ABSTRACT

The classical two dimensional theory of stability of parallel flow is extended to viscoelastic fluids. How the elasticity of the fluid affects the point of stability and determines the point of transition to turbulence is analyzed. In addition the magnification of disturbances is elucidated. This study is based on a viscoelastic constitutive equation which has been successful in predicting the experimental trends of various unsteady high shear rate laminar flows. A viscoelastic stability equation which is an extension to the Orr-Sommerfeld equation for a Newtonian fluid is derived and solved for a flow between parallel plates superimposed by a two-dimensional disturbance. A solution indicates that fluid elasticity minimally shifts the point of instability to lower values of Reynolds' number but to a greater degree than does the second-order/Maxwell stability equation. However, another result shows reduced disturbance magnification for turbulent flow at low wavenumber. The range of values of the disturbance wavenumber for which disturbances grow is diminished at these high Reynolds' numbers and low wavenumber. This may be a trend which offers an explanation to turbulent drag reduction by polymer additives based on viscoelastic properties. It is possible that the reduction in disturbance magnification reduces the turbulence level resulting in a reduction of Reynolds' stresses at the wall.

VISCOELASTIC EFFECTS ON PARALLEL FLOW STABILITY, TRANSITION TO TURBULENCE, AND TURBULENT FLOW

by
Michael Garofalo

A Thesis
Submitted to the Faculty of
New Jersey Institute of Technology
in Partial Fulfillment of the Requirements for the Degree of
Master of Science in Mechanical Engineering

Department of Mechanical and Industrial Engineering

January, 1995

APPROVAL PAGE

VISCOELASTIC EFFECTS ON PARALLEL FLOW STABILITY, TRANSITION TO TURBULENCE, AND TURBULENT FLOW

Michael Garofalo

Dr. Avraham Harnoy, Thesis Advisor | Date
Professor of Mechanical Engineering

Dr. John Tavantzis, Committee Member | Date
Professor of Mathematics

Dr. Demetrius Papageorgiou, Committee Member | Date
Professor of Mathematics

BIOGRAPHICAL SKETCH

Author: Michael Garofalo

Degree: Master of Science in Mechanical Engineering

Date: January, 1995

Undergraduate and Graduate Education:

- Master of Science in Mechanical Engineering,
  New Jersey Institute of Technology, 1995

- Bachelor of Science in Mechanical Engineering,
  New Jersey Institute of Technology, 1992

Major: Mechanical Engineering

ACKNOWLEDGMENT

The author expresses sincere gratitude to his advisor, Professor A. Harnoy for his guidance and moral support throughout this research.

I wish to thank Professor J. Tavantzis for his help in the mathematics of the research. His efforts were most rewarding.

I remember and greatly appreciate the help of Professor A. Cerkanowicz who obtained N.S.F. funding for computer tools necessary for this work.

Special thanks to Robert and Michael Mestice for their immense help.

I wish to thank the Guttenbergs for their support during my years at N.J.I.T. I am most appreciative of their help.

TABLE OF CONTENTS

TABLE OF CONTENTS
(Continued)

TABLE OF CONTENTS
(Continued)

LIST OF FIGURES

LIST OF FIGURES
(Continued)

LIST OF FIGURES
(Continued)

CHAPTER 1

INTRODUCTION

1.1 Goals of the Investigation

This study analyzes the role of the elasticity of the fluid in the stability of parallel flow, as well as on the magnification of disturbances. The classical linear theory of stability of parallel flow has been shown to be successful in explaining and predicting the transition to turbulent flow [1]. The theoretical investigations are based on the assumption that laminar flow is affected by certain small disturbances. The theory analyzes the behavior of such disturbances versus time when they are superimposed on the main flow. One of the main goals is to find the value of the critical Reynolds' number from stable flow where disturbances are magnified. Another important aspect to the role of viscoelasticity lies in comparing the magnification of disturbances for unsteady viscoelastic and non-viscoelastic flow at high Reynolds' numbers and at certain wavenumbers. The concern here is the range of values of the disturbance wavelength (at various Reynolds' numbers) for which the disturbance will grow. These goals require a solution to the well known Orr-Sommerfeld equation extended for viscoelastic fluids. Also called the stability differential equation which is derived from the Navier-Stokes equations for viscous fluid, the Orr-Sommerfeld equation has been extended to describe the role of viscoelasticity of a fluid at high shear rate, laminar flow, subjected to fluctuations. This was achieved by using a relatively new rheological constitutive equation [2].

The purpose is to elucidate the viscoelastic processes which affect the behavior of small disturbances and hence the stability at high shear rate, laminar flow. Also important to the investigation is the point of whether or not these disturbances are amplified or damped. The rheological constitutive equation can describe correctly the role of viscoelasticity of a fluid at high shear rate, laminar flow, subjected to fluctuations. Wherever shear rates are subjected to changes (fluctuations with time or along the flow lines), the relaxation time of the fluid must change the stress distribution which results in changes in the flow patterns (in comparison to the viscous flow). This can be demonstrated by experiments of oscillating, laminar shear flow between two parallel disks, where a phase lag is observed in shearing stress behind shear rate. One can visualize the fluid as a Maxwell model of a spring and a dashpot in series. Here the stress is not only a function of the instantaneous deformation rate but also of previous stresses. This phenomena is better known as the memory effect and is well demonstrated during the relaxation time, when stresses exist without any deformation rate. If the viscoelasticity changes the point of stability, it can result in a different point of transition from laminar to turbulent flow, resulting in a different shear at the wall and drag. Moreover, a change in the magnitude of amplitude of disturbances would affect the generation of turbulence in a fully developed turbulent flow resulting in a change in the Reynolds' stresses and friction between the fluid and a wall and drag between fluid and submerged bodies.

1.2 Importance of Drag Reducing Agents

Drag reduction techniques have important engineering applications which can contribute to considerable energy conservation and better equipment utilization. Understanding the drag reduction mechanism can provide the knowledge to develop better methods and viscoelastic agents to reduce drag. Thin viscoelastic liquid layers with enhanced rheological properties can serve as drag reducing agents near the surface of submerged bodies.

The effect of turbulent drag reduction by long-chain flexible polymer additives to viscous liquids was discovered by Toms (1946). He found that the pressure loss in a pipe can be reduced in half by a small polymer concentration of $(10\text{-}100)\times10^{-6}$ g/cm^3. Motivated by the potential for greater energy savings, a vast amount of research has been conducted in turbulent drag reduction. Although much empirical data has been gathered, in light of the complexity of turbulent flow, only tentative explanations have been suggested.

Most of the theories dealing with drag reduction mechanisms by polymers in turbulent boundary layers, rely on changes of the viscosity, owing to the elongation of the macromolecules [2]. These theories are successful in predicting drag reduction in a turbulent boundary layer, providing that there are sizable changes of viscosity (of a few orders magnitude) between the viscous sublayer and the turbulent layer from the wall. One must agree that in turbulent boundary layers, dominated by fluctuations of different frequencies, viscoelastic (memory) effects must play a role as well. Our purpose is to open a new avenue for investigating the role of the elasticity of the fluid in the drag reduction mechanism, which may play a significant role together with other effects, such as the elongational viscosity.

1.3 Rheological Constitutive Equation

This study is based on a relatively new viscoelastic rheological equation by Harnoy [3] that, on the one hand, is based on continuum mechanics principles and on the other hand, predicts correctly the trend of the experiments in unsteady, high shear rate laminar flow, while the previous conventional equations contradicted the experiments. Previous publications by different authors discussed the complete disagreement between analysis based on conventional viscoelastic equations and experimentation. The following three cases were studied in debth because of their importance in engineering: (a). Laminar boundary layer past submerged bodies [4]; (b). Squeeze film at constant approach velocity and (c). Squeeze film at constant force, where the resulting velocity is measured [5,6]. It has been shown that our relatively new equation predicts, for the first time, the trends of these three experiments.

The fluid equation in the present analysis represents the Maxwell model which is a spring and a dashpot in series at low Deborah number, $De = \lambda/\Delta t$, where λ is the relaxation time of the fluid and Δt is the characteristic time of flow. In order to decouple the relaxation effect from the normal stresses, our constitutive equation is described in a unique coordinate system which coincides with the principal axes of the strain-rate tensor.

The following equation is a first order approximation at low Deborah numbers where the equation reduces to the form

$$\tau'_{ij} = 2\,\mu\,(e_{ij} - \lambda\, D(e_{ij})/Dt) \qquad (1.1)$$

where τ'_{ij} is the deviatoric stress tensor, e_{ij} the strain rate tensor, μ is the viscosity, and λ is the relaxation time. Our time derivative D/Dt is defined in a rigid rectangular coordinate system (1,2,3) having its origin fixed at a fluid particle, moving with it, and having its directions coinciding with the three principal axes of the strain rate tensor.

The following equation describes the rate of change of the strain rate tensor, as seen by an observer positioned on the principal axes of the same tensor,

$$D(e_{ij})/Dt = \delta(e_{ij})/\delta t + \delta(e_{ij})/\delta x_\alpha[v_\alpha] - \Omega_{i\alpha}e_{\alpha j} + e_{i\alpha}\Omega_{\alpha j} \quad (1.2)$$

The vector Ω_{ij} is the angular velocity of the rigid coordinate system (1,2,3) attached to the principal axes and v_i are the velocity components of its origin. The difference between this and the well known time derivative of Jaumann is that in the latter, the angular velocity is of the fluid particle. The equation is further discussed and compared to the second-order and Maxwell model in chapter 4.

1.4 Overview of the Orr-Sommerfeld Equation

1.4.1 Principles of the Theory of Stability of Laminar Flow

The theoretical investigations of the process of transition are based on the assumption that laminar flows are affected by certain small disturbances. Whatever the origin of these disturbances (i.e. at pipe inlet, or due to wall roughness) the theory seeks to explain their behavior when they are superimposed on the main flow.

> *"The decisive question to answer in this connexion is whether the disturbances increase or die out with time. If the disturbances decay with time, the main flow is considered stable; on the other hand, if the disturbances increase with time the flow is considered unstable, and there exists the possibility of transition to a turbulent pattern. In this way a theory of stability is created, and its object is to predict the value of the critical Reynolds' number for a prescribed main flow. The basis of the theory of stability can be traced to O. Reynolds who supposed that the laminar pattern, being a solution of the differential equations of fluid dynamics, always represents a possible type of flow, but becomes unstable above a definite limit (precisely above the critical Reynolds' number) and changes into the turbulent pattern."[1]*

The theoretical investigation regarding the process of transition to turbulence for a

Newtonian fluid in parallel flow has been successful via the Orr-Sommerfeld equation. Before deriving the viscoelastic stability equation in Ch.2 the theory behind the development of Orr-Sommerfeld equation is explained.

1.4.2 Foundation of the Method of Small Disturbances

The theory of stability of laminar flows decomposes the motion into a mean flow (whose stability constitutes the subject of the investigation) and into a disturbance superimposed on it. We consider a steady mean flow described by its Cartesian velocity components U, V, and W, representing the velocity components in the x, y, and z directions respectively, and its pressure P. Adding the corresponding quantities for the non-steady disturbance u', v', w', and p', respectively gives the resultant motion velocity components

$$u = U + u', \quad v = V + v', \quad w = W + w' \tag{1.3}$$

and the pressure as:

$$p = P + p'. \tag{1.4}$$

It is assumed in most cases that the disturbance quantities are small as compared to the corresponding quantities of the main flow.

The method of small disturbances accepts only flows which are consistent with the equations of motion and analyzes the manner in which they develop in the flow as described by the appropriate differential equations.

Motion and Continuity Equations

In considering a two-dimensional incompressible mean flow and an equally two-dimensional disturbance, the resulting motion is described by the two-dimensional form of the Navier-

Stokes equations given by the motion equations

$$\delta u/\delta t + u\ \delta u/\delta x + v\ \delta u/\delta y = (1/\rho)X - (1/\rho)\delta p/\delta x + \upsilon\ (\delta^2 u/\delta x^2 + \delta^2 u/\delta y^2) \qquad (1.5)$$

$$\delta v/\delta t + u\ \delta v/\delta x + v\ \delta v/\delta y = (1/\rho)Y - (1/\rho)\delta p/\delta y + \upsilon\ (\delta^2 v/\delta x^2 + \delta^2 v/\delta y^2) \qquad (1.6)$$

and the continuity equation,

$$\delta u/\delta x + \delta v/\delta y = 0. \qquad (1.7)$$

Here υ represents the material viscosity. The problem is simplified by stipulating that the mean velocity U depends only on y, i.e., U = U(y), and the components V and W are supposed to be zero everywhere, or $V \equiv W \equiv 0$. Gravitational effects in the X and Y directions are considered equal to zero also. Such a situation is commonly referred to as a parallel flow problem. The flow in the boundary layer is also regarded as a good approximation to parallel flow because the dependence of the velocity U in the main flow on the x-coordinate is very much smaller than that on y. The pressure is assumed to be a function of x and y, or, P(x,y), because the pressure gradient $\delta P/\delta x$ maintains the flow. Thus we assume a mean flow with

$$U(y); \qquad V \equiv W \equiv 0; \qquad P(x,y). \qquad (1.8)$$

Parabolic Velocity Profile Assumption

Because this study considers the particular case of steady mean flow between parallel plates, it must also be assumed that the mean flow velocity profile is parabolic. This is shown by writing the Navier-Stokes equations for the mean flow

$$0 = -(1/\rho)P_x + \upsilon\ d^2U/dy^2 \qquad (1.9)$$

$$0 = -(1/\rho)P_y \qquad (1.10)$$

Because of the steady flow assumption we have

$$(1/\rho)P_x = \nu\, d^2U/dy^2 = \text{constant}. \tag{1.11}$$

If the mean flow velocity profile is written as

$$U(y) = 1 - y^2 \tag{1.12}$$

then d^2U/dy^2 is a constant and the boundary conditions,

$$y = \pm 1; \qquad U = 0, \tag{1.13}$$

are satisfied. This main flow is shown in fig.1.

Two-dimensional Disturbance Superimposed

Upon the mean flow we assume superimposed a two-dimensional disturbance which is a function of time and space. Its velocity components and pressure are, respectively,

$$u'(x,y,t), \qquad v'(x,y,t), \qquad p'(x,y,t). \tag{1.14}$$

So the resultant motion is described by

$$u = U + u'; \qquad v = v'; \qquad w = 0; \qquad p = P + p'. \tag{1.15}$$

The main flow is assumed a solution of the Navier-Stokes equations, and it is required that the resultant motion must also satisfy the Navier-Stokes equations. The task of the stability theory consists in determining whether the disturbance is amplified or whether it decays for a given mean motion. The flow, therefore, isconsidered unstable or stable depending on whether the former or the latter is the case.

We may now substitute equations 1.15 into the Navier-Stokes equations for a two-dimensional, incompressible, non-steady flow, (equations 1.5, 1.6, 1.7). Quadratic terms in the disturbance velocities may be neglected (as the fluctuating velocities are considered small). Also, if it is considered that the mean flow itself satisfies the Navier-Stokes equations, three equations for u', v', and p' are obtained

$$\delta u'/\delta t + U\, \delta u'/\delta x + v'\, dU/dy + (1/\rho)\, \delta p'/\delta x = v\, (d^2U/dy^2 + \Delta u') \tag{1.16}$$

$$\delta v'/\delta t + U\, \delta v'/\delta x + (1/\rho)\, \delta p'/\delta y = \upsilon\, (\Delta v') \tag{1.17}$$

$$\delta u'/\delta x + \delta v'/\delta y = 0. \tag{1.18}$$

Letting the small letters denote disturbance quantities, the primes differentiation with respect to y, and the subscripts differentiation with respect the parameter indicated, we may write

$$u_t + U\, u_x + U'\, v + (1/\rho)\, p_x = \upsilon\, (U'' + \Delta u) \tag{1.19}$$

$$v_t + U\, v_x \qquad + (1/\rho)\, p_y = \upsilon\, (\Delta v) \tag{1.20}$$

$$u_x + v_y = 0. \tag{1.21}$$

Introduction of Stream Function

A stream function has been established to model the disturbance quantities; and of much importance it is to serve the purpose of allowing for the determination of whether the disturbance becomes amplified or damped. The stream function must also satisfy continuity conditions (equation 1.21). If we write

$$u = \Psi_y \qquad v = -\Psi_x \tag{1.22}$$

then

$$\Psi_{yx} - \Psi_{xy} = 0, \tag{1.23}$$

and thus continuity is satisfied.

A stream function written as

$$\Psi(x,y,t) = 1/2\, \{\phi(y)\, e^{i(\alpha x - \beta t)} + \bar{\phi}(y)\, e^{-i(\alpha x - \bar{\beta} t)}\} \tag{1.24}$$

with $\bar{\phi}$ and $\bar{\beta}$ being complex conjugates of ϕ and β and where

$$\phi(y) = \phi(y)_r + i\, \phi(y)_i \tag{1.25}$$

$$\alpha = 2\pi/l \tag{1.26}$$

and $$\beta = \beta_r + i\,\beta_i \tag{1.27}$$

is shown to be an anzot and serves the purpose of determining whether the disturbance decays or becomes unstable. Here $\phi(y)$ represents the disturbance amplitude and is a complex eigenfunction of y, the α quantity is the wavenumber and is real (l is the wavelength), t is the time at which the disturbance begins, and β is the frequency of the disturbance and is also complex. It is advantageous and permissible (proof - App. p.67) that we may work with only $\phi(y)\, e^{i(\alpha x-\beta t)}$ and not its complex conjugate $\bar{\phi}(y)\, e^{i(\alpha x-\bar{\beta} t)}$. The disturbance velocities are then found to be, (with $\phi(y)$ now written as ϕ)

$$u = \phi'\, e^{i(\alpha x-\beta t)} \tag{1.28}$$

$$v = -i\,\alpha\,\phi\, e^{i(\alpha x-\beta t)} \tag{1.29}$$

Representing the stream function in different forms shows more clearly the critical point of the disturbance becoming amplified or damped. Rewriting

$$\Psi = (\phi_r + i\,\phi_i)\, e^{i(\alpha x-\beta r t)} e^{\beta i t} \tag{1.30}$$

and using the relations

$$e^{i\theta} = \cos\theta + i\sin\theta \tag{1.31}$$

$$e^{-i\theta} = \cos\theta - i\sin\theta \tag{1.32}$$

we have

$$\Psi = (\phi_r + i\,\phi_i)\,(\cos(\alpha\, x-\beta_r\, t) + i\sin(\alpha\, x-\beta_r\, t))\, e^{\beta i t} \tag{1.33}$$

If we consider only the real part of Ψ

$$\text{real } \Psi = (\phi_r \cos(\alpha\, x-\beta_r\, t) - \phi_i \sin(\alpha\, x-\beta_r\, t))\, e^{\beta i t} \tag{1.34}$$

it is clearly seen that the sign of β determines whether the stream function and thus the disturbance becomes amplified or damped. The purterbation velocities (real) are then found to be,

$$u = \delta\Psi/\delta y = (\phi_r' \cos(\alpha\, x-\beta_r\, t) - \phi_i' \sin(\alpha\, x-\beta_r\, t))\, e^{\beta it} \tag{1.35}$$

$$v = -\delta\Psi/\delta x = (\phi_r\ \sin(\alpha\, x-\beta_r\, t) + \phi_i\ \cos(\alpha\, x-\beta_r\, t))\ \alpha\ e^{\beta it} \tag{1.36}$$

The equations for the velocities u and v,

$$u = \Psi_y\ ;\ v = -\ \Psi_x$$

may be substituted into equations 1.19 and 1.20 giving

$$\Psi_{ty} + U\ \Psi_{xy} - U'\ \Psi_x + (1/\rho)p_x = \upsilon\ \Delta\ \Psi_y \tag{1.37}$$

$$-\Psi_{tx} - U\ \Psi_{xx} \qquad + (1/\rho)p_y = -\upsilon\ \Delta\ \Psi_x \tag{1.38}$$

Differentiating these equations with respect to y and x respectively and subtracting equations in order to eliminate pressure gives

$$(\Delta\Psi)_t + U\,(\Psi_{xyy} + \Psi_{xxx}) - U''\ \Psi_x = \upsilon\ \Delta^2\Psi \tag{1.39}$$

Equation 1.39 may be further simplified to yield the Orr-Sommerfeld equation.

The Orr-Sommerfeld Equation

Using the necessary derivatives of $\Psi = \phi(y)\ e^{i(\alpha x-\beta t)}$:

$$\Psi_{xyy} = i\ \alpha\ \phi''\ e^{i(\alpha x-\beta t)} \tag{1.40}$$

$$\Psi_{xxx} = -i\ \alpha^3\ \phi\ e^{i(\alpha x-\beta t)} \tag{1.41}$$

$$\Psi_x = i\ \alpha\ \phi\ e^{i(\alpha x-\beta t)} \tag{1.42}$$

$$\Psi_{xxxx} = \alpha^4\ \phi\ e^{i(\alpha x-\beta t)} \tag{1.43}$$

$$\Psi_{xxyy} = -\alpha^2\ \phi''\ e^{i(\alpha x-\beta t)} \tag{1.44}$$

$$\Psi_{yyyy} = \phi''''\ e^{i(\alpha x-\beta t)} \tag{1.45}$$

$$\Psi_{xx} = -\alpha^2\ \phi\ e^{i(\alpha x-\beta t)} \tag{1.46}$$

$$\Psi_{yy} = \phi''\ e^{i(\alpha x-\beta t)} \tag{1.47}$$

$$(\Delta \Psi)_t = -i\, \beta\, (\phi'' - \alpha^2 \phi)\, e^{i(\alpha x - \beta t)} \tag{1.48}$$

$$\Delta^2 \Psi = (\phi'''' - 2\alpha^2 \phi'' + \alpha^4 \phi)\, e^{i(\alpha x - \beta t)} \tag{1.49}$$

and the relations

$$c = (\beta_r + i\, \beta_i)/\alpha \; ; \;\; R = U_m b/v \tag{1.50}$$

where c is the wave velocity of propagation and is complex,

$$c = c_r + i\, c_i, \tag{1.51}$$

R denotes the Reynolds' number, is a characteristic of the mean flow, and eliminating $e^{i(\alpha x-\beta t)}$ gives the following ordinary, fourth-order, differential equation for the disturbance amplitude $\phi(y)$:

$$(U - c)(\alpha^2 \phi - \phi'') - U''\phi = (i/\alpha R)(\phi'''' - 2\, \alpha^2 \phi'' + \alpha^4 \phi) \tag{1.52}$$

This is the fundamental *differential equation for the disturbance (stability equation)* which forms the point of departure for the stability theory of laminar flows. It is commonly referred to as the *Orr-Sommerfeld equation*. Equation 1.52 has been cast in dimensionless form in that all lengths have been divided by half the channel length b and velocities have been divided by the maximum velocity U_m of the main flow. The primes denote differentiation with respect to the dimensionless coordinates y/b, and R as previously shown denotes the Reynolds' number which is a characteristic of the mean flow. For a boundary-layer flow in the case of flow between parallel plates, boundary conditions demand via the no slip condition that the components of the perturbation velocity must vanish at thewalls ($y = \pm 1$).

