ABSTRACT

ANALYTICAL SOLUTIONS OF OPENINGS FORMED BY INTERSECTION OF A CYLINDRICAL SHELL AND AN OBLIQUE NOZZLE UNDER INTERNAL PRESSURE

by
Hengming Cai

Since several decades ago, many authors have published their research results about local stress distributions of shells and shell-nozzles both analytically and numerically. However, there has not been a published paper which deals with analytical solutions of cylindrical shell and oblique nozzle, even though in the case of openings formed by intersection of a cylindrical shell and an oblique nozzle.

A comprehensive analytical study of local stress factors at the area of openings formed by intersection of a cylindrical shell and an oblique nozzle under internal pressure is presented in this dissertation.

By means of traditional approach in theory of elasticity, geometric equations, physical equations and equilibrium equations are derived and then simplified under the conditions of thin shell and internal pressure. The concepts of normalized forces and moments in the mid-surface are established to make all governing partial differential equations mathematically solvable.

This dissertation mathematically determines the exact geometric description of intersection formed by a cylindrical shell and an oblique nozzle. This result is not only the boundary conditions of the present study, but also a basis for analytical solutions of intersection formed by a cylindrical shell and an elliptical nozzle in the future.

Introducing the displacement function, this study combines the geometric equations, physical equations, equilibrium equations and boundary conditions to obtain the analytical solutions.

Finally, this dissertation calculates the results of five cases, which correspond to the intersection angles of 90°, 75°, 60°, 45° and 30° respectively. The results are presented in the forms of stress concentration factors (SCF) and described in the fourteen figures.

The typical calculations indicate:

1. When the intersection angle is 90°, the stress results are in good agreement with the existing literature [10].

2. At the neighborhood of point A, both of circumferential stresses and longitudinal stresses increase as the intersection angle decreases from 90° to 30°, and the closer to the 30°, the faster the increase becomes. Therefore, among all angles from 90° to 30°, the intersection angle 90° has the least local stresses.

3. At the neighborhood of point C, when the intersection angle varies from 90° to 30°, circumferential stresses remain virtually constant, however, longitudinal stresses are compressive and they remain constant on the outside surface, but, increase on the inside surface.

4. After consideration of all influential factors, it is suggested that the intersection angles from 90° to 60° should be the best choices. The intersection angles from 60° to 45° can be selected if the internal pressure is not too high. The intersection angles less than 45° should be avoided as practical as possible.

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A Dissertation
Submitted to the Faculty of
New Jersey Institute of Technology
in Partial Fulfillment of the Requirements for the Degree of
Doctor of Philosophy

Department of Mechanical Engineering

January 1997

APPROVAL PAGE

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This dissertation is dedicated to my mother who not only gave me my life,
but also went through all kinds of untold hardship for my growth
until the final minute of her life.

ACKNOWLEDGMENT

The author wishes to express his sincere appreciation to his dissertation advisor, Dr. Benedict C. Sun, for his essential guidance, friendship and moral support throughout this research.

Sincere gratitude to professor John Tavantzis for his important support in the mathematical area.

Special thanks to Dr. Rong-Yaw Chen, Dr. Bernard Koplik and Dr. Zhiming Ji for their serving as members of the committee and their kindly suggestions.

TABLE OF CONTENTS

Chapter		Page
1	INTRODUCTION	1
2	LITERATURE SURVEY	4
3	GENERAL ASSUMPTIONS	12
4	GEOMETRIC EQUATIONS	13
	4.1 Lame's Coefficients in Curvilinear Coordinate System	13
	4.2 Curvatures and Radius of Curvatures	15
	4.3 Geometric Equations for General Elastic Bodies	16
	4.4 Geometric Equations for General Shell Bodies	18
	4.5 Geometric Equations for Circular Cylindrical Shells	23
5	PHYSICAL EQUATIONS	26
6	EQUILIBRIUM EQUATIONS	34
7	BOUNDARY GEOMETRY	39
	7.1 Boundary Geometry	39
	7.2 Relationship between the Directions of All Kinds of coordinates	48
8	DERIVATION ANALYTICAL SOLUTIONS	57
	8.1 Derivation of the Basic Differential Equations	57
	8.2 Displacement Function and General Solutions of Basic Equations	62
	8.3 Application of the Boundary Conditions	70
	8.4 Analytical Solutions	76

TABLE OF CONTENTS
(Continued)

Chapter		Page
9	COMPARISON OF RESULTS	82
10	CONCLUSIONS	104
REFERENCES		106

LIST OF TABLES

Table **Page**

9.1 Comparison of the maximum values of stress concentration factors at different intersection angles at the point A and at the point C on the intersection curve 101

LIST OF FIGURES

Figure		Page
1.1	Flow chart of this study	3
4.1	Differential element of a general body	14
4.2	Differential element of a general shell body	19
4.3	Differential element of a cylindrical shell body	24
5.1	Membrane forces and bending moments on mid-surface	27
6.1	(a) Static equilibrium of forces on a differential element of a circular cylindrical shell body	36
6.1	(b) Static equilibrium of moments on a differential element of a circular cylindrical shell body	37
7.1	Cylindrical shell with oblique nozzle / vertical elliptical nozzle	40
7.2	Intersection of shell and oblique nozzle on mid-surface	43
7.3	Directions of unit vectors on intersection	49
8.1	Distribution of forces and moments on the intersection of the shell and nozzle	71
9.1	Stress concentration factors (SCF) at the point A on intersection when $\beta = 0.4$, $\gamma = 40$ and $\lambda = 90^0$	87
9.2	Stress concentration factors (SCF) at the point C on intersection when $\beta = 0.4$, $\gamma = 40$ and $\lambda = 90^0$	88
9.3	Circumferential stress concentration factors (SCF) on outside surface at point A on intersection when $\beta = 0.4$ and $\gamma = 40$	89
9.4	Circumferential stress concentration factors (SCF) on mid- surface at point A on intersection when $\beta = 0.4$ and $\gamma = 40$	90

LIST OF FIGURES
(Continued)

Figure **Page**

9.5 Circumferential stress concentration factors (SCF) on inside surface at point A on intersection when $\beta = 0.4$ and $\gamma = 40$ 91

9.6 Longitudinal stress concentration factors (SCF) on outside surface at point A on intersection when $\beta = 0.4$ and $\gamma = 40$ 92

9.7 Longitudinal stress concentration factors (SCF) on mid-surface at point A on intersection when $\beta = 0.4$ and $\gamma = 40$ 93

9.8 Longitudinal stress concentration factors (SCF) on inside surface at point A on intersection when $\beta = 0.4$ and $\gamma = 40$ 94

9.9 Longitudinal stress concentration factors (SCF) on outside surface at point C on intersection when $\beta = 0.4$ and $\gamma = 40$ 95

9.10 Longitudinal stress concentration factors (SCF) on inside surface at point C on intersection when $\beta = 0.4$ and $\gamma = 40$ 96

9.11 Longitudinal stress concentration factors (SCF) on mid-surface at point C on intersection when $\beta = 0.4$ and $\gamma = 40$ 97

9.12 Circumferential stress concentration factors (SCF) on outside surface at point C on intersection when $\beta = 0.4$ and $\gamma = 40$ 98

9.13 Circumferential stress concentration factors (SCF) on mid-surface at point C on intersection when $\beta = 0.4$ and $\gamma = 40$ 99

9.14 Circumferential stress concentration factors (SCF) on inside surface at point C on intersection when $\beta = 0.4$ and $\gamma = 40$ 100

9.15 Comparison of the maximum values of stress concentration factors at different intersection angles at point A on the intersection curve 102

9.16 Comparison of the maximum values of stress concentration factors at different intersection angles at point C on the intersection curve 103

NOMENCLATURES

Symbol		Description
H_i	=	Lame's coefficients, $i = 1, 2, 3$
k_i	=	Curvatures of shell in coordinate directions, $i = 1, 2, 3$
R_i	=	Radius of curvatures of shell in coordinate directions, $i = 1, 2, 3$
e_i	=	Normal strains of shell in coordinate directions, $i = 1, 2, 3$
e_{ij}	=	Shear strains of shell in coordinate directions, $i,j = 1, 2, 3$
u_i	=	Displacements of shell in coordinate directions, $i = 1, 2, 3$
u, v, w	=	Displacements on mid-surface of shell in coordinate directions
R	=	Mean radius of cylindrical shell, in
r	=	Mean radius of cylindrical nozzle, in
t	=	Thickness of cylindrical shell, in
σ_i	=	Normal stresses of shell in coordinate directions, $i = 1, 2, 3$, lb/in^2
τ_{ij}	=	Shear stresses of shell in coordinate directions, $i,j = 1, 2, 3$, lb/in^2
N_1	=	Longitudinal membrane force per unit length of mid-surface, lb/in
N_2	=	Circumferential membrane force per unit length of mid-surface, lb/in
S_{12} , S_{21}	=	Shear forces per unit length of mid-surface, lb/in
M_1	=	Longitudinal bending moment per unit length of mid-surface, lb-in/in
M_2	=	Circumferential bending moment per unit length of mid-surface, lb-in/in
M_{12}, M_{21}	=	Twisting moments per unit length of mid-surface, lb-in/in
Q_1 , Q_2	=	Transverse shear forces per unit length of mid-surface, lb/in

NOMENCLATURES
(Continued)

Symbol		Description
X ,Y, Z	=	External surface loading and volume loading, lb, lb/in^2 or lb/in^3
β	=	Ratio of nozzle radius to shell radius, r/R
γ	=	Ratio of shell radius to shell thickness, R/t
λ	=	Intersection angle between shell and nozzle, rad
E	=	Young's modules, $E = 30 \times 10^6$ psi for steel
μ	=	Poisson's ratio, $\mu = 0.3$ for steel
ν	=	A coefficient in the reference[7]
σ_1^m	=	Longitudinal membrane stress, lb $/in^2$
σ_1^b	=	Longitudinal bending stress, lb $/in^2$
σ_2^m	=	Circumferential membrane stress, lb $/in^2$
σ_2^b	=	Circumferential bending stress, lb $/in^2$
SCF	=	Stress concentration factor
∇^2	=	Laplacian differential operator
D	=	Flexural rigidity of thin wall shell, $D = Et^3 / [12(1-\mu^2)]$
p	=	Internal pressure, psi
Im	=	Imaginary part of a complex variable or function
Re	=	Real part of a complex variable or function
ε_1	=	Normal strain in the direction of coordinate ξ on mid-surface

NOMENCLATURES
(Continued)

Symbol		Description
ε_2	=	Normal strain in the direction of coordinate η on mid-surface
ε_{12}	=	Shear strain in the direction of coordinate η and perpendicular to the direction of coordinate ξ on mid-surface
χ_1	=	Change of the shell curvature k_1 in the direction of coordinate ξ , 1/in
χ_2	=	Change of the shell curvature k_2 in the direction of coordinate η , 1/in
χ_{12}	=	Change of the shell torsional curvature in the direction of coordinate
ξ	=	Longitudinal coordinate component on the mid-surface, in
η	=	Circumferential coordinate component on the mid-surface, in

CHAPTER 1

INTRODUCTION

Before the 1970s, plenty of efforts were made to obtain analytical solutions for this topic. Since numerical solutions have been achieved along with development of high speed computers during the past two decades, efforts to analytical solutions are still behind. But, as known to all, any numerical method must be based on mathematics and mechanics, therefore, theoretical analyses are always of importance. On the other hand, because of the user preparation time required, direct finite element procedures have not yet come into general design utilization, and the analytical studies still remain of great interest. The effort of this research attempts to make the theoretical study coincide with the recent developments` in this area.

Normally, an analytical study has to consist of four basic sections as follows:

1. The derivation of a series of equations and their general solutions,

2. Determination of corresponding boundary conditions,

3. Substitution of the boundary conditions into the general solutions to obtain the theoretical solutions of stress analysis,

4. Comparison of the theoretical solutions with the results of relevant researches to arrive at the conclusions for the present study.

A comprehensive literature survey indicates that for the same type of problems about shells and nozzles, many authors made efforts to solve them. Even though they may obtain some similar results, there are still some differences among the works of those authors.

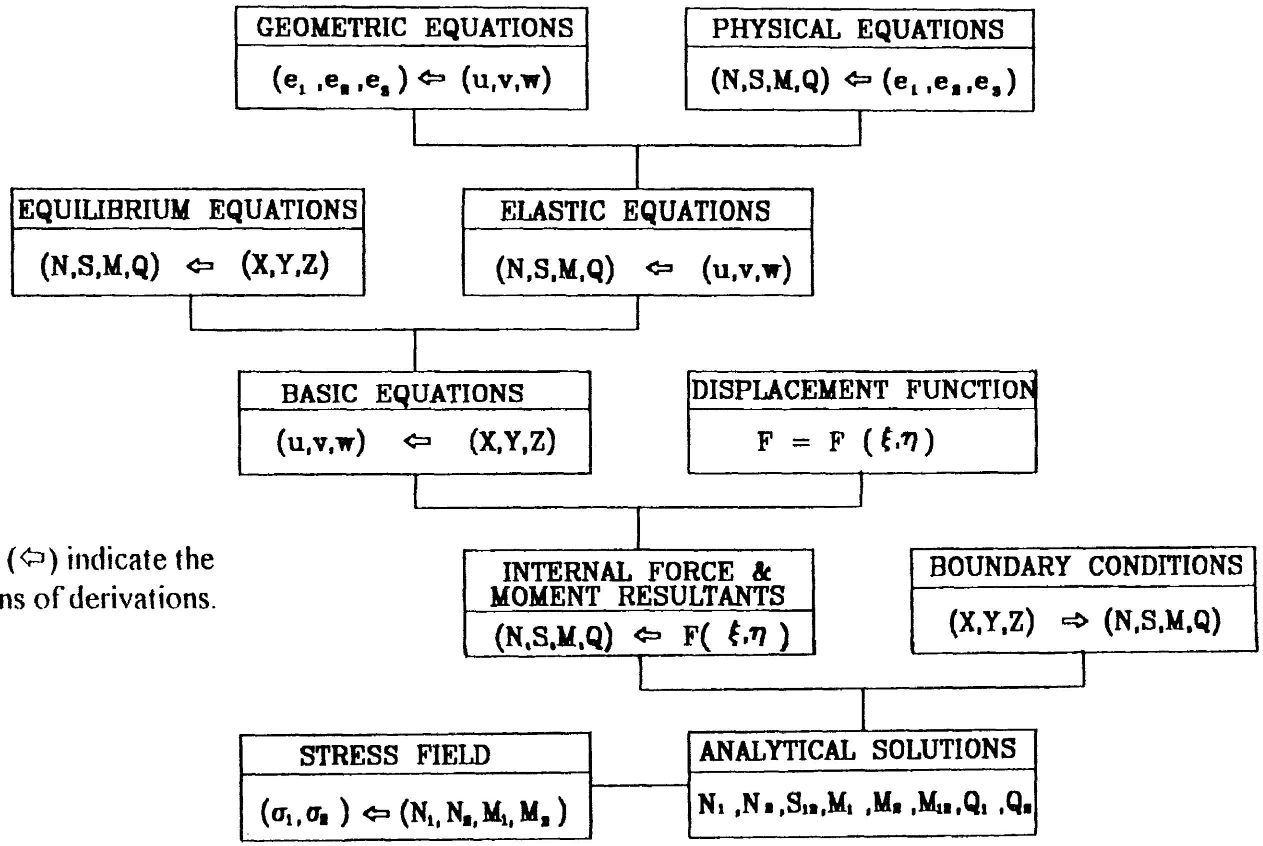
The main differences usually lie in their derivation of equations and boundary conditions. This study possesses features as follows:

1. The method to derive the equations is systematic and comprehensive (See Figure 1.1, the Flow Chart of This Study for a glance).

2. Application of fundamental theory of elasticity to solve the problem of openings formed by intersection of a cylindrical shell and an oblique nozzle.

3. This theoretical analysis is the first research which deals with the geometric analysis of intersection of a cylindrical shell and an oblique nozzle. The success of this study may establish the fundamental for the future research of intersection of cylindrical shells and elliptical nozzles.

The second and the third points are the main contribution to the theory of this area because, so far, there has not been a complete analytical study on openings formed by intersection of cylindrical shells and oblique nozzles, although many authors have explored the openings formed by intersection of cylindrical shells and cylindrical nozzles.



Note: Arrows (⇦) indicate the directions of derivations.

Figure 1.1 Flow chart of this study

CHAPTER 2

LITERATURE SURVEY

In 1920, A. Love [1] obtained three equations for the mid-surface displacements of a shell. These displacement relations are:

$$\frac{\partial^2 u}{\partial\xi^2}+\frac{1-\mu}{2}\frac{\partial^2 u}{\partial\Theta^2}+\frac{1+\mu}{2}\left[1-\frac{1}{12}\frac{(1-\mu)}{(1+\mu)}\left(\frac{t}{R}\right)\right]\frac{\partial^2 v}{\partial\xi\partial\Theta}+\mu\frac{\partial w}{\partial\xi}+\frac{1-\mu}{24}\left(\frac{t}{R}\right)^2\frac{\partial^3 w}{\partial\xi\partial\Theta^2}=0$$

$$\frac{1-\mu}{2}\left[1+\frac{1}{4}\left(\frac{t}{R}\right)^2\right]\frac{\partial^2 v}{\partial\xi^2}+\left[1+\frac{1}{12}\left(\frac{t}{R}\right)^2\right]\frac{\partial^2 v}{\partial\Theta^2}+\frac{1+\mu}{2}\frac{\partial^2 u}{\partial\xi\partial\Theta}+\frac{\partial w}{\partial\Theta}$$
$$-\frac{1}{12}\left(\frac{t}{R}\right)^2\left(\frac{3-\mu}{2}\frac{\partial^2 w}{\partial\xi^2}+\frac{\partial^2 w}{\partial\Theta^2}\right)=0$$

$$w+\frac{\partial v}{\partial\Theta}+\nu\frac{\partial u}{\partial\xi}+\frac{1}{12}\left(\frac{t}{R}\right)^2\left[\nabla^4 w-(2-\mu)\frac{\partial^3 v}{\partial\xi^2\partial\Theta}-\frac{\partial^3 v}{\partial\Theta^3}\right]=0.$$

This set of equations has become the basis of a huge quantity of analytical researches. However, it is very difficult to solve them directly. Many authors have made significant modifications to these equations to obtain certain types of applicable solutions.

W. Flügge[6], in 1932, obtained another set of similar equations:

$$\frac{\partial^2 u}{\partial\xi^2}+\frac{1-\mu}{2}\left[1+\frac{1}{12}\left(\frac{t}{R}\right)^2\right]\frac{\partial^2 u}{\partial\Theta^2}+\frac{1+\mu}{2}\frac{\partial^2 v}{\partial\xi\partial\Theta}+\nu\frac{\partial w}{\partial\xi}+\frac{1}{12}\left(\frac{t}{R}\right)^2\left(\frac{1-\mu}{2}\frac{\partial^2 w}{\partial\Theta^2}-\frac{\partial^2 w}{\partial\xi^2}\right)=0$$

$$\frac{1-\mu}{2}\left[1+\frac{1}{4}\left(\frac{t}{R}\right)^2\right]\frac{\partial^2 v}{\partial\xi^2}+\frac{\partial^2 v}{\partial\Theta^2}+\frac{1+\mu}{2}\frac{\partial^2 u}{\partial\xi\partial\Theta}+\frac{\partial w}{\partial\Theta}-\frac{3-\mu}{24}\left(\frac{t}{R}\right)^2\frac{\partial^3 w}{\partial\xi^2\partial\Theta}=0$$

$$w+\frac{\partial v}{\partial\Theta}+\nu\frac{\partial u}{\partial\xi}+\frac{1}{12}\left(\frac{t}{R}\right)^2\left(\nabla^4 w+2\frac{\partial^2 w}{\partial\Theta^2}+w+\frac{1-\mu}{2}\frac{\partial^3 u}{\partial\xi\partial\Theta^2}-\frac{\partial^3 u}{\partial\xi^3}-\frac{3-\mu}{2}\frac{\partial^3 v}{\partial\xi^2\partial\Theta}\right)=0.$$

This set of equations is relatively more convenient to be employed in practice so that L. Donnell in 1933 [2] and 1938 [3] made significant simplifications by omitting a number of terms and obtained a single eighth order equation for the shallow shell.

In 1958, L. Morley [4] proposed an equation which retained the accuracy of W. Flügge's equations and improved Donnell's equation:

$$\nabla^4(\nabla^2+1)^2 w+4\mu^4\frac{\partial^4 w}{\partial\xi^4}=0.$$

The significant advantage of L. Morley's equation is that it can be factored into the form as follows:

$$\left[\nabla^2(\nabla^2+1)+i\,2\mu^2\frac{\partial^2}{\partial\xi^2}\right]\left[\nabla^2(\nabla^2+1)-i\,2\mu^2\frac{\partial^2}{\partial\xi^2}\right]w=0.$$

It tremendously simplifies the calculation of roots of characteristic polynomials by means of separating differential operators. But L. Morley's equation is still recognized to be used in shallow cylindrical shell theory.

In 1971, J. Lekkerkerker [7] obtained an analytical solution of stress near the intersection of cylindrical shells with small nozzles, based on Donnell's shallow shell equation. His work consisted of two parts. The first part showed an insight into existing possibilities and difficulties while he made a survey of relevant literature. The second part contained a comprehensive process of theoretical analysis of the stress problem for a cylindrical shell with a circular hole which is intended to simulate a cylindrical shell with a branch pipe or nozzle.

The basic equation J. Lekkerkerker employed is Donnell's shallow shell equation:

$$\nabla^2\nabla^2\Psi + i\,4\nu^2 \frac{R^2}{r} \frac{\partial^2\Psi}{\partial\eta^2} = 0,$$

where Ψ is a complex function defined as follows:

$$\mathrm{Re}\,\Psi = w/r \quad \text{and} \quad \mathrm{Im}\,\Psi = \frac{\sqrt{12\,(1-\mu^2)}}{r\,E\,t^2}\,\psi,$$

and $\nu = \frac{1}{2}\frac{\sqrt[4]{12(1-\mu^2)}\ r}{\sqrt{Rt}}$.

The great advantage of Donnell's equation is that its differential operators are separable. From factorization, J. Lekkerkerker obtained the following solution of the basic equation in the form of Bessel functions J_k and Hankel functions $H_n^{(1)}$:

$$\Psi = \sum_{n=-\infty}^{+\infty} \sum_{k=-\infty}^{+\infty} i^k \left[\tilde{A}_n + (-1)^k \tilde{B}_n\right] e^{i(n-k)\psi} H_n^{(1)}\left(\nu\rho\sqrt{i}\right) J_k\left(\nu\rho\sqrt{i}\right),$$

where $\tilde{A}_n$ and $\tilde{B}_n$ are complex integration constants to be determined by the boundary conditions.

He verified the uniqueness of displacements and all types of symmetry. Finally the author obtained a complete set of solutions of stresses, strains and displacements by using the boundary conditions in the form of Fourier Series. The applicability of these solutions is restricted to a small diameter ratio, less than 0.25, as the author indicated in this paper.

In 1983, C. Steele and M. Steele [8] developed an analytical method for stress analysis in cylindrical vessel with external load in which Flügge-Conrad solutions where Sanders-Simmonds concentrated force solution were utilized for a local analysis to which asymptotic approximations for the effect of vessel length and continuity around the vessel circumference were added. To solve the shallow shell equation

$$\left(\nabla^2 \nabla^2 - 4 \frac{\partial^2}{\partial \xi^2} \right) \mathrm{w} = 0,$$

they employed the stress function with complex variables

$$\tilde{\mathrm{w}} = \mathrm{w} - i \, \frac{\Phi}{\mathrm{EtC}}$$

where $\mathrm{C} = \dfrac{\mathrm{t}}{\left[12(1-\mu^2)\right]^{1/2}}$, and Φ is the Airy's stress function.

The stress analysis was incorporated in a computer code, FAST, for which setup and run times were minimized for a given case of geometry and load. Some evidence indicated that Steele's method does not lead to the same results as those obtained by previous analytical methods. They still kept assumptions that the intersection between vessel and nozzle was approximated as a plane circle.

In 1986, C. Steele, M. Steele and A. Khathlan [9]proposed a computational approach for a large opening in a cylindrical vessel because they realized that their previous work was restricted to small size of nozzles only, $\frac{d}{D} \leq 0.5$. Their new approach may handle the geometry of vessel and nozzle to the range of diameter ratio of $0.5 \leq \frac{d}{D} < 1$.

In 1991, M. Xue, Y. Deng and K. Hwang [10] published their results on analytical solutions of cylindrical shell with large opening. Their contributions are:

1. The modified Morley's equation was employed instead of the shallow shell equation. The solutions of the modified equation are in the form of the double Fourier Series including Bessel Functions.

2. The accurate expression of boundary curve of large opening was utilized in the form of a power series. They proved that the previous curve geometric description was just the first term of their power series expansion.

3. The boundary conditions for general loading were also expanded in the form of power series and were truncated after the terms of the third power, which obviously improved the accuracy of the solution.

Their work, similar to all works of previous authors, is still only for the case of orthogonal intersection of a cylindrical vessel and a nozzle. Since Morley's equation can be factored, the simplified form of Morley's equation is obtained as

$$\left(\nabla^2\nabla^2 + \nabla^2 - i\,4\nu^2 \frac{\partial^2}{\partial\xi^2}\right) w = 0$$

in which w is the complex displacement- stress function.

To solve Morley's equation finally, the difficulty of its separability must be encountered. They used the modified Morley's equation

$$\left(\nabla^2\nabla^2 + \nabla^2 + \frac{1}{4} - i\ 4\nu^2 \frac{\partial^2}{\partial\xi^2}\right) W = 0\ .$$

Although the only difference from the Morley's equation is adding a small term, $\frac{W}{4}$, and the ratio between $\frac{W}{4}$ and $\nabla^2 W$ has the same or less order of $\left(\frac{t}{R}\right)$.

The new equation obtained a separability of differential operators, which is easy to be transformed as follows

$$\left(\nabla^2 + \frac{1}{2} + 2\nu\sqrt{i}\frac{\partial}{\partial\xi}\right)\left(\nabla^2 + \frac{1}{2} - 2\nu\sqrt{i}\frac{\partial}{\partial\xi}\right) W = 0.$$

Therefore, the solution of this equation can be obtained as follows:

$$W = \begin{cases} \displaystyle\sum_{k=0}^{\infty}\sum_{n=0}^{\infty} (-1)^k C_n F_{kn} \cos(2k\psi) & \left\{ \begin{array}{l} \text{for symmetric case about} \\ \psi = 0,\ \pi/2 \end{array} \right. \\ \displaystyle\sum_{k=1}^{\infty}\sum_{n=1}^{\infty} (-1)^k C_n F_{kn} \sin(2k\psi) & \left\{ \begin{array}{l} \text{for antisymmetric case about} \\ \psi = 0,\ \pi/2 \end{array} \right. \end{cases}$$

where F_{kn} is the Fourier's coefficients expressed with Bessel functions:

$$F_{kn} = \left\{ \begin{array}{ll} J_{-n}(\sqrt{-i}\ \nu\rho_1) H_n(\eta\rho_1), & k = 0 \\ \\ \left[J_{2k-n}(\sqrt{-i}\ \nu\rho_1) - J_{-2k-n}(\sqrt{-i}\ \nu\rho_1) \right] H_n(\nu\rho_1), & k > 0 \end{array} \right\} \text{ for symmetric case ,}$$

$$F_{kn} = \left[J_{2k-n}(\sqrt{-i}\nu\rho_1) - J_{-2k-n}(\sqrt{-i}\nu\rho_1) \right] H_n(\eta\rho_1) \qquad \text{for antisymmetric case,}$$

where C_n are the complex constants and $\eta = \left(-\frac{1}{2}\, i\ \nu^2 \right)^{1/2}$.

The components of generalized forces and moments were expressed as follows:

$$M_\xi = \frac{D}{R^2} \mathrm{Re}\left\{ \left[\frac{\partial^2}{\partial \rho_1{}^2} + \mu \left(\frac{1}{\partial} \frac{\partial}{\partial \rho_1} + \frac{1}{\rho_1{}^2} \frac{\partial^2}{\partial \psi^2} \right) \right] W \right\}$$

$$M_\psi = \frac{D}{R^2} \mathrm{Re}\left[\left(\frac{1}{\rho_1} \frac{\partial}{\partial \rho_1} + \frac{1}{\rho_1{}^2} \frac{\partial^2}{\partial \psi^2} + \mu \frac{\partial^2}{\partial \rho_1{}^2} \right) W \right]$$

$$M_{\rho_1 \psi} = \frac{(1-\mu)D}{R^2} \mathrm{Re}\left[\left(\frac{1}{\rho_1} \frac{\partial^2}{\partial \rho_1 \partial \psi} - \frac{1}{\rho_1{}^2} \frac{\partial}{\partial \psi} \right) W \right]$$

$$Q_{\rho_1} = \frac{D}{R^3}\mathrm{Re}\left[\frac{\partial}{\partial\rho_1}\left(\frac{\partial^2}{\partial\rho_1^{\,2}} + \frac{1}{\rho_1}\frac{\partial}{\partial\rho_1} + \frac{\partial^2}{\rho_1^{\,2}\partial\psi^2}\right)w\right]$$

$$Q_{\psi} = \frac{D}{R^3}\mathrm{Re}\left[\frac{\partial}{\rho_1\partial\psi}\left(\frac{\partial^2}{\partial\rho_1^{\,2}} + \frac{1}{\rho_1}\frac{\partial}{\partial\rho_1} + \frac{\partial^2}{\rho_1^{\,2}\partial\psi^2}\right)w\right]$$

$$T_{\rho_1} = \frac{E}{4v^2}\frac{t}{R}\mathrm{Im}\left[\left(\frac{\partial^2}{\rho_1^{\,2}\partial\psi^2} + \frac{1}{\rho_1}\frac{\partial}{\partial\rho_1}\right)w\right] + T^*_{\rho_1}$$

$$T_{\psi} = \frac{E}{4v^2}\frac{T}{R}\mathrm{Im}\left(\frac{\partial^2 w}{\partial\rho_1^{\,2}}\right) + T^*_{\psi}$$

$$T_{\rho_1\psi} = T_{\psi\rho_1} = \frac{E}{4v^2}\frac{T}{R}\mathrm{Im}\left[\frac{\partial}{\partial\rho_1}\left(\frac{\partial w}{\rho_1\partial\psi}\right)\right] + T^*_{\rho_1\psi}\ .$$

Finally, they made calculations and compared their results with those published by previous authors in numerical or analytical forms.

Literature [7], [8], [9] and [10] have made great progress in this area. But they all used the complex variable method to solve the relevant equations. The author of this dissertation attempts to explore a way to avoid using complex variables. Therefore, the present research is devoted to use the real variables and fundamental theory of elasticity to solve the stress analysis of a thin shell with an oblique nozzle due to internal pressure.

CHAPTER 3

GENERAL ASSUMPTIONS

To solve the problems analytically, one has to establish equations or relations between all variables. In deriving these equations, one always wishes to consider all factors as comprehensive and precise as possible, but, in many cases, if one considers all factors only from the point of view of precision and comprehensiveness, even if the equations are found, these equations may be too complicated to be solved at all in practice. A part of the reason for this is that mathematicians have proven that some equations do not have analytical solutions at all. In fact, for some equations, even though one could obtain solutions mathematically, but the solutions may be too superfluous to be used in engineering. Therefore, according to the characteristics of study objectives and the range in which the problems are to be solved, certain assumptions must be established to neglect certain less important factors to make the solutions possible and useful in practice. In this study, except for special declaration, the following assumptions are employed.

1. Shell body is fully elastic that follows the Hook's law.
2. Shell body is homogeneous and isotropic.
3. The normal strain, e_3 , perpendicular to the mid-surface is so small that it is negligible.
4. The shell is so long in axial direction that the influence of supporting constraints to the stress field can be neglected.
5. The influences of self-weight and temperature are negligible.

