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ABSTRACT

REDUCTION OF INERTIA-INDUCED FORCES IN A GENERAL SPATIAL MECHANISM

by

Sahidur Rahman

A computer-aided design procedure has been developed for minimizing the adverse effects of the inertia-induced forces by optimum mass redistribution amongst the links of high speed general spatial linkages. The evaluation of an optimality criterion for the mass redistribution of the mechanism will be carried out with the aid of a quadratic programming technique. This has been found to be successful in minimizing inertia-induced forces and torques. The validity of the optimization procedure will be demonstrated by application to one kind of spatial linkage.

No literature has been found on the balancing of a general spatial mechanism, since its kinematic equations are highly non-linear and therefore, are very difficult to solve. This is the first analysis of inertia-induced forces and torques in a general spatial mechanism. This method allows for the trade-offs necessary to achieve optimum dynamic response of the linkage in design stage. These trade-offs involve a balance among the shaking force, shaking moment, bearing reactions, and input torque fluctuations by mass distribution of the moving links. The results will be reduced to design procedures and guidelines. These have been outlined in a step-by-step fashion suitable for the non-specialist.
REDUCTION OF INERTIA-INDUCED FORCES IN A GENERAL SPATIAL MECHANISM

by

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To my father
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CHAPTER 1

INTRODUCTION

Balancing of shaking forces and moments in high-speed machinery has been a challenging problem for mechanism and machine designers. In recent years machines have been operated at higher and higher speeds. Smoothness of operation is frequently a dominant consideration in the design of high speed machines, but most mechanisms are not naturally smooth in their operation. The objective of balancing a mechanism is to eliminate or reduce the effect of the shaking force and shaking moment the mechanism exerts upon its frame and surroundings, in order that the mechanism will attain improved dynamic, wear, noise, precision of operation properties and extended fatigue life. The results of this study will provide the designer an enhanced control over dynamic properties of reciprocating machinery in the design stage. By this procedure, three sets of shaking force, shaking moment, bearing force, bearing moment and input torque for main directions X, Y, Z will be derived. The balancing condition is to be developed by combining the effects of all the inertia-induced forces and torques. The objective function is the summation of non-dimensionalized mean squared inertia-induced forces and torques with some weight factors. The designer will be given enough flexibility to adjust the weight factors depending upon different situations. The masses of the moving links will be kept constant. A quadratic programming technique will be developed and numerical example will be used to illustrate the methodology.

The dynamic balancing of machinery is essential for good high-speed performance. A considerable amount of research on balancing of shaking force and shaking moment in planar mechanisms has been carried out in recent years [22-30]. In contrast to rapid progress in balancing theory and techniques for planar linkages, the understanding of shaking force and shaking moment balancing of spatial linkages is very limited. Because
of their complexity, it is generally not practical to perform an analysis of spatial linkages by hand computation or by graphical methods. Spatial linkages have therefore attracted much research interest in recent years following the advent of the high speed digital computers. The complete shaking force and shaking moment balancing of spatial linkages is a very difficult problem.

When operated at high speeds, the mass distribution in the links of a mechanism give rise to forces and moments which are transmitted to the ground link of the machine. These forces and moments shake the foundation upon which the machine is mounted, causing vibration disturbing people and doing structural damage to the floor and often to the entire building.

Our objective is to optimally distribute a given amount of mass within a link so to reduce the shaking forces which disturb the foundation of a machine. Essentially what we are doing is to represent the moment of inertia of the links by a collection of point masses whose magnitudes are optimized to achieve the reduction in inertia-induced forces and moments.

The methodology is novel and has the advantage over previous methods in that it can be applied to spatial mechanisms rather than just to planar mechanisms. As an example, it is applied to a generalized slider crank mechanism, which contains different kinds of joints such as the cylindrical, spherical and prismatic types.

1.1 Background

In this chapter, existing techniques used for balancing high-speed mechanisms and machinery are discussed. There has been a need to develop the optimum balancing of general three-dimensional mechanisms.

An unbalanced linkage running at high speed transmits shaking forces and shaking moments to its foundation (frame). The shaking force is the resultant inertia force exerted on the frame and is equal to the vector sum of the inertia forces associated with the
moving links of the mechanism. The shaking moment about an axis in the frame is the vector sum of the inertia torques and the moments of the inertia forces about this axis. These forces and moments cause vibrations, fluctuations in the input torque and stresses, and therefore impose limitations on the performance of high-speed machinery.

1.1.1 Complete Balancing Techniques

Much literature [31] is available on the balancing of planar linkages. Complete shaking force and shaking moment balancing is important in the dynamic balancing of mechanism, both theoretically and practically. The major goal in this is full shaking-force balancing. Complete balancing of shaking forces can be achieved if the center of mass of the mechanism remains stationary. Various techniques have been developed for this purpose. "The static balancing method" consists of replacing the masses of the links by a statically equivalent system of point masses. By adding counterweights to the links, the center of mass of all the moving links can be brought to rest, i.e. to coincide with a point in the frame. "The method of principal vectors" consists of describing the motion of the center of mass of a mechanism analytically and then determining the parameters by which the total center of mass can be located at a stationary point. "The method of linearly-independent vectors" by Berkof [4] requires the ability to redistribute the masses of the links in such a way that the total center of mass becomes stationary. Lowen, Tepper, and Walker et. al. further developed this theory to a higher degree [39, 40, 43, 44]. They solved the problem of full shaking force balancing of general planar linkages by the method of inertial mass distribution [39, 43, 44]. Ning-Xin Chen extended this method to spatial linkages [6, 7]. Bagci made a special contribution on the "irregular force transmission mechanism" for both planar and spatial mechanisms [1, 2]. "The method of linearly-independent vectors" has been the most suitable method for full shaking force balancing of mechanisms and has been applied to both planar and spatial mechanisms. Therefore, the study of full shaking force balancing of mechanisms is satisfactory.
Nowadays, the complete shaking force and shaking moment balancing still remains a problem for some special planar mechanisms [22-30]. The complete shaking force and shaking moment balancing is much more complicated than the full shaking force balancing of a mechanism, and so only some special planar mechanisms could be completely balanced. When the shaking force of a mechanism is fully balanced, the shaking moment of the mechanism becomes a pure torque which is only relative to the rotations of the moving links of the mechanism, but not to the translations of mass centers of the links. "The method of linearly-independent vectors" of Berkof and Lowen is extended by Elliott and Tesar [9] to the shaking moment and driving torque functions. These tools are combined to completely eliminate shaking force and shaking moment with the addition of a physical negative mass. In addition to redistributing the masses, additional moving elements (cams, balance weights, etc.) can be introduced to eliminate shaking force and moments.

Investigation of the complete shaking force and shaking moment balancing of spatial mechanism has been very limited. In fact Yue-Qing's research [46-48] appears to be the only study in this field, and an encouraging achievement dealing with some types of mechanisms by the method of addition of balancing dyads. This paved the way to achieve the complete shaking force and shaking moment balancing of various kinds of spatial linkages.

1.1.2 Partial Balancing Techniques

Shaking force, shaking moment, inertia-induced joint reactions (bearing reactions) and input torque fluctuation are dynamic characteristics of mechanisms. Complete balancing of any one of these may result in an increased unbalance in the others. Hence partial balancing techniques permit desirable design trade-offs. In high speed mechanisms this is very essential. Some of the previously investigated techniques are described below.
In 1971, Berkof and Lowen [5] have presented a least-square theory for the optimization of the shaking moment of fully force-balanced planar four-bar linkages running at constant angular velocity. Sherwood [37] has used equivalent masses to minimize the kinetic-energy fluctuation of the coupler of a planar four-bar linkage having drag-linkage and crank-and-rocker proportions. Hockey [19] later presented an approach for the distribution of mass in the coupler to approximate a constant energy level for the four-bar linkage, which implies the driving torque remains near zero. Tricamo and Lowen [41, 42] introduced a two and three counterweight technique for simultaneously minimizing the maximum values of such dynamic reactions as the bearing force, the input moment and the shaking moment of a constant input-speed planar four-bar linkage, while additionally obtaining a prescribed maximum value of shaking force. In 1991 Kochev [29] performed optimum balancing of a well-known class of complex planar mechanisms which remain kinematically invariant (function cognates) with respect to the angular rearrangement of their sub linkages. His research revealed the potential of function cognate transformation for optimum balancing of such mechanisms. Providing complete shaking force balancing, he discussed two basic objectives: (i) minimization of the total balancing mass and (ii) minimization of shaking moment. However, the concept is rather general and may well contribute to other optimization problems, like minimization of a given joint reaction, balancing of flexibly mounted machines, etc.

Relatively little research has been devoted to techniques for the partial balancing of spatial linkages. Hockey [18] has minimized the fluctuation of kinetic energy and inertia forces of a spatial slider crank (RSKP) mechanism by optimizing the mass distribution. Symbol K denotes the universal joint. The exact solution of the optimized set of equations, which were obtained by assuming ten point masses in a particular configuration (to represent a three dimensional coupler), showed that for balancing purposes the coupler ideally should be a perfectly thin rod (an impractical proportion) rather than a three dimensional body. Hockey also obtained an approximate solution for a three dimensional
coupler. Sherwood [36] dealt with the distribution of mass in the links of a simple harmonic spatial slider crank mechanism in order to achieve constancy of total kinetic energy and inertia force and torque balance during the motion cycle. He replaced the coupler by three in-line point masses. For constancy of kinetic energy and inertia force balance, Sherwood obtained the condition that the center of mass of the piston and connecting rod should lie on the center of the crank pin. The inertia couple, however, was not completely balanced.

Very few methods [15, 16] can allow a trade-off among the shaking force, the shaking moment, bearing reactions, and the input torque in three-dimensional mechanisms. This method is limited to spherical mechanisms only. There is no general method which can solve this problem for general linkages, especially for spatial mechanisms. Both kinematic and dynamic properties of spatial mechanisms are much more complicated than those of planar mechanisms. Many balancing methods for planar mechanisms cannot be applied to spatial linkages. Therefore, techniques for shaking force and shaking moment balancing of general spatial mechanisms are still unavailable.

1.2 Motivation, Objective and Scope of Work

Kinematic and dynamic analyses of the generalized slider crank mechanism for a single cylinder engine were accomplished by Fischer and Rahman [11, 12] in 1993. In this mechanism the joint between frame and crank is cylindrical having one translational and one rotational degree of freedom. Both the joints between crank and connecting rod and connecting rod and slider (piston, in case of an engine) are spherical (ball) having three rotational degrees of freedom, one of which is passive, i.e. rotation about the connecting rod longitudinal axis. The joint between slider and frame is prismatic, having one translational degree of freedom. While conducting the dynamic analysis it is observed that due to the presence of offsets, the forces and torques acting on the joints deviate from the ideal case. Dynamic force and torque reactions in the mechanism were obtained using
dual-number ($\varepsilon \neq 0$, but $\varepsilon^2 = 0$) techniques as developed by Yang [45] in 1971. The particular formulation used in that study was developed by Pennock and Yang [33] in 1983.

The motivation for this research is to overcome certain difficulties involved in balancing the inertia effects occurring in high speed mechanisms. An analysis or design procedure should allow for trade-offs among various quantities and thus requires a new formulation of the dynamic problem. This is likely to involve lengthy calculations, such as matrix inversion or solution of simultaneous equations. Consequently for effective modeling of a linkage, efficient numerical procedures are required. An optimality criterion which can truly represent the dynamic characteristics of a linkage has to be developed and subsequently, an efficient optimization technique is required to yield a solution. The linkage balancing problem, although considered to be an old problem, certainly faces new challenges, particularly in light of the rational design of linkages. It therefore warrants an investigation from a global perspective, that is, a balancing of combined shaking force, shaking moment, bearing reactions and torque fluctuations in high speed linkages. The purpose of this investigation is to develop a balancing method which is capable of carrying out the trade-offs that are necessary to achieve optimum dynamic response of the linkage.

The objectives of the research are as follows:

(i) Determination of the inertia force and inertia torque associated with each moving link of the mechanism.

(ii) Determination of shaking force, shaking moment, bearing reactions and input torque as a function of joint variables.

(iii) Optimization of the mass distribution with respect to shaking force, shaking moment, bearing reactions and input torque fluctuation.

(iv) Application of these techniques for balancing a CSSP mechanism. This includes determination of inertia forces and torques due to the entire mechanism and
optimization of mass distribution for minimization of shaking force, shaking moment, bearing reactions and input torque fluctuation.

(v) Development of suitable computer-aided design procedures with the help of IMSL routines for the optimum mass distribution of high speed CSSP mechanism.

The result of this investigation will demonstrate that this method offers several advantages. The procedure is so general that it is applicable to many linkages with no restrictions. The method is efficient and can be easily utilized by practicing engineers without requiring any specialized skills.

1.3 Summary of Research

In this research a computer-aided design procedure has been developed for minimization of inertia-induced forces in a CSSP mechanism. Kinematic analysis data and dynamic force and torque equations used are from Fischer and Rahman [11, 12].

In Chapter 2 the kinematics of the mechanism are developed. For this purpose, one fixed coordinate system, three moving coordinate systems (each attached to a moving link) and four dual number transformation matrices are established. The mass distribution of each moving link (crank and connecting rod) is replaced by a dynamically equivalent system of four point masses. Vector coordinates of point masses relative to the fixed frame are determined by using "principle of transference" [21] and then direction cosines of the principal axes with respect to distal frame attached to each moving link. Vector coordinates of center of mass of the slider are determined by using the "principle of transference" only. Acceleration of point masses relative to the fixed frame are determined by the method as discussed in Fu, Gonzalez and Lee [14].

In Chapter 3 shaking forces, shaking moments, bearing reactions and input torques are determined as a function of crank rotation. All the forces and moments are expressed with respect to the fixed coordinate frame.
Chapter 4 deals with the minimization of inertia-induced forces in the mechanism. A quadratic objective function consisting of summation of non-dimensionalized, squared shaking force, shaking moment, bearing reactions and input torque is formulated over one complete cycle of rotation. Point masses are considered as designed variables. Design constraints are formulated as a set of equations linear in the design variables. The optimization of mass distribution is obtained by the application of IMSL routines.

In chapter 5 an example is presented to demonstrate the feasibility of the technique. Results are discussed with the help of tables and graphs.

Finally, Chapter 6 describes the general conclusions of this study and outlines the goals of future research.
CHAPTER 2

KINEMATICS OF THE MECHANISM

2.1 Coordinate Systems

Each link of the mechanism will be characterized by the relationship between the axes of its joints. As seen in figure 2.1, the link connecting axes $n$ and $n+1$ can be characterized by its length $a_n$, the shortest distance between axes $n$ and $n+1$, and twist angle $\alpha_n$, the angle between axes $n$ and $n+1$. On the distal end of each link $n$, there is a fixed coordinate frame $\{n+1\}$ such that the $i_{n+1}$ axis is aligned with a line of length $a_n$ and the $k_{n+1}$ axis is aligned with the axis of joint $n+1$. The displacements at each joint $n$ are the rotation $\theta_n$, representing the angles between the $i$-axes of frames $\{n\}$ and $\{n+1\}$ and the translation $s_n$, representing the shortest distance between those $i$-axes.

Figure 2.1 Generalized model of a link connecting two joints which are either cylindrical, prismatic or revolute.

As seen in figure 2.2, the crank of the generalized slider crank, designated as link 1, has a length $a_1$ and zero twist angle. The connecting rod is link 2, has length $a_2$ and also has zero twist angle. Link 3 is the slider, or piston, and it has zero length with a
twist angle $\alpha_3 = \pi/2$. The frame, or ground link, has a length $a_4$, the offset, and the twist angle $\alpha_4$ which would have a value of $3\pi/2$ for the planar case. At the joint between the crank and the frame, there occurs rotation through angle $\theta_1$ and translation through distance $s_1$. At each end of the connecting rod is a ball joint where the displacements are specified by two rotations, angles $\theta_2$ and $\eta_2$ at the connection with the crank, and angles $\theta_3$ and $\eta_3$ at the connection with the slider. The rotation of the connecting rod about its own axis is a redundant degree of freedom which is neglected. The displacement of the slider is a translation through distance $s_4$.  

![Diagram](image)

**Figure 2.2** The generalized slider-crank mechanism in which the cylinder and crankshaft axes are offset and non-perpendicular.

### 2.1.1 Fixed Coordinate Frame

As seen in figure 2.2, coordinate frame \{1\} is the fixed coordinate frame, it does not move when the mechanism works. All forces and torques are expressed in terms of frame \{1\}.  


2.1.2 Moving Coordinate Frames

The coordinate frame \{2\} is located at the distal end of the crank, frame \{3\} is located at the distal end of the connecting rod and frame \{4\} is located at the distal end of the slider. These are the moving coordinate frames. They move with respect to the fixed frame \{1\}.

2.1.3 Coordinate Transformation Matrices

A $3 \times 3$ dual-number matrix can be formulated to express the transformation between coordinate frames fixed on the distal ends of links comprising a mechanism. Referring to figure 2.1, one can trace the path from the position of frame \{n-1\} to the position of frame \{n\} as a rotation through the angle $\theta_n$ and translation through distance $s_n$ about the $k_n$ axis followed by rotation through angle $\alpha_n$ and translation through distance along the $i_n$ or $i_{n+1}$ axis. These displacements can be combined into the dual angles $\hat{\alpha}_n = \alpha_n + \varepsilon \alpha_n$ and $\hat{\theta}_n = \theta_n + \varepsilon s_n$, where letter $\varepsilon$ represents the dual number ($\varepsilon^2 = 0, \varepsilon \neq 0$). The transformation between the coordinate frames can be considered as a screw motion through dual angle $\hat{\theta}_n$ with respect to a $k$-axis followed by a screw motion through dual angle $\hat{\alpha}_n$ about an $i$-axis. All dual-number coordinate transformations are explained in detail in Appendix A. These screw motions $\hat{Z}(\hat{\theta}_n)$ and $\hat{X}(\hat{\alpha}_n)$ can be combined into a matrix $\hat{M}_n$ and expressed in $3 \times 3$ form as

$$\hat{M}_n = [Z(\hat{\theta}_n)][X(\hat{\alpha}_n)] = \begin{bmatrix} c\hat{\theta}_n & -s\hat{\theta}_n & 0 \\ s\hat{\theta}_n & c\hat{\theta}_n & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\hat{\alpha}_n & -s\hat{\alpha}_n \\ 0 & s\hat{\alpha}_n & c\hat{\alpha}_n \end{bmatrix}$$

which expands to

$$\hat{M}_n = \begin{bmatrix} c\theta_n & -c\alpha_n s\theta_n & s\alpha_n s\theta_n \\ s\theta_n & c\alpha_n c\theta_n & -s\alpha_n c\theta_n \\ 0 & s\alpha_n & c\alpha_n \end{bmatrix}$$
\[
\begin{bmatrix}
-s_n s\theta_n & a_n s\alpha_n s\theta_n - s_n c\alpha_n c\theta_n & a_n c\alpha_n s\theta_n + s_n s\alpha_n c\theta_n \\
-\frac{s_n \hat{c}\theta_n}{-a_n s\alpha_n \hat{c}\theta_n - s_n c\alpha_n s\theta_n - a_n c\alpha_n c\theta_n + s_n s\alpha_n s\theta_n}
\end{bmatrix} + \varepsilon
\]

\[\begin{bmatrix}
\frac{c\theta_n}{-a_n s\alpha_n c\theta_n} & -s\theta_n & 0 \\
\frac{s\theta_n}{-a_n s\alpha_n c\theta_n} & c\theta_n & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

For the crank, link 1, \(\alpha_i = 0\), this specializes to

\[\begin{bmatrix}
\frac{c\theta_1}{a_1 s\alpha_1} & -s\theta_1 & 0 \\
\frac{s\theta_1}{a_1 s\alpha_1} & c\theta_1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

and since the slider is constrained from rotating so that \(\theta_4 = 0\), we obtain

\[\begin{bmatrix}
1 & 0 & 0 \\
0 & c\alpha_4 & -s\alpha_4 \\
0 & s\alpha_4 & c\alpha_4
\end{bmatrix} + \varepsilon
\begin{bmatrix}
0 & -s_4 c\alpha_4 & s_4 s\alpha_4 \\
-s_4 a_4 s\alpha_4 & -a_4 c\alpha_4 \\
0 & a_4 c\alpha_4 & -a_4 s\alpha_4
\end{bmatrix}
\]

There are ball joints on the proximal ends of the connecting rod and the slider. Thus the motion at those joints requires an additional rotation through an angle \(\eta\) for its description so that the complete transformation through the joint and link takes the form

\[\hat{L} = [Z(\hat{\theta}_n)] \hat{\eta} [Y(\hat{\eta}_n)] \hat{X(\hat{\eta}_n)}]
\]

For the connecting rod, link 2, lengths \(s_2=0, e_2=0\) and twist angle \(\alpha_2=0\), so that

\[\begin{bmatrix}
\frac{c\theta_2 c \eta_2}{a_2 c \theta_2, s \eta_2} & -s\theta_2 & c\theta_2 s \eta_2 \\
\frac{s\theta_2 c \eta_2}{a_2 c \theta_2, s \eta_2} & c\theta_2 & s\theta_2 s \eta_2 \\
0 & 0 & c \eta_2
\end{bmatrix} + \varepsilon
\begin{bmatrix}
0 & a_2 c \theta_2, s \eta_2 & a_2 s \theta_2 \\
0 & a_2 s \theta_2, s \eta_2 & -a_2 c \theta_2 \\
0 & a_2 c \eta_2 & 0
\end{bmatrix}
\]

and for the slider, link 3, where lengths \(s_3=0, e_3=0, \alpha_3=0\) and twist angle \(\alpha_3=\pi/2\), we have

\[\begin{bmatrix}
\frac{c \theta_3 c \eta_3}{s \theta_3 c \eta_3} & c \theta_3 s \eta_3 & s \theta_3 \\
\frac{s \theta_3 c \eta_3}{s \theta_3 s \eta_3} & s \theta_3 s \eta_3 & -c \theta_3 \\
0 & c \eta_3 & 0
\end{bmatrix}
\]

All the joint variables, their derivatives and their second derivatives were developed by Fischer and Rahman [11].
2.2 Replacement of the Mass Distribution of a Link by Four Point Masses

The mass distribution of each moving link of the mechanism can be represented by four point masses [18, 38]. Magnitudes and the locations of the point masses are determined on the basis of dynamical equivalence. For systems to be dynamically equivalent they must have the same mass, the same center of mass, the same principal axes and same principal moments of inertia about the center of mass. The details of the replacement of the mass distribution of a moving link by four point masses is described below.

Let symbol \( m \) denotes the mass of the moving link and symbols \( I_{XX}, I_{YY} \) and \( I_{ZZ} \) the moments of inertia of the moving link about principal axes through the center of mass.

Let symbols \( m_1, m_2, m_3 \) and \( m_4 \) represent the point masses equivalent to the mass distribution of the moving link. In order to have same center of mass before and after mass distribution we use half point masses, each placed on the negative and positive side.

![Diagram of the mass distribution of the moving link](image)

**Figure 2.3** The mass distribution of the moving link

Let symbols \( m_1, m_2, m_3 \) and \( m_4 \) represent the point masses equivalent to the mass distribution of the moving link. In order to have same center of mass before and after mass distribution we use half point masses, each placed on the negative and positive side.
of each of the principal axes at equal distance from the origin. As shown below in figure 2.3, half point masses \( m_{1/2}, m_{2/2} \) and \( m_{3/2} \) are respectively located on the principal axes of the link at distances \( X_1, Y_2 \) and \( Z_3 \) and at distances \(-X_1, -Y_2, -Z_3\) from the center of mass. Mass \( m_4 \) lies at the center of mass of the moving link. We call this orientation a four-point mass system because of the symmetrical nature of location of the masses.

For the two systems to be dynamically equivalent, we obtain the following relations:

\[
m = m_1 + m_2 + m_3 + m_4
\]

\[
\begin{align*}
I_{xx} &= m_2 Y_2^2 + m_3 Z_3^2 \\
I_{yy} &= m_1 X_1^2 + m_3 Z_3^2 \\
I_{zz} &= m_1 X_1^2 + m_2 Y_2^2
\end{align*}
\]

We have four equations with seven unknowns (four point masses \( m_1, m_2, m_3, m_4 \) and three distances \( X_1, Y_2 \) and \( Z_3 \)). A solution of the system can be obtained by selecting values for three of the seven unknowns.

From equations (2.9) we obtain

\[
\begin{align*}
m_1 X_1^2 &= \frac{1}{2} (I_{xx} - I_{yy} + I_{zz}) \\
m_2 Y_2^2 &= \frac{1}{2} (I_{xx} - I_{yy} + I_{zz}) \\
m_3 Z_3^2 &= \frac{1}{2} (I_{xx} + I_{yy} - I_{zz})
\end{align*}
\]

If we assume values for \( m_1, m_2 \) and \( m_3 \) then equation (2.10) determines \( X_1, Y_2 \) and \( Z_3 \) and vice versa.