Thus:

$$u = v = 0; \qquad\qquad y = \pm 1 \tag{1.53}$$

The stream function is 1/2 the sum of $\phi(y)\, e^{i(\alpha x-\beta t)}$ and its complex conjugate $\bar{\phi}(y)\, e^{-i(\alpha x-\beta t)}$,

$$\Psi(x,y,t) = 1/2\, \{\phi(y)\, e^{i(\alpha x-\beta t)} + \bar{\phi}(y)\, e^{-i(\alpha x-\beta t)}\} \tag{1.54}$$

The stream function holds for all x and t so:

$$u = \Psi_y\, (y = \pm 1) = 0; \qquad \phi' = 0 + 0i \tag{1.55}$$

$$v = \Psi_x\, (y = \pm 1) = 0 \tag{1.56}$$

$$\Psi = \text{constant}\ (y = \pm 1) = 0; \qquad \phi = 0 + 0i \tag{1.57}$$

Writing these boundary conditions again

$$\phi = 0 + 0i\ ;\ (y = \pm 1) \qquad \phi' = 0 + 0i\ ;\ (y = \pm 1) \tag{1.58}$$

it is seen that some value for the disturbance amplitude, ϕ, at $y = 0$ must be chosen.

At $y = 0$ two cases are considered (with A being a real number);

Case 1: flow is antisymmetric; odd derivatives $y(0) = 0$; (1.59)

$$\phi = 1 + 0i$$

$$\phi' = 0 + 0i$$

$$\phi'' = A(1 + i)$$

$$\phi''' = 0 + 0i$$

Case 2: flow is symmetric; even derivatives $y(0) = 0$; (1.60)

$$\phi = 0 + 0i$$

$$\phi' = 1 + 0i$$

$$\phi'' = 0 + 0i$$

$$\phi''' = A(1 + i).$$

1.4.3 The Eigenvalue Problem

The stability problem is now reduced to an eigenvalue problem (equation 1.52) with the boundary conditions (equation 1.58) and one of the two cases considered (equations 1.59-60). When the mean flow U(y) is specified, equation 1.52 contains four parameters. They are α, R, c_r, c_i. Of these the Reynolds' number is specified as is the wavelength ($l = 2\ \pi/\alpha$), or wavenumber ($\alpha = 2\ \pi/l$), of the disturbance. So the Orr-Sommerfeld equation, together with the boundary conditions gives one eigenfunction $\phi(y)$ and one complex eigenconstant, $c = c_r + ic_i$, for each pair of values α, R. Here c_r represents the phase velocity of the prescribed disturbance whereas the sign of c_i determines whether the wave is amplified (when $c_i > 0$) or damped (when $c_i < 0$). For $c_i < 0$ the corresponding flow (U,R) is stable for the given value of α, whereas $c_i > 0$ denotes instability. The limiting case, $c_i = 0$, corresponds to neutral (indifferent) disturbances. The result of such an analysis for any prescribed laminar flow U(y) can be represented graphically in an α, R diagram because every point of this plane corresponds to a pair of values of c_r and c_i. In particular, the locus $c_i = 0$ separates the region of stable from that of unstable disturbances. This locus is called the *curve of neutral stability*. The point on this curve at which the Reynolds' number has its smallest value (tangent parallel to the α-axis) is of great interest since it indicates that value of the Reynolds' number below which all individual oscillations decay, whereas above that value at least some are amplified. This smallest Reynolds' number is called the *critical Reynolds' number or limit of stability* with respect to the type of laminar flow under consideration. With respect to flow between parallel plates, the curve of neutral stability for the Orr-Sommerfeld equation (antisymmetric case) is shown in fig.2. It is seen that the critical Reynolds' number was a calculated 5652.3. There is close agreement between this value and those found in fig.3 [10]. Numerical methods as discussed in the appendix have been used in this investigation for the solution of the Orr-Sommerfeld and viscoelastic stability equations.

CHAPTER 2

DERIVATION OF THE NEW VISCOELASTIC STABILITY EQUATION FOR PARALLEL FLOW

2.1 Utility of the Rheological Constitutive Equation

2.1.1 Development of the Equation

This study is based on a relatively new viscoelastic rheological equation by Harnoy [3] that, on the one hand, is based on continuum mechanics principles and on the other hand, predicts correctly the trend of the experiments in unsteady, high shear rate laminar flow, while the previous conventional equations contradicted the experiments. Previous publications by different authors discussed the complete disagreement between analysis based on conventional viscoelastic equations and experimentation. The following three cases were studied in debth because of their importance in engineering: (a). Laminar boundary layer past submerged bodies [4]; (b). Squeeze film at constant approach velocity and (c). Squeeze film at constant force, where the resulting velocity is measured [5,6]. It has been shown that the relatively new equation predicts, for the first time, the trends of these three experiments.

The fluid equation in the present analysis represents the Maxwell model which is a spring and a dashpot in series at low Deborah number, $De = \lambda/\Delta t$, where λ is the relaxation time of the fluid and λt is the characteristic time of flow. In order to decouple the relaxation effect from the normal stresses, the constitutive equation is described in a unique coordinate system which coincides with the principal axes of the strain-rate tensor.

The following equation is a first order approximation at low Deborah numbers where the equation reduces to the form

$$\tau'_{ij} = 2\,\mu\,(e_{ij} - \lambda\ D(e_{ij})/Dt) \tag{2.1}$$

where τ'_{ij} is the deviatoric stress tensor, e_{ij} the strain rate tensor, μ is the viscosity, and λ is the relaxation time. So we may write

$$\tau_{xx} = \mu\,(e_{xx} - \lambda\ D(e_{xx})/Dt) \tag{2.2}$$

$$\tau_{xy} = \mu\,(e_{xy} - \lambda\ D(e_{xy})/Dt) \tag{2.3}$$

$$\tau_{yy} = \mu\,(e_{yy} - \lambda\ D(e_{yy})/Dt) \tag{2.4}$$

$$\tau_{yx} = \mu\,(e_{yx} - \lambda\ D(e_{yx})/Dt) \tag{2.5}$$

The time derivative D/Dt is defined in a rigid rectangular coordinate system (1,2,3) having its origin fixed at a fluid particle, moving with it, and having its directions coinciding with the three principal axes of the strain rate tensor.

The following equation describes the rate of change of the strain rate tensor, as seen by an observer positioned on the principal axes of the same tensor,

$$D(e_{ij})/Dt = \delta(e_{ij})/\delta t + \delta(e_{ij})/\delta x_\alpha[v_\alpha] - \Omega_{i\alpha}e_{\alpha j} + e_{i\alpha}\Omega_{\alpha j} \tag{2.6}$$

The vector Ω_{ij} is the angular velocity of the rigid coordinate system (1,2,3) attached to the principal axes and v_i are the velocity components of its origin. The difference between this and the well known time derivative of Jaumann is that in the latter, the angular velocity is of the fluid particle.

Equations 2.1 and 2.6 show instability under a sudden elimination of stresses. This problem was resolved lately [8] by showing that the flow is unstable only at high values of De because equation 2.1 is a truncated infinite series of increasing powers of De and valid only at De << 1.

2.1.2 Demonstration of the Equation

The utility of our equation is demonstrated best in squeeze film flow. Tichy and Modest [5] present a squeeze film analysis, based on our constitutive equation. The results are in agreement with the trends of the two squeeze film experiments at steady velocity and under constant load. These results are encouraging as all previous theories, based on the second-order fluid equation, or other conventional equations, resulted in a direct contradiction to the experiments of Leider and Bird [8,9].

A squeeze film damper problem with a viscoelastic fluid has been solved. Experiments have shown that viscoelastic effects decrease load capacity (decrease lubrication effectiveness) but increase descent time (increase lubrication effectiveness). The Harnoy constitutive equation, for the first time predicted the correct experimental trends.

Another engineering application in which the constitutive equation predicts experiment correctly is that of relaxation effects in viscoelastic boundary layer flow. The equation shows what has been known from experiments, that drag reduction and delayed fluid separation results from viscoelastic flow past submerged bodies. In the calculation, the second-order equation has incorrectly predicted results from experimentation.

2.2 Derivation

2.2.1 Method of Small Disturbances

Mean Flow Considered

The following analysis is a derivation of the viscoelastic stability equation which is an extension of the Orr-Sommerfeld equation of stability for Newtonian fluid. Consider a two dimensional, incompressible, steady, parallel flow.The flow is defined by its Cartesian velocity coordinates U, V, in the x and y directions respectively and its pressure $P = P(x,y)$. It is assumed that the mean flow, $U = U(y)$ is in the direction of x and varying only as a function of y so that it is considered that $V = W = 0$. This situation is commonly referred to as a parallel flow problem. The flow in the boundary layer is also regarded as a good approximation to parallel flow because the dependence of the velocity U in the main flow on the x coordinate is very much smaller than that on y. The pressure is assumed to be a function of x and y, P(x,y), because the pressure gradient $\delta P/\delta x$ maintains the flow. Thus we assume a mean flow with

$$U(y); \quad V \equiv W \equiv 0; \quad P(x,y) \tag{2.7}$$

Parabolic Velocity Profile Assumption

Because this study considers the particular case ofsteady mean flow between parallel plates, it must also beassumed that the mean flow velocity profile is parabolic. This is shown by writing the Navier Stokes equations for the mean flow.

$$0 = -(1/\rho)\, P_x + \upsilon\, d^2U/dy^2 \tag{2.8}$$

$$0 = -(1/\rho)\, P_y \tag{2.9}$$

Because of the steady mean flow assumption we have

$$(1/\rho)\, P_x = \upsilon\, d^2U/dy^2 = \text{constan} \tag{2.10}$$

If the mean flow velocity profile is written as:

$$U(y) = 1 - y^2 \tag{2.11}$$

then d^2U/dy^2 is a constant and the boundary conditions

$$y = \pm 1: \quad U = 0 \tag{2.12}$$

are satisfied. This mean flow is shown in fig.1.

The assumption of a two dimensional flow is made to simplify the calculations. Also of importance is that the authenticity of the results lie in the fact that it has been shown that three dimensional flow is more stable than two dimensional flow for Newtonian fluids [10]. Also the results reflect the disturbance only at the beginning of the transition to turbulence from laminar.

Motion and Continuity Equations

The starting point of the mathematical analysis begins with the equations for a two-dimensional incompressible mean flow given by the equations of motion.

$$\rho\ Du/Dt = X - \delta p/\delta x + \delta \tau_{xx}/\delta x + \delta \tau_{xy}/\delta y \tag{2.13}$$

$$\rho\ Dv/Dt = Y - \delta p/\delta y + \delta \tau_{yx}/\delta x + \delta \tau_{yy}/\delta y \tag{2.14}$$

and the continuity equation

$$\delta u/\delta x + \delta v/\delta y = 0 \tag{2.15}$$

Viscoelastic Extension to the Navier-Stokes Equations

Noting that the equations for the strain rate tensors are equivalent to

$$e_{xx} = \delta u/\delta x \tag{2.16}$$

$$e_{xy} = \delta u/\delta y \tag{2.17}$$

$$e_{yx} = \delta v/\delta x \tag{2.18}$$

$$e_{yy} = \delta v/\delta y \tag{2.19}$$

and the equation for the time derivative is given by

$$D/Dt = \delta/\delta t + u\ \delta/\delta x + v\ \delta/\delta y \tag{2.20}$$

we may now substitute equation 2.1 into equations 2.13 and 2.14 which yields after neglecting gravitational effects,

$$\rho \; Du/Dt = -\delta p/\delta x + \mu \; \Delta u - \lambda \; \mu \; D/Dt(\Delta u) \tag{2.21}$$

$$\rho \; Dv/Dt = -\delta p/\delta y + \mu \; \Delta v - \lambda \; \mu \; D/Dt(\Delta v) \tag{2.22}$$

$$\delta u/\delta x + \delta v/\delta y = 0. \tag{2.23}$$

Equations 2.21 and 2.22 are an extension of the well known Navier-Stokes equation for a two-dimensional flow. For $\lambda = 0$ the equations are reduced to the Navier-Stokes equations for Newtonian flow.

Two-Dimensional Disturbance Superimposed

Now we may consider the mean flow (whose stability constitutes the subject of the investigation) being superimposed by an unsteady, two-dimensional disturbance which is very small in magnitude. Being a function of time and space, its velocity components and pressure are given as:

$$u' = u'(x,y,t),\; v' = v'(x, y,t), \text{ and } p' = p'(x,y,t). \tag{2.24}$$

Adding the mean flow and disturbance quantities gives the resultant motion velocity and pressure components,

$$u = U + u';\;\; v = v';\;\; w = 0;\;\; p = P + p'. \tag{2.25}$$

The assumption that the magnitude of the disturbance is very small as made above becomes very important, as it enables linearization of the equation, disregarding orders of u'^2 and $u'v'$. Linearization is justified since the study involves a short phase of time where the flow is in transition to turbulence from laminar.

The mean flow is assumed a solution of the Navier-Stokes equations, and it is required that the resultant motion must also satisfy the Navier-Stokes equations. The task of the stability theory consists in determining whether the disturbance is amplified or whether it decays for a given mean motion. The flow, therefore, is considered unstable or stable depending on whether the former or the latter is the case.

We may now substitute equations 2.25 into the Navier-Stokes equations for a two-dimensional, incompressible, non-steady flow extended to incorporate viscoelastic properties (equations 2.21, 2.22, and 2.23). Letting the small letters denote disturbance quantities, the primes differentiation with respect to y, and the subscripts differentiation with respect to the parameter indicated we may write,

$$u_t + U\,u_x + U''\,v + (1/\rho)\,p_x = \upsilon\,(U'' + \Delta u) - \lambda v\,(\Delta u_t + U\,\Delta u_x + U'''\,v) \qquad (2.26)$$

$$v_t + U\,v_x \qquad + (1/\rho)\,p_y = \upsilon\,(\Delta v) - \lambda v\,(\Delta v_t + U\,\Delta v_x) \qquad (2.27)$$

$$u_x + v_y = 0\,. \qquad (2.28)$$

Introduction of Stream Function

A stream function has been established to model the disturbance quantities; and of much importance it is to serve the purpose of allowing for the determination of whether the disturbance becomes amplified or damped. The stream function must also satisfy continuity conditions (equation 2.23). If we write

$$u = \Psi_y\,; \qquad v = -\,\Psi_x \qquad (2.29)$$

then

$$\Psi_{yx} - \Psi_{xy} = 0 \qquad (2.30)$$

and thus continuity is satisfied.

A stream function written as:

$$\Psi(x,y,t) = 1/2\ \{\phi(y)\ e^{i(\alpha x-\beta t)} + \bar{\phi}(y)e^{-i(\alpha x-\bar{\beta} t)}\} \tag{2.31}$$

with $\bar{\phi}$ and $\bar{\beta}$ being complex conjugates of ϕ and β where

$$\phi(y) = \phi(y)_r + i\ \phi(y)_i, \tag{2.32}$$

$$\alpha = 2\ \pi/l \tag{2.33}$$

and
$$\beta = \beta_r + i\ \beta_i \tag{2.34}$$

is shown to be an anzot and serves the purpose of determining whether the disturbance decays or becomes unstable. Here $\phi(y)$ represents the disturbance amplitude and is a complex eigenfunction of y, the α quantity is the wavenumber and is real (l is the wavelength), t is the time at which the disturbance begins, and β is the frequency of the disturbance and is also complex. It is advantageous and permissible (proof - App. p.67) that we may work with only $\phi(y)\ e^{i(\alpha x-\beta t)}$ and not its complex conjugate $\bar{\phi}(y)\ e^{i(\alpha x-\bar{\beta} t)}$.The disturbance velocities are then found to be, (with $\phi(y)$ now written as ϕ),

$$u = \phi'\ e^{i(\alpha x-\beta t)} \tag{2.35}$$

$$v = -i\ \alpha\ e^{i(\alpha x-\beta t)} \tag{2.36}$$

Representing the stream function in different forms shows more clearly the critical point of how this stream function serves the purpose of determining whether the disturbance becomes amplified or damped. Rewriting

$$\Psi = (\phi_r + \phi_i)\ e^{i(\alpha x-\beta t)}\ e^{\beta i t} \tag{2.37}$$

and using

$$e^{i\theta} = \cos\theta + i\sin\theta \tag{2.38}$$

$$e^{-i\theta} = \cos\theta - i\sin\theta \tag{2.39}$$

we have

$$\Psi = (\phi_r + i\,\phi_i)\,(\cos(\alpha\,x-\beta_r\,t) + i\,\sin(\alpha\,x-\beta_r\,t))\,e^{\beta i t} \tag{2.40}$$

If we consider only the real part of Ψ,

$$\text{real}\ \Psi = (\phi_r \cos(\alpha\,x-\beta_r\,t) - \phi_i \sin(\alpha\,x-\beta_r\,t))\,e^{\beta i t} \tag{2.41}$$

it is clearly seen that the sign of β determines whether the stream function and thus the disturbance becomes amplified or damped. The purterbation velocities (real) are thenfound to be

$$u = \delta\Psi/\delta y = (\phi_r' \cos(\alpha\,x-\beta_r\,t) - \phi_i' \sin(\alpha\,x-\,\beta_r\,t))\,e^{\beta i t}, \tag{2.42}$$

$$v = -\delta\Psi/\delta x = (\phi_r\ \sin(\alpha\,x-\beta_r\,t) + \phi_i\ \cos(\alpha\,x-\beta_r\,t))\,\alpha\,e^{\beta i t}. \tag{2.43}$$

The equations for the velocities u and v

$$u = \Psi_y\,;\ v = -\,\Psi_x,$$

may be substituted into equations 2.26 and 2.27 giving

$$\Psi_{ty} + U\,\Psi_{xy} - U'\,\Psi_x + (1/\rho)p_x = \upsilon\Delta\Psi_y - \lambda\,\upsilon(\Delta\Psi_{yt} + U\,\Delta\Psi_{xy} - U'''\,\Psi_x) \tag{2.44}$$

$$-\,\Psi_{tx} - U\,\Psi_{xx} \qquad + (1/\rho)p_y = -\,\upsilon\,\Delta\Psi_x + \qquad \lambda\upsilon\,(\Delta\Psi_{xt} + U\,\Delta\Psi_{xx}) \tag{2.45}$$

Differentiating these equations with respect to y and x respectively and subtracting equations in order to eliminate pressure gives

$$(\Delta\Psi)_t + U\,(\Psi_{xyy} + \Psi_{xxx}) - U''\,\Psi_x = \upsilon\,\Delta^2\Psi$$
$$-\,\lambda\,\upsilon\,(\Delta^2\Psi_t + U'\,\Delta\Psi_{xy} + U\,\Delta^2\Psi_x - U''''\Psi_x - U'''\Psi_{xy}) \tag{2.46}$$

Equation 2.46 may be further simplified to yield the viscoelastic stability equation.

The New Viscoelastic Stability Equation

Using the necessary derivatives of $\Psi = \phi(y)\,e^{i(\alpha x-\beta t)}$:

$$\Psi_{xyy} = i\,\alpha\,\phi''\,e^{i(\alpha x-\beta t)} \tag{2.47}$$

$$\Psi_{xxx} = -i\,\alpha^3\,\phi\,e^{i(\alpha x-\beta t)} \tag{2.48}$$

$$\Psi_x = i\ \alpha\ \phi\ e^{i(\alpha x - \beta t)} \tag{2.49}$$

$$\Psi_{xxxx} = \alpha^4\ \phi\ e^{i(\alpha x - \beta t)} \tag{2.50}$$

$$\Psi_{xxyy} = -\alpha^2\ \phi''\ e^{i(\alpha x - \beta t)} \tag{2.51}$$

$$\Psi_{yyyy} = \phi''''\ e^{i(\alpha x - \beta t)} \tag{2.52}$$

$$\Psi_{xx} = -\alpha^2\ \phi\ e^{i(\alpha x - \beta t)} \tag{2.53}$$

$$\Psi_{yy} = \phi''\ e^{i(\alpha x - \beta t)} \tag{2.54}$$

$$\Delta \Psi_t = -i\ \beta\ (\phi'' - \alpha^2\ \phi)\ e^{i(\alpha x - \beta t)} \tag{2.55}$$

$$\Delta^2 \Psi\ = (\phi'''' - 2\ \alpha^2\ \phi'' + \alpha^4\ \phi)\ e^{i(\alpha x - \beta t)} \tag{2.56}$$

$$\Delta^2 \Psi_t = -i\ \beta\ (\phi'''' - 2\ \alpha^2\ \phi'' + \alpha^4\ \phi)\ e^{i(\alpha x - \beta t)}, \tag{2.57}$$

$$\Delta \Psi_{xy} = i\ \alpha\ (\phi''' - \alpha^2\ \phi')\ e^{i(\alpha x - \beta t)} \tag{2.58}$$

$$\Delta^2 \Psi_x\ = i\ \alpha\ (\phi'''' - 2\ \alpha^2\ \phi'' + \alpha^4\ \phi)\ e^{i(\alpha x - \beta t)} \tag{2.59}$$

$$\Psi_{xy} = i\ \alpha\ \phi'\ e^{i(\alpha x - \beta t)} \tag{2.60}$$

the relations,

$$c = (\beta_r + i\ \beta_i)/\alpha\ ;\ \ R = U_m b/\upsilon\ ;\ \ \Gamma = \lambda U_m/b \tag{2.61}$$

where c is the wave velocity of propagation and is complex,

$$c = c_r + i\ c_i \tag{2.62}$$

R denotes the Reynolds' number, is a characteristic of the mean flow, Γ represents the elasticity number, and eliminating $e^{i(\alpha x - \beta t)}$ gives the following ordinary, fourth-order, differential equation for the amplitude $\phi(y)$:

$$\begin{aligned}(U - c)\ (\alpha^2\ \phi - \phi'') + U''\ \phi = \\ &(1/R)\ \{i/\alpha\ [\alpha^4\ \phi - 2\ \alpha^2\ \phi'' + \phi''''] + \Gamma\ (U-c)\ (\alpha^4\ \phi - 2\ \alpha^2\ \phi'' + \phi'''') \\ &- \Gamma\ [U'\ (\alpha^2\ \phi' - \phi''') + U'''\ \phi' + U''''\ \phi]\}\end{aligned} \tag{2.63}$$

This is the fundamental *differential equation for the disturbance (viscoelastic stability equation)* which forms the point of departure for the stability theory of laminar flows. It is the extension to the *Orr-Sommerfeld equation* for viscoelastic fluids. Equation 2.63 has been cast in dimensionless form in that all lengths have been divided by half the channel length b and velocities have been divided by the maximum velocity U_m of the mean flow. The primes denote differentiation with respect to the dimensionless coordinates y/b, R as previously shown denotes the Reynolds' number which is a characteristic of the mean flow, and Γ denotes the elasticity number which is a measure of fluid viscoelasticity. For a boundary-layer flow in the case of flow between parallel plates, boundary conditions demand via the no slip condition that the components of the perturbation velocity must vanish at the walls ($y = \pm 1$).