CHAPTER 4

GEOMETRIC EQUATIONS
STRAINS OF POINTS ON MID-SURFACE ARE EXPRESSED WITH DISPLACEMENTS OF CORRESPONDING POINTS ON MID-SURFACE

4.1 Lame's Coefficients in Curvilinear Coordinate System

From an element of general elastic body in the orthogonal curvilinear coordinates (ξ, η, ζ) and Cartesians coordinates (x, y, z) shown in Figure 4.1, the length of the arc PP_1 is

$$ds_1 = \sqrt{(dx)^2 + (dy)^2 + (dz)^2} = \sqrt{\left(\frac{\partial x}{\partial \xi}d\xi\right)^2 + \left(\frac{\partial y}{\partial \xi}d\xi\right)^2 + \left(\frac{\partial z}{\partial \xi}d\xi\right)^2}$$

$$= \sqrt{\left(\frac{\partial x}{\partial \xi}\right)^2 + \left(\frac{\partial y}{\partial \xi}\right)^2 + \left(\frac{\partial z}{\partial \xi}\right)^2}\, d\xi.$$

Let $H_1 = H_1(\xi, \eta, \zeta) = \sqrt{\left(\frac{\partial x}{\partial \xi}\right)^2 + \left(\frac{\partial y}{\partial \xi}\right)^2 + \left(\frac{\partial z}{\partial \xi}\right)^2}$

then

$$ds_1 = H_1\, d\xi \qquad (4.1)$$

Here, H_1 is known as Lame's coefficient [11]. Similarly the other two Lame's coefficients are:

$$H_2 = H_2(\xi, \eta, \zeta) = \sqrt{\left(\frac{\partial x}{\partial \eta}\right)^2 + \left(\frac{\partial y}{\partial \eta}\right)^2 + \left(\frac{\partial z}{\partial \eta}\right)^2}$$

$$H_3 = H_3(\xi, \eta, \zeta) = \sqrt{\left(\frac{\partial x}{\partial \zeta}\right)^2 + \left(\frac{\partial y}{\partial \zeta}\right)^2 + \left(\frac{\partial z}{\partial \zeta}\right)^2}.$$

Correspondingly, the other two lengths of arc are obtained as:

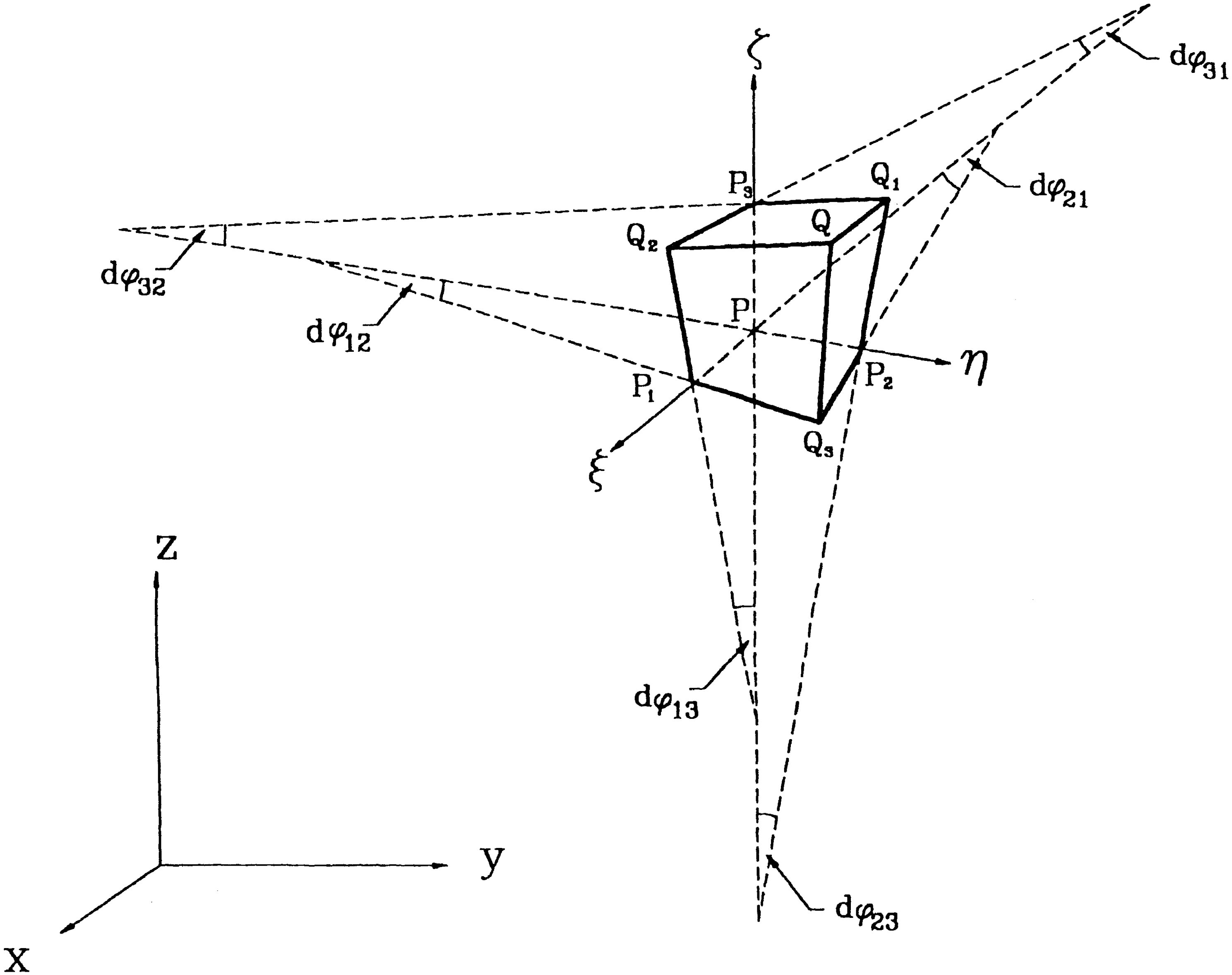


Figure 4.1 Differential element of a general body

$ds_2 = H_2\, d\eta$ for the arc pp_2 (4.2)

$ds_3 = H_3\, d\zeta$ for the arc pp_2 . (4.3)

Obviously, Lame's coefficient is a ratio of arc length increment to coordinate increment when each curvilinear coordinate changes independently.

4.2 Curvatures and Radius of Curvatures in General Orthogonal Curvilinear Coordinate System

As indicated in Figure 4.1, the angle between PP_3 and P_1Q_2 is

$$d\varphi_{13} = \frac{P_3Q_2 - PP_1}{PP_3} = \frac{\left(H_1 + \frac{\partial H_1}{\partial \zeta} d\zeta\right) d\xi - H_1\, d\xi}{H_3\, d\zeta} = \frac{1}{H_3} \frac{\partial H_1}{\partial \zeta} d\xi ,$$

then the curvature and radius of curvature of PP_1 in ξ–ζ plane are as follows

$$k_{13} = \frac{d\varphi_{13}}{H_1\, d\xi} = \frac{1}{H_1 H_3} \frac{\partial H_1}{\partial \zeta} \quad \text{and} \quad R_{13} = \frac{1}{k_{13}} . \tag{4.4}$$

Accordingly, the other curvatures and radii of curvature can be obtained as follows

$$k_{12} = \frac{1}{H_1 H_2} \frac{\partial H_1}{\partial \eta} \quad \text{and} \quad R_{12} = \frac{1}{k_{12}} \tag{4.5}$$

$$k_{23} = \frac{1}{H_2 H_3} \frac{\partial H_2}{\partial \zeta} \quad \text{and} \quad R_{23} = \frac{1}{k_{23}} \tag{4.6}$$

$$k_{21} = \frac{1}{H_2 H_1} \frac{\partial H_2}{\partial \xi} \quad \text{and} \quad R_{21} = \frac{1}{k_{21}} \tag{4.7}$$

$$k_{31} = \frac{1}{H_3 H_1} \frac{\partial H_3}{\partial \xi} \quad \text{and} \quad R_{31} = \frac{1}{k_{31}} \tag{4.8}$$

$$k_{32} = \frac{1}{H_3 H_2} \frac{\partial H_3}{\partial \eta} \quad \text{and} \quad R_{32} = \frac{1}{k_{32}} . \tag{4.9}$$

4.3 Geometric Equations for General Elastic Bodies

Here, u_1 , u_2 , u_3 are employed to express the displacements along the three directions of coordinates (ξ , η , ζ) respectively, e_1 , e_2 , e_3 are the corresponding normal strains respectively and e_{23} , e_{31} , e_{12} are the corresponding shear strains respectively. Now to express the strain components with displacements, consider the normal strain e_1 of PP_1 as an instance.

Due to u_1 , the normal strain component of PP_1 is

$$e_1' = \frac{\left(u_1 + \frac{\partial u_1}{\partial s_1} ds_1\right) - u_1}{ds_1} = \frac{\partial u_1}{\partial s_1} = \frac{1}{H_1}\frac{\partial u_1}{\partial \xi} ;$$

Due to u_2 , the normal strain component of PP_1 is

$$e_1'' = \frac{(R_{12} + u_2)d\varphi_{12} - R_{12}d\varphi_{12}}{R_{12}d\varphi_{12}} = \frac{u_2}{R_{12}} ;$$

Due to u_3 , the normal strain component of PP_1 is

$$e_1''' = \frac{(R_{13} + u_3)d\varphi_{13} - R_{13}d\varphi_{13}}{R_{13}d\varphi_{13}} = \frac{u_3}{R_{13}} ,$$

Therefore, by means of (4.4) and (4.5), the total normal strain component of PP_1 is obtained as follows

$$e_1 = e_1' + e_1'' + e_1''' = \frac{1}{H_1}\frac{\partial u_1}{\partial \xi} + \frac{u_2}{H_1 H_2}\frac{\partial H_1}{\partial \eta} + \frac{u_3}{H_1 H_3}\frac{\partial H_1}{\partial \zeta} . \qquad (4.10)$$

The shear strain e_{12}, in fact, is an angular increment because PP_1 and PP_2 rotate to each other, and it consists of four parts as follows. The first part is the angle by which PP_1 rotates to PP_2 in ξ–η plane due to u_2

$$e'_{12} = \frac{\left(u_2 + \frac{\partial u_2}{\partial s_1} ds_1\right) - u_2}{ds_1} = \frac{\partial u_2}{\partial s} = \frac{1}{H_1}\frac{\partial u_2}{\partial \xi};$$

The second part is the angle by which PP_1 rotates away from PP_2 in ξ-η plane due to u_1

$$e''_{12} = -\frac{u_1}{R_{12}},$$

The third part is the angle by which PP_2 rotates to PP_1 in ε–η surface due to u_1

$$e'''_{12} = \frac{\left(u_1 + \frac{\partial u_1}{\partial s_2} ds_1\right) - u_1}{ds_2} = \frac{\partial u_1}{\partial s_2} = \frac{1}{H_2}\frac{\partial u_1}{\partial \eta};$$

The last part is the angle by which PP_2 rotates away from PP_1 due to u_2 is

$$e''''_{12} = -\frac{u_2}{R_{21}},$$

Then the total shear strain of right angle P_1PP_2 is

$$e_{12} = e'_{12} + e''_{12} + e'''_{12} + e''''_{12} = \frac{1}{H_1}\frac{\partial u_2}{\partial \xi} - \frac{u_1}{H_1H_2}\frac{\partial H_1}{\partial \eta} + \frac{1}{H_1}\frac{\partial u_1}{\partial \eta} - \frac{u_2}{H_2H_1}\frac{\partial H_2}{\partial \xi}$$

$$= \frac{H_2}{H_1}\frac{\partial}{\partial \xi}\left(\frac{u_2}{H_2}\right) + \frac{H_1}{H_2}\frac{\partial}{\partial \eta}\left(\frac{u_1}{H_1}\right) \qquad (4.11)$$

where R_{12} and R_{21} have been substituted by (4.5) and (4.7). Similarly, the rest of components of normal strains and shear strains can be expressed with displacements as follows:

$$e_2 = \frac{1}{H_2}\frac{\partial u_2}{\partial \eta} + \frac{u_3}{H_2 H_3}\frac{\partial H_2}{\partial \zeta} + \frac{u_1}{H_2 H_1}\frac{\partial H_2}{\partial \xi} \qquad (4.12)$$

$$e_3 = \frac{1}{H_3}\frac{\partial u_3}{\partial \zeta} + \frac{u_1}{H_3 H_1}\frac{\partial H_3}{\partial \xi} + \frac{u_2}{H_3 H_2}\frac{\partial H_3}{\partial \eta} \qquad (4.13)$$

$$e_{23} = \frac{H_3}{H_2}\frac{\partial}{\partial \eta}\left(\frac{u_3}{H_3}\right) + \frac{H_2}{H_3}\frac{\partial}{\partial \zeta}\left(\frac{u_2}{H_2}\right) \qquad (4.14)$$

$$e_{31} = \frac{H_1}{H_3}\frac{\partial}{\partial \zeta}\left(\frac{u_1}{H_1}\right) + \frac{H_3}{H_1}\frac{\partial}{\partial \xi}\left(\frac{u_3}{H_3}\right) \ . \qquad (4.15)$$

Equations (4.10) through (4.15) are known as general geometric equations in orthogonal curvilinear coordinate system.

4.4 Geometric Equations for General Shell Bodies

The main difference of shells from general elastic bodies for us to consider lies in the concept of mid-surface of the shell wall. When the origin of an orthogonal curvilinear coordinate system is put in the mid-surface, the basic equations as well as boundary geometry will be considerably simplified. Now , Figure 4.2 shows a differential element of a general shell.

In the mid-surface, $\zeta = 0$, set the Lame's coefficients along the directions ξ and η to be A and B, that is

$$H_1(\xi, \eta, 0) = A \qquad \text{and} \qquad H_2(\xi, \eta, 0) = B, \qquad (4.16)$$

then for the point M in the mid-surface, the length of the arc is

$$MM_1 = A\, d\xi \ .$$

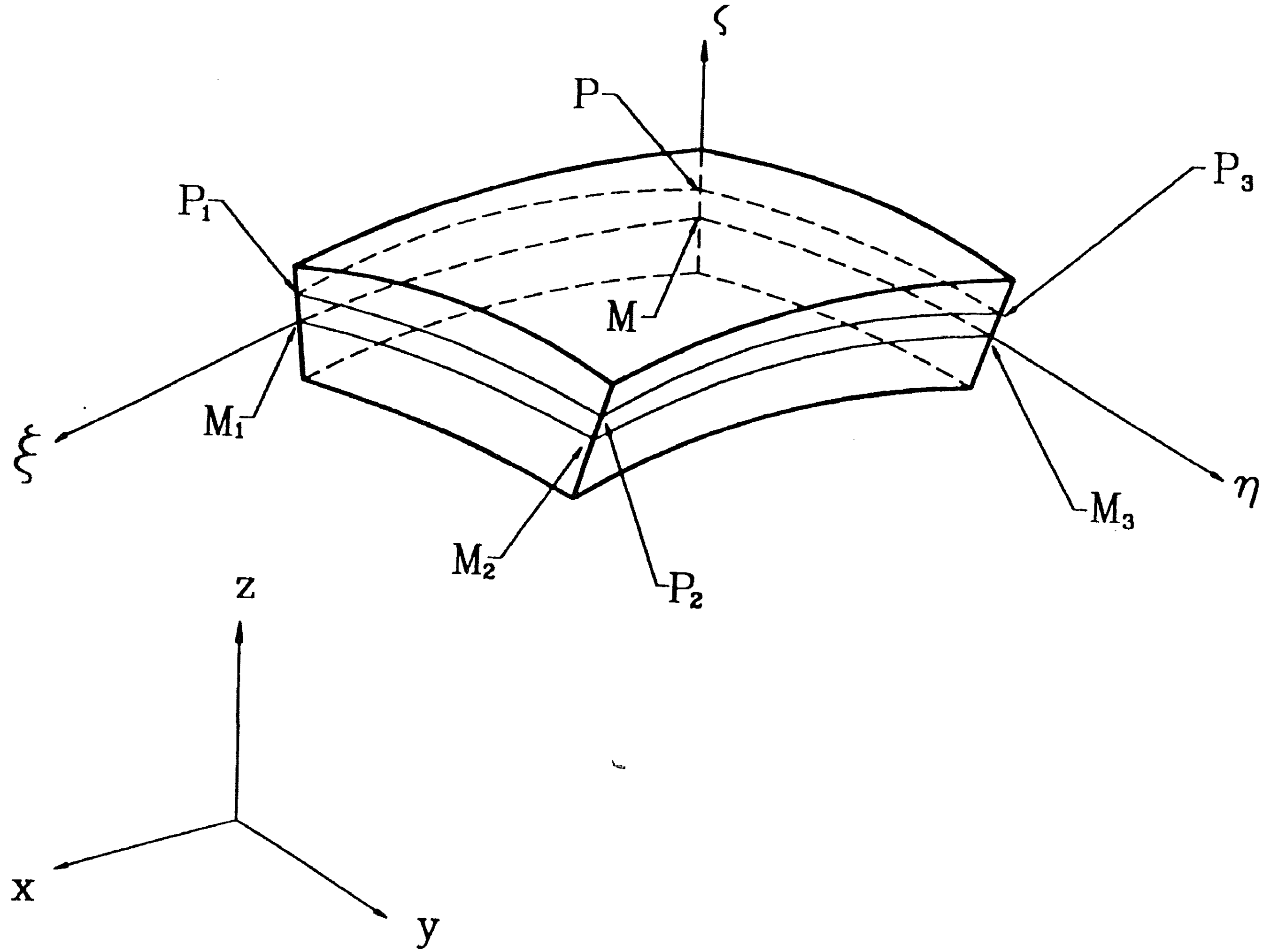


Figure 4.2 Differential element of a general shell body

If P is an arbitrary point in the shell body, the length of the arc $PP_1 = H_1\, d\xi$ and the following stands:

$$\frac{PP_1}{MM_1} = \frac{H_1\, d\xi}{A\, d\xi} = \frac{H_1}{A}.$$

From Figure 4.2, it is obvious that $\dfrac{PP_1}{MM_1} = \dfrac{R_1 + \zeta}{R_1}$,

therefore the following relation is obtained

$$\frac{H_1}{A} = 1 + \frac{\zeta}{R_1} = 1 + k_1\zeta \quad \text{then} \quad H_1 = A(1 + k_1\zeta). \tag{4.17}$$

Similarly, the other relation is also obtained

$$\frac{H_2}{B} = 1 + \frac{\zeta}{R_2} = 1 + k_2\zeta \quad \text{then} \quad H_2 = B(1 + k_2\zeta). \tag{4.18}$$

Moreover, according to the assumption in Chapter 3 that coordinate ζ is a straight line and its dimension is of length, the following is always correct:

$$H_3 = 1 \tag{4.19}$$

Now one can derive the geometric equations for a general shell body. According to (4.13), (4.14), (4.15) and the assumption in Chapter 3, e_3 is negligible, the following is obtained:

$$e_3 = 0 \qquad \text{that is,} \qquad \frac{\partial u_3}{\partial \zeta} = 0$$

which implies that the displacements in the normal direction of mid-surface does not change along with the coordinate ζ. Therefore, u_3 is a function of variables ξ and η only and it can be set as

$$u_3 = w = w(\xi, \eta). \tag{4.20}$$

Also according to the assumption in Chapter 3, we have

$$e_{31} = 0 \quad \text{and} \quad e_{23} = 0. \tag{4.21}$$

Substituting (4.14), (4.15), (4.18) and (4.19) into (4.21), we obtain

$$\frac{\partial}{\partial \zeta}\left[\frac{u_1}{A(1 + k_1\zeta)}\right] + \frac{1}{A^2(1 + k_1\zeta)^2}\frac{\partial w}{\partial \xi} = 0 \qquad \text{and}$$

$$\frac{\partial}{\partial \zeta}\left[\frac{u_2}{B(1 + k_2\zeta)}\right] + \frac{1}{B^2(1 + k_2\zeta)^2}\frac{\partial w}{\partial \eta} = 0.$$

Since w is not a variable along with coordinate ζ, when we integrate the two equations shown above with respect to ζ from 0 to ζ, the following can be obtained

$$\int_0^\zeta \left\{\frac{\partial}{\partial \zeta}\left[\frac{u_1}{A(1 + k_1\zeta)}\right] + \left[\frac{1}{A^2(1 + k_1\zeta)^2}\frac{\partial w}{\partial \xi}\right]\right\} d\zeta = 0 \qquad \text{that is}$$

$$\left[\frac{u_1}{A(1 + k_1\zeta)}\right]_0^\zeta - \left[\frac{1}{A^2 k_1(1 + k_1\zeta)}\right]_0^\zeta \frac{\partial w}{\partial \xi} = 0 \tag{4.22}$$

$$\int_0^\zeta \left\{\frac{\partial}{\partial \zeta}\left[\frac{u_2}{B(1 + k_2\zeta)}\right] + \frac{1}{B^2(1 + k_2\zeta)^2}\frac{\partial w}{\partial \eta}\right\} d\zeta = 0 \qquad \text{that is}$$

$$\left[\frac{u_2}{B(1 + k_2\zeta)}\right]_0^\zeta - \left[\frac{1}{B^2 k_2(1 + k_2\zeta)}\right]_0^\zeta \frac{\partial w}{\partial \eta} = 0. \tag{4.23}$$

Set the displacements of all points in the mid-surface along the directions ξ and η to be u and v respectively, that is

$$(u_1)_{\zeta=0} = u = u(\xi, \eta) \quad \text{and} \quad (u_2)_{\zeta=0} = v = v(\xi, \eta) \tag{4.24}$$

and solve u_1, u_2 from (4.22), (4.23), (4.24), then take u_3 from (4.20), the state equations of displacements of shell body are obtained:

$$u_1 = (1 + k_1\zeta)\,u - \frac{\zeta}{A}\frac{\partial w}{\partial \xi} \tag{4.25}$$

$$u_2 = (1 + k_2\zeta)\,v - \frac{\zeta}{B}\frac{\partial w}{\partial \eta} \tag{4.26}$$

$$u_3 = w \tag{4.27}$$

where, the displacements of each point in shell body (u_1, u_2 and u_3) have been expressed with the displacements of the corresponding point in mid-surface, u, v and w.

Now, substituting (4.17), (4.18), (4.19), (4.25), (4.26) and (4.27) into (4.10), (4.11) and (4.12) correspondingly, the geometric equations for the general shell body are obtained:

$$e_1 = \frac{1}{A(1 + k_1\zeta)}\frac{\partial}{\partial \xi}\left[(1 + k_1\zeta)\,u - \frac{\zeta}{A}\frac{\partial w}{\partial \xi}\right] + \frac{k_1}{1 + k_1\zeta}w + \frac{\frac{\partial}{\partial \eta}\left[A(1 + k_1\zeta)\right]}{AB(1 + k_1\zeta)(1 + k_2\zeta)}\left[(1 + k_2\zeta)\,v - \frac{\zeta}{B}\frac{\partial w}{\partial \eta}\right]. \tag{4.28}$$

$$e_2 = \frac{1}{B(1+k_2\zeta)}\frac{\partial}{\partial \eta}\left[(1 + k_2\zeta)\,v - \frac{\zeta}{B}\frac{\partial w}{\partial \zeta}\right] + \frac{k_2}{1 + k_2\zeta}w + \frac{\frac{\partial}{\partial \xi}\left[B(1 + k_2\zeta)\right]}{AB(1 + k_1\zeta)(1 + k_2\zeta)}\left[(1 + k_1\zeta)\,u - \frac{\zeta}{A}\frac{\partial w}{\partial \xi}\right]. \tag{4.29}$$

$$e_{12} = \frac{B(1 + k_2\zeta)}{A(1 + k_1\zeta)}\frac{\partial}{\partial \xi}\frac{(1 + k_2\zeta)\,v - \frac{\zeta}{B}\frac{\partial w}{\partial \eta}}{B(1 + k_2\zeta)}$$

$$+\frac{A(1+k_1\zeta)}{B(1+k_2\zeta)}\frac{\partial}{\partial\eta}\frac{(1+k_1\zeta)u-\dfrac{\gamma}{A}\dfrac{\partial w}{\partial\xi}}{A(1+k_1\zeta)}. \tag{4.30}$$

Here, the strains of all points in the shell have been expressed with the displacements of corresponding points in mid-surface.

4.5 Geometric Equations for Cylindrical Shell

Now we are going to derive the main results of this chapter, that is, geometric equations for circular cylindrical shells. Because huge quantities of cylindrical shells used in engineering are thin walled shells, that is, the wall thickness is small enough (usually $\gamma \geq 10$), a concept, so-called mid-surface, is introdued as shown in Figure 4.3.

Here, based on the geometric equations for general shell, the following special conditions are established to simplify our results:

a. In the case of circular cylindrical shell, $k_1 = 0$, and $k_2 = 1/R$,

b. For all points in the mid-surface in the coordinate system shown in Figure 4.3,

$$\frac{\partial x}{\partial\xi} = \frac{\partial y}{\partial\eta} = 1,\ \frac{\partial x}{\partial\eta} = \frac{\partial y}{\partial\xi} = 0 \ \text{ and } \ \zeta = 0,$$

therefore, from (4.16) and the definition of Lame's coefficients mentioned earlier,

A = B = 1,

c. Because of thin walled cylindrical shell, $\zeta \leq \frac{t}{2}$ and $t << R$, which means that

$$1+\frac{\zeta}{R} = 1 + k_2\zeta \cong 1 \quad (\frac{\zeta}{R} \text{ approaches zero})$$

Substituting these conditions shown above into (4.28) , (4.29) and (4.30), one obtains

$$e_1 = \varepsilon_1 + \chi_1\zeta \tag{4.31}$$

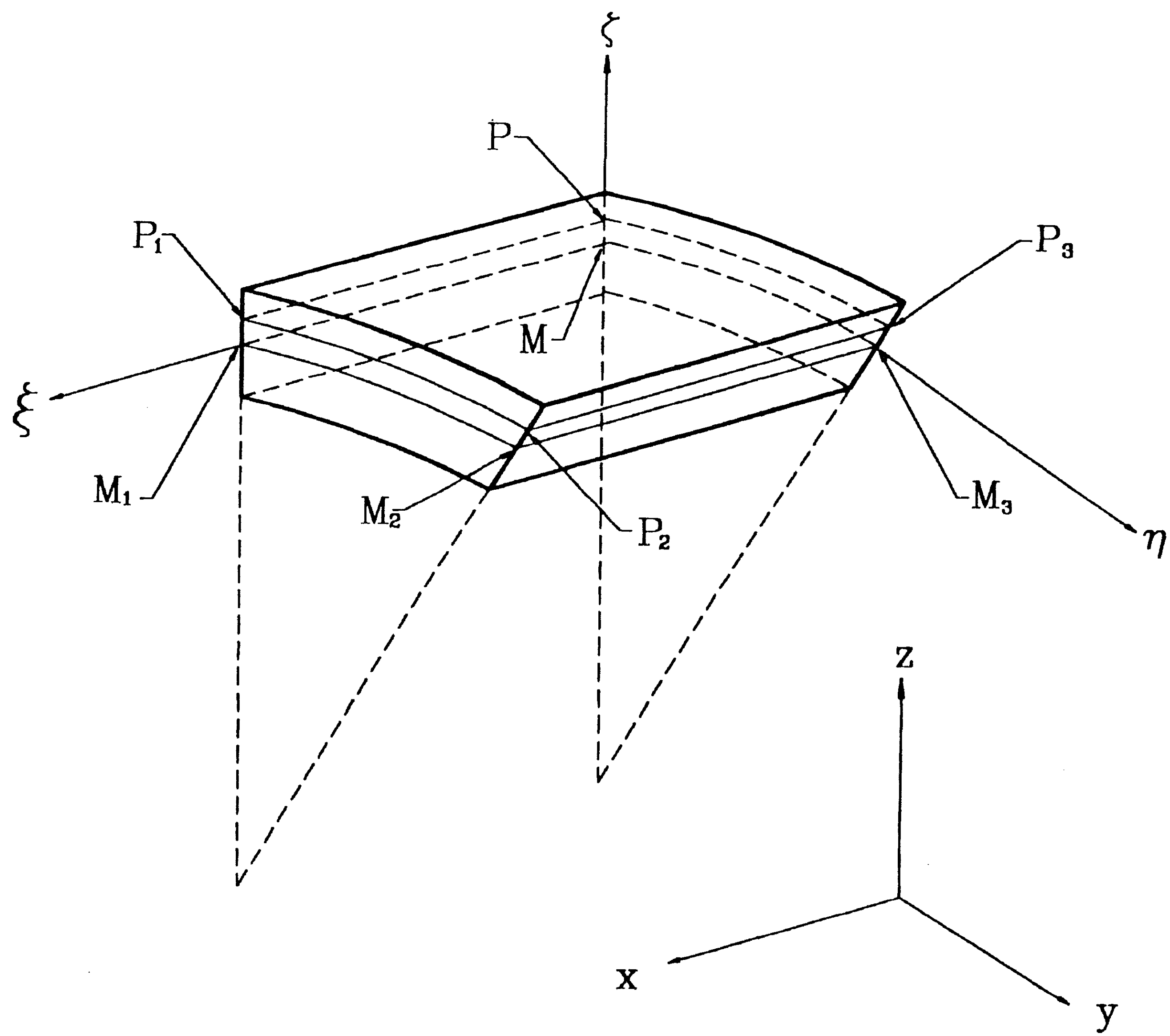


Figure 4.3 Differential element of a cylindrical shell body

$$e_2 = \varepsilon_2 + \chi_2 \zeta \tag{4.32}$$

$$e_{12} = \varepsilon_{12} + 2\chi_{12} \zeta \tag{4.33}$$

where

$$\varepsilon_1 = \frac{\partial u}{\partial \xi} \quad \text{and} \quad \chi_1 = -\frac{\partial^2 w}{\partial \xi^2} \tag{4.34}$$

$$\varepsilon_2 = \frac{\partial v}{\partial \eta} + \frac{w}{R} \quad \text{and} \quad \chi_2 = -\frac{\partial^2 w}{\partial \eta^2} \tag{4.35}$$

$$\varepsilon_{12} = \frac{\partial u}{\partial \eta} + \frac{\partial v}{\partial \xi} \quad \text{and} \quad \chi_{12} = -\frac{\partial^2 w}{\partial \xi \partial \eta} . \tag{4.36}$$

These are the geometric equations for a cylindrical shell by which the strains of all points in mid-surface have been expressed with the displacements of corresponding points in mid-surface.

CHAPTER 5

PHYSICAL EQUATIONS
RELATIONSHIP BETWEEN STRAINS AND INTERNAL FORCES ON THE MID-SURFACE OF THE SHELL

For a cylindrical thin shells, one may express stresses, strains and displacements of all points in shell body with generalized forces at the corresponding points on mid-surface. Therefore, the normal and shear stresses of all point in the shell body are to be simplified into the stress resultants, moment resultants and transverse force resultants at the corresponding points on the mid-surface.

In Figure 5.1, (a) and (b), the plane perpendicular to the ξ-axis is called ξ-plane and the plane perpendicular to the η-axis is called η-plane. In the ξ-plane, the normal stress σ_1 is simplified into two parts on the mid-surface: normal force per unit length N_1 and bending moment per unit length M_1. The shear stress τ_{12} is simplified into two parts on the mid-surface: shear force per unit length S_{12} and twisting moment per unit length M_{12}. Also the shear stress τ_{13} is simplified into a transverse force per unit length Q_1 on mid-surface.

Similarly, in the η-plane as shown in Figure 5.1 (a) and (b), the normal stress σ_2 is simplified into two parts on the mid-surface: normal force per unit length N_2 and bending moment per unit length M_2. The shear stress τ_{21} is simplified into two parts on the mid-surface: shear force per unit length S_{21} and twisting moment per unit length M_{21}. Also the shear stress τ_{23} is simplified into a transverse force per unit length Q_2 on the mid-surface.