Let

\[
m_1 = m_2 = m_3 = \frac{m}{4}
\]
Then from equation (2.10) we obtain
\[
\begin{align*}
X_1 &= \sqrt{\frac{2}{m} (-I_{xx} + I_{yy} + I_{zz})} \\
Y_2 &= \sqrt{\frac{2}{m} (I_{xx} - I_{yy} + I_{zz})} \\
Z_3 &= \sqrt{\frac{2}{m} (I_{xx} + I_{yy} - I_{zz})}
\end{align*}
\]
and from equation (2.8)
\[
m_4 = \frac{m}{4}
\] (2.13)

The mass distribution of the crank, link 1, is replaced by four point masses \(m_{11}, m_{12}, m_{13}\) and \(m_{14}\). Point mass \(m_{14}\) lies at the center of mass of the crank and the half point masses \(m_{11}/2\), \(m_{12}/2\) and \(m_{13}/2\) lie on the negative and positive side of the principal axes attached to the crank at distances \(l_{11}\), \(l_{12}\) and \(l_{13}\), respectively from its center of mass. The values of point masses \(m_{11}, m_{12}, m_{13}\) and \(m_{14}\) and distances \(l_{11}, l_{12}\) and \(l_{13}\) can be evaluated from the equations (2.11), (2.12), and (2.13) for the known values of moments of inertia of the crank.

The mass distribution of the connecting rod is replaced by four point masses \(m_{21}, m_{22}, m_{23}\) and \(m_{24}\). Point mass \(m_{24}\) lies at the center of mass of the connecting rod and the half point masses \(m_{21}/2\), \(m_{22}/2\) and \(m_{23}/2\) lie on the negative and positive side of the principal axes attached to the connecting rod at distances \(l_{21}, l_{22}\) and \(l_{23}\), respectively from its center of mass. The values of point masses \(m_{21}, m_{22}, m_{23}\) and \(m_{24}\) and distances \(l_{21}, l_{22}\) and \(l_{23}\) can be evaluated from the equations (2.11), (2.12) and (2.13) for the known values of moments of inertia of the connecting rod.

The mass distribution of the piston is replaced by a single point mass, \(m_4\). Point mass \(m_4\) lies at the center of the mass of the piston.
2.3 Kinematics of the Point Masses

Now for each moving link, four point masses and their locations with respect to center of mass can be computed using equations (2.11), (2.12) and (2.13). Having done that, the equations for position, velocity and acceleration of each point mass with respect to fixed coordinate frame can be formulated by the following method.

![Figure 2.4 Distance between two frames \(\{A\}\) and \(\{B\}\)](image)

If the transformation matrix \(\hat{T}_{p}\) describes unit vectors of frame \(\{B\}\) in terms of the unit vectors of frame \(\{A\}\) as shown in figure 2.4 and can be decomposed into

\[
\hat{T}_{p} = T_{p} + eT_{d}
\]  \hspace{1cm} (2.14)

then using the "principle of transference" developed by Hsia and Yang [21] the location of distal coordinate frame on each moving link with respect to the fixed frame can be found:

\[
[D] = [T_{p}]^T
\]  \hspace{1cm} (2.15)
Where the vector $D$ is the $3 \times 1$ primary-number column matrix describing displacement of origin of frame $\{B\}$ relative to origin of frame $\{A\}$ such that the displacement matrix

$$
[D] = \begin{bmatrix}
0 & -d_3 & d_2 \\
d_3 & 0 & -d_1 \\
-d_2 & d_1 & 0
\end{bmatrix}
$$

(2.16)

is the $3 \times 3$ form of vector

$$
D = \begin{bmatrix}
d_1 \\
d_2 \\
d_3
\end{bmatrix}
$$

(2.17)

Figure 2.5 Location of distal frame $\{n+1\}$ of the n-th link and four-point mass system of the moving link n with respect to fixed frame $\{1\}$. 
As shown in figure 2.5 let the position vector of the distal frame \{n\} relative to the fixed frame \{1\} be

\[
^1D^{n+1} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}
\]  
(2.18)

The time derivative of the position vector, i.e. the velocity vector, is

\[
^1\dot{D}^{n+1} = \begin{bmatrix} \dot{d}_1 \\ \dot{d}_2 \\ \dot{d}_3 \end{bmatrix}
\]  
(2.19)

The second time derivative of the position vector, i.e. the acceleration vector, is

\[
^1\ddot{D}^{n+1} = \begin{bmatrix} \ddot{d}_1 \\ \ddot{d}_2 \\ \ddot{d}_3 \end{bmatrix}
\]  
(2.20)

If the coordinates representing the location of center of mass of the moving link with respect to the distal coordinate frame\{n+1\} is

\[
^{n+1}\chi_{n-cm} = \begin{bmatrix} X_0 \\ Y_0 \\ Z_0 \end{bmatrix}
\]  
(2.21)

and the direction cosines between the centroidal principal coordinate frame\{n-cm\} and the distal frame\{n+1\} are represented by the matrix

\[
^{n+1}L_{n-cm} = \begin{bmatrix} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{23} \\ L_{31} & L_{32} & L_{33} \end{bmatrix}
\]  
(2.22)

then the following derivations give the location vectors of the point masses with respect to the distal coordinate frame\{n+1\}. 
Let us consider the point masses $m_I$. The vector locating the half point mass $m_I/2$, placed on the positive side of X-axis, relative to frame \{n-cm\} is

$$ n_{-cm}p_{1, P} = \begin{pmatrix} X_1 \\ 0 \\ 0 \end{pmatrix} \quad (2.23a) $$

$$ n_{+1}l_{1, P} = n_{+1}p_{n-cm} + n_{+1}L_{n-cm}p_{1, P} = \begin{pmatrix} X_0 \\ Y_0 \\ Z_0 \end{pmatrix} + \begin{bmatrix} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{23} \\ L_{31} & L_{32} & L_{33} \end{bmatrix} \begin{pmatrix} X_1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} X_0 + X_1L_{11} \\ Y_0 + X_1L_{21} \\ Z_0 + X_1L_{31} \end{pmatrix} \quad (2.24a) $$

Abbreviations $P$ and $N$ respectively represent positive and negative.

The vector locating the half point mass $m_I/2$, placed on the negative side of X-axis, relative to frame \{n-cm\} is

$$ n_{-cm}p_{1, N} = \begin{pmatrix} -X_1 \\ 0 \\ 0 \end{pmatrix} \quad (2.23b) $$

$$ n_{+1}l_{1, N} = n_{+1}p_{n-cm} + n_{+1}L_{n-cm}p_{1, N} = \begin{pmatrix} X_0 \\ Y_0 \\ Z_0 \end{pmatrix} + \begin{bmatrix} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{23} \\ L_{31} & L_{32} & L_{33} \end{bmatrix} \begin{pmatrix} -X_1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} X_0 - X_1L_{11} \\ Y_0 - X_1L_{21} \\ Z_0 - X_1L_{31} \end{pmatrix} \quad (2.24b) $$

We now consider the point masses $m_2$. The vector locating the half point mass $m_2/2$, placed on the positive side of Y-axis, relative to frame \{n-cm\} is

$$ n_{-cm}p_{2, P} = \begin{pmatrix} 0 \\ Y_2 \\ 0 \end{pmatrix} \quad (2.25a) $$

$$ n_{+1}l_{2, P} = n_{+1}p_{n-cm} + n_{+1}L_{n-cm}p_{2, P} = \begin{pmatrix} X_0 \\ Y_0 \\ Z_0 \end{pmatrix} + \begin{bmatrix} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{23} \\ L_{31} & L_{32} & L_{33} \end{bmatrix} \begin{pmatrix} 0 \\ Y_2 \\ 0 \end{pmatrix} = \begin{pmatrix} X_0 + Y_2L_{12} \\ Y_0 + Y_2L_{22} \\ Z_0 + Y_2L_{32} \end{pmatrix} \quad (2.26a) $$
The vector locating the half point mass $m_2/2$, placed on the negative side of Y-axis, relative to frame \( \{n\text{-cm}\} \) is

\[
\begin{bmatrix}
  n\text{-cm}p^{2,N} =
  \begin{bmatrix}
    0 \\
    -Y_2 \\
    0
  \end{bmatrix}
\end{bmatrix}
\]

(2.25b)

\[
\begin{bmatrix}
  n\text{-cm}p^{2,N} =
  \begin{bmatrix}
    L_{11} & L_{12} & L_{13} \\
    L_{21} & L_{22} & L_{23} \\
    L_{31} & L_{32} & L_{33}
  \end{bmatrix}
  \begin{bmatrix}
    X_0 - Y_2 L_{13} \\
    Y_0 - Y_2 L_{22} \\
    Z_0 - Y_2 L_{32}
  \end{bmatrix}
\end{bmatrix}
\]

(2.26b)

Let us now consider the point masses $m_3$. The vector locating the half point mass $m_3/2$, placed on the positive side of Z-axis, relative to frame \( \{n\text{-cm}\} \) is

\[
\begin{bmatrix}
  n\text{-cm}p^{3,P} =
  \begin{bmatrix}
    0 \\
    0 \\
    Z_3
  \end{bmatrix}
\end{bmatrix}
\]

(2.27a)

\[
\begin{bmatrix}
  n\text{-cm}p^{3,P} =
  \begin{bmatrix}
    L_{11} & L_{12} & L_{13} \\
    L_{21} & L_{22} & L_{23} \\
    L_{31} & L_{32} & L_{33}
  \end{bmatrix}
  \begin{bmatrix}
    X_0 + Z_3 L_{13} \\
    Y_0 + Z_3 L_{23} \\
    Z_0 + Z_3 L_{33}
  \end{bmatrix}
\end{bmatrix}
\]

(2.28a)

The vector locating the half point mass $m_3/2$, placed on the negative side of Z-axis, relative to frame \( \{n\text{-cm}\} \) is

\[
\begin{bmatrix}
  n\text{-cm}p^{3,N} =
  \begin{bmatrix}
    0 \\
    0 \\
    -Z_3
  \end{bmatrix}
\end{bmatrix}
\]

(2.27b)

\[
\begin{bmatrix}
  n\text{-cm}p^{3,N} =
  \begin{bmatrix}
    L_{11} & L_{12} & L_{13} \\
    L_{21} & L_{22} & L_{23} \\
    L_{31} & L_{32} & L_{33}
  \end{bmatrix}
  \begin{bmatrix}
    X_0 - Z_3 L_{13} \\
    Y_0 - Z_3 L_{23} \\
    Z_0 - Z_3 L_{33}
  \end{bmatrix}
\end{bmatrix}
\]

(2.28b)
We consider now point mass $m_4$. The vector locating the point mass $m_4$ relative to frame \{n-cm\} is

\[
_{n-cm}P^4 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\]  

(2.29)

\[
_{n+1}l_4 = \begin{bmatrix} X_0 \\ Y_0 \\ Z_0 \end{bmatrix}
\]  

(2.30)

The position vectors of the point masses $m_1, m_2, m_3$ and $m_4$ with respect to frame \{1\} are respectively,

\[
\{L_{1,P}\} = \{D\} + \left[_{n+1}T_R\right]_{n+1}l_{1,P}
\]  

(2.31a)

\[
\{L_{1,N}\} = \{D\} + \left[_{n+1}T_R\right]_{n+1}l_{1,N}
\]  

(2.31b)

\[
\{L_{2,P}\} = \{D\} + \left[_{n+1}T_R\right]_{n+1}l_{2,P}
\]  

(2.31c)

\[
\{L_{2,N}\} = \{D\} + \left[_{n+1}T_R\right]_{n+1}l_{2,N}
\]  

(2.31d)

\[
\{L_{3,P}\} = \{D\} + \left[_{n+1}T_R\right]_{n+1}l_{3,P}
\]  

(2.31e)

\[
\{L_{3,N}\} = \{D\} + \left[_{n+1}T_R\right]_{n+1}l_{3,N}
\]  

(2.31f)

\[
\{L_4\} = \{D\} + \left[_{n+1}T_R\right]_{n+1}l_4
\]  

(2.31g)
As shown in figure 2.6, symbol $l_i$ ($i=1, 2, 3$ and $4$) denotes the position vectors of the moving masses $m_i$ ($i=1, 2, 3$ and $4$) which are at rest in distal coordinate system $\{n+1\}$ which is moving (translating and rotating) relative to inertial coordinate system $\{1\}$. Symbol $L_i$ ($i=1, 2, 3$ and $4$) denotes the position vectors of the moving masses $m_i$ ($i=1, 2, 3$ and $4$) relative to inertial coordinate system $\{1\}$. The acceleration of a moving mass $m_i$ relative to coordinate frame $\{1\}$ can be expressed as

$$a = \ddot{l}_i + 2\omega \times \dot{l}_i + \omega \times (\omega \times l_i) + \frac{d\omega}{dt} \times l_i + \ddot{D}$$

(2.32)

where symbol $\omega$ is the angular velocity vector of the coordinate system $\{X_{n+1}Y_{n+1}Z_{n+1}\}$ with respect to fixed coordinate system $\{X_1Y_1Z_1\}$ (see texts such as Fu, Gonzalez and Lee [14]). The first term on the right-hand side of the equation (2.32) is the acceleration relative to the coordinate system $\{X_{n+1}Y_{n+1}Z_{n+1}\}$. The second term is
called the Coriolis acceleration. The third term is called the centripetal (toward the center) acceleration. The fourth term points directly toward and perpendicular to the axis of rotation. The last term is the linear acceleration of the frame \( \{n+1\} \) relative to the inertial frame \( \{1\} \). The \( 3 \times 3 \) skew-symmetric matrix expression of vector \( \omega \) can be found from the following equation as derived in Nikravesh [32].

\[
\bar{\omega} = \left[ \begin{array}{c}
I_{R}^{T}R_{n+1}^{T} \end{array} \right] = \left[ \begin{array}{ccc}
0 & -\omega_z & \omega_y \\
\omega_z & 0 & -\omega_x \\
-\omega_y & \omega_x & 0 \\
\end{array} \right]
\]  
(2.33)

where \( [R_{n+1}^{T}] \) and \( [R_{n+1}^{T}'] \) are respectively the rotation matrix and its derivative with respect to time. The components of vector \( \omega \) may be expressed as

\[
\omega = \begin{bmatrix}
\omega_x \\
\omega_y \\
\omega_z \\
\end{bmatrix}
\]  
(2.33)

### 2.3.1 Kinematics of the Crank

A crank of mass \( M_1 \) is replaced by four equal point masses, \( m_{11}, m_{12}, m_{13} \) and \( m_{14} \), such that

\[
m_{11} = m_{12} = m_{13} = m_{14} = \frac{M_1}{4}
\]  
(2.35)

The position vector of the point mass \( m_{11} \) is

\[
1-\cos p^{11} = \begin{bmatrix}
X_{11} \\
0 \\
0 \\
\end{bmatrix}
\]  
(2.36)

where

\[
X_{11} = \frac{1}{\sqrt{2m_{11}}}(I_{1y} + I_{1z} - I_{1x})
\]  
(2.37)
The position vector of the point mass $m_{I2}$ is

$$1-cm p^{12} = \begin{pmatrix} 0 \\ Y_{12} \\ 0 \end{pmatrix}$$  \hspace{1cm} (2.38)$$

where

$$Y_{12} = \frac{1}{\sqrt{2m_{12}}} (I_{Iz} + I_{Iy} - I_{Iy})$$  \hspace{1cm} (2.39)$$

The position vector of the point mass $m_{I3}$ is

$$1-cm p^{13} = \begin{pmatrix} 0 \\ 0 \\ Z_{13} \end{pmatrix}$$  \hspace{1cm} (2.40)$$

where

$$Z_{13} = \frac{1}{\sqrt{2m_{13}}} (I_{Ix} + I_{Iy} - I_{Iz})$$  \hspace{1cm} (2.41)$$

Mass $m_{I4}$ is located at the center of mass of the crank, so that

$$1-cm p^{14} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$  \hspace{1cm} (2.42)$$

As shown in figure 2.7, using the "principle of transference" we can find the location of the origin of the distal coordinate frame \{2\} with respect to the fixed frame \{1\}.
Figure 2.7 Crank is replaced by four point masses; $D_1$ is the position vector representing distance of distal frame {2} of the crank from frame {1}.

\[
[D_1] = [^2_1M_2][^1_2M_8]^T
\]

\[
= \begin{bmatrix}
-S_1 s\theta_1 & -S_1 c\theta_1 & a_1 s\theta_1 & c\theta_1 & s\theta_1 & 0 \\
S_1 c\theta_1 & -S_1 s\theta_1 & -a_1 c\theta_1 & -s\theta_1 & c\theta_1 & 0 \\
0 & a_1 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

\[
= \begin{bmatrix}
0 & -S_1 & a_1 s\theta_1 \\
S_1 & 0 & -a_1 c\theta_1 \\
-a_1 s\theta_1 & a_1 c\theta_1 & 0
\end{bmatrix}
\]

(2.43)

Thus, the position vector $\{D_1\}$ representing the distance from origin of fixed frame {1} to the origin of the coordinate frame at point 2 on the crank can be written as

\[
\{D_1\} = \begin{bmatrix}
a_1 c\theta_1 \\
a_1 s\theta_1 \\
S_1
\end{bmatrix}
\]

(2.44)
The time derivative of the position vector, i.e. the velocity vector, is

$$\{\dot{D}_i\} = \begin{cases} -a_i s \theta_i \dot{\theta}_i \\ a_i c \theta_i \dot{\theta}_i \\ \ddot{S}_i \end{cases}$$

(2.45)

The second time derivative of the position vector, i.e. the acceleration vector, is

$$\{\ddot{D}_i\} = \begin{cases} -a_i (c \theta_i \dot{\theta}_i^2 + s \theta_i \ddot{\theta}_i) \\ a_i (c \theta_i \dot{\theta}_i - s \theta_i \ddot{\theta}_i) \\ \dddot{S}_i \end{cases}$$

(2.46)

The vector representing the location of the center of mass of the crank with respect to the distal coordinate frame \{2\} is

$$^2r^{1-cm} = \begin{cases} X_{10} \\ Y_{10} \\ Z_{10} \end{cases}$$

(2.47)

and the direction cosines between the centrioidal principal coordinate frame \{1-cm\} and the distal coordinate frame \{2\} are represented by the matrix

$$^1_{1-cm}L = \begin{bmatrix} L_{1,11} & L_{1,12} & L_{1,13} \\ L_{1,21} & L_{1,22} & L_{1,23} \\ L_{1,31} & L_{1,32} & L_{1,33} \end{bmatrix}$$

(2.48)

Point masses \(m_{1L}, m_{12}, m_{13}\) and \(m_{14}\) are located on the principal axes \(X_1, Y_1, Z_1\) and origin of the centroidal coordinate frame respectively. Therefore, coordinates of half point masses \(m_{1L}/2, m_{12}/2, m_{13}/2\) and point mass \(m_{14}\) are respectively \((\pm X_{1L},0,0)\), \((0,\pm Y_{12},0)\), \((0,0,\pm Z_{13})\) and \((0,0,0)\).
The location of half point mass $m_{11}/2$, placed on the positive side of X-axis, with respect to the distal coordinate frame $\{2\}$ is

\[
\begin{bmatrix}
2d_{11,p} \end{bmatrix} = -r^{1-c_m} + c_m L^1_{m} e^{p_{11},p} = \begin{bmatrix} X_{10} \\ Y_{10} \\ Z_{10} \end{bmatrix} + \begin{bmatrix} L_{1,11} & L_{1,12} & L_{1,13} \\ L_{1,21} & L_{1,22} & L_{1,23} \\ L_{1,31} & L_{1,32} & L_{1,33} \end{bmatrix} \begin{bmatrix} X_{11} \\ 0 \\ 0 \end{bmatrix}
\]

\[
= \begin{bmatrix} X_{10} + L_{1,11} X_{11} \\ Y_{10} + L_{1,21} X_{11} \\ Z_{10} + L_{1,31} X_{11} \end{bmatrix} \tag{2.49a}
\]

Similarly, the location of half point mass $m_{11}/2$, placed on the negative side of X-axis, with respect to the distal coordinate frame $\{2\}$ is

\[
\begin{bmatrix}
2d_{11,N} \end{bmatrix} = \begin{bmatrix} X_{10} - L_{1,11} X_{11} \\ Y_{10} - L_{1,21} X_{11} \\ Z_{10} - L_{1,31} X_{11} \end{bmatrix} \tag{2.49b}
\]

The location of half point mass $m_{12}/2$, placed on the positive side of Y-axis, with respect to the distal coordinate frame $\{2\}$ is

\[
\begin{bmatrix}
2d_{12,p} \end{bmatrix} = -r^{1-c_m} + c_m L^1_{m} e^{p_{12},p} = \begin{bmatrix} X_{10} \\ Y_{10} \\ Z_{10} \end{bmatrix} + \begin{bmatrix} L_{1,11} & L_{1,12} & L_{1,13} \\ L_{1,21} & L_{1,22} & L_{1,23} \\ L_{1,31} & L_{1,32} & L_{1,33} \end{bmatrix} \begin{bmatrix} 0 \\ Y_{12} \\ 0 \end{bmatrix}
\]

\[
= \begin{bmatrix} X_{10} + L_{1,12} Y_{12} \\ Y_{10} + L_{1,22} Y_{12} \\ Z_{10} + L_{1,32} Y_{12} \end{bmatrix} \tag{2.50a}
\]
Similarly, the location of half point mass $m_{I2}/2$, placed on the negative side of Y-axis, with respect to the distal coordinate frame \{2\} is

$$\{^{2d}_{12,N}\} = \begin{bmatrix} X_{10} - L_{1,12}Y_{12} \\ Y_{10} - L_{1,22}Y_{12} \\ Z_{10} - L_{1,32}Y_{12} \end{bmatrix}$$

(2.50b)

The location of half point mass $m_{I3}/2$, placed on the positive side of Z-axis, with respect to the distal coordinate frame \{2\} is

$$\{^{2d}_{13,P}\} = r_{1-cm}L_{13,P}^{1,cm} + r_{1-cm}^{2,cm}L_{13,P}^{1,cm} = \begin{bmatrix} X_{10} \\ Y_{10} \\ Z_{10} \end{bmatrix} + \begin{bmatrix} L_{1,11} & L_{1,12} & L_{1,13} \\ L_{1,21} & L_{1,22} & L_{1,23} \\ L_{1,31} & L_{1,32} & L_{1,33} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ Z_{13} \end{bmatrix}$$

(2.51a)

Similarly, the location of half point mass $m_{I3}/2$, placed on the negative side of Z-axis, with respect to the distal coordinate frame \{2\} is

$$\{^{2d}_{13,N}\} = \begin{bmatrix} X_{10} + L_{1,13}Z_{13} \\ Y_{10} + L_{1,23}Z_{13} \\ Z_{10} + L_{1,33}Z_{13} \end{bmatrix}$$

(2.51b)

The location of point mass $m_{I4}$ with respect to the distal coordinate frame \{2\} is

$$\{^{2d}_{14}\} = r_{1-cm}X_{14} + r_{1-cm}^{2,cm}X_{14} = \begin{bmatrix} X_{10} \\ Y_{10} \\ Z_{10} \end{bmatrix}$$

(2.52)
The position vectors of the half point masses $m_{1/2}$, $m_{1/2}$, $m_{1/2}$ and point mass $m_{14}$ with respect to frame $\{1\}$ are respectively,

\[
\begin{align*}
\{D_{11,p}\} &= \{D_1\} + \left[\frac{1}{2} T_R^T\right] \{d_{11,p}\} \\
\{D_{11,N}\} &= \{D_1\} + \left[\frac{1}{2} T_R^T\right] \{d_{11,N}\} \\
\{D_{12,p}\} &= \{D_1\} + \left[\frac{1}{2} T_R^T\right] \{d_{12,p}\} \\
\{D_{12,N}\} &= \{D_1\} + \left[\frac{1}{2} T_R^T\right] \{d_{12,N}\} \\
\{D_{13,p}\} &= \{D_1\} + \left[\frac{1}{2} T_R^T\right] \{d_{13,p}\} \\
\{D_{13,N}\} &= \{D_1\} + \left[\frac{1}{2} T_R^T\right] \{d_{13,N}\} \\
\{D_{14}\} &= \{D_1\} + \left[\frac{1}{2} T_R^T\right] \{d_{14}\}
\end{align*}
\] (2.53a-g)

The acceleration of half point masses $m_{1/2}$, $m_{1/2}$, $m_{1/2}$ and point mass $m_{14}$ can respectively be written as follows. Since they are stationary with respect to distal coordinate frame $\{2\}$, first and second time derivatives of their distances from $\{2\}$ are zero.

\[
\begin{align*}
a_{11,p} &= \omega_1 \times (\omega_1 \times d_{11,p}) + \frac{d\omega_1}{dt} \times d_{11,p} + \ddot{D}_1 \\
a_{11,N} &= \omega_1 \times (\omega_1 \times d_{11,N}) + \frac{d\omega_1}{dt} \times d_{11,N} + \ddot{D}_1 \\
a_{12,p} &= \omega_1 \times (\omega_1 \times d_{12,p}) + \frac{d\omega_1}{dt} \times d_{12,p} + \ddot{D}_1 \\
a_{12,N} &= \omega_1 \times (\omega_1 \times d_{12,N}) + \frac{d\omega_1}{dt} \times d_{12,N} + \ddot{D}_1 \\
a_{13,p} &= \omega_1 \times (\omega_1 \times d_{13,p}) + \frac{d\omega_1}{dt} \times d_{13,p} + \ddot{D}_1 \\
a_{13,N} &= \omega_1 \times (\omega_1 \times d_{13,N}) + \frac{d\omega_1}{dt} \times d_{13,N} + \ddot{D}_1 \\
a_{14} &= \omega_1 \times (\omega_1 \times d_{14}) + \frac{d\omega_1}{dt} \times d_{14} + \ddot{D}_1
\end{align*}
\] (2.54a-g)
where vector \( \omega_1 = \begin{bmatrix} \omega_{1x} \\ \omega_{1y} \\ \omega_{1z} \end{bmatrix} \) represents the angular velocity of frame \{2\} with respect to frame \{1\} and as previously discussed, the elements of this vector can be determined using the relationship

\[
\omega_1 = \omega_1 = \begin{bmatrix} -s_\theta \dot{\theta}_1 & -c_\theta \dot{\theta}_1 & 0 \\ c_\theta \dot{\theta}_1 & -s_\theta \dot{\theta}_1 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\]

(2.55)

where

\[
\omega_1 = \begin{bmatrix} 0 & -\omega_{1z} & \omega_{1y} \\ -\omega_{1z} & 0 & -\omega_{1x} \\ -\omega_{1y} & \omega_{1z} & 0 \end{bmatrix}
\]

(2.56)

Since crank is rotating only about the Z-axis, other rotation components are zero.