Thus:

$$u = v = 0; \quad y = \pm 1. \tag{2.64}$$

The stream function is 1/2 the sum of $\phi(y)\, e^{i(\alpha x-\beta t)}$ and its complex conjugate $\bar{\phi}(y)\, e^{-i(\alpha x-\bar{\beta} t)}$,

$$\Psi\ (x,y,t) = 1/2\ \{\phi(y)\, e^{i(\alpha x-\beta t)} + \bar{\phi}(y)\, e^{-i(\alpha x-\bar{\beta} t)}\} \tag{2.65}$$

The stream function holds for all x and t so:

$$u = \Psi_y\ (y = \pm 1) = 0\ ; \quad \phi' = 0 + 0i. \tag{2.66}$$

$$v = \Psi_x\ (y = \pm 1) = 0 \tag{2.67}$$

$$\Psi = \text{constant}\ (y = \pm 1) = 0\ ; \quad \phi = 0 + 0i. \tag{2.68}$$

Writing these boundary conditions again

$$\phi = 0 + 0i\ ; \ (y = \pm 1) \tag{2.69}$$

$$\phi' = 0 + 0i\ ; \ (y = \pm 1),$$

it is seen that some value for the stream function at $y = 0$ must be chosen. At $y = 0$ two cases are considered (with A being a real number);

Case 1: flow is antisymmetric; (2.70)

odd derivatives $y(0) = 0$;

$$\phi = 1 + 0i,$$

$$\phi' = 0 + 0i,$$

$$\phi'' = A(1 + i),$$

$$\phi''' = 0 + 0i.$$

Case 2: flow is symmetric; (2.71)

even derivatives $y(0) = 0$;

$$\phi = 0 + 0i,$$

$$\phi' = 1 + 0i$$

$$\phi'' = 0 + 0i$$

$$\phi''' = A(1 + i).$$

2.2.2 The Eigenvalue Problem

The stability problem is now reduced to an eigenvalue problem (equation 2.63) with the boundary conditions (equation 2.69) and one of the two cases considered (equations 2.70-71). When the mean flow U(y) is specified, equation 2.63 contains four parameters. They are α, R, c_r, c_i. Of these the Reynolds' number is specified as is the wavelength ($l = 2\pi/\alpha$), or wavenumber ($\alpha = 2\pi/l$), of the disturbance. So the viscoelastic stability equation, together with the boundary conditions gives one eigenfunction $\phi(y)$ and one complex eigenconstant, $c = c_r + ic_i$, for each pair of values α, R. Here c_r represents the phase velocity of the prescribed disturbance whereas the sign of c_i determines whether the wave is amplified (when $c_i > 0$) or damped (when $c_i < 0$). For $c_i < 0$ the corresponding flow (U,R) is stable for the given value of α, whereas $c_i > 0$ denotes instability. The limiting case $c_i = 0$ corresponds to neutral (indifferent) disturbances.

The result of such an analysis for any prescribed laminar flow U(y) can be represented graphically in an α, R diagram because every point of this plane corresponds to a pair of values of c_r and c_i. In particular, the locus $c_i = 0$ separates the region of stable from that of unstable disturbances. This locus is called the *curve of neutral stability.* The point on this curve at which the Reynolds' number has its smallest value (tangent parallel to the α-axis) is of great interest since it indicates that value of the Reynolds' number below which all individual oscillations decay, whereas above that value at least some are amplified. This smallest Reynolds' number is called the *critical Reynolds' number or limit of stability* with respect to the type of laminar flow under consideration (flow between parallel plates considered here).The curves of neutral stability in figures 4-7 (antisymmetric case) are for fluids with various values of Γ shown vs the curve for the Orr-Sommerfeld equation which considers a Newtonian fluid only, $\Gamma = 0$. It is seen that the critical Reynolds' number decreases vs increasing values of Γ. Numerical methods as discussed in the appendix have been used in this investigation for the solution of the Orr-Sommerfeld and viscoelastic stability equations. There is close agreement between the work in this investigation (fig.2, critical Reynolds' number of 5652.3) and in previous work concerning the neutral stability curve for the Orr-Sommerfeld equation's antisymmetric (fig. 3). Comparison between the

Newtonian and Non-Newtonian curves of neutral stability is then valid because of the similar methods used for their solution.

CHAPTER 3

COMMENTS ON VISCOELASTIC STABILITY AND TRANSITION TO TURBULENCE

Important aspects to the curves of neutral stability for viscoelastic fluids calculated in this investigation can be explained by comparing those curves vs the neutralstability curve for a Newtonian fluid. Analysis of the antisymmetric, or odd case, shows that this type of prescribed disturbance induces the most dangerous of flows(or is most likely to progress the mean flow to a turbulent state). Results indicate that a Newtonian fluid undergoes transition to turbulence at a calculated Reynolds' number of 5652.3 when the antisymmetric case is considered (fig.2).When the symmetric, or even case is considered, transition to a turbulent state occurs at a calculated Reynolds' number of 7665 (fig.10). The aim is to find the critical Reynolds' number or limit of stability, so the case which produces the lowest critical Reynolds' number is the more important case and is therefore considered in depth. Another seemingly important difference between the curves of neutral stability for the odd and even cases is the value of the wavenumber at which disturbances begin to grow (or the value of the wavenumber at the critical Reynolds' number now denoted as the critical wavenumber). For the antisymmetric, or odd case, the critical wavenumber was calculated to be 1.0250 (fig.2) while for the symmetric, or even case, the critical wavenumber was found to be significantly higher (fig.10). This may give valuable insight into the curves of neutral stability for viscoelastic fluids.

As already stated, the antisymmetric case produces the most dangerous of flows and its results are seen in figures 4-7. Each figure shows two neutral stability curves. One for the Newtonian case vs a curve for a given value of elasticity number. It is seen that the critical Reynolds' number decreases with increasing values of Γ, or, elasticity number. An elasticity number of 10^{-5} is seen in figure 4 to produce a critical Reynolds' number of 5652.2, only slightly below 5652.3 (the critical Reynolds' number for Newtonian flow). Critical Reynolds' numbers of 5651.9, 5648.1, and 5610.9 correspond to elasticity numbers of 10^{-4}, 10^{-3}, and 10^{-2} respectively (figs. 4-7). This indicates that viscoelasticity plays a role which diminishes the value of the Reynolds' number at which tranition to turbulence occurs. However, for values of elasticity number up to 5 x 10^{-4}, a range including usual applications of drag reducing fluids, the critical Reynolds' number for the viscoelastic stability equation is 5650.1, only slightly below that for Newtonian fluids. This relationship between viscoelasticity and the critical Reynolds' number is seen more clearly in the semilog plot of figure 8. Viscoelasticity also diminishes the value of the critical Reynolds' number for the symmetric case (fig.11).

It is seen that the value of the critical wavenumber increases with increasing values of elasticity number. Elasticity numbers of 10^{-5} and 10^{-4} produce critical wavenumbers only slightly above 1.0250 (the critical wavenumber for Newtonian flow). Critical wavenumbers of 1.0258, and 1.0270 correspond to elasticity numbers of 10^{-3}, and 10^{-2} respectively showing that viscoelasticity plays a role which increases the value of the critical wavenumber. This relationship is seen more clearly in the semilog plot of figure 9. Although this relationship does not directly influence the transition to turbulence it may signify a more stable situation for turbulent flow as indicated by the high critical wavenumbers in the symmetric case for Newtonian and non-Newtonian fluids. Also, it was found that at high

Reynolds' number and low wavenumber, the value of disturbance growth rate (c imaginary) for the Newtonian fluid and for a fluid with an elasticity number of 10^{-3} (figure 21) become equal or intersect. It therefore is assumed that the neutral stability curves intersect.

Plots of the disturbance amplitude, or eigenfunction ϕ, are shown in figures 12 and 13. Both plots of the eigenfunction are at a Reynolds' number of 7000 and wavenumber of 1; the plot of figure 12 is for a Newtonian fluid and that of figure 13 is for a fluid with elasticity number of 10^{-3}. It can be seen from the plot of figure 14 that the values of the eigenfunction for the viscoelastic fluid ($\Gamma = 10^{-3}$) is greater than that of the Newtonian fluid ($\Gamma = 0$) at every point along the channel width. This shows that the disturbance for the viscoelastic fluid is more pronounced than in Newtonian fluid at this point in α, R space.

CHAPTER 4

THE VISCOELASTIC STABILITY EQUATION COMPARED TO THE SECOND-ORDER/MAXWELL MODEL STABILITY EQUATION

The second-order fluid of is described by the differential type of equation,

$$\tau_{ij} = \alpha_1 A^{(1)} + \alpha_2 A^{(2)} + \alpha_3 [A^{(1)}]^2 \tag{4.1}$$

where τ_{ij} is the stress tensor and α_i are fluid coefficients. $A^{(1)}$ and $A^{(2)}$ are rate of strain tensors,

$$A^{(1)} = v_{i,j} + v_{j,i} \tag{4.2}$$

$$A^{(2)} = d/dt(A^{(1)}{}_{ij}) + A^{(1)}{}_{im}\, v_{m,j} + A^{(1)}{}_{jm}\, v_{m,i} \tag{4.3}$$

v_i is the velocity vector. The coefficient α_1 is identical to the viscosity of the fluid μ. The convective time derivative is defined as

$$d/dt = \delta/\delta t + \delta/\delta x_i[v_i] \tag{4.4}$$

This is the conventional Jaumann' s time rate which is shown by Harnoy [4] to be valid only for low shear rates, $\lambda\, d(e_{ij})/dt$. Expanding equation 1.1 to an infinite series of increasing orders of time derivatives of e_{ij} yields,

$$\tau'_{ij} = 2\,\mu\,[e_{ij} - \lambda\, d(e_{ij})/dt + \lambda^2 d^2(e_{ij})/dt^2 + \ldots\ldots (-\lambda)^{n-1} d^{n-1}(e_{ij})/dt^{n-1} + \ldots] \tag{4.5}$$

It then follows from the expansion that $\lambda\, d(e_{ij})/dt \ll 1$, or $De \ll 1$. So if d/dt is the conventional Jaumann' s time rate, for a simple shear flow, the second-order equation is valid only for low shear rates (as is the new rheological equation). The the constant shear viscosity and normal stresses (corresponding to fluid parameters α_1 and α_2) are the only effects which enter into the second-order stability equation. The Maxwell fluid is described by the differential type of equation, effects which enter into the second-order stability equation. The Maxwell fluid is described by the differential type of equation,

$$\tau_{ij} = \alpha_1 A^{(1)} + \alpha_2 [(\delta \tau_{ij}/\delta t) + v_m \tau_{ij,m} - \tau_{im} v_{j,m} - \tau_{jm} v_{i,m}], \tag{4.6}$$

where τ_{ij} is the stress tensor and α_i are fluid coefficients. The model has a zero secondary normal stress difference, but the secondary normal stress difference does not enter into the stability equation, so the second-order and Maxwell models yield the same stability equation.

The primary function of the new fluid equation is to separate the normal stress and relaxation effects; the second-order and Maxwell rheological models do not separate these parameters. This was made possible by the introduction of a unique time derivative. Our time derivative D/Dt is defined in a rigid rectangular coordinate system (1,2,3) having its origin fixed at a fluid particle, moving with it, and having its directions coinciding with the three principal axes of the strain rate tensor. The following equation describes the rate of change of the strain rate tensor, as seen by an observer positioned on the principal axes of the same tensor.

$$D(e_{ij})/Dt = \delta(e_{ij})/\delta t + \delta(e_{ij})/\delta x_\alpha [v_\alpha] - \Omega_{i\alpha} e_{\alpha j} + e_{i\alpha} \Omega_{\alpha j}. \tag{4.7}$$

The vector Ω_{ij} is the angular velocity of the rigid, rectangular, coordinate system (1,2,3) attached to the principal axes and v_i are the velocity components of its origin. The difference between this and the well known time derivative of Jaumann is that in the latter, the angular velocity is of the fluid particle. The angular velocity of a fluid particle is given as:

$$w_{ij} = \Omega_{ij} + \overline{w}_{ij} \tag{4.8}$$

where w_{ij} is the angular velocity of a fluid particle relative to the coordinate system (1,2,3). An extension of the second-order equation which includes an additional fluid parameter yields the new rheological equation

$$\tau_{ij} = \alpha_1 A^{(1)}_{ij} + \alpha_2 D/Dt(A^{(1)}_{ij}) + \alpha_3(-\overline{w}_{i\alpha} A^{(1)}_{\alpha j} + A^{(1)}_{i\alpha} \overline{w}_{\alpha j}) + \alpha_4[A^{(1)}]^2 \qquad (4.9)$$

where α_1, α_2, α_3, and α_4 are fluid parameters which are functions of the strain-rate tensor invariants. It has been shown that at slow flow (as e_{ij} approaches 0), α_2 and α_3 coincide and so equation 4.7 converges to the second-order equation. The significance of the equations in the new rheological model(equations 4.5 - 4.7)are for high shear rate flows subjected to slow changes. All angular velocities are equal to zero for plane parallel flow.

Restating the solutions of the parallel flow problem for both the new rheological fluid equation and second-order/Maxwell models:

The viscoelastic stability equation,

$$(U-C)(\alpha^2 \phi - \phi'') + U'' \phi = (1/R)\{i/\alpha\,[\phi'''' - 2\alpha^2 \phi'' + \alpha^4 \phi]$$
$$+ \Gamma(U-C)(\phi'''' - 2\alpha^2 \phi'' + \alpha^4 \phi) - \Gamma[U'(\alpha^2 \phi' - \phi''') + U''' \phi' + U'''' \phi]\}$$

and the second-order/Maxwell model stability equation

$$(U-c)(\alpha^2 \phi - \phi'') + U'' \phi = (1/R)\{i/\alpha\,[\phi'''' - 2\alpha^2 \phi'' + \alpha^4 \phi]$$
$$+ \Gamma(U-c)(\phi'''' - 2\alpha^2 \phi'' + \alpha^4 \phi) - \Gamma U'''' \phi\}.$$

In the second-order/Maxwell model stability equation the ϕ' and ϕ''' terms do not enter into the equation as in the viscoelastic stability equation.

The viscoelastic stability equation and the second-order/Maxwell model stability equation show similar relationships with respect to transition to turbulence. The decrease in critical Reynolds' number with increasing elasticity number is slightly greater for the viscoelastic stability equation than for the second-order/Maxwell model viscoelastic stability equation. In neutral stability curves for the antisymmetric case it is seen that the critical Reynolds' number decreases with increasing values of elasticity number (the values were calculated using E, or, Γ/R for the second-order/Maxwell model). An elasticity number of 10^{-7} is seen in figure 15 to produce a critical Reynolds' number of 5650.8, only slightly below 5652.3 (the critical Reynolds' number for Newtonian flow). Critical Reynolds' numbers of 5638.9, and 5522.6 correspond to elasticity numbers of 10^{-6}, and 10^{-5} respectively (figs. 16-17).

So viscoelasticity, in the second-order/Maxwell model stability equation, as in the viscoelastic stability equation, diminishes the value of the Reynolds' number at which transition to turbulence occurs. Figure 18 shows the comparison between the two models (both plotted vs Γ). Figure 19 shows the relationship between c_i and both models at low values of Γ (at $R = 5652$, $\alpha = 1.025$). It shows that at all values of Γ the viscoelastic stability equation is less stable than the second-order/Maxwell model stability equation.

There is similarity between the second-order/Maxwell model stability equation and the viscoelastic stability equation with respect to the value of the critical wavenumber varying with elasticity number. Critical wavenumbers of 1.0250, 1.0247, 1.0252, and 1.0310 correspond to elasticity numbers of 10^{-8}, 10^{-7} ,10^{-6}, and 10^{-5} respectively. This shows that viscoelasticity, in the second-order/Maxwell model stability equation, increases the value of the critical wavenumber but not until higher values of elasticity number and until the critical Reynolds' number is diminished more when compared to the viscoelastic stability equation. This relationship is seen more clearly in the semilog plot of figure 20.

CHAPTER 5

SUMMARY AND CONCLUSIONS/COMMENTS ON TURBULENT FLOW

The decrease in critical Reynolds' number with elasticity number is slightly greater for the viscoelastic stability equation than the decrease for the second-order/Maxwell model stability equation. For values of elasticity number up to 5×10^{-4}, a range including usual applications of drag reducing fluids, the critical Reynolds' number for the viscoelastic stability equation and that for the second-order/Maxwell model stability equation are 5650.1 and 5651.0 respectively.

The increase in critical wavenumber with elasticity number was minimally greater for the viscoelastic stability equation. This increase, for either of the two stability equations, may give insight into why viscoelastic fluids do show delayed transition to turbulence vs Newtonian fluids and/or why turbulent flow pressure drop is lessened for viscoelastic fluids. This explanation may be likened to the phenomena of the symmetric case where the critical Reynolds' number is significantly higher as is the wavenumber; the symmetric case as already stated is the more stable of the flows.

Another important aspect to the curves of neutral stability (which may be a direct result of the wavenumber vs elasticity number increase) occurs for the viscoelastic stability equation occurs at high Reynolds' number and low wavenumber. For high shear rate flows (Reynolds' numbers greater than 8900), and low wavenumber (wavenumber = 0.8) it is seen that value of disturbance growth rate (c imaginary) for the Newtonian fluid and for a fluid with an elasticity number of 10^{-3} (figure 21) was equal. So it is assumed that the neutral stability curves at low wavenumber (high wavenumber would be an important investigation) for Newtonian fluids and for those with various measures of elasticity, or elasticity number, intersect. In this high range of Reynolds' numbers it is seen that faster damping of the disturbance occurs for the viscoelastic fluids than for Newtonian fluids. This does not mean that the flow remains laminar at such high Reynolds' numbers (as disturbances at various wavenumbers exist with some corresponding to positive growth rates which will increase the disturbance). But at some values of wavenumber the disturbance would decay for viscoelastic fluids but increase for the Newtonian. Again, Dr. Harnoy writes

> *"Moreover, a change in the magnitude of amplitude of disturbances would affect the generation of turbulence in a fully developed turbulent flow resulting in a change in the Reynolds' stresses and friction between the fluid and a wall and drag between fluid and submerged bodies."[3]*

APPENDIX A

FIGURES

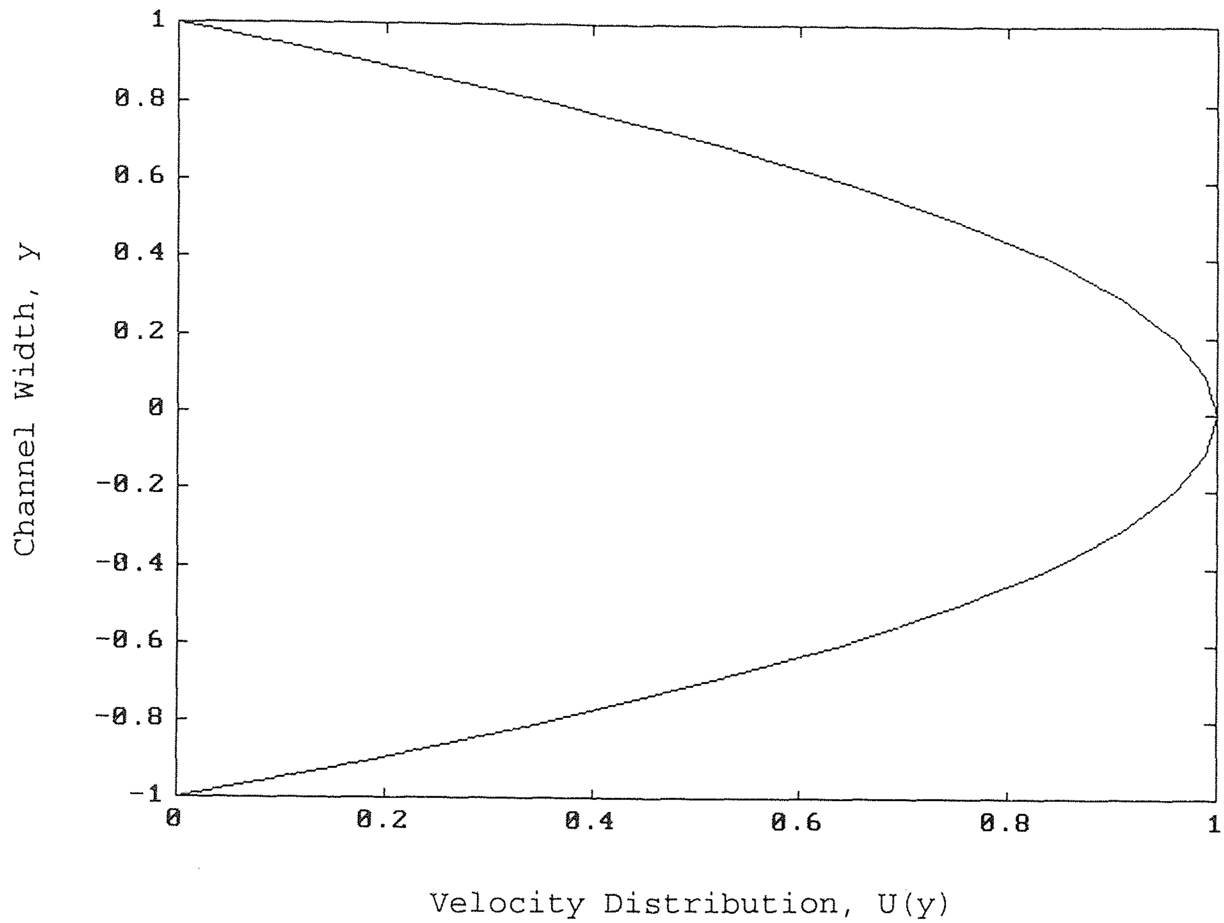
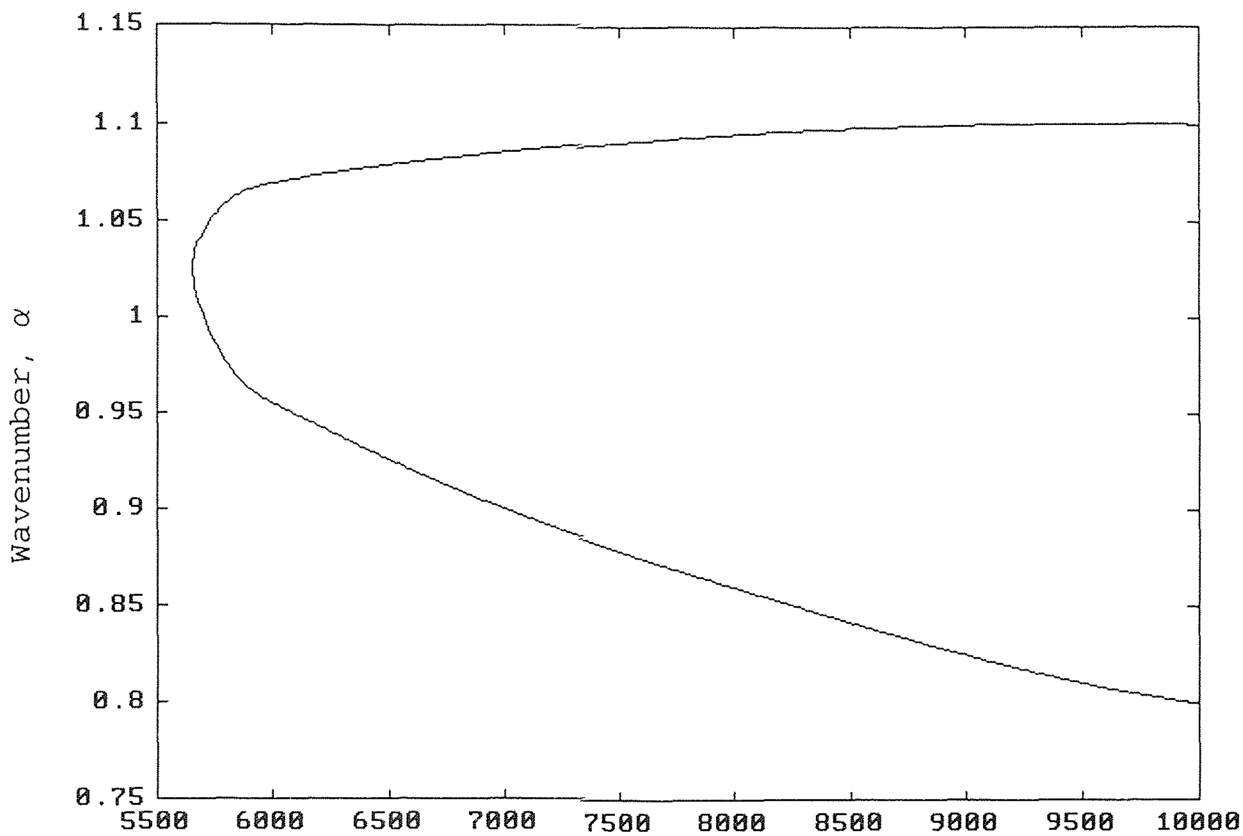