Figure 5.1 (a) Membrane forces resultants on mid-surface

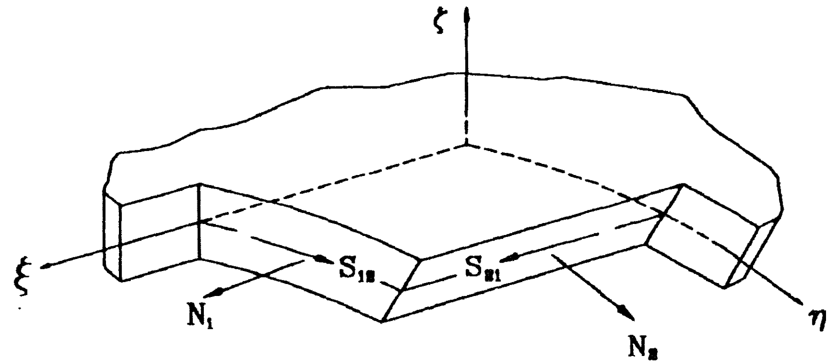
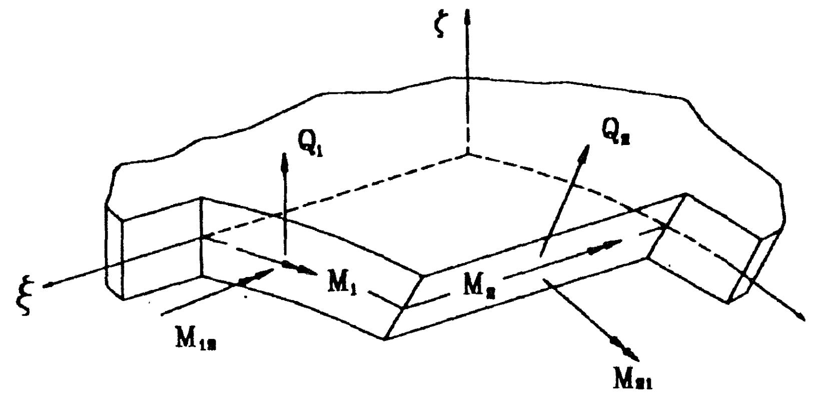


Figure 5.1 (b) Bending moment resultants on mid-surface



Among the these normal force, moment and transverse force resultants as shown in Figure 5.1, N_1 , N_2 , S_{12} and S_{21} are normally called membrane resultants, as shown in Figure 5.1 (a), and M_1 , M_2 , M_{12} , M_{21} , Q_1 and Q_2 are usually called bending resultants, as shown in Figure 5.1 (b). Since now on and in the sequel, they are called internal force and moment resultants . These internal force and moment resultants are derived as follows.

$$N_1 = \int_{-t/2}^{t/2} \frac{\sigma_1[(R + \zeta)d\eta]}{Rd\eta}\, d\zeta = \int_{-t/2}^{t/2} \sigma_1 \left(1 + \frac{\zeta}{R}\right) d\zeta \tag{5.1}$$

$$M_1 = \int_{-t/2}^{t/2} \zeta \frac{\sigma_1[(R + \zeta)d\eta]}{Rd\eta}\, d\zeta = \int_{-t/2}^{t/2} \sigma_1 \left(1 + \frac{\zeta}{R}\right) \zeta d\zeta \tag{5.2}$$

$$M_{12} = \int_{-t/2}^{t/2} \zeta \frac{\tau_{12}[(R + \zeta)d\eta]}{Rd\eta}\, d\zeta = \int_{-t/2}^{t/2} \tau_{12} \left(1 + \frac{\zeta}{R}\right) \zeta d\zeta \tag{5.3}$$

$$S_{12} = \int_{-t/2}^{t/2} \frac{\tau_{12}[(R + \zeta)d\eta]}{Rd\eta}\, d\zeta = \int_{-t/2}^{t/2} \tau_{12} \left(1 + \frac{\zeta}{R}\right) d\zeta \tag{5.4}$$

$$Q_1 = \int_{-t/2}^{t/2} \frac{\tau_{13}[(R + \zeta)d\eta]}{Rd\eta}\, d\zeta = \int_{-t/2}^{t/2} \tau_{13} \left(1 + \frac{\zeta}{R}\right) d\zeta \tag{5.5}$$

and

$$N_2 = \int_{-t/2}^{t/2} \frac{\sigma_2\, d\xi}{d\xi}\, d\zeta = \int_{-t/2}^{t/2} \sigma_2\, d\zeta \tag{5.6}$$

$$M_2 = \int_{-t/2}^{t/2} \zeta \frac{\sigma_2\, d\xi}{d\xi}\, d\zeta = \int_{-t/2}^{t/2} \sigma_2\, \zeta d\zeta \tag{5.7}$$

$$M_{21} = \int_{-t/2}^{t/2} \zeta \frac{\tau_{21}\, d\xi}{d\xi}\, d\zeta = \int_{-t/2}^{t/2} \tau_{21}\, \zeta d\zeta \tag{5.8}$$

$$S_{21} = \int_{-t/2}^{t/2} \frac{\tau_{21}\, d\xi}{d\xi}\, d\zeta = \int_{-t/2}^{t/2} \tau_{21}\ d\zeta \tag{5.9}$$

$$Q_2 = \int_{-t/2}^{t/2} \frac{\tau_{23}\, d\xi}{d\xi}\, d\zeta = \int_{-t/2}^{t/2} \tau_{23}\ d\zeta\ . \tag{5.10}$$

According to the assumption in Chapter 3 that the influence of normal stress σ_3 to strains is negligible, one obtains the relations between stresses and strains from Hook's law :

$$\sigma_1 = \frac{E}{1-\mu^2}(e_1 + \mu e_2)$$

$$\sigma_2 = \frac{E}{1-\mu^2}(e_2 + \mu e_1)$$

$$\tau_{12} = \tau_{21} = \frac{E}{2(1+\mu)}\ e_{12}\ .$$

Substituting (4.31), (4.32) and (4.33) into the above three equations gives:

$$\sigma_1 = \frac{E}{1-\mu^2}\left[(\varepsilon_1 + \mu\varepsilon_2) + (\chi_1 + \mu\chi_2)\,\zeta\right] \tag{5.11}$$

$$\sigma_2 = \frac{E}{1-\mu^2}\left[(\varepsilon_2 + \mu\varepsilon_1) + (\chi_2 + \mu\chi_1)\,\zeta\right] \tag{5.12}$$

$$\tau_{12} = \tau_{21} = \frac{E}{2(1+\mu)}(\varepsilon_{12} + 2\chi_{12}\zeta). \tag{5.13}$$

As far the shear stresses τ_{31} , τ_{13} , τ_{32} and τ_{23} , there are no simple formulas in use for them and the following is suggested [11]:

$$\tau_{31} = \tau_{13} = \frac{E}{2(1-\mu^2)}\left(\zeta^2 - \frac{t^2}{4}\right)\frac{\partial}{\partial\xi}\nabla^2 w \tag{5.14}$$

$$\tau_{32} = \tau_{23} = \frac{E}{2(1-\mu^2)}\left(\zeta^2 - \frac{t^2}{4}\right)\frac{\partial}{\partial\eta}\nabla^2 w. \tag{5.15}$$

Substituting Equations (5.11) through (5.15) into Equations (5.1) through (5.10) and considering that u , v and w are not the functions of the variable ζ , we obtain the physical equations for the cylindrical shell body as follows:

$$N_1 = \int_{-t/2}^{t/2} \frac{E}{1-\mu^2}\left[(\varepsilon_1 + \mu\varepsilon_2) + \zeta(\chi_1 + \mu\chi_2)\right]\left(1 + \frac{\zeta}{R}\right)d\zeta$$

$$= \frac{Et}{1-\mu^2}\left[(\varepsilon_1 + \mu\varepsilon_2) + \frac{t^2}{12R}(\chi_1 + \mu\chi_2)\right] \tag{5.16}$$

$$M_1 = \int_{-t/2}^{t/2} \frac{E}{1-\mu^2}\left[(\varepsilon_1 + \mu\varepsilon_2) + \zeta(\chi_1 + \mu\chi_2)\right]\left(1 + \frac{\zeta}{R}\right)\zeta d\zeta$$

$$= \frac{Et^3}{12(1-\mu^2)}\left[\frac{1}{R}(\varepsilon_1 + \mu\varepsilon_2) + (\chi_1 + \mu\chi_2)\right] \tag{5.17}$$

$$M_{12} = \int_{-t/2}^{t/2} \frac{E}{2(1+\mu)}(\varepsilon_{12} + 2\zeta\chi_{12})\left(1 + \frac{\zeta}{R}\right)\zeta d\zeta$$

$$= \frac{Et^3}{12(1+\mu)}\left(\frac{1}{2R}\varepsilon_{12} + \chi_{12}\right) \tag{5.18}$$

$$S_{12} = \int_{-t/2}^{t/2} \frac{E}{2(1+\mu)}(\varepsilon_{12} + 2\zeta\chi_{12})\left(1 + \frac{\zeta}{R}\right)d\zeta$$

$$= \frac{Et}{2(1+\mu)}\left(\varepsilon_{12} + \frac{t^2}{6R}\chi_{12}\right) \tag{5.19}$$

$$Q_1 = \int_{-t/2}^{t/2} \frac{E}{2(1-\mu^2)}\left[\left(\zeta^2 - \frac{t^2}{4}\right)\frac{\partial}{\partial\xi}\nabla^2 w\right]\left(1 + \frac{\zeta}{R}\right)d\zeta$$

$$= -\frac{Et^3}{12(1-\mu^2)}\frac{\partial}{\partial\xi}\nabla^2 w \tag{5.20}$$

$$N_2 = \int_{-t/2}^{t/2} \frac{E}{1-\mu^2}\left[(\varepsilon_2 + \mu\varepsilon_1) + \zeta(\chi_2 + \mu\chi_1)\right]d\zeta$$

$$= \frac{Et}{1 - \mu^2}(\varepsilon_2 + \mu\varepsilon_1) \tag{5.21}$$

$$M_2 = \int_{-t/2}^{t/2} \frac{E}{1 - \mu^2}\left[(\varepsilon_2 + \mu\varepsilon_1) + \zeta(\chi_2 + \mu\chi_1)\right] \zeta d\zeta$$

$$= \frac{Et^3}{12(1 - \mu^2)}(\chi_2 + \mu\chi_1) \tag{5.22}$$

$$M_{21} = \int_{-t/2}^{t/2} \frac{E}{2(1 + \mu)}(\varepsilon_{12} + 2\zeta\chi_{12}) \zeta d\zeta = \frac{Et^3}{12(1 + \mu)} \chi_{12} \tag{5.23}$$

$$S_{21} = \int_{-t/2}^{t/2} \frac{E}{2(1 + \mu)}(\varepsilon_{12} + 2\zeta\chi_{12}) d\zeta = \frac{Et}{2(1 + \mu)} \varepsilon_{12} \tag{5.24}$$

$$Q_2 = \int_{-t/2}^{t/2} \frac{E}{2(1 - \mu^2)}\left[\left(\zeta^2 - \frac{t^2}{4}\right)\frac{\partial}{\partial\eta}\nabla^2 w\right] d\zeta = - \frac{Et^3}{12(1 - \mu^2)} \frac{\partial}{\partial\eta}\nabla^2 w. \tag{5.25}$$

In the cases of thin walled cylindrical shells, because $\zeta \le \frac{t}{2}$ and $t \ll R$, replacing $1 + \frac{\zeta}{R}$ with 1 in (5.1), (5.2) and (5.3) before integration, (5.1), (5.2) and (5.3) become (5.26), (5.27) and (5.28) as shown below, instead of (5.16), (5.17) and (5.18). Therefore, finally, one arrives the simplified form of physical equations as follows:

$$N_1 = \frac{Et}{1 - \mu^2}(\varepsilon_1 + \mu\varepsilon_2) \tag{5.26}$$

$$N_2 = \frac{Et}{1 - \mu^2}(\varepsilon_2 + \mu\varepsilon_1) \tag{5.27}$$

$$M_1 = \frac{Et^3}{12(1 - \mu^2)}(\chi_1 + \mu\chi_2) \tag{5.28}$$

$$M_2 = \frac{Et^3}{12(1 - \mu^2)}(\chi_2 + \mu\chi_1) \tag{5.29}$$

$$M_{12} = M_{21} = \frac{Et^3}{12(1 + \mu)} \chi_{12} \tag{5.30}$$

$$S_{12} = S_{21} = \frac{Et}{2(1 + \mu)} \varepsilon_{12} \tag{5.31}$$

$$Q_1 = - \frac{Et^3}{12(1 - \mu^2)} \frac{\partial}{\partial \xi}\nabla^2 w \tag{5.32}$$

$$Q_2 = - \frac{Et^3}{12(1 - \mu^2)} \frac{\partial}{\partial \eta}\nabla^2 w. \tag{5.33}$$

Equations (5.26) through (5.33) are the main results of this chapter, which are called the physical equations for a cylindrical shell where the internal force and moment resultants have been expressed with strains of corresponding points on the mid-surface.

Now, we transform Equations (5.26) through (5.31) from the current forms of expressing internal forces and moment resultants with strains into the forms of expressing strains with internal force and moment resultants. Then, substitute the new forms of these equations into Equations (5.11) , (5.12) and (5.13), the following significant results are obtained :

$$\sigma_1 = \frac{N_1}{t} + \frac{12\, M_1}{t^3}\zeta , \qquad -\frac{t}{2} \le \zeta \le \frac{t}{2} \tag{5.34}$$

$$\sigma_2 = \frac{N_2}{t} + \frac{12\, M_2}{t^3}\zeta , \qquad -\frac{t}{2} \le \zeta \le \frac{t}{2} \tag{5.35}$$

$$\tau_{12} = \tau_{21} = \frac{S_{12}}{t} + \frac{12\, M_{12}}{t^3}\zeta\,, \qquad -\frac{t}{2} \le \zeta \le \frac{t}{2} \tag{5.36}$$

where, on the right hand side of each equation, the first term represents the membrane stress due to internal forces N_1 , N_2 and S_{12} , which are constants across the entire thickness of the shell, on the other hand, the second term represents the bending stress due to moments M_1 , M_2 and M_{12} , which linearly vary along the thickness of the shell and reach zero at the mid-surface of the shell.

CHAPTER 6

EQUILIBRIUM DIFFERENTIAL EQUATIONS RELATIONSHIPS BETWEEN THE INTERNAL FORCES AND LOADING ON THE MID-SURFACE OF THE SHELL

Figure 6.1 describes the same differential element from a cylindrical shell. The membrane stresses, such as normal stresses, shear stresses and transverse forces, are shown in Figure (a), and the bending moments and twisting moments are illustrated in Figure (b). X, Y and Z represent the loading per unit area of mid-surface along the longitudinal, circumferential and normal direction respectively, including volume loading and surface loading.

From Figure 6.1 (a) and according to $\sum F_\xi = 0$, the following is obtained:

$$\left[\left(N_1 + \frac{\partial N_1}{\partial \xi} d\xi\right) d\eta - N_1 d\eta\right] + \left[\left(S_{21} + \frac{\partial S_{21}}{\partial \eta} d\eta\right) d\xi - S_{21} d\xi\right] + X d\xi\, d\eta = 0,$$

according to $\sum F_\eta = 0$, the following is obtained:

$$\left[\left(S_{12} + \frac{\partial S_{12}}{\partial \xi} d\xi\right) d\eta - S_{12}\, d\eta\right] + \left[\left(N_2 + \frac{\partial N_2}{\partial \eta} d\eta\right) d\xi \left(\cos\frac{d\eta}{R}\right) - N_2\, d\xi\right]$$

$$+\left[\left(Q_2 + \frac{\partial Q_2}{\partial \eta} d\eta\right) d\xi \left(\sin\frac{d\eta}{R}\right)\right] + Y\, d\xi\, d\eta = 0\,.$$

and according to $\sum F_\zeta = 0$, the following is obtained:

$$\left[\left(Q_1 + \frac{\partial Q_1}{\partial \xi} d\xi\right) d\eta - Q_1\, d\eta\right] + \left[\left(Q_2 + \frac{\partial Q_2}{\partial \eta} d\eta\right) d\xi \left(\cos\frac{d\eta}{R}\right) - Q_2\, d\xi\right]$$

$$-\left(N_2 + \frac{\partial N_2}{\partial \eta} d\eta\right) d\xi \left(\sin \frac{d\eta}{R}\right) + Z\, d\xi\, d\eta = 0.$$

From Figure 6.1 (b) and according to $\sum M_\xi = 0$, the following is obtained:

$$\left[M_{12} d\eta - \left(M_{12} + \frac{\partial M_{12}}{\partial \xi} d\xi\right) d\eta\right] + \left[M_2 d\xi - \left(M_2 + \frac{\partial M_2}{\partial \eta} d\eta\right) d\xi\right]$$

$$\left(Q_2 + \frac{\partial Q_2}{\partial \eta} d\eta\right) d\xi \left(\cos \frac{d\eta}{R}\right) d\eta = 0,$$

according to $\sum M_\eta = 0$, the following is obtained:

$$\left[\left(M_{21} + \frac{\partial M_{21}}{\partial \eta} d\eta\right) d\xi \left(\cos \frac{d\eta}{R}\right) - M_{21}\, d\xi\right] + \left[\left(M_1 + \frac{\partial M_1}{\partial \xi} d\xi\right) d\eta - M_1\, d\eta\right]$$

$$-\left(Q_1 + \frac{\partial Q_1}{\partial \xi} d\xi\right) d\eta\, d\xi = 0,$$

and according to $\sum M_\zeta = 0$, the following is obtained:

$$\left(S_{12} + \frac{\partial S_{12}}{\partial \xi} d\xi\right) d\eta\, d\xi - \left(S_{21} + \frac{\partial S_{21}}{\partial \eta} d\eta\right) d\xi\, d\eta - \left(M_{21} + \frac{\partial M_{21}}{\partial \eta} d\eta\right) d\xi \left(\sin \frac{d\eta}{R}\right) = 0.$$

Considering $\sin\left(\frac{d\eta}{R}\right) \cong \frac{d\eta}{R}$ and $\cos\left(\frac{d\eta}{R}\right) \cong 1$ as well as neglecting all terms with third or higher order, then from all six equations shown above, one obtains the equilibrium equations for a cylindrical shell as follows:

$$\frac{\partial N_1}{\partial \xi} + \frac{\partial S_{21}}{\partial \eta} + X = 0 \qquad (6.1)$$

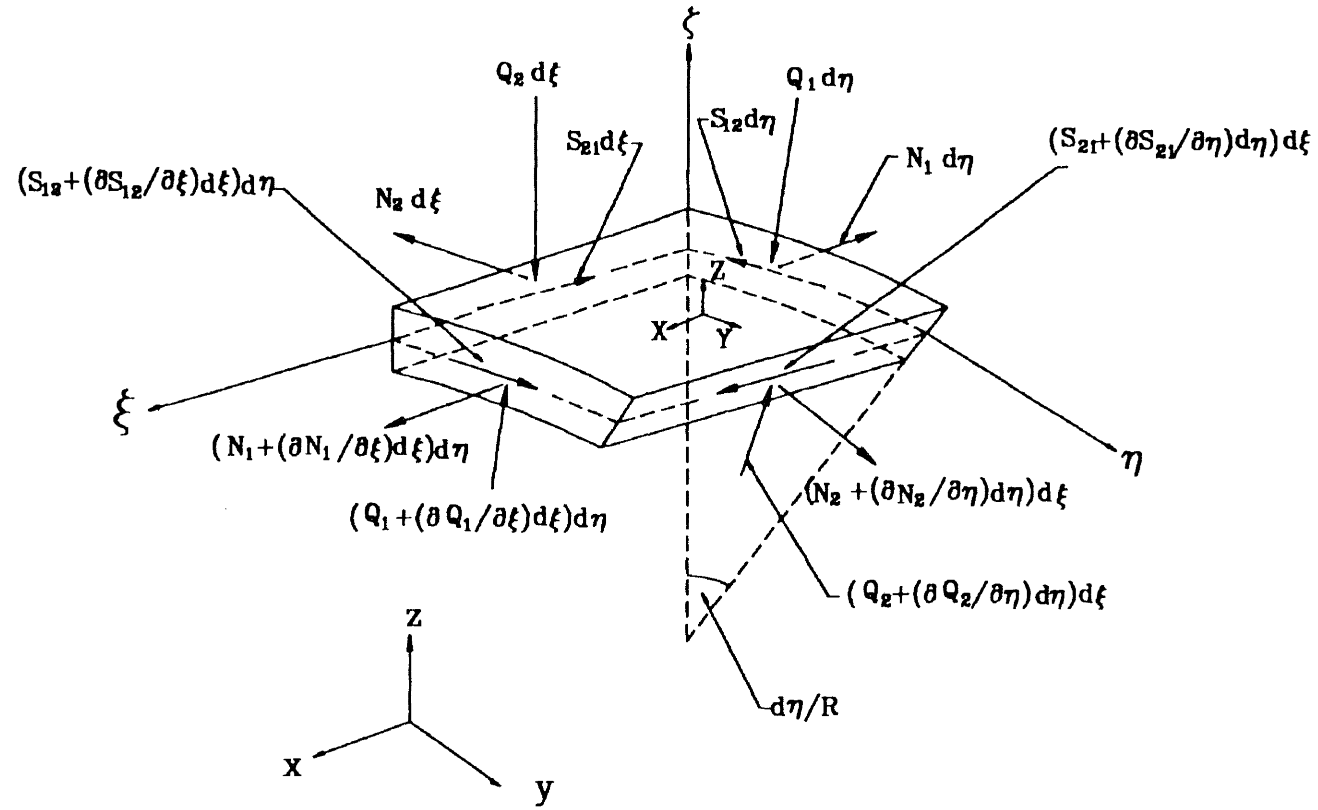


Figure 6.1 (a) Static equilibrium of forces on a differential element of a circular cylindrical shell body

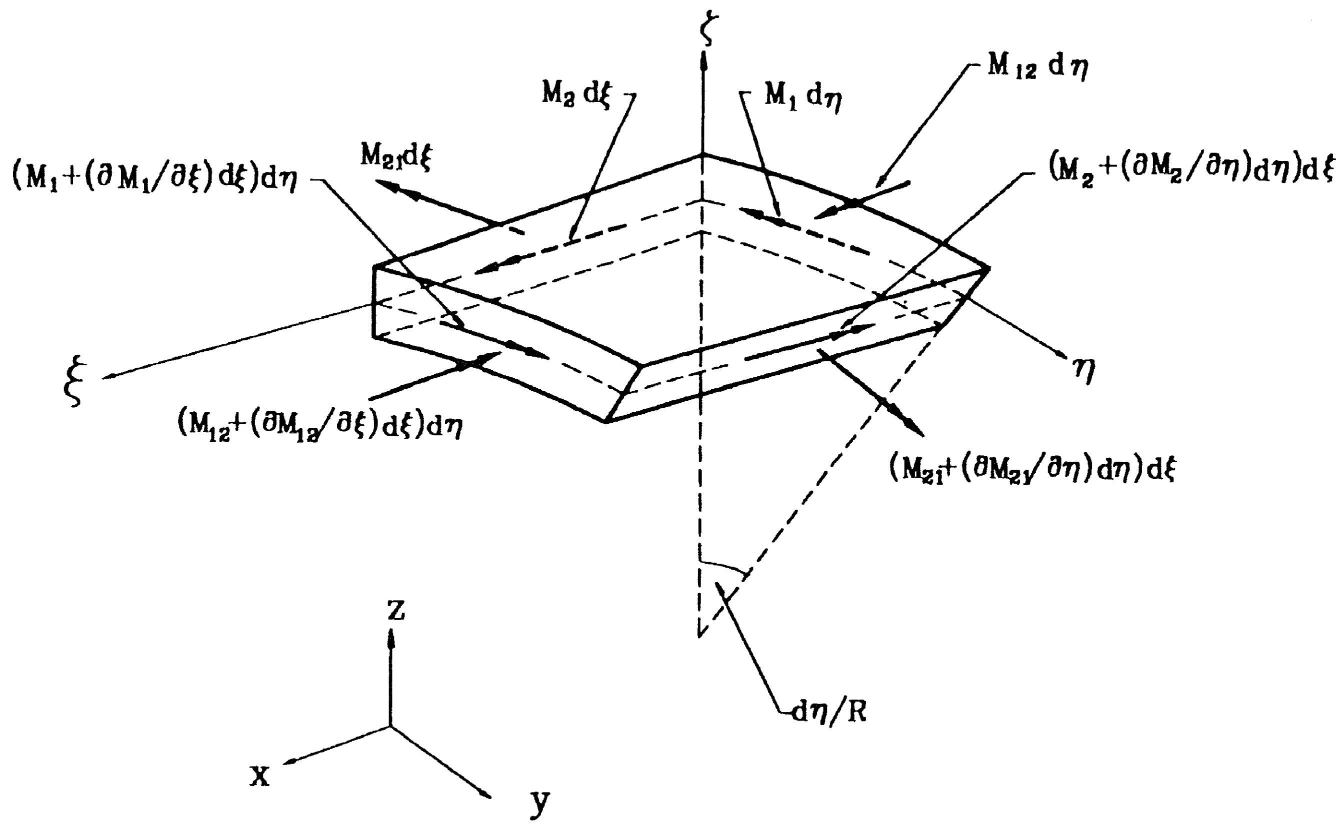


Figure 6.1 (b) Static equilibrium of moments on a differential element of a circular cylindrical shell body

$$\frac{\partial S_{12}}{\partial \xi} + \frac{\partial N_2}{\partial \eta} + \frac{Q_2}{R} + Y = 0 \tag{6.2}$$

$$\frac{\partial Q_1}{\partial \xi} + \frac{\partial Q_2}{\partial \eta} - \frac{N_2}{R} + Z = 0 \tag{6.3}$$

$$\frac{\partial M_1}{\partial \xi} + \frac{\partial M_{21}}{\partial \eta} - Q_1 = 0 \tag{6.4}$$

$$\frac{\partial M_{12}}{\partial \xi} + \frac{\partial M_2}{\partial \eta} - Q_2 = 0 \tag{6.5}$$

$$S_{12} - S_{21} - \frac{M_{21}}{R} = 0. \tag{6.6}$$

The Equation (6.6) is not a differential equation and it is always satisfied when the physical Equations (5.13) , (5.14) and (5.19) are substituted into it, therefore, it is an identity and it can be eliminated from the basic equations. Furthermore, the term $\frac{Q_2}{R}$ in Equation (6.2) represents the influence of the transverse force to the equilibrium in the circumferential direction and it is suggested to be negligible for simplification [11], so the reduced form of (6.2) is obtained as follows:

$$\frac{\partial S_{12}}{\partial \xi} + \frac{\partial N_2}{\partial \eta} + Y = 0 \tag{6.7}$$

Therefore, the equilibrium differential equations for the cylindrical shell are reduced to five equations, which are (6.1), (6.3), (6.4), (6.5) and (6.7).

CHAPTER 7

BOUNDARY GEOMETRY
INTERSECTION GEOMETRY AND GENERALIZED FORCES ON THE BOUNDARY OF INTERSECTION

7.1 Geometry of Intersection Curve

In the Cartesian coordinate system (x, y, z) shown in Figure 7.1, the equation of a cylindrical shell with radius R is

$$y^2 + z^2 = R^2 \tag{7.1}$$

In the Cartesian coordinate system (x_0 , y_0 , z_0) shown in Figure 7.1, the equation of a cylindrical nozzle with radius r is

$$y_0^2 + z_0^2 = r^2 \tag{7.2}$$

If one considers that the coordinate system (x, y, z) is developed by means of rotations of the coordinate system (x_0 , y_0 , z_0) about the axes x_0, y_0, and z_0 with angles δ , λ and Ω respectively, the relationship between the two coordinate systems can be expressed as follows:

$$\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} = [A] \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{7.3}$$

where the transformation matrix [A] has the form as follows [12]:

$$[A] = \begin{bmatrix} \cos\Omega\,\cos\lambda & \cos\Omega\,\sin\lambda\,\sin\delta - \sin\Omega\,\cos\delta & \cos\Omega\,\sin\lambda\,\cos\delta + \sin\Omega\,\sin\delta \\ \sin\Omega\,\cos\lambda & \sin\Omega\,\sin\lambda\,\sin\delta + \cos\Omega\,\cos\delta & \sin\Omega\,\sin\lambda\,\cos\delta - \cos\Omega\,\sin\delta \\ -\sin\lambda & \cos\lambda\,\sin\delta & \cos\lambda\,\cos\delta \end{bmatrix}.$$

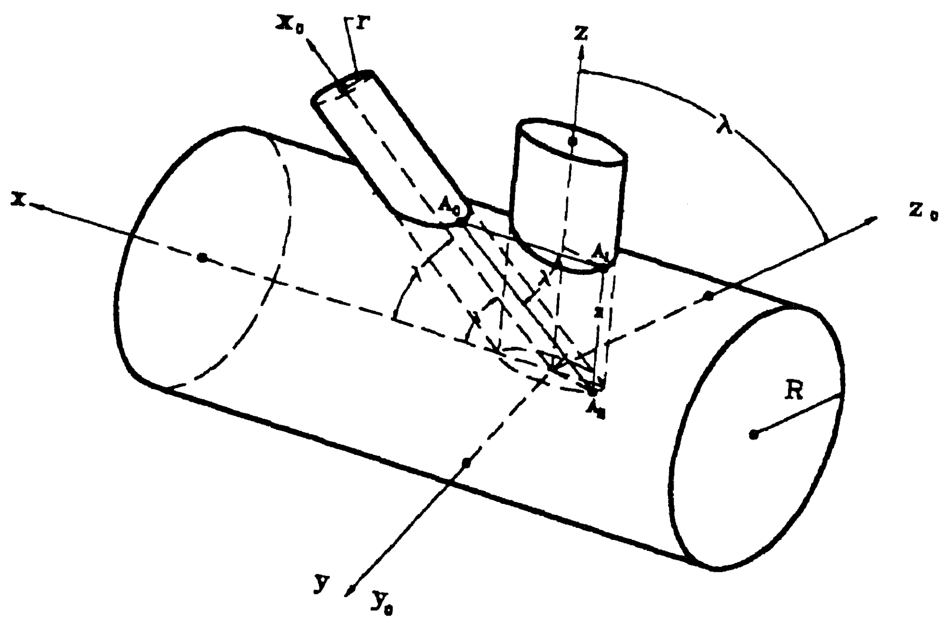


Figure 7.1 Cylindrical shell with oblique nozzle / vertical elliptical nozzle

In the case shown in Figure 7.1, since $\Omega = \delta = 0$, the transformation matrix is reduced to

$$[A] = \begin{bmatrix} \cos\lambda & 0 & \sin\lambda \\ 0 & 1 & 0 \\ -\sin\lambda & 0 & \cos\lambda \end{bmatrix}$$

therefore, according to (7.3), the relation between two coordinate systems is obtained as follows:

$$\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} = \begin{bmatrix} \cos\lambda & 0 & \sin\lambda \\ 0 & 1 & 0 \\ -\sin\lambda & 0 & \cos\lambda \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x\cos\lambda + z\sin\lambda \\ y \\ -x\sin\lambda + z\cos\lambda \end{pmatrix}. \tag{7.4}$$

Substituting (7.4) into (7.2) gives

$$y^2 + (-x\sin\lambda + z\cos\lambda)^2 = r^2 \tag{7.5}$$

then, (7.5) is the equation of the nozzle in the coordinates (x, y, z). For simplification, the coordinate system (x_0 , y_0 , z_0) will no longer be employed.

To demonstrate the concept of the intersection between the shell and the nozzle, by substituting $z = 0$ into (7.5), the intersection between the nozzle and the x-y plane is obtained as

$$y^2 + x^2\sin^2\lambda = r^2 \qquad \text{or} \qquad \frac{x^2}{\left(\dfrac{r}{\sin\lambda}\right)^2} + \frac{y^2}{r^2} = 1 \tag{7.6}$$

Obviously, (7.6) is an equation of an ellipse in the x-y plane. The length of semi-major axis of the ellipse is $\dfrac{r}{\sin\lambda}$, and the length of its semi-minor axis is r . If λ is equal to $\pi/2$, that is when the nozzle is perpendicular to the shell. This intersection ellipse in the x-y plane is reduces to a circle. Now that circle or ellipse are planar curves, as a space curve on the cylindrical surface, the intersection between the perpendicular nozzle and the shell can never be a circle or ellipse. Accordingly, as a space curve on the cylindrical surface, the intersection between the inclined nozzle and the shell , shown in Figure 7.2, can never be an ellipse either. Now, what kind of curve is the intersection between the inclined nozzle and the shell ? The answer is that this curve is determined by the two simultaneous equations consisting of Equations (7.1) and (7.5) as follows:

$$\begin{cases} y^2 + z^2 = R^2 \\ y^2 + (-x\sin\lambda + z\cos\lambda)^2 = r^2 \end{cases} \tag{7.7}$$

On the other hand, one may consider that if we take the ellipse expressed by (7.6) as a generating curve and develop an elliptical nozzle perpendicular to the x-y plane, the intersection between this elliptical nozzle and the shell can be determined by the two simultaneous equations consisting of (7.1) and (7.6) as follows:

$$\begin{cases} y^2 + z^2 = R^2 \\ \dfrac{x^2}{\left(\dfrac{r}{\sin\lambda}\right)^2} + \dfrac{y^2}{r^2} = 1 \end{cases} \tag{7.8}$$

Because the second equation in (7.8) has only two variables x and y, (7.8) looks simpler than (7.7). In fact, the distance between the point on the intersection of the inclined nozzle and the shell and the corresponding point on the intersection of elliptical cylindrical nozzle and the shell is equal to $z / \tan\lambda$ (see the triangle determined by three vertexes A_0, A_1 and A_2 shown in Figure 7.1). If $(x + z/\tan\lambda)$ is substituted into (7.8) to replace x, then (7.8) becomes the same as (7.7). This means that these two intersections on the shell surface are the same curves, but they are in different locations along the shell surface.

