}, \quad (4.27)$$

and thus  $Y_{O_{*0}}$  is a monotone increasing function of  $r_{*0}$ . Since a mass fraction is necessarily non-negative, (4.27) implies that there is a minimum critical radius of the reaction layer  $r_{*0c}$  for which  $Y_{O_{*0}} = 0$ , given by

$$r_{*0c} = a_0 \left( 1 + \text{Le}_O \frac{\ln \left( \frac{q}{l\nu_F W_F} \right)}{\ln \left( 1 + Y_{O_\infty} \frac{\nu_F W_F}{\nu_O W_O} \right)} \right). \quad (4.28)$$

This minimum critical radius is strictly greater than the drop radius.

A relation for  $\bar{D}a$  in terms of  $r_{*0}$  is given by combining (4.25) with the definition (4.20). To see this, we anticipate the result (shown later, in Chapter 6 at (6.53) and (6.56)) that  $T_1(r_{*0}^+)$  is given by

$$T_1(r_{*0}^+) = T_{s1} (1 - e^{-\frac{M_0}{r_{*0}}}), \quad (4.29)$$

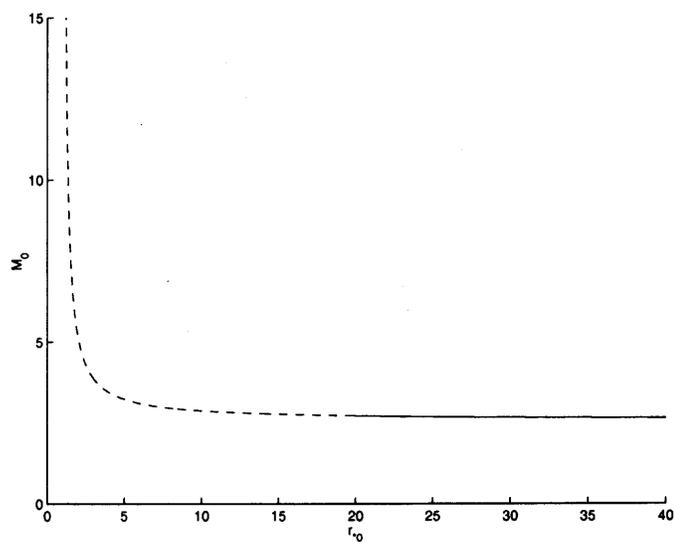
where  $T_{s1}$  is the correction to the drop vaporization temperature  $T_s$  at order  $O(\epsilon)$  and is known. Thus, on introducing a reduced Damkohler number  $\hat{D}a$  defined by

$$\ln \hat{D}a = \ln \bar{D}a + \ln \left( 2\Gamma(\nu_F + 1) \frac{\text{Le}_F^{\nu_F} (\nu_F W_F^{\nu_F+2})}{q^{\nu_F+1}} \right) \quad (4.30)$$

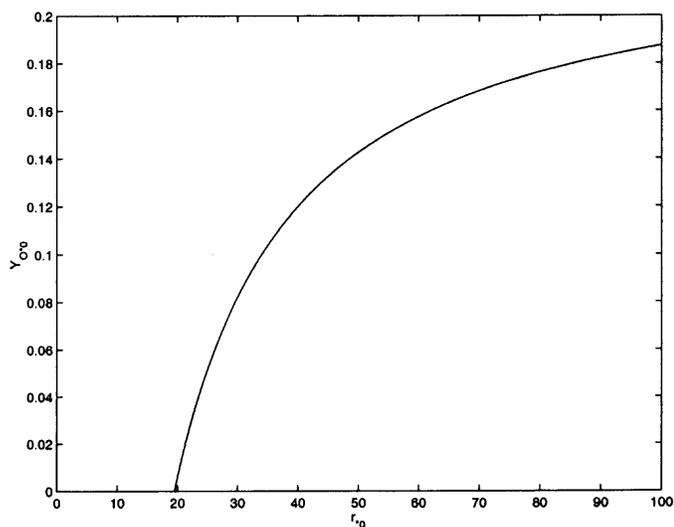
we have

$$\ln \hat{D}a = -T_{s1} (1 - e^{-\frac{M_0}{r_{*0}}}) + \ln \left( \frac{M_0^2}{r_{*0}^4 Y_{O_{*0}} \nu_O} \right). \quad (4.31)$$

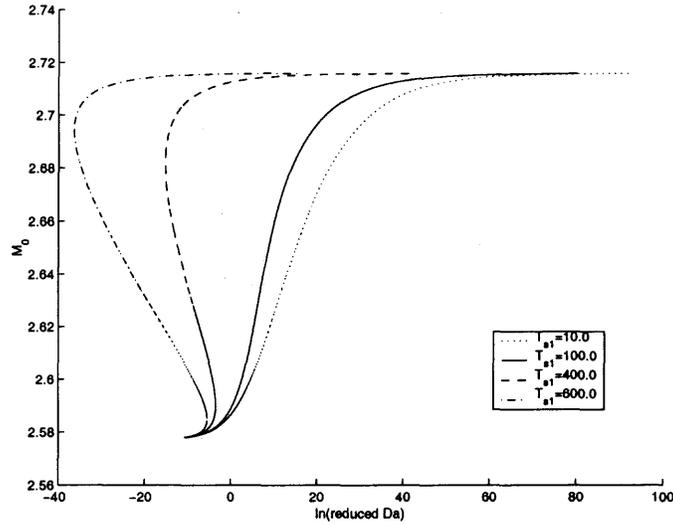
The solution branches are now defined parametrically for  $r_{*0} \in [r_{*0c}, \infty)$  by (4.26), (4.27), and (4.31). Figure 2(a-c) shows graphs of (a)  $M_0$  versus  $r_{*0}$ , (b)  $Y_{O_{*0}}$  versus  $r_{*0}$ , and (c)  $M_0$  versus  $\ln \hat{D}a$  for different values of  $T_{s1}$ . The drop radius is taken as fixed at its initial value  $a = 1$ , and values of the physico-chemical parameters are relevant to n-heptane as the liquid fuel. Note that the response for  $M_0$  and  $r_{*0}$  is multivalued over an interval of the Damkohler number when  $T_{s1}$  is positive and sufficiently large, but is monotone otherwise. In practice it is the Damkohler number,  $\bar{D}a$  or  $\hat{D}a$ , that is known, from known values of the physico-chemical and other parameters, and this determines the flame sheet radius  $r_{*0}$ .



**Figure 2a.** The steady state response relations, with fixed dimensionless drop radius  $a = 1$  and data for n-heptane fuel. Leading order mass flux  $M_0$  versus leading order flame sheet radius  $r_{*0}$ . The part of the curve for  $r_{*0}$  less than the critical minimum value  $r_{*0c}$  can not be attained, and is shown dashed.



**Figure 2b.** Leading order oxidizer mass fraction at the flame sheet  $Y_{O_{*0}}$  versus leading order flame sheet radius  $r_{*0}$ .  $Y_{O_{*0}} = 0$  at the critical minimum value  $r_{*0} = r_{*0c}$ .



**Figure 2c.** Mass flux  $M_0$  versus  $\ln \hat{D}a$  where  $\hat{D}a$  is the reduced Damkohler number. Response curves are shown for different values of the scaled vaporization temperature  $T_{s1}$ . The response is multivalued over an interval of the Damkohler number when  $T_{s1}$  is sufficiently large and positive.  $\hat{D}a$  is an increasing function of the initial dimensional drop radius  $a_0$ .

**Analysis for the temperature and mass fractions at order epsilon.** Construction of local Shvab-Zeldovich variables at this order leads to an additional two jump conditions, which we now derive.

When the expansions (4.2) are substituted into the energy equation (3.7) and the equation for the species mass fractions (3.8) we find at order  $O(\epsilon)$  that  $\theta_2$  and  $y_{i2}$  ( $i=F, O$ ) satisfy

$$\frac{M_0}{r_{*0}^2} \frac{\partial \theta_1}{\partial \xi} = \frac{\partial^2 \theta_2}{\partial \xi^2} + \frac{2}{r_{*0}} \frac{\partial \theta_1}{\partial \xi} - q \mathcal{R}_1 \quad (4.32)$$

$$\frac{M_0}{r_{*0}^2} \frac{\partial y_{i1}}{\partial \xi} = \frac{1}{Le_i} \left( \frac{\partial^2 y_{i2}}{\partial \xi^2} + \frac{2}{r_{*0}} \frac{\partial y_{i1}}{\partial \xi} \right) - \nu_i W_i \mathcal{R}_1, \quad i = F, O \quad (4.33)$$

where we have used the fact that the leading order mass flux at  $r = r_{*0}$  is constant and equal to  $M/r_{*0}^2$  and  $\mathcal{R}_1$  is the expansion of the reaction rate term at this order, the form of which will not concern us here.

Eliminating the rate term  $\mathcal{R}_1$  between equations (4.32) and (4.33), we find after simple manipulation that

$$\frac{\partial^2}{\partial \xi^2} \left( \theta_2 - \frac{q}{\text{Le}_i \nu_i W_i} y_{i2} \right) + \left( \frac{2}{r_{*0}} - \frac{M_0 \text{Le}_i}{r_{*0}^2} \right) \frac{\partial}{\partial \xi} \left( \theta_1 - \frac{q}{\text{Le}_i \nu_i W_i} y_{i1} \right) + \frac{M_0}{r_{*0}^2} (\text{Le}_i - 1) \frac{\partial}{\partial \xi} \theta_1 = 0, \quad (4.34)$$

where the first group in parentheses is the Shvab-Zeldovich variable at order  $O(\epsilon)$  and the second group is the Shvab-Zeldovich variable at order  $O(1)$  of equation (4.10). Equation (4.34) can be integrated once to give

$$\frac{\partial}{\partial \xi} \left( \theta_2 - \frac{q}{\text{Le}_i \nu_i W_i} y_{i2} \right) + \left( \frac{2}{r_{*0}} - \frac{M_0 \text{Le}_i}{r_{*0}^2} \right) \left( \theta_1 - \frac{q}{\text{Le}_i \nu_i W_i} y_{i1} \right) + \frac{M_0}{r_{*0}^2} (\text{Le}_i - 1) \theta_1 = B_{i2} \quad (4.35)$$

where  $B_{i2}$  is a constant, independent of  $\xi$ .

We now apply asymptotic matching conditions between the solution in the reaction layer and the outer solution to either side. These were given above for  $\theta_1$  and  $y_{i1}$  in (4.6) to (4.8), and, without reference to any specific properties of the outer solution, matching implies that  $\theta_2$  has the behavior

$$\theta_2 = -T_0''(r_{*0}^\pm) \frac{\xi^2}{2} - (r_{*1} T_0''(r_{*0}^\pm) + T_1'(r_{*0}^\pm, \theta)) \xi + O(1) \quad \text{as } \xi \rightarrow \pm\infty \quad (4.36)$$

with a similar expression for  $y_{i2}$ . As  $\xi \rightarrow \pm\infty$  the matching relations suggest at first that the left hand side of (4.35) behaves linearly in  $\xi$ , but the coefficient of  $\xi$  is seen to vanish from the differential equations (3.24) and (3.25) governing the leading order outer solution. Equating the two expressions that are found for the constant  $B_{i2}$  as  $\xi \rightarrow \infty$  and, separately, as  $\xi \rightarrow -\infty$ , and noting simplifications implied by (3.24), (3.25), and (4.12), we find the jump conditions

$$\left[ \frac{\partial T_1}{\partial r} \right]_{r_{*0}} + \frac{q}{\text{Le}_i \nu_i W_i} \left[ \frac{\partial Y_{i1}}{\partial r} \right]_{r_{*0}} + \frac{M_0}{r_{*0}^2} (\text{Le}_i - 1) [T_1]_{r_{*0}} = 0 \quad i = \text{F, O}. \quad (4.37)$$

### Analysis for the gas velocity, density, and pressure.

**Leading order analysis.** The continuity equation implies that the leading order gas mass flux is independent of  $\xi$ , and matching with the outer solution shows that the leading order gas density and velocity are both constant across the reaction layer with values 1 and  $u_{*0} = M_0/r_{*0}^2$  respectively. This establishes continuity of the density and radial velocity of the leading order outer solutions at  $r = r_{*0}$ , which was used in Chapter 3.

At leading order  $O(\epsilon^{-1})$  and within the reaction layer, the radial component of the momentum equation implies that

$$\frac{\partial p_0}{\partial \xi} = 2\text{Pr}\left(1 - \frac{\hat{\lambda}}{3}\right) \frac{\partial^2 u_1}{\partial \xi^2} \quad (4.38)$$

and the meridional component implies

$$\frac{\partial^2 v_1}{\partial \xi^2} = 0. \quad (4.39)$$

The solution in the reaction layer is subject to matching conditions with the solution in the near field outer region, which are readily seen to be

$$u_1(\xi, \theta) = u'_0(r_{*0}^\pm)\xi + r_{*1}u'_0(r_{*0}^\pm) + u_1(r_{*0}^\pm, \theta) + o(1) \quad \text{as } \xi \rightarrow \pm\infty \quad (4.40)$$

$$v_1(\xi, \theta) = v_1(r_{*0}^\pm, \theta) + o(1) \quad \text{as } \xi \rightarrow \pm\infty \quad (4.41)$$

$$p_0(\xi, \theta) = p_0(r_{*0}^\pm) + o(1) \quad \text{as } \xi \rightarrow \pm\infty. \quad (4.42)$$

Integration of (4.38) once with respect to  $\xi$  and application of the matching conditions shows that within the reaction layer

$$p_0(\xi, \theta) - 2\text{Pr}\left(1 - \frac{\hat{\lambda}}{3}\right) \frac{\partial u_1}{\partial \xi}(\xi, \theta) = \text{constant, independent of } \xi \text{ and } \theta \quad (4.43)$$

and that the outer solution satisfies the jump condition

$$[p_0(r)]_{r_{*0}} = 2\text{Pr}\left(1 - \frac{\hat{\lambda}}{3}\right) \left[ \frac{du_0}{dr} \right]_{r_{*0}}. \quad (4.44)$$

This last result shows that there is a jump in the near field pressure across the diffusion flame at the order of the Mach number squared, as found earlier in [18,19]. Integration of (4.39) twice with respect to  $\xi$  and application of the matching conditions (4.41) shows that within the reaction layer  $v_1$  is independent of  $\xi$  and that the outer solution satisfies the jump condition

$$[v_1(r, \theta)]_{r_{*0}} = 0. \quad (4.45)$$

**Analysis for the velocity, density, and pressure at the next order.** At order  $O(1)$  the continuity equation implies that within the reaction layer

$$\frac{\partial}{\partial \xi} \left( r_{*0}^2 u_1 + M_0 \rho_1 + \frac{2M_0}{r_{*0}} (r_{*1} + \xi) \right) = 0, \quad (4.46)$$

where we have used the leading order values of the density, 1, and radial velocity,  $u_{*0} = M_0/r_{*0}^2$ . The equation of state implies that, provided  $(Ma/Re)^2 = o(\epsilon)$ ,

$$\rho_1 = \theta_1. \quad (4.47)$$

On eliminating  $\theta_1$  between these equations and integrating once with respect to  $\xi$  we have

$$r_{*0}^2 u_1(\xi, \theta) + M_0 \theta_1(\xi, \theta) + \frac{2M_0}{r_{*0}} (r_{*1} + \xi) = \text{function of } \theta \text{ alone, independent of } \xi. \quad (4.48)$$

The matching conditions with the outer region, (4.6) and (4.40), suggest at first that the left hand side of (4.48) behaves linearly in  $\xi$  as  $\xi \rightarrow \pm\infty$ . However, from (3.41), (3.43), and (3.45) of Chapter 3, the leading order outer solution is such that the coefficient of  $\xi$  vanishes. Equating the limiting values of the left hand side that are implied by matching as  $\xi \rightarrow \infty$  and, separately, as  $\xi \rightarrow -\infty$ , we find that the outer solution satisfies

$$r_{*1} \left[ \frac{du_0}{dr} \right]_{r_{*0}} + [u_1]_{r_{*0}} - \frac{M_0}{r_{*0}^2} \left( r_{*1} \left[ \frac{dT_0}{dr} \right]_{r_{*0}} + [T_1]_{r_{*0}} \right) = 0. \quad (4.49)$$

A useful simplification of this results on recalling that the same relations from Chapter 3 used above imply that  $[u'_0]_{r_{*0}} = (M_0/r_0^2)[T'_0]_{r_{*0}}$ , so that the jump condition becomes

$$[u_1(r, \theta)]_{r_{*0}} - \frac{M_0}{r_{*0}^2}[T_1(r, \theta)]_{r_{*0}} = 0. \quad (4.50)$$

Equation (4.50) implies that there is no jump in the radial component of the mass flux across the reaction layer at order  $O(\epsilon)$ . To see this, note that in the outer regions  $\rho_1 \rightarrow -T_1$  as  $r \rightarrow r_{*0}^\pm$ , which follows from the equation of state at order  $O(\epsilon)$ , equation (6.6) of Chapter 6, and that the leading order density and radial velocity are continuous across the layer with values 1 and  $M_0/r^2$ , as stated after equation (4.46). We can therefore equate the jump in radial mass flux correction  $[\rho_0 u_1 + u_0 \rho_1]$  with the left hand side of (4.50). Alternatively, since overall mass is conserved in the mixture and the flow is steady, the normal mass fluxes entering and leaving the reaction layer are equal at successive orders of the  $\epsilon$ -parameter expansion. Since the mass flux and normal to the reaction layer are given by

$$\rho \mathbf{v} = (\rho_0 u_0 + \epsilon(\rho_0 u_1 + u_0 \rho_1) + \dots) \mathbf{e}_r + (\epsilon \rho_0 v_1 + \dots) \mathbf{e}_\theta \quad (4.51)$$

$$\hat{\mathbf{n}} \equiv \frac{\mathbf{e}_r + \frac{1}{r} \frac{dr}{d\theta} \mathbf{e}_\theta}{\sqrt{1 + \left(\frac{1}{r} \frac{dr}{d\theta}\right)^2}} = (1 + O(\epsilon^2)) \mathbf{e}_r + \epsilon \left( \frac{1}{r_{*0}} \frac{dr_{*1}}{d\theta} + O(\epsilon^2) \right) \mathbf{e}_\theta, \quad (4.52)$$

the meridional velocity contributes to the normal flux at order  $O(\epsilon^2)$ , and conservation at order  $O(\epsilon)$  leads again to (4.50).

Consideration of the vorticity facilitates derivation of two additional jump conditions. The vorticity equation (3.11) implies that within the reaction layer the leading order local vorticity  $\omega_1(\xi, \theta)$  satisfies

$$\frac{\partial^2 \omega_1}{\partial \xi^2} = 0, \quad (4.53)$$

and is subject to matching conditions with the near field outer solution, that

$$\omega_1(\xi, \theta) = \omega_1(r_{*0}^\pm, \theta) + o(1) \quad \text{as } \xi \rightarrow \pm\infty. \quad (4.54)$$

Integration of (4.53) twice with respect to  $\xi$  and application of the matching conditions shows that within the reaction layer  $\omega_1$  is independent of  $\xi$  and that the outer solution satisfies the jump condition

$$[\omega_1(r, \theta)]_{r_{*0}} = 0, \quad (4.55)$$

that is, the near field vorticity is continuous across the reaction layer to leading order.

Using this result, we find at the next order that  $\omega_2(\xi, \theta)$  satisfies

$$\frac{\partial^2 \omega_2}{\partial \xi^2} = 0, \quad (4.56)$$

and is subject to the matching conditions with the near field outer solution, that

$$\omega_2(\xi, \theta) = \omega_1'(r_{*0}^\pm, \theta)\xi + (r_{*1}\omega_1'(r_{*0}^\pm, \theta) + \omega_2(r_{*0}^\pm, \theta)) + o(1) \quad \text{as } \xi \rightarrow \pm\infty. \quad (4.57)$$

Integration of (4.56) with respect to  $\xi$  and application of the matching conditions shows that

$$\left[ \frac{\partial \omega_1}{\partial r}(r, \theta) \right]_{r_{*0}} = 0, \quad (4.58)$$

i.e., the radial component of the gradient of the near field vorticity is also continuous across the reaction layer to leading order.

The jump conditions (4.55) and (4.58) can be written in terms of the near field velocity and its derivatives at order  $O(\epsilon)$ , and this is used below. If we recall the relations (3.49) and (4.45), namely

$$\omega_1 = \frac{1}{r} \left( \frac{\partial(rv_1)}{\partial r} - \frac{\partial u_1}{\partial \theta} \right) \quad \text{and} \quad [v_1]_{r_{*0}} = 0, \quad (4.59)$$

then the first condition, (4.55), becomes

$$r_{*0} \left[ \frac{\partial v_1}{\partial r} \right]_{r_{*0}} - \left[ \frac{\partial u_1}{\partial \theta} \right]_{r_{*0}} = 0. \quad (4.60)$$

When  $\frac{\partial \omega_1}{\partial r}$  is written in terms of  $u_1$  and  $v_1$ , the second condition (4.58) can be simplified to

$$r_{*0} \left[ \frac{\partial^2 v_1}{\partial r^2} \right]_{r_{*0}} + 2 \left[ \frac{\partial v_1}{\partial r} \right]_{r_{*0}} - \left[ \frac{\partial^2 u_1}{\partial r \partial \theta} \right]_{r_{*0}} = 0. \quad (4.61)$$

Both of these jump conditions can be found from the radial and meridional components of the momentum equation at order  $O(1)$  but that approach is relatively lengthy. It does, however, provide one additional relation for the jump in the near field outer pressure  $p_1$ , which we give here for completeness.

At order  $O(1)$  the radial component of the momentum equation implies that

$$\frac{M_0}{r_{*0}^2} \frac{\partial u_1}{\partial \xi} = -\frac{\partial p_1}{\partial \xi} + 2\text{Pr}\left(1 - \frac{\hat{\lambda}}{3}\right) \left( \frac{\partial^2 u_2}{\partial \xi^2} + \frac{2}{r_{*0}} \frac{\partial u_1}{\partial \xi} - \frac{2M_0}{r_{*0}^4} \right) \quad (4.62)$$

where we have substituted the known leading order values of the gas density and radial velocity at  $r = r_{*0}$  and used the result that within the layer  $v_1$  is independent of  $\xi$ . Matching with the near field outer solution requires that variables within the layer have asymptotic behavior

$$u_2(\xi, \theta) = u_0''(r_{*0}^\pm) \frac{\xi^2}{2} + (r_{*1} u_0''(r_{*0}^\pm) + u_1'(r_{*0}^\pm, \theta)) \xi + O(1) \quad \text{as } \xi \rightarrow \pm\infty, \quad (4.63)$$

$$p_1(\xi, \theta) = p_0'(r_{*0}^\pm) \xi + r_{*1} p_0'(r_{*0}^\pm) + p_1(r_{*0}^\pm, \theta) + o(1) \quad \text{as } \xi \rightarrow \pm\infty, \quad (4.64)$$

with the asymptotic behavior for  $u_1(\xi, \theta)$  as  $\xi \rightarrow \pm\infty$  as given in (4.40).