### 2.3.2 Kinematics of Connecting Rod

A connecting rod of mass \( M_2 \) is replaced by four equal point masses, \( m_{21}, m_{22}, m_{23} \) and \( m_{24} \), such that

\[
m_{21} = m_{22} = m_{23} = m_{24} = \frac{M_2}{4}
\]

(2.57)

The position vector of the point mass \( m_{21} \) is

\[
2-\text{cm}P^{21} = \begin{bmatrix} X_{21} \\ 0 \\ 0 \end{bmatrix}
\]

(2.58)

where

\[
X_{21} = \frac{1}{\sqrt{2m_{21}}} \left( I_{2y} + I_{2z} - I_{2x} \right)
\]

(2.59)
The position vector of the point mass \( m_{22} \) is

\[
2-\text{cm} \mathbf{p}^{22} = \begin{bmatrix} 0 \\ Y_{22} \\ 0 \end{bmatrix}
\]

(2.60)

where

\[
Y_{22} = \sqrt{\frac{1}{2m_{22}} (I_{2x} + I_{2z} - I_{2y})}
\]

(2.61)

The position vector of the point mass \( m_{23} \) is

\[
2-\text{cm} \mathbf{p}^{23} = \begin{bmatrix} 0 \\ 0 \\ Z_{23} \end{bmatrix}
\]

(2.62)

where

\[
Z_{23} = \sqrt{\frac{1}{2m_{23}} (I_{2x} + I_{2y} - I_{2z})}
\]

(2.63)

Mass \( m_{24} \) is located at the center of mass of the connecting rod.

\[
2-\text{cm} \mathbf{p}^{24} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\]

(2.64)

First the position vectors from the origin of the fixed coordinate frame at point 1 to the moving point masses \( m_{21}, m_{22}, m_{23}, m_{24} \) will be formulated. Since point 3 on the connecting rod coincides with the point 4 of slider, the position of the distal coordinate frame on the connecting rod can be obtained from matrix \( {^d\tilde{\mathcal{M}}} \).
Figure 2.8  Connecting rod is replaced by four point masses; \( \mathbf{D}_2 \) is the position vector representing distance of distal frame \( \{3\} \) of the connecting rod from frame \( \{1\} \).

Using the "principle of transference" we can find the location of origin of distal coordinate frame \( \{3\} \) with respect to the fixed frame \( \{1\} \).

\[
[D_3] = [M_D][M_R]^T = 
\begin{bmatrix}
    0 & S_4 & 0 \\
    -S_4c\alpha_4 & -a_4s\alpha_4 & a_4c\alpha_4 \\
    S_4s\alpha_4 & -a_4c\alpha_4 & -a_4s\alpha_4
\end{bmatrix}
\begin{bmatrix}
    1 & 0 & 0 \\
    0 & c\alpha_4 & s\alpha_4 \\
    0 & -s\alpha_4 & c\alpha_4
\end{bmatrix}
\]

\[
= 
\begin{bmatrix}
    0 & S_4c\alpha_4 & -S_4s\alpha_4 \\
    -S_4c\alpha_4 & 0 & a_4 \\
    S_4s\alpha_4 & -a_4 & 0
\end{bmatrix}
\]  
(2.65)

Position vector \( \{D_2\} \) representing the distance from origin of fixed frame \( \{1\} \) to the origin of the coordinate frame at point 3 on the connecting rod or at point 4 on the slider can be written as

\[
\{D_2\} = 
\begin{bmatrix}
    -a_4 \\
    -S_4s\alpha_4 \\
    -S_4c\alpha_4
\end{bmatrix}
\]  
(2.66)
The velocity vector of the origin of the coordinate frame at point 3 with respect to
the fixed frame \{1\} can be obtained by differentiating equation (2.66) with respect to
time.

Therefore,

\[
\{ \dot{D}_3 \} = \begin{bmatrix} 0 \\ -\dot{s}_4 s \alpha_4 \\ -\dot{s}_4 c \alpha_4 \end{bmatrix} \tag{2.67}
\]

Differentiating equation (2.67) with respect to time, the acceleration vector can be
obtained.

Therefore,

\[
\{ \ddot{D}_3 \} = \begin{bmatrix} 0 \\ -\ddot{s}_4 s \alpha_4 \\ -\ddot{s}_4 c \alpha_4 \end{bmatrix} \tag{2.68}
\]

The location of the center of mass of the connecting rod with respect to the
coordinate frame \{3\} is represented by vector

\[
\bar{r}_{2-cm} = \begin{bmatrix} X_{20} \\ Y_{20} \\ Z_{20} \end{bmatrix} \tag{2.69}
\]

and the direction cosines between the centroidal principal coordinate frame\{2-cm\} and
frame \{3\} are represented by the matrix

\[
2-cm = \begin{bmatrix} L_{2,11} & L_{2,12} & L_{2,13} \\
L_{2,21} & L_{2,22} & L_{2,23} \\
L_{2,31} & L_{2,32} & L_{2,33} \end{bmatrix} \tag{2.70}
\]

Point masses \(m_{2,1}, m_{2,2}, m_{2,3}\) and \(m_{2,4}\) are located on the principal axes \(X_2, Y_2, Z_2\) and origin of the centroidal coordinate frame respectively. Therefore, coordinates of
half point masses \(m_{2,1}/2, m_{2,2}/2, m_{2,3}/2\) and point mass \(m_{2,4}\) are respectively \((\pm X_{21},0,0)\), \((0,\pm Y_{22},0)\), \((0,0,\pm Z_{23})\) and \((0,0,0)\).
The location of half point mass $m_{21/2}$, placed on the positive side of X-axis, with respect to the distal coordinate frame \{3\} is

$$\{d_{211, P}\} = r^{2-\text{cm}}_{+} + r^{2-\text{cm}}_{-}P_{21, P}$$

$$= \begin{pmatrix} X_{20} \\ Y_{20} \\ Z_{20} \end{pmatrix} + \begin{bmatrix} L_{211} & L_{212} & L_{213} \\ L_{221} & L_{222} & L_{223} \\ L_{231} & L_{232} & L_{233} \end{bmatrix} \begin{pmatrix} X_{21} \\ Y_{21} \\ Z_{21} \end{pmatrix} = \begin{pmatrix} X_{20} + L_{211}X_{21} \\ Y_{20} + L_{221}X_{21} \\ Z_{20} + L_{231}X_{21} \end{pmatrix}$$

(2.71a)

Similarly, the location of half point mass $m_{21/2}$, placed on the negative side of X-axis, with respect to the distal coordinate frame \{3\} is

$$\{d_{21N}\} = \begin{pmatrix} X_{20} - L_{211}X_{21} \\ Y_{20} - L_{221}X_{21} \\ Z_{20} - L_{231}X_{21} \end{pmatrix}$$

(2.71b)

The location of half point mass $m_{22/2}$, placed on the positive side of Y-axis, with respect to the distal coordinate frame \{3\} is

$$\{d_{221, P}\} = r^{2-\text{cm}}_{+} + r^{2-\text{cm}}_{-}P_{22, P}$$

$$= \begin{pmatrix} X_{20} \\ Y_{20} \\ Z_{20} \end{pmatrix} + \begin{bmatrix} L_{211} & L_{212} & L_{213} \\ L_{221} & L_{222} & L_{223} \\ L_{231} & L_{232} & L_{233} \end{bmatrix} \begin{pmatrix} X_{21} \\ Y_{21} \\ Z_{21} \end{pmatrix} = \begin{pmatrix} X_{20} + L_{212}Y_{22} \\ Y_{20} + L_{222}Y_{22} \\ Z_{20} + L_{232}Y_{22} \end{pmatrix}$$

(2.72a)
Similarly, the location of half point mass $m_{22}/2$, placed on the negative side of Y-axis, with respect to the distal coordinate frame \{3\} is

\[
\{\hat{3}d_{22,N}\} = \begin{bmatrix}
X_{20} - L_{2,12}Y_{22} \\
Y_{20} - L_{2,22}Y_{22} \\
Z_{20} - L_{2,32}Y_{22}
\end{bmatrix} \tag{2.72b}
\]

The location of half point mass $m_{23}/2$, placed on the positive side of Z-axis, with respect to the distal coordinate frame \{3\} is

\[
\{\hat{3}d_{23,P}\} = L_{2}^{2-cm} + 2-cmL_{2}^{2-cm}P_{23,P}^{23,P} = \begin{bmatrix}
X_{20} \\
Y_{20} \\
Z_{20}
\end{bmatrix} + \begin{bmatrix}
L_{2,11} & L_{2,12} & L_{2,13} \\
L_{2,21} & L_{2,22} & L_{2,23} \\
L_{2,31} & L_{2,32} & L_{2,33}
\end{bmatrix} \begin{bmatrix}
0 \\
0 \\
Z_{23}
\end{bmatrix}
\]

\[
= \begin{bmatrix}
X_{20} + L_{2,13}Z_{23} \\
Y_{20} + L_{2,23}Z_{23} \\
Z_{20} + L_{2,33}X_{23}
\end{bmatrix} \tag{2.73a}
\]

Similarly, the location of half point mass $m_{23}/2$, placed on the negative side of Z-axis, with respect to the distal coordinate frame \{3\} is

\[
\{\hat{3}d_{23,N}\} = \begin{bmatrix}
X_{20} - L_{2,13}Z_{23} \\
Y_{20} - L_{2,23}Z_{23} \\
Z_{20} - L_{2,33}X_{23}
\end{bmatrix} \tag{2.73b}
\]

The location of point mass $m_{24}$ with respect to the distal coordinate frame \{3\} is

\[
\{\hat{3}d_{24}\} = L_{2}^{2-cm} + 2-cmL_{2}^{2-cm}P_{24}^{2} = \begin{bmatrix}
X_{20} \\
Y_{20} \\
Z_{20}
\end{bmatrix}
\]

\[
\tag{2.74}
\]
The position vectors of the half point masses $m_{21}/2$, $m_{22}/2$, $m_{23}/2$ and point mass $m_{24}$ with respect to frame $\{1\}$ are respectively,

\[
\begin{align*}
\{D_{21,p}\} &= \{D_2\} + \left[ {^3T_R} \right]\{^3d_{21,p}\} \\
\{D_{21,N}\} &= \{D_2\} + \left[ {^3T_R} \right]\{^3d_{21,N}\} \\
\{D_{22,P}\} &= \{D_2\} + \left[ {^3T_R} \right]\{^3d_{22,P}\} \\
\{D_{22,N}\} &= \{D_2\} + \left[ {^3T_R} \right]\{^3d_{22,N}\} \\
\{D_{23,P}\} &= \{D_2\} + \left[ {^3T_R} \right]\{^3d_{23,P}\} \\
\{D_{23,N}\} &= \{D_2\} + \left[ {^3T_R} \right]\{^3d_{23,N}\} \\
\{D_{24}\} &= \{D_2\} + \left[ {^3T_R} \right]\{^3d_{24}\}
\end{align*}
\]

where

\[
\begin{bmatrix}
{^1T_R}\quad {^2T_R}\quad {^3T_R}
\end{bmatrix} =
\begin{bmatrix}
  c\theta_1 & -s\theta_1 & 0 & c\theta_2 c\eta_2 & -s\theta_2 & c\theta_2 s\eta_2 \\
  s\theta_1 & c\theta_1 & 0 & s\theta_2 c\eta_2 & c\theta_2 & s\theta_2 s\eta_2 \\
  0 & 0 & 1 & -s\eta_2 & 0 & c\eta_2
\end{bmatrix}
\]

\[
\begin{bmatrix}
  c\eta_2 c(\theta_1 + \theta_2) & -s(\theta_1 + \theta_2) & s\eta_2 c(\theta_1 + \theta_2) \\
  c\eta_2 s(\theta_1 + \theta_2) & c(\theta_1 + \theta_2) & s\eta_2 s(\theta_1 + \theta_2) \\
  -s\eta_2 & c(\theta_1 + \theta_2) & c\eta_2
\end{bmatrix}
\]

The accelerations of half point masses $m_{21}/2$, $m_{22}/2$, $m_{23}/2$, point mass $m_{24}$ can respectively be written as follows. Since they are stationary with respect to distal coordinate frame $\{3\}$, first and second time derivatives of their distances from $\{3\}$ are zero,

\[
\begin{align*}
a_{21,p} &= \omega_2 \times (\omega_2 \times ^3d_{21,p}) + \frac{d\omega_2}{dt} \times ^3d_{21,p} + \ddot{D}_2 \\
a_{21,N} &= \omega_2 \times (\omega_2 \times ^3d_{21,N}) + \frac{d\omega_2}{dt} \times ^3d_{21,N} + \ddot{D}_2 \\
a_{22,p} &= \omega_2 \times (\omega_2 \times ^3d_{22,p}) + \frac{d\omega_2}{dt} \times ^3d_{22,p} + \ddot{D}_2
\end{align*}
\]
\[ a_{22,N} = \omega_2 \times (\omega_2 \times d_{22,N}) + \frac{d\omega_2}{dt} \times d_{22,N} + \ddot{D}_2 \quad (2.77d) \]

\[ a_{23,p} = \omega_2 \times (\omega_2 \times d_{23,p}) + \frac{d\omega_2}{dt} \times d_{23,p} + \ddot{D}_2 \quad (2.77e) \]

\[ a_{23,N} = \omega_2 \times (\omega_2 \times d_{23,N}) + \frac{d\omega_2}{dt} \times d_{23,N} + \ddot{D}_2 \quad (2.77f) \]

\[ a_{24} = \omega_2 \times (\omega_2 \times d_{24}) + \frac{d\omega_2}{dt} \times d_{24} + \ddot{D}_2 \quad (2.77g) \]

where, \( \omega_2 = \begin{bmatrix} \omega_{2x} \\ \omega_{2y} \\ \omega_{2z} \end{bmatrix} \) represents the angular velocity of the origin of frame \{3\} with respect to frame \{1\} and the elements of this vector can be determined using the relationship

\[ \tilde{\omega}_2 = [I_R^T]^T \tilde{T}_R \]

\[
\begin{bmatrix}
-c \eta_2 s(\theta_1 + \theta_2)(\dot{\theta}_1 + \dot{\theta}_2) - s \eta_2 \dot{\eta}_2 c(\theta_1 + \theta_2) & -c(\theta_1 + \theta_2)(\dot{\theta}_1 + \dot{\theta}_2) \\
-c \eta_2 c(\theta_1 + \theta_2)(\dot{\theta}_1 + \dot{\theta}_2) - s \eta_2 \dot{\eta}_2 s(\theta_1 + \theta_2) & -s(\theta_1 + \theta_2)(\dot{\theta}_1 + \dot{\theta}_2) \\
-c \eta_2 \dot{\eta}_2 & 0 \\
-s \eta_2 s(\theta_1 + \theta_2)(\dot{\theta}_1 + \dot{\theta}_2) + c \eta_2 \dot{\eta}_2 c(\theta_1 + \theta_2) & s \eta_2 c(\theta_1 + \theta_2) & c \eta_2 s(\theta_1 + \theta_2) & -s \eta_2 \\
-s \eta_2 c(\theta_1 + \theta_2)(\dot{\theta}_1 + \dot{\theta}_2) + c \eta_2 \dot{\eta}_2 s(\theta_1 + \theta_2) & -s(\theta_1 + \theta_2) & c(\theta_1 + \theta_2) & 0 \\
-s \eta_2 \dot{\eta}_2 & s \eta_2 c(\theta_1 + \theta_2) & s \eta_2 s(\theta_1 + \theta_2) & c \eta_2 
\end{bmatrix}
\]

(2.78)

where

\[
\tilde{\omega}_2 = \begin{bmatrix} 0 & -\omega_{2z} & \omega_{2y} \\ -\omega_{2z} & 0 & -\omega_{2x} \\ -\omega_{2y} & \omega_{2x} & 0 \end{bmatrix} 
\]

(2.79)
After multiplying the matrices in equation (2.78) and then equating with the matrix in equation (2.79) angular velocity components can be determined.

### 2.3.3 Kinematics of the Slider

The slider motion is a translation along a straight line, no rotation is involved. Therefore, we can consider it as a single point mass.

The position vector locating the center of mass of the slider with respect to the fixed frame \( \{1\} \) is

\[
\{D_3\} = \begin{bmatrix}
-a_4 \\
-s_4 s \alpha_4 \\
-s_4 c \alpha_4
\end{bmatrix}
\]  

(2.80)

The velocity of the slider is

\[
\{\dot{D}_3\} = \begin{bmatrix}
0 \\
-\dot{s}_4 s \alpha_4 \\
-\dot{s}_4 c \alpha_4
\end{bmatrix}
\]  

(2.81)

The acceleration of the slider is

\[
\{\ddot{D}_3\} = \begin{bmatrix}
0 \\
-\ddot{s}_4 s \alpha_4 \\
-\ddot{s}_4 c \alpha_4
\end{bmatrix}
\]  

(2.82)
CHAPTER 3

DYNAMICS OF THE MECHANISM

3.1 Dynamics of the Point Masses

The inertia forces and torques exerted on the frame link by the moving links of the mechanism will be determined. By considering the inertia forces and external forces as applied forces acting on the system it is possible to apply d'Alembert's principle and reduce the analysis to the application of static equilibrium conditions. The mass distribution of the moving links will be replaced by a dynamically equivalent system of point masses. After calculation of their vector coordinates and accelerations, the inertia forces and torques will be obtained as well.

3.1.1 Definition of the Inertia Force

Newton's law of motion for a particle is given by

$$ F = m \ddot{P} $$  \hspace{1cm} (3.1)

where symbol $F$ represents the sum of the external forces acting on the particle, symbol $m$ is its mass and $\ddot{P}$ is the acceleration of the particle with respect to an inertial coordinate system. We can write the above equation in the form

$$ F - m \ddot{P} = 0 $$  \hspace{1cm} (3.2)

If we consider the term $-m \ddot{P}$ to represent the inertia force, then equation (3.2) states that the vector sum of external and internal forces vanishes (d'Alembert's principle).

3.1.2 Definition of the Inertia Torque

If a particle (point mass) moves relative to a fixed point, then the moment of inertia force about the fixed point is given by

$$ T = P \times (-m \ddot{P}) $$  \hspace{1cm} (3.3)
where $T$ is the inertia torque and $P$ is the position vector from the fixed point to the particle $m$.

3.2 Inertia Forces and Torques Exerted by the Moving Links on the Frame

In the CSSP mechanism moving links are crank, connecting rod and slider. When the mechanism runs in high speed the moving links exert huge amount of inertia-induced force and torque on the frame.

![Free body diagram of the crank](image)

**Figure 3.1** Free body diagram of the crank

3.2.1 Inertia Force and Torque Calculation for the Crank

The inertia force of the crank is given by

$$P_i = \frac{m_1 (a_{11,P} + a_{11,N}) + m_2 (a_{12,P} + a_{12,N}) + m_3 (a_{13,P} + a_{13,N})}{2} + m_4 a_{i4}$$  \hspace{1cm} (3.4)

and the inertia torque of the crank is given by

$$T_i = \frac{(D_{11,P} \times m_1 a_{11,P}) + (D_{12,P} \times m_2 a_{12,P}) + (D_{13,P} \times m_3 a_{13,P})}{2}$$

$$+ \frac{(D_{11,N} \times m_1 a_{11,N}) + (D_{12,N} \times m_2 a_{12,N}) + (D_{13,N} \times m_3 a_{13,N})}{2} + (D_{i4} \times m_4 a_{i4})$$  \hspace{1cm} (3.5)
3.2.2 Inertia Force and Torque Calculation for the Connecting Rod

The inertia force of the connecting rod is given by

\[ P_z = \frac{m_{21}(a_{21,P} + a_{21,N}) + m_{22}(a_{22,P} + a_{22,N}) + m_{23}(a_{23,P} + a_{23,N}) + m_{24}a_{24}}{2} \]  

(3.6)

The inertia torque of the connecting rod is given by

\[ T_z = \frac{(D_{21,P} \times m_{21}a_{21,N}) + (D_{22,P} \times m_{22}a_{22,N}) + (D_{23,P} \times m_{23}a_{23,P}) + (D_{24} \times m_{24}a_{24})}{2} \]  

(3.7)

Figure 3.2 Free body diagram of the connecting rod

Figure 3.3 Free body diagram of the slider (one point mass)
3.2.3 Inertia Force and Torque Calculation for the Slider

The inertia force of the slider is

\[ P_3 = m_3 \ddot{D}_3 \]  

(3.8)

The inertia torque of the slider is

\[ T_3 = D_3 \times m_3 \ddot{D}_3 \]  

(3.9)

3.3 Determination of Shaking Force and Shaking Moment

The shaking force is the sum of the forces exerted upon the frame by each moving link

\[
F = P_1 + P_2 + P_3 \\
= m_1 (a_{11,P} + a_{11,N}) + m_2 (a_{12,P} + a_{12,N}) + m_3 (a_{13,P} + a_{13,N}) + m_4 a_{14} \\
= m_21 (a_{21,P} + a_{21,N}) + m_22 (a_{22,P} + a_{22,N}) + m_23 (a_{23,P} + a_{23,N}) + m_24 a_{24} \\
+ m_3 \ddot{D}_3
\]  

(3.10)

The shaking moment is the sum of the torque exerted upon the frame by each moving link

\[
T = T_1 + T_2 + T_3 \\
= \frac{(D_{11,P} \times m_{11,a_{11,P}})}{2} + \frac{(D_{12,P} \times m_{12,a_{12,P}})}{2} + \frac{(D_{13,P} \times m_{13,a_{13,P}})}{2} \\
+ \frac{(D_{11,N} \times m_{11,a_{11,N}})}{2} + \frac{(D_{12,N} \times m_{12,a_{12,N}})}{2} + \frac{(D_{13,N} \times m_{13,a_{13,N}})}{2} + (D_{14} \times m_{14} a_{14}) \\
+ \frac{(D_{21,P} \times m_{21,a_{21,P}})}{2} + \frac{(D_{22,P} \times m_{22,a_{22,P}})}{2} + \frac{(D_{23,P} \times m_{23,a_{23,P}})}{2} \\
+ \frac{(D_{21,N} \times m_{21,a_{21,N}})}{2} + \frac{(D_{22,N} \times m_{22,a_{22,N}})}{2} + \frac{(D_{23,N} \times m_{23,a_{23,N}})}{2} + (D_{24} \times m_{24} a_{24}) \\
+ D_3 \times m_3 \ddot{D}_3
\]  

(3.11)
where all the variables except point masses $m_{11}, m_{12}, m_{13}, m_{14}, m_{21}, m_{22}, m_{23}, m_{24}$, and $m_3$ are known from kinematic analysis.

### 3.4 Determination of Bearing Force and Bearing Moment

The inertia forces and inertia torques of the moving links with respect to the fixed coordinate frame $\{1\}$ were formulated in section 3.2. They are to be converted with respect to the distal coordinate frame of each moving link to determine the bearing reaction forces and moments using the formulas developed by Fischer and Rahman [11, 12]. Then these reaction forces and torques will be expressed with respect to the moving coordinate frames located at the distal end of the moving links. To express those forces and torques with respect to fixed coordinate frame $\{1\}$ we shall premultiply the force and torque vectors by the rotational part of the transformation matrix expressed in terms of frame $\{1\}$. All these operations are mathematically expressed in the following equations.