Fig.1 Mean Flow Velocity Profile in Channel



Reynolds' Number, R

Fig.2 Neutral Stability Curve for a Newtonian Fluid, $\Gamma = 0$
(Viscoelastic Stability Equation)
(Antisymmetric Case)

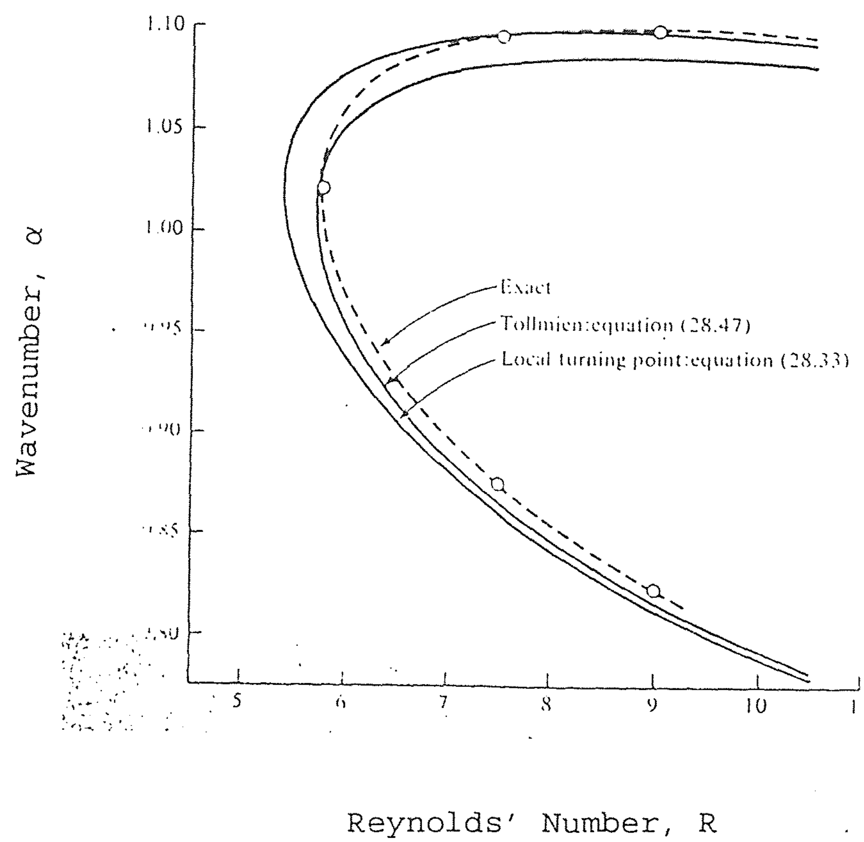


Fig.3 Neutral Stability Curve for a Newtonian Fluid, $\Gamma = 0$
(Viscoelastic Stability Equation)
(Antisymmetric Case) [10]

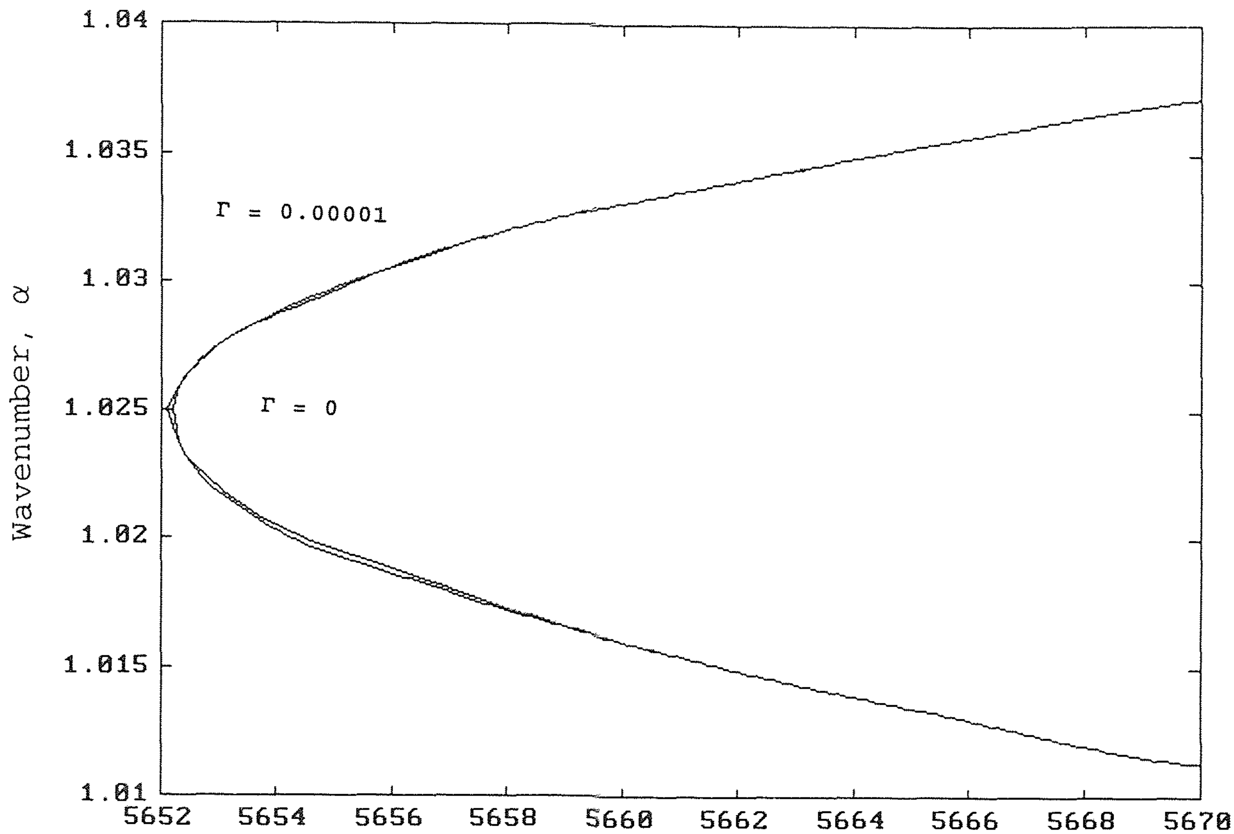


Fig.4 Neutral Stability Curve for a Newtonian Fluid, $\Gamma = 0$, and a Fluid With an Elasticity Number, $\Gamma = 10^{-5}$ (Viscoelastic Stability Equation) (Antisymmetric Case)

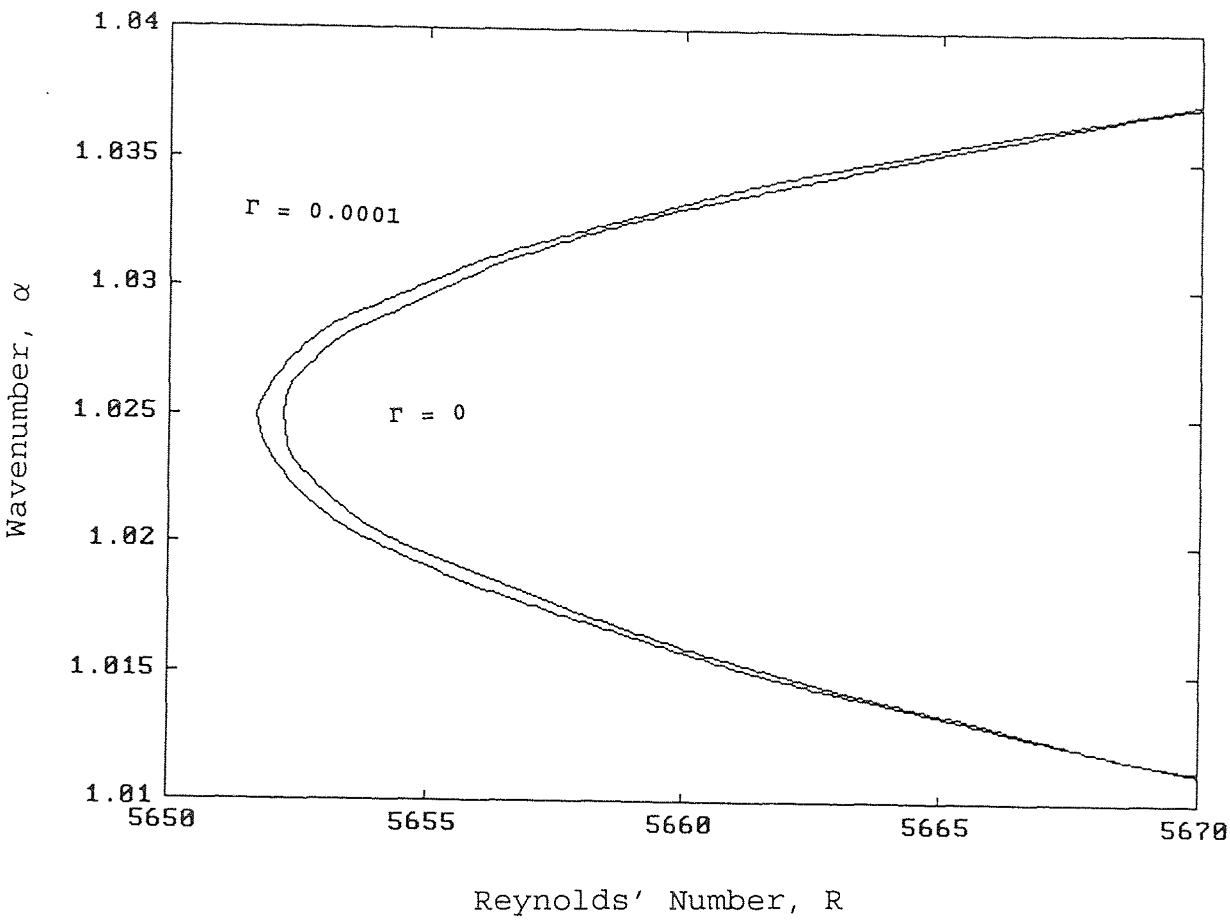


Fig.5 Neutral Stability Curve for a Newtonian Fluid, $\Gamma = 0$, and a Fluid With an Elasticity Number, $\Gamma = 10^{-4}$ (Viscoelastic Stability Equation) (Antisymmetric Case)

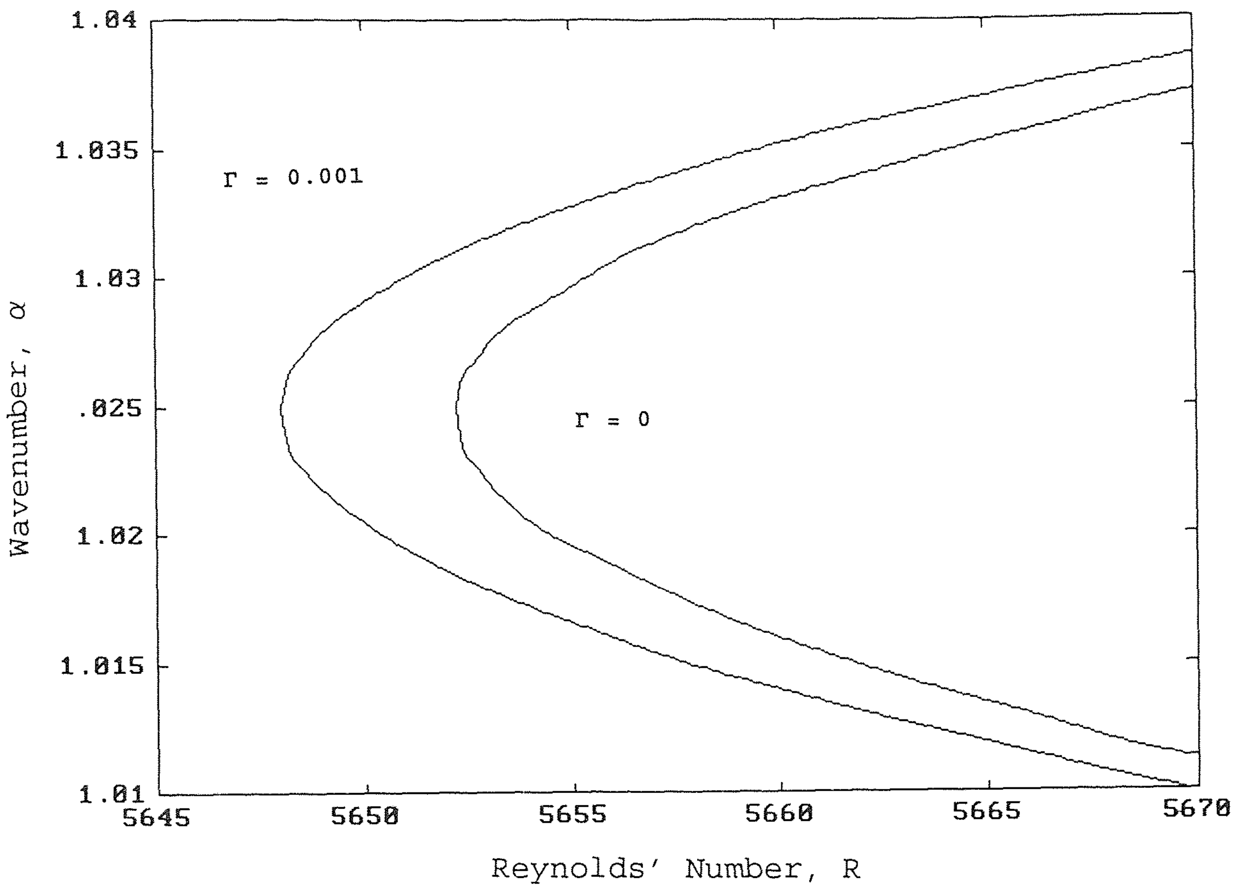


Fig.6 Neutral Stability Curve for a Newtonian Fluid, $\Gamma = 0$, and a Fluid With an Elasticity Number, $\Gamma = 10^{-3}$
(Viscoelastic Stability Equation)
(Antisymmetric Case)

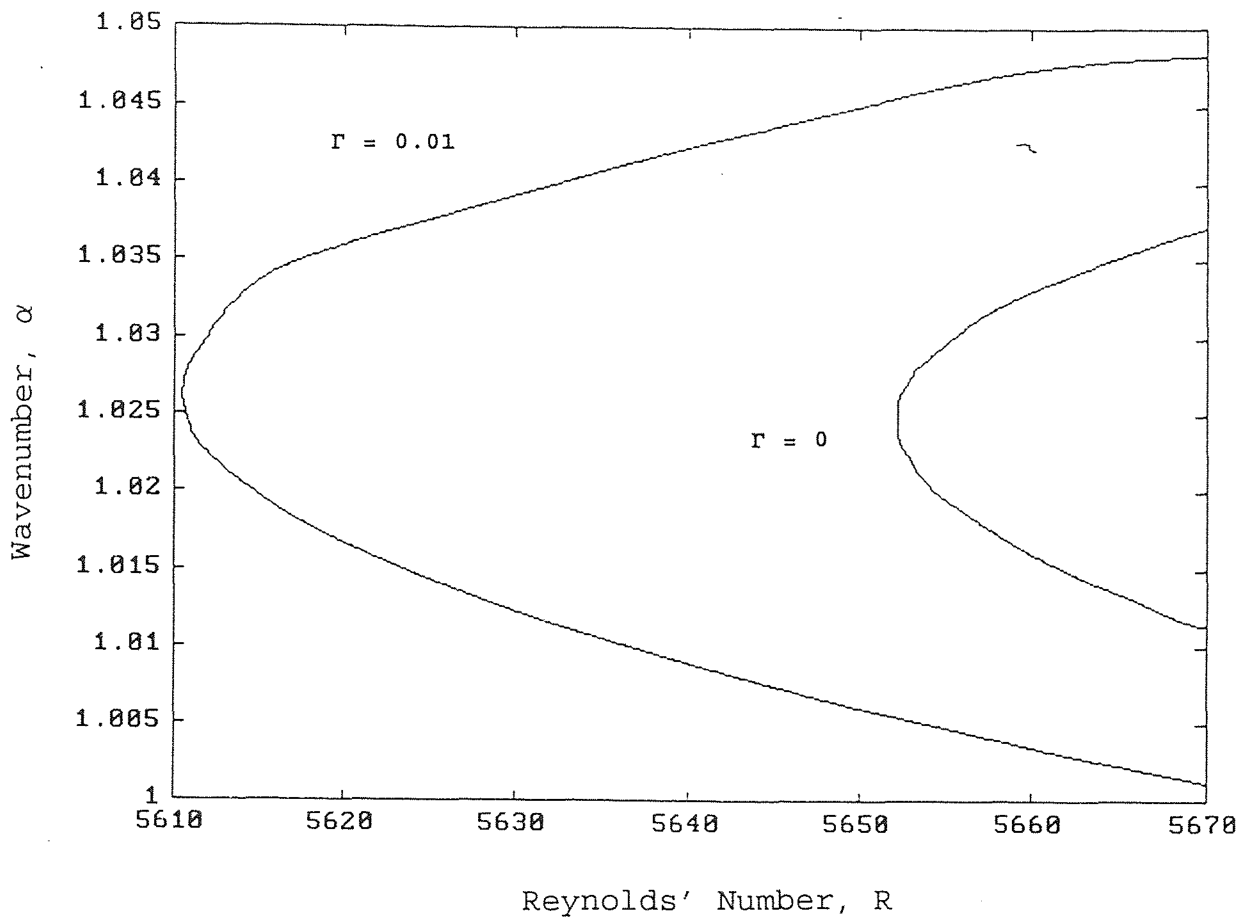


Fig.7 Neutral Stability Curve for a Newtonian Fluid, $\Gamma = 0$, and a Fluid With an Elasticity Number, $\Gamma = 10^{-2}$ (Viscoelastic Stability Equation) (Antisymmetric Case)

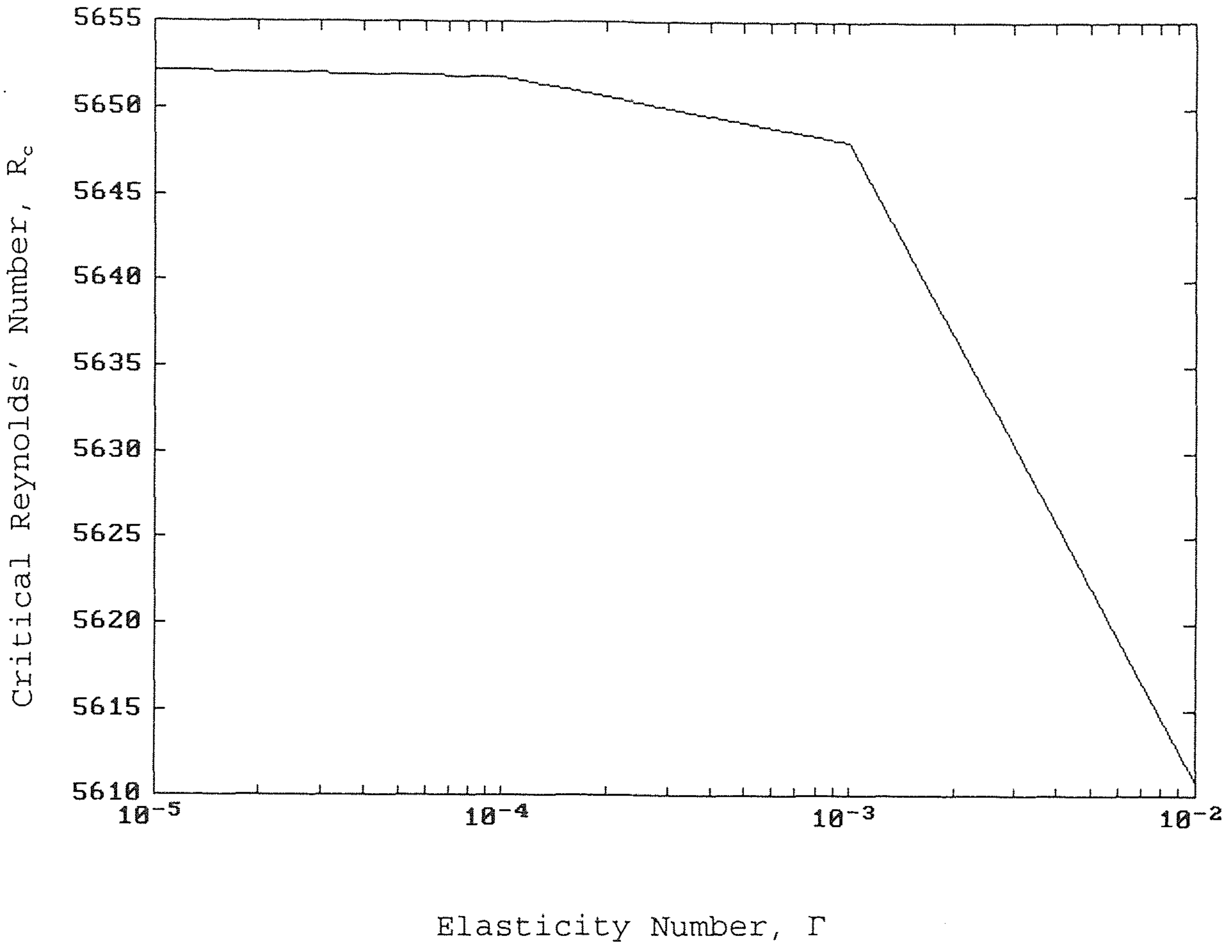


Fig.8 Plot of Critical Reynolds' Number, R_c vs Elasticity Number, Γ
(Viscoelastic Stability Equation)
(Antisymmetric Case)