Equation (4.62) can be integrated once with respect to  $\xi$  to give

$$p_1 + \frac{M_0}{r_{*0}^2} u_1 - 2\text{Pr}\left(1 - \frac{\hat{\lambda}}{3}\right) \left( \frac{\partial u_2}{\partial \xi} + \frac{2}{r_{*0}} u_1 - \frac{2M_0}{r_{*0}^4} \xi \right) = \text{function of } \theta \text{ alone, independent of } \xi. \quad (4.65)$$

The matching relations for  $u_1$ ,  $u_2$ , and  $p_1$  suggest at first that the left hand side behaves linearly in  $\xi$  as  $\xi \rightarrow \pm\infty$ , but the coefficient of  $\xi$  is seen to vanish from the radial component of the momentum equation in the outer regions at leading order, equation (3.26), evaluated at  $r = r_{*0}^+$  and  $r = r_{*0}^-$ . Next, evaluating the left hand side of (4.65) as  $\xi \rightarrow \infty$  and, separately, as  $\xi \rightarrow -\infty$ , we find that the variables in the outer regions must satisfy the jump condition

$$r_{*1} [p_0']_{r_{*0}} + [p_1]_{r_{*0}} + \frac{M_0}{r_{*0}^2} (r_{*1} [u_0']_{r_{*0}} + [u_1]_{r_{*0}}) - 2\text{Pr}\left(1 - \frac{\hat{\lambda}}{3}\right) \left( r_{*1} [u_0'']_{r_{*0}} + [u_1']_{r_{*0}} + \frac{2}{r_{*0}} (r_{*1} [u_0']_{r_{*0}} + [u_1]_{r_{*0}}) \right) = 0 \quad (4.66)$$

This can be simplified by forming the difference between the left and right hand limits of equation (3.26) as  $r \rightarrow r_{*0}^{\pm}$  and noting that the radial velocity in the outer regions is continuous across the reaction layer at leading order (i.e.,  $[u_0]_{r_{*0}} = 0$ ), which shows that

$$[p'_0]_{r_{*0}} + \frac{M_0}{r_{*0}^2} [u'_0]_{r_{*0}} - 2\text{Pr}(1 - \frac{\hat{\lambda}}{3}) \left( [u''_0]_{r_{*0}} + \frac{2}{r_{*0}} [u'_0]_{r_{*0}} \right) = 0. \quad (4.67)$$

We can now eliminate  $[p'_0]$  between the last two relations to find the jump condition in the pressure  $p_1$ , viz.,

$$[p_1]_{r_{*0}} + \frac{M_0}{r_{*0}^2} [u_1]_{r_{*0}} - 2\text{Pr}(1 - \frac{\hat{\lambda}}{3}) \left( \left[ \frac{\partial u_1}{\partial r} \right]_{r_{*0}} + \frac{2}{r_{*0}} [u_1]_{r_{*0}} \right) = 0. \quad (4.68)$$

**More precise choice of  $r_*$  and summary of jump conditions.** At this point we need to provide a more precise definition of the location of the center of the reaction layer,  $r = r_*$ , or equivalently  $\xi = 0$ . Although the choice is to some extent arbitrary, a definition that appears to be satisfactory is that the point is situated at the intersection of the near field outer solutions for the temperature (when each of these is extended into the domain of the reaction layer). In general, if a function  $F$  of  $r$  has an expansion

$$F(r, \theta; \epsilon) = F_0(r, \theta) + \epsilon F_1(r, \theta) + o(\epsilon) \quad \text{for } r < r_*(\theta; \epsilon), \text{ and for } r > r_*(\theta; \epsilon), \quad (4.69)$$

where

$$r = r_{*0}(\theta) + \epsilon r_{*1}(\theta) + o(\epsilon)$$

and all left and right hand limits exist as  $r \rightarrow r_*^{\pm}(\theta; \epsilon)$ , then equality of the left and right hand limits  $\lim_{r \rightarrow r_*^+(\theta; \epsilon)} F(r, \theta; \epsilon) = \lim_{r \rightarrow r_*^-(\theta; \epsilon)} F(r, \theta; \epsilon)$ , i.e., intersection or transversality of  $F$  to left and right of  $r = r_*$  implies that

$$[F_0]_{r_{*0}} = 0, \quad \text{and} \quad [F_1]_{r_{*0}} + r_{*1} \left[ \frac{\partial F_0}{\partial r} \right]_{r_{*0}} = 0. \quad (4.70)$$

We can conveniently consider  $F$  to consist of two different functions, each being defined to one side of their intersection at  $r = r_*$  and continued analytically to the other side.

From the second of equations (4.70) the choice of  $r_*$  is therefore such that

$$[T_1]_{r_{*0}} + r_{*1} \left[ \frac{dT_0}{dr} \right]_{r_{*0}} = 0 \quad (4.71)$$

When this condition is used to eliminate jumps in the temperature between (4.11) and (4.12), (4.12) becomes

$$[Y_{i1}]_{r_{*0}} + r_{*1} \left[ \frac{dY_{i0}}{dr} \right]_{r_{*0}} = 0 \quad i = F, O. \quad (4.72)$$

The jump  $[T_1]_{r_{*0}}$  can be eliminated between (4.37) and (4.71) to rewrite (4.37) as

$$\left[ \frac{\partial T_1}{\partial r} \right]_{r_{*0}} + \frac{q}{Le_i \nu_i W_i} \left[ \frac{\partial Y_{i1}}{\partial r} \right]_{r_{*0}} - r_{*1} \frac{M_0}{r_{*0}^2} (Le_i - 1) \left[ \frac{dT_0}{dr} \right]_{r_{*0}} = 0 \quad i = F, O. \quad (4.73)$$

For the velocity field, we have (4.45), viz.

$$[v_1]_{r_{*0}} = 0. \quad (4.74)$$

Jumps in the temperature are eliminated from (4.49) by using (4.71) to write (4.49) as

$$[u_1]_{r_{*0}} + r_{*1} \left[ \frac{du_0}{dr} \right]_{r_{*0}} = 0. \quad (4.75)$$

The jump conditions for the vorticity, (4.60) and (4.61), are

$$r_{*0} \left[ \frac{\partial v_1}{\partial r} \right]_{r_{*0}} - \left[ \frac{\partial u_1}{\partial \theta} \right]_{r_{*0}} = 0 \quad (4.76)$$

and

$$r_{*0} \left[ \frac{\partial^2 v_1}{\partial r^2} \right]_{r_{*0}} + 2 \left[ \frac{\partial v_1}{\partial r} \right]_{r_{*0}} - \left[ \frac{\partial^2 u_1}{\partial r \partial \theta} \right]_{r_{*0}} = 0, \quad (4.77)$$

respectively.

The above nine jump conditions, (4.71) to (4.77), will be used later in Chapter 6 to formulate the problem for the correction to the near field solution and reaction layer shape; some further manipulation of the jump conditions will be presented there. If the pressure field is required, then the jump conditions (4.44) and (4.68) are also needed.

We note that although the choice of location of  $r_*$  is to some extent arbitrary, our choice of (4.71) implies from the second of equations (4.70) that up to order  $O(\epsilon)$ ,  $r = r_*$  is situated at the intersection of the near field outer solutions for all dependent variables that are continuous across the reaction layer at leading order, i.e., the temperature, species mass fractions, gas mixture velocity, and density profiles, with the last result following immediately from the equation of state.

## CHAPTER 5

### THE FAR FIELD

Far from the drop, the flow field adjusts to meet the conditions of slow translation through the ambient, reaction-free atmosphere. In this far field region the solution is constructed by linearization about ambient free stream conditions with the translational Reynold's number  $Re = U_{\infty 0} a_0 / \nu_{\infty}$  as expansion parameter, as is familiar in the Oseen approximation. The single length scale that is appropriate to the far field is the viscous length scale  $\nu_{\infty} / U_{\infty 0}$ , as opposed to the near field length scale  $a_0$ , so that we introduce a scaled, far field distance  $\tilde{\mathbf{x}}$  defined by

$$\tilde{\mathbf{x}} = Re \mathbf{x}. \quad (5.1)$$

with associated polar coordinates  $(\tilde{r}, \theta)$ . From the nondimensional governing equations (2.40) to (2.44) of Chapter 2, the rescaled equations governing the far field are

$$\frac{\bar{\omega}}{Re} \frac{\partial \rho}{\partial t} + \tilde{\nabla} \cdot (\rho \mathbf{v}) = 0 \quad (5.2)$$

$$\rho \left( \frac{\bar{\omega}}{Re} \frac{\partial}{\partial t} + \mathbf{v} \cdot \tilde{\nabla} \right) T = \frac{1}{Pr} \tilde{\nabla} \cdot (\lambda \tilde{\nabla} T) \quad (5.3)$$

$$\rho \left( \frac{\bar{\omega}}{Re} \frac{\partial}{\partial t} + \mathbf{v} \cdot \tilde{\nabla} \right) Y_i = \frac{1}{Pr Le_i} \tilde{\nabla} \cdot (\rho D_i \tilde{\nabla} Y_i) \quad (5.4)$$

$$\rho \left( \frac{\bar{\omega}}{Re} \frac{\partial}{\partial t} + \mathbf{v} \cdot \tilde{\nabla} \right) \mathbf{v} = \frac{-1}{Ma^2} \tilde{\nabla} p + \tilde{\nabla} \cdot \sigma \quad (5.5)$$

$$p = \rho T \quad (5.6)$$

where spatial derivatives in the stress tensor  $\sigma$  are taken with respect to the far field variable  $\tilde{\mathbf{x}}$ .

In the Oseen approximation for slow, incompressible, viscous flow past a sphere, far field variations of the gas velocity about its free stream value are of the same order as the translational Reynold's number  $O(\text{Re})$ , while variations in the pressure are of order  $O(\text{ReMa}^2)$ . The same orders of magnitude occur here, and are implied by the quadratic nonlinearity of the advective term in the momentum equation and matching with the near field. It is special to the limit of nearly adiabatic burning, for which temperature variations are small everywhere outside the reaction layer, that far field variations in the temperature are of order  $O(\text{Re}^2)$ , and hence, from the equation of state, density variations are of the same order. Since this is an order of magnitude smaller than the  $O(\text{Re})$  variation in gas velocity, the continuity equation implies that the leading order correction to the free stream velocity is divergence-free, that is, incompressible, as is familiar from the classical problem of slow, viscous flow past a sphere [20].

Since the drop size and weight vary with time, the translational velocity of the free stream relative to the drop,  $v_\infty \hat{\mathbf{k}}$ , varies and is to be found as part of the solution to the problem. The rest-frame of the drop is, in fact, not an inertial frame, but the translation is slow and acceleration of the coordinate frame can be absorbed by adding a correction  $p_{v_\infty}$  to the pressure. Since  $\rho \rightarrow 1$  and  $v \rightarrow v_\infty$  as  $\tilde{r} \rightarrow \infty$ , the momentum equation (5.5) implies that

$$p_{v_\infty} = -\text{Ma}^2 \frac{\bar{\omega}}{\text{Re}} \frac{\partial v_\infty}{\partial t} \tilde{z}. \quad (5.7)$$

Note that the same correction to the pressure occurs in the near field but we have postponed discussion of it until here. However, the correction is of order  $O(\text{Ma}^2 \bar{\omega} / \text{Re})$ , and the orderings introduced at the end of Chapter 1 imply that it is thus  $o(\text{Ma}^2)$  and too small to modify the solution presented, by, for example, its appearance in the equation of state.

The expansion of the solution in the far field for the limit of nearly adiabatic burning is therefore given by setting

$$(T, Y_F, Y_O, \mathbf{v}, \rho, p) = (1, 0, Y_{O_\infty}, v_\infty \hat{\mathbf{k}}, 1, 1 + p_{v_\infty}) \\ + (\text{Re}^2 S, 0, \text{Re } \tilde{Y}_O, \text{Re } \mathbf{V}, \text{Re}^2 R, \text{Ma}^2 \text{Re } P_0) + \dots \quad (5.8)$$

Since the first bracketed term on the right hand side consists of the ambient free stream values, the perturbation quantities of the second term satisfy the boundary condition that each decays to zero as  $\tilde{r} \rightarrow \infty$ . Boundary conditions as  $\tilde{r} \rightarrow 0$  are given by matching with the near field.

Substituting the expansion (5.8) into the far field equations (5.2) to (5.6), we obtain the following linear system of equations governing variations about free stream values.

$$\tilde{\nabla} \cdot \mathbf{V} = 0 \quad (5.9)$$

$$v_\infty \text{Pr} \frac{\partial S}{\partial \tilde{z}} = \tilde{\nabla}^2 S \quad (5.10)$$

$$v_\infty \text{PrLe}_O \frac{\partial \tilde{Y}_O}{\partial \tilde{z}} = \tilde{\nabla}^2 \tilde{Y}_O \quad (5.11)$$

$$v_\infty \frac{\partial \mathbf{V}}{\partial \tilde{z}} = -\tilde{\nabla} P_0 + \tilde{\nabla}^2 \mathbf{V} \quad (5.12)$$

$$R + S = 0 \quad (5.13)$$

where  $\frac{\partial}{\partial \tilde{z}}$  denotes the directional derivative along the freestream vector  $\hat{\mathbf{k}}$ , i.e.

$$\frac{\partial}{\partial \tilde{z}} = \hat{\mathbf{k}} \cdot \tilde{\nabla} = \cos \theta \frac{\partial}{\partial \tilde{r}} - \frac{\sin \theta}{\tilde{r}} \frac{\partial}{\partial \theta}.$$

The equations (5.10) for  $S$  and (5.11) for  $\tilde{Y}_O$  are uncoupled from the system, with  $R = -S$ . In the nearly adiabatic limit, far field variations in oxidizer mass fraction,

temperature, and density are passive scalars which are advected with the free stream velocity  $\hat{\mathbf{k}}$ , while variations in gas velocity and pressure are governed by the equations of the Oseen approximation for incompressible viscous flow. Although a stream function can be introduced to find  $\mathbf{V}$ , the solution procedure here is simpler if we use the Helmholtz vector decomposition theorem to write  $\mathbf{V}$  as the sum of curl-free (longitudinal) and divergence-free (transverse) vectors and introduce a pair of velocity potentials. All dependent variables are then given by solution of either the modified Helmholtz equation or Laplace's equation, via multipole expansion.

To find the velocity field and pressure we write  $\mathbf{V}$  as the sum

$$\mathbf{V} = \mathbf{V}_L + \mathbf{V}_T \quad (5.14)$$

where  $\mathbf{V}_L$  and  $\mathbf{V}_T$  are such that

$$\tilde{\nabla} \wedge \mathbf{V}_L = 0, \quad \tilde{\nabla} \cdot \mathbf{V}_T = 0. \quad (5.15)$$

Equations (5.15) imply the existence of a scalar potential  $\phi$  and a solenoidal vector potential  $\mathbf{A}$  such that

$$\mathbf{V}_L = \tilde{\nabla} \phi \quad (5.16)$$

and

$$\mathbf{V}_T = \tilde{\nabla} \wedge \mathbf{A}, \quad \text{where } \tilde{\nabla} \cdot \mathbf{A} = 0. \quad (5.17)$$

Applying this decomposition of  $\mathbf{V}$  to the continuity equation (5.9), we see that  $\tilde{\nabla} \cdot \mathbf{V} = \tilde{\nabla} \cdot \mathbf{V}_L = 0$ , and since  $\mathbf{V}_L = \tilde{\nabla} \phi$ , the potential  $\phi$  satisfies Laplace's equation

$$\tilde{\nabla}^2 \phi = 0. \quad (5.18)$$

To find the transverse field  $\mathbf{V}_T$  and pressure perturbation  $P_0$ , we substitute the decomposition (5.14) into the momentum equation (5.12) and write all vectors in terms of their curl-free and divergence-free components. The vector identity  $\nabla^2 \mathbf{u} =$

$\nabla(\nabla \cdot \mathbf{u}) - \nabla \wedge \nabla \wedge \mathbf{u}$  applied to  $\mathbf{V}$  shows that, since  $\tilde{\nabla} \cdot \mathbf{V} = 0$  from the continuity equation,  $\tilde{\nabla}^2 \mathbf{V} = -\tilde{\nabla} \wedge \tilde{\nabla} \wedge \mathbf{V}_T$ , which puts the right hand side of (5.12) in the required form. For the left hand side we note that the directional derivative  $\frac{\partial}{\partial \tilde{z}} = \hat{\mathbf{k}} \cdot \tilde{\nabla}$  commutes with the operators grad and curl, so that

$$\frac{\partial \mathbf{V}_L}{\partial \tilde{z}} = (\hat{\mathbf{k}} \cdot \tilde{\nabla}) \tilde{\nabla} \phi = \tilde{\nabla} \left( (\hat{\mathbf{k}} \cdot \tilde{\nabla}) \phi \right) = \tilde{\nabla} \frac{\partial \phi}{\partial \tilde{z}} \quad (5.19)$$

and

$$\frac{\partial \mathbf{V}_T}{\partial \tilde{z}} = (\hat{\mathbf{k}} \cdot \tilde{\nabla}) \tilde{\nabla} \wedge \mathbf{A} = \tilde{\nabla} \wedge (\hat{\mathbf{k}} \cdot \tilde{\nabla} \mathbf{A}) = \tilde{\nabla} \wedge \left( \frac{\partial \mathbf{A}}{\partial \tilde{z}} \right), \quad (5.20)$$

where the first of these expressions is irrotational and the second is solenoidal. By the uniqueness of the Helmholtz decomposition, the irrotational and solenoidal components of equation (5.12) can be equated separately, so that

$$\tilde{\nabla} \left( P_0 + v_\infty \frac{\partial \phi}{\partial \tilde{z}} \right) = 0 \quad (5.21)$$

and

$$v_\infty \frac{\partial \mathbf{V}_T}{\partial \tilde{z}} = \tilde{\nabla}^2 \mathbf{V}_T. \quad (5.22)$$

Equation (5.21) can be integrated straight away, and from the boundary condition  $P_0 \rightarrow 0$  as  $\tilde{r} \rightarrow \infty$  we have

$$P_0 = -v_\infty \frac{\partial \phi}{\partial \tilde{z}}. \quad (5.23)$$

It is well-known that since the transverse component of the velocity field  $\mathbf{V}_T$  satisfies an equation of the form (5.22) and is axisymmetric with axis  $\hat{\mathbf{k}}$  it can be written in terms of a scalar potential  $\chi$ , such that

$$\mathbf{V}_T = -v_\infty \chi \hat{\mathbf{k}} + \tilde{\nabla} \chi. \quad (5.24)$$

A constructive proof for the existence of such a potential  $\chi$  is given in [21]. It follows that  $\chi$  satisfies

$$v_\infty \frac{\partial \chi}{\partial \tilde{z}} - \tilde{\nabla}^2 \chi = 0. \quad (5.25)$$

The velocity and pressure perturbations are therefore given by constructing two scalar potentials; the potential  $\phi$  is a solution of Laplace's equation (5.18), and the potential  $\chi$  satisfies (5.25), in terms of which

$$P_0 = v_\infty \left( -\cos\theta \frac{\partial\phi}{\partial\tilde{r}} + \frac{\sin\theta}{\tilde{r}} \frac{\partial\phi}{\partial\theta} \right) \quad (5.26)$$

and

$$U = \frac{\partial}{\partial\tilde{r}}(\phi + \chi) - v_\infty\chi \cos\theta \quad V = \frac{1}{\tilde{r}} \frac{\partial}{\partial\theta}(\phi + \chi) + v_\infty\chi \sin\theta \quad (5.27)$$

in polar coordinates, where  $\mathbf{V} = U\mathbf{e}_r + V\mathbf{e}_\theta$ . Both potentials tend to zero as  $\tilde{r} \rightarrow \infty$  to satisfy the boundary conditions there, and are to have the requisite behavior given by matching with the near field as  $\tilde{r} \rightarrow 0$ .

We now see that each of  $S$ ,  $Y_0$ , and  $\chi$  satisfies an equation of the same form, i.e.,

$$\alpha \frac{\partial G}{\partial \tilde{z}} - \tilde{\nabla}^2 G = 0 \quad (5.28)$$

and is to tend to zero at infinity. We therefore construct the general decaying solution of (5.28), which can be written in terms of modified spherical Bessel functions as follows. The substitution

$$G = \exp\left(\frac{\alpha\tilde{r}}{2} \cos\theta\right) \bar{G}(\tilde{r}, \theta) \quad (5.29)$$

transforms equation (5.28) into the modified Helmholtz equation

$$\left(\tilde{\nabla}^2 - \frac{\alpha^2}{4}\right) \bar{G} = 0, \quad (5.30)$$

for which a separable solution can be sought via eigenfunction expansion, with

$$\bar{G}(\tilde{r}, \theta) = \hat{G}(\tilde{r})\Phi(\theta). \quad (5.31)$$

Then

$$\frac{1}{\sin\theta} \frac{d}{d\theta} \left( \sin\theta \frac{d\Phi}{d\theta} \right) + p\Phi = 0 \quad (5.32)$$

$$\tilde{r}^2 \frac{d^2 \hat{G}}{d\tilde{r}^2} + 2\tilde{r} \frac{d\hat{G}}{d\tilde{r}} - \left( \left( \frac{\alpha\tilde{r}}{2} \right)^2 + p \right) \hat{G} = 0. \quad (5.33)$$

Equation (5.32) is the Legendre equation, which has bounded solutions for  $\theta \in [0, \pi]$  provided the eigenvalue  $p_n = n(n+1)$  where  $n = 0, 1, \dots$ , with

$$\Phi_n(\theta) = P_n(\cos \theta), \quad n = 0, 1, 2, \dots, \quad (5.34)$$

and  $P_n$  is the Legendre polynomial of degree  $n$ . Equation (5.33) has solutions that are modified spherical Bessel functions. If we put

$$z = \frac{\alpha\tilde{r}}{2} \quad \text{and} \quad \hat{G}(\tilde{r}) = \left( \frac{\pi}{2z} \right)^{\frac{1}{2}} w(z), \quad (5.35)$$

then  $w(z)$  satisfies the modified Bessel equation of order  $n + \frac{1}{2}$ , i.e.,

$$z^2 \frac{d^2 w}{dz^2} + z \frac{dw}{dz} - \left( z^2 + \left( n + \frac{1}{2} \right)^2 \right) w = 0, \quad (5.36)$$

which has linearly independent solutions  $I_{n+\frac{1}{2}}(z)$  and  $K_{n+\frac{1}{2}}(z)$ . The solution for  $\hat{G}(\tilde{r})$  with  $w = I_{n+\frac{1}{2}}(z)$  is analytic at  $\tilde{r} = 0$  but grows exponentially as  $\tilde{r} \rightarrow \infty$ , and is discarded since it does not satisfy the boundary condition of decay there, whereas the solution with  $w = K_{n+\frac{1}{2}}(z)$  is singular at  $\tilde{r} = 0$  and tends to zero exponentially as  $\tilde{r} \rightarrow \infty$ . In terms of elementary functions, together with its series representation about  $z = 0$ ,

$$\begin{aligned} \left( \frac{\pi}{2z} \right)^{\frac{1}{2}} K_{n+\frac{1}{2}}(z) &= \frac{\pi}{2} (-z)^n \left( \frac{1}{z} \frac{d}{dz} \right)^n \left( \frac{e^{-z}}{z} \right) \\ &= \frac{\pi e^{-z}}{(2z)^{n+1}} \frac{(2n)!}{n!} \sum_{j=0}^n \frac{n!(2n-j)!}{(n-j)!(2n)!} \frac{(2z)^j}{j!} \end{aligned}$$

so that, for example

$$K_{\frac{1}{2}}(z) = \left( \frac{\pi}{2z} \right) e^{-z}. \quad (5.37)$$

The general solution of equation (5.28) that satisfies the boundary condition of decay as  $\tilde{r} \rightarrow \infty$  is thus of the form

$$G(\tilde{r}, \theta) = \left( \frac{\pi}{\alpha\tilde{r}} \right)^{\frac{1}{2}} \exp \left( \frac{\alpha\tilde{r}}{2} \cos \theta \right) \sum_{n=0}^{\infty} G_n K_{n+\frac{1}{2}} \left( \frac{\alpha\tilde{r}}{2} \right) P_n(\cos \theta). \quad (5.38)$$

dependent variable	$\alpha$	coefficients
$S$	$v_\infty \text{Pr}$	$S_n$
$\tilde{Y}_O$	$v_\infty \text{PrLe}_O$	$\tilde{Y}_{On}$
$\chi$	$v_\infty$	$\chi_n$

The  $n^{\text{th}}$  term of this series has behavior

$$n^{\text{th}} \text{ term} = \frac{\pi G_n}{(\alpha \tilde{r})^{n+1}} \frac{(2n)!}{n!} (1 + O(\tilde{r})) P_n(\cos \theta) \quad \text{as } \tilde{r} \rightarrow 0 \quad (5.39)$$

and is associated with a  $n^{\text{th}}$  order multipole source (i.e., monopole, dipole, etc.) at the origin  $\tilde{r} = 0$ .