**Step 1:** Inertia forces and torques are expressed in terms of the distal coordinate frame:

\[
\begin{align*}
\{iF\} &= \left[ T_{12}^T \right] \{P_1\} \\
\{iF\} &= \left[ T_{13}^T \right] \{P_2\} \\
\{iF\} &= \left[ T_{14}^T \right] \{P_1\} \\
\end{align*}
\]

\[\tag{3.12}\]

\[
\begin{align*}
\{iM\} &= \left[ T_{21}^T \right] \{T_1\} \\
\{iM\} &= \left[ T_{31}^T \right] \{T_2\} \\
\{iM\} &= \left[ T_{41}^T \right] \{T_3\} \\
\end{align*}
\]

\[\tag{3.13}\]

**Step 2:** Equations developed by Fischer and Rahman [12] are used to determine the bearing reaction forces $F_1, F_2, F_3$ and $F_4$ and torques $M_1, M_2, M_3$ and $M_4$ in terms of the distal coordinate frames.
Step 3: The bearing reaction forces and torques determined in step 2 in terms of frame \{1\} are

\[
\begin{align*}
\{1F\} &= \{F_1\} \\
\{2F\} &= [1^TR][F_2] \\
\{3F\} &= [3^TR][F_3] \\
\{4F\} &= [4^TR][F_4]
\end{align*}
\] (3.14)

\[
\begin{align*}
\{1M\} &= \{M_1\} \\
\{2M\} &= [1^TR][M_2] \\
\{3M\} &= [3^TR][M_3] \\
\{4M\} &= [4^TR][M_4]
\end{align*}
\] (3.15)

Therefore, the sum of all bearing forces in terms of frame \{1\} can be expressed as

\[
R = \{1F\} + \{2F\} + \{3F\} + \{4F\}
\] (3.16)

Similarly, the sum of all bearing moments in terms of frame \{1\} can be expressed as

\[
M = \{1M\} + \{2M\} + \{3M\} + \{4M\}
\] (3.17)

3.5 Determination of Input Torque

The external torque required to operate the mechanism is the Z-component of moment \{1M\}. Let \(T_0\) represent the input torque. Then

\[
T_0 = \{1M\}_z
\] (3.18)
CHAPTER 4

MINIMIZATION OF INERTIA-INDUCED FORCES IN THE MECHANISM

4.1 Problem Formulation

An objective function is to be formulated for the purpose of minimizing the adverse effect of inertia-induced forces and torques. A quadratic objective function consisting of shaking forces, shaking moments, bearing reaction forces, bearing reaction torques and input torque is minimized by optimum mass redistribution of the links of the mechanism. The objective function involves the sum of the squared non-dimensionalized shaking force, shaking moment, bearing reactions and input torque over one cycle of operation of the mechanism. Then the magnitudes of the active point masses are chosen as design variables, and design constraint equations are formulated which are linear in the point masses.
4.2 Objective Function

Let \( \bar{F} \), \( \bar{T} \), \( \bar{R} \), \( \bar{M} \) and \( \bar{T}_0 \) denote the non-dimensionalized mean squared values of the shaking force, shaking moment, bearing forces, bearing moments and input torque, respectively for one complete cycle of operation (360 degrees rotation of input crank). Then

\[
\bar{F} = \frac{1}{2 \pi m^2 a^2 \theta_1^2} \int_0^{2\pi} (F_i \cdot F_i) d\theta_1
\]

\[
\bar{T} = \frac{1}{2 \pi m^2 a^2 \theta_1^2} \int_0^{2\pi} (T_i \cdot T_i) d\theta_1
\]

\[
\bar{R} = \frac{1}{2 \pi m^2 a^2 \theta_1^2} \int_0^{2\pi} (R_i \cdot R_i + R_2 \cdot R_2 + R_3 \cdot R_3 + R_4 \cdot R_4) d\theta_1
\]

\[
\bar{M} = \frac{1}{2 \pi m^2 a^2 \theta_1^2} \int_0^{2\pi} [(M_i - T_0) \cdot (M_i - T_0) + M_2 \cdot M_2 + M_3 \cdot M_3 + M_4 \cdot M_4] d\theta_1
\]

\[
\bar{T}_0 = \frac{1}{2 \pi m^2 a^2 \theta_1^2} \int_0^{2\pi} (T_0^2) d\theta_1
\]

In this analysis, integration over a complete cycle has been performed numerically by dividing the cycle into \( L \) equal intervals. The non-dimensionalized values then become as follows:

\[
\bar{F} = \frac{1}{L m^2 a^2 \theta_1^2} \sum_{i=1}^{L} (F_i \cdot F_i)
\]

\[
\bar{T} = \frac{1}{L m^2 a^2 \theta_1^2} \sum_{i=1}^{L} (T_i \cdot T_i)
\]

\[
\bar{R} = \frac{1}{L m^2 a^2 \theta_1^2} \sum_{i=1}^{L} [R_i \cdot R_i + R_2 \cdot R_2 + R_3 \cdot R_3 + R_4 \cdot R_4]
\]

\[
\bar{M} = \frac{1}{L m^2 a^2 \theta_1^2} \sum_{i=1}^{L} [(M_i - T_0) \cdot (M_i - T_0) + M_2 \cdot M_2 + M_3 \cdot M_3 + M_4 \cdot M_4]
\]

\[
\bar{T}_0 = \frac{1}{L m^2 a^2 \theta_1^2} \sum_{i=1}^{L} (T_0^2)
\]

where distance \( B_i \) denotes the length of each bearing. Expressions for the shaking force, shaking moment, bearing forces, bearing moments and input torque have been determined earlier.
In the formulation of the objective function, weight factors $W_1$, $W_2$, $W_3$, $W_4$ and $W_5$ are assigned to the shaking force, shaking moment, input torque, bearing force and bearing moment respectively. Weight factors are adjusted according to the designer's will, depending upon different circumstances and applications. Let the symbol $G$ represent the objective function which can be optimized using an IMSL package (described in the appendix B).

$$G = W_1F + W_2T + W_3T_0 + W_4R + W_5M$$

(4.3)

This quadratic function optimization algorithm QPROG is based on M.J.D. Powell's implementation of the Goldfarb and Idnani [17] dual quadratic programming (QP) algorithm for convex QP problems subject to general linear equality/inequality constraints, i.e., a problem of the form

$$\min_{x \in \mathbb{R}^n} \ g^T x + \frac{1}{2} x^T H x$$

subject to

$$A_1 x = b_1$$
$$A_2 x \geq b_2$$

given the vectors $b_1$, $b_2$ and $g$ and the matrices $H$ (Hessian Matrix), $A_1$ and $A_2$. Matrix $H$ is required to be positive definite. In this case, a unique vector $x$ solves the problem or the constraints are inconsistent. If $H$ is not positive definite, a positive definite perturbation of $H$ is used in place of $H$. For more details, see Powell [34, 35].

4.3 Design Variables

The mass distribution of each moving link is replaced by four point masses. One point mass lies at center of mass. The remaining three point masses are termed active point masses. The objective function consists only of active point masses and these will be chosen as design variables. By varying these point masses systematically it is possible to minimize the inertia-induced forces. The column vector of the design variables is then given as follows:
where \( x_1 = m_{11}, \ x_2 = m_{12}, \ x_3 = m_{13}, \ x_4 = m_{14}, \ x_5 = m_{21}, \ x_6 = m_{22}, \ x_7 = m_{23}, \ x_8 = m_{24} \) and \( x_9 = m_3 \).

4.4 Design Constraints

The design constraints will be a set of equations, linear in the design variables, which will allow the point masses to vary within prescribed limits. Each active point mass will be allowed to decrease a certain percentage of its original magnitude. The sum of the optimized active point masses of each link will be kept either less than or equal to the sum of the original active point masses for the same link. The constraint equations are given in their general form. For a particular problem these equations can be modified depending upon the characteristics of the linkage. The constraints are formulated as follows:

\[
\begin{align*}
   x_1 &\geq m_{11} (1 - w') \\
   x_2 &\geq m_{12} (1 - w') \\
   x_3 &\geq m_{13} (1 - w') \\
   x_4 &\geq m_{14} (1 - w') \\
   x_5 &\geq m_{21} (1 - w') \\
   x_6 &\geq m_{22} (1 - w') \\
   x_7 &\geq m_{23} (1 - w') \\
\end{align*}
\]
\begin{align*}
x_9 & \geq m_{2a} (1 - w') \quad (4.5h) \\
x_9 & = m_3 \quad (4.5i) \\
x_1 + x_2 + x_3 & \leq m_{11} + m_{12} + m_{13} \quad (4.5j) \\
x_5 + x_6 + x_7 & \leq m_{21} + m_{22} + m_{23} \quad (4.5k)
\end{align*}

where $0 < w' < 1$. 
CHAPTER 5

RESULTS

5.1 Example
A numerical example will be presented to demonstrate the effectiveness of the balancing method developed in this investigation. The designer has been given enough flexibility to adjust the weight factors involved in the objective function $G$ and to change the magnitude of the point masses of the crank and connecting rod, depending upon different circumstances and applications. The following numerical results are calculated setting all weight factors to 1. In this process the magnitude of the optimized point masses change (decrease or increase) slightly within the limits of the design constraints.

5.2 Dimensions of Example CSSP Mechanism and Other Necessary Data
Length of the crank $a_1 = 2.0$ inches
Mass of the crank $M_1 = 1.9$ lbs
Center of mass is at the midpoint of the crank
Length of the connecting rod $a_2 = 8.0$ inches
Mass of the connecting rod $M_2 = 7.6$ lbs
Center of mass is at the midpoint of the connecting rod
Mass of the slider $m_3 = 6.0$ lbs and its center of mass is at the joint between itself and connecting rod
Offset $a_4 = 1.0$ inch
Offset $s_I = 0.4$ inch
Offset $\alpha_4 = 250$ degrees
Acceleration due to gravity $= 386.4$ inches/second$^2$
Crank speed $= 3000$ RPM
Mass moments of Inertia (lb-sec\(^2\)-inch) of the crank about its center of mass

\[ I_{1XX} = 2.048826 \times 10^{-4}, \ I_{1YY} = 1.741503 \times 10^{-3}, \ I_{1ZZ} = 1.741503 \times 10^{-3} \]

Mass moments of Inertia (lb-sec\(^2\)-inch) of the connecting rod about its center of mass

\[ I_{2XX} = 8.195304 \times 10^{-4}, \ I_{2YY} = 0.105310, \ I_{2ZZ} = 0.105310 \]

Mass moments of Inertia (lb-sec\(^2\)-inch) of the slider about its center of mass

\[ I_{3XX} = 4.20548 \times 10^{-3}, \ I_{3YY} = 4.20548 \times 10^{-3}, \ I_{3ZZ} = 7.76397 \times 10^{-3} \]

The active point masses of the crank are \(m_{11} = m_{12} = m_{13} = m_{14} = 1.9/4\) lbs.

The active point masses of the connecting rod are \(m_{21} = m_{22} = m_{23} = m_{24} = 7.6/4\) lbs.

The magnitude of the active point masses of the crank and connecting rod were allowed to decrease by five percent while the sum of the point masses associated with each moving link was kept constant. The mass of the slider is considered as one point mass and kept constant.

### 5.3 Discussion of Results

Results are given in the form of tables and graphs. The variations of the objective function, shaking force, shaking moment, bearing force, bearing torque and input torque are shown. The improvements in the magnitude of inertia-induced forces are described below in detail.

Comparative values of the mass properties of the moving links before and after optimization, are tabulated in table 5.1. The crank and the connecting rod are modified to match the design characteristics obtained from the optimized values of point masses. It has been found that after optimization \(I_{XX}\) of the crank and connecting rod slightly decreases and \(I_{YY}\) and \(I_{ZZ}\) slightly increase.

Comparative values of the objective functions before and after optimization, are tabulated in table 5.2. Percentage variations of the objective function over a complete rotation of the crank are given in the form of a graph in figure 5.1. As shown in figure 5.1 the decrement of objective function \(G\) varies from +4.59\% to -0.128\%. It increases only
at crank angle 20 degrees. The average optimized value of the objective function has been decreased by 2.283%.

The shaking forces before and after optimization are compared and are tabulated in table 5.3. Percentage variations of the shaking force over a complete rotation of the crank are given in the form of a graph in figure 5.2. As shown in figure 5.2 the decrement of shaking force $F$ varies from $+0.7248\%$ to $+0.732\%$. It never increases and decreases very steadily. The average optimized value of the shaking force has been decreased by 0.728%.

Comparisons of the shaking moments before and after optimization are tabulated in table 5.4. Over a complete rotation of the crank percentage variations of the shaking moment are given in the form of a graph in figure 5.3. As shown in figure 5.3 the decrement of shaking moment $T$ varies from $+22.8\%$ to $-26.355\%$. Most of the time the decrement is in positive direction, it only goes negative a few times. The average optimized value of the shaking moment has been decreased by 5.389%.

Comparative values of the bearing forces before and after optimization, are tabulated in table 5.5. Over a complete rotation of the crank percentage variations of the bearing force are given in the form of a graph in figure 5.4. As shown in figure 5.4 the decrement of bearing force $R$ varies between $+2.936\%$ and $-1.33\%$. It increases only at an angle of 110 degrees. The average optimized value of the bearing force has been decreased by 1.3%.

The bearing moments before and after optimization are tabulated in table 5.6 to demonstrate the comparison. A graph in figure 5.5 shows the percentage variations of the bearing torque over a complete rotation of the crank. It is found that the decrement of bearing torque $M$ varies from $+3.535\%$ to $-0.26\%$. Most of the time the decrement is in positive direction, it only goes negative a few times. The average optimized value of the bearing torque has been decreased by 1.19%.
Comparative values of the input torques before and after optimization, are tabulated in table 5.7. Percentage variations of the input torque over a complete rotation of the crank are given in the form of a graph in figure 5.6. As shown in figure 5.6 the decrement of input torque $T_0$ varies from +35.552% to -2.725%. Most of the time the decrement is in positive direction, it only goes negative a few times. The average optimized value of the input torque has been decreased by 2.157%.

The average values of the objective function, shaking force, shaking moment, bearing force, bearing torque and input torque always decrease after optimization while the mass of the moving links remains unchanged. This demonstrates the effectiveness of this optimum balancing method.

The weight factors used in the objective function are $W_1 = W_2 = W_3 = W_4 = W_5 = 1.0$

<table>
<thead>
<tr>
<th>Mass moment of inertia (lb·sec²-inch)</th>
<th>Before optimization</th>
<th>After optimization</th>
</tr>
</thead>
<tbody>
<tr>
<td>Crank</td>
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<tr>
<td>$I_{XX}$</td>
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<td>1.899999E-03</td>
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<tr>
<td>$I_{ZZ}$</td>
<td>1.741503E-03</td>
<td>1.899999E-03</td>
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<tr>
<td>Connecting rod</td>
<td></td>
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</tr>
<tr>
<td>$I_{XX}$</td>
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Comparative values of the mass properties before and after optimization
Table 5.2

<table>
<thead>
<tr>
<th>Crank angle (degrees)</th>
<th>Before optimization</th>
<th>After optimization</th>
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Comparative values of the objective functions before and after optimization
Table 5.3

<table>
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Comparative values of the bearing forces before and after optimization
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Comparative values of the bearing torques before and after optimization.
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Comparative values of the input torques before and after optimization
Figure 5.1 Variation of Objective Function vs. Crank Angle

Figure 5.2 Variation of Shaking Force vs. Crank Angle
Figure 5.3 Variation of Shaking Moment vs. Crank Angle

Figure 5.4 Variation of Bearing Force vs. Crank Angle
Figure 5.5 Variation of Bearing Torque vs. Crank Angle

Figure 5.6 Variation of Input Torque vs. Crank Angle
CHAPTER 6

CONCLUSION

6.1 Conclusions

The principal objective of this dissertation has been the development of computer-aided design procedures for minimizing the adverse effect of the inertia-induced forces in a high-speed general spatial mechanism by optimum redistribution of the mass of the links.

To achieve this objective, the mass distribution of each moving link has been replaced by a dynamically equivalent system of point masses. Having calculated the vector coordinates with the help of the "principle of transference" and the accelerations, the shaking forces, shaking moments, bearing reactions and input torque are obtained. A quadratic objective function consisting of shaking forces, shaking moments, bearing reactions and input torque is then formulated. This function is generally a convex function. Choosing active point masses as design variables and forming the constraints as linear in the design variables, the optimum mass distribution is obtained using the IMSL routine.

The optimality criterion for the mass distribution of the links by using a quadratic programming technique has been found to be successful in minimizing the inertia-induced forces in a high-speed CSSP mechanism. An average decrease of 2.283% was achieved in the value of the objective function by allowing point masses to decrease up to five percent of their magnitudes while total mass of the links remained constant.

Much work has been conducted by many researchers to balance wide variety of mechanisms. Most of them [1, 4, 9, 22-30, 31, 39, 40, 43, 44] are on complete balancing of planar mechanisms. Relatively little research [2, 6, 7, 46-48] have been done on the complete balancing of spatial mechanisms because of its complicated kinematic and dynamic properties. Complete balancing of shaking forces can be achieved if the center of
mass of the mechanism remains stationary as developed by Berkof. Most of the time a stationary center of mass is obtained by adding counterweights to the moving links or by introducing additional moving elements as for example cams, etc. As a result of that, other dynamic characteristics deteriorate. Therefore, partial balancing techniques by optimum distribution of mass caught many researchers interest.

In the existing literature, optimal-mass distribution criteria have been used to minimize a few of the inertia-induced forces and has been limited to specific mechanisms. The least-square technique developed by Berkof and Lowen [5] for the optimization of the shaking moment of fully force-balanced four-bar linkages is applicable to planar mechanisms only.

Tricamo and Lowen's [41, 42] method for optimization of dynamic reactions such as the bearing force, the input moment and the shaking moment with prescribed maximum shaking force is restricted to only planar mechanisms. Tricamo and Lowen [41] describes a two-counterweight method for partially force balancing a four-bar linkage which allows the realization of a prescribed value for the maximum shaking force anywhere between zero and an inherent upper limit. In that paper it was found that a 50 percent reduction of the shaking force results in small amount of increases in bearing forces, shaking moments and input moments over a considerable portion of the design range. Tricamo and Lowen [42] introduces simultaneous optimization of the maximum values of the bearing forces, input moment and shaking moment of a constant speed four-bar linkage while additionally obtaining a prescribed maximum shaking force. The optimization technique determines the parameters of the three counterweights which must be attached to input link, coupler and output link.

Gill and Freudenstein [15,16] minimized the inertia-induced forces in general spherical four-bar mechanisms. Their method allows an optimum trade-off among shaking forces, shaking moments, bearing reactions and input torque fluctuations; but it is limited
to one particular type i.e. spherical mechanism in which kinematic and dynamic analyses are relatively simple.

Hockey [18, 19] and Sherwood's [36, 37] research on the optimum distribution of mass in the link of a spatial slider crank mechanism and a planar four-bar mechanism dealt with the minimization of fluctuation of kinetic energy and inertia force and torque balance during the motion cycle. Such an analysis is inadequate for high-speed machinery because it does not permit the trade-offs necessary for an effective design.

This is the first analysis which is capable of minimizing combined effects of all the inertia-induced forces and torques in a general spatial mechanism without any restriction, by optimum mass redistribution and allows for trade-offs among different inertia-induced forces and torques. In case of a general spatial mechanism kinematic and dynamic analyses are very complicated. Use of the dual number method and the "principle of transference" make this method generalized so that it can be used to design any spatial mechanism.

The results presented here, it is hoped, will be an aid to practicing engineers in the rational design of high-speed three dimensional mechanisms.

6.2 Future Work

Since this is the most generalized method of balancing of three dimensional mechanism, any other three dimensional mechanisms can be analyzed by similar means. Other types of design constraints, e.g. based on the yield strength of the link may be developed. Computer programs may be developed for synthesizing realistic proportions from optimized point mass distribution.
APPENDIX A

Dual-number Coordinate Transformations

The trigonometric functions of dual angle $\hat{\theta}$ can be obtained by using the Taylor expansion.

\[
\begin{align*}
sin \hat{\theta} &= \sin \theta + \varepsilon s \cos \theta \\
cos \hat{\theta} &= \cos \theta - \varepsilon s \sin \theta \\
tan \hat{\theta} &= \tan \theta + \varepsilon s \sec^2 \theta
\end{align*}
\]  

(A.1)

where $\hat{\theta} = \theta + \varepsilon s$, symbol $\theta$ being the primary component of dual angle $\hat{\theta}$ denotes rotational displacement and symbol $s$ being the dual component denotes translational displacement. All formal operations, except division by pure dual number, of dual numbers are the same as those of ordinary algebra followed by the setting $\varepsilon^2 = \varepsilon^3 = \varepsilon^4 = \ldots = 0$. All identities for ordinary trigonometry hold true for dual angle. Sines and cosines are respectively abbreviated as $s$ and $c$.

Screw motion through dual angle $\hat{\alpha}_n (\hat{\alpha}_n = \alpha_n + \varepsilon \alpha_n)$ about the X axis can be written in a $3 \times 3$ transformation matrix form as

\[
[X(\hat{\alpha}_n)] = \begin{bmatrix}
1 & 0 & 0 \\
0 & c\hat{\alpha}_n & -s\hat{\alpha}_n \\
0 & s\hat{\alpha}_n & c\hat{\alpha}_n
\end{bmatrix}
\]  

(A.2)

which, by separating primary and dual components, expands to

\[
[X(\hat{\alpha}_n)] = \begin{bmatrix}
1 & 0 & 0 \\
0 & c\alpha_n & -s\alpha_n \\
0 & s\alpha_n & c\alpha_n
\end{bmatrix} + \varepsilon \begin{bmatrix}
0 & 0 & 0 \\
\alpha_n & \alpha_n s & -\alpha_n s \alpha_n \\
0 & s \alpha_n & -s \alpha_n
\end{bmatrix}
\]  

(A.3)
Screw motion through dual angle $\hat{\eta}_n$ ($\hat{\eta}_n = \eta_n + \varepsilon_n$) about Y axis can be written in a $3 \times 3$ transformation matrix form as

$$[Y(\hat{\eta}_n)] = \begin{bmatrix} c\eta_n & 0 & s\hat{\eta}_n \\ 0 & 1 & 0 \\ -s\eta_n & 0 & c\hat{\eta}_n \end{bmatrix}$$

which, by separating primary and dual components, expands to

$$[Y(\hat{\eta}_n)] = \begin{bmatrix} c\eta_n & 0 & s\eta_n \\ 0 & 1 & 0 \\ -s\eta_n & 0 & c\eta_n \end{bmatrix} + \varepsilon \begin{bmatrix} -e_n s\eta_n & 0 & e_n c\eta_n \\ 0 & 0 & 0 \\ -e_n c\eta_n & 0 & -e_n s\eta_n \end{bmatrix}$$

Screw motion through dual angle $\hat{\theta}_n$ ($\hat{\theta}_n = \theta_n + \varepsilon_n$) about Z axis can be written in a $3 \times 3$ transformation matrix form as

$$[Z(\hat{\theta}_n)] = \begin{bmatrix} c\hat{\theta}_n & -s\hat{\theta}_n & 0 \\ s\hat{\theta}_n & c\hat{\theta}_n & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

which, by separating primary and dual components, expands to

$$[Z(\hat{\theta}_n)] = \begin{bmatrix} c\theta_n & -s\theta_n & 0 \\ s\theta_n & c\theta_n & 0 \\ 0 & 0 & 1 \end{bmatrix} + \varepsilon \begin{bmatrix} -s_n s \theta_n & -s_n c \theta & 0 \\ s_n c \theta_n & -s_n s \theta_n & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Screw motions $\hat{Z}(\hat{\theta}_n)$ followed by screw motion $\hat{X}(\hat{\alpha}_n)$ can be combined into a matrix $\hat{M}_n$ and expressed in $3 \times 3$ form as

$$\hat{M}_n = [Z(\hat{\theta}_n)][X(\hat{\alpha}_n)] = \begin{bmatrix} c\hat{\theta}_n & -s\hat{\theta}_n & 0 \\ s\hat{\theta}_n & c\hat{\theta}_n & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\hat{\alpha}_n & -s\hat{\alpha}_n \\ 0 & s\hat{\alpha}_n & c\hat{\alpha}_n \end{bmatrix}$$
which, by observing that $\varepsilon^2 = 0$, expands to

$$
M_{n+1} = \begin{bmatrix}
c\theta_n & -c\alpha_n s\theta_n & s\alpha_n s\theta_n \\
s\theta_n & c\alpha_n c\theta_n & -s\alpha_n c\theta_n \\
0 & s\alpha_n & c\alpha_n
\end{bmatrix}
$$

The motion at ball joint requires an additional rotation through an angle $\hat{\eta}_n$ for its description so that the complete transformation through the joint and link takes the form

$$
L_{n+1} = [Z(\hat{\theta}_n) \parallel Y(\hat{\eta}_n) \parallel X(\hat{\alpha}_n)]
$$

which can also be expanded and simplified for different link-joint parameters.
APPENDIX B

FORTRAN Program for optimum balancing of the CSSP mechanism

C**************************MAIN PROGRAM**************************
C PROGRAM FOR OPTIMUM BALANCING OF THE GENERALIZED SLIDER
C CRANK MECHANISM (CSSP). KINEMATICS IS PERFORMED USING
C QUICKER-DENAVIT-HARTENBERG METHOD.
C*

C**************************Declaration of the variables**************************
C Joint variables
REAL*8 THT1,THT2,ETA,ETA2,THT3,ETA3,ALP4,S4

C Derivatives of joint variables
REAL*8 THT1D,THT2D,ETA2D,THT3D,ETA3D,S4D

C Double derivatives of joint variables
REAL*8 THT2DD,ETA2DD,THT3DD,ETA3DD,S4DD

C Transformation matrices
REAL*8 M1R(3,3),M1D(3,3),L2R(3,3),L2D(3,3),
1 L3R(3,3),L3D(3,3),M4R(3,3),M4D(3,3)

C Differential operators of transformation matrices
C Q1=operator of M1, L2(wrt THT2) & L3(wrt THT3)
C Q2=operator of L2(wrt ETA2)
C Q3=operator of M4
C Q4=operator of L3(wrt ETA3)

REAL*8 Q1R(3,3),Q1D(3,3),Q2R(3,3),Q2D(3,3),
1 Q3R(3,3),Q3D(3,3),Q4R(3,3),Q4D(3,3)