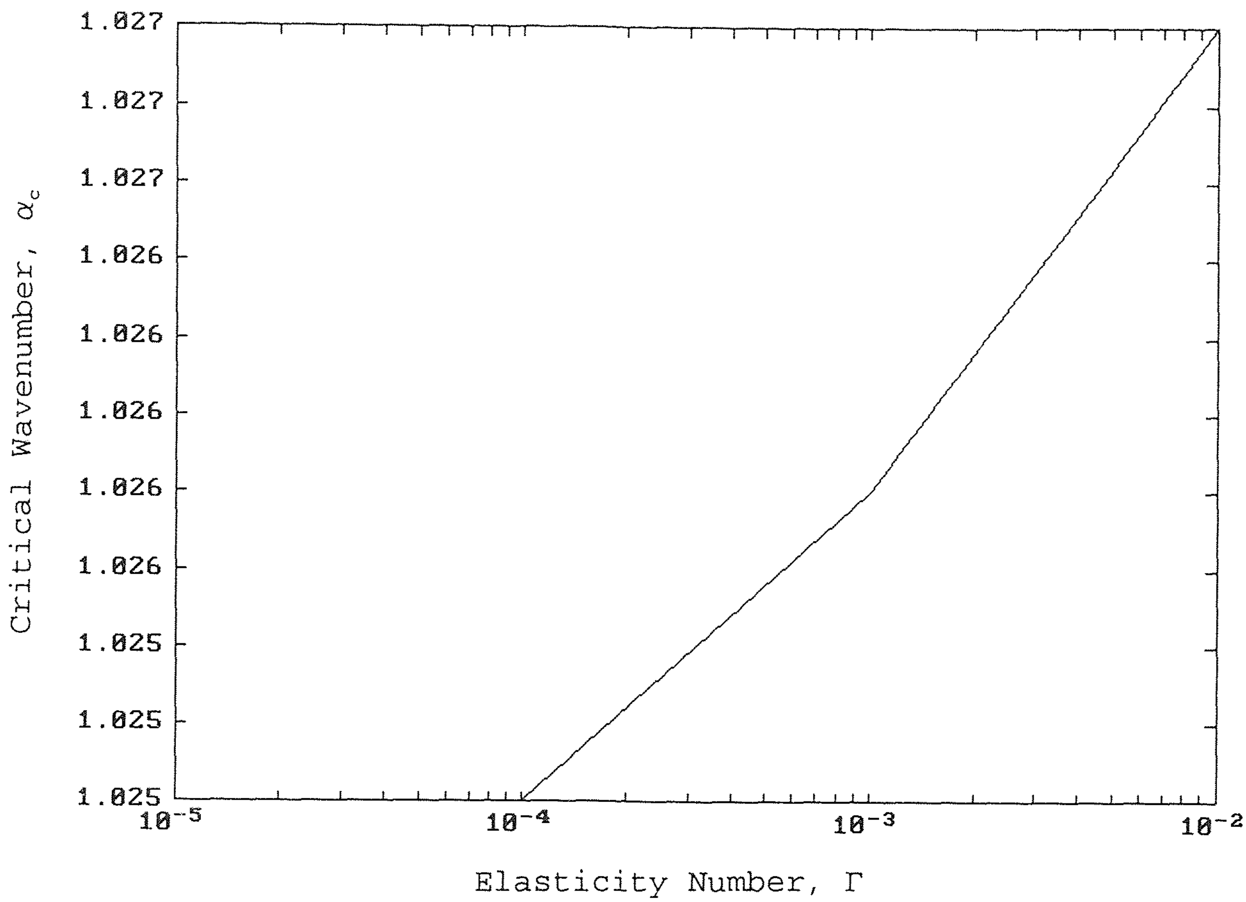


Fig.9 Plot of Critical Wavenumber, α_c vs
Elasticity Number, Γ
(Viscoelastic Stability Equation)
(Antisymmetric Case)

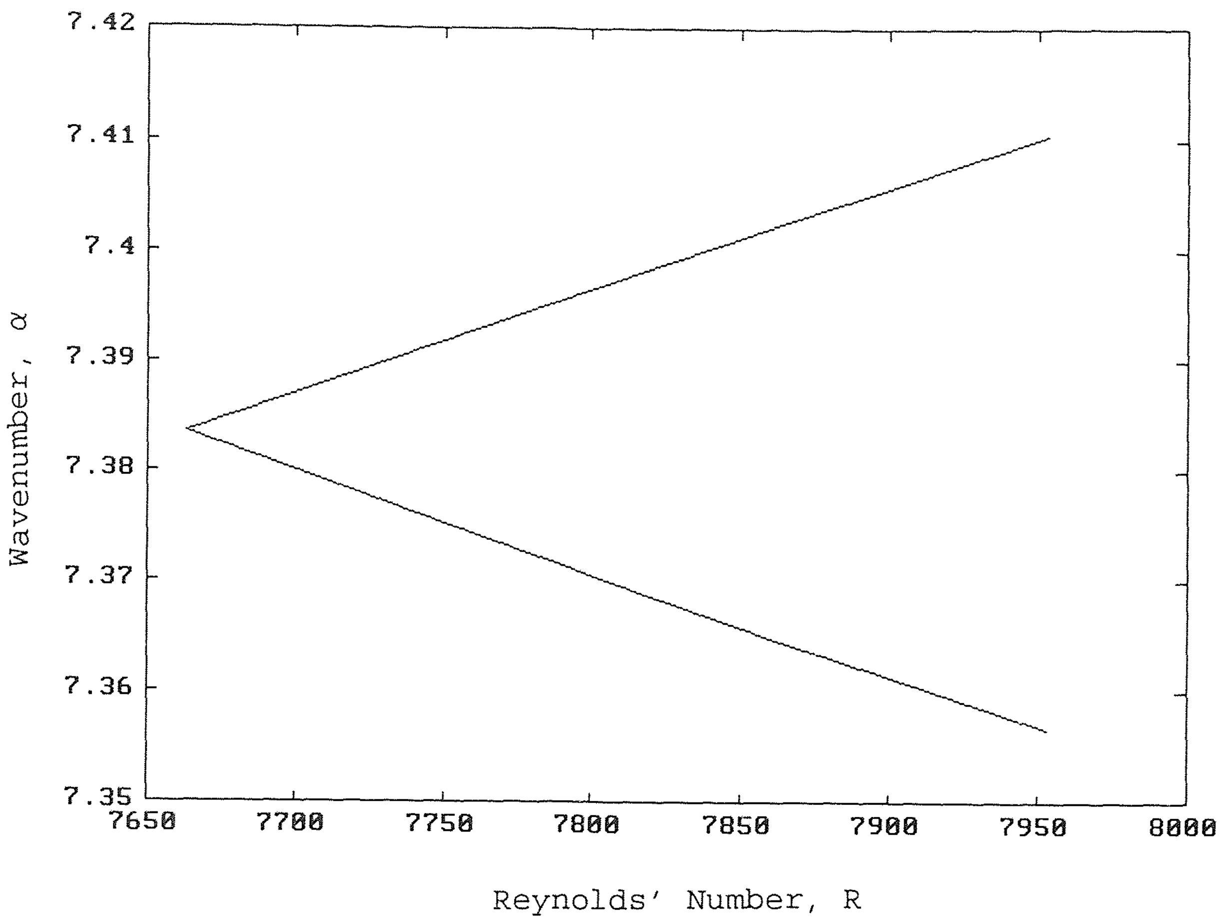


Fig.10 Neutral Stability Curve for a Newtonian Fluid, $\Gamma = 0$
(Viscoelastic Stability Equation)
(Symmetric Case)

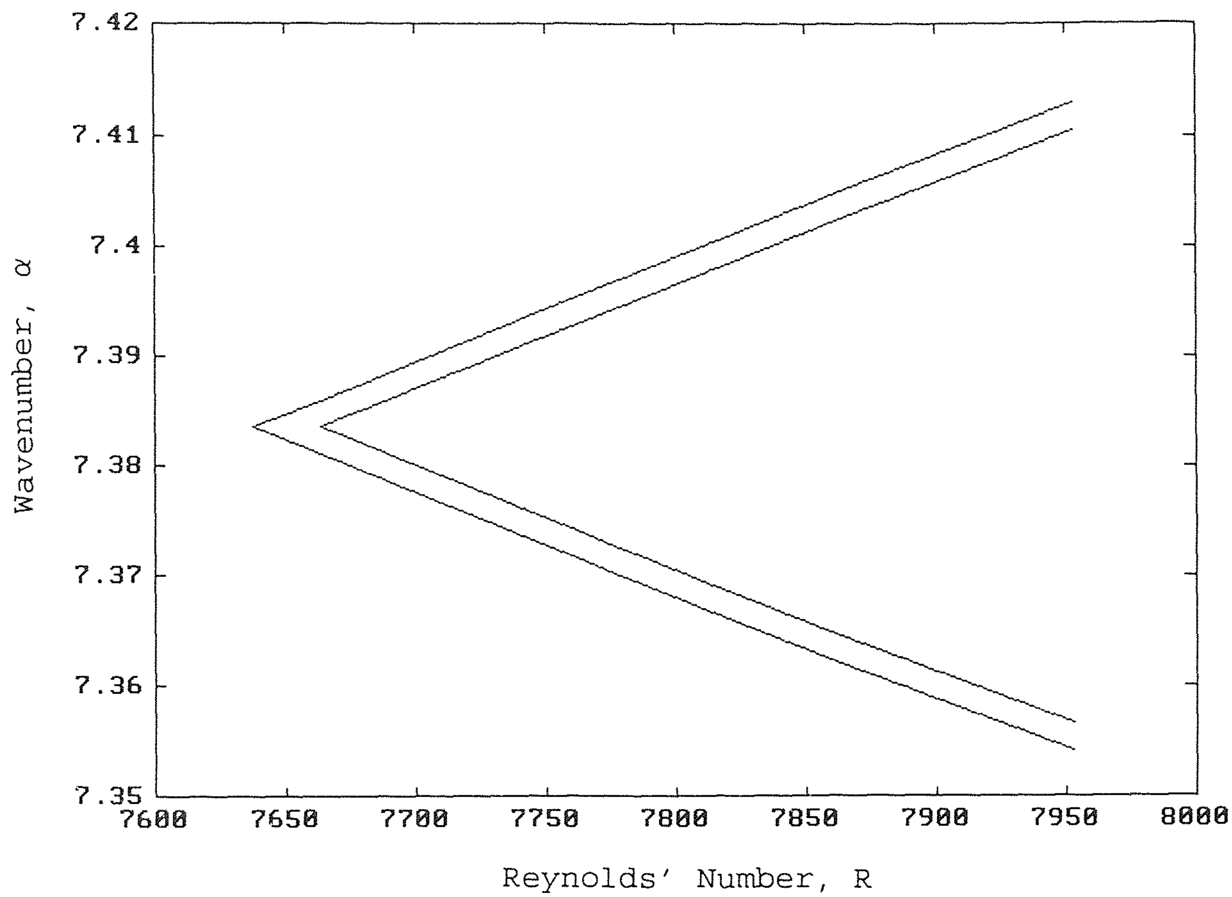


Fig.11 Neutral Stability Curve for a Newtonian Fluid, $\Gamma = 0$, and a Fluid With an Elasticity Number, $\Gamma = 10^{-3}$ (Viscoelastic Stability Equation) (Symmetric Case)

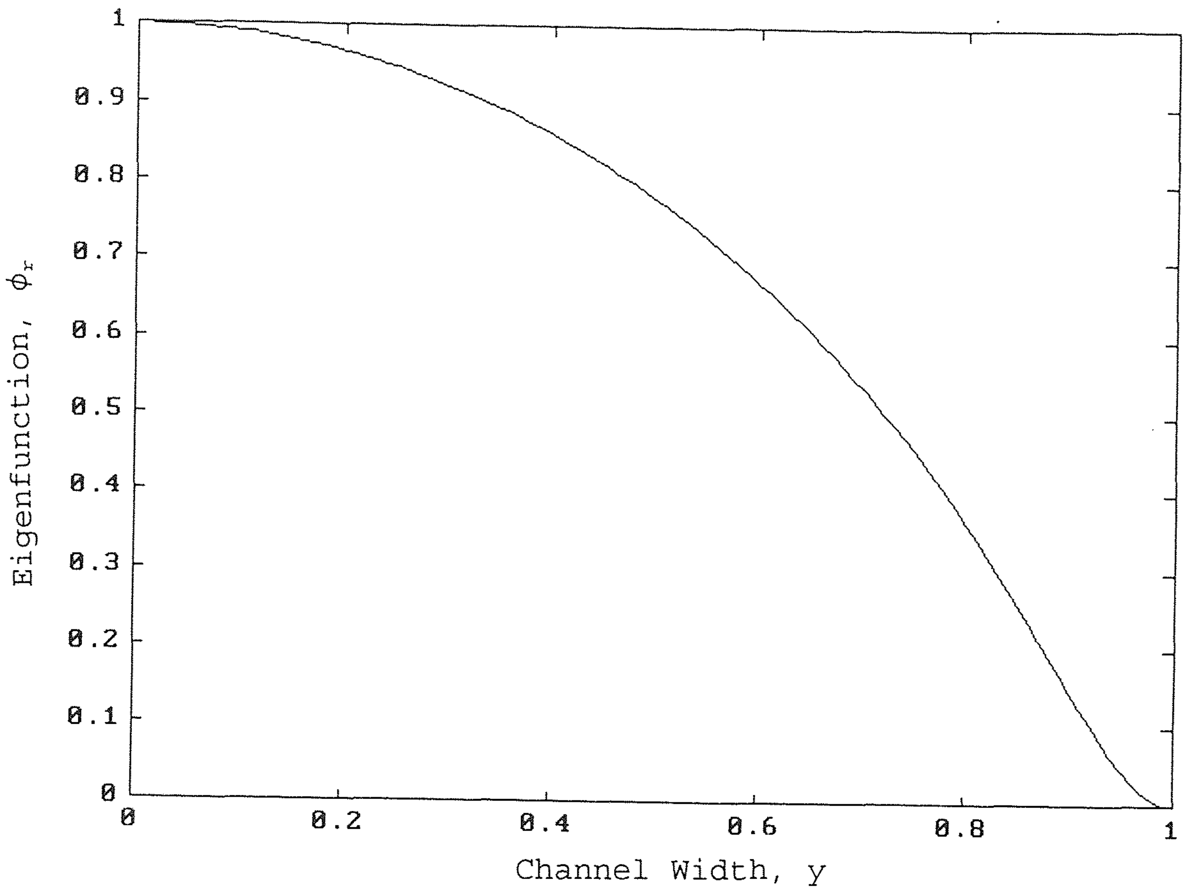


Fig.12 Disturbance Amplitude (Real Part), ϕ_r vs Channel Width, y

$R = 7000$, $\alpha = 1$, $\Gamma = 0$

(Viscoelastic Stability Equation)

(Antisymmetric Case)

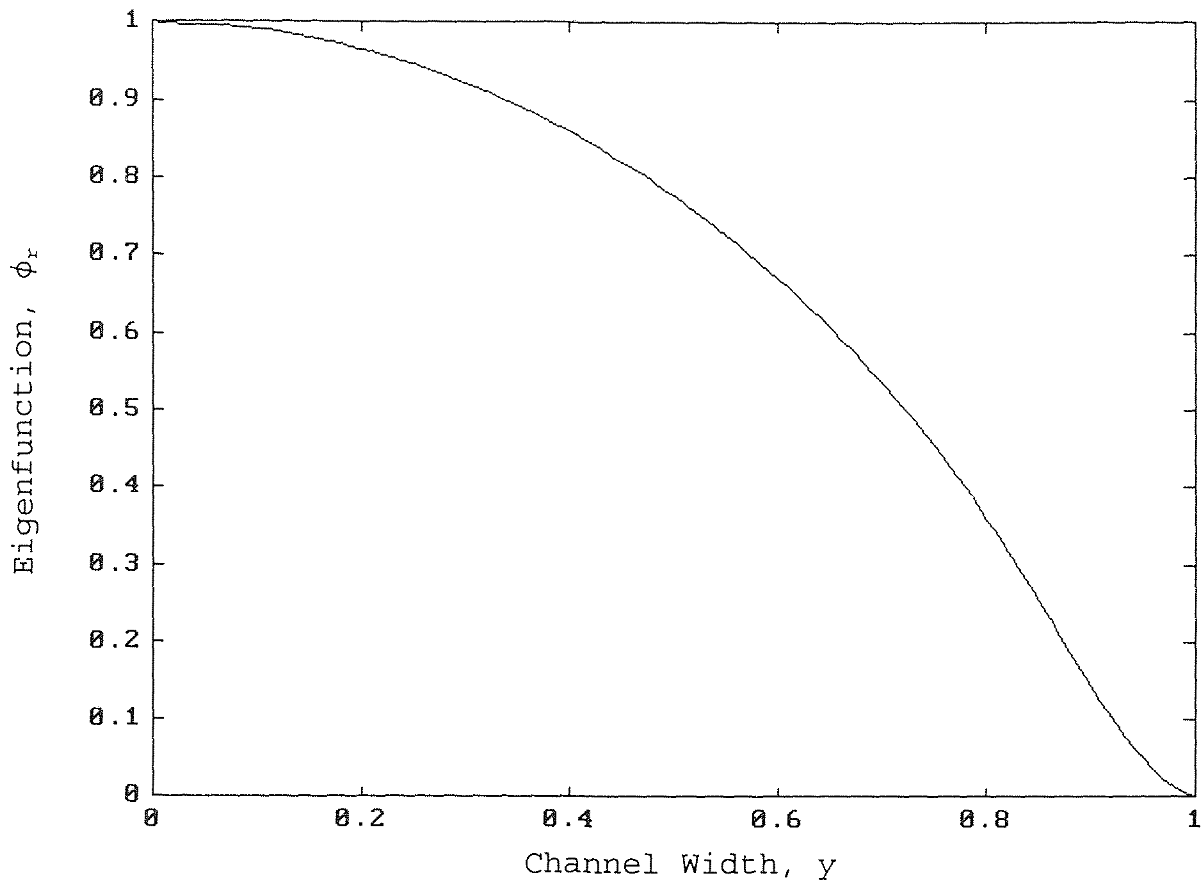


Fig.13 Disturbance Amplitude (Real Part), ϕ_r
vs Channel Width, y
$R = 7000$, $\alpha = 1$, $\Gamma = 10^{-3}$
(Viscoelastic Stability Equation)
(Antisymmetric Case)

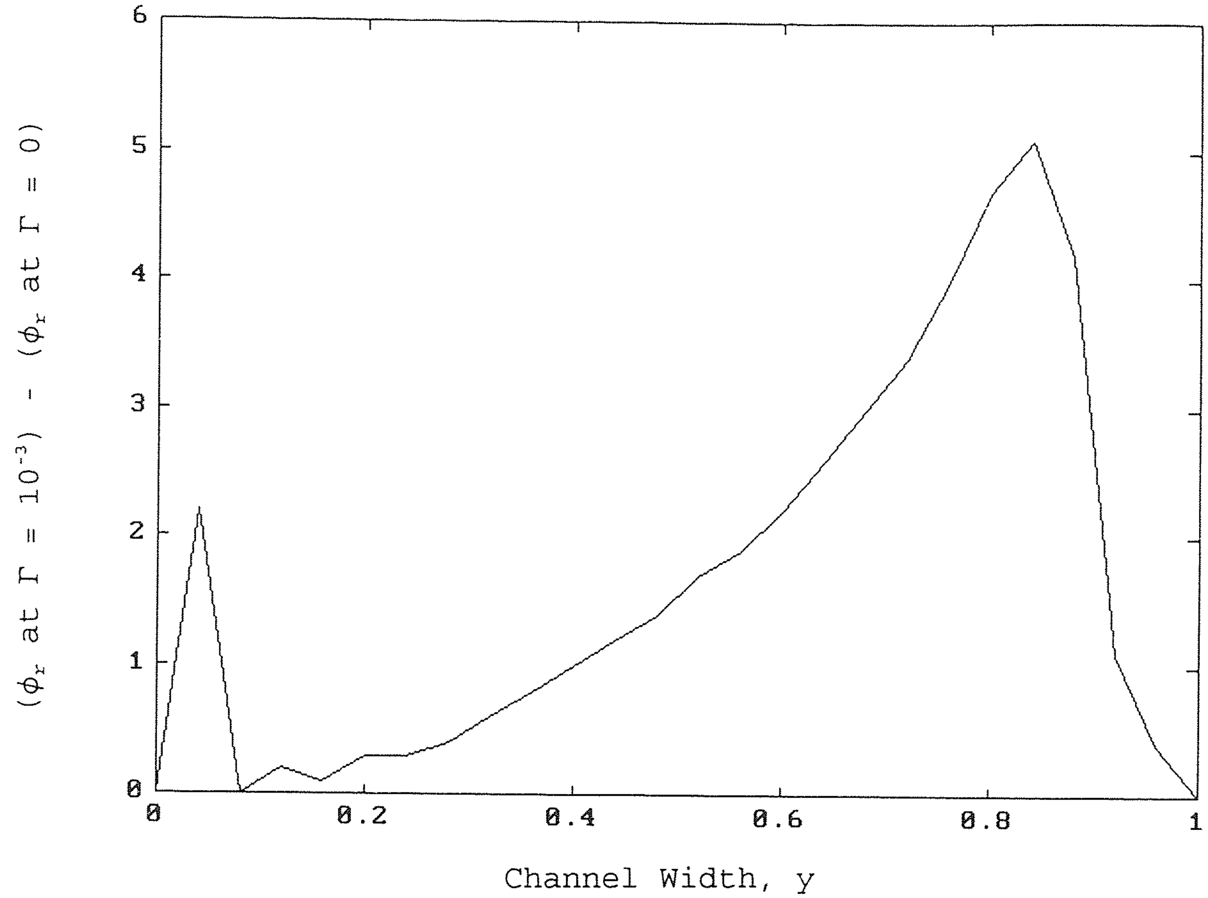


Fig.14 Difference Between Disturbance Amplitudes,
(ϕ_r at $\Gamma = 10^{-3}$) - (ϕ_r at $\Gamma = 0$) vs Channel Width, y
$R = 7000$, $\alpha = 1$, $\Gamma = 10^{-3}$
(Viscoelastic Stability Equation)
(Antisymmetric Case)

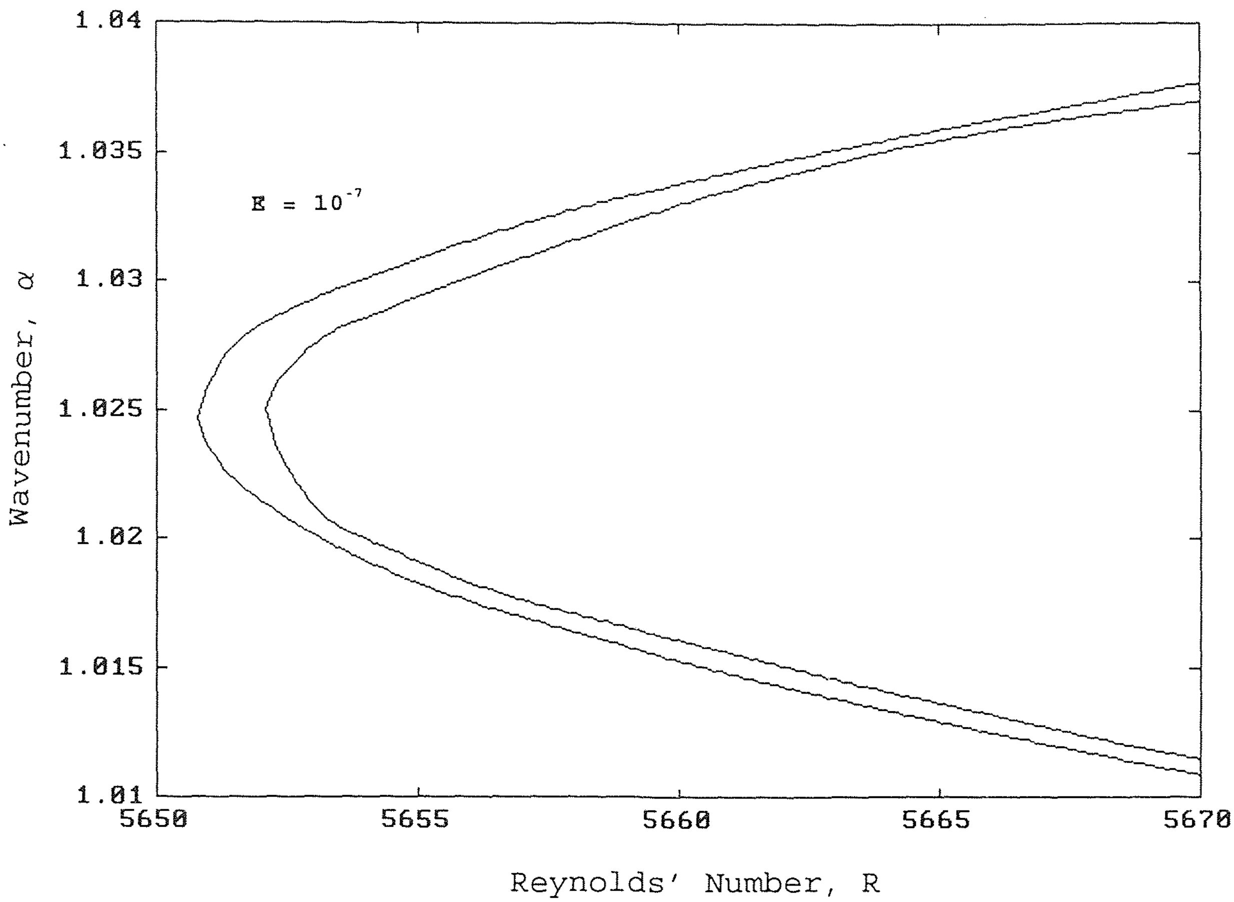


Fig.15 Neutral Stability Curve for a Newtonian Fluid, E = 0, and a Fluid With an Elasticity Number, E = 10^{-7} (Second-Order/Maxwell Model Stability Equation) (Antisymmetric Case)