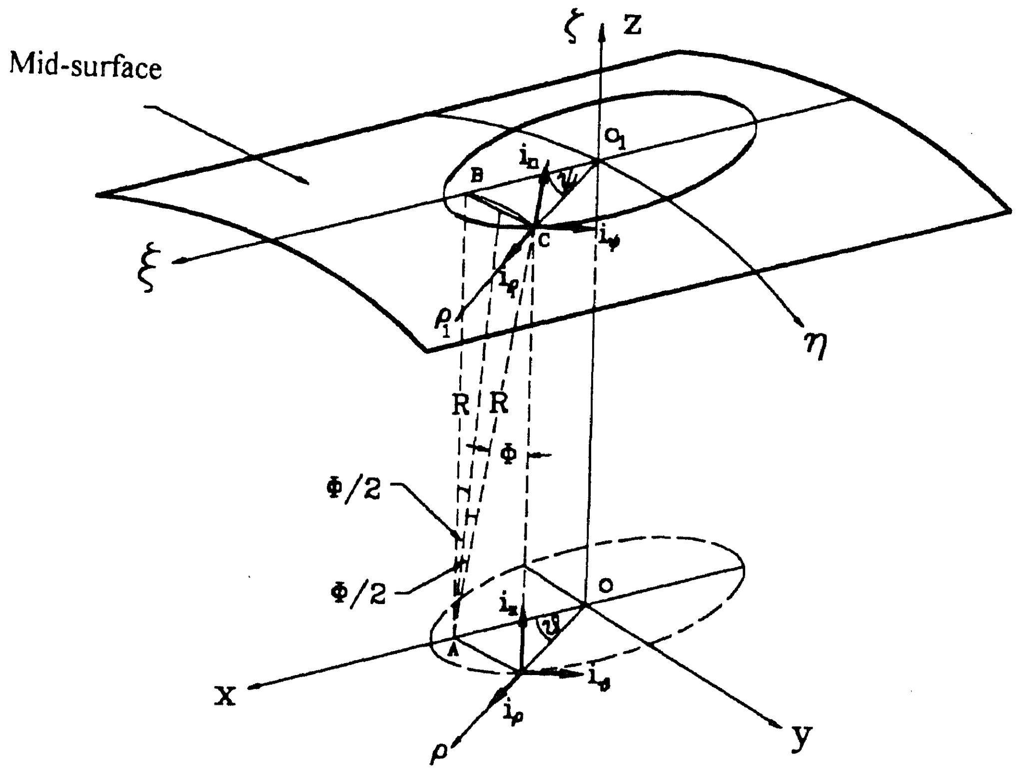


Figure 7.2 Intersection of shell and oblique nozzle on mid-surface

Therefore, from now on, one may take the intersection determined by (7.8) as the intersection between the nozzle and the shell because this brings tremendous simplification for the problem. Furthermore, to distinguish the oblique nozzle from the straight nozzle which have the same intersection on the shell surface, the inclined one is called oblique circular nozzle and the straight one is called straight elliptical nozzle respectively.

Now the cylindrical coordinate system (ρ, θ, z) is established which has the same origin and z-axis as the coordinates (x, y, z) as shown in Figure 7.2. In this cylindrical coordinates, the parametric equations of the intersection (7.8) are as follows:

$$x = \frac{r}{\sin\lambda}\cos\theta \tag{7.9}$$

$$y = r\sin\theta \tag{7.10}$$

$$z = \sqrt{R^2 - y^2} = \sqrt{R^2 - (r\sin\theta)^2} \tag{7.11}$$

where and in the sequel, for simplification, only the half of the shell above the x-y plane is considered.

Noting that $\rho = \sqrt{x^2 + y^2}$ and $\frac{r}{R} = \beta$, then, in the cylindrical coordinates (ρ, θ, z), the equation of the intersection curve (7.8) becomes as follows:

$$\rho = \frac{r}{\sin\lambda}\left(1 - \cos^2\lambda\ \sin^2\theta\right)^{1/2} \tag{7.12}$$

$$z = R\,(1 - \beta^2\ \sin^2\theta)^{1/2}\,. \tag{7.13}$$

The cylindrical coordinates (ρ, θ, z) in x-y plane is a polar coordinate system (ρ, θ). The other polar coordinate system (ρ_1, ψ) is also established to analyze the boundary conditions around the intersection area as shown in Fig. 7.2. The relation between the two kinds of polar coordinates about all points of the intersection curve can be determined as follows (see Figure 7.2).

Because $R \sin\Phi = \rho \sin\theta$, the following is obtained:

$$\sin\Phi = \frac{\rho}{R} \sin\theta \tag{7.14}$$

$$\cos\Phi = \left(1 - \sin^2\Phi\right)^{1/2} = \left[1 - \left(\frac{\rho}{R}\sin\theta\right)^2\right]^{1/2} . \tag{7.15}$$

The straight line segment BC in the isosceles triangle ABC is also a chord corresponding to the arc BC which is on the shell surface. The length of the straight line segment BC is

$$L_{BC} = 2\,R \sin\frac{\Phi}{2} = 2\,R\left(\frac{1 - \cos\Phi}{2}\right)^{1/2} = \sqrt{2}\,R\,(1 - \cos\Phi)^{1/2} ,$$

substituting (7.15) into the above expression gives

$$L_{BC} = \sqrt{2}\,R\left\{1 - \left[1 - \left(\frac{\rho}{R}\sin\theta\right)^2\right]^{1/2}\right\}^{1/2} .$$

Then, $\rho_1^2 = (L_{O,B})^2 + (L_{BC})^2 = (\rho\cos\theta)^2 + 2\,R^2\left\{1 - \left[1 - \left(\frac{\rho}{R}\sin\theta\right)^2\right]^{1/2}\right\}$

$$= \rho^2\cos^2\theta + 2\,R^2 - 2\,R^2\left[1 - \left(\frac{\rho}{R}\sin\theta\right)^2\right]^{1/2} ,$$

this yields $$\rho_1 = \left\{\rho^2\cos^2\theta + 2\,R^2 - 2\,R^2\left[1 - \left(\frac{\rho}{R}\sin\theta\right)^2\right]^{1/2}\right\}^{1/2} . \tag{7.16}$$

Also because $L_{O_1 B} = L_{OA}$, that is $\rho_1 \cos\psi = \rho \cos\theta$, one arrives

$$\cos\psi = \frac{\rho}{\rho_1}\cos\theta$$

then substituting (7.16) into the above gives

$$\cos\psi = \frac{\rho\cos\theta}{\left\{\rho^2\cos^2\theta + 2R^2 - 2R^2\left[1-\left(\frac{\rho}{R}\sin\theta\right)^2\right]^{1/2}\right\}^{1/2}} \tag{7.17}$$

and

$$\sin\psi = \left(1-\cos^2\psi\right)^{1/2} = \left\{\frac{2R^2 - 2R^2\left[1-\left(\frac{\rho}{R}\sin\theta\right)^2\right]^{1/2}}{\rho^2\cos^2\theta + 2R^2 - 2R^2\left[1-\left(\frac{\rho}{R}\sin\theta\right)^2\right]^{1/2}}\right\}^{1/2} \tag{7.18}$$

where

$$\psi = \arccos\left\{\frac{\rho\cos\theta}{\left\{\rho^2\cos^2\theta + 2R^2 - 2R^2\left[1-\left(\frac{\rho}{R}\sin\theta\right)^2\right]^{1/2}\right\}^{1/2}}\right\}$$

On the other hand, from Figure 7.2 and Equations (7.8), one may establish the relationship between the coordinate system (x, y, z) and the coordinate system (ξ, η, ζ) as follows:

$$\begin{cases} x = \xi \\ y = \sqrt{r^2 - \xi^2\sin^2\lambda} \\ z = \sqrt{R^2 - y^2} = \sqrt{R^2 - r^2 + \xi^2\sin^2\lambda} \end{cases} \tag{7.19}$$

$$\Phi = \arcsin\left(\frac{y}{R}\right) = \arcsin\left(\frac{\sqrt{r^2 - \xi^2 \sin^2\lambda}}{R^2}\right) = \arcsin\sqrt{\beta^2 - \xi^2\left(\frac{\sin\lambda}{R}\right)^2} \tag{7.20}$$

$$\eta = R\,\Phi = R\arcsin\sqrt{\beta^2 - \xi^2\left(\frac{\sin\lambda}{R}\right)^2}\ . \tag{7.21}$$

7.2 Relationship between the Directions of All Kinds of Coordinate Systems on the Boundary of Intersection Curve

Now, one can consider the generalized forces on the intersection between the shell and the nozzle. By means of substituting the geometrical equations of Chapter 4 into the physical equations of Chapter 5, all generalized forces can be expressed in the directions ξ, η and ζ, which will be reported in Chapter 8. But the given boundary forces, generally speaking, distribute in the three directions as below:

1. In the mid-surface, it is perpendicular to the intersection curve, as indicated by unit vector i_v, as shown in Figure 7.3.

2. In the mid-surface, it is tangent to the intersection curve, as indicated by the unit vector i_t, as shown in Figure 7.3.

3. In the normal direction of mid-surface, it is perpendicular to the intersection curve, as indicated by unit vector i_n, as shown in Figure 7.3.

The relation among the triads is determined as follows:

$$i_v \times i_t = i_n \tag{7.22}$$

To analyze the boundary conditions, two more sets of unit vectors are also established as shown in Figure 7.2 and Figure 7.3. Accordingly, we also have

$$i_\rho \times i_\theta = i_z \tag{7.23}$$

$$i_{\rho_1} \times i_\psi = i_n \,. \tag{7.24}$$

In (7.22) and (7.24), the two unit vectors i_n are the same.

The relation between the two sets of unit vectors in (7.23) and (7.24) is determined as follows:

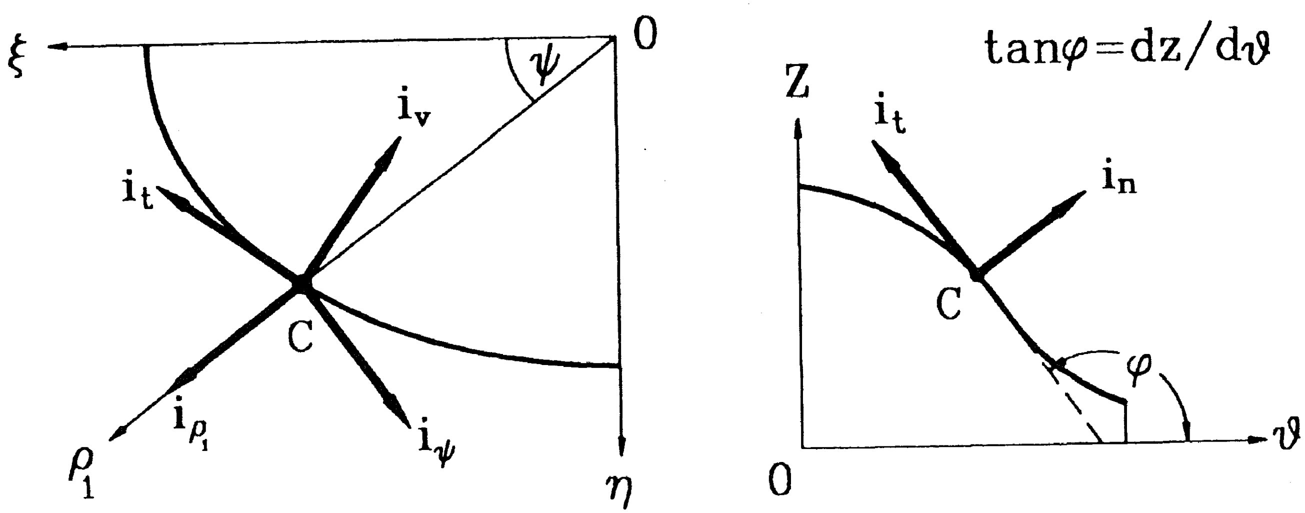


Figure 7.3 Directions of unit vectors on intersection

$$\begin{pmatrix} i_{\rho_1} \\ i_{\psi} \\ i_n \end{pmatrix} = [B] \begin{pmatrix} i_{\rho} \\ i_{\theta} \\ i_z \end{pmatrix} \tag{7.25}$$

where the transformation matrix [B] has the form as follows [11]:

$$[B] = \begin{bmatrix} \cos\theta\cos\psi + \sin\theta\sin\psi\cos\Phi & -\sin\theta\cos\psi + \cos\theta\sin\psi\cos\Phi & -\sin\psi\sin\Phi \\ -\cos\theta\sin\psi + \sin\theta\cos\psi\cos\Phi & \sin\theta\sin\psi + \cos\theta\cos\psi\cos\Phi & -\cos\psi\sin\Phi \\ \sin\theta\sin\Phi & \cos\theta\sin\Phi & \cos\Phi \end{bmatrix}$$

where all elements in [B] have been defined as before.

To simplify the calculation and speed up the computation, (7.15), (7.16), (7.17) and (7.18) are expanded into power series of trigonometric functions of variable θ as:

$$\rho = \frac{r}{\sin\lambda}\left(1 - \cos^2\lambda\,\sin^2\theta\right)^{1/2} = \frac{r}{\sin\lambda}\left(1 - \frac{1}{2}\cos^2\lambda\,\sin^2\theta + O(\cos^4\lambda)\right) \tag{7.26}$$

$$z = R\left(1 - \beta^2\sin^2\theta\right)^{1/2} = R\left[1 - \frac{1}{2}\beta^2\sin^2\theta + O(\beta^4)\right] \tag{7.27}$$

$$\sin\Phi = \frac{\rho}{R}\sin\theta = \frac{1}{R}\sin\theta\left[\frac{r}{\sin\lambda}\left(1 - \cos^2\lambda\,\sin^2\theta\right)^{1/2}\right]$$

$$= \frac{\beta}{\sin\lambda}\sin\theta\left[1 - \frac{1}{2}\cos^2\lambda\,\sin^2\theta + O(\cos^4\lambda)\right] \tag{7.28}$$

$$\cos\Phi = \left(1 - \sin^2\Phi\right)^{1/2} = \left[1 - \left(\frac{\rho}{R}\sin\theta\right)^2\right]^{1/2}$$

$$= \left\{1 - \frac{1}{R^2}\sin^2\theta\left[\frac{r}{\sin\lambda}\left(1 - \cos^2\lambda\,\sin^2\theta\right)\right]^2\right\}^{1/2}$$

$$= 1 - \frac{1}{2}\,\frac{\beta^2}{\sin^2\lambda}\sin^2\theta + O\left(\beta^4\right) \tag{7.29}$$

$$\Phi = \arcsin\left(\frac{\rho}{\mathrm{R}}\sin\theta\right) = \arcsin\left[\frac{\beta}{\sin\lambda}\sin\theta\left(1 - \cos^2\lambda\sin^2\theta\right)^{1/2}\right]$$

$$= \frac{\beta}{\sin\lambda}\sin\theta\left[1 - \frac{1}{2}\cos^2\lambda\sin^2\theta + O(\cos^6\lambda)\right] \tag{7.30}$$

$$\rho_1 = \left[\rho^2\cos^2\theta + 2\,\mathrm{R}^2(1 - \cos\Phi)\right]^{1/2}$$

$$= \left\{\frac{\mathrm{r}^2}{\sin^2\lambda}\left(1 - \cos^2\lambda\sin^2\theta\right)\cos^2\theta + 2\,\mathrm{R}^2 - 2\,\mathrm{R}^2\left[1 - \frac{1}{2}\left(\frac{\beta}{\mathrm{R}}\sin\theta\right)^2 + \ldots\right]\right\}^{1/2}$$

$$= \frac{\mathrm{r}}{\sin\lambda}\left(1 - \cos^2\lambda\sin^2\theta\cos^2\theta + \ldots\right)^{1/2}$$

$$= \frac{\mathrm{r}}{\sin\lambda}\left[1 - \frac{1}{2}\cos^2\lambda\sin^2\theta\cos^2\theta + O(\cos^4\theta)\right] \tag{7.31}$$

$$\cos\psi = \frac{\rho}{\rho_1}\cos\theta$$

$$= \frac{\mathrm{r}}{\sin\lambda}\left(1 - \frac{1}{2}\cos^2\lambda\sin^2\theta + \ldots\right)\left[\frac{\mathrm{r}}{\sin\lambda}\left(1 - \frac{1}{2}\cos^2\lambda\sin^2\theta\cos^2\theta\right)\right]^{-1}\cos\theta$$

$$= \left(1 - \frac{1}{2}\cos^2\lambda\sin^2\theta + \ldots\right)\left(1 + \frac{1}{2}\cos^2\lambda\sin^2\theta\cos^2\theta + \ldots\right)\cos\theta$$

$$= \cos\theta\left[1 - \frac{1}{2}\cos^2\lambda\sin^4\theta + O(\cos^4\lambda)\right] \tag{7.32}$$

$$\sin\psi = \left(1 - \cos^2\psi\right)^{1/2} = \left\{1 - \left[\cos\theta\left(1 - \frac{1}{2}\cos^2\lambda\sin^4\theta\right)\right]^2\right\}^{1/2}$$

$$= \sin\theta\left[1 + \frac{1}{2}\cos^2\lambda\sin^2\theta\cos^2\theta + O(\cos^4\lambda)\right]. \tag{7.33}$$

Now, substituting (7.28), (7.29), (7.31) and (7.32) into transformation matrix [B] in (7.25) gives that

$$B = \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{bmatrix} \tag{7.34}$$

where $B_{11} = 1 - \frac{1}{2}\frac{\beta^2}{\sin^2\lambda}\sin^4\theta + O(\cos^4\lambda)$

$$B_{12} = \frac{1}{2}A_0 \sin^3\theta\cos\theta\left[1 - \frac{1}{2}B_0 \sin^2\theta\cos^2\theta + O(\cos^4\lambda)\right]$$

where the constants $A_0 = \dfrac{\sin^2\lambda\cos^2\lambda - \beta^2}{\sin^2\lambda}$ and $B_0 = \dfrac{\beta^2\cos^2\lambda}{\sin^2\lambda\cos^2\lambda - \beta^2}$.

$$B_{13} = -\frac{\beta}{\sin\lambda}\sin^2\theta\left[1 - \frac{1}{2}\cos^2\lambda\sin^4\theta + O(\cos^4\theta)\right]$$

$$B_{21} = -\frac{1}{2}\sin^3\theta\cos\theta\left[\frac{\beta^2}{\sin^2\lambda} + \cos^2\lambda\ \cos^2\theta + O(\cos^4\lambda)\right]$$

$$B_{22} = 1 - \frac{1}{2}\frac{\beta^2}{\sin^2\lambda}\sin^2\theta\cos^2\theta + O(\cos^4\lambda)$$

$$B_{23} = -\frac{\beta}{\sin\lambda}\sin\theta\cos\theta\left[1 - \frac{1}{2}\cos^2\lambda\sin^2\theta + O(\cos^4\lambda)\right]$$

$$B_{31} = \frac{\beta}{\sin\lambda}\sin^2\theta\left[1 - \frac{1}{2}\cos^2\lambda\sin^2\theta + O(\cos^4\theta)\right]$$

$$B_{32} = \frac{\beta}{\sin\lambda}\sin\theta\cos\theta\left[1 - \frac{1}{2}\cos^2\lambda\sin^2\theta + O(\cos^4\lambda)\right]$$

$$B_{33} = 1 - \frac{1}{2}\frac{\beta^2}{\sin^2\lambda}\sin^2\theta + O\left(\frac{\beta^4}{\sin^4\lambda}\right).$$

From Fig. 7.3, one also obtains

$$\tan\varphi = \frac{dz}{d\theta} = \frac{d}{d\theta}\left[R\left(1 - \beta^2\sin^2\theta\right)^{1/2}\right]$$

$$= -R\beta^2\sin\theta\cos\theta\left(1 - \beta^2\sin^2\theta\right)^{-1/2}$$

$$= -R\beta^2 \sin\theta \cos\theta\left[1 + \frac{1}{2}\beta^2 \sin^2\theta \ + O(\beta^4)\right] \tag{7.35}$$

$$\cos\varphi = \left(1 + \tan^2\varphi\right)^{-1/2} = \left\{1 + \left[R\beta^2 \sin\theta \cos\theta\left(1 + \frac{1}{2}\beta^2 \sin^2\theta + \ldots\right)\right]^2\right\}^{-1/2}$$

$$= \left[1 + R^2\beta^4 \sin^2\theta \cos^2\theta + \ldots\right]^{-1/2}$$

$$= \left[1 - \frac{1}{2} R^2\beta^4 \sin^2\theta \cos^2\theta + O(\beta^4)\right] \tag{7.36}$$

$$\sin\varphi = \tan\varphi \cos\varphi$$

$$= \left[-R\beta^2 \sin\theta \cos\theta\left(1 + \frac{1}{2}\beta^2 \sin^2\theta + \ldots\right)\right]\left[1 - \frac{1}{2}R^2\beta^4 \sin^2\theta \cos^2\theta + \ldots\right]$$

$$= -R\beta^2 \sin\theta \cos\theta\left[1 + \frac{1}{2}\beta^2 \sin^2\theta \ + O(\beta^4)\right]. \tag{7.37}$$

By substituting (7.33) into (7.25), the triads of unit vectors i_{ρ_1}, i_ψ and i_n can also be expressed with the trigonometric functions of variable θ :

$$i_{\rho_1} = \left[1 - \frac{1}{2}\frac{\beta^2}{\sin^2\lambda}\sin^4\theta + (\cos^4\theta)\right] i_\rho$$

$$+\left\{\frac{1}{2}A_0 \sin^3\theta \cos\theta\left[1 - \frac{1}{2}B_0 \sin^2\theta \cos^2\theta + (\cos^4\theta)\right]\right\} i_\theta$$

$$+\left\{-\frac{\beta}{\sin\lambda}\sin^2\theta\left[1 - \frac{1}{2}\cos^2\lambda \sin^4\theta + (\cos^4\theta)\right]\right\} i_z$$

$$= \left[1 - \frac{1}{2}\frac{\beta^2}{\sin^2\lambda}\sin^4\theta + O(\cos^4\lambda)\right] i_\rho + \left(\frac{1}{2}A_0 \sin^3\theta \cos\theta\right) i_\theta$$

$$+\left[-\frac{\beta}{\sin\lambda}\sin^2\theta\right] i_z \tag{7.38}$$

$$\mathrm{i}_{\psi}=\left\{-\frac{1}{2}\sin^3\theta\cos\theta\left[\frac{\beta^2}{\sin^2\lambda}+\cos^2\lambda\ \cos^2\theta+O(\cos^4\lambda)\right]\right\}\mathrm{i}_{\rho}$$

$$+\left[1-\frac{1}{2}\frac{\beta^2}{\sin^2\lambda}\sin^2\theta\cos^2\theta+O(\cos^4\lambda)\right]\mathrm{i}_{\theta}$$

$$+\left\{-\frac{\beta}{\sin\lambda}\sin\theta\cos\theta\left[1-\frac{1}{2}\cos^2\lambda\sin^2\theta+O(\cos^4\lambda)\right]\right\}\mathrm{i}_z$$

$$=\left(-\frac{1}{2}\sin^3\theta\cos\theta\frac{\beta^2}{\sin^2\lambda}\right)\mathrm{i}_{\rho}+\left[1-\frac{1}{2}\frac{\beta^2}{\sin^2\lambda}\sin^2\theta\cos^2\theta+O(\cos^4\lambda)\right]\mathrm{i}_{\theta}$$

$$+\left(-\frac{\beta}{\sin\lambda}\sin\theta\cos\theta\right)\mathrm{i}_z \qquad (7.39)$$

$$\mathrm{i}_n=\left\{\frac{\beta}{\sin\lambda}\sin^2\theta\left[1-\frac{1}{2}\cos^2\lambda\sin^2\theta+\dots\right]\right\}\mathrm{i}_{\rho}$$

$$+\left\{\frac{\beta}{\sin\lambda}\sin\theta\cos\theta\left[1-\frac{1}{2}\cos^2\lambda\sin^2\theta+\dots\right]\right\}\mathrm{i}_{\theta}$$

$$+\left[1-\frac{1}{2}\frac{\beta^2}{\sin^2\lambda}\sin^2\theta+O\left(\beta^4\right)\right]\mathrm{i}_z$$

$$=\left(\frac{\beta}{\sin\lambda}\sin^2\theta\right)\mathrm{i}_{\rho}+\left(\frac{\beta}{\sin\lambda}\sin\theta\cos\theta\right)\mathrm{i}_{\theta}$$

$$+\left[1-\frac{1}{2}\frac{\beta^2}{\sin^2\lambda}\sin^2\theta+O\left(\beta^4\right)\right]\mathrm{i}_z\ . \qquad (7.40)$$

Now, from Figure 7.3, one obtains the follows:

$$\mathrm{i}_t=(-\cos\varphi)\,\mathrm{i}_{\theta}+(\sin\varphi)\,\mathrm{i}_z$$

$$=-\left[1-\frac{1}{2}R^2\beta^4\sin^2\theta\cos^2\theta+O(\beta^4)\right]\mathrm{i}_{\theta}$$

$$+\left\{-R\beta^2\sin\theta\cos\theta\left[1+\frac{1}{2}\beta^2\sin^2\theta+O(\beta^4)\right]\right\}\mathrm{i}_z$$

$$= -\left[1 - \frac{1}{2}R^2\beta^4 \sin^2\theta\cos^2\theta + O(\beta^4)\right] i_\theta$$

$$+ \left\{ - R\beta^2 \sin\theta\cos\theta\left[1 + \frac{1}{2}\beta^2\sin^2\theta + O(\beta^4)\right]\right\} i_z$$

$$= -\left[1 - \frac{1}{2}R^2\beta^4\sin^2\theta\cos^2\theta + O(\beta^4)\right] i_\theta + \left(-R\beta^2\sin\theta\cos\theta\right) i_z \tag{7.41}$$

$$i_v = i_t \times i_n = \begin{vmatrix} i_\rho & i_\theta & i_z \\ 0 & -\left[1 - \frac{1}{2}R^2\beta^4\sin^2\theta\cos^2\theta + \ldots\right] & -R\beta^2\sin\theta\cos\theta + \ldots \\ \frac{\beta}{\sin\lambda}\sin^2\theta + \ldots & \frac{\beta}{\sin\lambda}\sin\theta\cos\theta + \ldots & 1 - \frac{1}{2}\frac{\beta^2}{\sin^2\lambda}\sin^2\theta + \ldots \end{vmatrix}$$

$$= \left[-1 + \frac{1}{2}\frac{\beta^3}{\sin^2\lambda}\sin^2\theta + O(\beta^4)\right] i_\rho + \left(-\frac{R\beta^2}{\sin\lambda}\sin^3\theta\cos\theta\right) i_\theta$$

$$+ \left(\frac{\beta}{\sin\lambda}\sin^2\theta\right) i_z \tag{7.42}$$

Now, all the expressions to determine the relations of the directions between all generalized forces along the intersection of the shell and the nozzle are obtained as follows:

$$\cos(i_v , i_{\rho_1}) = \cos(i_t , i_\psi) = i_v \bullet i_{\rho_1}$$

$$= \left(-1 + \frac{1}{2}\frac{\beta^3}{\sin^2\lambda}\sin^2\theta + \ldots, \; -\frac{R\beta^3}{\sin\lambda}\sin^3\theta\cos\theta + \ldots, \; \frac{\beta}{\sin\lambda}\sin^2\theta + \ldots\right)$$

$$\bullet \left(1 - \frac{1}{2}\frac{\beta^3}{\sin^3\lambda}\sin^4\theta + \ldots, \; \frac{1}{2}A_0\sin^3\theta\cos\theta + \ldots, \; -\frac{\beta}{\sin\lambda}\sin^2\theta + \ldots\right)$$

$$= -1 + \frac{1}{2}\frac{\beta^2}{\sin^2\lambda}\sin^2\theta\cos^2\theta + O(\beta^4) \tag{7.43}$$

$$\cos(i_v, i_\psi) = -\cos(i_t, i_{\rho_1}) = i_v \bullet i_\psi$$

$$= \left(-1 + \frac{1}{2}\frac{\beta^3}{\sin^2\lambda}\sin^2\theta + \ldots,\ -\frac{R\beta^3}{\sin\lambda}\sin^3\theta\cos\theta + \ldots,\ \frac{\beta}{\sin\lambda}\sin^2\theta + \ldots\right)$$

$$\bullet\left(-\frac{1}{2}\frac{\beta^2}{\sin^2\lambda}\sin^3\theta\cos\theta + \ldots,\ 1 - \frac{1}{2}\frac{\beta^2}{\sin^2\lambda}\sin^2\theta\cos^2\theta + \ldots,\ -\frac{\beta}{\sin\lambda}\sin\theta\cos\theta + \ldots\right)$$

$$= -\frac{\beta^2 + R\beta^3\sin\lambda}{2\sin^2\lambda}\sin^3\theta\cos\theta + O(\beta^4) \qquad (7.44)$$

CHAPTER 8

DERIVATION OF ANALYTICAL SOLUTIONS

8.1 Derivation of the Basic Differential Equations

By means of substituting the geometrical equations (4.31) , (4.32) and (4.33) of Chapter 4 into the physical equations (5.26) through (5.31) of Chapter 5 , the following is obtained:

$$N_1 = \frac{Et}{1-\mu^2}\left[\frac{\partial u}{\partial \xi} + \mu\left(\frac{\partial u}{\partial \eta} + \frac{w}{R}\right)\right] \tag{8.1}$$

$$N_2 = \frac{Et}{1-\mu^2}\left[\left(\frac{\partial v}{\partial \eta} + \frac{w}{R}\right) + \mu\frac{\partial u}{\partial \xi}\right] \tag{8.2}$$

$$S_{12} = S_{21} = \frac{Et}{2(1+\mu)}\left(\frac{\partial u}{\partial \eta} + \frac{\partial v}{\partial \xi}\right) \tag{8.3}$$

$$M_1 = -D\left(\frac{\partial^2 w}{\partial \xi^2} + \mu\frac{\partial^2 w}{\partial \eta^2}\right) \tag{8.4}$$

$$M_2 = -D\left(\frac{\partial^2 w}{\partial \eta^2} + \mu\frac{\partial^2 w}{\partial \xi^2}\right) \tag{8.5}$$

$$M_{12} = M_{21} = -(1-\mu)D\frac{\partial^2 w}{\partial \xi \partial \eta} . \tag{8.6}$$

Then, substituting (8.1) and (8.6) into (6.4) and (6.5) of the equilibrium differential equations of Chapter 6, one obtains

$$Q_1 = -D\frac{\partial}{\partial \xi}\nabla^2 w \tag{8.7}$$

$$Q_2 = -D\frac{\partial}{\partial \eta}\nabla^2 w \tag{8.8}$$

and substituting the expressions (8.1) through (8.8) into the (6.1) , (6.7) and (6.3) of the equilibrium equations in Chapter 6., one obtains the basic differential equations as follows:

$$\left(\frac{\partial^2}{\partial \xi^2} + \frac{1-\mu}{2}\frac{\partial^2}{\partial \eta^2}\right)u + \frac{1+\mu}{2}\frac{\partial^2 v}{\partial \xi \partial \eta} + \frac{\mu}{R}\frac{\partial w}{\partial \xi} = -\frac{1-\mu^2}{Et}X \tag{8.9}$$

$$\frac{1+\mu}{2}\frac{\partial^2 u}{\partial \xi \partial \eta} + \left(\frac{\partial^2}{\partial \eta^2} + \frac{1-\mu}{2}\frac{\partial^2}{\partial \xi^2}\right)v + \frac{1}{R}\frac{\partial w}{\partial \eta} = -\frac{1-\mu^2}{Et}Y \tag{8.10}$$

$$\frac{\mu}{R}\frac{\partial u}{\partial \xi} + \frac{1}{R}\frac{\partial v}{\partial \eta} + \frac{w}{R^2} + \frac{t^2}{12}\nabla^4 w = \frac{1-\mu^2}{Et}Z \tag{8.11}$$

where the relations between displacements on mid-surface and loading have been established.