C Derivatives, transpose, multiplications, etc of
C transformation matrices
REAL*8 M1Q1R(3,3),M1Q1D(3,3),TM1R(3,3),TM1D(3,3),
1 H2R(3,3),H2D(3,3),M1Q2R(3,3),M1Q2D(3,3),H2DR(3,3),
1 H2DD(3,3),M1L2R(3,3),M1L2D(3,3),TM1L2R(3,3),
1 TM1L2D(3,3),H6R(3,3),H6D(3,3),H3R(3,3),H3D(3,3),
1 H7R(3,3),H7D(3,3),H3DR(3,3),H3DD(3,3),H5R(3,3),
1 H5D(3,3),H5Q3R(3,3),H5Q3D(3,3),TH5R(3,3),TH5D(3,3),
1 H4DDR(3,3),H4DDD(3,3),BR(3,3),BD(3,3),A(6,5),V(6),
1 AT(5,6),ATA(5,5),IATA(5,5),DELT(5,1A(5,6),
1 L3M4R(3,3),L3M4D(3,3),TL3M4R(3,3),TL3M4D(3,3),
1 TL3R(3,3),TM4R(3,3),L3RD(3,3),L3RDD(3,3),
1 TM4L3R(3,3),TM4L3RD(3,3),TM4L3RDD(3,3)

C L3RD & L3RDD are the diff and double diff of TL3R
C Joint angles in degrees

70
REAL*8 TH1, TH2, TH3, ET2, ET3, AL4

C Velocity components
REAL*8 V2XP, V2XD, V2YP, V2YD, V2ZP, V2ZD

C Velocity and acceleration matrices
REAL*8 VM(6,5), VMT(5,6), VMTVM(5,5), IVMTVM(5,5),
1 IVM(5,6), VV(6), AV(6), DOT(5), DDOT(5)

C Mass properties (Radii of gyration, C.G.'s, masses & MOI)
C of the moving links
REAL*8 K1X, K1Y, K1Z, K2X, K2Y, K2Z, K3X, K3Y, K3Z,
1 K1XCG, K1YCG, K1ZCG, K2XCG, K2YCG, K2ZCG,
1 G1, G2, G3, M1, M2, M3, I1X, I1Y, I1Z, I2X, I2Y, I2Z,
1 I3X, I3Y, I3Z, NU11X, NU11Y, NU11Z, NU12X, NU12Y, NU12Z

C Momentum components and their derivatives
REAL*8 H2XP, H2XD, H2YP, H2YD, H2ZP, H2ZD,
1 H2DYP, H2DXD, H2DYP, H2DYP, H2DP, H2DP

C Inertia force and torque components
REAL*8 R1P(3), R2P(3), R3P(3), R1D(3), R2D(3), R3D(3),
1 M1R1P(3), M1R1D1(3), M1R1D2(3), M1R1D(3),
1 M1L2R2P(3), M1L2R2D1(3), M1L2R2D2(3), M1L2R2D(3),
1 TM4R3P(3), NUR1XP, NUR2XP, NUR1YP, NUR2YP, NUR1ZP, NUR2ZP,
1 NUR1XD, NUR2XD, NUR1YD, NUR2YD, NUR1ZD, NUR2ZD

C Reaction force and torque components
REAL*8 T1I, T1J, T1K, F1I, F1J, F1K, F2I, F2J, F2K,

C Location of point masses and their derivatives
REAL*8 D(3), DD(3), DDD(3), LL(3), LL1(3), LL2(3),
1 L11DD(3), L12DD(3), L13DD(3), L14(3), L14D(3), L14DD(3),
1 L11I(3), L11D1(3), temp,
1 L11I1(3), L12I(3), L121D(3), L12DD1(3), L131(3),
1 L131D(3), L13DD1(3), L12D(3), L11D2(3),
1 L11D2(3), L122(3), L12D2(3), L12DD2(3), L132(3),
1 L13D2(3), L13DD2(3), L12DD(3),
1 L22DD(3), L23DD(3), L24(3), L24D(3), L24DD(3),
1 L21I(3), L21D1(3), L21DD1(3), L221(3), L22D1(3),
1 L22D1(3), L231(3), L23D1(3), L23D1(3),
1 L212(3), L21D2(3), L21DD2(3), L222(3), L22D2(3),
1 L22DD2(3), L232(3), L23D2(3), L23DD2(3)
REAL*8 X10, Y10, Z10, X1, Y1, Z1, L1(3,3),
1 X20, Y20, Z20, X2, Y2, Z2, L2(3,3)
REAL*8 M1RD(3,3), M1RDD(3,3)
REAL*8 PI, DEGRAD
real*8 m11, m12, m13, m14, m21, m22, m23, m24, m(9)
C Variables required for optimization

real*8 px(9),fx(9),fxsq(9,9),py(9),fy(9),fysq(9,9),
1 px(9),fx(9),fxsq(9,9),q(9),tx(9),txsq(9,9),
1 qy(9),ty(9),tysq(9,9),qz(9),tz(9),tisq(9,9),
1 flx(9),f1xsq(9,9),fly(9),f1ysq(9,9),
1 flz(9),f1szq(9,9),tlx(9),tlxsq(9,9),
1 t1y(9),t1ysq(9,9),tlz(9),tlzsq(9,9),
1 f2x(9),f2xsq(9,9),f2y(9),f2ysq(9,9),
1 f2z(9),f2szq(9,9),f3x(9),f3xsq(9,9),
1 f3y(9),f3ysq(9,9),f3z(9),f3szq(9,9),
1 f4x(9),f4xsq(9,9),f4y(9),f4ysq(9,9),
1 f4z(9),f4zsq(9,9),f4zsq(9,9),
1 f4i(9),f4isq(9,9),f4zsq(9,9),
1 f4zsq(9,9),f4zsq(9,9),
1 f4zsq(9,9),f4zsq(9,9),
1 f4zsq(9,9),f4zsq(9,9),
1 f4zsq(9,9),f4zsq(9,9),
1 F(9,9),OF(9,9),C1,C2,C3,OBJFUN,NUOBJFUN,
1 mfx,nufx,mfy,nufy,mfz,nufz,mtx,ntx,mt,nty,
1 mtz,ntz,mf1x,nuf1x,mf1y,nuf1y,mf1z,nuf1z,
1 mt1x,nt1x,mt1y,nt1y,mt1z,nt1z,mt2x,nuf2x,
1 mt2y,nuf2y,mf2z,nuf2z,mf3x,nuf3x,mf3y,nuf3y,
1 mf3z,nuf3z,mf4x,nuf4x,mf4y,nuf4y,mt4x,nu4x,
1 mt4y,nu4y,mt4z,nu4z,shf,nushf,sht,nushf,
1 brf,nubrf,brt,nubrt,int,nuint,DOBJFUN,dshf,
1 dsht,dbrf,dbrt, dint

integer lda,ldh,ncon,neq,nvar
parameter (ncon=11,neq=3,nvar=9,lda=ncon,ldh=nvar)
integer iact(nvar),k,nact,nout
real aa(lda,nvar),alamda(nvar),b(ncon),diag,gg(nvar),
1 h(ldh,ldh),sol(nvar)
external qprog,umach

data aa/1.0,0.0,0.0,0.0,1.0,0.0,0.0,0.0,0.0,0.0,0.0,0.0,0.0,0.0,0.0,0.0,0.0,0.0,0.0,
1 1.0,0.0,0.0,0.0,0.0,1.0,0.0,0.0,0.0,0.0,0.0,0.0,0.0,0.0,0.0,0.0,0.0,0.0,0.0,
1 1.0,0.0,0.0,0.0,0.0,0.0,1.0,0.0,0.0,0.0,0.0,0.0,0.0,0.0,0.0,0.0,0.0,0.0,0.0,
1 0.0,0.0,0.0,0.0,0.0,0.0,0.0,1.0,0.0,0.0,0.0,0.0,0.0,0.0,0.0,0.0,0.0,0.0,0.0,
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1 0.0,0.0,0.0,0.0,0.0,0.0,0.0,0.0,0.0,0.0,0.0,0.0,1.0,0.0,0.0,0.0,0.0,0.0,0.0 /

C Weight factors and bearing length

PARAMETER (W1=1,W2=1, W3=1,W4=1, W5=1,BL=0.4)

C LINK PARAMETERS ARE AS FOLLOWS

PARAMETER (A1=2.0,A2=8.0,A4=1.0,AL4=250.0,S1=0.4)
PARAMETER (G=386.4,RPM=3000.0,THT1DD=0.0, S1D=0.0,S1DD=0.0)
OPEN (6,FILE="dint.dat",STATUS='new')
PI=4.0*ATAN(1.0)
DEGRAD=PI/180.0

C Mass properties of the original links

G1=-1.0
G2=-4.0
G3=0.0
M1=1.9/G
M2=7.6/G
M3=6.0/G

C Radii of gyration about C.G. & about distal coordinate frame

K1XCG=0.2041241
K1YCG=0.595119
K1ZCG=0.595119
K1X=0.2041241
K1Y=1.1636867
K1Z=1.1636867
K2XCG=0.2041241
K2YCG=2.3139072
K2ZCG=2.3139072
K2X=0.2041241
K2Y=4.6210569
K2Z=4.6210569
K3X=0.5204165
K3Y=0.5204165
K3Z=0.7071067

C Mass moment of inertia about C.G.

I1X=M1*K1XCG**2
I1Y=M1*K1YCG**2
I1Z=M1*K1ZCG**2
I2X=M2*K2XCG**2
I2Y=M2*K2YCG**2
I2Z=M2*K2ZCG**2
I3X=M3*K3X**2
I3Y=M3*K3Y**2
I3Z=M3*K3Z**2
X10=G1
X20=G2
C Magnitude of point masses
m11=M1/4.0
m12=m11
m13=m11
m14=M1-(m11+m12+m13)
m21=M2/4.0
m22=m21
m23=m21
m24=M2-(m21+m22+m23)

m(1)=m11
m(2)=m12
m(3)=m13
m(4)=m14
m(5)=m21
m(6)=m22
m(7)=m23
m(8)=m24
m(9)=M3

C Direction cosines between distal & principal coordinate frames
L1(1,1)=1.0
L1(2,2)=1.0
L1(3,3)=1.0
L2(1,1)=1.0
L2(2,2)=1.0
L2(3,3)=1.0

C Location of point masses
X1=((I1Y+I1Z-I1X)/(2.0*m11))**0.5
Y1=((I1X+I1Z-I1Y)/(2.0*m12))**0.5
Z1=((I1X+I1Y-I1Z)/(2.0*m13))**0.5
X2=((I2Y+I2Z-I2X)/(2.0*m21))**0.5
Y2=((I2X+I2Z-I2Y)/(2.0*m22))**0.5
Z2=((I2X+I2Y-I2Z)/(2.0*m23))**0.5

C Initialization of the objective function OF(9,9)
DO 100 12=1,9
DO 110 J2=1,9
OF(I2,J2)=0
110 CONTINUE
100 CONTINUE

do 41 I3=1,2

C INITIAL ESTIMATE FOR DISPLACEMENTS
\[ S_4 = (A_2^2 - (A_1 + A_4)^2)^{0.5} \]
\[ \text{temp} = S_4 / (A_4 + A_1) \]
\[ THT2 = \pi - \text{ATAN}(\text{temp}) \]
\[ \text{C} \quad THT2 = \pi - \text{ATAN2}(S_4, (A_4 + A_1)) \]
\[ THT3 = 2\pi - THT2 \]

\[ \text{DO 400 J=1,37} \]
\[ \text{TH1} = J \times 10.0 - 10.0 \]
\[ THT1 = \text{TH1} \times \text{DEGRAD} \]
\[ \text{ALP4} = A_4 \times \text{DEGRAD} \]
\[ THT1D = 6.0 \times \text{RPM} \times \text{DEGRAD} \]

\[ \text{C} \quad \text{STEP 1} \]
\[ I = 1 \]
\[ \text{10 IF (I.LE.150) THEN} \]

\[ \text{C} \quad \text{SPECIFICATION OF M1,L2,L3,M4 MATRICES RESPECTIVELY} \]
\[ M1R(1,1) = \cos(THT1) \]
\[ M1R(1,2) = -\sin(THT1) \]
\[ M1R(1,3) = 0.0 \]
\[ M1R(2,1) = \sin(THT1) \]
\[ M1R(2,2) = \cos(THT1) \]
\[ M1R(2,3) = 0.0 \]
\[ M1R(3,1) = 0.0 \]
\[ M1R(3,2) = 0.0 \]
\[ M1R(3,3) = 1.0 \]
\[ M1D(1,1) = -S_1 \times \sin(THT1) \]
\[ M1D(1,2) = -S_1 \times \cos(THT1) \]
\[ M1D(1,3) = A_1 \times \sin(THT1) \]
\[ M1D(2,1) = S_1 \times \cos(THT1) \]
\[ M1D(2,2) = -S_1 \times \sin(THT1) \]
\[ M1D(2,3) = -A_1 \times \cos(THT1) \]
\[ M1D(3,1) = 0.0 \]
\[ M1D(3,2) = A_1 \]
\[ M1D(3,3) = 0.0 \]

\[ \text{C} \]
\[ L2R(1,1) = \cos(THT2) \times \cos(ETA2) \]
\[ L2R(1,2) = -\sin(THT2) \]
\[ L2R(1,3) = \cos(THT2) \times \sin(ETA2) \]
\[ L2R(2,1) = \sin(THT2) \times \cos(ETA2) \]
\[ L2R(2,2) = \cos(THT2) \]
\[ L2R(2,3) = \sin(THT2) \times \sin(ETA2) \]
\[ L2R(3,1) = -\sin(ETA2) \]
\[ L2R(3,2) = 0.0 \]
\[ L2R(3,3) = \cos(ETA2) \]
\[ L2D(1,1) = 0.0 \]
L2D(1,2) = A2 * COS(THT2) * SIN(ETA2)  
L2D(1,3) = A2 * SIN(THT2)  
L2D(2,1) = 0.0  
L2D(2,2) = A2 * SIN(THT2) * SIN(ETA2)  
L2D(2,3) = -A2 * COS(THT2)  
L2D(3,1) = 0.0  
L2D(3,2) = A2 * COS(ETA2)  
L2D(3,3) = 0.0

C

L3R(1,1) = COS(THT3) * COS(ETA3)  
L3R(1,2) = COS(THT3) * SIN(ETA3)  
L3R(1,3) = SIN(THT3)  
L3R(2,1) = SIN(THT3) * COS(ETA3)  
L3R(2,2) = SIN(THT3) * SIN(ETA3)  
L3R(2,3) = -COS(THT3)  
L3R(3,1) = -SIN(ETA3)  
L3R(3,2) = COS(ETA3)  
L3R(3,3) = 0.0

M4R(1,1) = 1.0  
M4R(1,2) = 0.0  
M4R(1,3) = 0.0  
M4R(2,1) = 0.0  
M4R(2,2) = COS(ALP4)  
M4R(2,3) = -SIN(ALP4)  
M4R(3,1) = 0.0  
M4R(3,2) = SIN(ALP4)  
M4R(3,3) = COS(ALP4)

M4D(1,1) = 0.0  
M4D(1,2) = -S4 * COS(ALP4)  
M4D(1,3) = S4 * SIN(ALP4)  
M4D(2,1) = S4  
M4D(2,2) = -A4 * SIN(ALP4)  
M4D(2,3) = -A4 * COS(ALP4)  
M4D(3,1) = 0.0  
M4D(3,2) = A4 * COS(ALP4)
M4D(3,3) = -A4*SIN(ALP4)

C

C SPECIFICATION OF PARTIAL DERIVATIVE OPERATORS
Q1R(1,2) = -1.0
Q1R(2,1) = 1.0
Q2R(1,3) = COS(THT2)
Q2R(2,3) = SIN(THT2)
Q2R(3,1) = -COS(THT2)
Q2R(3,2) = -SIN(THT2)
Q3D(1,2) = -1.0
Q3D(2,1) = 1.0
Q4R(1,3) = COS(THT3)
Q4R(2,3) = SIN(THT3)
Q4R(3,1) = -COS(THT3)
Q4R(3,2) = -SIN(THT3)

C

CALL PRODUCTDUAL (M1R,M1D,Q1R,Q1D,M1Q1R,M1Q1D)
CALL TRANSPOSE (M1R,M1D,TM1R,TM1D)
CALL PRODUCTDUAL (M1Q1R,M1Q1D,TM1R,TM1D,H2R,H2D)
CALL PRODUCTDUAL (M1R,M1D,Q2R,Q2D,M1Q2R,M1Q2D)
CALL PRODUCTDUAL (M1Q2R,M1Q2D,TM1R,TM1D,H2DR,H2DD)
CALL PRODUCTDUAL (M1R,M1D,L2R,L2D,M1L2R,M1L2D)
CALL TRANSPOSE (M1L2R,M1L2D,TM1L2R,TM1L2D)
CALL PRODUCTDUAL (M1L2R,M1L2D,Q1R,Q1D,H6R,H6D)
CALL PRODUCTDUAL (H6R,H6D,TM1L2R,TM1L2D,H3R,H3D)
CALL PRODUCTDUAL (M1L2R,M1L2D,Q4R,Q4D,H7R,H7D)
CALL PRODUCTDUAL (H7R,H7D,TM1L2R,TM1L2D,H3DR,H3DD)
CALL PRODUCTDUAL (M1L2R,M1L2D,L3R,L3D,H5R,H5D)
CALL PRODUCTDUAL (H5R,H5D,Q3R,Q3D,H5Q3R,H5Q3D)
CALL TRANSPOSE (H5R,H5D,TH5R,TH5D)
CALL PRODUCTDUAL (H5Q3R,H5Q3D,TH5R,TH5D,H4DDR,H4DDD)
CALL PRODUCTDUAL (H5R,H5D,M4R,M4D,BR,BD)
CALL PRODUCTDUAL (L3R,L3D,M4R,M4D,L3M4R,L3M4D)
CALL TRANSPOSE (L3M4R,L3M4D,TL3M4R,TL3M4D)

C SPECIFICATION OF A & V MATRICES
A(1,1) = H2R(1,2)
A(1,2) = H2DR(1,2)
A(1,3) = H3R(1,2)
A(1,4) = H3DR(1,2)
A(1,5) = H4DDR(1,2)
A(2,1) = H2R(1,3)
A(2,2) = H2DR(1,3)
A(2,3) = H3R(1,3)
A(2,4) = H3DR(1,3)
A(2,5) = H4DDR(1,3)
A(3,1) = H2R(2,3)
A(3,2) = H2DR(2,3)
A(3,3) = H3R(2,3)
A(3,4) = H3DR(2,3)
A(3,5) = H4DDR(2,3)
A(4,1) = H2D(1,2)
A(4,2) = H2DD(1,2)
A(4,3) = H3D(1,2)
A(4,4) = H3DD(1,2)
A(4,5) = H4DDD(1,2)
A(5,1) = H2D(1,3)
A(5,2) = H2DD(1,3)
A(5,3) = H3D(1,3)
A(5,4) = H3DD(1,3)
A(5,5) = H4DDD(1,3)
A(6,1) = H2D(2,3)
A(6,2) = H2DD(2,3)
A(6,3) = H3D(2,3)
A(6,4) = H3DD(2,3)
A(6,5) = H4DDD(2,3)

V(1) = BR(2,1) + BR(1,1) + BR(2,2) - 2.0
V(2) = BR(3,1) + BR(1,1) + BR(3,3) - 2.0
V(3) = BR(3,2) + BR(2,2) + BR(3,3) - 2.0
V(4) = BD(2,1)
V(5) = BD(3,1)
V(6) = BD(3,2)

C Determination of joint variables
CALL TRANS (A,AT,6,5)
CALL MULTI (AT,A,ATA,5,5,6)
CALL INVERSE (ATA,IATA)
CALL MULTI (IATA,AT,IA,5,6,5)
CALL MULMAVEC (IA,V,DELTA,5,6)

C
IF ( ABS(DELTA(1)) .LE. 0.0000001 .AND.
1   ABS(DELTA(2)) .LE. 0.0000001 .AND.
1   ABS(DELTA(3)) .LE. 0.0000001 .AND.
1   ABS(DELTA(4)) .LE. 0.0000001 .AND.
1   ABS(DELTA(5)) .LE. 0.0000001 ) THEN

TH2 = THT2/DEGRAD
TH3 = THT3/DEGRAD
ET2 = ETA2/DEGRAD
ET3=ETA3/DEGRAD

**C Derivative of joint variables**

\[
VM(1,1) = -\sin(\eta_2)
\]

\[
VM(1,3) = \sin(\theta_2)\cos(\eta_2)
\]

\[
VM(2,5) = \sin(\theta_3)
\]

\[
VM(3,3) = \cos(\theta_2)
\]

\[
VM(3,4) = 1.0
\]

\[
VM(4,1) = A_2\cos(\eta_2)
\]

\[
VM(4,3) = A_2\sin(\theta_2)\sin(\eta_2)
\]

\[
VM(4,5) = -\cos(\theta_3)
\]

\[
VM(5,1) = \cos(\eta_2)
\]

\[
VM(5,2) = 1.0
\]

\[
VM(5,3) = \sin(\theta_2)\sin(\eta_2)
\]

\[
VM(6,3) = -A_2\cos(\eta_2)
\]

\[
VV(1) = \theta_1D\sin(\eta_2)
\]

\[
VV(2) = S_1D\sin(\eta_2) - A_1\theta_1D\sin(\theta_2)\cos(\eta_2)
\]

\[
VV(4) = -\theta_1D(A_1\cos(\theta_2) + A_2\cos(\eta_2))
\]

\[
VV(5) = -\theta_1D\cos(\eta_2)
\]

\[
VV(6) = -\theta_1D A_1\sin(\theta_2)\sin(\eta_2) - S_1D\cos(\eta_2)
\]

**CALL TRANS (VM,VMT,6,5)**

**CALL MULTI (VMT,VM,VMTVM,5,5,6)**

**CALL INVERSE (VMTVM,IVMTVM)**

**CALL MULTI (IVMTVM,VMT,IVM,5,6,5)**

**CALL MULMAVEC (IVM,VV,DOT,5,6)**

\[
\theta_2D = DOT(1)
\]

\[
\theta_3D = DOT(2)
\]

\[
\eta_2D = DOT(3)
\]

\[
\eta_3D = DOT(4)
\]

\[
S_4D = DOT(5)
\]

**C Double derivative of joint variables**

\[
AV(1) = \theta_1D\thetaD\sin(\eta_2) + \eta_2D(\theta_1D + \theta_2D)\cos(\eta_2)
\]

\[
1 - \eta_2D(\theta_2D\cos(\theta_2)\cos(\eta_2) - \eta_2D(\theta_2D\sin(\theta_2)\sin(\eta_2))
\]

\[
AV(2) = A_1(\theta_1D\sin(\theta_2)\sin(\eta_2) + \eta_2D\theta_1D\theta_2D - \theta_1DD\theta_2D\theta_2D\cos(\eta_2) - \theta_2DD\eta_2D\cos(\eta_2))
\]

\[
+ S_1DD\sin(\eta_2) + S_1D\eta_2D(\cos(\eta_2) - S_4D/th_3D\cos(\theta_3)
\]

\[
AV(3) = \theta_2D\eta_2D\sin(\eta_2)
\]

\[
AV(4) = A_1\sin(\eta_2) + A_2D(\theta_1D + \theta_2D) - A_2D(\theta_1D + \theta_2D) - A_2\theta_2D\sin(\eta_2)
\]

\[
A_2D(\cos(\eta_2) + \sin(\eta_2)\eta_2D)
\]

\[
1 - \eta_2D(\theta_2D(\cos(\eta_2) - \sin(\eta_2)\eta_2D)
\]

\[
AV(5) = \theta_2D\eta_2D\sin(\eta_2)
\]

\[
AV(6) = \theta_2D\eta_2D\sin(\eta_2)
\]
\[
\begin{align*}
1 & \quad \cos(\eta_2)\eta_2 - \sin(\theta_3)\theta_3 = S_{4D} \\
A & \quad \text{AV}(5) = \theta_1 \eta_2 - \theta_1 \eta_2 + \sin(\eta_2) \\
B & \quad \text{AV}(6) = \sin(\eta_2) \eta_2 - A_1 \theta_1 \eta_2 + S_{4D} \cos(\eta_2) \\
C & \quad \text{CALL MULMAVEC (IVM, AV, DDOT, 5, 6)} \\
D & \quad T\theta_2 D = \text{DDOT}(1) \\
E & \quad T\theta_3 D = \text{DDOT}(2) \\
F & \quad \eta_2 D = \text{DDOT}(3) \\
G & \quad \eta_3 D = \text{DDOT}(4) \\
H & \quad S_{4D} = \text{DDOT}(5) \\
I & \quad \text{C Inertia forces and torques of original links} \\
J & \quad V_2 X P = 0.0 \\
K & \quad V_2 X D = -S_{4D} \sin(\theta_3) \\
L & \quad V_2 Y P = -\eta_3 D \\
M & \quad V_2 Y D = S_{4D} \cos(\theta_3) \\
N & \quad V_2 Z P = -\theta_3 D \\
O & \quad V_2 Z D = 0.0 \\
P & \quad H_2 X P = -S_{4D} \sin(\theta_3) \\
Q & \quad H_2 X D = 0.0 \\
R & \quad H_2 Y P = -G_2 \theta_3 D + S_{4D} \cos(\theta_3) \\
S & \quad H_2 Y D = -(K_{2Y}^2) \eta_3 D \\
T & \quad H_2 Z P = \eta_3 D G_2 \\
U & \quad H_2 Z D = -(K_{2Z}^2) \theta_3 D + G_2 S_{4D} \cos(\theta_3) \\
V & \quad \text{C} \\
W & \quad H_2 X P = -S_{4D} \sin(\theta_3) - S_{4D} \theta_3 D \cos(\theta_3) \\
X & \quad H_2 X D = 0.0 \\
Y & \quad H_2 Y P = -G_2 \theta_3 D - S_{4D} \cos(\theta_3) + S_{4D} \cos(\theta_3) \\
Z & \quad H_2 Y D = -(K_{2Y}^2) \eta_3 D \\
aa & \quad H_2 Z P = \eta_3 D G_2 \\
b & \quad H_2 Z D = -(K_{2Z}^2) \theta_3 D - G_2 (S_{4D} \cos(\theta_3) - \sin(\theta_3) \eta_3 D S_{4D}) \\
c & \quad \text{C} \\
d & \quad R_{1P}(1) = -M_1 (\theta_1 D^2) (A_1 + G_1) \\
e & \quad R_{1D}(1) = 0.0 \\
f & \quad R_{1P}(2) = M_1 \theta_1 D (A_1 + G_1) \\
g & \quad R_{1D}(2) = -M_1 G_1 S_{1D} \\
h & \quad R_{1P}(3) = M_1 S_{1D} \\
i & \quad R_{1D}(3) = M_1 \theta_1 D ((K_{1Z}^2) + G_1 A_1)
\end{align*}
\]
CALL MULMAVEC (M1R,R1P,M1R1P,3,3)
CALL MULMAVEC (M1R,R1D,M1R1D1,3,3)
CALL MULMAVEC (M1D,R1P,M1R1D2,3,3)
CALL ADDVEC (M1R1D1,M1R1D2,M1R1D)