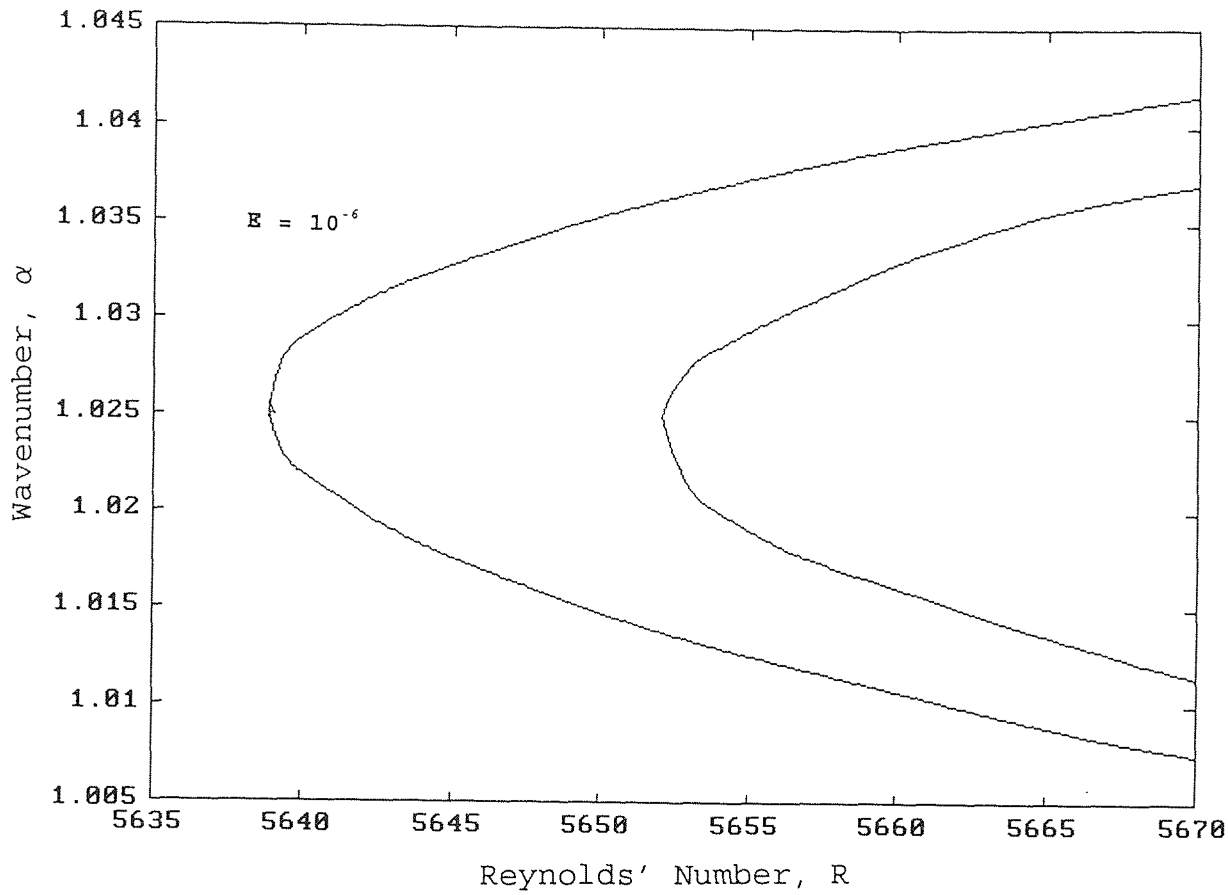


Fig.16 Neutral Stability Curve for a Newtonian Fluid, E = 0, and a Fluid With an Elasticity Number, $E = 10^{-6}$ (Second-Order/Maxwell Model Stability Equation) (Antisymmetric Case)

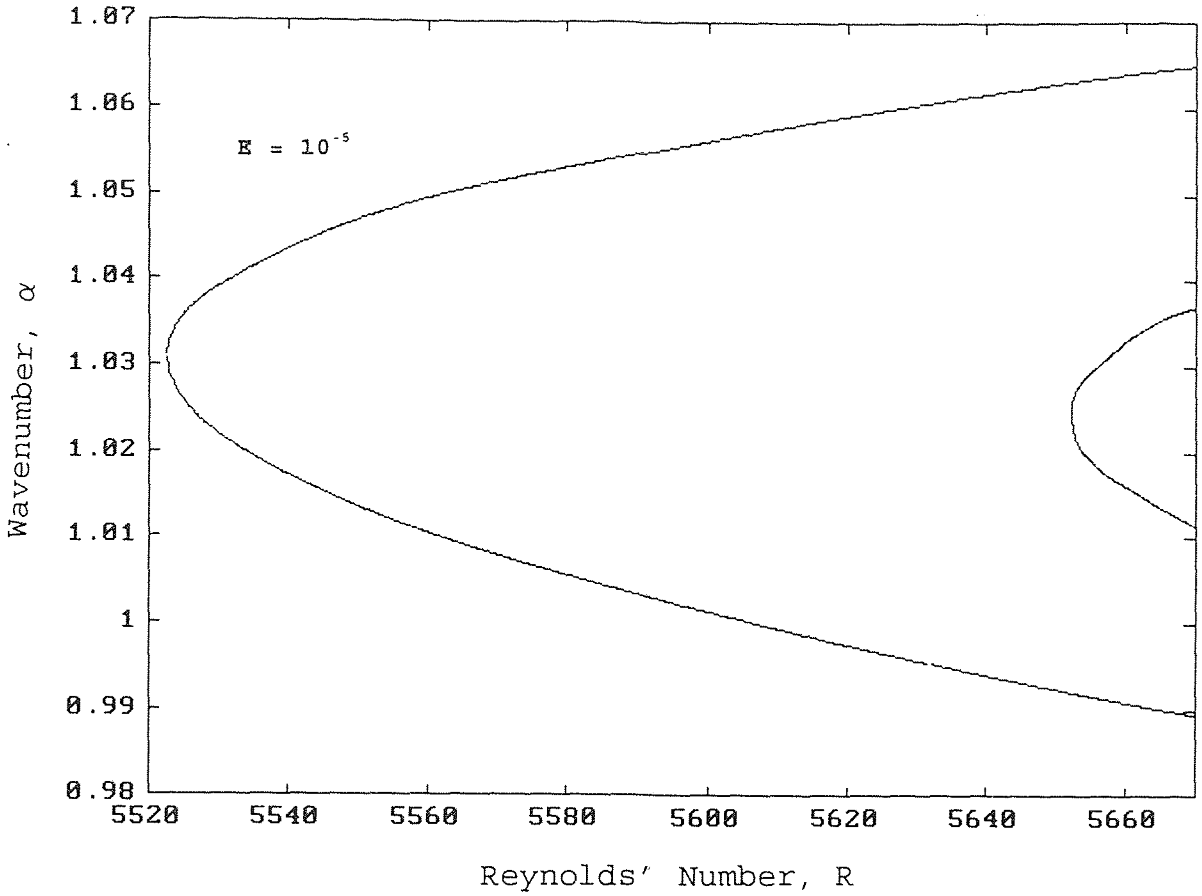


Fig.17 Neutral Stability Curve for a Newtonian Fluid, $E = 0$, and a Fluid With an Elasticity Number, $E = 10^{-5}$
(Second-Order/Maxwell Model Stability Equation)
(Antisymmetric Case)

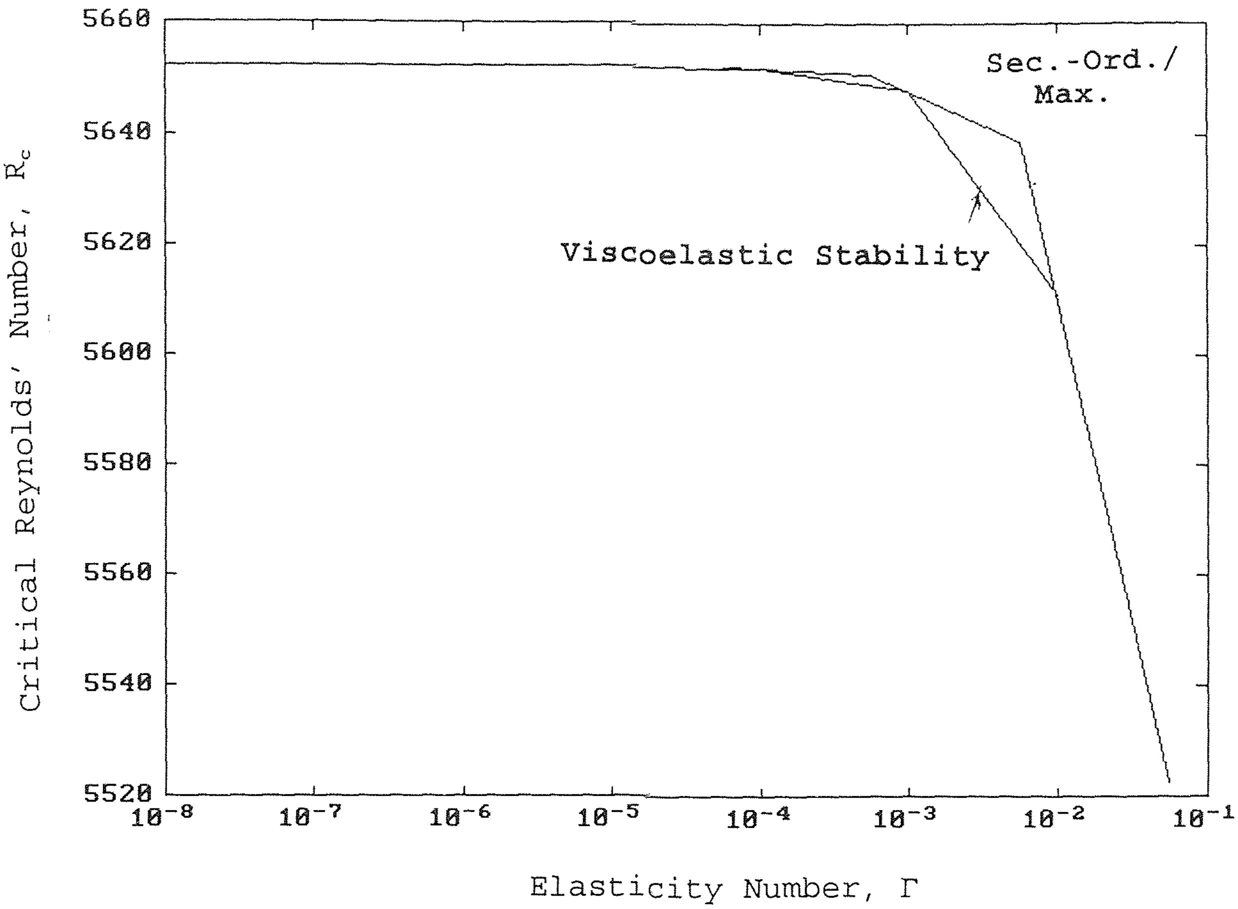


Fig.18 Comparison of Critical Reynolds' Number, R_c between the Viscoelastic and Second-Order/Maxwell Model Stability Equations (Antisymmetric Case)

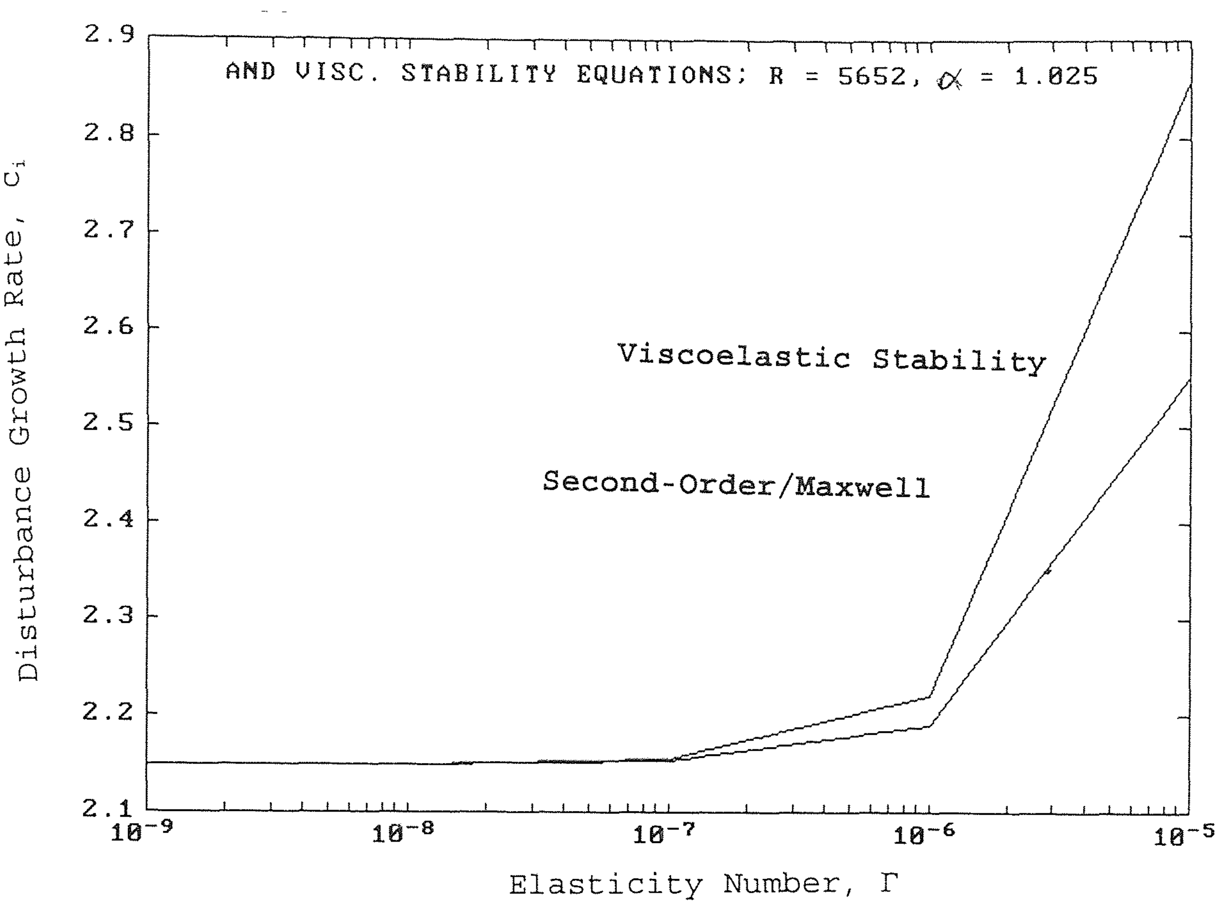


Fig.19 Comparison of Disturbance Growth Rate, c_i between the Viscoelastic and Second-Order/Maxwell Model Stability Equations at $R = 5650$, $\alpha = 1.025$ (Antisymmetric Case)

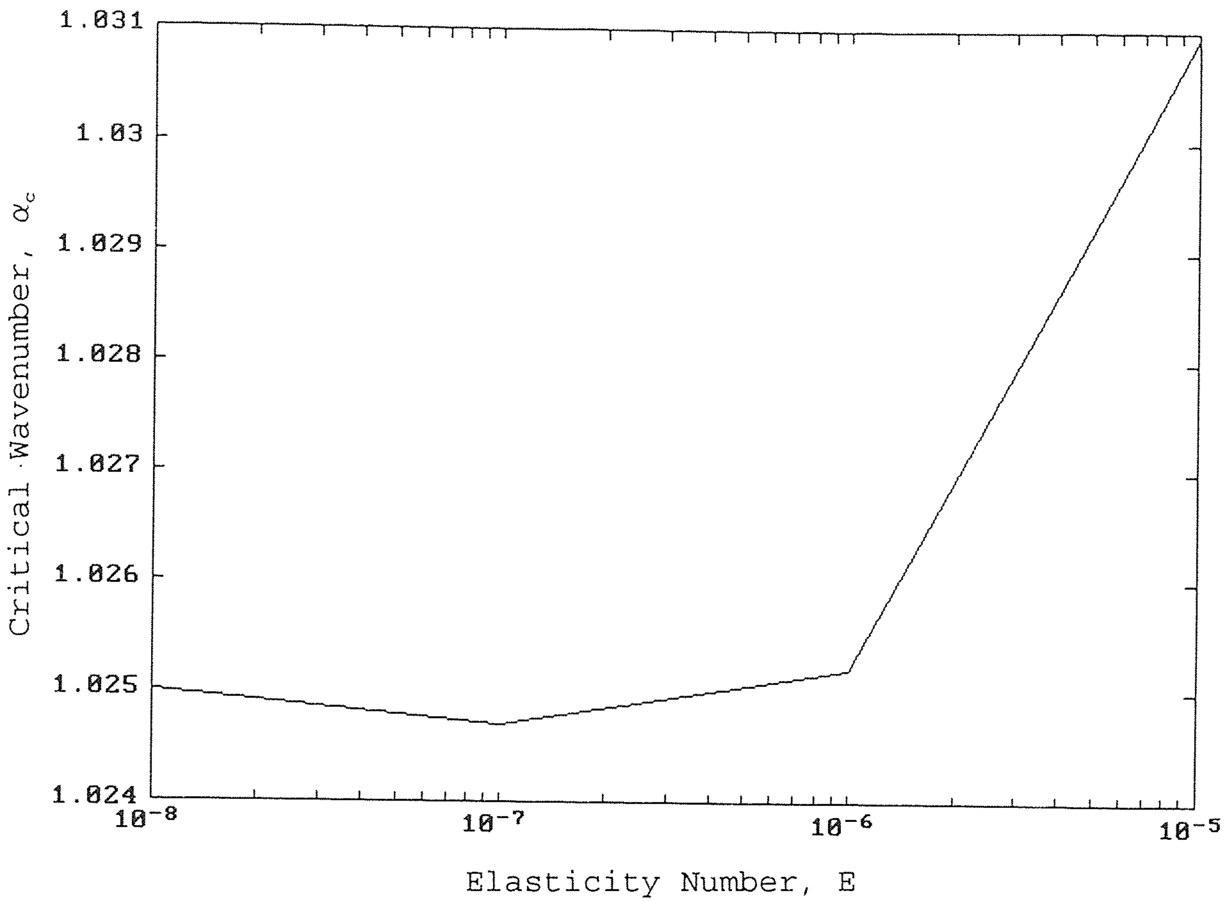


Fig.20 Plot of Critical Wavenumber, α_c vs Elasticity Number, E
(Second-Order/Maxwell Model Stability Equation)
(Antisymmetric Case)

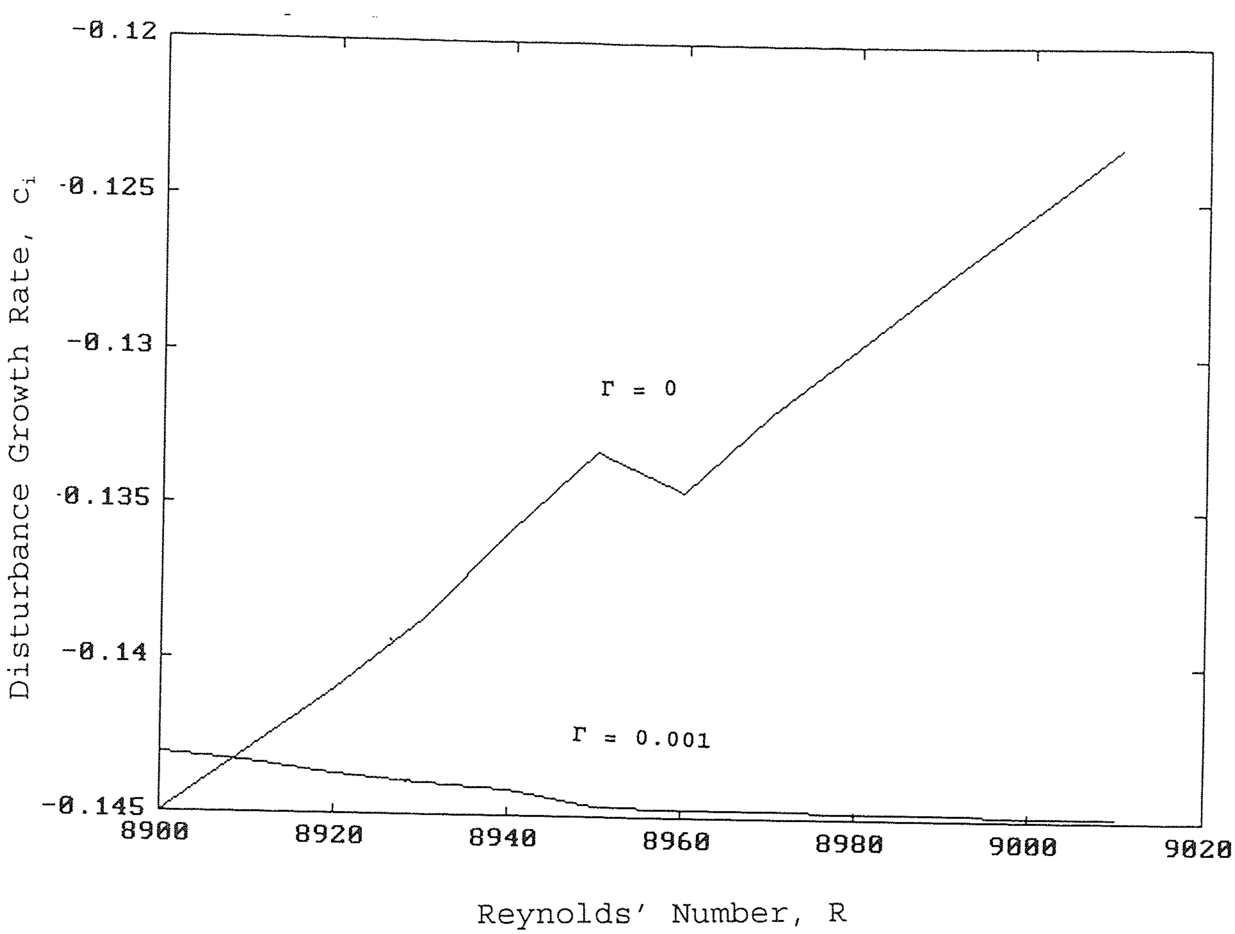


Fig.21 Disturbance Growth Rate, c_i vs Reynolds' Number, R for Fluids of $\Gamma = 0$ and $\Gamma = 10^{-3}$ at Wavenumber, $\alpha = 0.8$ (Viscoelastic Stability Equation)

APPENDIX B

NUMERICAL ANALYSIS AND OUTLINE OF PROGRAM USED TO EVALUATE THE VISCOELASTIC STABILITY EQUATION

B.1 Numerical Analysis

The numerical methods used in the solution to the viscoelastic stability equation and an outline of the program is discussed. Being a fourth-order initial-value problem, a numerical solution to the viscoelastic stability equation requires that it must be reduced to a first-order system.