Now, let us start with the case of normal loading acting on the circular cylindrical shell where the loading, such as internal pressure, is always perpendicular to the mid-surface, which is the most popular loading to pressure vessels. In this case, $X = Y = 0$, then, Equations (8.9) and (8.10) are reduced to

$$\left(\frac{\partial^2}{\partial \xi^2} + \frac{1-\mu}{2}\frac{\partial^2}{\partial \eta^2}\right)u + \frac{1+\mu}{2}\frac{\partial^2 v}{\partial \xi \partial \eta} + \frac{\mu}{R}\frac{\partial w}{\partial \xi} = 0 \tag{8.12}$$

$$\frac{1+\mu}{2}\frac{\partial^2 u}{\partial \xi \partial \eta} + \left(\frac{\partial^2}{\partial \eta^2} + \frac{1-\mu}{2}\frac{\partial^2}{\partial \xi^2}\right)v + \frac{1}{R}\frac{\partial w}{\partial \eta} = 0\,. \tag{8.13}$$

Partially differentiated with respect to η , (8.12) becomes

$$\left(\frac{\partial^3}{\partial \xi^2 \partial \eta} + \frac{1-\mu}{2}\frac{\partial^3}{\partial \eta^3}\right)u + \frac{1+\mu}{2}\frac{\partial^3 v}{\partial \xi \partial \eta^2} + \frac{\mu}{R}\frac{\partial^2 w}{\partial \xi \partial \eta} = 0\,. \tag{8.14}$$

Partially differentiated with respect to ξ and multiplied by μ , (8.13) becomes

$$\frac{\mu(1+\mu)}{2}\frac{\partial^3 u}{\partial \xi^2 \partial \eta} + \mu\left(\frac{\partial^3}{\partial \xi \partial \eta^2} + \frac{1-\mu}{2}\frac{\partial^3}{\partial \xi^3}\right)v + \frac{\mu}{R}\frac{\partial^2 w}{\partial \xi \partial \eta} = 0\,. \tag{8.15}$$

Subtracting (8.15) from (8.14) , the following is obtained:

$$\frac{\partial}{\partial\eta}\left[(2+\mu)\frac{\partial^2}{\partial\xi^2}+\frac{\partial^2}{\partial\eta^2}\right]u+\frac{\partial}{\partial\xi}\left(\frac{\partial^2}{\partial\eta^2}-\mu\frac{\partial^2}{\partial\xi^2}\right)v=0. \tag{8.16}$$

From the information provided by (8.16), one can imagine that if there exists a function $F = F(\xi, \eta)$ which makes that

$$u=\frac{\partial}{\partial\xi}\left(\frac{\partial^2}{\partial\eta^2}-\mu\frac{\partial^2}{\partial\xi^2}\right)F \tag{8.17}$$

$$v=-\frac{\partial}{\partial\eta}\left[(2+\mu)\frac{\partial^2}{\partial\xi^2}+\frac{\partial^2}{\partial\eta^2}\right]F \tag{8.18}$$

and noting that the total differential operators of the function F in (8.17) is the same as the total differential operators of the displacement v in (8.16). And the total differential operators of the function F in (8.18) is exactly the total differential operators of the displacement u in (8.16) with a minus sign, it is obvious that the equation (8.16) will be fully satisfied.

Our purpose is to make the possible function F satisfy (8.12) and (8.13). Because Equation (8.16) came from the combination of (8.12) and (8.13), as long as the function F satisfies either (8.12) or (8.13), the function F will satisfy both of (8.12) and (8.13). Now substitute (8.17) and (8.18) into (8.12):

$$\left(\frac{\partial^2}{\partial\xi^2}+\frac{1-\mu}{2}\frac{\partial^2}{\partial\eta^2}\right)\left[\frac{\partial}{\partial\xi}\left(\frac{\partial^2}{\partial\eta^2}-\mu\frac{\partial^2}{\partial\xi^2}\right)F\right]+$$

$$\frac{1+\mu}{2}\frac{\partial^2}{\partial\xi\partial\eta}\left\{-\frac{\partial}{\partial\eta}\left[\frac{\partial^2}{\partial\eta^2}+(2+\mu)\frac{\partial^2}{\partial\xi^2}\right]F\right\}+\frac{\mu}{R}\frac{\partial w}{\partial\xi}=0$$

that is

$$-\mu\frac{\partial}{\partial\xi}\left(\frac{\partial^4}{\partial\xi^4}+2\frac{\partial^4}{\partial\xi^2\partial\eta^2}+\frac{\partial^4}{\partial\eta^4}\right)F+\frac{\mu}{R}\frac{\partial w}{\partial\xi}=0$$

thus $$w = \int \left[R \frac{\partial}{\partial \xi} \left(\frac{\partial^4}{\partial \xi^4} + 2 \frac{\partial^4}{\partial \xi^2 \partial \eta^2} + \frac{\partial^4}{\partial \eta^4} \right) F \right] d\xi .$$

Since only one of the expressions of displacement w , instead of all possible expressions of w, is needed , the arbitrary function from integration has been ignored. then one obtains

$$w = R \nabla^4 F \tag{8.19}$$

where the double Laplacian operator $\nabla^4 = \nabla^2 \nabla^2 = \frac{\partial^4}{\partial \xi^4} + 2 \frac{\partial^4}{\partial \xi^2 \partial \eta^2} + \frac{\partial^4}{\partial \eta^4}$.

Now it is clear that (8.19) is the requirement for the function F to satisfy both (8.12) and (8.13).

To make function F a displacement function, it is required that function F must satisfy not only (8.12) and (8.13), but also (8.14) simultaneously; otherwise, F still can not be a qualified displacement function. To see the necessary conditions for the function F to become a displacement function, substitute (8.17) , (8.18) and (8.19) into (8.11) obtains the following:

$$\frac{\mu}{R} \frac{\partial}{\partial \xi} \left[\frac{\partial}{\partial \xi} \left(\frac{\partial^2}{\partial \eta^2} - \mu \frac{\partial^2}{\partial \xi^2} \right) F \right] + \frac{1}{R} \frac{\partial}{\partial \eta} \left\{ - \frac{\partial}{\partial \eta} \left[\frac{\partial^2}{\partial \eta^2} + (2 + \mu) \frac{\partial^2}{\partial \xi^2} \right] F \right\}$$
$$+ \frac{1}{R^2} \left(R \nabla^4 F \right) + \frac{t^2}{12} \nabla^4 \left(R \nabla^4 F \right) = \frac{1 - \mu^2}{Et} Z$$

therefore it is obtained that

$$\nabla^8 F + \frac{Et}{DR^2} \frac{\partial^4 F}{\partial \xi^4} = \frac{Z}{RD} \tag{8.20}$$

where $\nabla^8 = \nabla^4 \nabla^4$ and $D = \dfrac{Et^3}{12(1 - \mu^2)}$ is known as flexural rigidity of thin shell.

Now, it has been determined that the function F can be employed as a displacement function as long as (8.17), (8.18), (8.19) and (8.20) are satisfied simultaneously. Before the form of function F is selected, one may express all internal force and moment resultants with the possible displacement function F. By means of substituting (8.17), (8.18) and (8.19) into (8.1) through (8.8), the following is obtained:

$$N_1 = Et\frac{\partial^4 F}{\partial \xi^2 \partial \eta^2} \tag{8.21}$$

$$N_2 = Et\frac{\partial^4 F}{\partial \xi^4} \tag{8.22}$$

$$S_{12} = S_{21} = -Et\frac{\partial^4 F}{\partial \xi^3 \partial \eta} \tag{8.23}$$

$$M_1 = -RD\left(\frac{\partial^2}{\partial \xi^2} + \mu\frac{\partial^2}{\partial \eta^2}\right)\nabla^4 F \tag{8.24}$$

$$M_2 = -RD\left(\frac{\partial^2}{\partial \eta^2} + \mu\frac{\partial^2}{\partial \xi^2}\right)\nabla^4 F \tag{8.25}$$

$$M_{12} = M_{21} = -(1 - \mu)RD\frac{\partial^2}{\partial \xi \partial \eta}\nabla^4 F \tag{8.26}$$

$$Q_1 = -RD\frac{\partial}{\partial \xi}\nabla^6 F \tag{8.27}$$

$$Q_2 = -RD\frac{\partial}{\partial \eta}\nabla^6 F \ . \tag{8.28}$$

8.2 Displacement Function and General Solutions of Basic Equations

If one attempt to use Fourier's series as the form of the displacement function F, that is

$$F = F(\xi, \eta) = f_0(\xi) + \sum_{n=1}^{\infty} f_n(\xi) \cos\left(\frac{n\pi}{R}\eta\right) \tag{8.29}$$

the expressions (8.21) through (8.28) become as follows:

$$N_1 = -Et \sum_{n=1}^{\infty} \frac{n^2\pi^2}{R^2} f_n''(\xi) \cos\left(\frac{n\pi}{R}\eta\right) \tag{8.30}$$

$$N_2 = Et\left[f_0^{(4)}(\xi) + \sum_{n=1}^{\infty} f_n^{(4)}(\xi) \cos\left(\frac{n\pi}{R}\eta\right)\right] \tag{8.31}$$

$$S_{12} = S_{21} = Et \sum_{n=1}^{\infty} \frac{n\pi}{R} f_n'''(\xi) \sin\left(\frac{n\pi}{R}\eta\right) \tag{8.32}$$

$$M_1 = -RD\left\{f_0^{(6)}(\xi) + \sum_{n=1}^{\infty}\left[f_n^{(6)}(\xi) - \frac{(2+\mu)n^2\pi^2}{R^2} f_n^{(4)}(\xi) + \frac{(1+2\mu)n^4\pi^4}{R^4} f_n''(\xi)\right.\right.$$
$$\left.\left. - \frac{\mu n^6\pi^6}{R^6} f_n(\xi)\right] \cos\left(\frac{n\pi}{R}\eta\right)\right\} \tag{8.33}$$

$$M_2 = -RD\left\{\mu f_0^{(6)}(\xi) + \sum_{n=1}^{\infty}\left[\mu f_n^{(6)}(\xi) - \frac{(1+2\mu)n^2\pi^2}{R^2} f_n^{(4)}(\xi) + \frac{(2+\mu)n^4\pi^4}{R^4} f_n''(\xi)\right.\right.$$
$$\left.\left. - \frac{n^6\pi^6}{R^6} f_n(\xi)\right] \cos\left(\frac{n\pi}{R}\eta\right)\right\} \tag{8.34}$$

$$M_{12} = M_{21} = (1-\mu)RD \sum_{n=1}^{\infty} \frac{n\pi}{R}\left[f_n^{(5)}(\xi) - \frac{2n^2\pi^2}{R^2} f_n'''(\xi) + \frac{n^4\pi^4}{R^4} f_n'(\xi)\right] \sin\left(\frac{n\pi}{R}\eta\right) \tag{8.35}$$

$$Q_1 = -RD\left\{f_0^{(7)}(\xi) + \sum_{n=1}^{\infty}\left[f_n^{(7)}(\xi) - \frac{3n^2\pi^2}{R^2} f_n^{(5)}(\xi) + \frac{3n^4\pi^4}{R^4} f_n'''(\xi)\right.\right.$$
$$\left.\left. - \frac{n^6\pi^6}{R^6} f_n'(\xi)\right] \cos\left(\frac{n\pi}{R}\eta\right)\right\} \tag{8.36}$$

$$Q_2 = RD\sum_{n=1}^{\infty}\frac{n\pi}{R}\left[f_n^{(6)}(\xi)-\frac{3n^2\pi^2}{R^2}f_n^{(4)}(\xi)+\frac{3n^4\pi^4}{R^4}f_n''(\xi)-\frac{n^6\pi^6}{R^6}f_n(\xi)\right]\sin\left(\frac{n\pi}{R}\eta\right) \quad (8.37)$$

Substituting (8.29) into the governing equation (8.20) gives

$$\left(\nabla^8 + \frac{Et}{DR^2}\frac{\partial^4}{\partial\xi^4}\right)\left[f_0(\xi)+\sum_{n=1}^{\infty}f_n(\xi)\cos\left(\frac{n\pi}{R}\eta\right)\right] = \frac{Z}{RD},$$

when Z = p = constant in the case of internal pressure, the above becomes

$$\left[\frac{d^8}{d\xi^8}+\frac{Et}{DR^2}\frac{d^4}{d\xi^4}\right]f_0(\xi)+\sum_{n=1}^{\infty}\left\{\left[\frac{d^8}{d\xi^8}-4\frac{n^2\pi^2}{R^2}\frac{d^6}{d\xi^6}+\left(6\frac{n^4\pi^4}{R^4}+\frac{Et}{DR^2}\right)\frac{d^4}{d\xi^4}\right.\right.$$

$$\left.\left.-4\frac{n^6\pi^6}{R^6}\frac{d^2}{d\xi^2}+\frac{n^8\pi^8}{R^8}\right]f_n(\xi)\right\}\cos\left(\frac{n\pi}{R}\eta\right)=\frac{p}{RD} \quad (8.38)$$

where the constant $\frac{p}{RD}$ can be expanded into the same form of Fourier series as the form on the left side of the equation. Assume this Fourier cosine series has the form as follows:

$$\frac{p}{RD}=\frac{J_0}{2}+\sum_{n=1}^{\infty}J_n\cos\left(\frac{n\pi}{R}\eta\right) \qquad -R\le\eta\le R$$

then $\quad J_0=\frac{2}{R}\int_0^R\frac{p}{RD}\,d\eta=\frac{2p}{RD}$

and $\quad J_n=\frac{2}{R}\int_0^R\left[\frac{p}{RD}\cos\left(\frac{n\pi\eta}{R}\right)\right]d\eta=0 \qquad n = 1, 2, 3, ...$

Therefore, from (8.38), two equations are obtained as follows:

$$\left[\frac{d^8}{d\xi^8}+\frac{Et}{DR^2}\frac{d^4}{d\xi^4}\right]f_0(\xi)=\frac{p}{RD} \quad (8.39)$$

$$\left[\left(\frac{d^2}{d\xi^2}-\frac{n^2\pi^2}{R^2}\right)^4+\frac{Et}{DR^2}\frac{d^4}{d\xi^4}\right]f_n(\xi)=0 \qquad n = 1, 2, 3, ... \quad (8.40)$$

(8.39) is a non-homogeneous eighth-order ordinary differential equation and its general solution consists of the summation of a particular solution and a linear combination of solutions of its homogeneous equation. The particular solution of equation (8.39) can be visually found as follows

$$f_0^* = \frac{pR}{24Et}\xi^4 \tag{8.41}$$

and the solution of its homogeneous equation can be determined by its characteristic equation:

$$S_0^8 + 4\,G^4\,S_0^4 = 0$$

where $G = \left(\dfrac{Et}{4\,D\,R^2}\right)^{1/4}$ and this characteristic equation has two pairs of conjugate complex roots and four repeated real roots as follows:

$$S_{1,2} = G\,(1 \pm i)\,,\quad S_{3,4} = -\,G\,(1 \pm i) \quad \text{and} \quad S_{5,6,7,8} = 0.$$

Therefore, the complete solution of equation (8.39) is obtained as follows:

$$\begin{aligned} f_0(\xi) = {} & \frac{pR}{24Et}\xi^4 + k_{01}\cosh(G\xi)\sin(G\xi) + k_{02}\cosh(G\xi)\cos(G\xi) \\ & + k_{03}\sinh(G\xi)\sin(G\xi) + k_{04}\sinh(G\xi)\cos(G\xi) \\ & + k_{05} + k_{06}\xi + k_{07}\xi^2 + k_{08}\xi^3 \end{aligned} \tag{8.42}$$

where k_{0i}, $i = 1, 2, \ldots, 8$ are arbitrary constants to be determined by boundary conditions. But, noting (8.30) through (8.37), one can find that k_{05}, k_{06}, k_{07} and k_{08} will never appear in any member of those internal force and moment resultants, so they can be reasonably set as follows:

$$k_{05} = k_{06} = k_{07} = k_{08} = 0\,.$$

And the solution of the homogeneous equation (8.40) can be determined by means of its characteristic equation as follows

$$\left(S_n^2 - \frac{n^2\pi^2}{R^2} \right)^4 + \frac{Et}{DR^2} S_n^4 = 0 \qquad n = 1, 2, 3, \ldots$$

The four pairs of conjugate complex roots of this characteristic equation are as follows

$a_n \pm b_n i$, $-a_n \pm b_n i$, $c_n \pm d_n i$ and $-c_n \pm d_n i$.

Therefore, the solution of equation (8.40) is obtained as follows:

$$f_n(\xi) = e^{a_n\xi}\left[k_{n1}\cos(b_n\xi) + k_{n2}\sin(b_n\xi)\right] + e^{-a_n\xi}\left[k_{n3}\cos(b_n\xi) + k_{n4}\sin(b_n\xi)\right]$$
$$+ e^{c_n\xi}\left[k_{n5}\cos(d_n\xi) + k_{n6}\sin(d_n\xi)\right] + e^{-c_n\xi}\left[k_{n7}\cos(d_n\xi) + k_{n8}\sin(d_n\xi)\right]$$
$$n = 1, 2, 3, \ldots \qquad (8.43)$$

where a_n , b_n , c_n and d_n , n=1, 2, 3, ... are as follows

$$a_n = \frac{\sqrt{2}}{4}\left(M + \sqrt{\sqrt{16N^4 + M^4} + 4N^2}\right), \quad b_n = \frac{\sqrt{2}}{4}\left(M + \sqrt{\sqrt{16N^4 + M^4} - 4N^2}\right)$$
$$c_n = \frac{\sqrt{2}}{4}\left(M - \sqrt{\sqrt{16N^4 + M^4} + 4N^2}\right), \quad d_n = \frac{\sqrt{2}}{4}\left(M - \sqrt{\sqrt{16N^4 + M^4} - 4N^2}\right) \qquad (8.44)$$

and $\quad M = \left(\frac{Et}{DR^2}\right)^{1/4} \quad$ and $\quad N = \frac{n\pi}{R} \qquad n = 1, 2, 3, \ldots$

where k_{ni} , $i = 1, 2, \ldots, 8$ are arbitrary constants to be determined by the boundary conditions. Noting that

$a_n > 0$ and $c_n < 0 \qquad n = 1, 2, 3, \ldots$

we realize that the value of $f_n(\xi)$ and its derivatives will approach infinite as n increases. Since the force and moment resultants can not be infinite as ξ increases, the following is reasonable and necessary:

$$k_{n1} = k_{n2} = k_{n7} = k_{n8} = 0, \qquad \text{for } \xi > 0$$

$$k_{n3} = k_{n4} = k_{n5} = k_{n6} = 0, \qquad \text{for } \xi < 0. \tag{8.45}$$

To compute the internal force and moment resultants in (8.30) through (8.37), we need to calculate all derivatives of the functions $f_0(\xi)$ and $f_n(\xi)$ first. Now, according to (8.42) and (8.43), all derivatives possibly needed are calculated as follows :

$$f_0'(\xi) = \frac{pR}{6Et}\xi^3 + k_{01}\left\{G\left[\sinh(G\xi)\sin(G\xi) + \cosh(G\xi)\cos(G\xi)\right]\right\}$$

$$+ k_{02}\left\{G\left[\sinh(G\xi)\cos(G\xi) - \cosh(G\xi)\sin(G\xi)\right]\right\}$$

$$+ k_{03}\left\{G\left[\cosh(G\xi)\sin(G\xi) + \sinh(G\xi)\cos(G\xi)\right]\right\}$$

$$+ k_{04}\left\{G\left[\cosh(G\xi)\cos(G\xi) - \sinh(G\xi)\sin(G\xi)\right]\right\},$$

$$f_0''(\xi) = \frac{pR}{2Et}\xi^2 + k_{01}\left[2G^2\sinh(G\xi)\cos(G\xi)\right] + k_{02}\left[-2G^2\sinh(G\xi)\sin(G\xi)\right]$$

$$+ k_{03}\left[2G^2\cosh(G\xi)\cos(G\xi)\right] + k_{04}\left[-2G^2\cosh(G\xi)\sin(G\xi)\right],$$

$$f_0'''(\xi) = \frac{pR}{Et}\xi + k_{01}\left\{2G^3\left[\cosh(G\xi)\cos(G\xi) - \sinh(G\xi)\sin(G\xi)\right]\right\}$$

$$+ k_{02}\left\{-2G^3\left[\cosh(G\xi)\sin(G\xi) + \sinh(G\xi)\cos(G\xi)\right]\right\}$$

$$+ k_{03}\left\{2G^3\left[\sinh(G\xi)\cos(G\xi) - \cosh(G\xi)\sin(G\xi)\right]\right\}$$

$$+ k_{04}\left\{-2G^3\left[\sinh(G\xi)\sin(G\xi) + \cosh(G\xi)\cos(G\xi)\right]\right\},$$

$$f_0^{(4)}(\xi) = \frac{pR}{Et} + k_{01}\left[-4G^4\cosh(G\xi)\sin(G\xi)\right] + k_{02}\left[-4G^4\cosh(G\xi)\cos(G\xi)\right]$$

$$+ k_{03}\left[-4G^4\sinh(G\xi)\sin(G\xi)\right] + k_{04}\left[-4G^4\sinh(G\xi)\cos(G\xi)\right],$$

$$f_0^{(5)}(\xi) = k_{01}\left\{-4G^5\left[\sinh(G\xi)\sin(G\xi) + \cosh(G\xi)\cos(G\xi)\right]\right\}$$

$$+ k_{02} \left\{- 4G^5 \left[\sinh(G\xi) \cos(G\xi) - \cosh(G\xi) \sin(G\xi)\right]\right\}$$

$$+ k_{03} \left\{- 4G^5 \left[\cosh(G\xi) \sin(G\xi) + \sinh(G\xi) \cos(G\xi)\right]\right\}$$

$$+ k_{04} \left\{- 4G^5 \left[\cosh(G\xi) \cos(G\xi) - \sinh(G\xi) \sin(G\xi)\right]\right\},$$

$$f_0^{(6)}(\xi) = k_{01} \left[- 8G^6 \sinh(G\xi) \cos(G\xi)\right] + k_{02} \left[8G^6 \sinh(G\xi) \sin(G\xi)\right]$$

$$+ k_{03} \left[- 8G^6 \cosh(G\xi) \cos(G\xi)\right] + k_{04} \left[8G^6 \cosh(G\xi) \sin(G\xi)\right],$$

$$f_0^{(7)}(\xi) = k_{01} \left\{- 8G^7 \left[\cosh(G\xi) \cos(G\xi) - \sinh(G\xi) \sin(G\xi)\right]\right\}$$

$$+ k_{02} \left\{8G^7 \left[\cosh(G\xi) \sin(G\xi) + \sinh(G\xi) \mathrm{con}(G\xi)\right]\right\}$$

$$+ k_{03} \left\{- 8G^7 \left[\sinh(G\xi) \cos(G\xi) - \cosh(G\xi) \sin(G\xi)\right]\right\}$$

$$+ k_{04} \left\{8G^7 \left[\sinh(G\xi) \sin(G\xi) + \cosh(G\xi) \cos(G\xi)\right]\right\}.$$

For $n = 1, 2, 3, \ldots$, the derivatives of the function $f_n(\xi)$ are calculated as follows:

$$f_n^{(i)}(\xi) = e^{a_n\xi}\left[A_{i1} \cos(b_n\xi) + A_{i2} \sin(b_n\xi)\right] + e^{-a_n\xi}\left[A_{i3} \cos(b_n\xi) + A_{i4} \sin(b_n\xi)\right]$$

$$+ e^{c_n\xi}\left[B_{i1} \cos(d_n\xi) + B_{i2} \sin(d_n\xi)\right] + e^{-c_n\xi}\left[B_{i3} \cos(d_n\xi) + B_{i4} \sin(d_n\xi)\right],$$

$$i = 1, 2, 3, \ldots, 7$$

where

$A_{11} = a_n k_{n1} + b_n k_{n2}$, $\qquad A_{12} = -b_n k_{n1} + a_n k_{n2}$

$A_{13} = -a_n k_{n3} + b_n k_{n4}$, $\qquad A_{14} = -b_n k_{n3} - a_n k_{n4}$,

$B_{11} = c_n k_{n5} + d_n k_{n6}$, $\qquad B_{12} = -d_n k_{n5} + c_n k_{n6}$,

$B_{13} = -c_n k_{n7} + d_n k_{n8}$, $\qquad B_{14} = -d_n k_{n7} - c_n k_{n8}$.

$A_{21} = (a_n^2 - b_n^2) k_{n1} + 2a_n b_n k_{n2}$, $\qquad A_{22} = -2a_n b_n k_{n1} + (a_n^2 - b_n^2) k_{n2}$,

$A_{23} = (a_n^2 - b_n^2) k_{n3} - 2a_n b_n k_{n4}$, $\qquad A_{24} = 2a_n b_n k_{n3} + (a_n^2 - b_n^2) k_{n4}$,

$$B_{21} = (c_n^2 - d_n^2)\,k_{n5} + 2c_n d_n\,k_{n6}\,, \qquad B_{22} = -2c_n d_n\,k_{n5} + (c_n^2 - d_n^2)\,k_{n6}\,,$$

$$B_{23} = (c_n^2 - d_n^2)\,k_{n7} - 2c_n d_n\,k_{n8}\,, \qquad B_{24} = 2c_n d_n\,k_{n7} + (c_n^2 - d_n^2)\,k_{n8}\,,$$

$$A_{31} = (a_n^3 - 3a_n b_n^2)\,k_{n1} + (3a_n^2 b_n - b_n^3)\,k_{n2}\,,$$

$$A_{32} = -(3a_n^2 b_n - b_n^3)\,k_{n1} + (a_n^3 - 3a_n b_n^2)\,k_{n2}\,,$$

$$A_{33} = -(a_n^3 - 3a_n b_n^2)\,k_{n3} + (3a_n^2 b_n - b_n^3)\,k_{n4}\,,$$

$$A_{34} = -(3a_n^2 b_n - b_n^3)\,k_{n3} - (a_n^3 - 3a_n b_n^2)\,k_{n4}\,,$$

$$B_{31} = (c_n^3 - 3c_n d_n^2)\,k_{n5} + (3c_n^2 d_n - d_n^3)\,k_{n6}\,,$$

$$B_{32} = -(3c_n^2 d_n - d_n^3)\,k_{n5} + (c_n^3 - 3c_n d_n^2)\,k_{n6}\,,$$

$$B_{33} = -(c_n^3 - 3c_n d_n^2)\,k_{n7} + (3c_n^2 d_n - d_n^3)\,k_{n8}\,,$$

$$B_{34} = -(3c_n^2 d_n - d_n^3)\,k_{n7} - (c_n^3 - 3c_n d_n^2)\,k_{n8}\,,$$

$$A_{41} = (a_n^4 - 6a_n^2 b_n^2 + b_n^4)\,k_{n1} + (4a_n^3 b_n - 4a_n b_n^3)\,k_{n2}\,,$$

$$A_{42} = -(4a_n^3 b_n - 4a_n b_n^3)\,k_{n1} + (a_n^4 - 6a_n^2 b_n^2 + b_n^4)\,k_{n2}\,,$$

$$A_{43} = (a_n^4 - 6a_n^2 b_n^2 + b_n^4)\,k_{n3} - (4a_n^3 b_n - 4a_n b_n^3)\,k_{n4}\,,$$

$$A_{44} = (4a_n^3 b_n - 4a_n b_n^3)\,k_{n3} + (a_n^4 - 6a_n^2 b_n^2 + b_n^4)\,k_{n4}\,,$$

$$B_{41} = (c_n^4 - 6c_n^2 d_n^2 + d_n^4)\,k_{n5} + (4c_n^3 d_n - 4c_n d_n^3)\,k_{n6}\,,$$

$$B_{42} = -(4c_n^3 d_n - 4c_n d_n^3)\,k_{n5} + (c_n^4 - 6c_n^2 d_n^2 + d_n^4)\,k_{n6}\,,$$

$$B_{43} = (c_n^4 - 6c_n^2 d_n^2 + d_n^4)\,k_{n7} - (4c_n^3 d_n - 4c_n d_n^3)\,k_{n8}\,,$$

$$B_{44} = (4c_n^3 d_n - 4c_n d_n^3)\,k_{n7} + (c_n^4 - 6c_n^2 d_n^2 + d_n^4)\,k_{n8}\,,$$

$$A_{51} = (a_n^5 - 10a_n^3 b_n^2 + 5a_n b_n^4)\,k_{n1} + (5a_n^4 b_n - 10a_n^2 b_n^3 + b_n^5)\,k_{n2}\,,$$

$$A_{52} = -(5a_n^4 b_n - 10a_n^2 b_n^3 + b_n^5)\,k_{n1} + (a_n^5 - 10a_n^3 b_n^2 + 5a_n b_n^4)\,k_{n2}\,,$$

$$A_{53} = -(a_n^5 - 10a_n^3 b_n^2 + 5a_n b_n^4)\,k_{n3} + (5a_n^4 b_n - 10a_n^2 b_n^3 + b_n^5)\,k_{n4}\,,$$

$$A_{54} = -(5a_n^4b_n - 10a_n^2b_n^3 + b_n^5)\,k_{n3} - (a_n^5 - 10a_n^3b_n^2 + 5a_nb_n^4)\,k_{n4}\,,$$

$$B_{51} = (c_n^5 - 10c_n^3d_n^2 + 5c_nd_n^4)\,k_{n5} + (5c_n^4d_n - 10c_n^2d_n^3 + d_n^5)\,k_{n6}\,,$$

$$B_{52} = -(5c_n^4d_n - 10c_n^2d_n^3 + d_n^5)\,k_{n5} + (c_n^5 - 10c_n^3d_n^2 + 5c_nd_n^4)\,k_{n6}\,,$$

$$B_{53} = -(c_n^5 - 10c_n^3d_n^2 + 5c_nd_n^4)\,k_{n7} + (5c_n^4d_n - 10c_n^2d_n^3 + d_n^5)\,k_{n8}\,,$$

$$B_{54} = -(5c_n^4d_n - 10c_n^2d_n^3 + d_n^5)\,k_{n7} - (c_n^5 - 10c_n^3d_n^2 + 5c_nd_n^4)\,k_{n8}\,,$$

$$A_{61} = (a_n^6 - 15a_n^4b_n^2 + 15a_n^2b_n^4 - b_n^6)\,k_{n1} + (6a_n^5b_n - 20a_n^3b_n^3 + 6a_nb_n^5)\,k_{n2}\,,$$

$$A_{62} = -(6a_n^5b_n - 20a_n^3b_n^3 + 6a_nb_n^5)\,k_{n1} + (a_n^6 - 15a_n^4b_n^2 + 15a_n^2b_n^4 - b_n^6)\,k_{n2}\,,$$

$$A_{63} = (a_n^6 - 15a_n^4b_n^2 + 15a_n^2b_n^4 - b_n^6)\,k_{n3} - (6a_n^5b_n - 20a_n^3b_n^3 + 6a_nb_n^5)\,k_{n4}\,,$$

$$A_{64} = (6a_n^5b_n - 20a_n^3b_n^3 + 6a_nb_n^5)\,k_{n3} + (a_n^6 - 15a_n^4b_n^2 + 15a_n^2b_n^4 - b_n^6)\,k_{n4}\,,$$

$$B_{61} = (c_n^6 - 15c_n^4d_n^2 + 15c_n^2d_n^4 - d_n^6)\,k_{n5} + (6c_n^5d_n - 20c_n^3d_n^3 + 6c_nd_n^5)\,k_{n6}\,,$$

$$B_{62} = -(6c_n^5d_n - 20c_n^3d_n^3 + 6c_nd_n^5)\,k_{n5} + (c_n^6 - 15c_n^4d_n^2 + 15c_n^4d_n^2 - d_n^6)\,k_{n6}\,,$$

$$B_{63} = (c_n^6 - 15c_n^4d_n^2 + 15c_n^2d_n^4 - d_n^6)\,k_{n7} - (6c_n^5d_n - 20c_n^3d_n^3 + 6c_nd_n^5)\,k_{n8}\,,$$

$$B_{64} = (6c_n^5d_n - 20c_n^3d_n^3 + 6c_nd_n^5)\,k_{n7} + (c_n^6 - 15c_n^4d_n^2 + 15c_n^4d_n^2 - d_n^6)\,k_{n8}\,,$$

$$A_{71} = (a_n^7 - 21a_n^5b_n^2 + 35a_n^3b_n^4 - 7a_nb_n^6)\,k_{n1} + (7a_n^6b_n - 35a_n^4b_n^3 + 21a_n^2b_n^5 - b_n^7)\,k_{n2}\,,$$

$$A_{72} = -(7a_n^6b_n - 35a_n^4b_n^3 + 21a_n^2b_n^5 - b_n^7)\,k_{n1} + (a_n^7 - 21a_n^5b_n^2 + 35a_n^3b_n^4 - 7a_nb_n^6)\,k_{n2}\,,$$

$$A_{73} = -(a_n^7 - 21a_n^5b_n^2 + 35a_n^3b_n^4 - 7a_nb_n^6)\,k_{n3} + (7a_n^6b_n - 35a_n^4b_n^3 + 21a_n^2b_n^5 - b_n^7)\,k_{n4}\,,$$

$$A_{74} = -(7a_n^6b_n - 35a_n^4b_n^3 + 21a_n^2b_n^5 - b_n^7)\,k_{n3} - (a_n^7 - 21a_n^5b_n^2 + 35a_n^3b_n^4 - 7a_nb_n^6)\,k_{n4}\,,$$

$$B_{71} = (c_n^7 - 21c_n^5d_n^2 + 35c_n^3d_n^4 - 7c_nd_n^6)\,k_{n5} + (7c_n^6d_n - 35c_n^4d_n^3 + 21c_n^2d_n^5 - d_n^7)\,k_{n6}\,,$$

$$B_{72} = -(7c_n^6d_n - 35c_n^4d_n^3 + 21c_n^2d_n^5 - d_n^7)\,k_{n5} + (c_n^7 - 21c_n^5d_n^2 + 35c_n^3d_n^4 - 7a_nb_n^6)\,k_{n6}$$

$$B_{73} = -(c_n^7 - 21c_n^5d_n^2 + 35c_n^3d_n^4 - 7c_nd_n^6)\,k_{n7} + (7c_n^6d_n - 35c_n^4d_n^3 + 21c_n^2d_n^5 - d_n^7)\,k_{n8}\,,$$

$$B_{74} = -(7c_n^6d_n - 35c_n^4d_n^3 + 21c_n^2d_n^5 - d_n^7)\,k_{n7} - (c_n^7 - 21c_n^5d_n^2 + 35c_n^3d_n^4 - 7a_nb_n^6)\,k_{n8}\,.$$

8.3 Application of the Boundary Conditions

All possible components of loading N_v, S_t, Q_n, M_v, and M_t as well as the internal force and moment resultants are shown in Figure 8.1. By taking equilibrium, the relations between the loading and internal force and moment resultants can be obtained as follows:

From $\sum F_{\rho_1} = 0$, the following is obtained:

$$N_v\, ds\, \cos(v, \rho_1) - S_t\, ds\, \cos(t, \rho_1) =$$

$$(N_1 d\eta + S_{21} d\xi)\cos\psi + (N_2 d\xi + S_{12} d\eta)\sin\psi. \tag{8.46}$$

From $\sum F_{\psi} = 0$, the following is obtained:

$$N_v\, ds\, \cos(v, \psi) + S_t\, ds\, \cos(t, \psi) =$$

$$-(N_1 d\eta + S_{21} d\xi)\sin\psi + (N_2 d\xi + S_{12} d\eta)\cos\psi. \tag{8.47}$$

From $\sum F_n = 0$, the following is obtained:

$$Q_n\, ds = Q_1\, d\eta + Q_2\, d\xi\ . \tag{8.48}$$

From $\sum M_{\rho_1} = 0$, the following is obtained:

$$M_v\, ds\, \cos(v, \rho_1) + M_t\, ds\, \cos(t, \rho_1) =$$

$$(M_1 d\eta - M_{21} d\xi)\sin\psi - (M_2 d\xi - M_{12} d\eta)\cos\psi. \tag{8.49}$$

From $\sum M_{\psi} = 0$, the following is obtained:

$$M_v\, ds\, \cos(v, \psi) - M_t\, ds\, \cos(t, \psi) =$$

$$(M_1 d\eta - M_{21} d\xi)\cos\psi + (M_2 d\xi - M_{12} d\eta)\sin\psi \tag{8.50}$$

where $ds = \sqrt{(d\xi)^2 + (d\eta)^2} = \sqrt{1 + \left(\dfrac{d\eta}{d\xi}\right)^2}\ d\xi$.