C
R2P(1)=M2*(H2DXP-V2ZP*H2YP+V2YP*H2ZP)
R2D(1)=M2*(H2DXD-V2ZP*H2YD-V2ZD*H2YP+V2YP*H2ZD+V2YD*H2ZP)
R2P(2)=M2*(H2DYP-V2XP*H2ZP+V2ZP*H2XP)
R2D(2)=M2*(H2DYD-V2XP*H2ZD-V2XD*H2YP+V2YP*H2XD+V2ZD*H2XP)
R2P(3)=M2*(H2DZP-V2YP*H2XP+V2XP*H2YP)
R2D(3)=M2*(H2DZD-V2YP*H2XD-V2YD*H2XP+V2XP*H2YD+V2XD*H2YP)
C CALL MULMAVEC (M1L2R,R2P,M1L2R2P,3,3)
C CALL MULMAVEC (M1L2R,R2D,M1L2R2D1,3,3)
C CALL MULMAVEC (M1L2D,R2P,M1L2R2D2,3,3)
C CALL ADDVEC (M1L2R2D1,M1L2R2D2,M1L2R2D)
CALL MULMAVEC (TL3M4R,R2P,M1L2R2P,3,3)
CALL MULMAVEC (TL3M4R,R2D,M1L2R2D1,3,3)
CALL MULMAVEC (TL3M4D,R2P,M1L2R2D2,3,3)
CALL ADDVEC (M1L2R2D1,M1L2R2D2,M1L2R2D)

R3P(1)=0.0
R3D(1)=0.0
R3P(2)=0.0
R3D(2)=M3*G3*S4DD
R3P(3)=-M3*S4DD
R3D(3)=0.0
CALL TRANS (L3R,TL3R,3,3)
CALL TRANS (M4R,TM4R,3,3)
CALL MULMAVEC (TM4R,R3P,TM4R3P,3,3)
CALL MULTI (TM4R,TL3R,TM4L3R,3,3,3)

C Joint forces and torques of the original links
F3J=-(R2P(2)+R2D(3)/A2)
F3K=R2D(2)/A2-R2P(3)
T4I=-R3D(1)
T4J=-R3D(2)
T4K=-R3D(3)
F3I=(F3J*COS(THT3)-F4K-R3P(3))/SIN(THT3)
F4J=-(COS(THT3)*SIN(ETA3)*F3I+SIN(THT3)*SIN(ETA3)*F3J+F4K*R3P(2))
F4I=-(COS(THT3)*COS(ETA3)*F3I+SIN(THT3)*COS(ETA3)*F3J+F4K*R3P(1))
F2J=-(F3J+R2P(2))*COS(THT2)-R2D(2)*SIN(ETA2)*SIN(THT2)/A2
F2I=-(F3I+R2P(1))*SIN(THT2)*COS(ETA2)
\[ F_{2I} = (F_{3J} + R_{2P}(2)) \sin(THT2) - R_{2D}(2) \sin(ETA2) \cos(THT2)/A2 \]
\[ - (F_{3I} + R_{2P}(1)) \cos(THT2) \cos(ETA2) \]
\[ F_{2K1} = -(F_{2I} \cos(THT2) \sin(ETA2) + F_{2J} \sin(THT2) \sin(ETA2) + F_{3K} + R_{2P}(3)) \]
\[ F_{2K2} = (F_{2I} \cos(THT2) \cos(ETA2) + F_{2J} \sin(THT2) \cos(ETA2) + F_{3I} + R_{2P}(1)) \]
\[ \text{temp} = F_{2K2}/F_{2K1} \]
\[ \text{ETA} = \text{ATAN} (\text{temp}) \]
\[ \text{if} (\text{ABS} (\sin(\text{ETA})) \text{ GT. ABS} (\sin(\text{ETA}))) \text{ then} \]
\[ F_{2K} = F_{2K1}/\cos(\text{ETA}) \]
\[ \text{else} \]
\[ F_{2K} = F_{2K2}/\sin(\text{ETA}) \]
\[ \text{endif} \]
\[ F_{1K} = -(F_{2K} + R_{1P}(3)) \]
\[ F_{1I} = -(F_{2I} + R_{1P}(1)) \cos(THT1) + (F_{2J} + R_{1P}(2)) \sin(THT1) \]
\[ F_{1J} = -(F_{2I} + R_{1P}(1)) \sin(THT1) - (F_{2J} + R_{1P}(2)) \cos(THT1) \]
\[ T_{1K} = \cos(THT1) \ast A_1 \ast F_{1J} - \sin(THT1) \ast A_1 \ast F_{1I} - R_{1D}(3) \]
\[ T_{1J} = S_1 \ast F_{1I} - R_{1D}(1) \ast \sin(THT1) - (A_1 \ast F_{1K} + R_{1D}(2)) \ast \cos(THT1) \]
\[ T_{1I} = -S_1 \ast F_{1J} - R_{1D}(1) \ast \cos(THT1) + (A_1 \ast F_{1K} + R_{1D}(2)) \ast \sin(THT1) \]

C**********************************************************************
C USE OF POINT MASSES
C**********************************************************************
C For Crank (link 1)
D(1) = A1 * \cos(THT1)
DD(1) = -A1 * \sin(THT1) * THT1D
DDD(1) = -A1 * (\cos(THT1) * THT1D**2 + \sin(THT1) * THT1DD)
D(2) = A1 * \sin(THT1)
DD(2) = A1 * \cos(THT1) * THT1D
DDD(2) = A1 * (\cos(THT1) * THT1DD - \sin(THT1) * THT1D**2)
D(3) = S1
DD(3) = S1D
DDD(3) = S1DD
C Derivative and double derivative of M1R
M1RD(1, 1) = -\sin(THT1) * THT1D
M1RD(1, 2) = -\cos(THT1) * THT1D
M1RD(2, 1) = \cos(THT1) * THT1D
M1RD(2, 2) = -\sin(THT1) * THT1D
M1RDD(1, 1) = -\sin(THT1) * THT1DD - \cos(THT1) * (THT1D**2)
M1RDD(1, 2) = -\cos(THT1) * THT1DD + \sin(THT1) * (THT1D**2)
M1RDD(2, 1) = \cos(THT1) * THT1DD - \sin(THT1) * (THT1D**2)
M1RDD(2, 2) = -\sin(THT1) * THT1DD - \cos(THT1) * (THT1D**2)
C Point mass m11
\[ LL1(1) = X10 + X1*L1(1,1) \]
\[ LL1(2) = Y10 + X1*L1(1,2) \]
\[ LL1(3) = Z10 + X1*L1(1,3) \]
\[ LL2(1) = X10 - X1*L1(1,1) \]
\[ LL2(2) = Y10 - X1*L1(1,2) \]
\[ LL2(3) = Z10 - X1*L1(1,3) \]

CALL KINEMATIC (M1R,M1RD,M1RDD,D,DD,DDD,LL1, 
1 L111,L11D1,L11DD1)
CALL KINEMATIC (M1R,M1RD,M1RDD,D,DD,DDD,LL2, 
1 L112,L11D2,L11DD2)
CALL ADDVEC (L11DD1,L11DD2,L11DD)

C Point mass m12
\[ LL1(1) = X10 + Y1*L1(2,1) \]
\[ LL1(2) = Y10 + Y1*L1(2,2) \]
\[ LL1(3) = Z10 + Y1*L1(2,3) \]
\[ LL2(1) = X10 - Y1*L1(2,1) \]
\[ LL2(2) = Y10 - Y1*L1(2,2) \]
\[ LL2(3) = Z10 - Y1*L1(2,3) \]

CALL KINEMATIC (M1R,M1RD,M1RDD,D,DD,DDD,LL1, 
1 L121,L12D1,L12DD1)
CALL KINEMATIC (M1R,M1RD,M1RDD,D,DD,DDD,LL2, 
1 L122,L12D2,L12DD2)
CALL ADDVEC (L12DD1,L12DD2,L12DD)

C Point mass m13
\[ LL1(1) = X10 + Z1*L1(3,1) \]
\[ LL1(2) = Y10 + Z1*L1(3,2) \]
\[ LL1(3) = Z10 + Z1*L1(3,3) \]
\[ LL2(1) = X10 - Z1*L1(3,1) \]
\[ LL2(2) = Y10 - Z1*L1(3,2) \]
\[ LL2(3) = Z10 - Z1*L1(3,3) \]

CALL KINEMATIC (M1R,M1RD,M1RDD,D,DD,DDD,LL1, 
1 L131,L13D1,L13DD1)
CALL KINEMATIC (M1R,M1RD,M1RDD,D,DD,DDD,LL2, 
1 L132,L13D2,L13DD2)
CALL ADDVEC (L13DD1,L13DD2,L13DD)

C Point mass m14
\[ LL(1) = X10 \]
\[ LL(2) = Y10 \]
\[ LL(3) = Z10 \]

CALL KINEMATIC (M1R,M1RD,M1RDD,D,DD,DDD,LL, 
1 L14,L14D,L14DD)
C New inertia forces and torques of crank using point masses

\[ \text{NUR1XP} = (-m_{11} \cdot L_{11DD(1)} - m_{12} \cdot L_{12DD(1)} - m_{13} \cdot L_{13DD(1)})/2 \]

\[ \text{NUR1YP} = (-m_{11} \cdot L_{11DD(2)} - m_{12} \cdot L_{12DD(2)} - m_{13} \cdot L_{13DD(2)})/2 \]

\[ \text{NUR1ZP} = (-m_{11} \cdot L_{11DD(3)} - m_{12} \cdot L_{12DD(3)} - m_{13} \cdot L_{13DD(3)})/2 \]

\[ \text{NUR1XD} = -m_{11} \cdot (L_{111(2)} \cdot L_{11DD1(2)} - L_{111(3)} \cdot L_{11DD1(3)})/2 \]

\[ \text{NUR1YD} = -m_{11} \cdot (L_{111(3)} \cdot L_{11DD1(1)} - L_{111(1)} \cdot L_{11DD1(3)})/2 \]

\[ \text{NUR1ZD} = -m_{11} \cdot (L_{111(1)} \cdot L_{11DD1(2)} - L_{111(2)} \cdot L_{11DD1(1)})/2 \]

For Connecting rod (link 2)

\[ D(1) = -A_4 \]

\[ D(2) = -S_4 \cdot \sin(ALP_4) \]

\[ D(3) = -S_4 \cdot \cos(ALP_4) \]

\[ DD(1) = 0.0 \]

\[ DD(2) = -S_4 \cdot \sin(ALP_4) \]

\[ DD(3) = -S_4 \cdot \cos(ALP_4) \]

\[ DDD(1) = 0.0 \]

\[ DDD(2) = -S_4 \cdot \sin(ALP_4) \]

\[ DDD(3) = -S_4 \cdot \cos(ALP_4) \]

C********Derivative of (TM4R)(TL3R)=TM4L3RD***************

\[ L3RD(1,1) = -\cos(THT3) \cdot \sin(ETA3) \cdot ETA3D \]

\[ L3RD(1,2) = -\sin(THT3) \cdot \cos(ETA3) \cdot THT3D \]

\[ L3RD(1,3) = \cos(ETA3) \cdot ETA3D \]

\[ L3RD(2,1) = \cos(THT3) \cdot \cos(ETA3) \cdot THT3D \]
I  -SIN(THT3)*SIN(ETA3)*ETA3D
L3RD(2,1)=COS(THT3)*COS(ETA3)*ETA3D
I  -SIN(THT3)*SIN(ETA3)*THT3D
L3RD(2,3)=-SIN(ETA3)*ETA3D
L3RD(2,2)=COS(THT3)*SIN(ETA3)*THT3D
I  +SIN(THT3)*COS(ETA3)*ETA3D
L3RD(3,1)=COS(THT3)*THT3D
L3RD(3,3)=0.0
L3RD(3,2)=SIN(THT3)*THT3D
CALL MULTI (TM4R,L3RD,TM4L3RD,3,3,3)
C**********Double derivative of (TM4R)(TL3R)=TM4L3RDD**********
L3RDD(1,1)=2*SIN(THT3)*SIN(ETA3)*THT3D*ETA3D
I  -COS(THT3)*COS(EAT3)*((THT3D**2)+(ETA3D**2))
I  -SIN(THT3)*COS(ETA3)*THT3DD
I  -COS(THT3)*SIN(ETA3)*ETA3DD
L3RDD(1,3)=SIN(ETA3)*(ETA3D**2)-COS(ETA3)*ETA3DD
L3RDD(1,2)=-2*SIN(ETA3)*COS(THT3)*THT3D*ETA3D
I  -SIN(THT3)*COS(EAT3)*((THT3D**2)+(ETA3D**2))
I  -SIN(THT3)*SIN(ETA3)*ETA3DD
I  +COS(THT3)*COS(ETA3)*THT3DD
L3RDD(2,1)=-2*SIN(THT3)*COS(ETA3)*THT3D*ETA3D
I  -COS(THT3)* SIN(EAT3)*((THT3D**2)+(ETA3D**2))
I  +COS(THT3)*SIN(ETA3)*THT3DD
I  +SIN(THT3)*COS(ETA3)*ETA3DD
L3RDD(2,3)=-COS(ETA3)*(ETA3D**2)-SIN(ETA3)*ETA3DD
L3RDD(2,2)=2*COS(THT3)*COS(ETA3)*THT3D*ETA3D
I  -SIN(THT3)*COS(EAT3)*((THT3D**2)+(ETA3D**2))
I  +COS(THT3)*SIN(ETA3)*THT3DD
I  +SIN(THT3)*COS(ETA3)*ETA3DD
L3RDD(3,1)=-SIN(THT3)*(THT3D**2)+COS(THT3)*THT3DD
L3RDD(3,3)=0.0
L3RDD(3,2)=COS(THT3)*(THT3D**2)+SIN(THT3)*THT3DD
CALL MULTI (TM4R,L3RDD,TM4L3RDD,3,3,3)

C*******************************************************************************
C Point mass m21
LL1(1)=X20+X2*L2(1,1)
LL1(2)=Y20+X2*L2(1,2)
LL1(3)=Z20+X2*L2(1,3)
LL2(1)=X20-X2*L2(1,1)
LL2(2)=Y20-X2*L2(1,2)
LL2(3)=Z20-X2*L2(1,3)
CALL CORRLS (TM4L3R,TM4L3RD,TM4L3RDD,D,DD,DDD,LL1,
L211,L21D1,L21DD1)
CALL CORRLS (TM4L3R, TM4L3RD, TM4L3RDD, D, DD, DDD, LL2, 
1 L212, L21D2, L21DD2)
CALL ADDVEC (L21DD1, L21DD2, L21DD)

C Point mass m22
LL1(1)=X20+Y2*L2(2,1)
LL1(2)=Y20+Y2*L2(2,2)
LL1(3)=Z20+Y2*L2(2,3)
LL2(1)=X20-Y2*L2(2,1)
LL2(2)=Y20-Y2*L2(2,2)
LL2(3)=Z20-Y2*L2(2,3)
CALL CORRLS (TM4L3R, TM4L3RD, TM4L3RDD, D, DD, DDD, LL1, 
1 L221, L22D1, L22DD1)
CALL CORRLS (TM4L3R, TM4L3RD, TM4L3RDD, D, DD, DDD, LL2, 
1 L222, L22D2, L22DD2)
CALL ADDVEC (L22DD1, L22DD2, L22DD)

C Point mass m23
LL1(1)=X20+Z2*L2(3,1)
LL1(2)=Y20+Z2*L2(3,2)
LL1(3)=Z20+Z2*L2(3,3)
LL2(1)=X20-Z2*L2(3,1)
LL2(2)=Y20-Z2*L2(3,2)
LL2(3)=Z20-Z2*L2(3,3)
CALL CORRLS (TM4L3R, TM4L3RD, TM4L3RDD, D, DD, DDD, LL1, 
1 L231, L23D1, L23DD1)
CALL CORRLS (TM4L3R, TM4L3RD, TM4L3RDD, D, DD, DDD, LL2, 
1 L232, L23D2, L23DD2)
CALL ADDVEC (L23DD1, L23DD2, L23DD)

C Point mass m24
LL(1)=X20
LL(2)=Y20
LL(3)=Z20
CALL CORRLS (TM4L3R, TM4L3RD, TM4L3RDD, D, DD, DDD, LL, 
1 L24, L24D, L24DD)

C New inertia forces and torques of connecting rod using point masses
NUR2XP=(-m21*L21DD(1)-m22*L22DD(1)-m23*L23DD(1))/2
1 -m24*L24DD(1)
C NUR2XP=-M2*L24DD(1)
NUR2YP=(-m21*L21DD(2)-m22*L22DD(2)-m23*L23DD(2))/2
1 -m24*L24DD(2)
C NUR2YP=-M2*L24DD(2)
NUR2ZP=(-m21*L21DD(3)-m22*L22DD(3)-m23*L23DD(3))/2
C NUR2ZP = -M2*L24DD(3)
NUR2XD = -m21*(L211(2)*L21DD1(3)-L211(3)*L21DD1(2))/2
1 -m21*(L212(2)*L21DD2(3)-L212(3)*L21DD2(2))/2
1 -m22*(L221(2)*L22DD1(3)-L221(3)*L22DD1(2))/2
1 -m22*(L222(2)*L22DD2(3)-L222(3)*L22DD2(2))/2
1 -m23*(L231(2)*L23DD1(3)-L231(3)*L23DD1(2))/2
1 -m23*(L232(2)*L23DD2(3)-L232(3)*L23DD2(2))/2
1 -m24*(L24(2)*L24DD(3)-L24(3)*L24DD(2))
NUR2YD = -m21*(L211(3)*L21DD1(1)-L211(1)*L21DD1(3))/2
1 -m21*(L212(3)*L21DD2(1)-L212(1)*L21DD2(3))/2
1 -m22*(L221(3)*L22DD1(1)-L221(1)*L22DD1(3))/2
1 -m22*(L222(3)*L22DD2(1)-L222(1)*L22DD2(3))/2
1 -m23*(L231(3)*L23DD1(1)-L231(1)*L23DD1(3))/2
1 -m23*(L232(3)*L23DD2(1)-L232(1)*L23DD2(3))/2
1 -m24*(L24(3)*L24DD(1)-L24(1)*L24DD(3))
NUR2ZD = -m21*(L211(1)*L21DD1(2)-L211(2)*L21DD1(1))/2
1 -m21*(L212(1)*L21DD2(2)-L212(2)*L21DD2(1))/2
1 -m22*(L221(1)*L22DD1(2)-L221(2)*L22DD1(1))/2
1 -m22*(L222(1)*L22DD2(2)-L222(2)*L22DD2(1))/2
1 -m23*(L231(1)*L23DD1(2)-L231(2)*L23DD1(1))/2
1 -m23*(L232(1)*L23DD2(2)-L232(2)*L23DD2(1))/2
1 -m24*(L24(1)*L24DD(2)-L24(2)*L24DD(1))

C********************************************************OPTIMIZATION EQUATIONS********************************************************
px(1) = -L11DD(1)
px(2) = -L12DD(1)
px(3) = -L13DD(1)
px(4) = -L14DD(1)
px(5) = -L21DD(1)
px(6) = -L22DD(1)
px(7) = -L23DD(1)
px(8) = -L24DD(1)
px(9) = 0.0
py(1) = -L11DD(2)
py(2) = -L12DD(2)
py(3) = -L13DD(2)
py(4) = -L14DD(2)
py(5) = -L21DD(2)
py(6) = -L22DD(2)
py(7) = -L23DD(2)
py(8) = -L24DD(2)
py(9) = M3*S4DD*SIN(ALP4)
pz(1) = -L11DD(3)
pz(2) = -L12DD(3)
pz(3) = -L13DD(3)
pz(4) = -L14DD(3)
pz(5) = -L21DD(3)
pz(6) = -L22DD(3)
pz(7) = -L23DD(3)
pz(8) = -L24DD(3)
pz(9) = M3 * S4DD * COS(ALP4)

qx(1) = -(L111(2) * L11DD1(3) - L111(3) * L11DD1(2)) / 2
1   -(L112(2) * L11DD2(3) - L112(3) * L11DD2(2)) / 2
qx(2) = -(L121(2) * L12DD1(3) - L121(3) * L12DD1(2)) / 2
1   -(L122(2) * L12DD2(3) - L122(3) * L12DD2(2)) / 2
qx(3) = -(L131(2) * L13DD1(3) - L131(3) * L13DD1(2)) / 2
1   -(L132(2) * L13DD2(3) - L132(3) * L13DD2(2)) / 2
qx(4) = -(L14(2) * L14DD(3) - L14(3) * L14DD(2))
1   -(L14(2) * L14DD(3) - L14(3) * L14DD(2))
qx(5) = -(L211(2) * L21DD1(3) - L211(3) * L21DD1(2)) / 2
1   -(L212(2) * L21DD2(3) - L212(3) * L21DD2(2)) / 2
qx(6) = -(L221(2) * L22DD1(3) - L221(3) * L22DD1(2)) / 2
1   -(L222(2) * L22DD2(3) - L222(3) * L22DD2(2)) / 2
qx(7) = -(L231(2) * L23DD1(3) - L231(3) * L23DD1(2)) / 2
1   -(L232(2) * L23DD2(3) - L232(3) * L23DD2(2)) / 2
qx(8) = -(L24(2) * L24DD(3) - L24(3) * L24DD(2))
qx(9) = 0.0

qy(1) = -(L111(3) * L11DD1(1) - L111(1) * L11DD1(3)) / 2
1   -(L112(3) * L11DD2(1) - L112(1) * L11DD2(3)) / 2
qy(2) = -(L211(3) * L21DD1(1) - L211(1) * L21DD1(3)) / 2
1   -(L212(3) * L21DD2(1) - L212(1) * L21DD2(3)) / 2
qy(3) = -(L131(3) * L13DD1(1) - L131(1) * L13DD1(3)) / 2
1   -(L132(3) * L13DD2(1) - L132(1) * L13DD2(3)) / 2
qy(4) = -(L14(3) * L14DD(1) - L14(1) * L14DD(3))
1   -(L14(3) * L14DD(1) - L14(1) * L14DD(3))
qy(5) = -(L211(3) * L21DD1(1) - L211(1) * L21DD1(3)) / 2
1   -(L212(3) * L21DD2(1) - L212(1) * L21DD2(3)) / 2
qy(6) = -(L221(3) * L22DD1(1) - L221(1) * L22DD1(3)) / 2
1   -(L222(3) * L22DD2(1) - L222(1) * L22DD2(3)) / 2
qy(7) = -(L231(3) * L23DD1(1) - L231(1) * L23DD1(3)) / 2
1   -(L232(3) * L23DD2(1) - L232(1) * L23DD2(3)) / 2
qy(8) = -(L24(3) * L24DD(1) - L24(1) * L24DD(3))
qy(9) = A4 * fz(9)

qz(1) = -(L111(1) * L11DD1(2) - L111(2) * L11DD1(1)) / 2
1   -(L112(1) * L11DD2(2) - L112(2) * L11DD2(1)) / 2
qz(2) = -(L121(1) * L12DD1(2) - L121(2) * L12DD1(1)) / 2
1   -(L122(1) * L12DD2(2) - L122(2) * L12DD2(1)) / 2
qz(3) = -(L131(1) * L13DD1(2) - L131(2) * L13DD1(1)) / 2
1   -(L132(1) * L13DD2(2) - L132(2) * L13DD2(1)) / 2
qz(4) = -(L14(1) * L14DD(2) - L14(2) * L14DD(1))
\[
q_z(5) = -\frac{(L211(1) \cdot L21DD1(2) - L211(2) \cdot L21DD1(1))}{2}
\]