Comparing equations (5.10), (5.11), and (5.25) with equation (5.28) we see that the general decaying solutions for  $S$ ,  $\tilde{Y}_O$ , and  $\chi$  are given by introducing coefficients and setting the value of  $\alpha$  as follows:

The general, decaying solution of Laplace's equation for the potential  $\phi$  is

$$\phi = \sum_{n=0}^{\infty} \frac{\phi_n}{\tilde{r}^{n+1}} P_n(\cos \theta). \quad (5.40)$$

We see below that only the first term in the series expansion is non-zero at the order of calculation, i.e., in the far field, the drop behaves like a point monopole source or sink for temperature, oxidizer, and the transverse and longitudinal velocity potentials.

#### Matching with the solution in the near field

The far field and near field solutions can be matched with an intermediate variable  $\eta$  such that

$$\tilde{r} = \text{Re}r = \delta \eta \quad (5.41)$$

$$\text{where } \eta = O(1) \text{ and } \text{Re} \ll \delta \ll 1. \quad (5.42)$$

Thus, the limit of matching between the far field and near field solutions is such that

$$\text{the far field coordinate } \tilde{r} = \delta\eta \rightarrow 0, \text{ while the near field coordinate } = \frac{\delta\eta}{\text{Re}} \rightarrow \infty. \quad (5.43)$$

Matching the velocity field. In the near field, from the scaling (3.2) and expansion (3.19) of Chapter 3, when written in terms of the intermediate variable  $\eta$  the gas mixture velocity is

$$\begin{aligned} u &= \frac{1}{\text{RePr}} u_0 \left( \frac{\delta\eta}{\text{Re}} \right) + \frac{\epsilon}{\text{RePr}} u_1 \left( \frac{\delta\eta}{\text{Re}}, \theta \right) + \dots \\ v &= \frac{\epsilon}{\text{RePr}} v_1 \left( \frac{\delta\eta}{\text{Re}}, \theta \right) + \dots \end{aligned} \quad (5.44)$$

In the far field, from (5.8) and in terms of  $\eta$ , the gas mixture velocity is

$$\begin{aligned} u &= v_\infty \cos \theta + \text{Re } U(\delta\eta, \theta) + \dots \\ v &= -v_\infty \sin \theta + \text{Re } V(\delta\eta, \theta) + \dots \end{aligned} \quad (5.45)$$

The components  $U$  and  $V$  of (5.45) are known in terms of the potentials  $\phi$  and  $\chi$  from (5.27) and their general solutions as generated from (5.38) and (5.40), which contain the unknown coefficients  $\phi_n$  and  $\chi_n$  for  $n = 0, 1, \dots$ . The most singular term induced by each of these coefficients as  $\tilde{r} \rightarrow 0$  in the far field expansion (5.45) is of order  $O(\text{Re}\delta^{-(n+2)})$ , but terms of this type can appear in the near field expansion (5.44) only for  $n = 0$ . Thus,  $\phi_n = \chi_n = 0$  for  $n \geq 1$ , and in terms of  $\tilde{r}$  the far field solution is

$$\begin{aligned} u &= v_\infty \cos \theta - \text{Re} \left\{ \frac{\phi_0}{\tilde{r}^2} \right. \\ &\quad \left. + \frac{\pi\chi_0}{v_\infty \tilde{r}^2} \exp \left( -\frac{v_\infty \tilde{r}}{2} (1 - \cos \theta) \right) \left( 1 + \frac{v_\infty \tilde{r}}{2} (1 + \cos \theta) \right) \right\} + \dots \quad (5.46) \\ v &= -v_\infty \sin \theta + \text{Re} \frac{\pi\chi_0}{2\tilde{r}} \exp \left( -\frac{v_\infty \tilde{r}}{2} (1 - \cos \theta) \right) \sin \theta + \dots \end{aligned}$$

If we recall the leading order near field solution  $u_0(r) = M_0/r^2$  of (3.45) in Chapter 3, then matching the expressions for the radial velocity in (5.44) and (5.47)

at order  $O(\text{Re}/\delta^2)$  implies that

$$\phi_0 + \frac{\pi\chi_0}{v_\infty} = -\frac{M_0}{\text{Pr}}. \quad (5.47)$$

Matching the radial and meridional components of the velocity at orders  $O(1)$  and  $O(\text{Re}/\delta)$  implies that the near field solutions  $u_1$  and  $v_1$  have the behavior

$$u_1(r, \theta) = \left(\frac{\text{Re}}{\epsilon}\right) \text{Pr} \left(v_\infty - \frac{\pi\chi_0}{r} + o(r^{-1})\right) \cos \theta \quad \text{as } r \rightarrow \infty \quad (5.48)$$

and

$$v_1(r, \theta) = \left(\frac{\text{Re}}{\epsilon}\right) \text{Pr} \left(-v_\infty + \frac{\pi\chi_0}{2r} + o(r^{-1})\right) \sin \theta \quad \text{as } r \rightarrow \infty. \quad (5.49)$$

Matching the temperature. In the near field, from the expansion (3.17) and leading order solution for  $r > r_{*0}$  (3.43) that  $T_0(r) = 1$ , when written in terms of  $\eta$  the temperature is

$$T = 1 + \epsilon T_1 \left(\frac{\delta\eta}{\text{Re}}, \theta\right) + \dots \quad (5.50)$$

In the far field, from (5.8) and in terms of  $\eta$ , the temperature is

$$T = 1 + \text{Re}^2 S(\delta\eta, \theta) + \dots \quad (5.51)$$

The general solution for  $S$  is given by (5.38), which contains the unknown coefficients  $S_n$ . The most singular term induced by  $S_n$  as  $\tilde{r} \rightarrow 0$  in the far field expansion (5.51) is of order  $O(\text{Re}^2 \delta^{-(n+1)})$  for  $n = 0, 1, \dots$ , but terms of this type can appear in the near field expansion (5.50) only for  $n = 0$ . Thus,  $S_n = 0$  for  $n \geq 1$ , and in terms of  $\tilde{r}$  the far field solution is

$$T = 1 + \text{Re}^2 \frac{\pi S_0}{v_\infty \text{Pr} \tilde{r}} \exp\left(-\frac{v_\infty \text{Pr} \tilde{r}}{2}(1 - \cos \theta)\right) + \dots \quad (5.52)$$

Matching the expressions for the temperature in (5.50) and (5.52) at order  $O(\text{Re}^2/\delta)$  implies that the near field temperature correction  $T_1$  has the behavior

$$T_1(r, \theta) = \left(\frac{\text{Re}}{\epsilon}\right) \left(\frac{\pi S_0}{v_\infty \text{Pr} r} + o(r^{-1})\right) \quad \text{as } r \rightarrow \infty. \quad (5.53)$$

Matching the oxidizer mass fraction. In the near field, from the expansion (3.18) of Chapter 3, in terms of  $\eta$  the oxidizer mass fraction is

$$Y_{\text{O}} = Y_{\text{O}_0} \left( \frac{\delta\eta}{\text{Re}} \right) + \epsilon Y_{\text{O}_1} \left( \frac{\delta\eta}{\text{Re}}, \theta \right) + \dots, \quad (5.54)$$

where the solution for  $Y_{\text{O}_0}(r)$  is given by (4.17). In the far field, from (5.8) and in terms of  $\eta$ , the oxidizer mass fraction is

$$Y_{\text{O}} = Y_{\text{O}_\infty} + \text{Re} \tilde{Y}_{\text{O}}(\delta\eta, \theta) + \dots \quad (5.55)$$

The general solution for  $\tilde{Y}_{\text{O}}$  is generated from (5.38), which contains the unknown coefficients  $\tilde{Y}_{\text{O}n}$ . The most singular term induced by  $\tilde{Y}_{\text{O}n}$  as  $\tilde{r} \rightarrow 0$  in the far field expansion (5.55) is of order  $O(\text{Re}\delta^{-(n+1)})$  for  $n = 0, 1, \dots$ , but terms of this type can appear in the near field expansion (5.54) only for  $n = 0$ . Thus,  $\tilde{Y}_{\text{O}n} = 0$  for  $n \geq 1$ , and in terms of  $\tilde{r}$  the far field solution is

$$Y_{\text{O}} = Y_{\text{O}_\infty} + \text{Re} \frac{\pi \tilde{Y}_{\text{O}_0}}{v_\infty \text{PrLe}_O \tilde{r}} \exp \left( -\frac{v_\infty \text{PrLe}_O \tilde{r}}{2} (1 - \cos \theta) \right) + \dots \quad (5.56)$$

Matching the expressions for the oxidizer mass fraction in (5.54) and (5.56) confirms at order  $O(1)$  that  $Y_{\text{O}_0}(r)$  tends to the free stream value  $Y_{\text{O}_\infty}$  as  $r \rightarrow \infty$ . Next, using the known form of  $Y_{\text{O}_0}(r)$  from (4.17) and matching at order  $O(\text{Re}/\delta)$  we find that

$$\tilde{Y}_{\text{O}_0} = -\frac{v_\infty \text{PrLe}_O^2 M_0}{\pi} \left( Y_{\text{O}_\infty} + \frac{\nu_o W_{\text{O}}}{\nu_{\text{F}} W_{\text{F}}} \right), \quad (5.57)$$

and at order  $O(\text{Re})$  the near field correction  $Y_{\text{O}_1}$  has the behavior

$$Y_{\text{O}_1}(r, \theta) = \left( \frac{\text{Re}}{\epsilon} \right) \frac{v_\infty \text{PrLe}_O^2 M_0}{2} \left( Y_{\text{O}_\infty} + \frac{\nu_o W_{\text{O}}}{\nu_{\text{F}} W_{\text{F}}} \right) (1 - \cos \theta) + o(1) \quad \text{as } r \rightarrow \infty. \quad (5.58)$$

Matching the pressure. In the near field, from the expansion (3.21) of Chapter 3, the pressure is given in terms of  $\eta$  by

$$p = 1 + \left( \frac{\text{Ma}}{\text{RePr}} \right)^2 \left( p_0 \left( \frac{\delta\eta}{\text{Re}} \right) + \epsilon p_1 \left( \frac{\delta\eta}{\text{Re}}, \theta \right) + \dots \right), \quad (5.59)$$

where the solution for  $p_0(r)$  is given by (3.45). In the far field, from (5.8) and in terms of  $\eta$ , the pressure is

$$p = 1 + \text{Ma}^2 \text{Re} P_0(\delta\eta, \theta) + \dots, \quad (5.60)$$

where  $P_0 = -v_\infty \frac{\partial \phi}{\partial \bar{z}}$  from (5.23), and  $\phi$  is known up to the constant  $\phi_0$  from the general solution (5.40) and result of matching the velocity field. Thus, in terms of  $\tilde{r}$ , the far field pressure is

$$p = 1 + \text{Ma}^2 \text{Re} \frac{v_\infty \phi_0}{\tilde{r}^2} \cos \theta + \dots \quad (5.61)$$

Since from (3.45) we find that in (5.59)  $p_0(r) = -M_0^2/2r^4 + c_{p0+}$ , matching the near and far field expressions implies at order  $O(\text{Ma}^2/\text{Re}^2)$  that the constant

$$c_{p0+} = 0. \quad (5.62)$$

The constant  $c_{p0-}$  of (3.42) can now be found by using the jump condition (4.44).

At the next order,  $O(\text{Ma}^2 \text{Re}/\delta^2)$ , matching implies that the near field pressure  $p_1(r, \theta)$  of (5.59) has the behavior

$$p_1(r, \theta) = \left( \frac{\text{Re}}{\epsilon} \right) \text{Pr}^2 \frac{v_\infty \phi_0}{r^2} \cos \theta + o(r^{-2}) \quad \text{as } r \rightarrow \infty. \quad (5.63)$$

No new information is obtained by matching the near field and far field expressions for the density  $\rho$ , which is constructed from the temperature profile and equation of state. Also,  $Y_F = 0$  everywhere outside the reaction layer  $r = r_{*0}$ .

**Transfer of energy to infinity.** The integral for the dimensionless rate of energy transfer to infinity,  $E_\infty$ , was given in the last of relations (2.40) of Chapter 1, and is

$$E_\infty = \frac{1}{4\pi} \int_{S_\infty} (\text{Pe} \rho \mathbf{v} T - \lambda \nabla T) \cdot \mathbf{e}_r \, dS. \quad (5.64)$$

The first term in the integrand is the contribution due to advection of heat while the second is the contribution due to conduction, and  $S_\infty$  is the sphere at infinity,  $\tilde{r} = \infty$ .

In the advective contribution, the leading order radial flux is due to the unperturbed free stream and integrates to zero. The correction is due to perturbation from free stream values and, from (5.47), is

$$E_{\infty a} = -\lim_{\tilde{r} \rightarrow \infty} \frac{\text{Pr}}{2} \int_0^\pi \left( \frac{\phi_0}{\tilde{r}^2} + \frac{\pi \chi_0}{v_\infty \tilde{r}^2} \exp\left(-\frac{v_\infty \tilde{r}}{2}(1 - \cos \theta)\right) \left[1 + \frac{v_\infty \tilde{r}}{2}(1 + \cos \theta)\right] \right) \tilde{r}^2 \sin \theta d\theta + O(\text{Re}). \quad (5.65)$$

Here, the component proportional to  $\phi_0$  is due to the longitudinal velocity  $\mathbf{V}_L$  and is distributed uniformly over the sphere, whereas the component proportional to  $\chi_0$  is due to the transverse velocity  $\mathbf{V}_T$  and for large  $\tilde{r}$  is concentrated in the wake  $\theta \simeq 0$ . Evaluating the integral, we find that in the limit

$$E_{\infty a} = -\text{Pr} \left( \phi_0 + \frac{\pi \chi_0}{v_\infty} \right) + O(\text{Re}) = M_0 + O(\text{Re}) \quad (5.66)$$

where the last result follows from the matching relation (5.47). We note, however, that the relative contributions from the longitudinal and transverse components of the velocity field vary with time, and this point will be discussed further in Chapter 7.

The contribution due to conduction of heat, from (5.53), is given by

$$E_{\infty c} = \lim_{\tilde{r} \rightarrow \infty} \text{Re} \frac{S_0}{4\text{Pr}} \int_0^\pi \frac{1}{\tilde{r}^2} \exp\left(-\frac{\text{Pr} v_\infty \tilde{r}}{2}(1 - \cos \theta)\right) \left[1 + \frac{\text{Pr} v_\infty \tilde{r}}{2}(1 - \cos \theta)\right] \tilde{r}^2 \sin \theta d\theta + O(\text{Re}^2). \quad (5.67)$$

In the limit, the integral is zero, so that the leading order transfer of heat to infinity is due to convection alone and is

$$E_\infty = M_0 + O(\text{Re}). \quad (5.68)$$

## CHAPTER 6

### THE NEAR FIELD AT FIRST ORDER

Analysis of the near field beyond leading order gives the correction to the flow field, including the reaction layer shape and the drag exerted on the drop as it translates through the uniform free stream of the far field. The governing equations for the near field were stated in Chapter 3 at (3.6) to (3.10), together with the boundary conditions at the drop surface (3.12) to (3.16); the expansion of the solution in the near field is given at (3.17) to (3.21), and the leading order solution is given in (3.37) to (3.45). The jump conditions across the reaction layer were derived in Chapter 4 and summarized at (4.71) to (4.77), and matching conditions with the far field were found in Chapter 5.

**Governing equations.** At order  $O(\epsilon)$  the governing equations for the near field, away from the reaction layer, are:

Continuity equation

$$\frac{1}{r^2} \frac{\partial}{\partial r} (r^2(\rho_0 u_1 + \rho_1 u_0)) + \frac{\rho_0}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta v_1) = 0 \quad (6.1)$$

Energy equation

$$M_0 \frac{\partial T_1}{\partial r} + r^2(\rho_0 u_1 + \rho_1 u_0) \frac{dT_0}{dr} - \left\{ \frac{\partial}{\partial r} \left( r^2 \frac{\partial T_1}{\partial r} \right) + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial T_1}{\partial \theta} \right) \right\} = 0 \quad (6.2)$$

Species equation,  $i = F, O$ ,

$$M_0 \frac{\partial Y_{i1}}{\partial r} + r^2(\rho_0 u_1 + \rho_1 u_0) \frac{dY_{i0}}{dr} - \frac{1}{Le_i} \left\{ \frac{\partial}{\partial r} \left( r^2 \frac{\partial Y_{i1}}{\partial r} \right) + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial Y_{i1}}{\partial \theta} \right) \right\} = 0 \quad (6.3)$$

Momentum equation

Radial component

$$\begin{aligned} \frac{M_0}{r^2} \frac{\partial u_1}{\partial r} + (\rho_0 u_1 + \rho_1 u_0) \frac{du_0}{dr} &= -\frac{\partial p_1}{\partial r} \\ &+ \text{Pr} \left\{ 2\left(1 - \frac{\hat{\lambda}}{3}\right) \frac{\partial}{\partial r} \left( \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 u_1) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta v_1) \right) \right. \\ &\left. - \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \left( \frac{\partial}{\partial r} (r v_1) - \frac{\partial u_1}{\partial \theta} \right) \right) \right\} \end{aligned} \quad (6.4)$$

Meridional component

$$\begin{aligned} \frac{M_0}{r^2} \left( \frac{\partial v_1}{\partial r} + \frac{v_1}{r} \right) &= -\frac{1}{r} \frac{\partial p_1}{\partial \theta} \\ &+ \text{Pr} \left\{ 2\left(1 - \frac{\hat{\lambda}}{3}\right) \frac{1}{r} \frac{\partial}{\partial \theta} \left( \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 u_1) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta v_1) \right) \right. \\ &\left. + \frac{1}{r} \frac{\partial}{\partial r} \left( \frac{\partial}{\partial r} (r v_1) - \frac{\partial u_1}{\partial \theta} \right) \right\} \end{aligned} \quad (6.5)$$

Equation of state

$$\rho_0 T_1 + T_0 \rho_1 = 0 \quad (6.6)$$

We note that the above equations have been written in a form that is sufficiently general for them to apply for both  $r < r_{*0}$  and  $r > r_{*0}$ . However, they simplify further when  $r > r_{*0}$ , since then in the leading order solution  $T_0 = \rho_0 = 1$  and  $Y_{F_0} = 0$ , and at order  $O(\epsilon)$  there is no leakage of fuel through the reaction layer so that  $Y_{F_1} = 0$ .

**Boundary conditions at  $r = a$ .** The boundary conditions at the drop surface  $r = a$  are given in (3.12) to (3.16), and imply at order  $O(\epsilon)$  that

$$T_1 = T_{s1} \quad (6.7)$$

$$\frac{\partial T_1}{\partial r} = l(\rho_0 u_1 + \rho_1 u_0) \quad (6.8)$$

$$\frac{\partial Y_{F1}}{\partial r} - \frac{M_0 \text{Le}_F}{a^2} Y_{F1} = -\text{Le}_F (1 - Y_{F_0}) (\rho_0 u_1 + \rho_1 u_0) \quad (6.9)$$

$$\frac{\partial Y_{O1}}{\partial r} - \frac{M_0 \text{Le}_O}{a^2} Y_{O1} = \text{Le}_O Y_{O_0} (\rho_0 u_1 + \rho_1 u_0) \quad (6.10)$$

$$v_1 = 0 \quad (6.11)$$

We note the introduction of  $T_{s1}$  in equation (6.7), which is the scaled fuel vaporization temperature  $T_s$  at order  $O(\epsilon)$ . The absolute temperature of fuel vaporization is known for a given fuel and pressure and is independent of the activation energy  $\epsilon$ , so that it may appear that  $T_{s1}$  should be set to zero. However, our choice of temperature scaling in Chapter 1 is based on the absolute temperature at infinity. This is regarded as known and controllable, and in the nearly adiabatic burning limit is such that the difference between the scaled reaction temperature,  $T_*$ , and the scaled temperature at infinity, unity, is of order  $O(\epsilon)$ , i.e.,  $T_* - 1 = O(\epsilon)$ . The value of  $T_{s1} \neq 0$  is related, by solution of the field equations and boundary conditions, to  $(T_* - 1)/\epsilon = O(1)$ , and can be regarded as adjustable by control of the absolute temperature at infinity.

**Jump conditions at  $r = r_{*0}$ .** Further manipulation of some of the jump conditions at  $r = r_{*0}$  is included here. This eases the computations and shows that the operators for the jump conditions are independent.