When the loading is the internal pressure p only,

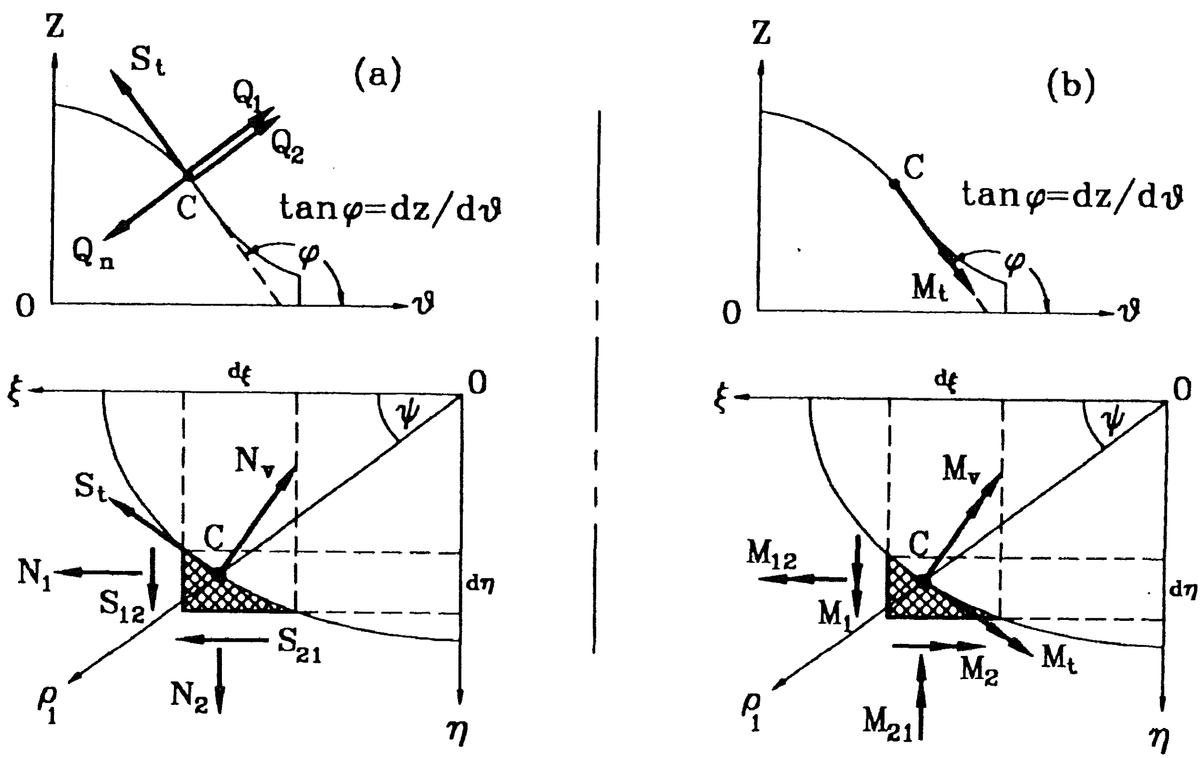


Figure 8.1 Distribution of forces and moments on the intersection of the shell and nozzle

$N_v = S_t = M_v = M_t = 0$, then from (8.46), (8.47), (8.48), (8.49) and (8.50), one obtains:

$$N_2 = -S_{12}\frac{d\eta}{d\xi} \tag{8.51}$$

$$S_{21} = -N_1\frac{d\eta}{d\xi} \tag{8.52}$$

$$Q_1\frac{d\eta}{d\xi} + Q_2 = Q_n\left(\sqrt{1 + \left(\frac{d\eta}{d\xi}\right)^2}\right) \tag{8.53}$$

$$M_{21} = M_1\frac{d\eta}{d\xi} \tag{8.54}$$

$$M_2 = M_{12}\frac{d\eta}{d\xi} \; . \tag{8.55}$$

Furthermore, noting that $S_{12} = S_{21}$ and $M_{12} = M_{21}$, then from (8.51), (8.52), (8.54) and (8.55) when $\xi \neq 0$ or $\frac{d\eta}{d\xi} \neq 0$, the following two equations are obtained::

$$N_1 = N_2\left(\frac{d\eta}{d\xi}\right)^{-2} \tag{8.56}$$

$$M_1 = M_2\left(\frac{d\eta}{d\xi}\right)^{-2} \; . \tag{8.57}$$

In the same manner, one obtains from (8.53):

$$Q_1 = Q_n\left(\sqrt{1 + \left(\frac{d\eta}{d\xi}\right)^{-2}}\right) - Q_2\left(\frac{d\eta}{d\xi}\right)^{-1} \; . \tag{8.58}$$

To determine the coefficients in (8.42) and (8.43), one can consider some special points on the intersection curve of the shell and nozzle. For instance, from Figure 7.2 , at the point $\xi = 0$, variable η arrives at its non-zero extreme value, which means

$$\left(\frac{d\eta}{d\xi}\right)_{\xi=0} = 0 \quad . \tag{8.59}$$

Noting $Q_n = -pR$ at $\xi = 0$ [10] and substituting (8.59) into (8.51) through (8.55) and considering (8.31), (8.32), (8.34), (8.35) and (8.37), we obtain that

$$f_0^{(4)}(0) = 0 \tag{8.60}$$

$$f_n^{(4)}(0) = 0 \qquad n = 1, 2, 3, \ldots \tag{8.61}$$

$$f_n'''(0) = 0 \qquad n = 1, 2, 3, \ldots \tag{8.62}$$

$$f_n^{(6)}(0) - \frac{3n^2\pi^2}{R^2} f_n^{(4)}(0) + \frac{3n^4\pi^4}{R^4} f_n''(0) - \frac{n^6\pi^6}{R^6} f_n(0)$$

$$= \begin{cases} \dfrac{-4pR}{n^2\pi^2 D} & n = 2k-1 \\ 0 & n = 2k \end{cases} \qquad k = 1, 2, 3, \ldots \tag{8.63}$$

where the Fourier's expansion of $Q_n = -pR$ is utilized as follows:

$$RD \sum_{n=1}^{\infty} \frac{n\pi}{R}\left[f_n^{(6)}(0) - \frac{3n^2\pi^2}{R^2} f_n^{(4)}(0) + \frac{3n^4\pi^4}{R^4} f_n''(0) - \frac{n^6\pi^6}{R^6} f_n(0)\right]\sin\left(\frac{n\pi}{R}\eta\right)$$

$$= -pR = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{R}\eta\right)$$

and $b_n = \dfrac{2}{R}\displaystyle\int_0^R \left[-pR\sin\left(\frac{n\pi}{R}\eta\right)\right] d\eta = \begin{cases} \dfrac{-4pR}{n\pi}, & n = 2k-1 \\ 0, & n = 2k \end{cases} \qquad k = 1, 2, 3, \ldots$

$$f_n^{(5)}(0) - \frac{2n^2\pi^2}{R^2} f_n'''(0) + \frac{n^4\pi^4}{R^4} f_n'(0) = 0 \qquad n = 1, 2, 3, \ldots \tag{8.64}$$

$$f_0^{(6)}(0) = 0 \tag{8.65}$$

$$\mu f_n^{(6)}(0) - \frac{(1 + 2\mu)n^2\pi^2}{R^2} f_n^{(4)}(0) + \frac{(2 + \mu)n^4\pi^4}{R^4} f_n''(0)$$

$$- \frac{n^6\pi^6}{R^6} f_n(0) = 0 \qquad n = 1, 2, 3, \ldots \tag{8.66}$$

From Figure 7.2, at the point $(\xi, \eta) = \left(\frac{r}{\sin\lambda}, 0\right)$, $\frac{d\eta}{d\xi} \rightarrow \infty$, which means that

$$\left(\frac{d\eta}{d\xi}\right)^{-1}_{\xi = r/\sin\lambda} = \left(\frac{d\eta}{d\xi}\right)^{-2}_{\xi = r/\sin\lambda} = 0 \qquad n = 1, 2, 3, \ldots \tag{8.67}$$

Noting $Q_n = 0$ at $\xi = r / \sin\lambda$ [10] and substituting (8.67) into (8.56) and (8.57) and considering (8.30) and (8.33), one arrives:

$$f_n''\left(\frac{r}{\sin\lambda}\right) = 0 \qquad n = 1, 2, 3, \ldots \tag{8.68}$$

$$f_0^{(6)}\left(\frac{r}{\sin\lambda}\right) = 0 \tag{8.69}$$

$$f_n^{(6)}\left(\frac{r}{\sin\lambda}\right) - \frac{(2 + \mu)n^2\pi^2}{R^2} f_n^{(4)}\left(\frac{r}{\sin\lambda}\right) + \frac{(1 + 2\mu)n^4\pi^4}{R^4} f_n''\left(\frac{r}{\sin\lambda}\right)$$

$$- \mu \frac{n^6\pi^6}{R^6} f_n\left(\frac{r}{\sin\lambda}\right) = 0 \qquad n = 1, 2, 3, \ldots \tag{8.70}$$

substituting (8.67) into (8.58) and considering (8.36), one can obtain:

$$f_0^{(7)}\left(\frac{r}{\sin\lambda}\right) = 0 \tag{8.71}$$

$$f_n^{(7)}\left(\frac{r}{\sin\lambda}\right) - \frac{3n^2\pi^2}{R^2} f_n^{(5)}\left(\frac{r}{\sin\lambda}\right) + \frac{3n^4\pi^4}{R^4} f_n'''\left(\frac{r}{\sin\lambda}\right)$$

$$-\frac{n^6\pi^6}{R^6}f_n'\left(\frac{r}{\sin\lambda}\right)=0 \qquad\qquad n=1,2,3,\ldots \qquad (8.72)$$

Now there are twelve equations, among which four equations, (8.60), (8.65), (8.69) and (8.71), describe the function $f_0(\xi)$ and eight equations, (8.61), (8.62), (8.63), (8.64), (8.66), (8.68), (8.70) and (8.72) describe the function $f_n(\xi)$. These twelve equations are sufficient to solve for the necessary coefficients in solutions (8.42) and (8.43).

8.4 Analytical Solutions

Now, by substituting (8.42) into (8.60), (8.65), (8.69) and (8.71) to determine the coefficients in (8.42), the following is obtained:

$$k_{01} = \left\{ \frac{SS}{CC} + \frac{CS}{SC} \left[\frac{SC(CS+SC) - SS(CC-SS)}{CS(CC-SS) - SC(SS+CC)} \right] \right\} \frac{pR}{4G^4Et} \tag{8.73}$$

$$k_{02} = \frac{pR}{4G^4Et} \tag{8.74}$$

$$k_{03} = 0 \tag{8.75}$$

$$k_{04} = \frac{SC(CS+SC) - SS(CC-SS)}{CS(CC-SS) - SC(SS+CC)} \frac{pR}{4G^4Et} \tag{8.76}$$

where $CS = \cosh\left(G \frac{r}{\sin\lambda}\right) \sin\left(G \frac{r}{\sin\lambda}\right)$, $CC = \cosh\left(G \frac{r}{\sin\lambda}\right) \mathrm{Cos}\left(G \frac{r}{\sin\lambda}\right)$

$SC = \sinh\left(G \frac{r}{\sin\lambda}\right) \cos\left(G \frac{r}{\sin\lambda}\right)$, $SS = \sinh\left(G \frac{r}{\sin\lambda}\right) \sin\left(G \frac{r}{\sin\lambda}\right)$.

and $G = \left(\frac{Et}{4DR^2}\right)^{1/4}$.

To determine the coefficients in (8.43), substituting (8.43) into (8.61), (8.62), (8.63), (8.64), (8.66), (8.68), (8.70) and (8.72) gives:

$$\begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} & K_{15} & K_{16} & K_{17} & K_{18} \\ K_{21} & K_{22} & K_{23} & K_{24} & K_{25} & K_{26} & K_{27} & K_{28} \\ K_{31} & K_{32} & K_{33} & K_{34} & K_{35} & K_{36} & K_{37} & K_{38} \\ K_{41} & K_{42} & K_{43} & K_{44} & K_{45} & K_{46} & K_{47} & K_{48} \\ K_{51} & K_{52} & K_{53} & K_{54} & K_{55} & K_{56} & K_{57} & K_{58} \\ K_{61} & K_{62} & K_{63} & K_{64} & K_{65} & K_{66} & K_{67} & K_{68} \\ K_{71} & K_{72} & K_{73} & K_{74} & K_{75} & K_{76} & K_{77} & K_{78} \\ K_{81} & K_{82} & K_{83} & K_{84} & K_{85} & K_{86} & K_{87} & K_{88} \end{bmatrix} \begin{pmatrix} k_{n1} \\ k_{n2} \\ k_{n3} \\ k_{n4} \\ k_{n5} \\ k_{n6} \\ k_{n7} \\ k_{n8} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -4pR/(n^2\pi^2 D) \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$n = 2k - 1 , \qquad k = 1, 2, 3, \ldots \qquad (8.77)$$

where, the elements of this matrix are as follows:

$$K_{11} = A_{n4} , \quad K_{12} = B_{n4} , \quad K_{13} = A_{n4} , \quad K_{14} = -B_{n4} ,$$

$$K_{15} = C_{n4} , \quad K_{16} = D_{n4} , \quad K_{17} = C_{n4} , \quad K_{18} = -D_{n4} ,$$

$$K_{21} = A_{n3} , \quad K_{22} = B_{n3} , \quad K_{23} = -A_{n3} , \quad K_{24} = B_{n3} ,$$

$$K_{25} = C_{n3} , \quad K_{26} = D_{n3} , \quad K_{27} = -C_{n3} , \quad K_{28} = D_{n3} ,$$

$$K_{31} = A_{n6} - \frac{3n^2\pi^2}{R^2} A_{n4} + \frac{3n^4\pi^4}{R^4} A_{n2} - \frac{n^6\pi^6}{R^6} , \quad K_{32} = B_{n6} - \frac{3n^2\pi^2}{R^2} B_{n4} + \frac{3n^4\pi^4}{R^4} B_{n2}$$

$$K_{33} = A_{n6} - \frac{3n^2\pi^2}{R^2} A_{n4} + \frac{3n^4\pi^4}{R^4} A_{n2} - \frac{n^6\pi^6}{R^6} , \quad K_{34} = -\left(B_{n6} - \frac{3n^2\pi^2}{R^2} B_{n4} + \frac{3n^4\pi^4}{R^4} B_{n2} \right),$$

$$K_{35} = C_{n6} - \frac{3n^2\pi^2}{R^2} C_{n4} + \frac{3n^4\pi^4}{R^4} C_{n2} - \frac{n^6\pi^6}{R^6} , \quad K_{36} = D_{n6} - \frac{3n^2\pi^2}{R^2} D_{n4} + \frac{3n^4\pi^4}{R^4} D_{n2} ,$$

$$K_{37} = C_{n6} - \frac{3n^2\pi^2}{R^2} C_{n4} + \frac{3n^4\pi^4}{R^4} C_{n2} - \frac{n^6\pi^6}{R^6} , \quad K_{38} = -\left(D_{n6} - \frac{3n^2\pi^2}{R^2} D_{n4} + \frac{3n^4\pi^4}{R^4} D_{n2} \right),$$

$$K_{41} = A_{n5} - \frac{2n^2\pi^2}{R^2} A_{n3} + \frac{n^4\pi^4}{R^4} A_{n1} , \qquad K_{42} = B_{n5} - \frac{2n^2\pi^2}{R^2} B_{n3} + \frac{n^4\pi^4}{R^4} B_{n1} ,$$

$$K_{43} = -\left(A_{n5} - \frac{2n^2\pi^2}{R^2} A_{n3} + \frac{n^4\pi^4}{R^4} A_{n1} \right), \qquad K_{44} = B_{n5} - \frac{2n^2\pi^2}{R^2} B_{n3} + \frac{n^4\pi^4}{R^4} B_{n1} ,$$

$$K_{45} = C_{n5} - \frac{2n^2\pi^2}{R^2} C_{n3} + \frac{n^4\pi^4}{R^4} C_{n1} , \qquad K_{46} = D_{n5} - \frac{2n^2\pi^2}{R^2} D_{n3} + \frac{n^4\pi^4}{R^4} D_{n1} ,$$

$$K_{47} = -\left(C_{n5} - \frac{2n^2\pi^2}{R^2} C_{n3} + \frac{n^4\pi^4}{R^4} C_{n1} \right), \qquad K_{48} = D_{n5} - \frac{2n^2\pi^2}{R^2} D_{n3} + \frac{n^4\pi^4}{R^4} D_{n1} ,$$

$$K_{51} = \mu A_{n6} - \frac{(1+2\mu)n^2\pi^2}{R^2} A_{n4} + \frac{(2+\mu)n^4\pi^4}{R^4} A_{n2} - \frac{n^6\pi^6}{R^6} ,$$

$$K_{52} = \mu B_{n6} - \frac{(1+2\mu)n^2\pi^2}{R^2} B_{n4} + \frac{(2+\mu)n^4\pi^4}{R^4} B_{n2} ,$$

$$K_{53} = \mu A_{n6} - \frac{(1+2\mu)n^2\pi^2}{R^2} A_{n4} + \frac{(2+\mu)n^4\pi^4}{R^4} A_{n2} - \frac{n^6\pi^6}{R^6},$$

$$K_{54} = -\left[\mu B_{n6} - \frac{(1+2\mu)n^2\pi^2}{R^2} B_{n4} + \frac{(2+\mu)n^4\pi^4}{R^4} B_{n2}\right],$$

$$K_{55} = \mu C_{n6} - \frac{(1+2\mu)n^2\pi^2}{R^2} C_{n4} + \frac{(2+\mu)n^4\pi^4}{R^4} C_{n2} - \frac{n^6\pi^6}{R^6},$$

$$K_{56} = \mu D_{n6} - \frac{(1+2\mu)n^2\pi^2}{R^2} D_{n4} + \frac{(2+\mu)n^4\pi^4}{R^4} D_{n2},$$

$$K_{57} = \mu C_{n6} - \frac{(1+2\mu)n^2\pi^2}{R^2} C_{n4} + \frac{(2+\mu)n^4\pi^4}{R^4} C_{n2} - \frac{n^6\pi^6}{R^6},$$

$$K_{58} = -\left[\mu D_{n6} - \frac{(1+2\mu)n^2\pi^2}{R^2} D_{n4} + \frac{(2+\mu)n^4\pi^4}{R^4} D_{n2}\right],$$

$$K_{61} = e^{a_n r_m}\left[A_{n2}\cos(b_n r_m) - B_{n2}\sin(b_n r_m)\right],$$

$$K_{62} = e^{a_n r_m}\left[B_{n2}\cos(b_n r_m) + A_{n2}\sin(b_n r_m)\right],$$

$$K_{63} = e^{-a_n r_m}\left[A_{n2}\cos(b_n r_m) + B_{n2}\sin(b_n r_m)\right],$$

$$K_{64} = e^{-a_n r_m}\left[-B_{n2}\cos(b_n r_m) + A_{n2}\sin(b_n r_m)\right],$$

$$K_{65} = e^{c_n r_m}\left[C_{n2}\cos(d_n r_m) - D_{n2}\sin(d_n r_m)\right],$$

$$K_{66} = e^{c_n r_m}\left[D_{n2}\cos(d_n r_m) + C_{n2}\sin(d_n r_m)\right],$$

$$K_{67} = e^{-c_n r_m}\left[C_{n2}\cos(d_n r_m) + D_{n2}\sin(d_n r_m)\right],$$

$$K_{68} = e^{-c_n r_m}\left[-D_{n2}\cos(d_n r_m) + C_{n2}\sin(d_n r_m)\right],$$

$$K_{71} = e^{a_n r_m}\left\{\left[A_{n6}\cos(b_n r_m) - B_{n6}\sin(b_n r_m)\right] - \frac{(2+\mu)n^2\pi^2}{R^2}\left[A_{n4}\cos(b_n r_m) - B_{n4}\sin(b_n r_m)\right]\right.$$

$$\left. + \frac{(1+2\mu)n^4\pi^4}{R^4}\left[A_{n2}\cos(b_n r_m) - B_{n2}\sin(b_n r_m)\right] - \frac{\mu n^6\pi^6}{R^6}\cos(b_n r_m)\right\}$$

$$K_{72} = e^{a_n r_m}\left\{\left[B_{n6}\cos(b_n r_m) + A_{n6}\sin(b_n r_m)\right] - \frac{(2+\mu)n^2\pi^2}{R^2}\left[B_{n4}\cos(b_n r_m) + A_{n4}\sin(b_n r_m)\right]\right.$$

$$\left. + \frac{(1+2\mu)n^4\pi^4}{R^4}\left[B_{n2}\cos(b_n r_m) + A_{n2}\sin(b_n r_m)\right] - \mu\frac{n^6\pi^6}{R^6}\sin(b_n r_m)\right\},$$

$$K_{73} = e^{-a_n r_m}\left\{\left[A_{n6}\cos(b_n r_m) + B_{n6}\sin(b_n r_m)\right] - \frac{(2+\mu)n^2\pi^2}{R^2}\left[A_{n4}\cos(b_n r_m) + B_{n4}\sin(b_n r_m)\right]\right.$$

$$\left. + \frac{(1+2\mu)n^4\pi^4}{R^4}\left[A_{n2}\cos(b_n r_m) + B_{n2}\sin(b_n r_m)\right] - \mu\frac{n^6\pi^6}{R^6}\cos(b_n r_m)\right\},$$

$$K_{74} = e^{-a_n r_m}\left\{\left[-B_{n6}\cos(b_n r_m) + A_{n6}\sin(b_n r_m)\right] - \frac{(2+\mu)n^2\pi^2}{R^2}\left[-B_{n4}\cos(b_n r_m) + A_{n4}\sin(b_n r_m)\right]\right.$$

$$\left. + \frac{(1+2\mu)n^4\pi^4}{R^4}\left[-B_{n2}\cos(b_n r_m) + A_{n2}\sin(b_n r_m)\right] - \mu\frac{n^6\pi^6}{R^6}\sin(b_n r_m)\right\},$$

$$K_{75} = e^{c_n r_m}\left\{\left[C_{n6}\cos(d_n r_m) - D_{n6}\sin(d_n r_m)\right] - \frac{(2+\mu)n^2\pi^2}{R^2}\left[C_{n4}\cos(d_n r_m) - D_{n4}\sin(d_n r_m)\right]\right.$$

$$\left. + \frac{(1+2\mu)n^4\pi^4}{R^4}\left[C_{n2}\cos(d_n r_m) - D_{n2}\sin(d_n r_m)\right] - \frac{\mu n^6\pi^6}{R^6}\cos(d_n r_m)\right\},$$

$$K_{76} = e^{c_n r_m}\left\{\left[D_{n6}\cos(d_n r_m) + C_{n6}\sin(d_n r_m)\right] - \frac{(2+\mu)n^2\pi^2}{R^2}\left[D_{n4}\cos(d_n r_m) + C_{n4}\sin(d_n r_m)\right]\right.$$

$$\left. + \frac{(1+2\mu)n^4\pi^4}{R^4}\left[D_{n2}\cos(d_n r_m) + C_{n2}\sin(d_n r_m)\right] - \mu\frac{n^6\pi^6}{R^6}\sin(d_n r_m)\right\},$$

$$K_{77} = e^{-c_n r_m}\left\{\left[C_{n6}\cos(d_n r_m) + D_{n6}\sin(d_n r_m)\right] - \frac{(2+\mu)n^2\pi^2}{R^2}\left[C_{n4}\cos(d_n r_m) + D_{n4}\sin(d_n r_m)\right]\right.$$

$$\left. + \frac{(1+2\mu)n^4\pi^4}{R^4}\left[C_{n2}\cos(d_n r_m) + D_{n2}\sin(d_n r_m)\right] - \mu\frac{n^6\pi^6}{R^6}\cos(d_n r_m)\right\},$$

$$K_{78} = e^{-c_n r_m}\left\{\left[-D_{n6}\cos(d_n r_m) + C_{n6}\sin(d_n r_m)\right] - \frac{(2+\mu)n^2\pi^2}{R^2}\left[-D_{n4}\cos(d_n r_m) + C_{n4}\sin(d_n r_m)\right]\right.$$

$$\left. + \frac{(1+2\mu)n^4\pi^4}{R^4}\left[-D_{n2}\cos(d_n r_m) + C_{n2}\sin(d_n r_m)\right] - \mu\frac{n^6\pi^6}{R^6}\sin(d_n r_m)\right\}.$$

$$K_{81}=e^{a_n r_m}\left\{\left[A_{n7}\cos(b_n r_m)-B_{n7}\sin(b_n r_m)\right]-\frac{3n^2\pi^2}{R^2}\left[A_{n5}\cos(b_n r_m)-B_{n5}\sin(b_n r_m)\right]\right.$$

$$\left.+\frac{3n^4\pi^4}{R^4}\left[A_{n3}\cos(b_n r_m)-B_{n3}\sin(b_n r_m)\right]-\frac{n^6\pi^6}{R^6}\left[A_{n1}\cos(b_n r_m)-B_{n1}\sin(b_n r_m)\right]\right\},$$

$$K_{82}=e^{a_n r_m}\left\{\left[B_{n7}\cos(b_n r_m)+A_{n7}\sin(b_n r_m)\right]-\frac{3n^2\pi^2}{R^2}\left[B_{n5}\cos(b_n r_m)+A_{n5}\sin(b_n r_m)\right]\right.$$

$$\left.+\frac{3n^4\pi^4}{R^4}\left[B_{n3}\cos(b_n r_m)+A_{n3}\sin(b_n r_m)\right]-\frac{n^6\pi^6}{R^6}\left[B_{n1}\cos(b_n r_m)+A_{n1}\sin(b_n r_m)\right]\right\},$$

$$K_{83}=e^{-a_n r_m}\left\{\left[-A_{n7}\cos(b_n r_m)-B_{n7}\sin(b_n r_m)\right]-\frac{3n^2\pi^2}{R^2}\left[-A_{n5}\cos(b_n r_m)-B_{n5}\sin(b_n r_m)\right]\right.$$

$$\left.+\frac{3n^4\pi^4}{R^4}\left[-A_{n3}\cos(b_n r_m)-B_{n3}\sin(b_n r_m)\right]-\frac{n^6\pi^6}{R^6}\left[-A_{n1}\cos(b_n r_m)-B_{n1}\sin(b_n r_m)\right]\right\},$$

$$K_{84}=e^{-a_n r_m}\left\{\left[B_{n7}\cos(b_n r_m)-A_{n7}\sin(b_n r_m)\right]-\frac{3n^2\pi^2}{R^2}\left[B_{n5}\cos(b_n r_m)-A_{n5}\sin(b_n r_m)\right]\right.$$

$$\left.+\frac{3n^4\pi^4}{R^4}\left[B_{n3}\cos(b_n r_m)-A_{n3}\sin(b_n r_m)\right]-\frac{n^6\pi^6}{R^6}\left[B_{n1}\cos(b_n r_m)-A_{n1}\sin(b_n r_m)\right]\right\},$$

$$K_{85}=e^{c_n r_m}\left\{\left[C_{n7}\cos(d_n r_m)-D_{n7}\sin(d_n r_m)\right]-\frac{3n^2\pi^2}{R^2}\left[C_{n5}\cos(d_n r_m)-D_{n5}\sin(d_n r_m)\right]\right.$$

$$\left.+\frac{3n^4\pi^4}{R^4}\left[C_{n3}\cos(d_n r_m)-D_{n3}\sin(d_n r_m)\right]-\frac{n^6\pi^6}{R^6}\left[C_{n1}\cos(d_n r_m)-D_{n1}\sin(d_n r_m)\right]\right\},$$

$$K_{86}=e^{c_n r_m}\left\{\left[D_{n7}\cos(d_n r_m)+C_{n7}\sin(d_n r_m)\right]-\frac{3n^2\pi^2}{R^2}\left[D_{n5}\cos(d_n r_m)+C_{n5}\sin(d_n r_m)\right]\right.$$

$$\left.+\frac{3n^4\pi^4}{R^4}\left[D_{n3}\cos(d_n r_m)+C_{n3}\sin(d_n r_m)\right]-\frac{n^6\pi^6}{R^6}\left[D_{n1}\cos(d_n r_m)+C_{n1}\sin(d_n r_m)\right]\right\},$$

$$K_{87}=e^{-c_n r_m}\left\{\left[-C_{n7}\cos(d_n r_m)-D_{n7}\sin(d_n r_m)\right]-\frac{3n^2\pi^2}{R^2}\left[-C_{n5}\cos(d_n r_m)-D_{n5}\sin(d_n r_m)\right]\right.$$

$$+\frac{3n^4\pi^4}{R^4}\left[-C_{n3}\cos(d_n r_m)-D_{n3}\sin(d_n r_m)\right]-\frac{n^6\pi^6}{R^6}\left[-C_{n1}\cos(d_n r_m)-D_{n1}\sin(d_n r_m)\right]\Big\},$$

$$K_{88}=e^{-c_n r_m}\Big\{\left[D_{n7}\cos(d_n r_m)-C_{n7}\sin(d_n r_m)\right]-\frac{3n^2\pi^2}{R^2}\left[D_{n5}\cos(d_n r_m)-C_{n5}\sin(d_n r_m)\right]$$

$$+\frac{3n^4\pi^4}{R^4}\left[D_{n3}\cos(d_n r_m)-C_{n3}\sin(d_n r_m)\right]-\frac{n^6\pi^6}{R^6}\left[D_{n1}\cos(d_n r_m)-C_{n1}\sin(d_n r_m)\right]\Big\},$$

where all constants are as follows:

$A_{n1}=a_n$, $B_{n1}=b_n$, $A_{n2}=a_n^2-b_n^2$, $B_{n2}=2a_nb_n$,

$A_{n3}=a_n\left(a_n^2-3b_n^2\right)$, $B_{n3}=\left(3a_n^2-b_n^2\right)b_n$,

$A_{n4}=a_n^4-6a_n^2b_n^2+b_n^4$, $B_{n4}=4a_nb_n\left(a_n^2-b_n^2\right)$,

$A_{n5}=a_n\left(a_n^4-10a_n^2b_n^2+5b_n^4\right)$, $B_{n5}=\left(5a_n^4-10a_n^2b_n^2+b_n^4\right)b_n$,

$A_{n6}=a_n^6-15a_n^4b_n^2+15a_n^2b_n^4-b_n^6$, $B_{n6}=2a_nb_n\left(3a_n^4-10a_n^2b_n^2+3b_n^4\right)$,

$A_{n7}=a_n\left(a_n^6-21a_n^4b_n^2+35a_n^2b_n^4-7b_n^6\right)$, $B_{n7}=\left(7a_n^6-35a_n^4b_n^2+21a_n^2b_n^4-b_n^6\right)b_n$,

$C_{n1}=c_n$, $D_{n1}=d_n$

$C_{n2}=c_n^2-d_n^2$, $D_{n2}=2c_nd_n$,

$C_{n3}=c_n\left(c_n^2-3d_n^2\right)$, $D_{n3}=\left(3c_n^2-d_n^2\right)d_n$,

$C_{n4}=c_n^4-6c_n^2d_n^2+d_n^4$, $D_{n4}=4c_nd_n\left(c_n^2-d_n^2\right)$,

$C_{n5}=c_n\left(c_n^4-10c_n^2d_n^2+5d_n^4\right)$, $D_{n5}=\left(5c_n^4-10c_n^2d_n^2+d_n^4\right)d_n$,

$C_{n6}=c_n^6-15c_n^4d_n^2+15c_n^2d_n^4-d_n^6$, $D_{n6}=2c_nd_n\left(3c_n^4-10c_n^2d_n^2+3d_n^4\right)$,

$C_{n7}=c_n\left(c_n^6-21c_n^4d_n^2+35c_n^2d_n^4-7d_n^6\right)$, $D_{n7}=\left(7c_n^6-35c_n^4d_n^2+21c_n^2d_n^4-d_n^6\right)d_n$,

and $r_m=r/\sin\lambda$.

where a_n , b_n , c_n and d_n have been determined earlier in (8.44) and (8.45).