The classical procedure dictates that we first must convert a general mth-order differential equation of the form,

$$y^{(m)}(t) = f(t, y, y', .., y^{(m-1)}), \quad a \le t \le b \tag{B.1}$$

with initial conditions,

$$\begin{aligned} y(a) &= \alpha_1, \\ y'(a) &= \alpha_2, \\ y^{(m-1)}(a) &= \alpha_m, \end{aligned} \tag{B.2}$$

into a system of equations in the form,

$$\begin{aligned} du_1/dt &= f_1(t, u_1, u_2, ...u_m), \\ du_2/dt &= f_2(t, u_1, u_2, ...u_m), \\ &\vdots \\ du_m/dt &= f_m(t, u_1, u_2, ...u_m), \end{aligned} \tag{B.3}$$

or $a \le t \le b$, with the initial conditions,

$$u_1(a) = \alpha_1, \qquad (B.4)$$
$$u_2(a) = \alpha_2,$$
$$\vdots$$
$$u_m(a) = \alpha_m.$$

The object is to find m functions $u_1, u_2, \ldots, u_m$ that satisfy the system of differential equations as well as the initial conditions into a system of equations in the form B.3 and B.4,

$$u_1(t) = y(t), \qquad (B.5)$$
$$u_2(t) = y'(t),$$
$$\vdots$$
$$u_m(t) = y^{(m-1)}(t).$$

Using this notation, we obtain the first-order system,

$$du_1/dt = dy/dt = u_2, \qquad (B.6)$$
$$du_2/dt = dy'/dt = u_3,$$
$$\vdots$$
$$du_{m-1}/dt = dy^{(m-2)}/dt = u_m,$$

and,

$$du_m/dt = dy^{(m-1)}/dt = y^{(m)} = \qquad (B.7)$$
$$f(t,y,y',..,y^{(m-1)}) = f(t,u_1,u_2,\ldots,u_m),$$

with initial conditions,

$$u_1(a) = y(a) = \alpha_1, \qquad \text{(B.8)}$$
$$u_2(a) = y'(a) = \alpha_2,$$
$$\vdots$$
$$u_m(a) = y^{(m-1)}(a) = \alpha_m.$$

For the viscoelastic stability equation we have

$$\phi_j' = f_j = (y,\ \phi_1,\ \phi_2,\ \phi_3,\ \phi_4,\ c) \qquad \text{(B.9)}$$

$$\phi_j' = d\phi_j/dy \qquad 0 \le y \le 1.$$

Here ϕ_j' is also a function of y (the channel width) and c (the velocity of the wave); with the initial conditions and case 1 considered (antisymmetric):

$$\phi_1(0) = 1 + 0i = \alpha_1, \qquad \text{(B.10)}$$
$$\phi_2(0) = 0 + 0i = \alpha_2,$$
$$\phi_3(0) = A(1 + i) = \alpha_3,$$
$$\phi_4(0) = 0 + 0i = \alpha_4,$$
$$c = C(1 + i) = \alpha_5,$$

or with case 2 considered (symmetric):

$$\phi_1(0) = 0 + 0i = \alpha_1, \qquad \text{(B.11)}$$
$$\phi_2(0) = 1 + 0i = \alpha_2,$$
$$\phi_3(0) = 0 + 0i = \alpha_3,$$
$$\phi_4(0) = A(1 + i) = \alpha_4,$$
$$c = C(1 + i) = \alpha_5.$$

Here A and C are initial estimates and with,

$$u_1(t) = \phi(y), \qquad \text{(B.12)}$$
$$u_2(t) = \phi'(y),$$
$$u_3(t) = \phi''(y),$$
$$u_4(t) = \phi'''(y),$$

the differential equation is transformed into the system,

$$u_1'(y) = u_2(y) = f_1 \qquad \text{(B.13)}$$
$$u_2'(y) = u_3(y) = f_2$$
$$u_3'(y) = u_4(y) = f_3$$
$$u_4'(y) = u_5(y) = f_4.$$

Also note that

$$f_4 = \phi'''' = f(y, u_5, u_4, u_3, u_2, u_1, c). \qquad \text{(B.14)}$$

The system may be solved as follows:

Begin again with the initial conditions (case 1):

$$\phi_1(0) = 1 + 0i = \alpha_1, \qquad \text{(B.15)}$$
$$\phi_2(0) = 0 + 0i = \alpha_2,$$
$$\phi_3(0) = A(1 + i) = \alpha_3,$$
$$\phi_4(0) = 0 + 0i = \alpha_4,$$
$$c = C(1 + i) = \alpha_5.$$

These give initial values for ϕ and its derivatives and the eigenvalues are evaluated ($0 \le y \le 1$) using the Runge-Kutta fourth-order method [11]. The boundary conditions hold that the real and imaginary parts of ϕ and ϕ' at the wall ($y = 1$)

must equal zero. Or we may write

$$\phi_1(1) = 0 + 0i, \qquad (B.16)$$

$$\phi_2(1) = 0 + 0i.$$

We may denote the calculated values of ϕ and ϕ' after integrating (using the initial values of the real numbers A and C) as,

$$\phi(1, A_0, C_0) \qquad (B.17)$$

$$\phi'(1, A_0, C_0).$$

We may also find after varying these initial guesses by an amount of + h and - h:

$$(B.18)$$

$$\delta\phi/\delta A(1, A_0, C_0), \qquad \delta\phi/\delta C(1, A_0, C_0),$$

$$\delta\phi'/\delta A(1, A_0, C_0), \qquad \delta\phi'/\delta C(1, A_0, C_0).$$

These may be calculated as

$$(A.19)$$

$$\delta\phi/\delta A(1, A_0, C_0) = (\phi(1, A_0 + h, C_0) - \phi(1, A_0 - h, C_0))/2h$$

$$\delta\phi/\delta C(1, A_0, C_0) = (\phi(1, A_0, C_0 + h) - \phi(1, A_0, C_0 - h))/2h$$

$$\delta\phi'/\delta A(1, A_0, C_0) = (\phi(1, A_0 + h, C_0) - \phi(1, A_0 - h, C_0))/2h$$

$$\delta\phi'/\delta C(1, A_0, C_0) = (\phi(1, A_0, C_0 + h) - \phi(1, A_0, C_0 - h))/2h.$$

We may then implement the Newton-Raphson method written in matrix form which will yield the new approximations to A and C, denoted here as X_1, and the values of all derivatives. We have,

$$X_1 = X_0 - (\delta f/\delta X(X_0))^{-1} \quad f(X_0), \tag{B.20}$$

where,

(B.21)

$$X_1 = \begin{vmatrix} A_1 \\ C_1 \end{vmatrix} \qquad X_0 = \begin{vmatrix} A_0 \\ C_0 \end{vmatrix}$$

and,

(B.22)

$$\delta f/\delta X(X_0) = \begin{vmatrix} \delta\phi/\delta A(1, A_0, C_0) & \delta\phi/\delta C(1, A_0, C_0) \\ \delta\phi'/\delta A(1, A_0, C_0) & \delta\phi'/\delta C(1, A_0, C_0) \end{vmatrix}$$

$$f(X_0) = \begin{vmatrix} \phi(1, A_0, C_0) \\ \phi'(1, A_0, C_0) \end{vmatrix}. \tag{B.23}$$

The process may be repeated n times until the solution X_n converges to some tolerance of the values of the eigenfunctions at the boundary (y = 1); again stated (for case 1):

(B.24)

$$\phi_1(1) = 0 + 0i,$$

$$\phi_2(1) = 0 + 0i.$$

B.2 Program Outline

Output approximations W_j to $\phi_j(y)$ at the (N + 1) number of y values.

Step 1 Set variables to implicit double precision.

Dimension necessary vectors

Input constants; R, Reynolds' number, R (real)

F, wavenumber, α (real)

G, viscoelastic constant, Γ (real)

N, step size, (real)

H, Newton-Raphson variation,

$H \approx 10^{-3} - 10^{-10}$ (imaginary).

Calculate; Dy = (1-0)/N;

Set y = 0;

Step 2 for j = 1,2,3,4 set initial values = $\alpha_{j,0}$

case 1;

add both + h,-h to ϕ'' and c (or α_3 and α_5) and solve using Step 3 for each of the five variations)

case 2;

add both + h,-h to ϕ''' and c (or α_4 and α_5) and solve using Step 3 for each of the five variations)

Step 3 for i = 0 to N

Step 4 Runge-Kutta fourth-order method

Step 5 output ($W_{j,i}$)

Step 6 Newton-Raphson method

If solution converges then stop

output c, $W_{j,i}$

If solution does not converge, then repeat using the new approximations ($W_{j,i}$) for α_3 and α_5.

** The program contains loops that allow for the variation of α and R while convergence continues

APPENDIX C

STREAM FUNCTION COMPONENTS

Since,

$$\Psi(x,y,t) = 1/2 \left\{\phi(y)\ e^{i(\alpha x-\beta t)} + \phi(y)\ e^{-i(\alpha x-\beta t)}\right\}, \tag{C.25}$$

and,

$$\beta = \beta_r + i\, \beta_i, \tag{C.26}$$

we have,

$$\Psi(x,y,t) = 1/2 \left\{\phi(y)\ e^{i(\alpha x-(\beta r+i\beta i)t)} + \phi(y)\ e^{-i(\alpha x-(\beta r-i\beta i)t)}\right\}, \tag{C.27}$$

$$\Psi(x,y,t) = 1/2 \left\{\phi(y)\ e^{\beta it}\, e^{i(\alpha x-\beta rt)} + \phi(y)\ e^{\beta it}\, e^{-i(\alpha x-\beta rt)}\right\}. \tag{C.28}$$

It is clearly seen from equation C.28 that the sign of β_i in each component points to stability conditions ; we have stability if $c_i < 0$, instability if $c_i > 0$, and neutral stability when $c_i = 0$. We therefore need to work with only one component in order to determine the sign of c_i. For convenience $\phi(y)\ e^{i(\alpha x-\beta t)}$ is chosen and we need not be concerned with its conjugate $\phi(y)\ e^{-i(\alpha x-\beta t)}$.