Since (4.75) holds for all  $\theta$ , it can be differentiated to give  $[\frac{\partial u_1}{\partial \theta}]_{r_{*0}} + \frac{dr_{*1}}{d\theta} [\frac{du_0}{dr}]_{r_{*0}} = 0$ . The quantity  $[\frac{\partial u_1}{d\theta}]$  can then be eliminated from (4.76), which becomes

$$\left[ \frac{\partial v_1}{\partial r} \right]_{r_{*0}} + \frac{1}{r_{*0}} \frac{dr_{*1}}{d\theta} \left[ \frac{du_0}{dr} \right]_{r_{*0}} = 0. \quad (6.12)$$

To recast (4.77), recall from (4.74) that  $[v_1]_{r_{*0}} = 0$  and hence  $[\frac{\partial v_1}{\partial \theta}]_{r_{*0}} = 0$ . Also, from the jump condition (4.50) and the discussion below it, we have  $[\rho_0 u_1 + u_0 \rho_1]_{r_{*0}} = 0$ . By taking the limit of the continuity equation (6.1) at order  $O(\epsilon)$  as  $r \rightarrow r_{*0}^\pm$  we then find that

$$\left[ \frac{\partial}{\partial r} (\rho_0 u_1 + u_0 \rho_1) \right]_{r_{*0}} = 0. \quad (6.13)$$

The leading order solution given in (3.41) to (3.45) of Chapter 3 implies that  $\rho_0 = 1/T_0$  and  $u_0 = M_0 T_0 / r^2$ , and from the equation of state (6.6) we have  $\rho_1 = -\rho_0 T_1 / T_0$ . These results hold for  $r < r_{*0}$  and  $r > r_{*0}$  but, in addition, for  $r > r_{*0}$  we have  $T_0 = 1$ .

We therefore find that

$$\rho_0 u_1 + u_0 \rho_1 = \frac{1}{T_0} \left( u_1 - \frac{M_0}{r^2} T_1 \right),$$

and when this is substituted in (6.13), on recalling (4.50) and (4.71), we can recast (6.13) as

$$\left[ \frac{\partial u_1}{\partial r} \right]_{r_{*0}} - \frac{M_0}{r_{*0}^2} \left[ \frac{\partial T_1}{\partial r} \right]_{r_{*0}} - \left( r_{*1} \frac{2M_0}{r_{*0}^3} + u_1(r_{*0}^+) - \frac{M_0}{r_{*0}^2} T_1(r_{*0}^+) \right) \left[ \frac{dT_0}{dr} \right]_{r_{*0}} = 0.$$

When this last expression is differentiated with respect to  $\theta$  it can be used to eliminate  $\left[ \frac{\partial^2 u_1}{\partial r \partial \theta} \right]_{r_{*0}}$  from (4.77), while (6.12) can be used to eliminate  $\left[ \frac{\partial v_1}{\partial r} \right]_{r_{*0}}$  and from the leading order solution we have  $\left[ \frac{du_0}{dr} \right]_{r_{*0}} = \frac{M_0}{r_{*0}^2} \left[ \frac{dT_0}{dr} \right]$ . The result is that (4.77) can now be recast as

$$\left[ \frac{\partial^2 v_1}{\partial r^2} \right]_{r_{*0}} - \frac{M_0}{r_{*0}^3} \left[ \frac{\partial^2 T_1}{\partial r \partial \theta} \right]_{r_{*0}} - \left( \frac{4M_0}{r_{*0}^4} \frac{dr_{*1}}{d\theta} + \frac{1}{r_{*0}} \frac{\partial u_1}{\partial \theta} \Big|_{r_{*0}^+} - \frac{M_0}{r_{*0}^3} \frac{\partial T_1}{\partial \theta} \Big|_{r_{*0}^+} \right) \left[ \frac{dT_0}{dr} \right]_{r_{*0}} = 0. \quad (6.14)$$

We next write all variables of the leading order solution in terms of the temperature  $T_0(r)$  by using the results of Chapter 3, and recall that since  $T_0 = 1$  for  $r > r_{*0}$ ,  $\left[ \frac{dT_0}{dr} \right]_{r_{*0}} = -T_0'(r_{*0}^-)$ . The jump conditions become:

$$[T_1]_{r_{*0}} - r_{*1} T_0'(r_{*0}^-) = 0 \quad (6.15)$$

$$[u_1]_{r_{*0}} - r_{*1} \frac{M_0}{r_{*0}^2} T_0'(r_{*0}^-) = 0 \quad (6.16)$$

$$[v_1]_{r_{*0}} = 0. \quad (6.17)$$

$$\left[ \frac{\partial v_1}{\partial r} \right]_{r_{*0}} - \frac{dr_{*1}}{d\theta} \frac{M_0}{r_{*0}^3} T_0'(r_{*0}^-) = 0 \quad (6.18)$$

$$\left[ \frac{\partial^2 v_1}{\partial r^2} \right]_{r_{*0}} - \frac{M_0}{r_{*0}^3} \left[ \frac{\partial^2 T_1}{\partial r \partial \theta} \right]_{r_{*0}} + \left( \frac{4M_0}{r_{*0}^4} \frac{dr_{*1}}{d\theta} + \frac{1}{r_{*0}} \frac{\partial u_1}{\partial \theta} \Big|_{r_{*0}^+} - \frac{M_0}{r_{*0}^3} \frac{\partial T_1}{\partial \theta} \Big|_{r_{*0}^+} \right) T_0'(r_{*0}^-) = 0. \quad (6.19)$$

$$\frac{q}{\text{Le}_i \nu_i W_i} [Y_{i1}]_{r_{*0}} + r_{*1} T'_0(r_{*0}^-) = 0 \quad i = \text{F, O}. \quad (6.20)$$

$$\left[ \frac{\partial T_1}{\partial r} \right]_{r_{*0}} + \frac{q}{\text{Le}_i \nu_i W_i} \left[ \frac{\partial Y_{i1}}{\partial r} \right]_{r_{*0}} + r_{*1} \frac{M_0}{r_{*0}^2} (\text{Le}_i - 1) T'_0(r_{*0}^-) = 0 \quad i = \text{F, O}. \quad (6.21)$$

Here, from the leading order solution in the nearly adiabatic limit, per (3.37), (3.40), and (4.15), we have

$$T'_0(r_{*0}^-) = \frac{M_0}{r_{*0}^2} \frac{q}{\nu_F W_F}. \quad (6.22)$$

The boundary value problem for the correction to the near field solution at order  $O(\epsilon)$  is linear. The governing equations and the jump conditions at  $r = r_{*0}$  are homogeneous, where the correction to the reaction layer position  $r_{*1}$  is an unknown to be determined as part of the solution. The only inhomogeneities occur: (i) in the boundary condition (6.7) for the temperature on the drop surface, and (ii) via matching with the far field, for all dependent variables. However, since the far field solution is a separable, eigenfunction expansion, the far field matching conditions are separable in  $r$  and  $\theta$ , and the dependence on  $\theta$  contains  $P_0(\cos \theta) = 1$ ,  $P_1(\cos \theta) = \cos \theta$  and  $-\frac{d}{d\theta} P_1(\cos \theta) = \sin \theta$  alone. Moreover, the same eigenfunctions  $P_n(\cos \theta)$  occur in the near field solution at order  $O(\epsilon)$ , as we now show.

**Expansion in eigenfunctions.** Separable near field solutions are found at this order by an eigenfunction expansion, which takes the form

$$\phi(r, \theta) = \sum_{n=0}^{\infty} \phi_n(r) P_n(\cos \theta) \quad (6.23)$$

for  $T_1, Y_{i1}, u_1, \rho_1, p_1$ , and  $r_{*1}$ . That is, it applies for the correction to the reaction layer shape  $r_{*1}$  and all dependent variables except the meridional velocity  $v_1$ , for which the expansion is

$$v_1(r, \theta) = \sum_{n=1}^{\infty} v_{1n}(r) \frac{-d}{d\theta} P_n(\cos \theta). \quad (6.24)$$

The system of ordinary differential equations governing the dependence of the solution on  $r$  is therefore

Continuity equation

$$\frac{d}{dr}(r^2(\rho_0 u_{1n} + u_0 \rho_{1n})) + n(n+1)r\rho_0 v_{1n} = 0, \quad (6.25)$$

Energy equation

$$M_0 \frac{dT_{1n}}{dr} + r^2(\rho_0 u_{1n} + u_0 \rho_{1n}) \frac{dT_0}{dr} - \frac{d}{dr}(r^2 \frac{dT_{1n}}{dr}) + n(n+1)T_{1n} = 0 \quad (6.26)$$

Species equation,  $i = F, O$ ,

$$M_0 \frac{dY_{i1n}}{dr} + r^2(\rho_0 u_{1n} + u_0 \rho_{1n}) \frac{dY_{i0}}{dr} - \frac{1}{Le_i} \frac{d}{dr}(r^2 \frac{dY_{i1n}}{dr}) + \frac{1}{Le_i} n(n+1)Y_{i1n} = 0 \quad (6.27)$$

Momentum equation. Radial component, for  $n = 0, 1, \dots$ ,

$$\begin{aligned} \frac{dp_{1n}}{dr} = & -\left(\frac{M_0}{r^2} \frac{du_{1n}}{dr} + \rho_0 \frac{du_0}{dr} u_{1n} + u_0 \frac{d\rho_0}{dr} \rho_{1n}\right) \\ & + \text{Pr} \left\{ 2\left(1 - \frac{\hat{\lambda}}{3}\right) \frac{d}{dr} \left( \frac{1}{r^2} \frac{d}{dr}(r^2 u_{1n}) \right) \right. \\ & \left. + n(n+1) \left( 2\left(1 - \frac{\hat{\lambda}}{3}\right) \frac{d}{dr} \left( \frac{v_{1n}}{r} \right) - \frac{1}{r^2} \frac{d}{dr}(r v_{1n}) - \frac{u_{1n}}{r^2} \right) \right\} \end{aligned} \quad (6.28)$$

Momentum equation. Meridional component, for  $n = 1, 2, \dots$ ,

$$\begin{aligned} p_{1n} = & \frac{M_0}{r^2} \left( r \frac{dv_{1n}}{dr} + v_{1n} \right) \\ & + \text{Pr} \left\{ 2\left(1 - \frac{\hat{\lambda}}{3}\right) \frac{1}{r^2} \frac{d}{dr}(r^2 u_{1n}) \right. \\ & \left. - \frac{d^2}{dr^2}(r v_{1n}) - \frac{du_{1n}}{dr} + n(n+1)2\left(1 - \frac{\hat{\lambda}}{3}\right) \frac{v_{1n}}{r} \right\} \end{aligned} \quad (6.29)$$

Equation of state

$$\rho_0 T_{1n} + T_0 \rho_{1n} = 0 \quad (6.30)$$

The conditions on the drop surface  $r = a$  become

$$T_{10} = T_{s1} \quad \text{and} \quad T_{1n} = 0 \quad \text{for } n = 1, 2, \dots \quad (6.31)$$

$$\frac{dT_{1n}}{dr} = l(\rho_0 u_{1n} + u_0 \rho_{1n}) \quad (6.32)$$

$$\frac{dY_{F1n}}{dr} - \frac{\text{Le}_F M_0}{a^2} Y_{F1n} = -\text{Le}_F (1 - Y_{F0})(\rho_0 u_{1n} + u_0 \rho_{1n}) \quad (6.33)$$

$$\frac{dY_{O1n}}{dr} - \frac{\text{Le}_O M_0}{a^2} Y_{O1n} = \text{Le}_O Y_{O0}(\rho_0 u_{1n} + u_0 \rho_{1n}) \quad (6.34)$$

$$v_{1n} = 0 \quad \text{for } n = 1, 2, \dots \quad \text{Also} \quad M_1 = a^2(\rho_0 u_{10} + u_0 \rho_{10})|_{r=a} \quad (6.35)$$

We note that the second of relations (6.35) is not strictly a boundary condition, since  $M_1$ , as the correction to the drop mass flux, is an unknown to be determined as part of the solution.  $M_1$  depends only on the  $n = 0$  radial component of the near field solution since, in the integral over the drop surface,  $\int_S P_n(\cos \theta) dS = 0$  for  $n \geq 1$ .

Expression of the jump conditions at  $r = r_{*0}$  is simplified on recalling the point (ii) made below equation (4.24) in the analysis of the reaction layer, where it was shown that the temperature correction  $T_1$  is independent of  $\theta$  as  $r \rightarrow r_{*0}^+$ . Thus, in the eigenfunction expansion of  $T_1$  for  $r > r_{*0}$ ,

$$T_{1n}(r_{*0}^+) = 0 \quad \text{for } n \geq 1. \quad (6.36)$$

There is also no leakage of fuel through the reaction layer at  $O(\epsilon)$ , so that  $Y_{F1} = 0$  when  $r > r_{*0}$  and hence  $Y_{F1n} = 0$  there for all  $n$ . The jump conditions at  $r = r_{*0}$ , (6.15) to (6.21), therefore become

$$T_{1n}(r_{*0}^-) + r_{*1n} T_0'(r_{*0}^-) = \begin{cases} T_{10}(r_{*0}^+) & \text{for } n = 0 \\ 0 & \text{for } n \geq 1 \end{cases} \quad (6.37)$$

$$u_{1n}(r_{*0}^-) + r_{*1n} \frac{M_0}{r_{*0}^2} T_0'(r_{*0}^-) = u_{1n}(r_{*0}^+) \quad \text{for } n \geq 0 \quad (6.38)$$

$$v_{1n}(r_{*0}^-) = v_{1n}(r_{*0}^+) \quad \text{for } n \geq 1 \quad (6.39)$$

$$v'_{1n}(r_{*0}^-) - r_{*1n} \frac{M_0}{r_{*0}^3} T_0'(r_{*0}^-) = v'_{1n}(r_{*0}^+) \quad \text{for } n \geq 1 \quad (6.40)$$

$$v''_{1n}(r_{*0}^-) + \frac{M_0}{r_{*0}^3} T_{1n}'(r_{*0}^-) + r_{*1n} \frac{4M_0}{r_{*0}^4} T_0'(r_{*0}^-) = v''_{1n}(r_{*0}^+) - \frac{1}{r_{*0}} u_{1n}(r_{*0}^+) \quad \text{for } n \geq 1 \quad (6.41)$$

$$Y_{i_{1n}}(r_{*0}^-) - r_{*1n} \frac{\text{Le}_i \nu_i W_i}{q} T_0'(r_{*0}^-) = \begin{cases} 0 & \text{for } i = \text{F}, n \geq 0 \\ Y_{O_{1n}}(r_{*0}^+) & \text{for } i = \text{O}, n \geq 0 \end{cases} \quad (6.42)$$

$$T_{1n}'(r_{*0}^-) + \frac{q}{\text{Le}_i \nu_i W_i} Y_{1n}'(r_{*0}^-) - r_{*1n} \frac{M_0}{r_{*0}^2} (\text{Le}_i - 1) T_0'(r_{*0}^-) = \begin{cases} T_{10}'(r_{*0}^+) & i = \text{F}, n = 0 \\ (T_{10}'(r_{*0}^+) \\ + \frac{q}{\text{Le}_O \nu_O W_O} Y_{O_{1d}}'(r_{*0}^+)) & i = \text{O}, n = 0 \\ 0 & i = \text{F}, n \geq 1 \\ \frac{q}{\text{Le}_i \nu_i W_i} Y_{O_{1n}}'(r_{*0}^+) & i = \text{O}, n \geq 1 \end{cases} \quad (6.43)$$

The matching conditions with the far field solution were found in Chapter 5, and in terms of the eigenfunction expansion of the solution in the near field require that as  $r \rightarrow \infty$

$$T_{10} = \left( \frac{\text{Re}}{\epsilon} \right) \frac{1}{\text{Pr}} \left( \frac{\pi S_0}{r} + o(r^{-1}) \right), \quad T_{1n} = o(1) \text{ for } n \geq 1 \quad (6.44)$$

$$Y_{O_{1n}} = \begin{cases} (-1)^n \left( \frac{\text{Re}}{\epsilon} \right) \frac{v_\infty \text{Pr} \text{Le}_O^2 M_0}{2} \left( Y_{O_\infty} + \frac{\nu_O W_O}{\nu_F W_F} \right) + o(1) & \text{for } n = 0, 1 \\ o(1) & \text{for } n \geq 2 \end{cases}$$

$$u_{1n} = \begin{cases} \left( \frac{\text{Re}}{\epsilon} \right) \text{Pr} \left( 1 - \frac{\pi \chi_0}{r} + o(r^{-1}) \right) & \text{for } n = 1 \\ o(1) & \text{for } n = 0 \text{ and } n \geq 2 \end{cases} \quad (6.45)$$

$$v_{11} = \left( \frac{\text{Re}}{\epsilon} \right) \text{Pr} \left( -1 + \frac{\pi \chi_0}{2r} + o(r^{-1}) \right), \quad v_{1n} = o(1) \text{ for } n \geq 2 \quad (6.46)$$

$$p_{11} = \left( \frac{\text{Re}}{\epsilon} \right) \text{Pr}^2 \frac{\phi_0}{r^2} + o(r^{-2}), \quad p_{1n} = o(r^{-2}) \text{ for } n = 0 \text{ and } n \geq 2 \quad (6.47)$$

As noted after equation (6.21), the only inhomogeneities that occur enter via boundary and matching data, and at this order are confined to terms of the eigenfunction expansion with  $n = 0$  and  $n = 1$  alone. Inhomogeneities for the temperature occur at the drop surface and via matching with the far field only when  $n = 0$ , from (6.31) and (6.44). Inhomogeneities for the other dependent variables occur via

matching with the far field, from (6.45) to (6.47); for the oxidizer mass fraction this occurs for  $n = 0$  and  $n = 1$ , while for the velocity and pressure it occurs only for  $n = 1$ . Since the problem for  $n \geq 2$  is linear and homogeneous, and numerical studies show that the operator is invertible, the solution for these values of  $n$  is zero and we need only consider the solution for  $n = 0$  and  $n = 1$ .

**The radial component,  $n = 0$ .**

When  $n = 0$  the continuity equation (6.25) implies that  $r^2(\rho_0 u_{10} + u_0 \rho_{10}) =$  constant. By using the equation of state (6.30) with the jump conditions (6.37) and (6.38) for  $n = 0$ , this constant takes the same value for both  $r < r_{*0}$  and  $r > r_{*0}$  and equals  $M_1$  from (6.35). The same result can be found directly from conservation of mass across the reaction layer at order  $O(\epsilon)$ , as shown immediately after equation (4.50). Thus

$$\rho_0 u_{10} + u_0 \rho_{10} = \frac{M_1}{r^2}. \quad (6.48)$$

It is now straightforward to write down the general solution of the governing equations that satisfies the boundary conditions on the drop surface and the matching conditions at infinity in terms of  $M_1$ ,  $T_{s1}$ , and  $r_{*10}$ . In fact, the problem for the radial component  $n = 0$  at order  $O(\epsilon)$  is a linearization of the problem for the near field at leading order, with respect to the dependent variables and parameters  $M$  and  $T_s$ , so that the way in which the system uncouples and its solution procedure are similar. The temperature is to satisfy (6.26) subject to boundary conditions (6.31) and (6.32) at  $r = a$  and (6.44) as  $r \rightarrow \infty$ ; the species mass fractions, are to satisfy (6.27) subject to boundary conditions (6.33) and (6.34) at  $r = a$  with the oxidizer satisfying (6.45) as  $r \rightarrow \infty$  and zero fuel mass fraction for all  $r > r_{*0}$ . The density and gas velocity are then found from (6.48) and the equation of state (6.30), and the pressure is found from the momentum equation (6.29).

When  $a < r < r_{*0}$ . The solution can be written

$$T_{10} = T_{s1} + M_1 l \left( \frac{1}{a_0} - \frac{1}{r} \right) e^{M_0 \left( \frac{1}{a_0} - \frac{1}{r} \right)} \quad (6.49)$$

$$Y_{F10} = e^{-Le_F M_0 \left( \frac{1}{r} - \frac{1}{r_{*0}} \right)} \left( \frac{M_1 Le_F}{r} + c_{Y_{F1-}} \right) \quad (6.50)$$

$$Y_{O10} = e^{-Le_O M_0 \left( \frac{1}{r} - \frac{1}{r_{*0}} \right)} \left( -\frac{M_1 Le_O Y_{O_{*0}}}{r} + c_{Y_{O1-}} \right) \quad (6.51)$$

with

$$\rho_{10} = -\frac{T_{10}}{T_0^2}, \quad u_{10} = \frac{1}{r^2} (M_0 T_{10} + M_1 T_0). \quad (6.52)$$

Here  $c_{Y_{i1-}}$ ,  $i = F, O$ , are constants which are to be found.

When  $r_{*0} < r < \infty$ . The solution can be written

$$T_{10} = \kappa \left( 1 - e^{-\frac{M_0}{r}} \right) \quad (6.53)$$

$$Y_{O10} = -M_1 Le_O \left( Y_{O_\infty} + \frac{\nu_O W_O}{\nu_F W_F} \right) \frac{e^{-\frac{Le_O M_0}{r}}}{r} + c_{Y_{O1+}} \left( 1 - e^{-\frac{Le_O M_0}{r}} \right) \\ + \left( \frac{Re}{\epsilon} \right) \frac{\nu_\infty Pr Le_O^2 M_0}{2} \left( Y_{O_\infty} + \frac{\nu_O W_O}{\nu_F W_F} \right) e^{-\frac{Le_O M_0}{r}} \quad (6.54)$$

Here  $\kappa$  has been introduced as a parameter and is determined by the jump conditions at  $r = r_{*0}$ . The matching condition (6.44) for the temperature with the far field implies that the constant  $S_0$  of (5.52) is given in terms of  $\kappa$  by

$$S_0 = \left( \frac{\epsilon}{Re} \right) \frac{\nu_\infty M_0 Pr}{\pi} \kappa. \quad (6.55)$$

The remaining five independent jump conditions for  $n = 0$  are (6.37), (6.42), and (6.43) where the last two conditions hold for  $i = F$  and  $i = O$ . These determine the three constants  $c_{Y_{F1-}}$ ,  $c_{Y_{O1-}}$ , and  $c_{Y_{O1+}}$ , and the parameter  $\kappa$ , and provide one relation between the two unknowns  $M_1$  and  $r_{*10}$  and the known value of  $T_{s1}$ . We find that

$$\kappa = T_{s1}, \quad (6.56)$$

and

$$M_1 \left( \frac{1}{a_0} - \frac{1}{r_{*0}} \right) + r_{*10} \frac{M_0}{r_{*0}^2} = -T_{s1} \frac{\nu_F W_F}{q} e^{-\frac{M_0}{r_{*0}}}. \quad (6.57)$$

An independent relation between  $M_1$  and  $r_{*10}$  needs to be found to completely determine the solution for  $n = 0$ . We believe that it can be derived by further analysis of the reaction layer but omit this point, since (i) the radial problem is not our main interest, and (ii) details of the analysis are lengthy.