CHAPTER 9

NUMERICAL CALCULATIONS AND COMPARISON OF STRESS RESULTS

To verify the analytical solutions of this dissertation, a series of numerical calculations of samples has been carried out by the method of this dissertation with a Fortran program.

One of the samples is given by the literature [10] which is limited to the case of the shell with an orthogonal nozzle and its geometric parameters are described as follows:

$\beta = r / R = 0.4$, $\gamma = R / t = 40$, $p = 100$ psi and $\lambda = 90^0$,

and from Table 9.1, one can see the largest stress concentration factor, SCF, on the inside surface of the shell at point A in circumferential direction is 9.86. This value deviates about 9% from the literature [10].

All the calculation results of stress concentration factors at points A and C are shown in the 16 figures, Figure 9.1 through Figure 9.16 , as well as in Table 9.1.

The stress concentration factors can be calculated in the forms as follows:

1. The longitudinal stress concentration factors on the inside surface at the area of points on the intersection of shell and nozzle :

$$SCF = \frac{\sigma_1^m - \sigma_1^b}{p R / t} \tag{9.1}$$

2. The longitudinal stress concentration factors on the mid-surface at the area of points on the intersection of shell and nozzle:

$$SCF = \frac{\sigma_1^m}{pR/t} \tag{9.2}$$

3. The longitudinal stress concentration factors on the outside surface at the area of points on the intersection of shell and nozzle:

$$SCF = \frac{\sigma_1^m + \sigma_1^b}{pR/t} \tag{9.3}$$

4. The circumferential stress concentration factors on the inside surface at the area of points on the intersection of shell and nozzle:

$$SCF = \frac{\sigma_2^m - \sigma_2^b}{pR/t} \tag{9.4}$$

5. The circumferential stress concentration factors on the mid-surface at the area of points on the intersection of shell and nozzle:

$$SCF = \frac{\sigma_2^m}{pR/t} \tag{9.5}$$

6. The circumferential stress concentration factors on the outside surface at the area of points on the intersection of shell and nozzle:

$$SCF = \frac{\sigma_2^m + \sigma_2^b}{pR/t} \tag{9.6}$$

where σ_1^m and σ_1^b are the longitudinal membrane and bending stresses, and σ_2^m and σ_2^b are the circumferential membrane and bending stresses, all of which can be afforded by Equations (5.34) and (5.35) in Chapter 5 of this dissertation as follows:

$$\sigma_1^m = \frac{N_1}{t} \tag{9.7}$$

$$\sigma_1^b = \frac{6\,M_1}{t^2} \tag{9.8}$$

$$\sigma_2^m = \frac{N_2}{t} \tag{9.9}$$

$$\sigma_2^b = \frac{6\,M_2}{t^2} \tag{9.10}$$

This dissertation carries out the numerical calculations in an extensive range of intersection angles from $30°$ to $90°$. The purpose of these calculations is to try to find out the stress distribution at the areas of points on openings formed by the intersection of a cylindrical shell and an oblique nozzle so as to propose better suggestions for the design of shells and nozzles.

The intersection angles selected in the computation are 90°, 75°, 60°, 45° and 30° respectively. For each case, the longitudinal stress concentration factors and circumferential stress concentration factors on outside surface, mid-surface and inside surface corresponding to the point A and the point C have been calculated. All sixty curves calculated by this dissertation are illustrated in twelve figures from Figure 9.3 to Figure 9.14 where point A and point C are defined as shown in all figures, respectively.

These figures and the table indicate:

1. At the point A (see the figures for the location), the values of the circumferential and longitudinal stresses increase as the intersection angles change from 90° to 30° and the closer to the 30°, the faster the increase becomes. Therefore, any change of the intersection angle from 90° must cause the increase of the circumferential and longitudinal stresses.

2. At the point C (see the figures for the location), the values of the circumferential and longitudinal stresses decrease as the intersection angles change from 90° to 30° and the closer to the 30°, the faster the decrease becomes. Therefore, any change of the intersection angle from 90° must cause the decrease of the circumferential and longitudinal stresses.

The above phenomena can be explained that as the intersection angle is changing, the curvatures of intersection curve at the point A and the point C are changed. At point A, the value of curvatures increases as the intersection angle decreases from 90° to 30° and at point C, the value of curvatures decreases as the intersection angle decreases from 90° to 30°. Therefore, we can conclude that stress concentration factors increase as the value of curvatures of intersection curve increases. In other words, the stress concentration depends on the curvature of the opening curve.

It is also observed that at point C, the stress concentration factors always decrease with the intersection angle decreases., that is, any change of the intersection angle from 90° can only improve the stress situation.

On the other hand, the success of this study proves that the geometric analysis of description of the intersection curve and its results is true. This analysis is very helpful not only for the research of this dissertation, but also for the future study of stress analysis of the intersection of a cylindrical shell and elliptical nozzle.

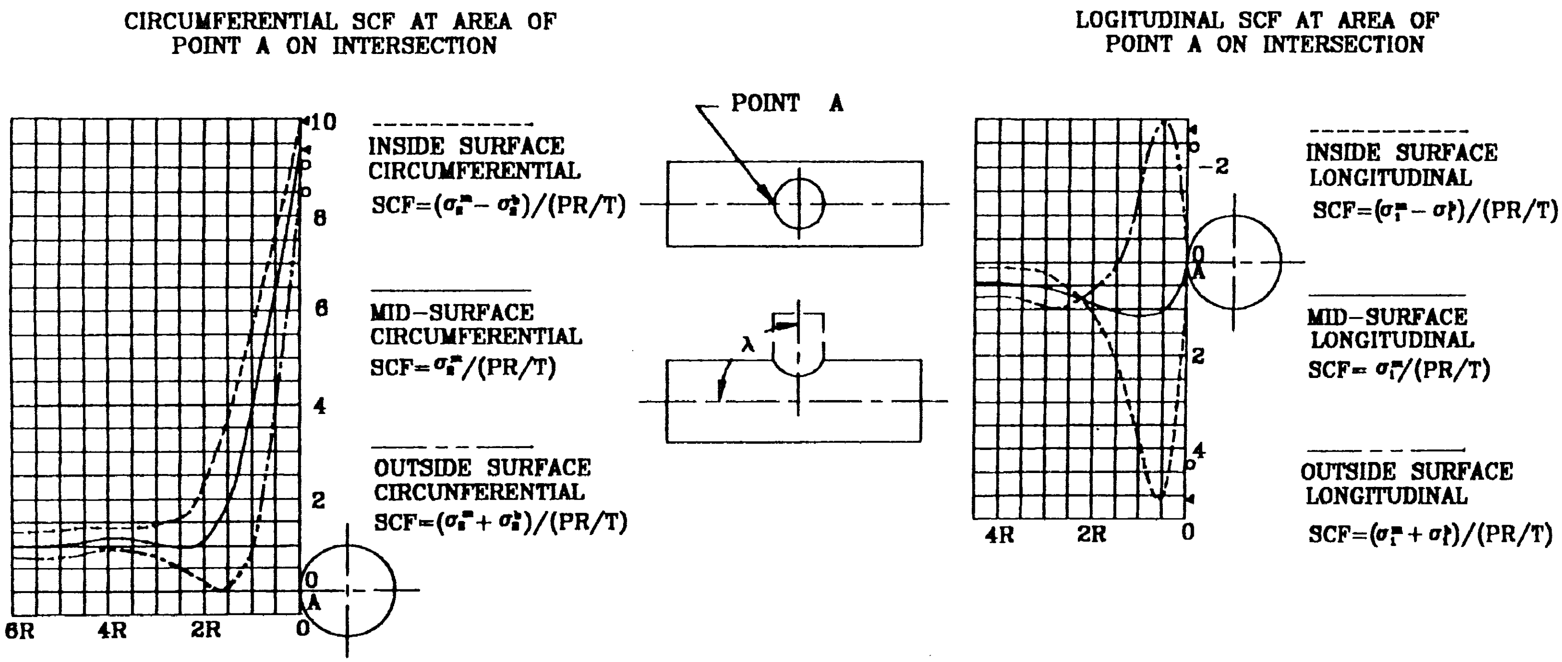


Figure 9.1 Stress concentration factors (SCF) at the point A on intersection when $\beta = 0.4$, $\gamma = 40$ and $\lambda = 90^0$

o : The maximum values of literature [10]

▼ : The maximum values of this study

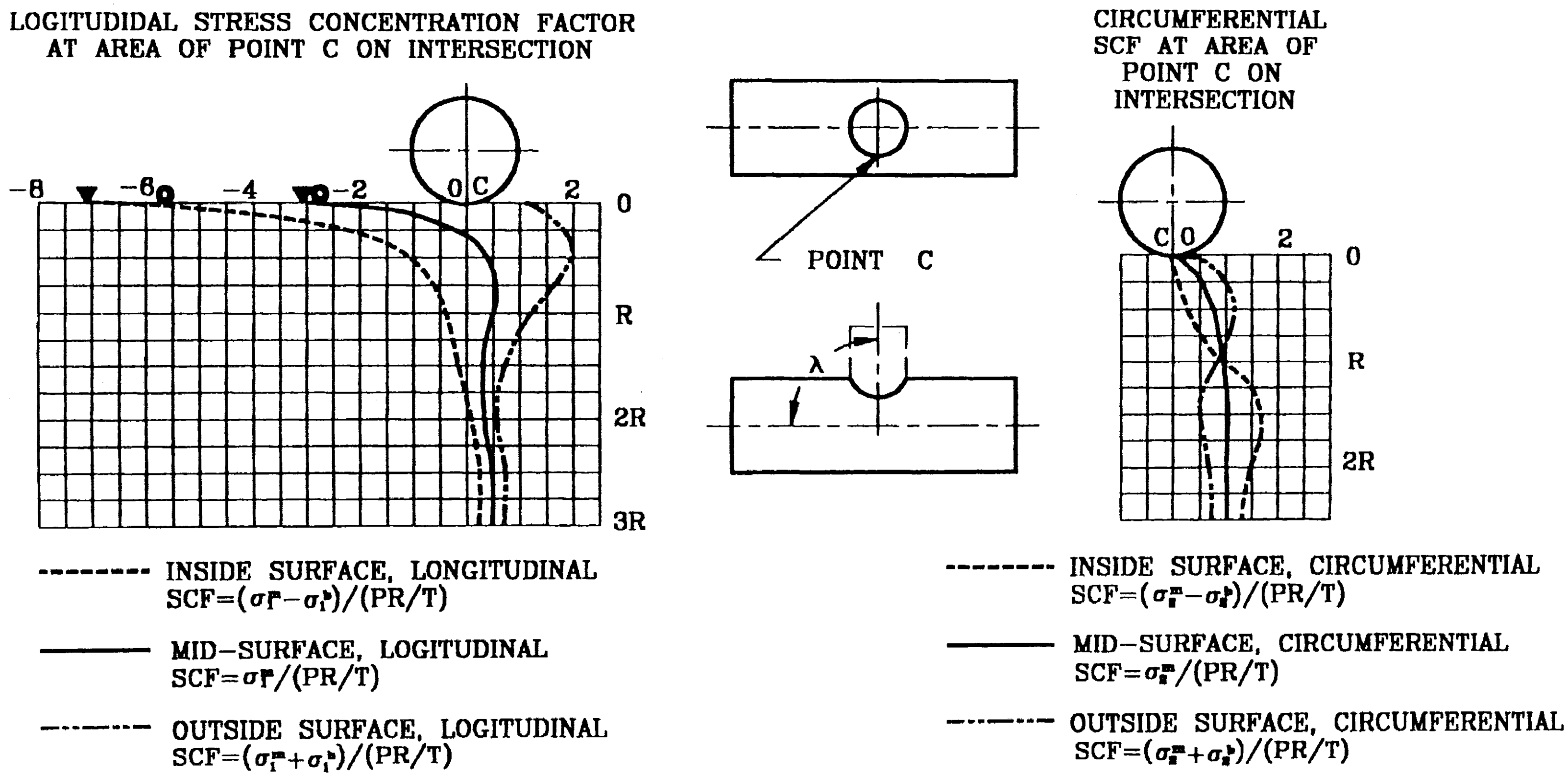


Figure 9.2 Stress concentration factors (SCF) at the point C on intersection when $\beta = 0.4$, $\gamma = 40$ and $\lambda = 90^0$

● : The maximum values of literature [10] .

▼ : The maximum values of this study

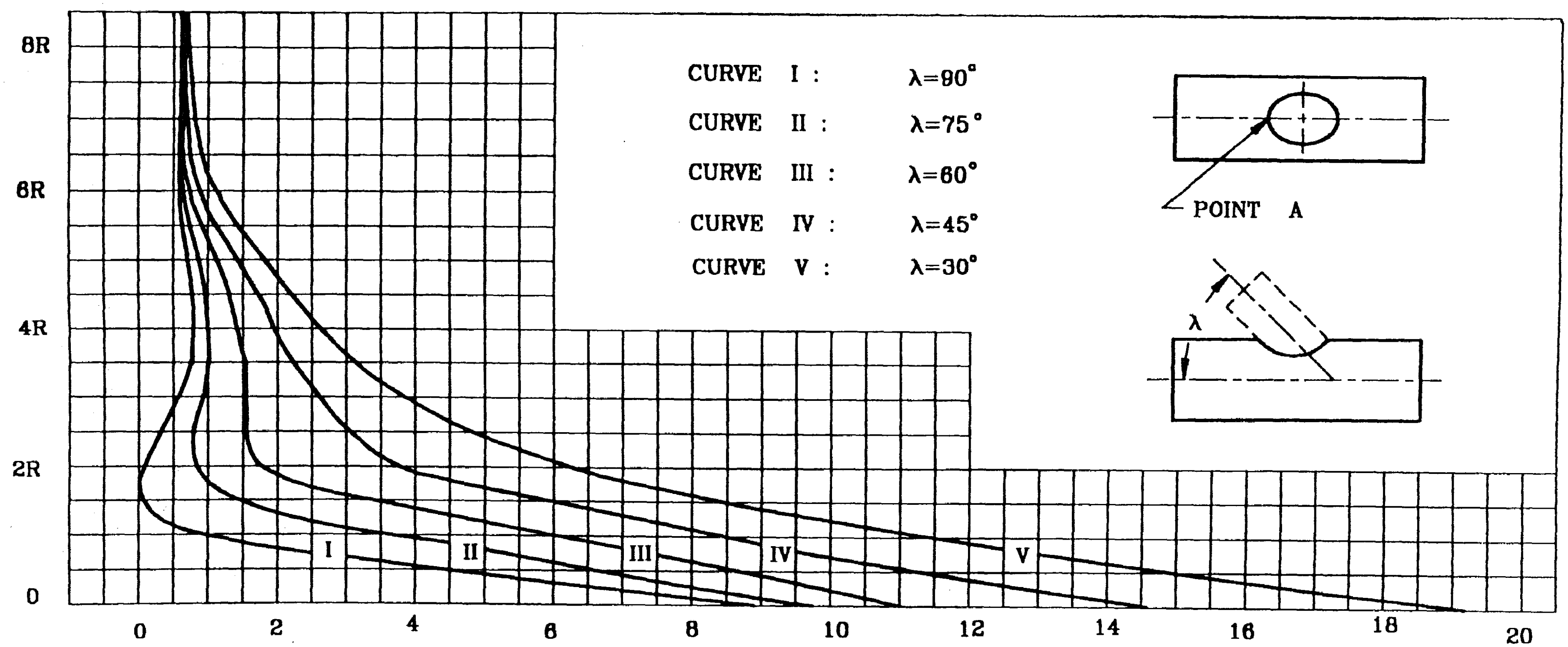


Figure 9.3 Circumferential stress concentration factors (SCF) on outside surface at point A on intersection when $\beta = 0.4$ and $\gamma = 40$

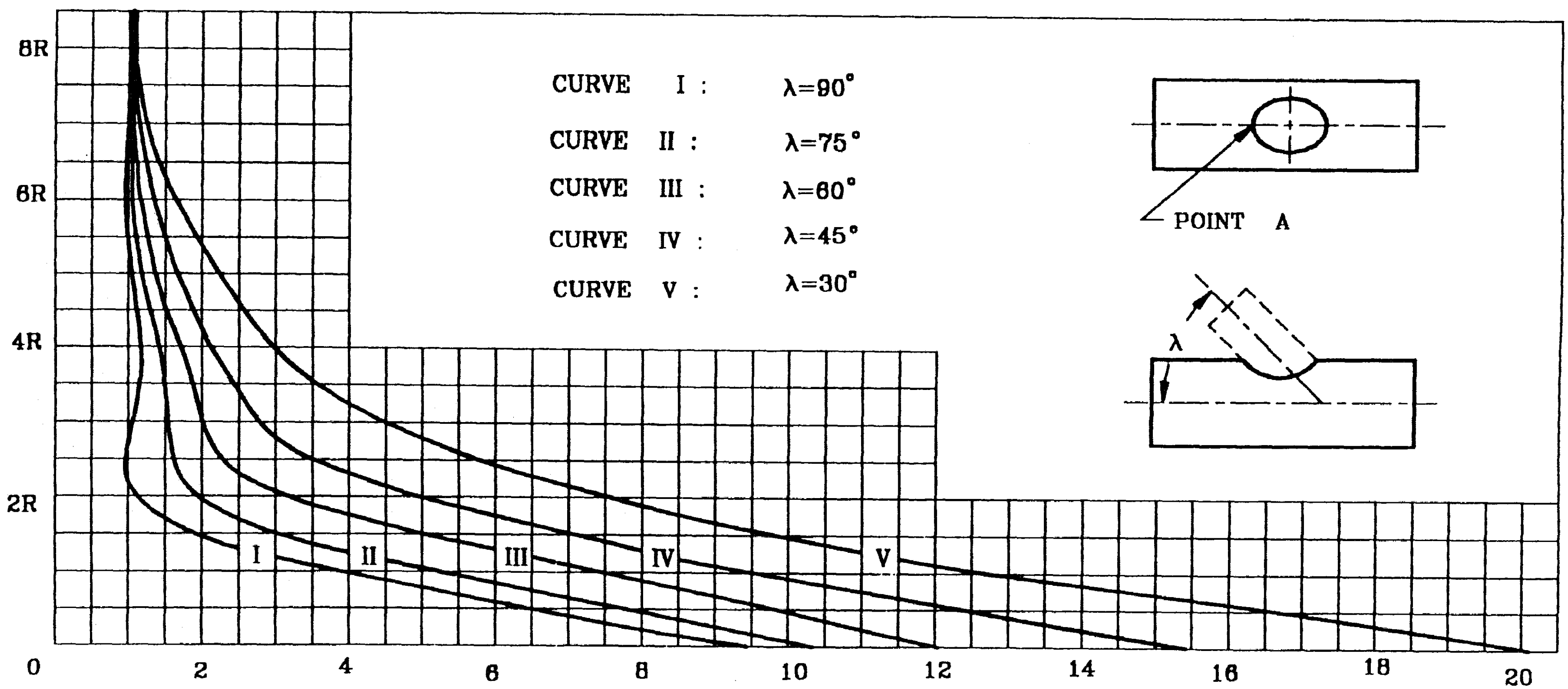


Figure 9.4 Circumferential stress concentration factors (SCF) on mid- surface at point A on intersection when $\beta = 0.4$ and $\gamma = 40$

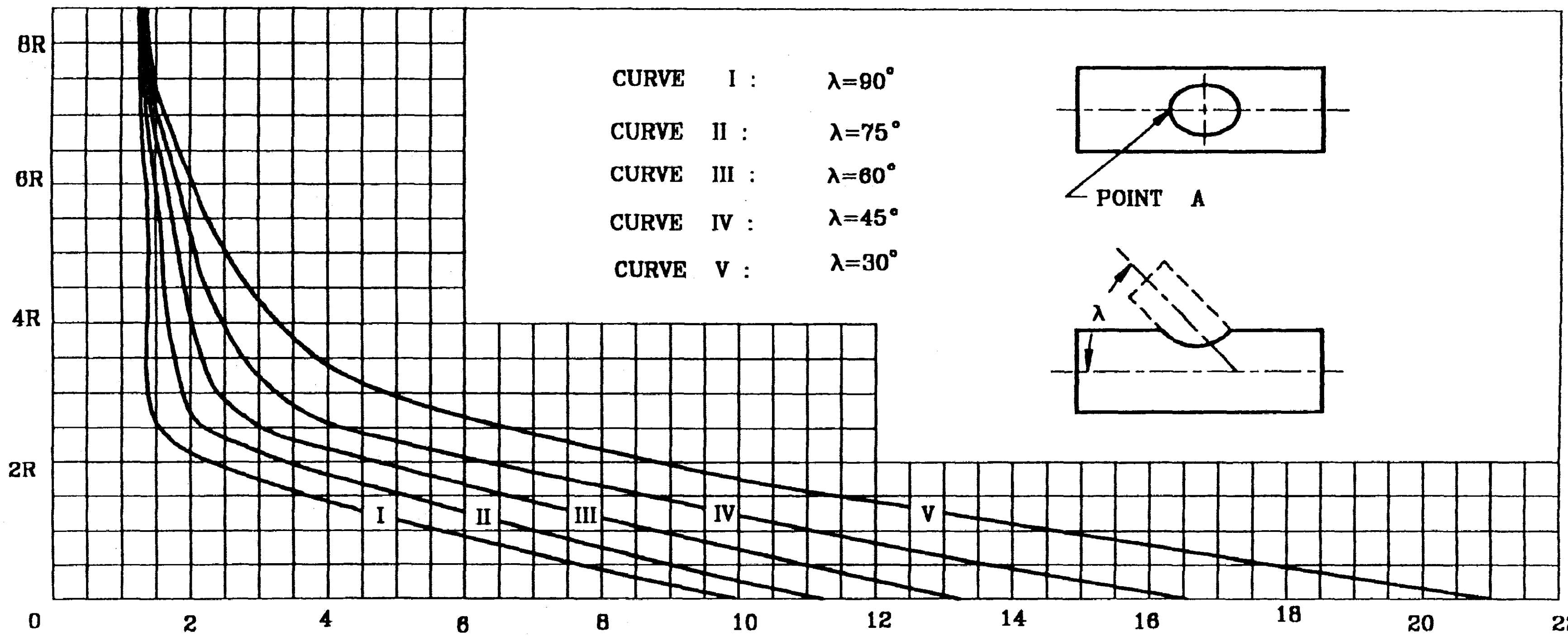


Figure 9.5 Circumferential stress concentration factors (SCF) on inside surface at point A on intersection when $\beta = 0.4$ and $\gamma = 40$

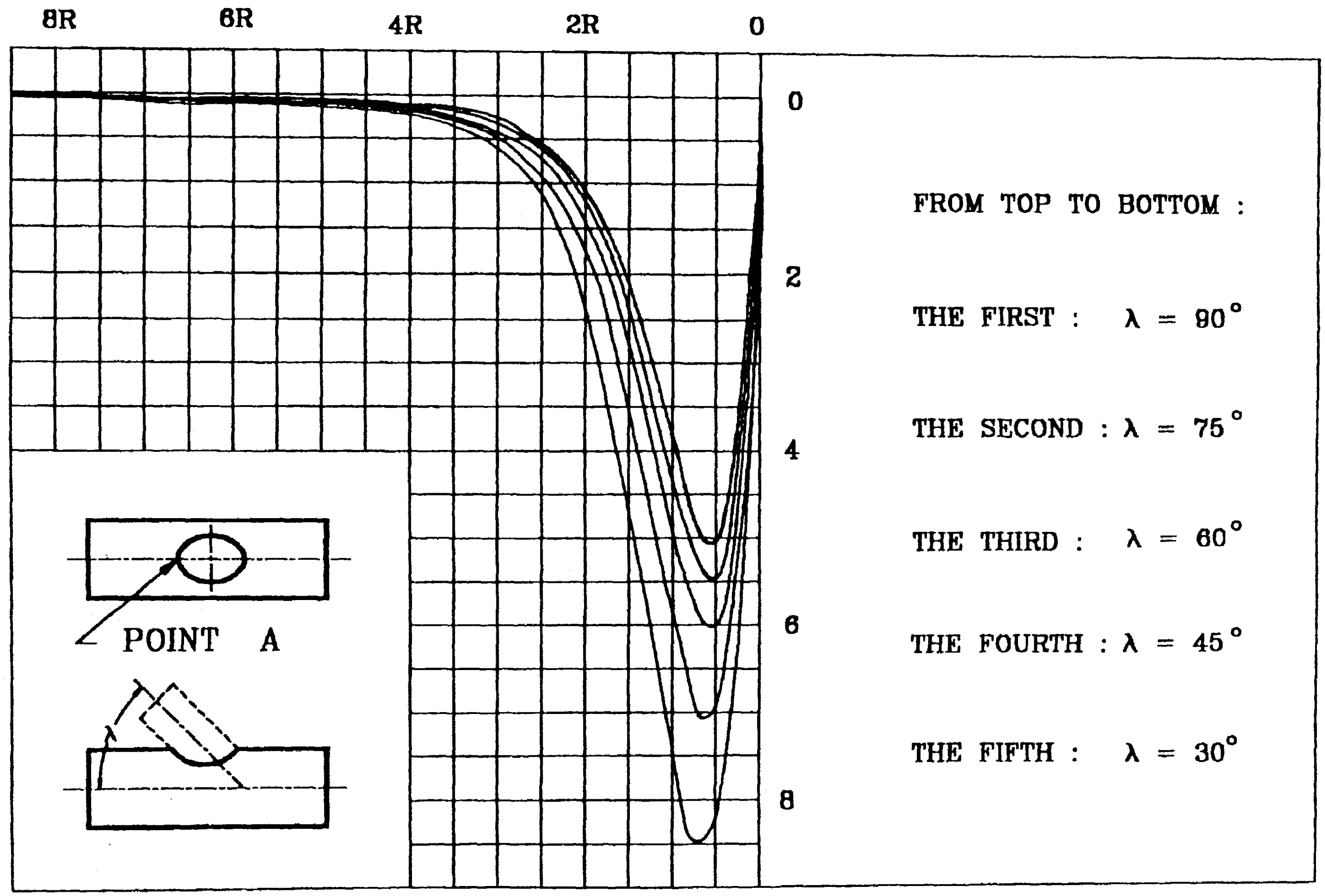


Figure 9.6 Longitudinal stress concentration factors (SCF) on outside surface at point A on intersection when $\beta = 0.4$ and $\gamma = 40$

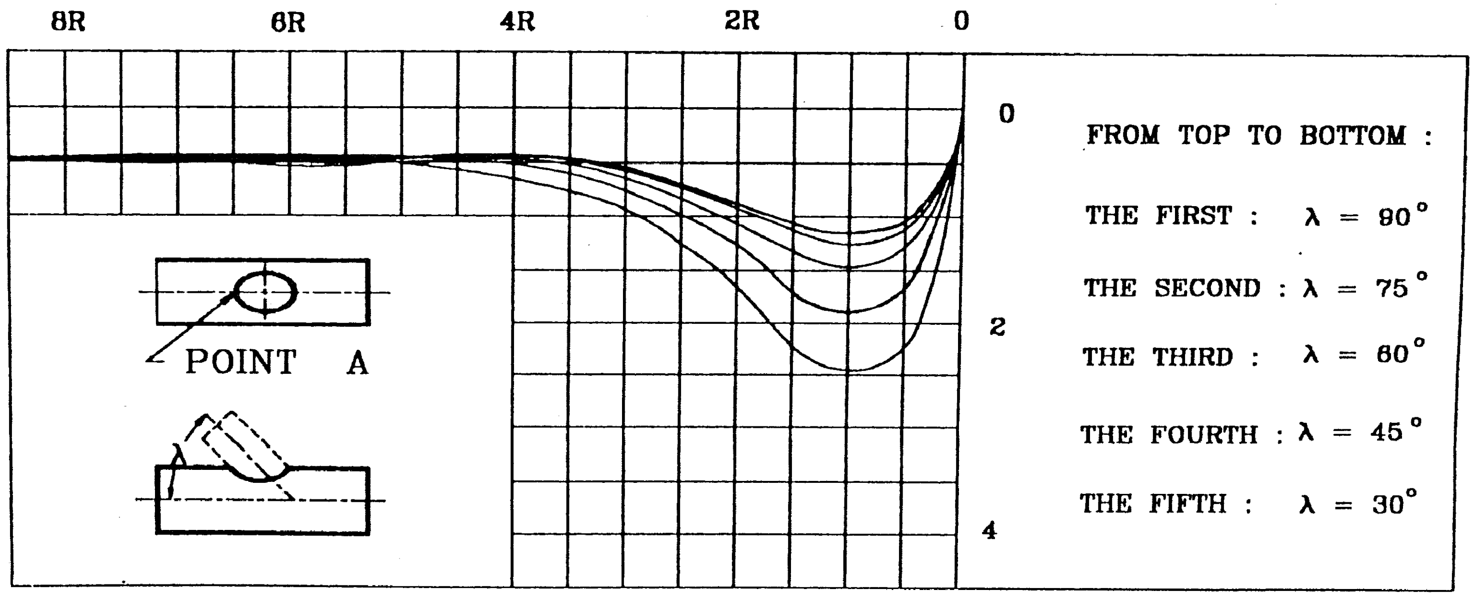


Figure 9.7 Longitudinal stress concentration factors (SCF) on mid-surface at point A on intersection when $\beta = 0.4$ and $\gamma = 40$