\[
q_z(6) = -\frac{(L221(1) \cdot L22DD1(2) - L221(2) \cdot L22DD1(1))}{2}
\]

\[
q_z(7) = -\frac{(L231(1) \cdot L23DD1(2) - L231(2) \cdot L23DD1(1))}{2}
\]

\[
q_z(8) = -\frac{(L24(1) \cdot L24DD(2) - L24(2) \cdot L24DD(1))}{2}
\]

\[
q_z(9) = A4 \cdot f_y(9)
\]

\[
\text{DO 30 K=1,4}
\]

\[
f_x(K) = TM1R(1,1) \cdot px(K) + TM1R(1,2) \cdot py(K) + TM1R(1,3) \cdot pz(K)
\]

\[
f_y(K) = TM1R(2,1) \cdot px(K) + TM1R(2,2) \cdot py(K) + TM1R(2,3) \cdot pz(K)
\]

\[
f_z(K) = TM1R(3,1) \cdot px(K) + TM1R(3,2) \cdot py(K) + TM1R(3,3) \cdot pz(K)
\]

\[
t_x(K) = TM1R(1,1) \cdot qx(K) + TM1R(1,2) \cdot gy(K) + TM1R(1,3) \cdot qz(K)
\]

\[
t_y(K) = TM1R(2,1) \cdot qx(K) + TM1R(2,2) \cdot gy(K) + TM1R(2,3) \cdot qz(K)
\]

\[
t_z(K) = TM1R(3,1) \cdot qx(K) + TM1R(3,2) \cdot qy(K) + TM1R(3,3) \cdot qz(K)
\]

\[
30 \ \text{CONTINUE}
\]

\[
\text{DO 40 K1=5,8}
\]

\[
f_x(K1) = TM1L2R(1,1) \cdot px(K1) + TM1L2R(1,2) \cdot py(K1) + TM1L2R(1,3) \cdot pz(K1)
\]

\[
f_y(K1) = TM1L2R(2,1) \cdot px(K1) + TM1L2R(2,2) \cdot py(K1) + TM1L2R(2,3) \cdot pz(K1)
\]

\[
f_z(K1) = TM1L2R(3,1) \cdot px(K1) + TM1L2R(3,2) \cdot py(K1) + TM1L2R(3,3) \cdot pz(K1)
\]

\[
t_x(K1) = TM1L2R(1,1) \cdot qx(K1) + TM1L2R(1,2) \cdot qy(K1) + TM1L2R(1,3) \cdot qz(K1)
\]

\[
t_y(K1) = TM1L2R(2,1) \cdot qx(K1) + TM1L2R(2,2) \cdot qy(K1) + TM1L2R(2,3) \cdot qz(K1)
\]

\[
t_z(K1) = TM1L2R(3,1) \cdot qx(K1) + TM1L2R(3,2) \cdot qy(K1) + TM1L2R(3,3) \cdot qz(K1)
\]

\[
40 \ \text{CONTINUE}
\]

\[
f_x(9) = M4R(1,1) \cdot px(9) + M4R(1,2) \cdot py(9) + M4R(1,3) \cdot pz(9)
\]

\[
f_y(9) = M4R(2,1) \cdot px(9) + M4R(2,2) \cdot py(9) + M4R(2,3) \cdot pz(9)
\]

\[
f_z(9) = M4R(3,1) \cdot px(9) + M4R(3,2) \cdot py(9) + M4R(3,3) \cdot pz(9)
\]

\[
t_x(9) = M4R(1,1) \cdot qx(9) + M4R(1,2) \cdot qy(9) + M4R(1,3) \cdot qz(9)
\]

\[
t_y(9) = M4R(2,1) \cdot qx(9) + M4R(2,2) \cdot qy(9) + M4R(2,3) \cdot qz(9)
\]

\[
t_z(9) = M4R(3,1) \cdot qx(9) + M4R(3,2) \cdot qy(9) + M4R(3,3) \cdot qz(9)
\]

\[
f_3y(5) = -(f_y(5) + t_z(5)/A2)
\]

\[
f_3y(6) = -(f_y(6) + t_z(6)/A2)
\]

\[
f_3y(7) = -(f_y(7) + t_z(7)/A2)
\]

\[
f_3y(8) = -(f_y(8) + t_z(8)/A2)
\]

\[
f_3z(5) = -(f_z(5) + t_y(5)/A2)
\]

\[
f_3z(6) = -(f_z(6) + t_y(6)/A2)
\]

\[
f_3z(7) = -(f_z(7) + t_y(7)/A2)
\]

\[
f_3z(8) = -(f_z(8) + t_y(8)/A2)
\]

\[
f_3x(5) = \cos(THT3) \cdot f_3y(5)/\sin(THT3)
\]

\[
f_3x(6) = \cos(THT3) \cdot f_3y(6)/\sin(THT3)
\]

\[
f_3x(7) = \cos(THT3) \cdot f_3y(7)/\sin(THT3)
\]

\[
f_3x(8) = \cos(THT3) \cdot f_3y(8)/\sin(THT3)
\]
\[ f_{4x}(5) = (\sin(\eta_3) f_{3z}(5) - \cos(\theta_3) \cos(\eta_3) f_{3x}(5) - \sin(\theta_3) \cos(\eta_3) f_{3y}(5)) \]

\[ f_{4x}(6) = (\sin(\eta_3) f_{3z}(6) - \cos(\theta_3) \cos(\eta_3) f_{3x}(6) - \sin(\theta_3) \cos(\eta_3) f_{3y}(6)) \]

\[ f_{4x}(7) = (\sin(\eta_3) f_{3z}(7) - \cos(\theta_3) \cos(\eta_3) f_{3x}(7) - \sin(\theta_3) \cos(\eta_3) f_{3y}(7)) \]

\[ f_{4x}(8) = (\sin(\eta_3) f_{3z}(8) - \cos(\theta_3) \cos(\eta_3) f_{3x}(8) - \sin(\theta_3) \cos(\eta_3) f_{3y}(8)) \]

\[ f_{4x}(9) = (\cos(\theta_3) \cos(\eta_3) f_{3x}(9) - f_{x}(9)) \]

\[ f_{4y}(5) = -(\cos(\eta_3) f_{3z}(5) + \cos(\theta_3) \sin(\eta_3) f_{3x}(5) + \sin(\theta_3) \sin(\eta_3) f_{3y}(5)) \]

\[ f_{4y}(6) = -(\cos(\eta_3) f_{3z}(6) + \cos(\theta_3) \sin(\eta_3) f_{3x}(6) + \sin(\theta_3) \sin(\eta_3) f_{3y}(6)) \]

\[ f_{4y}(7) = -(\cos(\eta_3) f_{3z}(7) + \cos(\theta_3) \sin(\eta_3) f_{3x}(7) + \sin(\theta_3) \sin(\eta_3) f_{3y}(7)) \]

\[ f_{4y}(8) = -(\cos(\eta_3) f_{3z}(8) + \cos(\theta_3) \sin(\eta_3) f_{3x}(8) + \sin(\theta_3) \sin(\eta_3) f_{3y}(8)) \]

\[ f_{4y}(9) = (\cos(\theta_3) \sin(\eta_3) f_{3x}(9) + f_{y}(9)) \]

\[ t_{4x}(9) = T_{4I} \]

\[ t_{4y}(9) = T_{4J} \]

\[ t_{4z}(9) = T_{4K} \]

\[ f_{2x}(5) = (-(f_{3x}(5) - f_{x}(5)) \cos(\theta_2) \cos(\eta_2) + (f_{y}(5) + f_{y}(5)) \sin(\eta_2)) \]

\[ f_{2x}(6) = (-(f_{3x}(6) - f_{x}(6)) \cos(\theta_2) \cos(\eta_2) + (f_{y}(6) + f_{y}(6)) \sin(\eta_2)) \]

\[ f_{2x}(7) = (-(f_{3x}(7) - f_{x}(7)) \cos(\theta_2) \cos(\eta_2) + (f_{y}(7) + f_{y}(7)) \sin(\eta_2)) \]

\[ f_{2x}(8) = (-(f_{3x}(8) - f_{x}(8)) \cos(\theta_2) \cos(\eta_2) + (f_{y}(8) + f_{y}(8)) \sin(\eta_2)) \]

\[ f_{2x}(9) = -(f_{3x}(9) \cos(\theta_2) \cos(\eta_2)) \]

\[ f_{2y}(5) = (-(f_{3x}(5) - f_{x}(5)) \sin(\theta_2) \cos(\eta_2) - (f_{y}(5) + f_{y}(5)) \sin(\eta_2)) \]

\[ f_{2y}(6) = (-(f_{3x}(6) - f_{x}(6)) \sin(\theta_2) \cos(\eta_2) - (f_{y}(6) + f_{y}(6)) \sin(\eta_2)) \]

\[ f_{2y}(7) = (-(f_{3x}(7) - f_{x}(7)) \sin(\theta_2) \cos(\eta_2) - (f_{y}(7) + f_{y}(7)) \sin(\eta_2)) \]

\[ f_{2y}(8) = (-(f_{3x}(8) - f_{x}(8)) \sin(\theta_2) \cos(\eta_2) - (f_{y}(8) + f_{y}(8)) \sin(\eta_2)) \]

\[ f_{2y}(9) = -(f_{3y}(9) \sin(\theta_2) \cos(\eta_2)) \]

\[ \text{if} \ (\text{ABS} (\text{COS} (\eta_3)) \ < \ \text{ABS} (\text{SIN} (\eta_3))) \ \text{then} \]

\[ f_{2z}(5) = -(f_{z}(5) + f_{3z}(5) + f_{2y}(5) \sin(\theta_2) \sin(\eta_2) + f_{x}(5) \cos(\theta_2) \sin(\eta_2)) / \cos(\eta_3) \]

\[ f_{2z}(6) = -(f_{x}(6) + f_{3z}(6) + f_{2y}(6) \sin(\theta_2) \sin(\eta_2) + f_{x}(6) \cos(\theta_2) \sin(\eta_2)) / \cos(\eta_3) \]
\[ f_2x(6) \times \cos(THT2) \times \sin(ETA2) / \cos(ETA2) \]
\[ f_2z(7) = -(fz(7) + f_3z(7) + f_2y(7) \times \sin(THT2) \times \sin(ETA2) + \]
\[ f_2x(7) \times \cos(THT2) \times \sin(ETA2) / \cos(ETA2) \]
\[ f_2z(8) = -(fz(8) + f_3z(8) + f_2y(8) \times \sin(THT2) \times \sin(ETA2) + \]
\[ f_2x(8) \times \cos(THT2) \times \sin(ETA2) / \cos(ETA2) \]
\[ f_2z(9) = -(f_2y(9) \times \sin(THT2) \times \sin(ETA2) + \]
\[ f_2x(9) \times \cos(THT2) \times \sin(ETA2) / \cos(ETA2) \]
\[ \text{else} \]
\[ f_2z(5) = (fz(5) + f_3z(5) + f_2y(5) \times \sin(THT2) \times \cos(ETA2) + \]
\[ f_2x(5) \times \cos(THT2) \times \cos(ETA2) / \sin(ETA2) \]
\[ f_2z(6) = (fz(6) + f_3z(6) + f_2y(6) \times \sin(THT2) \times \cos(ETA2) + \]
\[ f_2x(6) \times \cos(THT2) \times \cos(ETA2) / \sin(ETA2) \]
\[ f_2z(7) = (fz(7) + f_3z(7) + f_2y(7) \times \sin(THT2) \times \cos(ETA2) + \]
\[ f_2x(7) \times \cos(THT2) \times \cos(ETA2) / \sin(ETA2) \]
\[ f_2z(8) = (fz(8) + f_3z(8) + f_2y(8) \times \sin(THT2) \times \cos(ETA2) + \]
\[ f_2x(8) \times \cos(THT2) \times \cos(ETA2) / \sin(ETA2) \]
\[ f_2z(9) = (f_2y(9) \times \sin(THT2) \times \cos(ETA2) + \]
\[ f_2x(9) \times \cos(THT2) \times \cos(ETA2) / \sin(ETA2) \]
\[ \text{endif} \]

DO 50 K2=5,8
\[ f_3x(K2) = M1L2R(1,1) \times f_3x(K2) + M1L2R(1,2) \times f_3y(K2) + M1L2R(1,3) \times f_3z(K2) \]
\[ f_3y(K2) = M1L2R(2,1) \times f_3x(K2) + M1L2R(2,2) \times f_3y(K2) + M1L2R(2,3) \times f_3z(K2) \]
\[ f_3z(K2) = M1L2R(3,1) \times f_3x(K2) + M1L2R(3,2) \times f_3y(K2) + M1L2R(3,3) \times f_3z(K2) \]
50 CONTINUE

DO 60 K3=5,8
\[ f_2x(K3) = M1R(1,1) \times f_2x(K3) + M1R(1,2) \times f_2y(K3) + M1R(1,3) \times f_2z(K3) \]
\[ f_2y(K3) = M1R(2,1) \times f_2x(K3) + M1R(2,2) \times f_2y(K3) + M1R(2,3) \times f_2z(K3) \]
\[ f_2z(K3) = M1R(3,1) \times f_2x(K3) + M1R(3,2) \times f_2y(K3) + M1R(3,3) \times f_2z(K3) \]
60 CONTINUE

DO 70 K4=5,8
\[ f_4x(K4) = TM4R(1,1) \times f_4x(K4) + TM4R(1,2) \times f_4y(K4) \]
\[ f_4y(K4) = TM4R(2,1) \times f_4x(K4) + TM4R(2,2) \times f_4y(K4) \]
\[ f_4z(K4) = TM4R(3,1) \times f_4x(K4) + TM4R(3,2) \times f_4y(K4) \]
70 CONTINUE

\[ t_4x(9) = TM4R(1,1) \times t_4x(9) + TM4R(1,2) \times t_4y(9) + TM4R(1,3) \times t_4z(9) \]
\[ t_4y(9) = TM4R(2,1) \times t_4x(9) + TM4R(2,2) \times t_4y(9) + TM4R(2,3) \times t_4z(9) \]
\[ t_4z(9) = TM4R(3,1) \times t_4x(9) + TM4R(3,2) \times t_4y(9) + TM4R(3,3) \times t_4z(9) \]

\[ f_1x(1) = -fz(1) \]
\[ f_1x(2) = -fz(2) \]
\[ f_1x(3) = -fz(3) \]
\[ f_1x(4) = -fz(4) \]
\[ f_1x(5) = -fz(5) \]
\[ f_1x(6) = -fz(6) \]
\[ f_{1z}(7) = -f_{2z}(7) \]
\[ f_{1z}(8) = -f_{2z}(8) \]
\[ f_{1z}(9) = -f_{2z}(9) \]

\[ f_{1x}(1) = (-f_{1x}(1) \cdot \cos(THT 1) + f_{y}(1) \cdot \sin(THT 1)) \]
\[ f_{1x}(2) = (-f_{1x}(2) \cdot \cos(THT 1) + f_{y}(2) \cdot \sin(THT 1)) \]
\[ f_{1x}(3) = (-f_{x}(3) \cdot \cos(THT 1) + f_{y}(3) \cdot \sin(THT 1)) \]
\[ f_{1x}(4) = (-f_{x}(4) \cdot \cos(THT 1) + f_{y}(4) \cdot \sin(THT 1)) \]
\[ f_{1x}(5) = (-f_{x}(5) \cdot \cos(THT 1) + f_{y}(5) \cdot \sin(THT 1)) \]
\[ f_{1x}(6) = (-f_{x}(6) \cdot \cos(THT 1) + f_{y}(6) \cdot \sin(THT 1)) \]
\[ f_{1x}(7) = (-f_{x}(7) \cdot \cos(THT 1) + f_{y}(7) \cdot \sin(THT 1)) \]
\[ f_{1x}(8) = (-f_{x}(8) \cdot \cos(THT 1) + f_{y}(8) \cdot \sin(THT 1)) \]
\[ f_{1x}(9) = (-f_{x}(9) \cdot \cos(THT 1) + f_{y}(9) \cdot \sin(THT 1)) \]

\[ f_{1y}(1) = -(f_{x}(1) \cdot \sin(THT 1) + f_{y}(1) \cdot \cos(THT 1)) \]
\[ f_{1y}(2) = -(f_{x}(2) \cdot \sin(THT 1) + f_{y}(2) \cdot \cos(THT 1)) \]
\[ f_{1y}(3) = -(f_{x}(3) \cdot \sin(THT 1) + f_{y}(3) \cdot \cos(THT 1)) \]
\[ f_{1y}(4) = -(f_{x}(4) \cdot \sin(THT 1) + f_{y}(4) \cdot \cos(THT 1)) \]
\[ f_{1y}(5) = -(f_{x}(5) \cdot \sin(THT 1) + f_{y}(5) \cdot \cos(THT 1)) \]
\[ f_{1y}(6) = -(f_{x}(6) \cdot \sin(THT 1) + f_{y}(6) \cdot \cos(THT 1)) \]
\[ f_{1y}(7) = -(f_{x}(7) \cdot \sin(THT 1) + f_{y}(7) \cdot \cos(THT 1)) \]
\[ f_{1y}(8) = -(f_{x}(8) \cdot \sin(THT 1) + f_{y}(8) \cdot \cos(THT 1)) \]
\[ f_{1y}(9) = -(f_{x}(9) \cdot \sin(THT 1) + f_{y}(9) \cdot \cos(THT 1)) \]

\[ t_{lz}(1) = (f_{1y}(1) \cdot A_{1} \cdot \cos(THT 1) - f_{1x}(1) \cdot A_{1} \cdot \sin(THT 1) - t_{z}(1)) \]
\[ t_{lz}(2) = (f_{1y}(2) \cdot A_{1} \cdot \cos(THT 1) - f_{1x}(2) \cdot A_{1} \cdot \sin(THT 1) - t_{z}(2)) \]
\[ t_{lz}(3) = (f_{1y}(3) \cdot A_{1} \cdot \cos(THT 1) - f_{1x}(3) \cdot A_{1} \cdot \sin(THT 1) - t_{z}(3)) \]
\[ t_{lz}(4) = (f_{1y}(4) \cdot A_{1} \cdot \cos(THT 1) - f_{1x}(4) \cdot A_{1} \cdot \sin(THT 1) - t_{z}(4)) \]
\[ t_{lz}(5) = (f_{1y}(5) \cdot A_{1} \cdot \cos(THT 1) - f_{1x}(5) \cdot A_{1} \cdot \sin(THT 1)) \]
\[ t_{lz}(6) = (f_{1y}(6) \cdot A_{1} \cdot \cos(THT 1) - f_{1x}(6) \cdot A_{1} \cdot \sin(THT 1)) \]
\[ t_{lz}(7) = (f_{1y}(7) \cdot A_{1} \cdot \cos(THT 1) - f_{1x}(7) \cdot A_{1} \cdot \sin(THT 1)) \]
\[ t_{lz}(8) = (f_{1y}(8) \cdot A_{1} \cdot \cos(THT 1) - f_{1x}(8) \cdot A_{1} \cdot \sin(THT 1)) \]
\[ t_{lz}(9) = (f_{1y}(9) \cdot A_{1} \cdot \cos(THT 1) - f_{1x}(9) \cdot A_{1} \cdot \sin(THT 1)) \]

\[ t_{lx}(1) = (-f_{1x}(1) \cdot S_{1} - t_{x}(1) \cdot \cos(THT 1) + f_{lz}(1) \cdot A_{1} + t_{y}(1) \cdot \sin(THT 1)) \]
\[ t_{lx}(2) = (-f_{1x}(2) \cdot S_{1} - t_{x}(2) \cdot \cos(THT 1) + f_{lz}(2) \cdot A_{1} + t_{y}(2) \cdot \sin(THT 1)) \]
\[ t_{lx}(3) = (-f_{1x}(3) \cdot S_{1} - t_{x}(3) \cdot \cos(THT 1) + f_{lz}(3) \cdot A_{1} + t_{y}(3) \cdot \sin(THT 1)) \]
\[ t_{lx}(4) = (-f_{1x}(4) \cdot S_{1} - t_{x}(4) \cdot \cos(THT 1) + f_{lz}(4) \cdot A_{1} + t_{y}(4) \cdot \sin(THT 1)) \]
\[ t_{lx}(5) = t_{lz}(5) - f_{1y}(5) \cdot S_{1} \]
\[ t_{lx}(6) = t_{lz}(6) - f_{1y}(6) \cdot S_{1} \]
\[ t_{lx}(7) = t_{lz}(7) - f_{1y}(7) \cdot S_{1} \]
\[ t_{lx}(8) = t_{lz}(8) - f_{1y}(8) \cdot S_{1} \]
\[ t_{lx}(9) = t_{lz}(9) - f_{1y}(9) \cdot S_{1} \]

\[ t_{ly}(1) = f_{1x}(1) \cdot S_{1} - f_{lz}(1) \cdot A_{1} - t_{x}(1) \cdot \sin(THT 1) - t_{y}(1) \cdot \cos(THT 1) \]
\[\begin{align*}
t_{1y}(2) &= f_{1x}(2)S_1 - f_{1z}(2)A_1 - tx(2)\sin(THT1) - ty(2)\cos(THT1) \\
t_{1y}(3) &= f_{1x}(3)S_1 - f_{1z}(3)A_1 - tx(3)\sin(THT1) - ty(3)\cos(THT1) \\
t_{1y}(4) &= f_{1x}(4)S_1 - f_{1z}(4)A_1 - tx(4)\sin(THT1) - ty(4)\cos(THT1) \\
t_{1y}(5) &= f_{1x}(5)S_1 - f_{1z}(5)A_1 \\
t_{1y}(6) &= f_{1x}(6)S_1 - f_{1z}(6)A_1 \\
t_{1y}(7) &= f_{1x}(7)S_1 - f_{1z}(7)A_1 \\
t_{1y}(8) &= f_{1x}(8)S_1 - f_{1z}(8)A_1 \\
t_{1y}(9) &= f_{1x}(9)S_1 - f_{1z}(9)A_1 \\
\end{align*}\]

CALL SQUARE (fx,fxsq)
CALL SQUARE (fy,fysq)
CALL SQUARE (fz,fzsq)
CALL SQUARE (tx,txsq)
CALL SQUARE (ty,tysq)
CALL SQUARE (tz,tzsq)
CALL SQUARE (flx,f1xsq)
CALL SQUARE (fly,flysq)
CALL SQUARE (flz,f1zsq)
CALL SQUARE (tlx,t1xsq)
CALL SQUARE (tly,tlysq)
CALL SQUARE (t1z,t1zsq)
CALL SQUARE (f2x,f2xsq)
CALL SQUARE (f2y,f2ysq)
CALL SQUARE (f2z,f2zsq)
CALL SQUARE (f3x,f3xsq)
CALL SQUARE (f3y,f3ysq)
CALL SQUARE (f3z,f3zsq)
CALL SQUARE (f4x,f4xsq)
CALL SQUARE (f4y,f4ysq)
CALL SQUARE (t4x,t4xsq)
CALL SQUARE (t4y,t4ysq)
CALL SQUARE (t4z,t4zsq)

\[\begin{align*}
C1 &= 37(M1**2)(A1**2)(THT1D**4) \\
C2 &= C1*(A1**2) \\
C3 &= C1*(BL**2) \\
DO 80 I1=1,9 \\
DO 90 J1=1,9 \\
fsq(I1,J1) &= (fxsq(I1,J1)+fysq(I1,J1)+fzsq(I1,J1))/C1 \\
tsq(I1,J1) &= (txsq(I1,J1)+tysq(I1,J1)+tzsq(I1,J1))/C2 \\
rsq(I1,J1) &= (flxsq(I1,J1)+flysq(I1,J1)+flzsq(I1,J1)) \\
1 &= f2xsq(I1,J1)+f2ysq(I1,J1)+f2zsq(I1,J1) \\
1 &= f3xsq(I1,J1)+f3ysq(I1,J1)+f3zsq(I1,J1) \\
1 &= f4xsq(I1,J1)+f4ysq(I1,J1)+f4zsq(I1,J1))/C1 \\
msq(I1,J1) &= (t1xsq(I1,J1)+t1ysq(I1,J1)) \\
\end{align*}\]
$t_0 \text{sq}(I_1, J_1) = t_1 \text{zsg}(I_1, J_1) / C_2$

$F(I_1, J_1) = W_1 * f_\text{sq}(I_1, J_1) + W_2 * t_\text{sq}(I_1, J_1) + W_3 * r_\text{sq}(I_1, J_1) + W_4 * m_\text{sq}(I_1, J_1) + W_5 * t_0 \text{sq}(I_1, J_1)$

$OF(I_1, J_1) = OF(I_1, J_1) + F(I_1, J_1)$

$h(I_1, J_1) = 2 * OF(I_1, J_1)$

CONTINUE

write (6, 1) $F(1, 1)$
write (6, 1) $OF(6, 8), OF(8, 6), OF(9, 9)$
write (6, 1) $fx(1), fx(2), fx(3)$
write (6, 1) $L_11 DD(1), L_12 DD(1), L_13 DD(1)$
write (6, 1) $fx(4), fx(5), fx(6)$
write (6, 1) $fx(7), fx(8), fx(9)$
write (6, 1) $fxsq(1, 1), fxsq(1, 2), fxsq(1, 3)$
write (6, 1) $fxsq(1, 4), fxsq(1, 5), fxsq(1, 6)$
write (6, 1) $tsq(1, 7), tsq(1, 8), tsq(1, 9)$
write (6, 1) $NUR2XP, NUR2YP, NUR2ZP$
write (6, 1) $M1L2R2P(1), M1L2R2P(2), M1L2R2P(3)$