The mass fraction terms can now be written as

$$Y_{F10} = \begin{cases} \text{Le}_F e^{-\text{Le}_F M_0 \left( \frac{1}{r} - \frac{1}{r_{*0}} \right)} \left( r_{*10} \frac{M_0}{r_{*0}^2} + M_1 \left( \frac{1}{r} - \frac{1}{r_{*0}} \right) \right) & a_0 < r < r_{*0} \\ 0 & r_{*0} < r < \infty \end{cases} \quad (6.58)$$

$$Y_{O10} = \begin{cases} \left( Y_{O\infty} + \frac{\nu_o W_o}{\nu_F W_F} \right) \text{Le}_O e^{-\frac{\text{Le}_O M_0}{r}} \left( \left( \frac{\text{Re}}{\epsilon} \right) \frac{\nu_\infty \text{Pr} \text{Le}_O M_0}{2} - \frac{M_1}{r} \right) \\ + \frac{\nu_o W_o}{\nu_F W_F} \text{Le}_O e^{-\text{Le}_O M_0 \left( \frac{1}{r} - \frac{1}{r_{*0}} \right)} \left( r_{*10} \frac{M_0}{r_{*0}^2} + M_1 \left( \frac{1}{r} - \frac{1}{r_{*0}} \right) \right) & a_0 < r < r_{*0} \\ \left( Y_{O\infty} + \frac{\nu_o W_o}{\nu_F W_F} \right) \text{Le}_O e^{-\frac{\text{Le}_O M_0}{r}} \left( \left( \frac{\text{Re}}{\epsilon} \right) \frac{\nu_\infty \text{Pr} \text{Le}_O M_0}{2} - \frac{M_1}{r} \right) & r_{*0} < r < \infty \end{cases} \quad (6.59)$$

The pressure term  $p_{10}$  is found by substitution of the above profiles in the momentum equation (6.29) with  $n = 0$ , and is given by

$$p_{10} = \begin{cases} -\frac{M_0}{2r^4} (M_0 T_{s1} + 2M_1 (T_{s0} - l)) + \frac{12M_1}{M_0^3} e^{M_0 \left( \frac{1}{a_0} - \frac{1}{r} \right)} \left\{ 4 \left( 1 + \left( \frac{M_0}{r} \right) \right. \right. \\ \left. \left. + \frac{1}{2} \left( \frac{M_0}{r} \right)^2 + \frac{1}{6} \left( \frac{M_0}{r} \right)^3 + \frac{1}{24} \left( \frac{M_0}{r} \right)^4 + \frac{1}{48} (1 - 2\text{Pr} (1 - \frac{\lambda}{3})) \left( \frac{M_0}{r} \right)^5 \right\} \\ - \left( \frac{M_0}{a_0} + 2 \right) \left( 1 + \left( \frac{M_0}{r} \right) + \frac{1}{2} \left( \frac{M_0}{r} \right)^2 + \frac{1}{6} \left( \frac{M_0}{r} \right)^3 \right. \\ \left. + \frac{1}{12} (1 - 2\text{Pr} (1 - \frac{\lambda}{3})) \left( \frac{M_0}{r} \right)^4 \right\} + c_{p10} & a_0 < r < r_{*0} \\ -\frac{M_0}{2r^4} (M_0 T_{s1} + 2M_1) + \frac{12T_{s1}}{M_0^2} \left\{ e^{-\frac{M_0}{r}} \left( 1 + \left( \frac{M_0}{r} \right) + \frac{1}{2} \left( \frac{M_0}{r} \right)^2 + \frac{1}{6} \left( \frac{M_0}{r} \right)^3 \right. \right. \\ \left. \left. + \frac{1}{12} (1 - 2\text{Pr} (1 - \frac{\lambda}{3})) \left( \frac{M_0}{r} \right)^4 \right) - 1 \right\} & r_{*0} < r < \infty. \end{cases} \quad (6.60)$$

Here  $p_{10} = O(r^{-4})$  as  $r \rightarrow \infty$ , and the constant  $c_{p10-}$  can be found directly from the jump condition (4.68).

**The departure from spherical symmetry,  $n = 1$ .**

We now formulate the two point boundary value problem for the departure of the solution from spherical symmetry. Although the only non-zero components of the eigenfunction expansion of the solution for  $n \geq 0$  occur for  $n = 1$  we leave  $n$  general throughout much of the presentation below.

The differential operator. The differential operator is of order ten. Recall that we can use the leading order solution for both  $r < r_{*0}$  and  $r > r_{*0}$  to write the density  $\rho_0$  and gas velocity  $u_0$  in terms of the temperature  $T_0$ , with  $\rho_0 = 1/T_0$  and  $u_0 = M_0 T_0 / r^2$ . Also, the density component  $\rho_{1n}$  at order  $O(\epsilon)$  can be eliminated in favor the temperature from the equation of state (6.30) as  $\rho_{1n} = -T_{1n}/T_0^2$ , and the pressure  $p_{1n}$  is found in terms of the other dependent variables from the meridional component of the momentum equation. We therefore formulate the boundary value problem in terms of  $T_{1n}, Y_{i1n}$  ( $i=F,O$ ),  $u_{1n}$  and  $v_{1n}$  as dependent variables.

If the pressure  $p_{1n}$  is eliminated between the radial and meridional components of the momentum equation (6.29) and (6.30), we have

$$\frac{d^2 \psi_n}{dr^2} - \frac{\alpha}{r^2} \frac{d\psi_n}{dr} - \left( \frac{n(n+1)}{r^2} - \frac{2\alpha}{r^3} \right) \psi_n = \frac{\alpha}{r^2 T_0} \left( \frac{dT_0}{dr} u_{1n} - M_0 \frac{d}{dr} \left( \frac{T_0}{r^2} \right) T_{1n} \right). \quad (6.61)$$

Here we have introduced the parameter  $\alpha = M_0/\text{Pr}$ , and  $\psi_n$  is defined by

$$\psi_n = \frac{d}{dr}(r v_{1n}) + u_{1n} \quad (6.62)$$

which is related to the vorticity, from (3.49), (6.23), and (3.19) by

$$\omega_1 = \sum_{n=1}^{\infty} \frac{1}{r} \frac{d\psi_n}{dr} \frac{d}{d\theta} P_n(\cos \theta). \quad (6.63)$$

This form of (6.61) will be useful for  $r > r_{*0}$ , but for  $r < r_{*0}$  we need to consider the velocity components  $u_{1n}$  and  $v_{1n}$  independently, for which (6.61) can be written

$$\begin{aligned} & \frac{d^3 v_{1n}}{dr^3} + \left( \frac{3}{r} - \frac{\alpha}{r^2} \right) \frac{d^2 v_{1n}}{dr^2} + \frac{1}{r} \frac{d^2 u_{1n}}{dr^2} - \frac{n(n+1)}{r^2} \frac{dv_{1n}}{dr} - \frac{\alpha}{r^3} \frac{du_{1n}}{dr} \\ & - \left( \frac{n(n+1)}{r^3} - \frac{2\alpha}{r^4} \right) (v_{1n} + u_{1n}) = \frac{\alpha}{r^3 T_0} \left( \frac{dT_0}{dr} u_{1n} - M_0 \frac{d}{dr} \left( \frac{T_0}{r^2} \right) T_{1n} \right). \end{aligned} \quad (6.64)$$

The continuity equation (6.25) and energy equation (6.26), respectively, become

$$\frac{du_{1n}}{dr} - \frac{M_0}{r^2} \frac{dT_{1n}}{dr} + \frac{T_0}{r^2} \frac{d}{dr} \left( \frac{r^2}{T_0} \right) u_{1n} + \frac{M_0}{r^2 T_0} \frac{dT_0}{dr} T_{1n} = -\frac{n(n+1)}{r} v_{1n}, \quad (6.65)$$

and

$$\frac{d^2 T_{1n}}{dr^2} + \left( \frac{2}{r} - \frac{M_0}{r^2} \right) \frac{dT_{1n}}{dr} - \left( \frac{n(n+1)}{r^2} - \frac{M_0}{r^2 T_0} \frac{dT_0}{dr} \right) T_{1n} = \frac{1}{T_0} \frac{dT_0}{dr} u_{1n}. \quad (6.66)$$

From the last three equations we see that the differential operator factors so that there is a sixth order operator acting on  $u_{1n}$ ,  $v_{1n}$ , and  $T_{1n}$  alone. However, these three dependent variables are coupled to  $Y_{i1n}$  via the boundary and jump conditions. Further, the mass fraction terms  $Y_{i1n}$  are coupled to  $u_{1n}$ ,  $v_{1n}$ , and  $T_{1n}$  by the two second order differential equations, with  $i = F, O$ ,

$$\frac{d^2 Y_{i1n}}{dr^2} + \left( \frac{2}{r} - \frac{Le_i M_0}{r^2} \right) \frac{dY_{i1n}}{dr} - \frac{n(n+1)}{r^2} Y_{i1n} = Le_i \frac{dY_{i0}}{dr} \left( \frac{1}{T_0} u_{1n} - \frac{M_0}{r^2 T_0} T_{1n} \right). \quad (6.67)$$

Some manipulation enables (6.64) to (6.67) to be written as a first order system. An expression for the second derivative of  $u_{1n}$  is given by differentiating (6.65) once. This also contains the second derivative of  $T_{1n}$  which can be eliminated using (6.66). Second and first derivatives of  $u_{1n}$  can now be eliminated from (6.64). It is then straightforward to rewrite the system in the form of ten first order equations. We define

$$\begin{aligned} x_1 &= T_{1n}, & x_2 &= \frac{dT_{1n}}{dr}, & x_3 &= u_{1n}, & x_4 &= v_{1n}, & x_5 &= \frac{dv_{1n}}{dr}, & x_6 &= \frac{d^2 v_{1n}}{dr^2}, \\ x_7 &= Y_{F1n}, & x_8 &= \frac{dY_{F1n}}{dr}, & x_9 &= Y_{O1n}, & x_{10} &= \frac{dY_{O1n}}{dr} \end{aligned} \quad (6.68)$$

then

$$\frac{dx_i}{dr} = A_{ij}x_j \quad i, j = 1, \dots, 10, \quad r \in (a, r_{*0}) \cup (r_{*0}, \infty) \quad (6.69)$$

where

$$A_{ij} = \begin{cases} A_{ij}^- & \text{for } a < r < r_{*0} \\ A_{ij}^+ & \text{for } r_{*0} < r < \infty. \end{cases} \quad (6.70)$$

The entries of the matrix  $A_{ij}^-$  are given in the Appendix in terms of the leading order solution for  $T_0$  and  $Y_{i0}$ ,  $i = F, O$ , for  $r < r_{*0}$  of (3.37) to (3.39). The entries  $A_{ij}^+$  are given by the same formal expressions in terms of  $T_0$  and  $Y_{i0}$ ,  $i = F, O$ , but for  $r > r_{*0}$  the leading order solution of (3.43) and (3.44) is simpler, since  $T_0 = 1$  and  $Y_{F0} = 0$ , so that the matrix is relatively sparse.

In fact, a general solution can be constructed analytically for  $r > r_{*0}$ , and we consider this next.

Solution for  $r > r_{*0}$ . Since  $T_0 = 1$  for  $r > r_{*0}$ , the equation (6.66) for  $T_{1n}$  is uncoupled. It is also linear, second order, and homogeneous, with far field matching condition (6.44) that  $T_{1n} \rightarrow 0$  as  $r \rightarrow \infty$ , and the condition (6.36) that  $T_{1n}(r_{*0}^+) = 0$  which is specific to the limit of nearly adiabatic burning. Thus

$$T_{1n} = 0 \text{ for all } r > r_{*0} \text{ and } n \geq 1, \quad (6.71)$$

and from the equation of state  $\rho_{1n} = 0$  also.

The differential equations and matching conditions for the velocity field are now uncoupled from the temperature, and it becomes convenient to consider the quantity  $\psi_n$ , for which the governing equation (6.61) is homogeneous. The continuity equation (6.65) also simplifies, and if  $v_{1n}$  is eliminated between it and the definition (6.62) of  $\psi_n$ , the result can be written as

$$\frac{d^2 U_n}{dr^2} - \frac{n(n+1)}{r^2} U_n = -n(n+1)\psi_n, \quad (6.72)$$

where we have introduced  $U_n = r^2 u_{1n}$ . Once a solution for  $\psi_n$  and  $U_n$  has been constructed, the velocity components are found from the definition of  $U_n$  and the continuity equation, that is,

$$u_{1n} = \frac{U_n}{r^2} \quad \text{and} \quad v_{1n} = \frac{-1}{n(n+1)} \frac{1}{r} \frac{dU_n}{dr}. \quad (6.73)$$

Equation (6.61) becomes

$$\frac{d^2 \psi_n}{dr^2} - \frac{\alpha}{r^2} \frac{d\psi_n}{dr} - \left( \frac{n(n+1)}{r^2} - \frac{2\alpha}{r^3} \right) \psi_n = 0, \quad (6.74)$$

which, on setting  $\psi_n(r) = x^n e^{-x} y(x)$  with  $x = \alpha/r$ , is transformed to the confluent hypergeometric equation in  $y(x)$ . A pair of linearly independent solutions consists of  $y_1(x) = \Phi(n, 2n+2, x)$  and  $y_2(x) = e^x \Psi(n+2, 2n+2, -x)$ , where  $y_2(x)$  is a (finite) polynomial in  $x^{-1}$ . The general solution for  $\psi_n$  is therefore

$$\psi_n = A_{\Phi n} e^{-\frac{\alpha}{r}} \left( \frac{\alpha}{r} \right)^n \Phi(n, 2n+2, \frac{\alpha}{r}) + A_{\Psi n} \left( \frac{\alpha}{r} \right)^n \Psi(n+2, 2n+2, -\frac{\alpha}{r}) \quad (6.75)$$

where  $A_{\Phi 1}$  and  $A_{\Psi 1}$  are constants. If we restrict our attention to the only non zero component, with  $n = 1$ , for which

$$\Phi(1, 4, x) = \frac{6}{x^3} \left( e^x - 1 - x - \frac{x^2}{2!} \right) \quad \text{and} \quad \Psi(3, 4, -x) = -\frac{1}{x^3}, \quad (6.76)$$

then the general solution (6.75) is

$$\psi_1 = 6A_{\Phi 1} e^{-\frac{\alpha}{r}} \left( \frac{r}{\alpha} \right)^2 \left( e^{\frac{\alpha}{r}} - \left( 1 + \left( \frac{\alpha}{r} \right) + \frac{1}{2!} \left( \frac{\alpha}{r} \right)^2 \right) \right) - A_{\Psi 1} \left( \frac{r}{\alpha} \right)^2. \quad (6.77)$$

The matching relations with the far field (6.45) and (6.46) imply that this is to be such that  $\psi_1 = (\text{Re}/\epsilon) o(1)$  as  $r \rightarrow \infty$ , so that the constant  $A_{\Psi 1} = 0$  while the constant  $A_{\Phi 1}$  is as yet undetermined.

We can now construct the general solution of equation (6.72) for  $U_1$  using variation of parameters, and then from equations (6.73) find the general solution for  $u_{11}$  and  $v_{11}$ . This takes the form

$$\begin{aligned} u_{11}(r) &= A_{\Phi 1} u_{\Phi 1}(r) + \frac{c_{21}}{r^3} + c_{11} \\ v_{11}(r) &= A_{\Phi 1} v_{\Phi 1}(r) + \frac{c_{21}}{2r^3} - c_{11} \end{aligned} \quad (6.78)$$

where  $c_{11}$  and  $c_{21}$  are arbitrary constants generated by the homogeneous part of the solution for  $U_1$ , and the functions  $u_{\Phi 1}$  and  $v_{\Phi 1}$  are generated by the inhomogeneous part. Explicitly, in terms of

$$u = \frac{r}{\alpha} \quad \text{where} \quad \alpha = \frac{M_0}{\text{Pr}} \quad (6.79)$$

we find

$$\begin{aligned} u_{\Phi 1} &= 1 - \frac{6u^2}{5} \left( 1 - e^{-\frac{1}{u}} \left( 1 + \frac{1}{u} - \frac{1}{3u^2} + \frac{1}{6u^3} - \frac{1}{6u^4} \right) \right) + \frac{1}{5u^3} E_1 \left( \frac{1}{u} \right) \\ v_{\Phi 1} &= -1 + \frac{12u^2}{5} \left( 1 - e^{-\frac{1}{u}} \left( 1 + \frac{1}{u} + \frac{1}{12u^2} - \frac{1}{24u^3} + \frac{1}{24u^4} \right) \right) + \frac{1}{10u^3} E_1 \left( \frac{1}{u} \right), \end{aligned} \quad (6.80)$$

where  $E_1(z) = \int_z^\infty \frac{e^{-u}}{u} du$  is an exponential integral. As  $r \rightarrow \infty$  the solution for  $u_{11}$  and  $v_{11}$  has the behavior

$$\begin{aligned} u_{11} &= A_{\Phi 1} \left( \frac{\alpha}{r} \right) (1 + O(r^{-1})) + \frac{c_{21}}{r^3} + c_{11} \\ v_{11} &= -\frac{A_{\Phi 1}}{2} \left( \frac{\alpha}{r} \right) (1 + O(r^{-1})) + \frac{c_{21}}{2r^3} - c_{11}. \end{aligned} \quad (6.81)$$

The matching conditions (6.45) and (6.46) with the far field therefore imply that

$$c_{11} = \left( \frac{\text{Re}}{\epsilon} \right) \text{Pr} v_\infty \quad \text{and} \quad A_{\Phi 1} = - \left( \frac{\text{Re}}{\epsilon} \right) \text{Pr}^2 \frac{\pi \chi_0}{M_0}. \quad (6.82)$$

We can now construct the general solution for the component  $Y_{O_{11}}$  of the oxidizer mass fraction, which satisfies the differential equation (6.67). Since  $T_{11} = 0$ , on defining

$$\beta = \frac{M_0 \text{Le}_O}{2} \quad \text{and} \quad \gamma_0 = M_0 \text{Le}_O^2 \left( Y_{O_\infty} + \frac{\nu_o W_O}{\nu_F W_F} \right) \quad (6.83)$$

and using the leading order solution, equation (6.67) becomes

$$\left( \frac{d^2}{dr^2} + 2 \left( \frac{1}{r} - \frac{\beta}{r^2} \right) \frac{d}{dr} - \frac{2}{r^2} \right) Y_{O_{11}} = \gamma_0 \frac{u_{11}(r)}{r^2} e^{-\frac{2\beta}{r}}. \quad (6.84)$$

From the form of the solution (6.78) for  $u_{11}$ , the general solution for  $Y_{O_{11}}$  can be written as

$$Y_{O_{11}}(r) = B_1 g_{H1}(r) + \gamma_0 \left( A_{\Phi_1} g_{A1}(r) + c_{21} g_{c21}(r) - \frac{c_{11}}{2} e^{-\frac{2\beta}{r}} \right). \quad (6.85)$$

Here, the terms multiplied by  $\gamma_0$  are the inhomogeneous part of the solution, which can be found by variation of parameters. If we define

$$z = \frac{\beta}{r} \quad (6.86)$$

then (6.84) has a pair of linearly independent homogeneous solutions  $z^{\frac{1}{2}} e^{-z} I_{\frac{3}{2}}(z)$  and  $z^{\frac{1}{2}} e^{-z} K_{\frac{3}{2}}(z)$  where  $I_{\frac{3}{2}}(z)$  and  $K_{\frac{3}{2}}(z)$  are modified Bessel functions of the first and third kind. Equivalently, we can use the pair

$$\frac{e^{-2z}}{z}(1+z) \quad \text{and} \quad \frac{1}{z}(1-z). \quad (6.87)$$

Then, in terms of  $z$ , we find

$$g_{c21}(z) = \frac{3}{8\beta^3} \left( \frac{e^{-2z}}{z}(1+z - \frac{2z^3}{3}) - \frac{1}{z}(1-z) \right), \quad (6.88)$$

and

$$\begin{aligned} g_{A1}(z) = & \frac{e^{-2z}}{2\gamma^2 z^2} \left( -\frac{3}{5} + (\gamma^2 + \frac{2}{5})z \right) \\ & + \frac{e^{-(\gamma+2)z}}{2\gamma^2 z^2} \left( \frac{3}{5} - \left[ \frac{3}{10}\gamma^4 + \frac{6}{5}\gamma^3 + \frac{9}{10}\gamma^2 - \frac{1}{5}\gamma + \frac{2}{5} \right] z - \frac{\gamma^3}{10}z^2 + \frac{\gamma^4}{10}z^3 \right) \\ & + \frac{e^{-2z}}{z} \left( \left[ \frac{3}{40}\gamma^3 + \frac{3}{8}\gamma^2 + \frac{1}{2}\gamma \right] (1+z) - \frac{\gamma^3}{20}z^3 \right) E_1(\gamma z) \\ & + \frac{1}{z}(1-z) \left\{ \frac{2}{5\gamma^2} E_1(2z) - \left[ \frac{3}{40}\gamma^3 + \frac{3}{8}\gamma^2 + \frac{1}{2}\gamma + \frac{2}{5\gamma^2} \right] E_1((\gamma+2)z) \right. \\ & \quad - \left( \left[ \frac{3}{40}\gamma^3 + \frac{3}{8}\gamma^2 + \frac{1}{2}\gamma \right] \ln \left( 1 + \frac{2}{\gamma} \right) \right. \\ & \quad \left. \left. + \frac{2}{5\gamma^2} \ln \left( 1 + \frac{\gamma}{2} \right) - \left[ \frac{3}{20}\gamma^2 + \frac{3}{5}\gamma - \frac{1}{20} + \frac{1}{5\gamma} \right] \right) \right\}. \end{aligned}$$

In the homogeneous solution, a component generated by  $K_{\frac{3}{2}}(z)$  grows like  $r$  as  $r \rightarrow \infty$  and is discarded since it does not satisfy the boundary condition, whereas a

component generated by  $I_{\frac{3}{2}}(z)$  decays like  $r^{-2}$ . We choose

$$g_{H1}(z) = \frac{3}{\beta^2} \left(\frac{\pi}{2}\right)^{\frac{1}{2}} z^{\frac{1}{2}} e^{-z} I_{\frac{3}{2}}(z) = \frac{3}{2\beta^2} \left( \frac{e^{-2z}}{z} (1+z) - \frac{1}{z} (1-z) \right). \quad (6.89)$$

As  $r \rightarrow \infty$ , the solution just constructed has the behavior

$$Y_{O_{11}} = \frac{B_1}{r^2} (1 + o(1)) + \gamma_0 \left( -\frac{A_{\Phi 1} \alpha}{2r} (1 + o(1)) + \frac{c_{21}}{4r^3} (1 + o(1)) - \frac{c_{11}}{2} (1 + o(1)) \right). \quad (6.90)$$

This has to satisfy the matching condition (6.45) with the far field solution. On recalling the definition (6.83) of  $\gamma_0$  and the result of matching the velocity field given by (6.82), we find that no new information is provided.

**Drag force exerted on the drop.** Once the solution of the two point boundary value problem has been constructed, the flow field at the drop surface is known and the drag force exerted on the drop can be found. From the second of relations (2.40) of Chapter 2 the dimensionless drag is

$$\mathbf{F} = -\frac{1}{6\pi} \int_{S_a} \text{Re} \rho \mathbf{v} \left( \mathbf{v} - \bar{\omega} \frac{d\mathbf{a}}{dt} \right) \cdot \mathbf{e}_r + \frac{\text{Re}}{\text{Ma}^2} p \mathbf{e}_r - \sigma \cdot \mathbf{e}_r \, dS \quad (6.91)$$

where  $S_a$  is the drop surface. Recall that in (6.91) the gas velocity has been nondimensionalized by the initial translation speed  $U_{\infty 0}$  but not rescaled to the near field velocity  $\tilde{\mathbf{v}} = \text{Pe} \mathbf{v}$ . The first term in the integrand of (6.91) is the contribution to the force experienced by the drop due to the momentum of material vaporizing from its surface, in which the drop regression rate  $\frac{d\mathbf{a}}{dt}$  is a higher order effect. The second term in the integrand is the 'pressure drag', and the third term is the 'viscous drag'. The size of the first two contributions depends on the order at which the gas velocity, density, and pressure depart from spherical symmetry, as represented by the terms  $u_{11}, v_{11}, \rho_{11}$ , and  $p_{11}$  in the near field expansion.