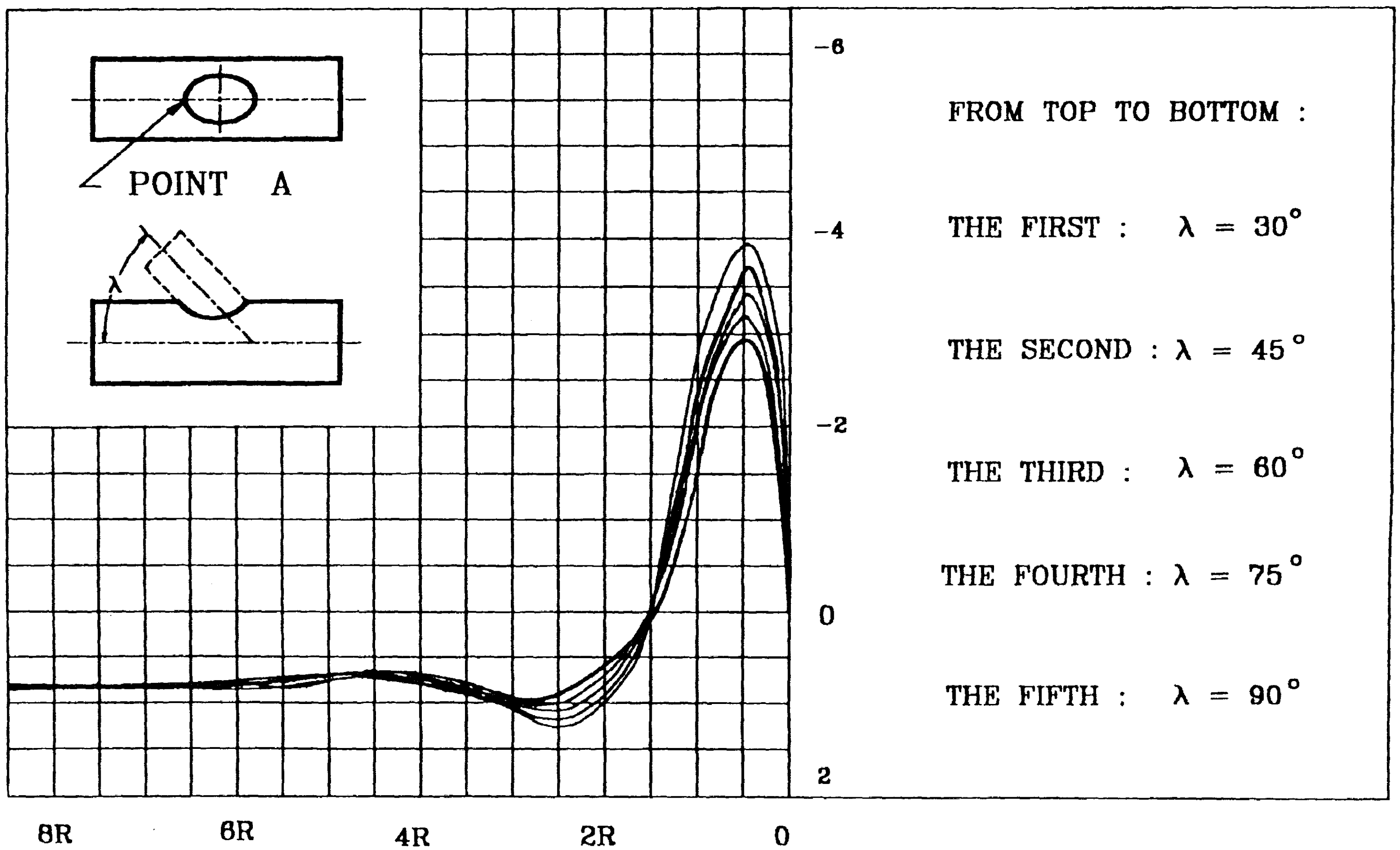


Figure 9.8 Longitudinal stress concentration factors (SCF) on inside surface at point A on intersection when $\beta = 0.4$ and $\gamma = 40$

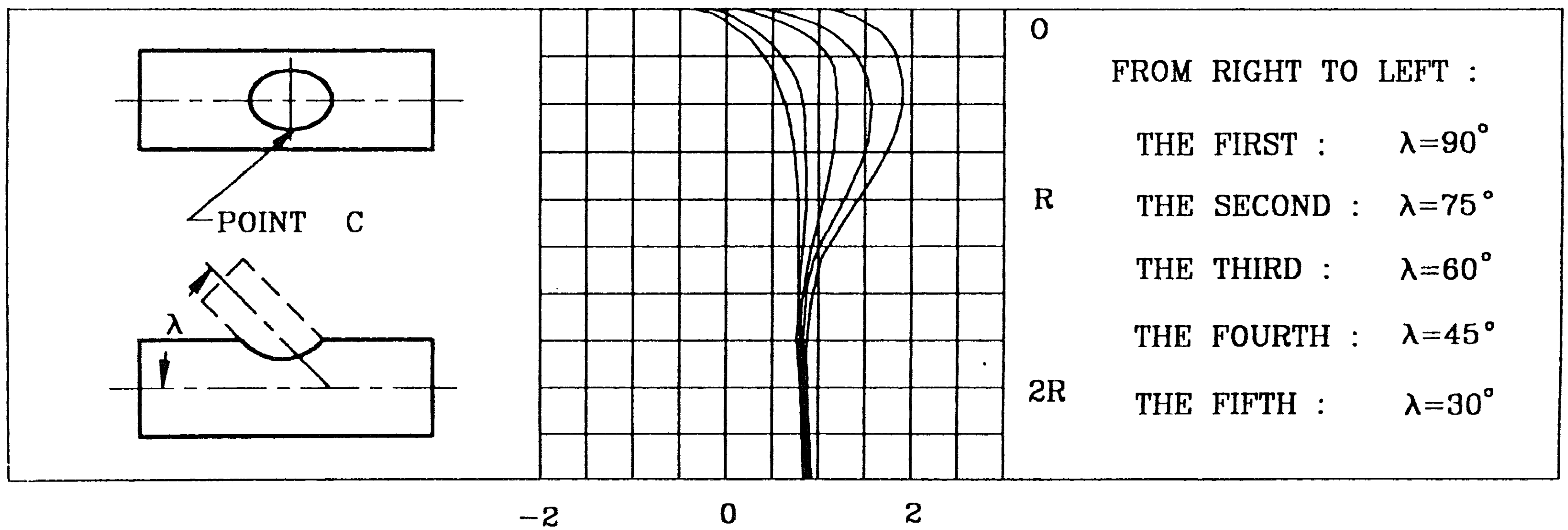


Figure 9.9 Longitudinal stress concentration factors (SCF) on outside surface at point C on intersection when $\beta = 0.4$ and $\gamma = 40$

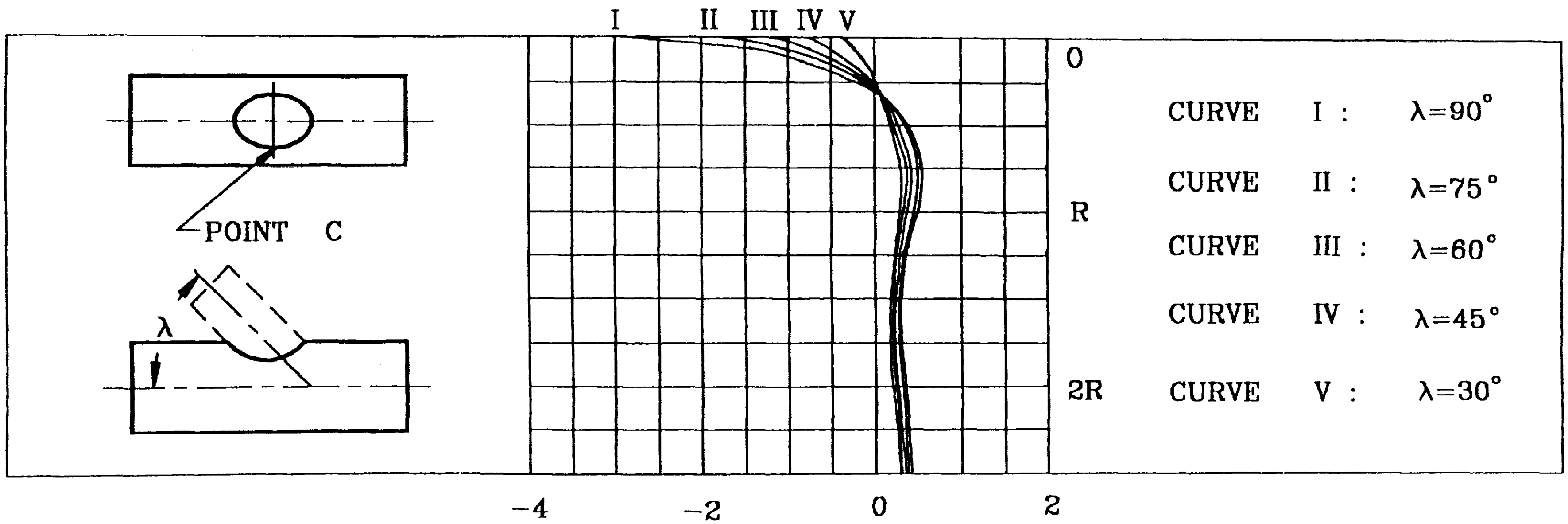


Figure 9.10 Longitudinal stress concentration factors (SCF) on inside surface at point C on intersection when $\beta = 0.4$ and $\gamma = 40$

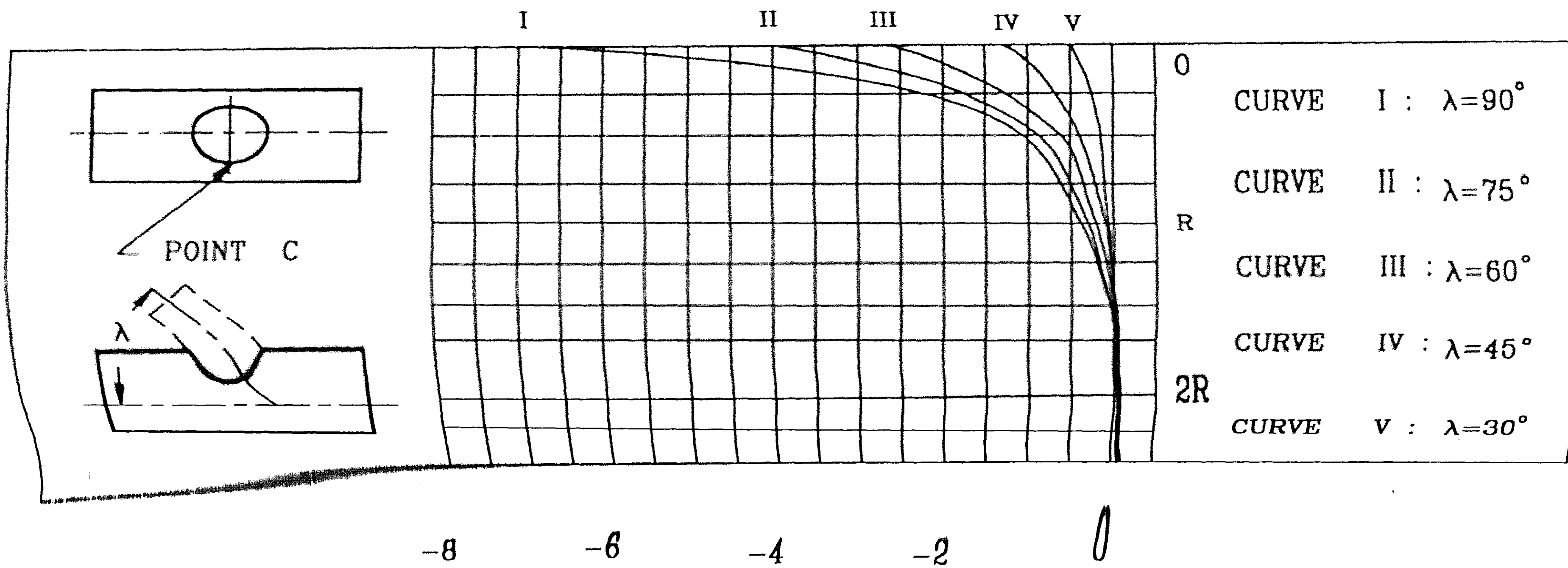


Figure 9.11 Longitudinal stress concentration factors (SCF) on mid-surface at point C on intersection when $\beta = 0.4$ and $\gamma = 40$

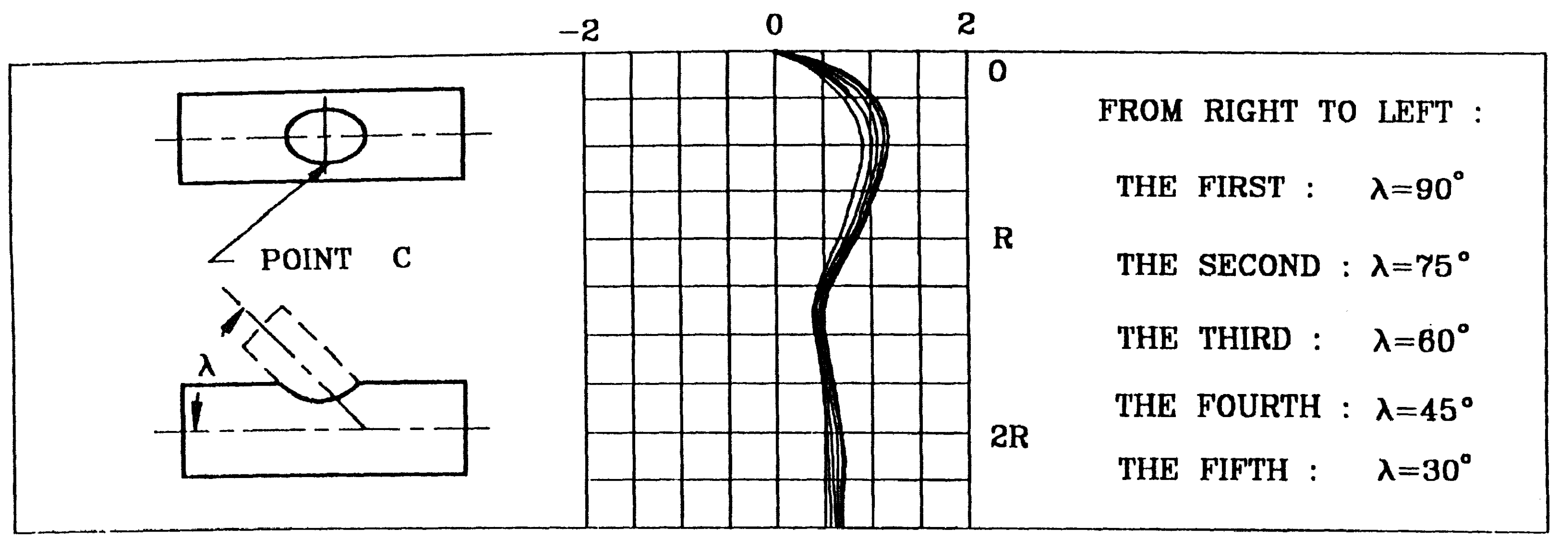


Figure 9.12 Circumferential stress concentration factors (SCF) on outside surface at point C on intersection when $\beta = 0.4$ and $\gamma = 40$

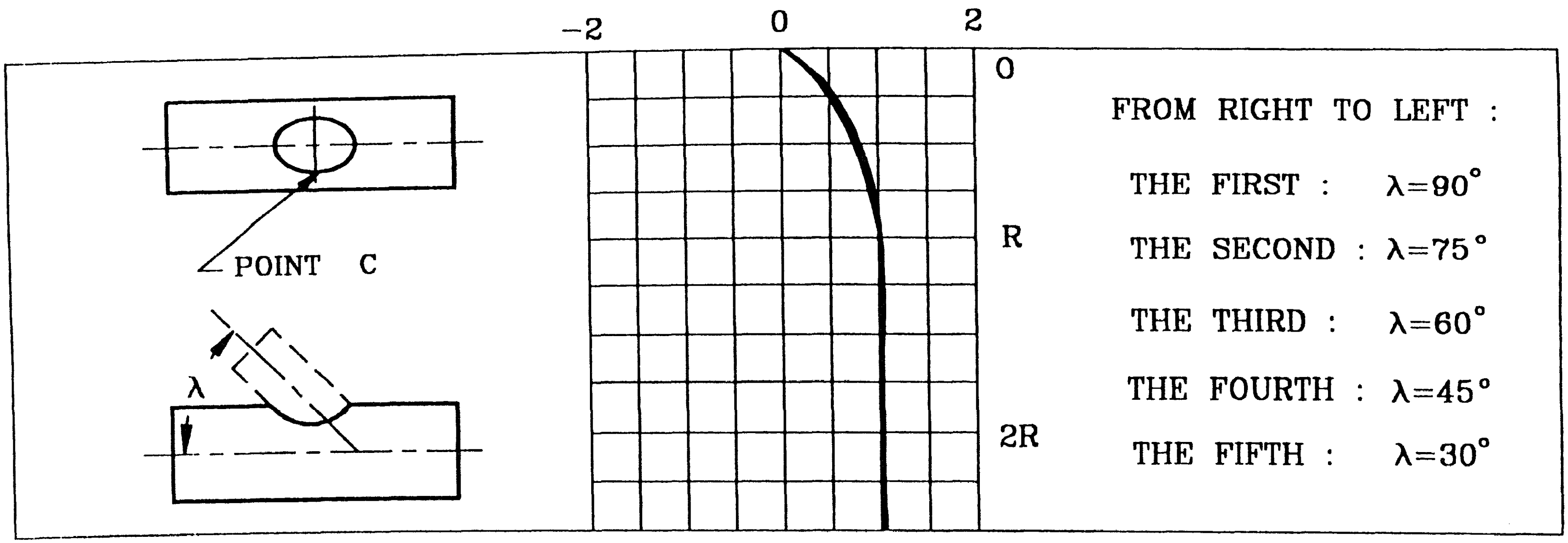


Figure 9.13 Circumferential stress concentration factors (SCF) on mid-surface at point C on intersection when $\beta = 0.4$ and $\gamma = 40$

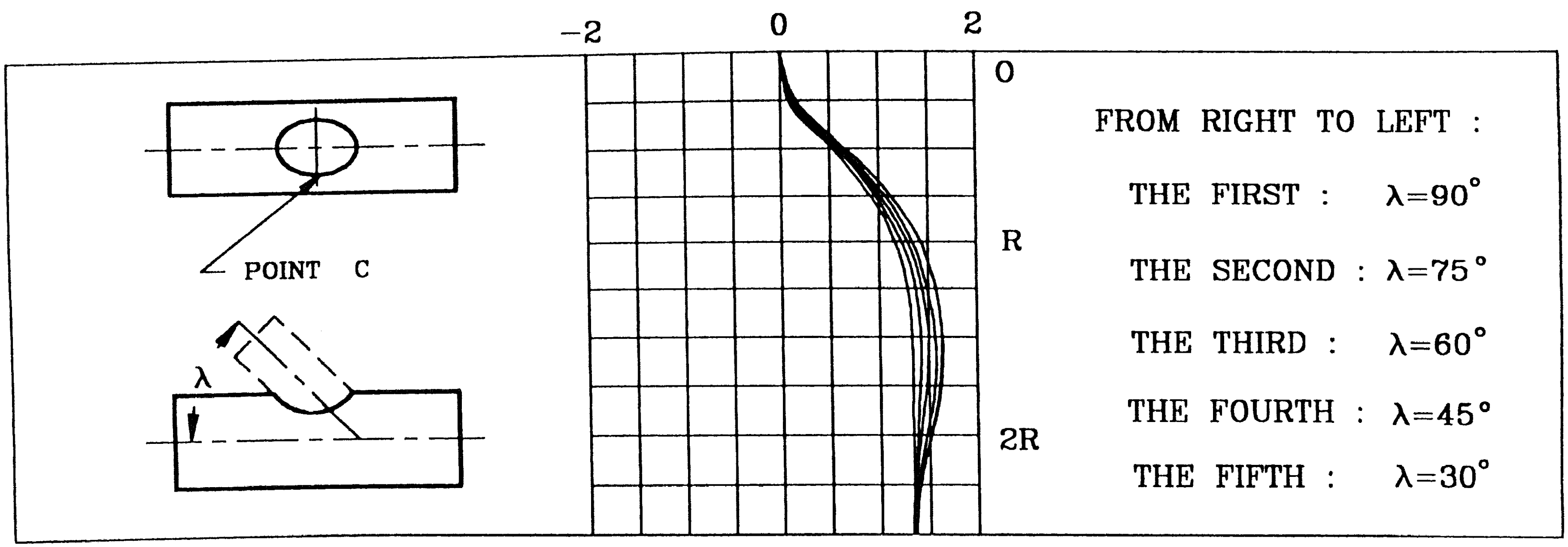


Figure 9.14 Circumferential stress concentration factors (SCF) on inside surface at point C on intersection when $\beta = 0.4$ and $\gamma = 40$

Table 9.1 The maximum values of stress concentration factors at different intersection angles at the point A and point C on the intersection curve

Intersection angles	The Maximum Values of Stress Concentration Factors							
	At the Point A				At the Point C			
	Circumferential SCF		Longitudinal SCF		Circumferential SCF		Longitudinal SCF	
	Inside	Outside	Inside	Outside	Inside	Outside	Inside	Outside
90°	9.03 (1)	8.50 (1)	4.22 (1)	−2.41 (1)	1.55 (1)	1.45 (1)	−5.78 (1)	1.30 (1)
	9.86	8.85	4.95	−2.85	1.65	1.25	−7.26	1.23
	9% (2)	4% (2)	17% (2)	18% (2)	6% (2)	−14% (2)	25% (2)	−5% (2)
75°	11.29	9.47	5.65	−3.15	1.55	1.20	−3.90	0.35
60°	13.29	10.92	6.05	−3.35	1.45	1.15	−2.35	0.15
45°	16.42	14.54	7.10	−3.55	1.35	1.05	−1.25	−0.25
30°	21.01	19.35	8.45	−3.85	1.25	0.85	−0.35	−0.35

Notes: (1) The values are from the literature [10].
(2) The percentage of errors in the table are between this study and the literature [10] at the intersection angle of 90 by using the latter as a standard.

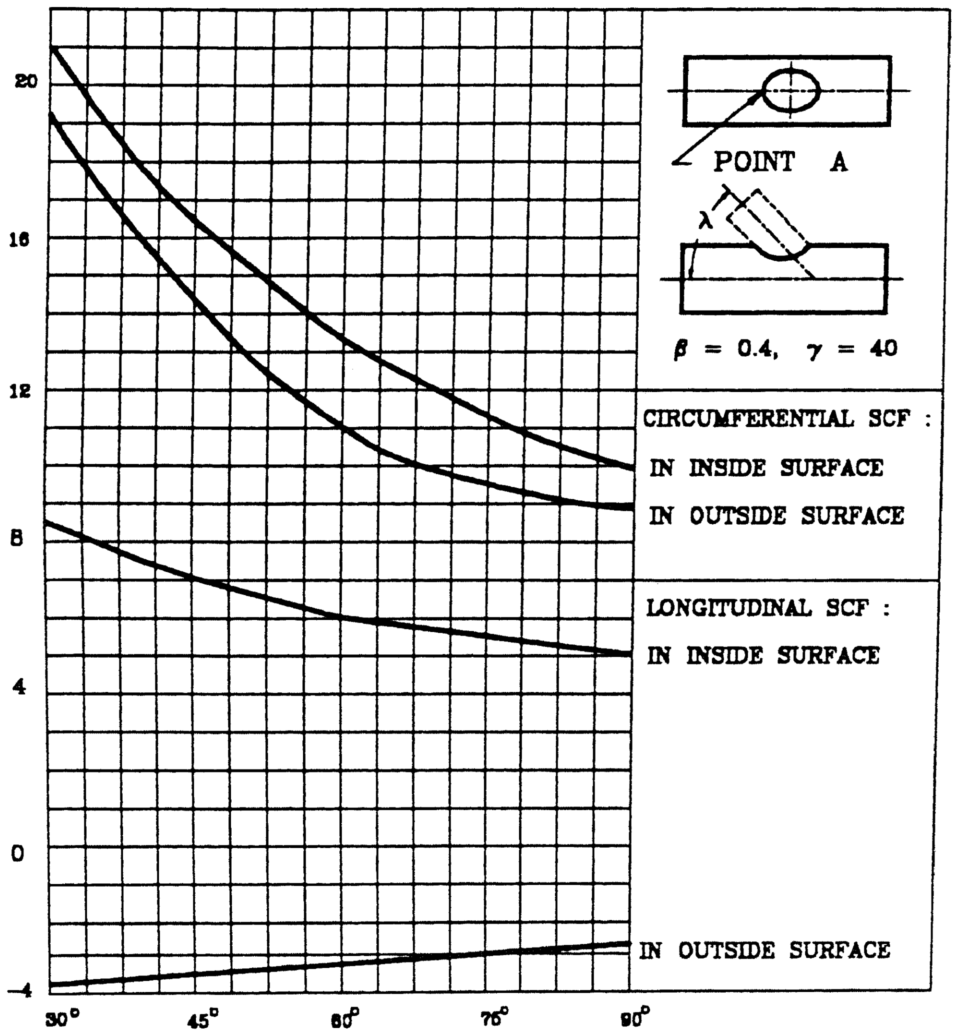


Figure 9.15 The maximum values of stress concentration factors at different intersection angles at point A on the intersection curve

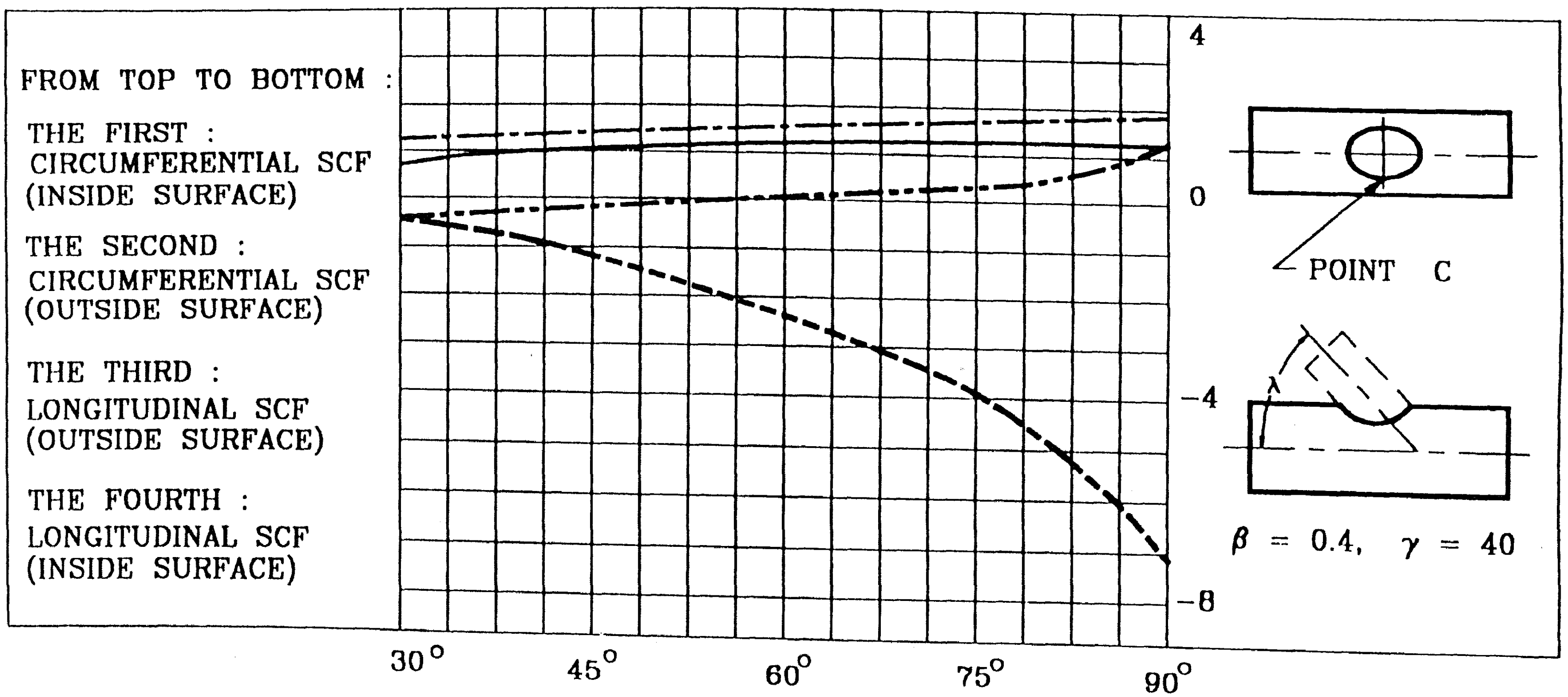


Figure 9.16 The maximum values of stress concentration factors at different intersection angles at point C on the intersection curve

CHAPTER 10

CONCLUSIONS

The comprehensive analyses and the extensive calculations of this dissertation indicate that the achievement of the theoretical derivations of this dissertation has been proven and this analytical solutions can provide helpful results for analysis and design work of shells and nozzles under the conditions shown in this dissertation.

The following conclusions can be achieved:

1. When the intersection angle is 90°, the stress results are in good agreement with the existing literature [10].

2. At the neighborhood of point A, both of circumferential stresses and longitudinal stresses increase as the intersection angle decreases from 90° to 30°, and the closer to the 30°, the faster the increase becomes. Therefore, among all angles from 90° to 30°, the intersection angle 90° has the least local stresses.

3. At the neighborhood of point C, when the intersection angle varies from 90° to 30°, circumferential stresses remain virtually constant, however, longitudinal stresses are compressive and they remain constant on the outside surface, but, increase on the inside surface.

4. After consideration of all influential factors, it is suggested that the intersection angles from 90° to 60° should be the best choices. The intersection angles from 60° to 45° can be selected if the internal pressure is not too high. The intersection angles less than 45° should be avoided as practical as possible.

5. The geometric analysis of the intersection curves provides a theoretical basis for the future stress analysis of intersection of a cylindrical shell and an elliptical nozzle.

REFERENCES

1. A. E. H. Love, *A Treatise on the Mathematical Theory of Elasticity*, University Press, Cambridge, 1920.

2. L. H. Donnell, "Stability of Thin walled Tubes under Torsion", NACA TR 479 (1933).

3. L. H. Donnell, "A Discussion of Thin Shell Theory", Proc. 5th Int. Congr. for Appl. Mech., 1938 pp. 66- 70.

4. L. S. D. Morley, " An Improvement on Donnell's Approximation for Thin-Walled Circular Cylinders", Q. Jl. Mech. Appl. Math. 12. 89 1959.

5. I. Ichino and H. Tkahashi, "Theory of Non-Symmetrical Bending State for Cylindrical Shells", Bull. J. S. M. E. 8, 169 , 1965.

6. J. G. Simmonds, "A Set of Simple, Accurate Equations for Circular Cylindrical Elastic Shells", International Journal of Solids and Structures, Vol. 2, 1966, pp. 525- 541.

7. J. G. Lekkerkerker, "The Determination of Elastic Stresses Near Cylinder-to-Cylinder Intersection", Nuclear Engineering and Design, Vol. 20, 1972, pp. 57-84.

8. C. R. Steele and M. L. Steele, "Stress Analysis of Nozzles in Cylindrical Vessels with External Load", ASME Journal of Pressure Vessel Technology, Vol. 105, 1983, pp. 191- 200.

9. C. R. Steele, M. L. Steele and A. Khathlah, "An Efficient Computational Approach for a Large Opening in a Cylindrical Vessel", ASME Journal of Pressure Vessel Technology, Vol. 108, 1986, pp. 436- 442.

10. M. -D. Xue, Y. Deng and K. -C. Hwang, "Some Results on the Analytical Solution of Cylindrical Shells with Large Openings", Journal of Pressure Vessel Technology, Vol. 113, pp. 297- 307, May 1991.

11. Zhilun Xu, *Theory of Elasticity*, Vol. 2 of 2nd Edition, People's Educational Press, Beijing , China, 1982, pp. 5, pp. 207-208, pp. 250.

12. K. S. Fu, R. C. Gonzalez and C. S. G. Lee, *Robotics*, McGraw-Hill Book Company, New York, 1987, pp. 24-25.