ELSE
THT2 = THT2 + DELTA(1)
ETA2 = ETA2 + DELTA(2)
THT3 = THT3 + DELTA(3)
ETA3 = ETA3 + DELTA(4)
S4 = S4 + DELTA(5)
I = I + 1
GO TO 10
ENDIF
ELSE
WRITE(6, 2) I
STOP
ENDIF

C 1 FORMAT('SLIDER VELOCITY=', F10.5, 2X, 'WHEN CRANK ANGLE=')
C 1, F7.2, 1X, 'S1=', F7.2, 1X, 'NO OF ITERATIONS=', I3)
1 FORMAT(3F20.5)
2 FORMAT('THE METHOD FAILS AFTER', I3, 'ITERATIONS')

C Determination of the objective function
OBJFUN = 0.0
NUOBJFUN = 0.0
if (I3 .EQ. 2) then
do 42 I4 = 1, 9
do 43 J4 = 1, 9
if (I4 .NE. J4) then
\[ F(I4,J4) = 2.0 \times F(I4,J4) \]

\[ \text{OBJFUN} = \text{OBJFUN} + m(I4) \times m(J4) \times F(I4,J4) \]

\[ \text{NUOBJFUN} = \text{NUOBJFUN} + \text{sol}(I4) \times \text{sol}(J4) \times F(I4,J4) \]

43 \text{ continue}
42 \text{ continue}

\( \text{call dotprod}(m, fx, mfx) \)
\( \text{C} \quad \text{call dotprod}(\text{sol}, fx, nufx) \)
\( \text{call dotprod}(m, fy, mfy) \)
\( \text{C} \quad \text{call dotprod}(\text{sol}, fy, nufy) \)
\( \text{call dotprod}(m, fz, mfz) \)
\( \text{C} \quad \text{call dotprod}(\text{sol}, fz, nufz) \)
\( \text{call dotprod}(m, tx, mtx) \)
\( \text{C} \quad \text{call dotprod}(\text{sol}, tx, nutx) \)
\( \text{call dotprod}(m, ty, mty) \)
\( \text{C} \quad \text{call dotprod}(\text{sol}, ty, nuty) \)
\( \text{call dotprod}(m, tz, mtz) \)
\( \text{C} \quad \text{call dotprod}(\text{sol}, tz, nutz) \)
\( \text{call dotprod}(m, flx, mflx) \)
\( \text{C} \quad \text{call dotprod}(\text{sol}, flx, nuflx) \)
\( \text{call dotprod}(m, fly, mfly) \)
\( \text{C} \quad \text{call dotprod}(\text{sol}, fly, nufly) \)
\( \text{call dotprod}(m, flz, mflz) \)
\( \text{C} \quad \text{call dotprod}(\text{sol}, flz, nuflz) \)
\( \text{call dotprod}(m, tlx, mtlx) \)
\( \text{C} \quad \text{call dotprod}(\text{sol}, tlx, nutlx) \)
\( \text{call dotprod}(m, tly, mtly) \)
\( \text{C} \quad \text{call dotprod}(\text{sol}, tly, nutly) \)
\( \text{call dotprod}(m, tlz, mtlz) \)
\( \text{C} \quad \text{call dotprod}(\text{sol}, tlz, nutlz) \)
\( \text{call dotprod}(m, f2x, mf2x) \)
\( \text{C} \quad \text{call dotprod}(\text{sol}, f2x, nuf2x) \)
\( \text{call dotprod}(m, f2y, mf2y) \)
\( \text{C} \quad \text{call dotprod}(\text{sol}, f2y, nuf2y) \)
\( \text{call dotprod}(m, f2z, mf2z) \)
\( \text{C} \quad \text{call dotprod}(\text{sol}, f2z, nuf2z) \)
\( \text{call dotprod}(m, f3x, mflz) \)
\( \text{C} \quad \text{call dotprod}(\text{sol}, f3x, nuf3x) \)
\( \text{call dotprod}(m, f3y, mf3y) \)
\( \text{C} \quad \text{call dotprod}(\text{sol}, f3y, nuf3y) \)
\( \text{call dotprod}(m, f3z, mf3z) \)
\( \text{C} \quad \text{call dotprod}(\text{sol}, f3z, nuf3z) \)
\( \text{call dotprod}(m, f4x, mf4x) \)
\( \text{C} \quad \text{call dotprod}(\text{sol}, f4x, nuf4x) \)
call dotprod(m,f4y,mf4y)
C call dotprod(sol,f4y,nuf4y)
call dotprod(m,t4x,mt4x)
C call dotprod(sol,t4x,nut4x)
call dotprod(m,t4y,mt4y)
C call dotprod(sol,t4y,nut4y)
call dotprod(m,t4z,mt4z)
C call dotprod(sol,t4z,nut4z)

shf=(mf1x**2+mf3y**2+mfz**2)**0.5
sht=(mt1x**2+mt3y**2+mtz**2)**0.5
brf=((mf1x+mflx+mf3x+mf4x)**2
1 +(mf1y+mf2y+mf3y+mf4y)**2
1 +(mf1z+mf2z+mf3z)**2)**0.5
brt=((mt1x+mt4x)**2
1 +(mt1y+mt4y)**2
1 +(mt1z+mt4z)**2)**0.5
int=mt1z

C write (6,*) OBJFUN,NUOBJFUN
C write (6,*) mfx,nufx
C write (6,*) mfy,nufy
C write (6,*) mfz,nufz
C write (6,*) flx(3),flx(1),m(3)*fx(3),sol(3)*fx(3)

nufx=0.0
nufy=0.0
nufz=0.0
nutx=0.0
nuty=0.0
nutz=0.0

nuf1x=0.0
nuf1y=0.0
nuf1z=0.0
nut1x=0.0
nut1y=0.0
nut1z=0.0
nuf2x=0.0
nuf2y=0.0
nuf2z=0.0
nuf3x=0.0
nuf3y=0.0
nuf3z=0.0
nuf4x=0.0
nuf4y=0.0
nut4x=0.0
nut4y=0.0
nut4z=0.0

do 233 i10=1,9
    nufx=nufx+sol(i10)*fx(i10)
nufy=nufy+sol(i10)*fy(i10)
nufz=nufz+sol(i10)*fz(i10)
nutx=nutx+sol(i10)*tx(i10)
nuty=nuty+sol(i10)*ty(i10)
nutz=nutz+sol(i10)*tz(i10)
nuf1x=nuf1x+sol(i10)*flx(i10)
nuf1y=nuf1y+sol(i10)*f1y(i10)
nuf1z=nuf1z+sol(i10)*f1z(i10)
nut1x=nut1x+sol(i10)*t1x(i10)
nut1y=nut1y+sol(i10)*t1y(i10)
nut1z=nut1z+sol(i10)*t1z(i10)
nuf2x=nuf2x+sol(i10)*f2x(i10)
nuf2y=nuf2y+sol(i10)*f2y(i10)
nuf2z=nuf2z+sol(i10)*f2z(i10)
nuf3x=nuf3x+sol(i10)*f3x(i10)
nuf3y=nuf3y+sol(i10)*f3y(i10)
nuf3z=nuf3z+sol(i10)*f3z(i10)
nuf4x=nuf4x+sol(i10)*f4x(i10)
nuf4y=nuf4y+sol(i10)*f4y(i10)
nuf4z=nuf4z+sol(i10)*f4z(i10)
233 continue

nushf=(((nufx**2)+(nufy**2)+(nufz**2))**0.5
nusht=(((nutx**2)+nuty**2+nutz**2))**0.5
nubrf=(((nuf1x+nuf2x+nuf3x+nuf4x)**2
l +((nuf1y+nuf2y+nuf3y+nuf4y)**2
l +((nuf1z+nuf2z+nuf3z)**2)**0.5
l +((nut1x+nut4x)**2
l +((nut1y+nut4y)**2
l +((nut1z+nut4z)**2)**0.5

nuint=nut1z

DOBJFUN=(OBJFUN-NUOBJFUN)*100/OBJFUN
dshf=(shf-nushf)*100/shf
dsht=(sht-nusht)*100/sht
dbrf=(brf-nubrf)*100/brf
dbrt=(brt-nubrt)*100/brt
dint=(int-nuint)*100/int
write(6,*), TH1, dint
endif
400 CONTINUE
C write (6, 1) OF(6,8), OF(8,6), OF(9,9)
call qprog (nvar,ncon,neq,aa,lda,b,gg,h,ldh,diag,sol,
1 nact,iact,alamda)
call umach (2,nout)
C write (nout,999) (sol(k),k=1,nvar)
C 999 format ('the solution vector is',/,'sol=(',9F12.6,')')
C write(6,*), m(1), m(2), m(3), m(4), m(5), m(6), m(7), m(8), m(9)
41 continue

NUI1X=(Y1**2)* sol(2)+(Z1**2)* sol(3)
NUI2X=(Y2**2)* sol(6)+(Z2**2)* sol(7)
NUI1Y=(X1**2)* sol(1)+(Z1**2)* sol(3)
NUI2Y=(X2**2)* sol(5)+(Z2**2)* sol(7)
NUI1Z=(X1**2)* sol(1)+(Y1**2)* sol(2)
NUI2Z=(X2**2)* sol(5)+(Y2**2)* sol(6)
C write(6,*), I1X, NUI1X
C write(6,*), I1Y, NUI1Y
C write(6,*), I1Z, NUI1Z
C write(6,*), I2X, NUI2X
C write(6,*), I2Y, NUI2Y
C write(6,*), I2Z, NUI2Z

STOP
END

C*****************************************SUBROUTINES START HERE*****************************************

SUBROUTINE PRODUCTDUAL (AR, AD, BR, BD, RESLTR, RESLTD)
real*8 AR(3,3), AD(3,3), BR(3,3), BD(3,3), TEMP1(3,3)
real*8 RESLTR(3,3), RESLTD(3,3), TEMP2(3,3)
CALL MULTI (AR, BR, RESLTR, 3, 3, 3)
CALL MULTI (AD, BR, TEMP1, 3, 3, 3)
CALL MULTI (AR, BD, TEMP2, 3, 3, 3)
CALL ADDMAT (TEMP1, TEMP2, RESLTD)
RETURN
END
C
SUBROUTINE CORRLS (MTRX, MTRXD, MTRXDD, D, DD, DDD,
LL,L,LD,LDD)
real*8 MTRX(3,3),MTRXD(3,3),MTRXDD(3,3),D(3),DD(3),
DDD(3),LL(3),L(3),LD(3),LDD(3),
TMTRX(3,3),TMTRXD(3,3),OMEGA(3,3),
ALPHA1(3,3),ALPHA2(3,3),ALPHA(3,3),
WX,WY,WZ,ALPX,ALPY,ALPZ,MTRXL(3),MTRXDL(3)
CALL KINEMATIC (MTRX,MTRXD,MTRXDD,D,DD,DDD,
LL,L,LD,LDD)
CALL MULMAVEC (MTRX,LL,MTRXL,3,3)
CALL MULMAVEC (MTRXD,LL,MTRXDL,3,3)
CALL TRANS (MTRX,TMTRX,3,3)
CALL TRANS (MTRXD,TMTRXD,3,3)
CALL MULTI (MTRXD,TMTRX,OMEGA,3,3,3)
CALL MULTI (MTRXDD,TMTRX,ALPHA1,3,3,3)
CALL MULTI (MTRXD,TMTRXD,ALPHA2,3,3,3)
CALL ADDMAT (ALPHA1,ALPHA2,ALPHA)
WX=OMEGA(3,2)
WY=OMEGA(1,3)
WZ=OMEGA(2,1)
ALPX=ALPHA(3,2)
ALPY=ALPHA(1,3)
ALPZ=ALPHA(2,1)
LDD(1)=WY*(WX*MTRXL(2)-WY*MTRXL(1))
1  -WZ*(WZ*MTRXL(1)-WX*MTRXL(3))
C 1  +2*WY*MTRXDL(3)-2*WZ*MTRXDL(2)
1  +ALPY*MTRXL(3)-ALPZ*MTRXL(2)
1  +DDD(1)
LDD(2)=WZ*(WY*MTRXL(3)-WZ*MTRXL(2))
1  -WX*(WX*MTRXL(2)-WY*MTRXL(1))
C 1  +2*WZ*MTRXDL(1)-2*WX*MTRXDL(3)
1  +ALPZ*MTRXL(1)-ALPX*MTRXL(3)
1  +DDD(2)
LDD(3)=WX*(WZ*MTRXL(1)-WX*MTRXL(3))
1  -WY*(WZ*MTRXL(3)-WZ*MTRXL(2))
C 1  +2*WX*MTRXDL(2)-2*WY*MTRXDL(1)
1  +ALPX*MTRXL(2)-ALPY*MTRXL(1)
1  +DDD(3)
RETURN
END
C
SUBROUTINE KINEMATIC (MTRX,MTRXD,MTRXDD,D,DD,DDD,
real*8 MTRX(3,3),MTRXD(3,3),MTRXDD(3,3),D(3),DD(3),
1 DDD(3),LL(3),L(3),LD(3),LDD(3),
1 MTRXL(3),MTRXDL(3),MTRXDDL(3)
CALL MULMAVEC (MTRX,LL,MTRXL,3,3)
CALL ADDVEC (D,MTRXL,L)
CALL MULMAVEC (MTRXD,LL,MTRXDL,3,3)
CALL ADDVEC (DD,MTRXDL,LD)
CALL MULMAVEC(MTRXDD,LL,MTRXDDL,3,3)
CALL ADDVEC (DDD,MTRXDDL,LDD)
RETURN
END

SUBROUTINE SQUARE (A,ASQ)
real*8 A(9),ASQ(9,9)
DO 10 I=1,9
   DO 20 J=1,9
      ASQ(I,J)=A(I)*A(J)
      IF (I .NE. J) THEN
         ASQ(I,J)=2*ASQ(I,J)
      ENDIF
20 CONTINUE
10 CONTINUE
RETURN
END

SUBROUTINE ADDMAT (A,B,C)
real*8 A(3,3),B(3,3),C(3,3)
DO 20 I=1,3
   DO 10 J=1,3
      C(I,J)=A(I,J)+B(I,J)
10 CONTINUE
20 CONTINUE
RETURN
END

SUBROUTINE ADDVEC (A,B,C)
REAL*8 A(3),B(3),C(3)
DO 10 I=1,3
   C(I)=A(I)+B(I)
10 CONTINUE
RETURN
END

SUBROUTINE TRANSPOSE (AR,AD,TANSAR,TANSAD)
real*8 AR(3,3), AD(3,3), TANSAR(3,3), TANSAD(3,3)
CALL TRANS (AR, TANSAR, 3, 3)
CALL TRANS (AD, TANSAD, 3, 3)
RETURN
END

SUBROUTINE TRANS (A, AT, m, n)
real*8 A(m, n), AT(n, m)
DO 1, I=1, m
   DO 1, J=1, n
   AT(J, I)=A(I, J)
1 RETURN
END

SUBROUTINE MULTI (A, B, C, l, m, n)
real*8 A(l, n), B(n, m), C(l, m)
DO 1, I=1, l
   DO 1, J=1, m
      C(I, J)=0.0
   DO 1, K=1, n
      C(I, J)=C(I, J)+A(I, K)*B(K, J)
1 RETURN
END

SUBROUTINE MULMAVEC (A, B, C, l, m)
real*8 A(l, m), B(m), C(l)
DO 1, I=1, l
   C(I)=0.0
   DO 1, J=1, m
      C(I)=C(I)+A(I, J)*B(J)
1 RETURN
END

SUBROUTINE INVERSE (S, B)
REAL*8 S(5, 5), B(5, 5), C(5, 10)
REAL*8 TEMP

SUBROUTINE dotprod(a, b, c)
real*8 a(9), b(9), c
   c=0.0
   do 10 i=1, 9
      c=c+a(i)*b(i)
10 continue
   return
END

SUBROUTINE INVERSE (S, B)
C SPECIFICATION OF C MATRIX (AUGMENTED ATA MATRIX)

DO 10 L = 1, 5
DO 20 J = 1, 5
C(L, J) = S(L, J)
20 CONTINUE
10 CONTINUE

DO 11 I = 1, 5
C(I, I + 5) = 1.0
11 CONTINUE

C TO CHECK WHETHER PIVOT C(1,1) IS NONZERO

IF (C(1, 1).EQ.0.00000000) THEN
DO 30 J = 1, 10
TEMP = C(1, J)
C(1, J) = C(2, J)
C(2, J) = TEMP
30 CONTINUE
ELSE
DO 61 I = 2, 5
DO 71 J = 2, 5
C(I, J) = C(I, J) - (C(1, 1) * C(1, J) / C(1, 1))
71 CONTINUE
61 CONTINUE
GO TO 75
END IF

IF (C(1, 1).EQ.0.00000000) THEN
DO 31 J = 1, 10
TEMP = C(1, J)
C(1, J) = C(3, J)
C(3, J) = TEMP
31 CONTINUE
ELSE
DO 62 I = 2, 5
DO 72 J = 2, 5
C(I, J) = C(I, J) - (C(I, 1) * C(1, J) / C(I, 1))
72 CONTINUE
62 CONTINUE
GO TO 75
END IF

IF (C(1, 1).EQ.0.00000000) THEN
DO 32 J = 1, 10
TEMP = C(1, J)
C(1, J) = C(4, J)
C(4, J) = TEMP
32 CONTINUE
ELSE
END
DO 63 I=2,5
DO 73 J=2,5
C(I,J)=C(I,J)-(C(I,1)*C(1,J)/C(1,1))
73 CONTINUE
63 CONTINUE
GO TO 75
END IF
IF (C(1,1).EQ.0.00000000) THEN
DO 33 J=1,10
TEMP=C(1,J)
C(1,J)=C(5,J)
C(5,J)=TEMP
33 CONTINUE
ELSE
DO 64 I=2,5
DO 74 J=2,5
C(I,J)=C(I,J)-(C(I,1)*C(1,J)/C(1,1))
74 CONTINUE
64 CONTINUE
GO TO 75
ENDIF
IF (C(1,1).EQ.0.00000000) THEN
WRITE(6,*)'MATRIX IS SINGULAR'
STOP
ELSE
C DETERMINATION OF NEW C(I,J)
DO 60 I=2,5
DO 70 J=2,5
C(I,J)=C(I,J)-(C(I,1)*C(1,J)/C(1,1))
70 CONTINUE
60 CONTINUE
END IF
IF (C(1,1).EQ.0.00000000) THEN
WRITE(6,*)'MATRIX IS SINGULAR'
STOP
ELSE
C TO CHECK PIVOT C(2,2)
IF (C(2,2).EQ.0.00000000) THEN
DO 100 J=2,10
TEMP=C(2,J)
C(2,J)=C(3,J)
C(3,J)=TEMP
100 CONTINUE
ELSE
DO 111 I=3,5
DO 121 J=3,6
  C(I,J)=C(I,J)-(C(I,2)*C(2,J)/C(2,2))
121 CONTINUE
111 CONTINUE
   GO TO 114
END IF
IF( C(2,2).EQ.0.00000000) THEN
  DO 80 J=2,10
  TEMP=C(2,J)
  C(2,J)=C(4,J)
  C(4,J)=TEMP
80 CONTINUE
ELSE
  DO 112 I=3,5
  DO 122 J=3,6
  C(I,J)=C(I,J)-(C(I,2)*C(2,J)/C(2,2))
122 CONTINUE
112 CONTINUE
   GO TO 114
END IF
IF( C(2,2).EQ.0.00000000) THEN
  DO 90 J=2,10
  TEMP=C(2,J)
  C(2,J)=C(5,J)
  C(5,J)=TEMP
90 CONTINUE
ELSE
  DO 113 I=3,5
  DO 123 J=3,6
  C(I,J)=C(I,J)-(C(I,2)*C(2,J)/C(2,2))
123 CONTINUE
113 CONTINUE
   GO TO 114
END IF
IF( C(2,2).EQ.0.00000000) THEN
  WRITE(6,*) 'MATRIX IS SINGULAR'
  STOP
ELSE
  C DETERMINATION OF NEW C(I,J)
  DO 110 I=3,5
  DO 120 J=3,6
  C(I,J)=C(I,J)-(C(I,2)*C(2,J)/C(2,2))
120 CONTINUE
110 CONTINUE
END IF
114  \( C(3,7) = -\frac{C(3,2)}{C(2,2)} \)
\( C(4,7) = -\frac{C(4,2)}{C(2,2)} \)
\( C(5,7) = -\frac{C(5,2)}{C(2,2)} \)

C TO CHECK PIVOT \( C(3,3) \)

IF \( (C(3,3) \not= 0.00000000) \) THEN
DO 150 J=3,10
TEMP = C(3,J)
C(3,J) = C(4,J)
C(4,J) = TEMP
150 CONTINUE
ELSE
DO 161 I=4,5
DO 171 J=4,7
C(I,J) = C(I,J) - (C(I,3)*C(3,J)/C(3,3))
171 CONTINUE
161 CONTINUE
GO TO 430
END IF

IF \( (C(3,3) \not= 0.00000000) \) THEN
DO 130 J=3,10
TEMP = C(3,J)
C(3,J) = C(5,J)
C(5,J) = TEMP
130 CONTINUE
ELSE
DO 162 I=4,5
DO 172 J=4,7
C(I,J) = C(I,J) - (C(I,3)*C(3,J)/C(3,3))
172 CONTINUE
162 CONTINUE
GO TO 430
END IF

WRITE(6,*)'MATRIX IS SINGULAR'
STOP
ELSE
C DETERMINATION OF NEW \( C(I,J) \)
DO 160 I=4,5
DO 170 J=4,7
C(I,J) = C(I,J) - (C(I,3)*C(3,J)/C(3,3))
170 CONTINUE
160 CONTINUE
END IF

430  \( C(4,8) = -\frac{C(4,3)}{C(3,3)} \)
\( C(5,8) = -\frac{C(5,3)}{C(3,3)} \)
C TO CHECK PIVOT C(4,4)
  IF (C(4,4).EQ.0.00000000) THEN
    DO 200 J=4,10
    TEMP=C(4,J)
    C(4,J)=C(5,J)
    C(5,J)=TEMP
 200  CONTINUE
  ELSE
    DO 220 J=5,8
    C(5,J)=C(5,J)-(C(5,4)*C(4,J)/C(4,4))
  220  CONTINUE
    GO TO 190
  END IF
  IF (C(4,4).EQ.0.00000000) THEN
    WRITE(6,*)'MATRIX IS SINGULAR'
    STOP
  ELSE
    C DETERMINATION OF NEW C(I,J)
    DO 180 J=5,8
    C(5,J)=C(5,J)-(C(5,4)*C(4,J)/C(4,4))
  180  CONTINUE
  END IF
  190 C(5,9)=-C(5,4)/C(4,4)

C TO CHECK PIVOT C(5,5)
  IF (C(5,5).EQ.0.00000000) THEN
    WRITE(6,*)'MATRIX IS SINGULAR'
    STOP
  ELSE
    C(5,10)=1.0
    C(4,10)=-C(4,5)/C(5,5)
    C(4,9)=1.0-(C(5,9)*C(4,5)/C(5,5))
    C(4,8)=C(4,8)-(C(5,8)*C(4,5)/C(5,5))
    C(4,7)=C(4,7)-(C(5,7)*C(4,5)/C(5,5))
    C(4,6)=C(4,6)-(C(5,6)*C(4,5)/C(5,5))
    C(3,10)=-C(3,5)/C(5,5)
    C(3,9)=-C(5,9)*C(3,5)/C(5,5)
    C(3,8)=1.0-(C(5,8)*C(3,5)/C(5,5))
    C(3,7)=C(3,7)-(C(5,7)*C(3,5)/C(5,5))
    C(3,6)=C(3,6)-(C(5,6)*C(3,5)/C(5,5))
    C(2,10)=-C(2,5)/C(5,5)
    C(2,9)=-C(5,9)*C(2,5)/C(5,5)
    C(2,8)=-C(5,8)*C(2,5)/C(5,5)
    C(2,7)=1.0-(C(5,7)*C(2,5)/C(5,5))
    C(2,6)=C(2,6)-(C(5,6)*C(2,5)/C(5,5))
    C(1,10)=-C(1,5)/C(5,5)
C(1,9) = -C(5,9)*C(1,5)/C(5,5)
C(1,8) = -C(5,8)*C(1,5)/C(5,5)
C(1,7) = -C(5,7)*C(1,5)/C(5,5)
C(1,6) = 1.0 - (C(5,6)*C(1,5)/C(5,5))
DO 250 I = 3, 1, -1
DO 260 J = 5, 10
   C(I,J) = C(I,J) - (C(I,4)*C(4,J)/C(4,4))
260 CONTINUE
250 CONTINUE
DO 270 I = 2, 1, -1
DO 280 J = 4, 10
   C(I,J) = C(I,J) - (C(I,3)*C(3,J)/C(3,3))
280 CONTINUE
270 CONTINUE
DO 300 J = 3, 10
   C(1,J) = C(1,J) - (C(1,2)*C(2,J)/C(2,2))
300 CONTINUE
C TO DETERMINE ELEMENTS OF B
DO 310 I = 1, 5
DO 320 J = 1, 5
   B(I,J) = C(I,J+5)/C(I,I)
320 CONTINUE
310 CONTINUE
END IF
RETURN
END
REFERENCES


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