Each contribution to the drag occurs at the same order, and we find, on putting  $\mathbf{F} = F\hat{\mathbf{k}}$ , that in general

$$F = \frac{2}{9\text{Pr}^2} \left( \frac{\epsilon}{\text{Re}} \right) \left\{ \text{Pr}2a \left[ a \left( 1 - \frac{\hat{\lambda}}{3} \right) \frac{\partial u_{11}}{\partial r} - a \frac{\partial v_{11}}{\partial r} + \left( 1 - \frac{2\hat{\lambda}}{3} \right) (u_{11} + v_{11}) \right] - a^2 p_{11} - a^2 [2\rho_0 u_0 (u_{11} - v_{11}) + u_0^2 \rho_{11}] \right\}_{r=a} + o(1). \quad (6.92)$$

Since we have assumed no internal motion within the drop and that its surface is everywhere at the same vaporization temperature,  $v_{11} = \rho_{11} = 0$  in (6.92). If the meridional component of the momentum equation (6.30) is used to eliminate the pressure  $p_{11}$ , the expression for the leading order drag becomes

$$F = \frac{2}{9\text{Pr}^2} \left( \frac{\epsilon}{\text{Re}} \right) \left\{ \text{Pr}a \left[ a \frac{\partial u_{11}}{\partial r} + a^2 \frac{\partial^2 v_{11}}{\partial r^2} - 2u_{11} \right] - M_0 \left( 2u_{11} + a \frac{\partial v_{11}}{\partial r} \right) \right\}_{r=a}. \quad (6.93)$$

**Transfer of energy to the drop surface.** From the third of the relations (2.40) of Chapter 2, the dimensionless rate of energy transfer to the drop surface is

$$E_a = \frac{-1}{4\pi} \int_{S_a} \text{Pe}\rho(\mathbf{v} - \bar{\omega} \frac{d\mathbf{a}}{dt}) \cdot \mathbf{e}_r T - \lambda \nabla T \cdot \mathbf{e}_r \, dS, \quad (6.94)$$

where, as in (6.91), the gas velocity is to be rescaled to  $\tilde{\mathbf{v}} = \text{Pe}\mathbf{v}$ , and the drop regression rate is a higher order effect. If the integral is evaluated and expressed using the terms of the near field expansion, we find

$$E_a = -a^2 \left( \rho_0 u_0 T_0 - \frac{\partial T_0}{\partial r} + \epsilon (\rho_0 u_0 T_{10} + \rho_0 u_{10} T_0 + \rho_{10} u_0 T_0 - \frac{\partial T_{10}}{\partial r}) \right)_{r=a} + o(\epsilon), \quad (6.95)$$

which, from the solution in the near field is

$$E_a = - (M_0(T_{s0} - l) + \epsilon(M_0 T_{s1} + M_1(T_{s0} - l))) + o(\epsilon). \quad (6.96)$$

as a function of the underlying parameters. Up to order  $O(\epsilon)$ , this is  $E_a = -M(T_s - l)$ .

Referring back to the result (5.68) at the end of Chapter 5 for the rate of transfer of energy to infinity, we find that  $E_a + E_\infty = M_0(1 - T_{s0} + l) + o(1)$ , which,

from the relation (4.15) of Chapter 4 for the nearly adiabatic burning limit, becomes

$$E_a + E_\infty = M_0 \left( \frac{q}{\nu_F W_F} \right) + o(1). \quad (6.97)$$

This provides a simple check on the analysis. The right hand side is the dimensionless rate of production of heat due to the reaction written in terms of the burning rate and heat release, while the left hand side is the net flux of heat outwards to infinity and inwards to the drop. The two are equal due to overall conservation of energy between chemical and thermal forms, with the kinetic energy of the gas being negligible.

## CHAPTER 7

### NUMERICAL EXAMPLES AND CONCLUSION

The system we solve to describe the evolution consists of the quasisteady response relation (4.31) with the relations (4.26) and (4.27) of Chapter 4, together with the ODE's (2.52) and (2.53) of Chapter 2, and the expression (6.93) of Chapter 6 for the force  $F\hat{\mathbf{k}}$  exerted by the fluid on the drop. Evaluation of  $F$  necessitates solution of the two point boundary value problem formulated in Chapter 6. The reduced Damkohler number  $\hat{Da}$  that appears in the quasisteady response relation (4.31) is fixed and is a known function of the initial drop radius and other physico-chemical parameters.

**Evolution of drop radius and flame sheet radius.** Solution of (4.31), with (4.26) and (4.27), gives a relation between the drop radius  $a$  and the leading order flame sheet radius  $r_{*0}$ . The evolution of  $a$  and  $r_{*0}$  in time is then given by appending the ODE (2.52), and this system for  $a$  and  $r_{*0}$  is closed.

The response relation (4.31) is

$$\ln \hat{Da} = -T_{s1}(1 - e^{-\frac{M_0}{r_{*0}}}) + \ln \left( \frac{M_0^2}{r_{*0}^4 Y_{O_{*0}} \nu_o} \right). \quad (7.1)$$

where, on introducing  $\hat{q} = q/(l\nu_F W_F)$ ,  $M_0$  and  $Y_{O_{*0}}$  are given by

$$M_0 = \frac{\ln \left( \frac{\hat{q}}{l\nu_F W_F} \right)}{\frac{1}{a} - \frac{1}{r_{*0}}}, \quad (7.2)$$

and

$$Y_{O_{*0}} = -\frac{\nu_o W_O}{\nu_F W_F} + \left( Y_{O_\infty} + \frac{\nu_o W_O}{\nu_F W_F} \right) e^{-\frac{Le_O M_0}{r_{*0}}}. \quad (7.3)$$

Expansion of (7.1) as  $a$  tends to zero shows that  $r_{*0}$  also tends to zero, in such a way that

$$r_{*0} \sim \left( \frac{\ln \hat{q}}{\hat{Da} Y_{O_\infty} \nu_o} \right)^{1/4} a^{1/2} + \left\{ 2 - T_{s1} \ln \hat{q} + \nu_o Le_O \left( 1 + \frac{\nu_o W_O}{Y_{O_\infty} \nu_F W_F} \right) \left( \frac{\ln^5 \hat{q}}{\hat{Da} Y_{O_\infty} \nu_o} \right)^{1/4} \right\} \frac{a}{4} + o(a). \quad (7.4)$$

This indicates a change in the behavior as the limit of infinite Damkohler number is approached. At finite values of  $\hat{D}a$  we see from (7.4) that  $r_{*0} = O(a^{1/2})$ , whereas at infinite Damkohler number  $r_{*0} = O(a)$  as  $a \rightarrow 0$ .

The limit of infinite Damkohler number is referred to as the Burke-Schumann limit, and was alluded to earlier in discussion of the quasisteady response relation in Chapter 4. In this limit the oxidizer mass fraction at the flame sheet  $Y_{O_{*0}}$  is zero and  $r_{*0}$  is equal to the minimum critical radius  $r_{*0c}$  throughout the evolution, where

$$r_{*0c} = a \left( 1 + Le_O \frac{\ln \hat{q}}{\ln \left( 1 + Y_{O_\infty} \frac{\nu_F W_F}{\nu_O W_O} \right)} \right). \quad (7.5)$$

In this limit  $r_{*0}$  is directly proportional to  $a$ , so that from (7.2) the mass flux  $M_0$  is also directly proportional to  $a$  throughout the drop's lifetime. The ODE (2.52) can then be written

$$\frac{d}{dt}(a^2) = -\frac{2M_0}{\alpha_l a} = -c_\infty \quad (7.6)$$

where  $c_\infty$  is a constant given by

$$c_\infty = \frac{2}{\alpha_l} \left( 1 + Y_{O_\infty} \frac{\nu_F W_F}{\nu_O W_O} \right)^{1/Le_O} \ln \hat{q} \quad (7.7)$$

which depends on the physico-chemical parameters alone. In this limit, evolution of the drop radius is given by a version of the d-squared law, since the square of its diameter decreases linearly in time, with a simple expression for the drop lifetime  $\tau_l$ . That is, since  $a(0) = 1$ ,

$$a^2 = 1 - c_\infty t \quad \text{and} \quad \tau_l = c_\infty^{-1}. \quad (7.8)$$

At finite, fixed Damkohler number, (4.31) implies that the ratio  $r_{*0}/a$  is no longer constant, so that some departure from the d-squared law is to be anticipated, although its numerical value may be small, and in this case the perturbation expansion of (7.4) enables us to predict some aspects of the evolution at

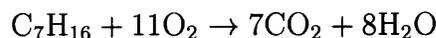
the end of the drop's lifetime analytically. From (7.2), at finite Damkohler number  $M_0/a = \ln \hat{q}(1 + O(a^{1/2}))$  as  $a \rightarrow 0$ , so that

$$\lim_{a \rightarrow 0} \frac{d}{dt}(a^2) = -c_f \quad \text{where} \quad c_f = \frac{2}{\alpha_l} \ln \hat{q}. \quad (7.9)$$

Hence, in the final stages and irrespective of whether the Damkohler number is finite or not, the drop radius tends to zero with the behavior of a square root in time,  $(\tau_l - t)^{1/2}$ , but the constant of proportionality, which is  $c_f$  or  $c_\infty$ , differs by an order  $O(1)$  amount. The slope of a plot of  $a^2$  versus time  $t$  therefore differs by order unity at the point where  $a = 0$ .

The limiting behavior of  $r_{*0}$  shows a more marked difference. At infinite Damkohler number, from (7.5) and (7.8), the flame sheet radius tends to zero like a square root in time, whereas at finite Damkohler number, from (7.4) and (7.9), the flame sheet radius tends to zero like  $(\tau_l - t)^{1/4}$ . This implies a relatively rapid collapse of the flame sheet during the final phase of the evolution.

**Parameter values.** We consider heptane ( $C_7H_{16}$ ) as the liquid fuel, which burns in air at 1 atmosphere pressure with an ambient temperature at or near an adiabatic flame temperature of 3273 K (i.e., 3000 C). Physico-chemical data for heptane can be found in [22] and [23], and [22] includes formulae for estimation of molecular transport coefficients at elevated temperatures. The overall reaction is



Dimensional values of the molecular transport coefficients at infinity are:  $\nu_\infty = 3.42 \text{ cm}^2 \text{ s}^{-1}$ ,  $\kappa_\infty = 5.6 \text{ cm}^2 \text{ s}^{-1}$ ,  $D_{F\infty} = 29.43 \text{ cm}^2 \text{ s}^{-1}$ , and  $D_{O\infty} = 12.74 \text{ cm}^2 \text{ s}^{-1}$ . The ambient density  $\rho_\infty = 1.1766 \times 10^{-4} \text{ g cm}^{-3}$  and the equilibrium sound speed  $a_\infty = 952 \text{ m s}^{-1}$ . Dimensionless groupings that appear in the analysis have the

estimated values given in Table 1.

$$\begin{array}{ll}
 l = \frac{L}{c_p T_\infty} = 0.0721 & Y_{O_\infty} = 0.232 \\
 T_{s0} = \frac{T_s}{T_\infty} = 0.1233 & \frac{\rho_\infty}{\rho_l} = 0.193 \times 10^{-3} \\
 \frac{q}{\nu_F W_F} = 1 - T_{s0} + l = 0.9488 & \frac{\rho_l W_O}{\nu_F W_F} = 3.513 \\
 \epsilon = \frac{R^0 T_\infty}{E} = 0.216 & \text{Pr} = 0.61 \\
 \text{Le}_F = 0.20 & \text{Le}_O = 0.46
 \end{array}$$

**Table 1.** Estimated values of dimensionless groupings for heptane burning in air at 1 atm and 3273 K.

The range of the initial droplet radius for which the analysis can be expected to apply can be estimated as follows. We choose the dimensional time scale  $t_0$  and initial velocity  $U_{\infty 0}$  to be

$$t_0 = \frac{\rho_l a_0^2}{\rho_\infty \kappa_\infty} \quad \text{and} \quad U_{\infty 0} = \frac{\rho_l a_0^2 g}{\rho_\infty \nu_\infty}, \quad (7.10)$$

where  $g$  is the gravitational acceleration. Then  $t_0$  is a dimensional estimate of the lifetime of a burning droplet, from ignition to final disappearance at zero radius, and is such that, from the definition of  $\alpha_l$  in Chapter 2,  $\alpha_l = 1$ . The expression for  $U_{\infty 0}$  is the dimensional value of the terminal speed for free-fall of a rigid sphere in inert, isothermal, viscous flow with Stokes law drag. This is relevant to cases where any momentum which the drop may have had on its initial injection into the flow has decayed, so that the last part of the inert warm-up stage that the drop experiences just before ignition occurs under conditions of slow relative translation.

Provided the kinematic viscosity and thermal diffusivity are of similar magnitude, as is the case here,  $t_0$  is also of the same order of magnitude as the time scale of approach to terminal speed for the Stokes law sphere.

We can now express the dimensionless parameter groupings in terms of  $a_0$  as follows

$$\begin{aligned}\bar{\omega} &= \frac{a_0}{U_{\infty 0} t_0} = \left(\frac{\rho_{\infty}}{\rho_l}\right)^2 \frac{\kappa_{\infty} \nu_{\infty}}{a_0^3 g}, & \text{Re} &= \frac{U_{\infty 0} a_0}{\nu_{\infty}} = \frac{\rho_l a_0^3 g}{\rho_{\infty} \nu_{\infty}^2}, \\ \text{Fr} &= \frac{U_{\infty 0}^2}{a_0 g} = \left(\frac{\rho_l}{\rho_{\infty}}\right)^2 \frac{a_0^3 g}{\nu_{\infty}^2} = \frac{1}{\bar{\omega} \text{Pr}}, & \beta_g &= \frac{\text{Pe} \rho_l}{\text{Fr} \rho_{\infty}} = \frac{1}{\bar{\omega} \text{Fr}} = \text{Pr}.\end{aligned}\quad (7.11)$$

For terrestrial applications  $g = 981 \text{ cm s}^{-2}$ . Referring to the scaling relations at the end of Chapter 2, we have already ensured that  $\bar{\omega} = O(\text{Fr}^{-1})$  and that  $\rho_{\infty}/\rho_l \sim \bar{\omega} \text{Re}$  since  $\alpha_l = 1$ , but we need to ensure further that  $\bar{\omega}$ ,  $\text{Re}$ , and  $\bar{\omega}/\text{Re}$  are small. Values of these last three groupings are given for an initial radius  $a_0$  in the range  $1 \mu\text{m}$  to  $1000 \mu\text{m}$  in Table 2. We infer from Table 2 that our model applies for an initial drop radius in the range from about  $20 \mu\text{m}$  to  $100 \mu\text{m}$ .

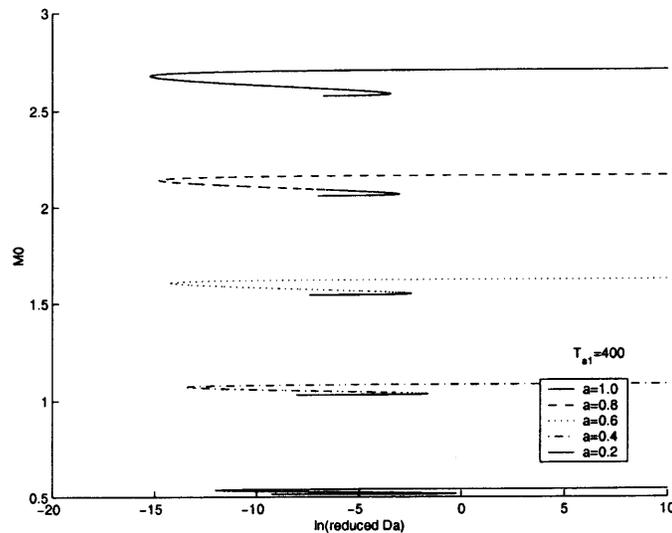
$a_0 \mu\text{m}$	$\text{Re} \sim a_0^3$	$\bar{\omega} \sim a_0^{-3}$	$\bar{\omega}/\text{Re} \sim a_0^{-6}$
$10^3$	$0.4 \times 10^3$	$7 \times 10^{-7}$	$1.75 \times 10^{-9}$
$10^2$	$0.4 \times 10^0$	$7 \times 10^{-4}$	$1.75 \times 10^{-3}$
$10^1$	$0.4 \times 10^{-3}$	$7 \times 10^{-1}$	$1.75 \times 10^3$
$10^0$	$0.4 \times 10^{-6}$	$7 \times 10^2$	$1.75 \times 10^9$

**Table 2.** Estimated values of the Reynolds number  $\text{Re}$ , Strouhal number  $\bar{\omega}$ , and ratio  $\bar{\omega}/\text{Re}$  versus initial drop radius  $a_0$ .

**Numerical results.** The parameter values above were used with an initial drop radius of  $a_0 = 100 \mu\text{m}$ , which implies that the reduced Damkohler number  $\hat{\text{Da}}$  is fixed and such that  $\ln \hat{\text{Da}} \simeq 7$ .

The quasisteady response for  $M_0$  versus  $\ln \hat{\text{Da}}$  at the initial radius is shown for different values of the temperature perturbation  $T_{s1}$  in Figure 3(a). The response is monotone for  $T_{s1}$  less than a critical value of about 200 but is S-shaped for greater values of  $T_{s1}$ . We recall that with increase of  $T_{s1}$  the reaction sheet temperature

increases relative to the temperature at infinity, and there is a weak but increasing conduction of heat outward from the flame sheet as  $T_{s1}$  increases above zero. Figure 3(b) shows how the  $M_0$  versus  $\ln \hat{D}a$  response varies as the dimensionless drop radius  $a$  decreases, with  $T_{s1}$  fixed at  $T_{s1} = 400$ . From this we see that when  $\ln \hat{D}a = 7$  the system state is at a point that is always on the upper branch of the S-shaped response and is bounded well away from its upper turning point. Figure 3(c) shows oxidizer mass fraction at the flame sheet  $Y_{O,*_0}$  versus flame sheet radius  $r_{*0}$  at different values of drop radius  $a$ , as given by the quasisteady relation (7.2).



**Figure 3a.** Leading order mass flux  $M_0$  versus  $\ln \hat{D}a$ , where  $\hat{D}a$  is the reduced Damkohler number, at fixed  $T_{s1}$  for different values of the dimensionless drop radius

$$a = 1(-0.2)0.2. \text{ Data for n-heptane fuel. } T_{s1} = 400$$

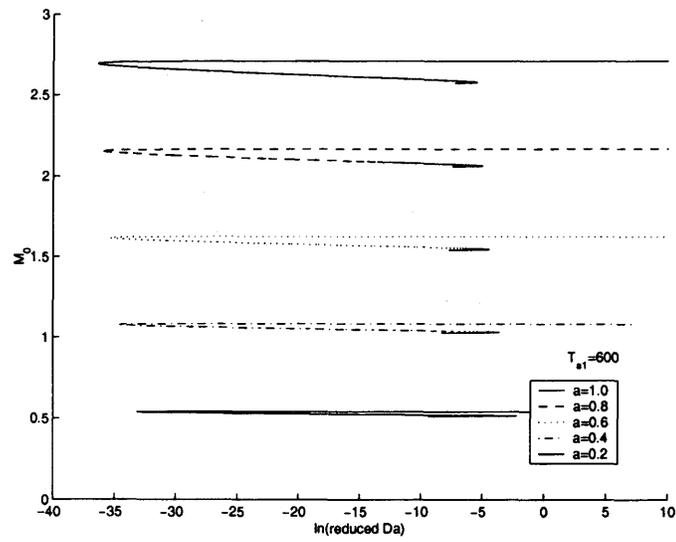


Figure 3b. Per Fig. 3(a) with  $T_{s_1} = 600$

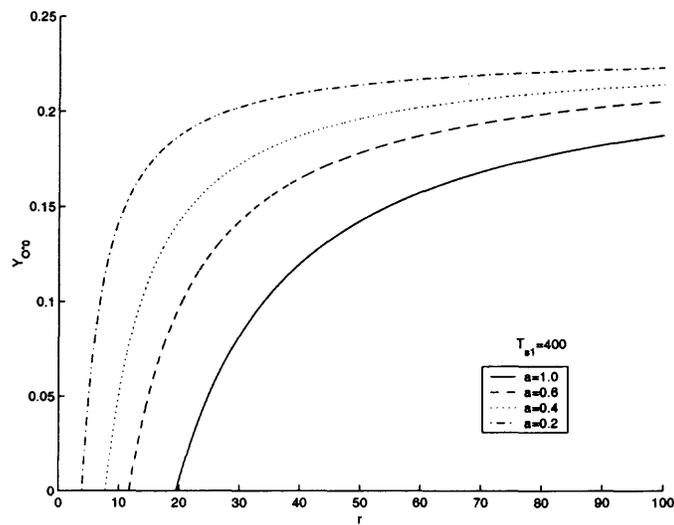
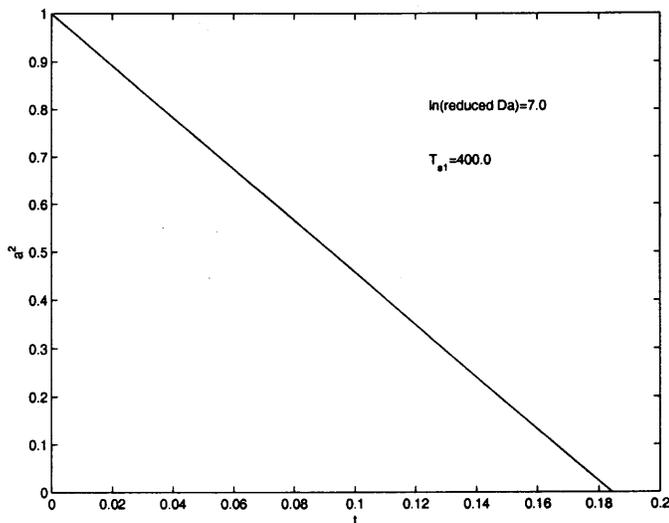


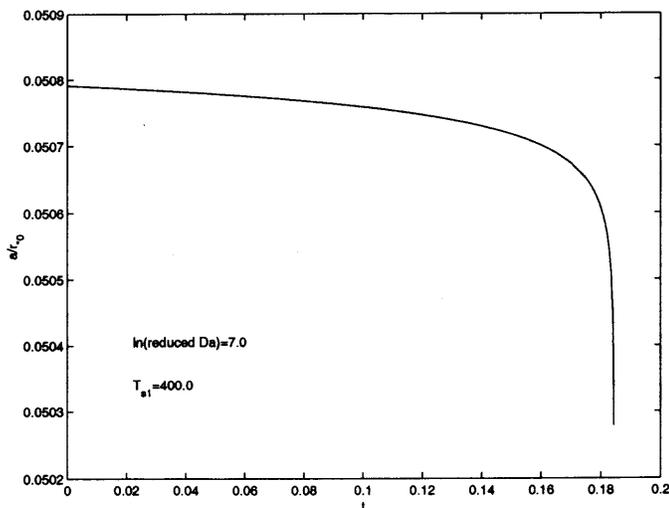
Figure 3c. Oxidizer mass fraction at the flame sheet  $Y_{O_{*0}}$  versus flame sheet radius  $r_{*0}$  at different values of drop radius  $a$

The group of Figures 4(a) to 4(g) show data for different aspects of the time evolution of the drop system with  $T_{s1} = 400$  and  $\ln \hat{Da} = 7$ . The first of these, Figure 4(a), shows  $a^2$  versus  $t$ . The plot very closely follows a straight line as would be predicted by a d-squared law. There is a very slight decrease in magnitude of the slope toward the end of the drop's lifetime, i.e., the time rate of change of  $a^2$  slows down, but this is so small as to be imperceptible. Figure 4(b) shows variation in the ratio of drop radius to flame sheet radius,  $a/r_{*0}$ , versus time  $t$ . The ratio decreases monotonically but remains nearly constant throughout the first 70% to 80% of the motion, and then deviates according to the expansion (7.4) for finite Damkohler number over the last 20% of the motion.

Figure 4(c) shows the scaled translational speed  $v_\infty$  versus  $t$ . The direction of motion of the drop is always downward, in the direction of gravitational acceleration as is to be expected, at a moderate speed that increases slightly with time during the first part of the motion. However, in the final stage as the drop radius decreases to zero,  $v_\infty$  increases and grows without bound at the end of the drop's lifetime. Figure 4(d) shows the graph of  $F$  versus  $t$ , where  $F\hat{\mathbf{k}}$  is the dimensionless force exerted by the fluid on the drop. The sign of  $F$  is seen to be negative throughout the motion, so that this net force like the drop's weight acts downwards. Both the net force  $F$  and the speed  $v_\infty$  show moderate increase in magnitude during the first 70% to 80% of the motion, but then increase rapidly to grow without bound in the final stage as the drop vanishes.



**Figure 4a.** Various computed quantities are shown in Figure 4(a) to Figure 4(h) for a heptane fuel drop that burns and falls under the action of gravity. The temperature perturbation  $T_{s1} = 400$ , the drop has initial radius  $a_0 = 100\mu\text{m}$ , the adiabatic flame temperature is  $3273\text{K}$ , and (for terrestrial gravity)  $g = 981 \text{ cm s}^{-2}$ . Other parameters, such as  $\ln \hat{Da} = 7.0$ , can be estimated from these values. The first figure shows  $a^2$  versus  $t$



**Figure 4b.** Ratio of drop radius to flame sheet radius  $a/r_{*0}$  versus time  $t$

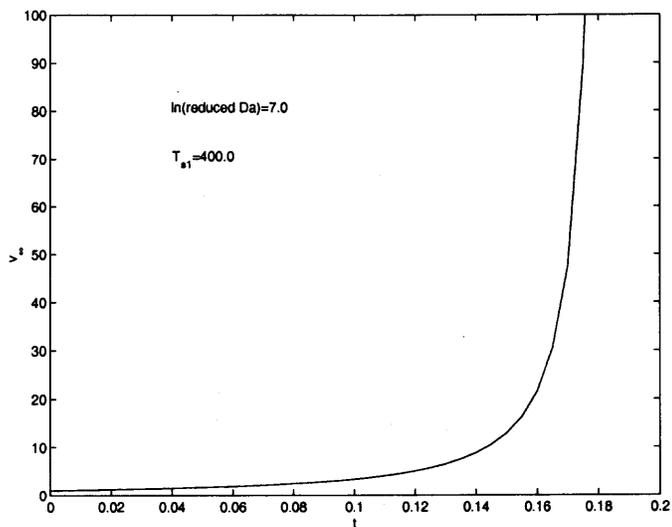


Figure 4c.  $v_\infty$  versus  $t$

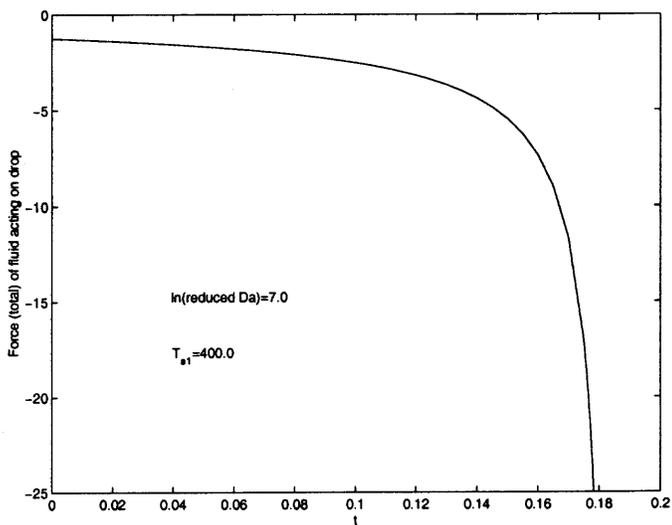


Figure 4d. Force (total)  $F$  of fluid acting on drop versus  $t$

Figure 4(e) shows the separate normalized contributions to the force exerted by the fluid on the drop. These are  $F_v$ ,  $F_p$ , and  $F_e$  normalized by their sum  $F = F_v + F_p + F_e$ . Here  $F_v$  is the contribution due to the effects of viscosity, and since  $F$  is negative, the data show that  $F_v$  is positive throughout the motion so that this 'viscous drag' always acts upwards and opposes the drop's motion, as theory predicts it must.  $F_p$  is the contribution due to departure of the pressure at the drop surface from spherical symmetry or 'pressure drag'. The data show that  $F_p$  is negative throughout the drop's lifetime, so that the pressure drag is found to act in the same downward direction as the motion.  $F_e$  is the contribution due to the momentum flux of matter expelled from the drop surface as it vaporizes and burns. The data show that this contribution to the net force, like the pressure drag, always acts in the downward direction implying that more momentum is expelled from the downstream part of the drop surface and this acts so as to propel the drop along its course. From the figure we see that this contribution of expelled momentum flux  $F_e$  dominates the total force  $F$  exerted by the fluid on the drop throughout all but the very earliest stage of the motion. Each of the separate contributions to  $F$  grows without bound in the final stage as the drop vanishes.

The solid line of Figure 4(f) shows the rate of transfer of energy from the drop to infinity  $E_\infty$  versus time  $t$ . We recall from Chapter 5 that this dimensionless total is equal to the dimensionless drop mass flux or burning rate  $M_0$ . There is therefore a net transfer of energy outward, which decreases monotonically in time and finally tends to zero with square root behavior as the drop vanishes.

Separate contributions to  $E_\infty$  are: (i)  $E_{\infty T}$  which is due to the transverse (i.e., vortical) component of the gas velocity in the far field, and (ii)  $E_{\infty L}$  which is due to the longitudinal (curl-free) component. It was pointed out in Chapter 5 that far from the drop the vortical component of the energy flux is concentrated entirely within the drop wake ( $\theta \simeq 0$ ), and the data in Figure 4(f) show that the integrated

flux  $E_{\infty T}$  is negative. Hence, although the direction of the net transfer of energy is outward from the drop to infinity, within the wake and at large distances from the drop there is an opposed (inward) transfer of energy toward the drop.

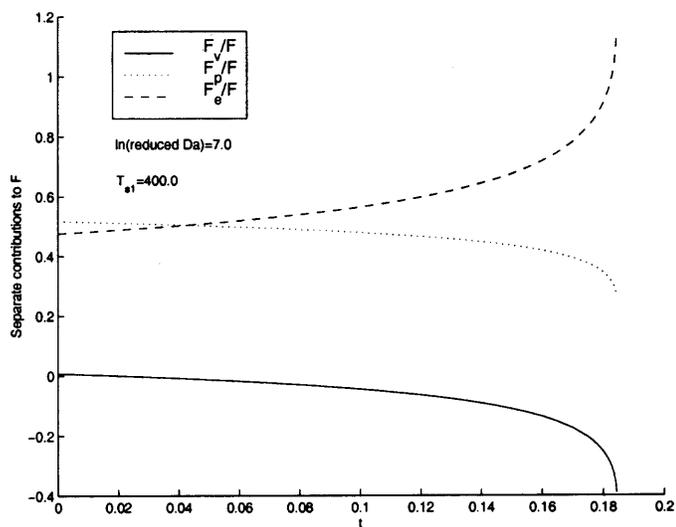


Figure 4e. Separate contributions to  $F$  versus  $t$

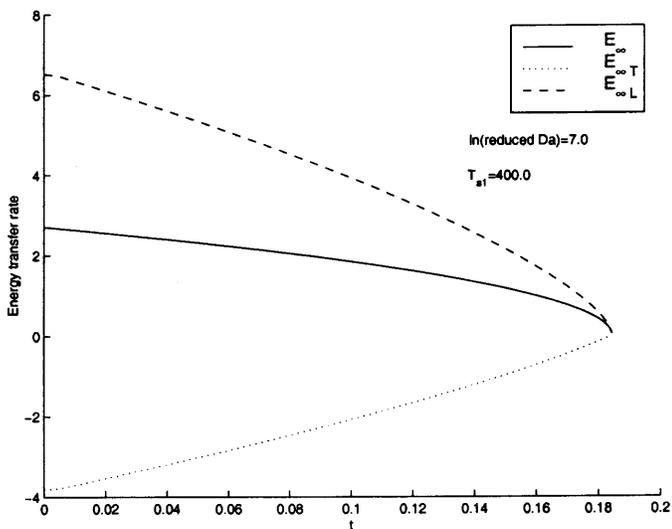


Figure 4f. Energy transfer rate  $E$  versus  $t$

Figure 4(g) shows the departure of the flame sheet from spherical symmetry, as represented by the coefficient  $r_{*11}$ , versus time. The flame sheet radius is given by  $r_* = r_{*0} + \epsilon(r_{*10} + r_{*11} \cos \theta) + o(\epsilon)$  and is assumed to be situated well within the near field where the gas velocity has a strong radial outflow. Note that there is a stagnation point in the upstream direction from the drop ( $\theta = \pi$ ), and this is in fact situated in the overlap (matching) region between the near field and far field. The prediction that  $r_{*11} < 0$  implies that the limaçon-shape of the flame sheet is more pointed toward the upstream direction and flat in the downstream direction.

Figure 4(h) shows  $M_0/4\pi a^2$ , which is the mass flux *per unit area* vaporized from the drop, versus time  $t$ . Note that this is due to the leading order radial component of velocity at the drop surface alone. The increase in this quantity as  $t \rightarrow \tau_l$  indicates that the final stage of acceleration of the drop is accompanied by more vigorous *overall* burning. The two features of drop mass approaching zero and expelled momentum flux  $F_e$  tending to infinity ( $F_e$  is given by integration of a flux per unit area, but is directed, and acts as a 'thrust') are sufficient to make the speed  $v_\infty$  tend to infinity - these are accompanied by more vigorous burning all over the drop surface, with this third feature being the precise cause of mass loss.

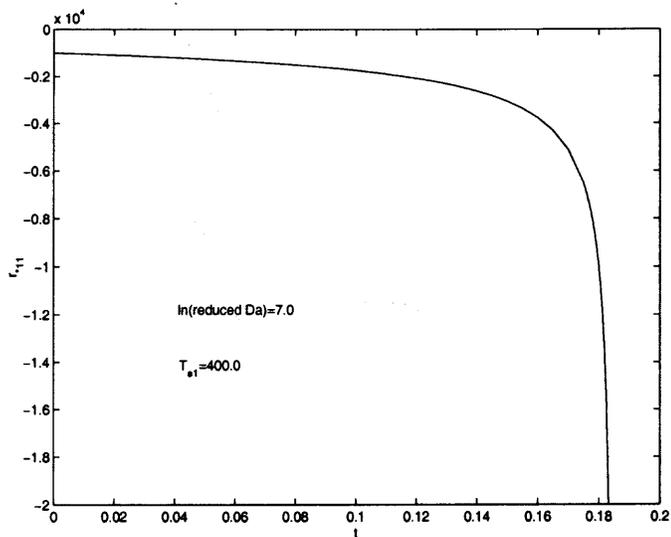


Figure 4g. Departure of the flame sheet from spherical symmetry, as represented by the coefficient  $r_{*11}$ , versus time  $t$

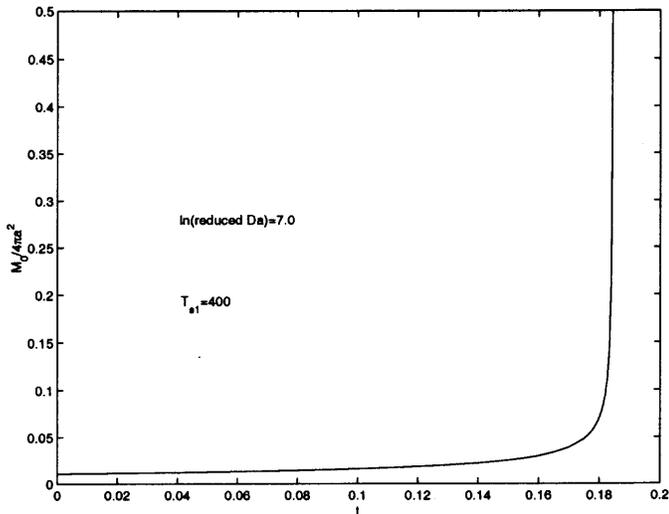


Figure 4h. Mass flux per unit area vaporized from drop surface  $M_0/4\pi a^2$  versus time  $t$

Figures 5(a) to 5(g) show data that parallels the data of Figures 4(a) to 4(g), respectively, but at a lower value of the temperature perturbation  $T_{s1} = 100$ . The Damkohler number and initial dimensional drop radius thus remain fixed at the values used for Figure 4 of  $a_0 = 100\mu\text{m}$  and  $\ln \hat{D}a = 7$ . We see first from Figure 3(a) that at this lower value of  $T_{s1}$  the quasisteady response for  $M_0$  versus  $\ln \hat{D}a$  is now monotone, not S-shaped, at  $t = 0$ , and it turns out that the  $M_0$  versus  $\ln \hat{D}a$  response remains monotone throughout the evolution.

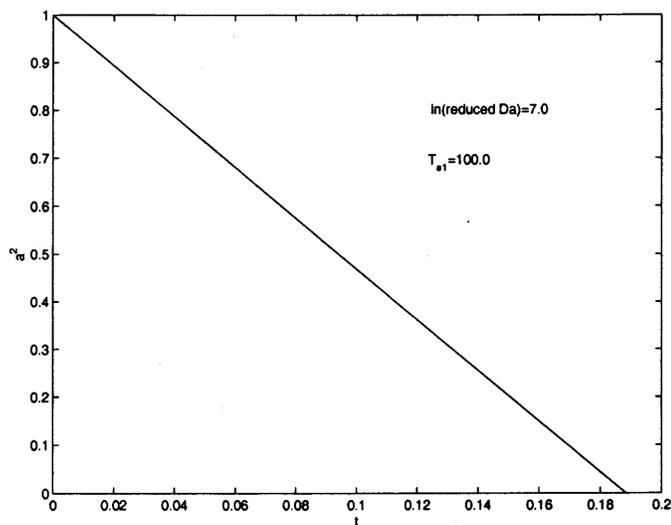
Figure 5(a) indicates that, despite the theoretical departure of the plot of  $a^2$  versus  $t$  from a straight line, the computed departure is barely noticeable for our example of what we believe to be relevant and typical parameter values. Comparison of Figures 4(a) and 5(a) shows that the drop lifetime is slightly extended at lower  $T_{s1}$ , and this is intuitively reasonable since the flame sheet temperature is reduced and burning is less vigorous. The flame sheet is also located further away from the drop surface, as is seen by comparing the different scales of the vertical axes in Figures 5(b) and 4(b). At both values of  $T_{s1}$  the ratio  $a/r_{*0}$  remains fairly constant throughout most of the motion but rapidly decreases to zero near the end of the drop's lifetime.

Figures 5(c) to 5(e) show similar behavior to their counterparts, i.e., Figures 4(c) to 4(e). The drop translates downwards, in the direction of gravitational acceleration, with a slow monotone increase in its speed during much of the motion. There is then a phase of rapid (downward) acceleration which sets in at about 70% to 80% through the drop's lifetime, and the translational speed is predicted to grow without bound in the limit as the drop radius tends to zero. The separate contributions to the force exerted by the fluid on the drop are again dominated by the expelled momentum flux term  $F_e$ , which is directed downward and propels the drop along at greater rate in its final moments. Note the consistency that as  $t \rightarrow \tau_l$  the drop

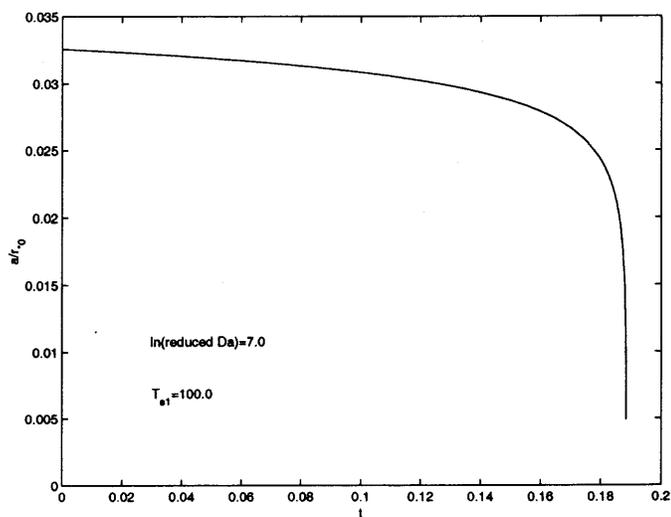
radius, mass, and burning rate tend to zero while the net force  $F$  and speed  $v_\infty$  tend to infinity.

There is little quantitative difference to be found between the translational speed and contributions to  $F$  for the two different values of  $T_{s1}$  used in Figures 4 and 5. Comparing Figures 5(f) to 5(h) with Figures 4(f) and 4(h) we see that there is also little change at this lower value of  $T_{s1}$  in the rate of energy transfer to infinity  $E_\infty$  or in the drop shape and rate of mass loss. The contribution to  $E_\infty$  of the curl-free component  $E_{\infty L}$  is of greater magnitude than  $E_{\infty T}$  in both cases, per Figures 5(f) and 4(f) with a resultant net transfer of energy from the drop to infinity. As discussed after equation (5.66) in Chapter 5, the flux of energy that integrates to  $E_{\infty L}$  is distributed uniformly over the sphere  $S_\infty$ , i.e., it is independent of direction or the meridional angle  $\theta$ , while the flux that integrates to  $E_{\infty T}$  is concentrated in the wake and is directed inwards against the net transfer.

It is known that the spherical diffusion flame can be unstable on a rapid, diffusional-thermal time scale that is based on drop size divided by near field gas velocity and a length scale equal to drop size. Much of the recent analytical work in this area is due to either J.S. Kim or M. Matalon and S. Cheatham. We suspect that the value of the Damkohler number used here ( $\hat{D}a = e^7$ ) would be such that the flame sheet is stable to disturbances of this type, but in the absence of stability results for the nearly adiabatic burning limit we are clearly unable to be certain. Also, we note that analysis of a diffusional-thermal instability generally adopts Lewis numbers close to one, and predicts instability at Lewis numbers far away from one, whereas our estimated Lewis numbers given in Table 1 are  $Le_F = 0.20$  and  $Le_O = 0.46$ .



**Figure 5a.** Figure 5(a) to Figure 5(h) show computed quantities per Figure 4(a) to Figure 4(h), respectively, but with smaller value of the temperature perturbation  $T_{s1} = 100$ ; i.e., the first figure shows  $a^2$  versus  $t$



**Figure 5b.** Ratio of drop radius to flame sheet radius  $a/r_{*0}$  versus time  $t$

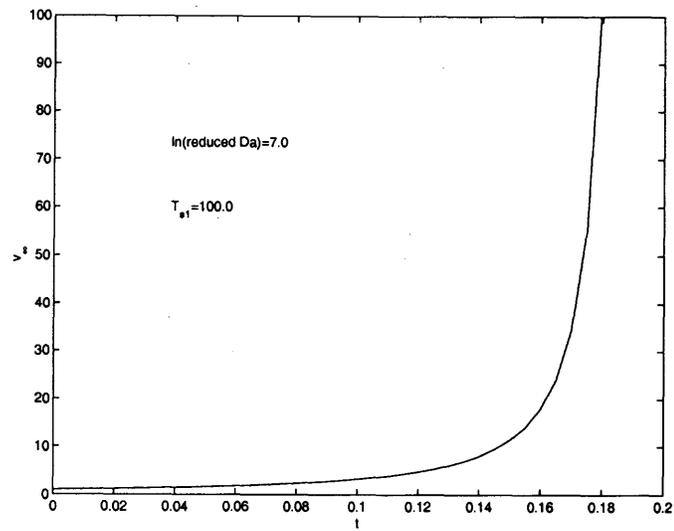


Figure 5c.  $v_\infty$  versus  $t$

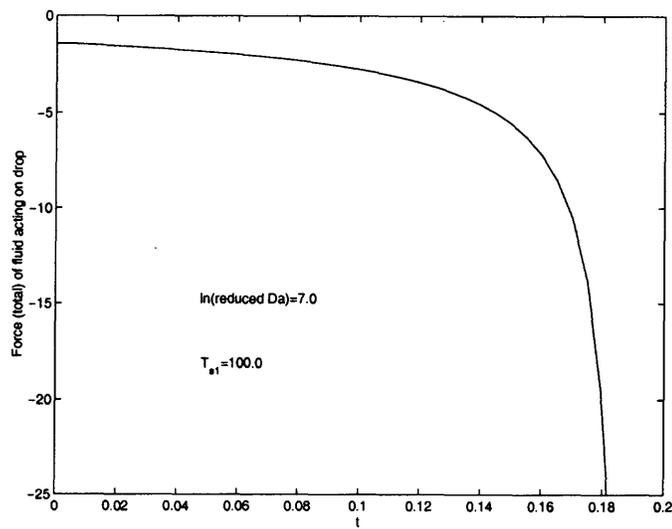


Figure 5d. Force (total)  $F$  of fluid acting on drop versus  $t$

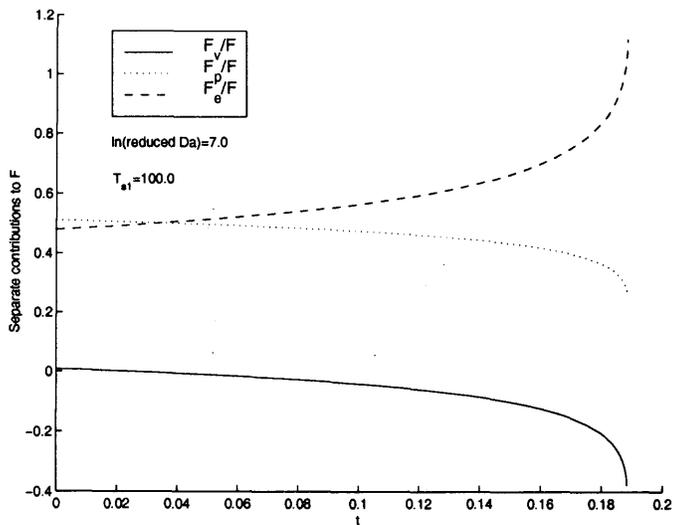


Figure 5e. Separate contributions to  $F$  versus  $t$

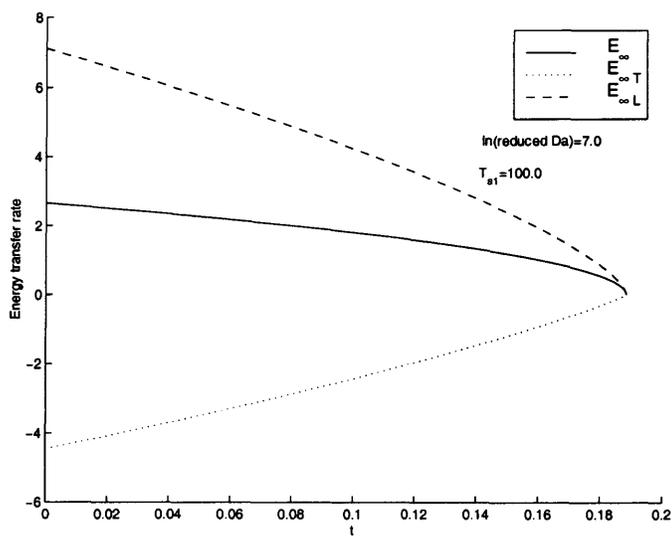


Figure 5f. Energy transfer rate  $E$  versus  $t$

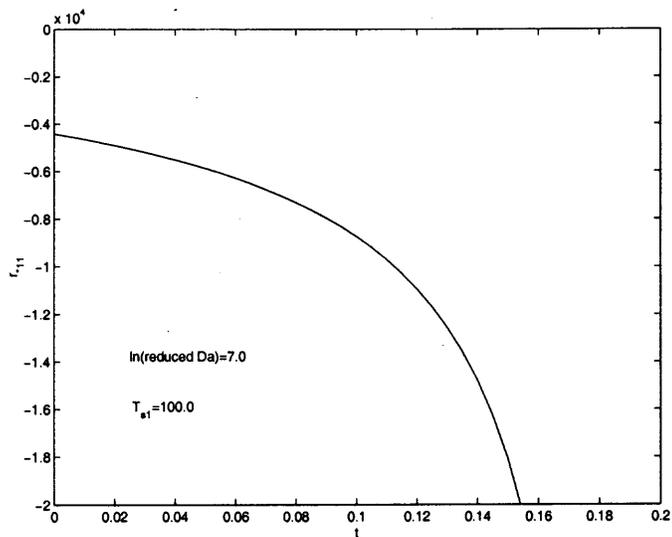


Figure 5g.  $r_{*11}$ , versus time

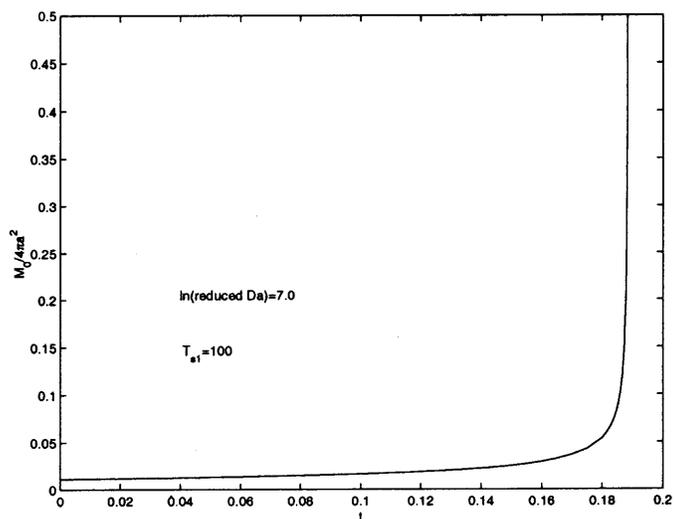


Figure 5h. Mass flux per unit area vaporized from drop surface  $M_0/4\pi a^2$  versus time  $t$

## APPENDIX A

### BOUNDARY VALUE PROBLEM

Here we provide some details on the expression of the two point boundary value problem of Chapter 6 in a form suitable for computation.

**The differential operator as a first order system.** We write equations (6.64) to (6.67) as a system of ten first order equations in terms of the variables  $x_i$  ( $i = 1, \dots, 10$ ), which are defined at (6.68) as

$$\begin{aligned} x_1 = T_{1n}, \quad x_2 = \frac{dT_{1n}}{dr}, \quad x_3 = u_{1n}, \quad x_4 = v_{1n}, \quad x_5 = \frac{dv_{1n}}{dr}, \quad x_6 = \frac{d^2v_{1n}}{dr^2}, \\ x_7 = Y_{F1n}, \quad x_8 = \frac{dY_{F1n}}{dr}, \quad x_9 = Y_{O1n}, \quad x_{10} = \frac{dY_{O1n}}{dr}. \end{aligned} \quad (\text{A.1})$$

Recall that here  $n = 1$ . Then

$$\frac{dx_i}{dr} = a_{ij}^-(r)x_j \quad i, j = 1, \dots, 10, \quad r \in (a_0, r_{*0}) \quad (\text{A.2})$$

where the non-zero entries of the matrix  $A^- = \{a_{ij}^-\}$  as functions of  $r$ , after some simplification and with a prime denoting a derivative with respect to  $r$ , are

$$\begin{aligned} a_{12}^- = a_{45}^- = a_{56}^- = a_{78}^- = a_{910}^- = 1 \\ a_{21}^- = \frac{1}{r^2} \left( n(n+1) - M_0 \frac{T_0'}{T_0} \right), \quad a_{22}^- = \frac{M_0}{r^2} - \frac{2}{r}, \quad a_{23}^- = \frac{T_0'}{T_0}, \\ a_{31}^- = -\frac{M_0}{r^2} \frac{T_0'}{T_0}, \quad a_{32}^- = \frac{M_0}{r^2}, \quad a_{33}^- = \frac{T_0'}{T_0} - \frac{2}{r}, \quad a_{34}^- = -\frac{n(n+1)}{r}, \\ a_{61}^- = -\frac{M_0}{r^4 T_0} \left\{ 2T_0' \left( 3 - \frac{M_0}{r} \left( 1 - \frac{1}{\text{Pr}} \right) \right) + \frac{T_0}{r} \left( n(n+1) - \frac{2M_0}{\text{Pr}} \frac{1}{r} \right) \right\}, \\ a_{62}^- = \frac{M_0}{r^4} \left( 6 - \frac{M_0}{r} \left( 1 - \frac{1}{\text{Pr}} \right) \right), \\ a_{63}^- = \frac{1}{r^2 T_0} \left\{ 2T_0' \left( 3 - \frac{M_0}{r} \left( 1 - \frac{1}{\text{Pr}} \right) \right) - \frac{T_0}{r} \left( 6 - n(n+1) + \frac{4M_0}{\text{Pr}} \frac{1}{r} \right) \right\}, \\ a_{64}^- = \frac{1}{r^2 T_0} \left\{ n(n+1) \left( T_0' - \frac{2}{r} T_0 \right) - \frac{M_0}{\text{Pr}} (n(n+1) + 2) \frac{1}{r^2} T_0 \right\}, \end{aligned} \quad (\text{A.3})$$

$$\begin{aligned}
a_{65}^- &= \frac{2n(n+1)}{r^2}, a_{66}^- = \frac{1}{r} \left( \frac{M_0}{\text{Pr}} \frac{1}{r} - 3 \right), \\
a_{81}^- &= -\frac{M_0 \text{Le}_F}{r^2 T_0} \frac{dY_{F_0}}{dr}, a_{83}^- = \frac{\text{Le}_F}{T_0} \frac{dY_{F_0}}{dr}, \\
a_{87}^- &= \frac{n(n+1)}{r^2}, a_{88}^- = -\left( \frac{2}{r} - \frac{M_0 \text{Le}_F}{r^2} \right), \\
a_{101}^- &= -\frac{M_0 \text{Le}_O}{r^2 T_0} \frac{dY_{O_0}}{dr}, a_{103}^- = \frac{\text{Le}_O}{T_0} \frac{dY_{O_0}}{dr}, \\
a_{109}^- &= \frac{n(n+1)}{r^2}, a_{1010}^- = -\left( \frac{2}{r} - \frac{M_0 \text{Le}_O}{r^2} \right).
\end{aligned}$$

The system simplifies when  $r > r_{*0}$ . The general solution is given in closed form in terms of elementary functions and an exponential integral in Chapter 6. However, the governing equations for  $r > r_{*0}$  can also be formed readily from the system for  $r < r_{*0}$  by noting that  $T_{1n} = 0$  and  $Y_{F_{1n}} = 0$  for  $n = 1$  as a consequence of matching across the flame zone, and by noting that the leading order temperature  $T_0 = 1$ . Thus, for  $r > r_{*0}$  we have

$$x_1 = x_2 = x_7 = x_8 = 0 \quad (\text{A.4})$$

so that the system is sixth order, with

$$\frac{dx_i}{dr} = a_{ij}^+(r) x_j \quad i, j = 3, \dots, 6, 9, 10, \quad r > r_{*0} \quad (\text{A.5})$$

where the  $6 \times 6$  matrix  $a_{ij}^+$  is given by setting  $T_0 = 1$  in  $a_{ij}^-$  for  $i, j = 3, \dots, 6, 9, 10$ .

**Boundary conditions at the drop surface.** The boundary conditions at the drop surface,  $r = a$ , are given in equations (6.31) to (6.35). In terms of the variables  $x_i$  of (A.1) they define the following five boundary operators  $B_i$  ( $i = 1, \dots, 5$ ), and five

boundary conditions which are homogeneous: at the left end-point  $r = a$ ,

$$\begin{aligned}
B_1 x &\equiv x_1 = 0 \\
B_2 x &\equiv x_2 - \frac{l}{T_{s0}} x_3 = 0 \\
B_3 x &\equiv x_4 = 0 \\
B_4 x &\equiv x_8 - \frac{M_0 \text{Le}_F}{a^2} x_7 + \frac{\text{Le}_F}{T_{s0}} \left( \frac{q}{l\nu_F W_F} \right)^{-\text{Le}_F} x_3 = 0 \\
B_5 x &\equiv x_{10} - \frac{M_0 \text{Le}_O}{a^2} x_9 - \frac{\text{Le}_O}{T_{s0}} Y_{O*0} \left( \frac{q}{l\nu_F W_F} \right)^{-\text{Le}_O} x_3 = 0.
\end{aligned} \tag{A.6}$$

**Jump conditions at  $r = r_{*0}$ .** The jump conditions are given in equations (6.37) to (6.43) of Chapter 6. When these are written in terms of the  $x_i$ , a set of nine boundary operators  $B_i$  ( $i = 6, \dots, 14$ ) acting on the solution  $x_i(r_{*0}^-)$  at the left of  $r = r_{*0}$  is defined. The boundary conditions can be regarded as inhomogeneous,  $B_i x = \beta_{i-5}$  for  $i = 6, \dots, 14$ , where the  $\beta_i$  are linear combinations of the four unknown constants ( $A_{\Phi 1}, c_{21}, r_{*11}, B_1$ ) introduced at (6.78) and (6.85) in the solution for  $r > r_{*0}$ . These linear combinations are known functions of the data  $(a, r_{*0}, v_\infty)$  of the leading order problem, since quantities such as  $M_0, Y_{O*0}$ , and the temperature profile  $T_0(r)$  are, in turn, known in terms of them. Note that  $c_{11}$  is not included among the four unknown constants, since, per the first of the matching relations (6.82) it is a known function of  $v_\infty$ , whereas  $A_{\Phi 1}$  is included and once found it will determine the coefficient  $\chi_0$  of the far field solution per the second of the matching relations (6.82). The boundary conditions are: at the right end-point  $r = r_{*0}^-$ ,

$$\begin{aligned}
B_6x &\equiv x_1 = \beta_1 \\
B_7x &\equiv x_3 = \beta_2 \\
B_8x &\equiv x_4 = \beta_3 \\
B_9x &\equiv x_5 = \beta_4 \\
B_{10}x &\equiv x_6 + \frac{M_0}{r_{*0}^3}x_2 = \beta_5 \\
B_{11}x &\equiv x_7 = \beta_6 \\
B_{12}x &\equiv x_9 = \beta_7 \\
B_{13}x &\equiv x_2 + \left( \frac{q}{Le_F \nu_F W_F} \right) x_8 = \beta_8 \\
B_{14}x &\equiv x_2 + \left( \frac{q}{Le_O \nu_O W_O} \right) x_{10} = \beta_9.
\end{aligned} \tag{A.7}$$

An indication that the boundary value problem is well-posed is given by observing that it consists of a tenth order differential operator acting on the solution for  $r \in (a, r_{*0})$  while the solution for  $r > r_{*0}$  is described by the four unknowns  $(A_{\Phi 1}, c_{21}, r_{*11}, B_1)$ . There is a total of fourteen boundary data at the end-points,  $r = a$  and  $r = r_{*0}^-$ , provided the boundary data are independent, which is in fact the case, as we see.

The linear combinations of the  $\beta_i$  in terms of the four unknowns  $(A_{\Phi 1}, c_{21}, r_{*11}, B_1)$  form the algebraic system

$$\begin{array}{c}
\left| \begin{array}{cccc}
0 & 0 & a_{13} & 0 \\
a_{21} & a_{22} & a_{23} & 0 \\
a_{31} & a_{32} & 0 & 0 \\
a_{41} & a_{42} & a_{43} & 0 \\
a_{51} & a_{52} & a_{53} & 0 \\
0 & 0 & a_{63} & 0 \\
a_{71} & a_{72} & a_{73} & a_{74} \\
0 & 0 & a_{83} & 0 \\
a_{91} & a_{92} & a_{93} & a_{94}
\end{array} \right|
\begin{array}{c}
A_{\Phi 1} \\
c_{21} \\
r_{*11} \\
B_1
\end{array}
=
\begin{array}{c}
\left| \begin{array}{c}
\beta_1 \\
\beta_2 - c_{11} \\
\beta_3 + c_{11} \\
\beta_4 \\
\beta_5 + \mu_5 c_{11} \\
\beta_6 \\
\beta_7 + \mu_7 c_{11} \\
\beta_8 \\
\beta_9 + \mu_9 c_{11}
\end{array} \right|.
\end{array}
\tag{A.8}$$

If we omit the notation of left and right hand limits by writing  $T'_0(r_{*0})$  for  $T_0(r_{*0}^-)$  and  $u_{\Phi 1}(r_{*0})$  for  $u_{\Phi 1}(r_{*0}^+)$  etc., then with  $n = 1$ , the matrix elements  $a_{ij}$  are given in terms of the data  $(a, r_{*0}, v_{\infty})$  by

$$\begin{aligned}
a_{13} &= -T'_0(r_{*0}), & a_{21} &= u_{\Phi n}(r_{*0}), & a_{22} &= \frac{1}{r_{*0}^{n+2}}, & a_{23} &= -\frac{M_0}{r_{*0}^2} T'_0(r_{*0}), \\
a_{31} &= v_{\Phi n}(r_{*0}), & a_{32} &= \frac{1}{(n+1)r_{*0}^{n+2}}, \\
a_{41} &= v'_{\Phi n}(r_{*0}), & a_{42} &= \frac{-(n+2)}{(n+1)r_{*0}^{n+3}}, & a_{43} &= \frac{M_0}{r_{*0}^3} T'_0(r_{*0}), \\
a_{51} &= v''_{\Phi n}(r_{*0}) - \frac{T'_0(r_{*0})}{r_{*0}} u_{\Phi n}(r_{*0}), & a_{52} &= \frac{(n+2)(n+3)}{(n+1)r_{*0}^{n+4}} - \frac{T'_0(r_{*0})}{r_{*0}^{n+3}}, \\
a_{53} &= \frac{-4M_0}{r_{*0}^4} T'_0(r_{*0}), & a_{63} &= \left( \frac{\text{Le}_F \nu_F W_F}{q} \right) T'_0(r_{*0}), \\
a_{71} &= \gamma_0 g_{An}(r_{*0}), & a_{72} &= \gamma_0 g_{c2n}(r_{*0}), & a_{73} &= \left( \frac{\text{Le}_O \nu_O W_O}{q} \right) T'_0(r_{*0}), \\
a_{74} &= g_{Hn}(r_{*0}), & a_{83} &= \frac{M_0}{r_{*0}^2} (\text{Le}_F - 1) T'_0(r_{*0}), \\
a_{91} &= \left( \frac{q}{\text{Le}_O \nu_O W_O} \right) \gamma_0 g'_{An}(r_{*0}), & a_{92} &= \left( \frac{q}{\text{Le}_O \nu_O W_O} \right) \gamma_0 g'_{c2n}(r_{*0}), \\
a_{93} &= \frac{M_0}{r_{*0}^2} (\text{Le}_O - 1) T'_0(r_{*0}), & a_{94} &= \left( \frac{q}{\text{Le}_O \nu_O W_O} \right) g'_{Hn}(r_{*0}).
\end{aligned}
\tag{A.9}$$

The  $\mu_i$  are also known functions of the data  $(a, r_{*0}, v_\infty)$ , given by

$$\mu_5 = \frac{T'_0(r_{*0})}{r_{*0}}, \quad \mu_7 = \frac{\gamma_0}{2} e^{-\frac{Le_0 M_0}{r_{*0}}}, \quad \mu_9 = \left( \frac{q}{Le_0 \nu_0 W_0} \right) \frac{M_0 Le_0}{r_{*0}^2} \frac{\gamma_0}{2} e^{-\frac{Le_0 M_0}{r_{*0}}}.$$

The algebraic system (A.8) comprises nine equations in the four unknowns  $(A_{\Phi 1}, c_{21}, r_{*11}, B_1)$ , and there are therefore five conditions for the system to have a consistent solution. In the matrix on the left-hand side, the first and third row vectors are linearly independent by inspection, and the second row vector is linearly independent of these two provided

$$\Delta \equiv a_{21}a_{32} - a_{22}a_{31} \neq 0. \quad (\text{A.10})$$

This condition can be verified immediately after finding the solution for  $r > r_{*0}$ , that is, before the shooting method is applied to solve the two point boundary value problem on  $r \in (a, r_{*0})$ . Numerical studies show that this condition is always satisfied; assuming that the first three row vectors are linearly independent, and therefore span a three-dimensional subspace of the four-dimensional solution space, we can find four conditions for the consistency of the system (A.8) except for the seventh and ninth equations, for which the row vectors lie outside the three-dimensional subspace. We order these consistency conditions on the sixth, eighth, fourth, and fifth equations respectively; consistency of the seventh and ninth

equations gives a fifth condition, and these are labeled  $\mathcal{F}_i$ ,  $i = 1, \dots, 5$ . Thus

$$\begin{aligned}
\mathcal{F}_1 &\equiv a_{63}\beta_1 - a_{13}\beta_6 = 0 \\
\mathcal{F}_2 &\equiv a_{83}\beta_1 - a_{13}\beta_8 = 0 \\
\mathcal{F}_3 &\equiv \left( (a_{42}a_{31} - a_{41}a_{32})\frac{a_{23}}{a_{13}} + \Delta\frac{a_{43}}{a_{13}} \right) \beta_1 + (a_{41}a_{32} - a_{42}a_{31})\beta_2 + \\
&\quad (a_{42}a_{21} - a_{41}a_{22})\beta_3 - \Delta\beta_4 + \{a_{42}(a_{21} + a_{31}) - a_{41}(a_{22} + a_{32})\}c_{11} = 0 \\
\mathcal{F}_4 &\equiv \left( (a_{52}a_{31} - a_{51}a_{32})\frac{a_{23}}{a_{13}} + \Delta\frac{a_{53}}{a_{13}} \right) \beta_1 + (a_{51}a_{32} - a_{52}a_{31})\beta_2 + \\
&\quad (a_{52}a_{21} - a_{51}a_{22})\beta_3 - \Delta\beta_5 + \\
&\quad \{a_{52}(a_{21} + a_{31}) - a_{51}(a_{22} + a_{32}) + \Delta\mu_5\}c_{11} = 0 \\
\mathcal{F}_5 &\equiv \left( [(a_{74}a_{92} - a_{72}a_{94})a_{31} - (a_{74}a_{91} - a_{71}a_{94})a_{32}]\frac{a_{23}}{a_{13}} + (a_{74}a_{93} - a_{73}a_{94})\frac{\Delta}{a_{13}} \right) \beta_1 \\
&\quad - \{(a_{74}a_{92} - a_{72}a_{94})a_{31} - (a_{74}a_{91} - a_{71}a_{94})a_{32}\}\beta_2 \\
&\quad + \{(a_{74}a_{92} - a_{72}a_{94})a_{21} - (a_{74}a_{91} - a_{71}a_{94})a_{22}\}\beta_3 + \Delta a_{94}\beta_7 - \Delta a_{74}\beta_9 \\
&\quad + \{(a_{74}a_{92} - a_{72}a_{94})(a_{21} + a_{31}) - (a_{74}a_{91} - a_{71}a_{94})(a_{22} + a_{32})\} \\
&\quad + \Delta(\mu_7 a_{94} - \mu_9 a_{74})\}c_{11} = 0.
\end{aligned} \tag{A.11}$$

**The shooting method.** We introduce five ‘false’ boundary operators at  $r = a$ , which are independent of the given boundary operators  $B_i$ ,  $i = 1, \dots, 5$  of (A.7). The five ‘false’ boundary conditions at  $r = a$  are, in general, inhomogeneous and are used together with the five given homogeneous boundary conditions as a set of initial conditions to integrate the tenth order system (A.2) from  $r = a$  to  $r = r_{*0}$ . Suitable ‘false’ boundary conditions are: at  $r = a$ ,

$$\begin{aligned}
B_{f1}y &\equiv x_2 = c_1 \\
B_{f2}y &\equiv x_5 = c_2 \\
B_{f3}y &\equiv x_6 = c_3 \\
B_{f4}y &\equiv x_7 = c_4 \\
B_{f5}y &\equiv x_9 = c_5.
\end{aligned} \tag{A.12}$$

At the end of the integration, the boundary operators  $B_i x = \beta_{i-5}$  ( $i = 6, \dots, 14$ ) of (A.8) are evaluated and the corresponding  $\beta_i$  ( $i = 1, \dots, 9$ ) and  $\mathcal{F}_i$ , ( $i = 1, \dots, 5$ ) of (A.12) are known. Each  $\beta_i$  is a function of the five parameters  $c_j$  of (A.13), and hence the functionals  $\mathcal{F}_i$  are also functions of the  $c_j$ 's. The shooting method is to iterate on the five parameters  $c_j$  of (A.13) until the five constraints  $\mathcal{F}_i = 0$  are satisfied.

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