APPENDIX D

FORTRAN PROGRAM USED TO EVALUATE THE VISCOELASTIC STABILITY EQUATION

```
      PROGRAM OV
** SET VARIABLES TO DOUBLE PRECISION
      IMPLICIT DOUBLE PRECISION (D,F,G,R,U,Y)
      IMPLICIT DOUBLE COMPLEX (A-C,H,K,P,S,V-X,Z)

** THIS FORTRAN PROGRAM USES THE RUNGE-KUTTA, NEWTON-RAPHSON
** METHODS TO SOLVE THE ORR-SOMMERFELD EQUATION  AND THE EXTENSION
** TO VISCOELASTIC FLOW
**
** SET INITIAL VALUES AND CONSTANTS; NUMBER OF Y INTERVALS = N,
** MAXIMUM NUMBER OF TIMES PROGRAM CALCULATES VALUES FOR PHI AND
** ITS DERIVATIVES = LN WITH LL AS COUNTER (LL = 1, FIRST STEP IN
** NEWTON-RAPHSON WITH NEW VALUES FOR PHI'' AND C;LL = 2, SECOND STEP
** IN N.R.WITH VALUES FOR PHI'' + A SMALL CHANGE = H;LL = 3, THIRD
** STEP IN N.R WITH VALUES FOR PHI'' - H;LL = 4,FOURTH STEP WITH
** VALUES FOR C + H;LL = 5, FIFTH STEP WITH VALUES FOR
** C - H;LL = 6, FIRST STEP REPEATS WITH NEW VALUES FOR PHI'' AND
** C CALCULATED BY NEWTON-RAPHSON METHOD)
**
** SET VARIOUS LOOP COUNTERS; M,MM,MH...
**
** SET REAL CONSTANTS;WAVE NUMBER = F, MAXIMUM VELOCITY = UM,
** REYNOLDS NUMBER = R, DIMENSIONLESS VISCOELASTIC MEASURE = G,
** INITIAL VALUE FOR Y, (Y = 0)
**
** SET COMPLEX VARIABLES AND CONSTANTS; INITIAL GUESS FOR
** PHI'' = A,INITIAL GUESS FOR C = C, COMPLEX NUMBER I = BB,
** SMALL CHANGE IN VALUES FOR PHI'' AND C = H
**
** INITIALIZE VALUES OF TERMS MULTIPLIED BY PHI AND ITS DERIVATIVES
** WHERE
** VALUE OF TERMS TIMES PHI     AT Y =       W1R1
**  "       "        "    PHI'    AT Y =       W2R1
**  "       "        "    PHI''   AT Y =       W3R1
**                        PHI'''  AT Y =       W4R1
**
** VALUE OF TERMS TIMES PHI   AT Y + DY/2 =  W1R2
**  "       "        "    PHI'  AT Y + DY/2 =  W2R2
**  "       "        "    PHI'' AT Y + DY/2 =  W3R2
**  "       "        "    PHI'''AT Y + DY/2 =  W4R2
**
** VALUE OF TERMS TIMES PHI   AT Y + DY/2 =  W1R3
**  "       "        "
**  "       "        "
** VALUE OF TERMS TIMES PHI   AT Y + DY =    W1R4
**  "       "        "
**
** INITIALIZE VALUES FOR INVERSE OF F | dPHI /dA,dPHI /dC |
**                                    | dPHI'/dA,dPHI'/dC |
** INITIALIZE VALUES FOR VISCOELASTIC CONSTANT TERMS V1-V4
** WHICH DEPEND TIME Y ,Y + DY/2, OR Y + DY; AND Z TERMS

      PARAMETER (N = 1250)
      PARAMETER (LN = 11)
      PARAMETER (IMR = 1)
      PARAMETER (IMAC = 10)
      INTEGER M,MM,MR,JO,JL,JMR,JMAC,IO,MNR,MAC,I,J,JJ,LL,E1,E2,E3,E4
      DATA F,R,G,Y/1.025D0,.56510D4,1D-8,0D0/
      DATA A,C,H,BB/(-1.672796D0,-.276727D-4),(.2656344D0,.3109655D-5),(6D-4,6
      DATA W1R1,W2R1,W3R1,W4R1,W1R2,W2R2,W3R2,W4R2/(0D0,0D0),(0D0,0D0),(0D0,0D
```

```
      DATA W1R3,W2R3,W3R3,W4R3,W1R4,W2R4,W3R4,W4R4/(0D0,0D0),(0D0,0D0),(0D0,0D
      DATA PD0A,PD1A,PD0C,PD1C,PDB/(0D0,0D0),(0D0,0D0),(0D0,0D0),(0D0,0D0),(0D
      DATA V1,V2,V3,V4/(0D0,0D0),(0D0,0D0),(0D0,0D0),(0D0,0D0)/
      DATA ZZ11,ZZ12,ZZ21,ZZ22,RWSMALL/(0D0,0D0),(0D0,0D0),(0D0,0D0),(0D0,0D0)
      DATA CONW3R1,CONW3R2,CONW3R3,CONW3R4,LL/(0D0,0D0),(0D0,0D0),(0D0,0D0),(0

** DIMENSION NECESSARY VECTORS; W(PHI-PHI''',Y STEPS,N.R.STEPS)
**                             W(5 = C     ,Y STEPS,N.R.STEPS)
**   *W (5, DOES NOT CHANGE WITH Y)
**
** K1-K4, RUNGE-KUTTA VECTORS

      DIMENSION W(5,N+1,LN+1),WRM1(IMR,IMAC,LN),WRM3(IMR,IMAC,LN),WRC(IMR,IMAC
      DIMENSION K1(5,N+1,LN+1),K2(5,N+1,LN+1),K3(5,N+1,LN+1),K4(5,N+1,LN+1)

** CALCULATE STEP SIZE = DY

      DY = 1D0/N

** ENTER INITIAL VALUES FOR PHI'' AND C

**         READ (5,*) A,C

**         READ(5,*) R

**         READ(5,*) HV

** PRINT CONSTANTS

**         WRITE(6,*) '  VALUES OF CONSTANTS'
**         WRITE(6,*) ' F  =',  F,'           R =', R
**         WRITE(6,*) ' DY =', DY,'          UM =', UM
**         WRITE(6,*) '      H =', H

** INITIALIZE ARRAYS TO ZERO, START NEWTON R. LOOP, AND INPUT INITIAL
** CONDITIONS (AT Y = 0) W (1,1,1) = PHI (0) = 1
**                       W (2,1,1) = PHI' (0) = 0
**                       W (3,1,1) = PHI'' (0) = A
**                       W (4,1,1) = PHI''' (0) = 0
**                       W (5,1,1) =     C       = C
**
** INITIALIZE ARRAYS TO ZERO

      DO 20,JO = 1,5
      DO 30,IO = 1,N+1
      DO 40, MNR = 1,LN+1
      W(JO,IO,MNR) = (0D0,0D0)
      K1(JO,IO,MNR) = (0D0,0D0)
      K2(JO,IO,MNR) = (0D0,0D0)
      K3(JO,IO,MNR) = (0D0,0D0)
      K4(JO,IO,MNR) = (0D0,0D0)
      40 CONTINUE
      30 CONTINUE
      20 CONTINUE
      DO 50000 JMR = 1,IMR
      DO 50100 JMAC = 1,IMAC
      DO 50200 JL = 1,LN
      OWRMC (JMR,JMAC,JL) = 0
      WRM1 (JMR,JMAC,JL) = (0D0,0D0)
      WRM3 (JMR,JMAC,JL) = (0D0,0D0)
```

```
   WRC (JMR,JMAC,JL) = (0D0,0D0)
   50300
   50200
   50100
   50000

** 30000  LOOP ALLOWS FOR H TO CHANGE
** 70 LOOP BEGINS NEWTON-RAPHSON METHOD G=-8 5652
   F = F-(1D-3)
**      5522.6 1.0308  1.0311 AC = 1.0310
**      5522.7 1.0298  1.0319
**      5522.8 1.0293  1.0322
**      5523   1.0288  1.0327
**      5524   1.0271  1.0345
**      5525   1.0259  1.0357
**      5526
** 10-5 5527   1.0240  1.0373
**      5528   1.0233  1.0380
**      5529   1.0226  1.0386
**      5530   1.0220  1.0391  -6    +5
**      5540   1.0172  1.0435 -48   +44
**      5580   1.0056  1.0531  -116  +96
**      5620   0.9975  1.0592  +81    +61
**      5670   0.9894  1.0650  -81   -58
**                             -278  +215
** 4hr. - MG*IMR = 300
   DO 29300,MG = 15,15

   IF (MG.LT.1.1) THEN
   ELSE

   IF (MG.LT.2.1) THEN
   G = 1D-2
   A = (-1.628D0,-.276D-4)
   C = (.265D0,.311D-5)
   F = 1.025D0-(1D-3)
   R = .56290D4
   ELSE

   IF (MG.LT.3.1) THEN
   A = (-1.67274D0,-.40287D-4)
   C = (.265618D0,.65124D-5)
   F = 1.025D0-(1D-3)
   R = .56560D4
   ELSE

   IF (MG.LT.4.1) THEN
   A = (-1.67267D0,-.52889D-4)
   C = (.2656017D0,.991228D-5)
   F = 1.025D0-(1D-3)
   R = .56580D4
   ELSE

   IF (MG.LT.5.1) THEN
   A = (-1.672615D0,-.65479D-4)
   C = (.26558D0,.13309D-4)
   F = 1.025D0-(1D-3)
   R = .56600D4
   ELSE
```

```
      ENDIF
      ENDIF
      ENDIF
      ENDIF
       ENDIF
       ENDIF
       ENDIF
       ENDIF
       ENDIF
       ENDIF
       ENDIF
       ENDIF
       ENDIF
       ENDIF
       ENDIF
       ENDIF
       ENDIF
       ENDIF
       ENDIF
       ENDIF
       ENDIF
       ENDIF
       ENDIF
       ENDIF
       ENDIF
       ENDIF
       ENDIF
       ENDIF
       ENDIF
       ENDIF
       ENDIF
       ENDIF
       ENDIF
       ENDIF
      DO 29200,MF = 1,1

      F = F+(1D-3)

      DO 29100,MR = 1,IMR

      DO 29000,MAC = 1,IMAC


      IF (MAC.LT.1.1) THEN
      H = (6D-4,6D-4)
**          H = HV
      ELSE

      IF (MAC.LT.2.1) THEN
      H = (1D-7,1D-7)
**          H = HV
      ELSE

      IF (MAC.LT.3.1) THEN
**          H = HV
      H = (5D-9,5D-9)
      ELSE

**          H = HV
      H = (8D-10,8D-10)
```

```
      ENDIF
      ENDIF
      ENDIF

      IF (MAC.GT.1.1) THEN
      A = W(3,1,LN)
      C = W(5,1,LN)
      ELSE
      ENDIF

**           IF (MF.GT.1.1.AND.MAC.LT.1.1) THEN
**           A = WRM3(JMR,JMAC,JL)
**           C = WRC(JMR,JMAC,JL)
**           ELSE
**           ENDIF


      DO 70,LL = 1,LN

** RESET INITIAL CONDITIONS (AT Y = 0)
** THE A AND C VALUES ARE TAKEN FOR THE FIRST FIVE LL
** NEWTON-RAPHSON VALUES ARE TAKEN AFTER THE FIRST FIVE LL

      W(1,1,LL) = (1D0,0D0)

      W(2,1,LL) = (0D0,0D0)

      IF(LL.LT.6) THEN
      W(3,1,LL) = A
      ELSE
      ENDIF

      W(4,1,LL) = (0D0,0D0)

      IF(LL.LT.6) THEN
      DO 80, M = 1,N+1
      W(5,M,LL) = C
      80 CONTINUE
      ELSE
      DO 90, M = 1,N+1
      W(5,M,LL) = W(5,1,LL)
      90 CONTINUE
      ENDIF

** PRINT INITIAL CONDITIONS

**           WRITE (6,*) 'VALUES OF PHI AND ITS DERIVITIVES AT y = 0'
**           WRITE (6,*) ' '
**           WRITE (6,*) 'PHI =     ',W(1,1,LL)
**           WRITE (6,*) 'PHID   = ',W(2,1,LL)
**           WRITE (6,*) 'PHIDD  = ',W(3,1,LL)
**           WRITE (6,*) 'PHIDDD = ',W(4,1,LL)
**           WRITE (6,*) 'C      = ',W(5,1,LL)

** NEWTON-RAPHSON METHOD 100-140 LOOPS, ADDS OR SUBTRACTS H
** DURING PROPER LL

      DO 100, M = 1,LN,5
```

```
   IF (LL.EQ.M) THEN
   ENDIF
   100 CONTINUE

   DO 110, M = 2,LN,5
   IF (LL.EQ.M)   THEN
   W (3,1,LL) = W (3,1,LL) + H
   ENDIF
   110 CONTINUE

   DO 120, M = 3,LN,5
   IF (LL.EQ.M)   THEN
   W (3,1,LL) = W (3,1,LL) - H
   ENDIF
   120 CONTINUE

   DO 130, M = 4,LN,5
   IF (LL.EQ.M)   THEN
   W (5,1,LL) = W (5,1,LL) + H
**        DO 140, MM = 1,N+1
**        W (5,MM,LL) = W (5,MM,LL) + H
**        140 CONTINUE
   ENDIF
   130 CONTINUE

   DO 150, M = 5,LN,5
   IF (LL.EQ.M)  THEN
   W (5,1,LL) = W (5,1,LL) - H
**        DO 160, MM = 1,N+1
**        W (5,MM,LL) = W (5,MM,LL) - H
**        160 CONTINUE
   ENDIF
   150 CONTINUE

** THE RUNGE-KUTTA ROUTINE, RESET Y TO 0
** 1000 LOOP, Y STEPS

   Y = 0D0


   DO 1000,I = 1,N
**        WRITE (6,*) '              Y =  ', Y


** VISCOELASTIC TERM, (1/(I/AL + G (U-C))), THEN MULTIPLIED BY
** W1-W4 VALUES

   V1 = 1D0/((BB/F)+(G*((1D0-(Y)**2)-W(5,1,LL))))

   V2 = 1D0/((BB/F)+(G*((1D0-(Y+DY/2D0)**2)-W(5,1,LL))))

**        V3 = -1D0/((BB/F)+(G*(UM*(1D0-(Y+DY/2D0)**2)-W(5,1,LL))))
**        THIS IS THE SAME AS V2 SO WE MAY SET V3 = V2

   V3 = V2

   V4 = 1D0/((BB/F)+(G*((1D0-(Y+DY)**2)-W(5,1,LL))))

** W1-W4 TERMS
```

```
**V.S.          W4R1 = V1*(G*(2D0*(Y)))
      W4R1 = 0D0
      CONW3R1= (2D0*G*F**2)*((1D0-(Y)**2)-W(5,1,LL))
      W3R1 = V1*(-R*((1D0-(Y)**2)-W(5,1,LL))+2D0*BB*F+CONW3R1)
**V.S.          W2R1 = V1*(G*(-2D0*(Y))*F**2)
      W2R1 = 0D0
**          WRITE(6,*) ' W1R1 AND W3R1', W1R1,'    ', W3R1

      W1R1 = V1*((R*F**2)*((1D0-(Y)**2)-W(5,1,LL))-2D0*R-BB*F**3-(G*(F**4))*((


**V.S.          W4R2 = V2*(G*(2D0*(Y+DY/2D0)))
      W4R2 = 0D0
      CONW3R2 = (2D0*G*F**2)*((1D0-(Y+DY/2D0)**2)-W(5,1,LL))
      W3R2 = V2*(-R*((1D0-(Y+DY/2D0)**2)-W(5,1,LL))+2D0*BB*F+CONW3R2)
**V.S.          W2R2 = V2*(G*(-2D0*(Y+DY/2D0))*F**2)
      W2R2 = 0D0
      W1R2 = V2*((R*F**2)*((1D0-(Y+DY/2D0)**2)-W(5,1,LL))-2D0*R-BB*F**3-(G*(F*
**          WRITE(6,*) ' W1R2 AND W3R2', W1R2,'   ', W3R2


**V.S.          W4R3 = V3*(G*(2D0*(Y+DY/2D0)))
      W4R3 = 0D0
      CONW3R3 = (2D0*G*F**2)*((1D0-(Y+DY/2D0)**2)-W(5,1,LL))
      W3R3 = V3*(-R*((1D0-(Y+DY/2D0)**2)-W(5,1,LL))+2D0*BB*F+CONW3R3)
**V.S.          W2R3 = V3*(G*(-2D0*(Y+DY/2D0))*F**2)
      W2R3 = 0D0
      W1R3 = V3*((R*F**2)*((1D0-(Y+DY/2D0)**2)-W(5,1,LL))-2D0*R-BB*F**3-(G*(F*
```

```
**V.S.          W4R4 = V4*(G*(2D0*(Y+DY)))

   W4R4 = 0D0

   CONW3R4 = (2D0*G*F**2)*((1D0-(Y+DY)**2)-W(5,1,LL))

   W3R4 = V4*(-R*((1D0-(Y+DY)**2)-W(5,1,LL))+2D0*BB*F+CONW3R4)

**V.S.          W2R4 = V4*(G*(-2D0*(Y+DY))*F**2)

   W2R4 = 0D0

   W1R4 = V4*((R*F**2)*((1D0-(Y+DY)**2)-W(5,1,LL))-2D0*R-BB*F**3-(G*(F**4))

** RUNGE-KUTTA LOOPS FOR CALCULATING K1 (1-4)-K2 (1-4)

   DO 1500, E1 = 1,3
   K1(E1,I,LL) = DY*W(E1+1,I,LL)
**       WRITE(6,*) ' K1', E1,' =', K1(E1,I,LL)
   1500 CONTINUE

   K1(4,I,LL) = DY*(W3R1*W(3,I,LL)+W1R1*W(1,I,LL)+W2R1*W(2,I,LL)+W4R1*W(4,I
**         WRITE(6,*) ' K1',4,' =', K1(4,I,LL)


   DO 2000, E2 = 1,3
   K2(E2,I,LL) = DY*(W(E2+1,I,LL)+0.5*K1(E2+1,I,LL))
**         WRITE(6,*) ' K2', E2,' =', K2 (E2,I,LL)
   2000 CONTINUE

   K2(4,I,LL) = DY*(W3R2*(W(3,I,LL)+K1(3,I,LL))+W1R2*(W(1,I,LL)+K1(1,I,LL))
**         WRITE(6,*) ' K2',4,' =',K2 (4,I,LL)


   DO 2500, E3 = 1,3
   K3(E3,I,LL) = DY*(W(E3+1,I,LL)+0.5*K2(E3+1,I,LL))
**         WRITE(6,*) ' K3', E3,' =', K3 (E3,I,LL)
   2500 CONTINUE

   K3(4,I,LL) = DY*(W3R3*(W(3,I,LL)+K2(3,I,LL))+W1R3*(W(1,I,LL)+K2(1,I,LL))
**         WRITE(6,*) ' K3',4,' =',K3 (4,I,LL)


   DO 3000, E4 = 1,3
   K4(E4,I,LL) = DY*(W(E4+1,I,LL)+0.5*K3(E4+1,I,LL))
**         WRITE(6,*) ' K4', E4,' =', K4 (E4,I,LL)
   3000 CONTINUE

   K4(4,I,LL) = DY*(W3R4*(W(3,I,LL)+K3(3,I,LL))+W1R4*(W(1,I,LL)+K3(1,I,LL))
**         WRITE(6,*) ' K4',4,' =',K4 (4,I,LL)

** RUNGE-KUTTA CALCULATION OF W (1-4) AND CALCULATE Y

   DO 3500, J = 1,4
   W(J,I+1,LL) = W(J,I,LL) + (K1(J,I,LL)+2*K2(J,I,LL)+2*K3(J,I,LL)+K4(J,I,L
   3500 CONTINUE
```

```
  Y=I*DY
**        DO 3310, MPG = 1,IMR,10
**        IF (MPG.EQ.MR) THEN
**        DO 3300, MPL = 1,LN,5
**        DO 3400, MPH = 1,N+1,10
**        IF (LL.EQ.MPL.AND.I.EQ.(MPH-1)) THEN
**        WRITE(6,*) '  Y           = ',Y,MAC,LL
**        WRITE(6,*) '                '
**        DO 3600, J = 1,1
**        WRITE(6,*) ' PHI ',J-1,' = ',W (J,I+1,LL)
**        3600
**        ELSE
**        ENDIF
**        3400
**        3300
**        ELSE
**        ENDIF
**        3310

***       IF (I.EQ.N) THEN
***       WRITE(6,*) '             Y =  ',Y
***       ELSE
***       ENDIF

   1000 CONTINUE

*** NEWTON-RAPHSON CONTINUED, FIRST PRINT W (Y = 0) AND W (Y = 1)
*** FOR THE FIVE LL CASES

   DO 1100,M = 1,LN,5
   IF (LL.EQ.M) THEN
***       WRITE(6,*) '  VALUES WITH NO ADDED H = ',H,' RUN ',LL/6+1
   DO 10000, J = 1,5
***       WRITE(6,*) ' W', J,'        Y = 0',LL,'= ',W (J,1,LL)
   10000 CONTINUE
   DO 11000, JJ = 1,5
***       WRITE(6,*) ' W', JJ,'       Y = 1',LL,'= ',W(JJ,N + 1,LL)
   11000 CONTINUE
   ENDIF
   1100 CONTINUE

   DO 1110, M = 2,LN,5
   IF (LL .EQ. M) THEN
***       WRITE(6,*) '  VALUES WITH W (3,1) OR PHIDD +',H,'RUN ',LL/6+1
   DO 12000, J = 1,5
***       WRITE(6,*) ' W', J,'        Y = 0',LL,'= ',W (J,1,LL)
   12000 CONTINUE
   DO 13000,JJ = 1,5
***       WRITE(6,*) ' W', JJ,'       Y = 1',LL,' =', W (JJ,N + 1,LL)
   13000 CONTINUE
   ENDIF
   1110 CONTINUE

   DO 1120, M = 3,LN,5
   IF (LL .EQ. M) THEN
***       WRITE(6,*) '  VALUES WITH W (3,1) OR PHIDD -',H,'RUN ',LL/6+1
   DO 14000, J = 1,5
***       WRITE(6,*) ' W', J,'        Y = 0',LL,'= ',W (J,1,LL)
   14000 CONTINUE
```

```
      DO 15000,JJ = 1,5
**         WRITE(6,*) ' W', JJ,'          Y = 1',LL,' =',W (JJ,N + 1,LL)
      15000 CONTINUE
      ENDIF
      1120 CONTINUE

      DO 1130,M = 4,LN,5
      IF (LL .EQ. M) THEN
**         WRITE(6,*) '  VALUES WITH W (5,1)  OR  C +',H,'RUN ', LL/6+1
      DO 16000, J = 1,5
**         WRITE(6,*) ' W', J,'           Y = 0',LL,'= ',W (J,1,LL)
      16000 CONTINUE
      DO 17000,JJ = 1,5
**         WRITE(6,*) ' W', JJ,'          Y = 1',LL,' =',W (JJ,N + 1,LL)
      17000 CONTINUE
      ENDIF
      1130  CONTINUE

      DO 1140,M = 5,LN,5
      IF (LL .EQ. M) THEN
**         WRITE(6,*) '  VALUES WITH W (5,1)  OR  C -',H,'RUN ', LL/6+1
      DO 18000, J = 1,5
**         WRITE(6,*) ' W', J,'           Y = 0',LL,'= ',W (J,1,LL)
      18000 CONTINUE
      DO 19000,JJ = 1,5
**         WRITE(6,*) ' W', JJ,'          Y = 1',LL,' =',W (JJ,N + 1,LL)
      19000 CONTINUE
      ENDIF
      1140  CONTINUE

** NEWTON-RAPHSON METHOD,CALCULATION OF NEW PHI'' AND C
** VALUES OR W (3,1,6 OR 11 OR 16...) AND W (5, 1,6 OR 11 OR 16...)
** PD0A = CHANGE IN PHI WITH PHI''
** PD1A = CHANGE IN PHI' WITH PHI''
** PD0C = CHANGE IN PHI WITH C
** PD1C = CHANGE IN PHI' WITH C
** PDB,ZZ11,ZZ12,ZZ21,ZZ22, NEEDED TERMS

      DO 1150,M = 5,LN,5
      IF (LL.EQ.M) THEN

      PD0A=(W(1,N+1,LL-3)-W(1,N+1,LL-2))/(2*H)
      PD1A=(W(2,N+1,LL-3)-W(2,N+1,LL-2))/(2*H)
      PD0C=(W(1,N+1,LL-1)-W(1,N+1,LL  ))/(2*H)
      PD1C=(W(2,N+1,LL-1)-W(2,N+1,LL  ))/(2*H)
      PDB=(PD1C*PD0A-PD0C*PD1A)/PD0A

      ZZ11=W(1,N+1,LL-4)*(1/PD0A+(PD0C*PD1A)/(PDB*PD0A**2))
      ZZ12=W(2,N+1,LL-4)*(-PD0C/(PDB*PD0A))
      W(3,1,LL+1)=W(3,1,LL-4)-(ZZ11+ZZ12)
      DO 19100, MM = 2,5
      W(3,1,LL+MM) = W(3,1,LL+1)
      19100 CONTINUE

      ZZ21=W(1,N+1,LL-4)*(-PD1A/(PDB*PD0A))
      ZZ22=W(2,N+1,LL-4)*(1/PDB)
      W(5,1,LL+1)=W(5,1,LL-4)-(ZZ21+ZZ22)
      DO 19200, MM = 2,5
      W(5,1,LL+MM) = W(5,1,LL+1)
      19200 CONTINUE
```

```
**         WRITE(6,*) ' PHIDD (Y=0) FOR RUN',  LL/5+1,' =',W (3,1,LL+1)
**         WRITE(6,*) '   C         FOR RUN',  LL/5+1,' =',W (5,1,LL+1)
**         WRITE(6,*) 'PD0A      =',PD0A
**         WRITE(6,*) 'PD1A      =',PD1A
**         WRITE(6,*)  PD0C,PD1C,PDB
**         WRITE(6,*)  ZZ11,ZZ12,ZZ21,ZZ22
   ELSE
   ENDIF
   1150 CONTINUE

**         DO 19300,MOP = 1,LN,5
**         IF (LL.EQ.MOP) THEN
**         WRITE(6,*) 'F = ',F,' R = ',R,' H = ',H
**         WRITE(6,*) ' W 1 OR PHI  AT Y=1 RUN',MOP/5+1,W(1,N+1,MOP)
**         WRITE(6,*) ' W 2 OR PHID AT Y=1 RUN',MOP/5+1,W(2,N+1,MOP)
**         WRITE(6,*) 'THEIR ABSOLUTE VALUES'
**         WRITE(6,*) 'W 1 ;',CDABS(W(1,N+1,LL))
**         WRITE(6,*) 'W 2 ;',CDABS(W(2,N+1,LL))

**         IF (CDABS(W(1,N+1,LL)).LT..01D0.AND.CDABS(W(2,N+1,LL)).LT..05D0)
**         WRITE(6,*) ' CONVERGENCE ;F = ',F,' R = ',R,' H = ',H
**         GOTO 44000

**         ELSE
**         ENDIF
**         ELSE
**         ENDIF
**         19300 CONTINUE

   70 CONTINUE


   DO 20000,M = 1,LN,5
   WRM1(MR,MAC,M)=W(1,N+1,M)
   WRM3(MR,MAC,M)=W(3,1,M)
   WRC  (MR,MAC,M)=W(5,1,M)
**         WRITE(6,*) ' F = ',F,' R = ',R,'H = ',H
**         WRITE(6,*) '         PHIDD AT Y=0 RUN',M/5+1,W(3,1,M)
**         WRITE(6,*) '            C  FOR    RUN',M/5+1,W(5,1,M)
**         WRITE(6,*) ' W 1 OR PHI  AT Y=1 RUN',M/5+1,W(1,N+1,M
**         WRITE(6,*) ' W 1 ABS VAL FOR    RUN',M/5+1         ,CDABS(W(1,N+1,
**         WRITE(6,*) ' W 2 ABS VAL FOR    RUN',M/5+1         ,CDABS(W(2,N+1,
   20000 CONTINUE

   29000

   29100 CONTINUE



   DO 34000 JMR = 1,IMR
   RWSMALL = CDABS(WRM1(JMR,1,1))
   DO 34100 JMAC = 1,IMAC
   DO 34200 JL = 1,LN,5
   IF(CDABS(WRM1(JMR,JMAC,JL)).LT.RWSMALL) THEN
   RWSMALL = CDABS(WRM1(JMR,JMAC,JL))
   OWRMC (JMR,JMAC,JL) = 1
   ENDIF
   34200
```

```
      34100
      34000

      DO 34600, JMR = 1,IMR,1
      DO 34700, JMAC = IMAC,1,-1
      DO 34800, JL = LN,1,-5

      IF (OWRMC (JMR,JMAC,JL).EQ.1) THEN
      WRITE (6,*) JMR,JMAC,JL
      WRITE (6,*) 'R=',R,'F=',F
      WRITE (6,*) 'G=',G
      WRITE (6,*) 'PHI ABS(Y=1)',CDABS(WRM1(JMR,JMAC,JL))
      WRITE (6,*) 'PHI     (Y=1)',WRM1(JMR,JMAC,JL)
      WRITE (6,*) 'PHIDD   (Y=0)',WRM3(JMR,JMAC,JL)
      WRITE (6,*) 'C=             ',WRC(JMR,JMAC,JL)

      IF (JMR.GT.1.1) THEN
***          IF (DIMAG(WRC(JMR,JMAC,JL))*DIMAG(WRCP).LT.0D0) THEN
***
*** WRITE(6,*) JMR-1,JMAC,JL
*** WRITE(6,*) 'F=',F
*** WRITE(6,*) 'R critical =',RC+1D0,'G=',G
*****RITE(6,*) 'PHI ABS(Y=1)',CDABS(WRM1(JMR-1,JMAC,JL))
*** WRITE(6,*) 'PHI     (Y=1)',WRM1(JMR-1,JMAC,JL)
*** WRITE(6,*) 'PHIDD   (Y=0)',WRM3(JMR-1,JMAC,JL)
*** WRITE(6,*) 'C=             ',WRC(JMR-1,JMAC,JL)
***          A=WRM3(JMR-1,JMAC,JL)
***          C=WRC(JMR-1,JMAC,JL)
***          R=RC+2D0
***          RC=RC+2D0
***          GOTO 29300
***          ELSE
***          ENDIF
      ELSE
      ENDIF

***          WRCP=WRC(JMR,JMAC,JL)
      GOTO 34600
      ENDIF

      34800
      34700
      34600


      29200
      29300
***          30000
      44000 STOP
      END
```

```
 File  Edit  Search  Options                                            Help
áááááááááááááááááááááááááááááááááá NVFU0 áááááááááááááááááááááááááááááááááááá¢
/1 = 1D0/((BB/F)+(G*((1D0-(Y)**2)-W(5,1,LL))))

/2 = 1D0/((BB/F)+(G*((1D0-(Y+DY/2D0)**2)-W(5,1,LL))))

      V3 = -1D0/((BB/F)+(G*(UM*(1D0-(Y+DY/2D0)**2)-W(5,1,LL))))
      THIS IS THE SAME AS V2 SO WE MAY SET V3 = V2

/3 = V2

/4 = 1D0/((BB/F)+(G*((1D0-(Y+DY)**2)-W(5,1,LL))))

V1-W4 TERMS



V4R1 = V1*(G*(2D0*(Y)))

CONW3R1 = (2D0*G*F**2)*((1D0-(Y)**2)-W(5,1,LL))

V3R1 = V1*(-R*((1D0-(Y)**2)-W(5,1,LL))+2D0*BB*F+CONW3R1)

MS-DOS Editor  <F1=Help> Press ALT to activate menus          N 00589:085
 File  Edit  Search  Options                                            Help
áááááááááááááááááááááááááááááááááá NVFU0 áááááááááááááááááááááááááááááááááááá¢
W3R1 = V1*(-R*((1D0-(Y)**2)-W(5,1,LL))+2D0*BB*F+CONW3R1)

W2R1 = V1*(G*(-2D0*(Y))*F**2)

       WRITE(6,*) ' W1R1 AND W3R1', W1R1,'    ', W3R1


W1R1 = V1*((R*F**2)*((1D0-(Y)**2)-W(5,1,LL))-2D0*R-BB*F**3-(G*(F**4))*((1D0-(

W4R2 = V2*(G*(2D0*(Y+DY/2D0)))

CONW3R2 = (2D0*G*F**2)*((1D0-(Y+DY/2D0)**2)-W(5,1,LL))

W3R2 = V2*(-R*((1D0-(Y+DY/2D0)**2)-W(5,1,LL))+2D0*BB*F+CONW3R2)

W2R2 = V2*(G*(-2D0*(Y+DY/2D0))*F**2)

W1R2 = V2*((R*F**2)*((1D0-(Y+DY/2D0)**2)-W(5,1,LL))-2D0*R-BB*F**3-(G*(F**4))*

MS-DOS Editor  <F1=Help> Press ALT to activate menus          N 00608:085
 File  Edit  Search  Options                                            Help
áááááááááááááááááááááááááááááááááá NVFU0 áááááááááááááááááááááááááááááááááááá¢
LL))+2D0*BB*F+CONW3R1)
```

W1R1,'   ', W3R1


```
5,1,LL))-2D0*R-BB*F**3-(G*(F**4))*((1D0-(Y)**2)-W(5,1,LL)))

D0)**2)-W(5,1,LL))

(5,1,LL))+2D0*BB*F+CONW3R2)

**2)-W(5,1,LL))-2D0*R-BB*F**3-(G*(F**4))*((1D0-(Y+DY/2D0)**2)-W(5,1,LL)))
```

-DOS Editor <F1=Help> Press ALT to activate menus   °   N 00608:120

File Edit Search Options   Help

áááááááááááááááááááááááááááááááááá NVFU0 áááááááááááááááááááááááááááááááááááááá¢

```
 W4R3 = V3*(G*(2D0*(Y+DY/2D0)))

 CONW3R3 = (2D0*G*F**2)*((1D0-(Y+DY/2D0)**2)-W(5,1,LL))

 W3R3 = V3*(-R*((1D0-(Y+DY/2D0)**2)-W(5,1,LL))+2D0*BB*F+CONW3R3)

 W2R3 = V3*(G*(-2D0*(Y+DY/2D0))*F**2)

 W1R3 = V3*((R*F**2)*((1D0-(Y+DY/2D0)**2)-W(5,1,LL))-2D0*R-BB*F**3-(G*(F**4)

 W4R4 = V4*(G*(2D0*(Y+DY)))

 CONW3R4 = (2D0*G*F**2)*((1D0-(Y+DY)**2)-W(5,1,LL))

 W3R4 = V4*(-R*((1D0-(Y+DY)**2)-W(5,1,LL))+2D0*BB*F+CONW3R4)
```

-DOS Editor <F1=Help> Press ALT to activate menus   °   N 00634:083

File Edit Search Options   Help

áááááááááááááááááááááááááááááááááá NVFU0 áááááááááááááááááááááááááááááááááááááá¢

```
      W3R3 = V3*(-R*((1D0-(Y+DY/2D0)**2)-W(5,1,LL))+2D0*BB*F+CONW3R3)

      W2R3 = V3*(G*(-2D0*(Y+DY/2D0))*F**2)

      W1R3 = V3*((R*F**2)*((1D0-(Y+DY/2D0)**2)-W(5,1,LL))-2D0*R-BB*F**3-(G*(
```

```
W4R4 = V4*(G*(2D0*(Y+DY)))

CONW3R4 = (2D0*G*F**2)*((1D0-(Y+DY)**2)-W(5,1,LL))

W3R4 = V4*(-R*((1D0-(Y+DY)**2)-W(5,1,LL))+2D0*BB*F+CONW3R4)

W2R4 = V4*(G*(-2D0*(Y+DY))*F**2)

W1R4 = V4*((R*F**2)*((1D0-(Y+DY)**2)-W(5,1,LL))-2D0*R-BB*F**3-(G*(F**4
```



```
MS-DOS Editor  &lt;F1=Help&gt; Press ALT to activate menus        N 00638:001

 File  Edit  Search  Options                                     Help
áááááááááááááááááááááááááááááááááá NVFU0 áááááááááááááááááááááááááááááááááááá¢
)-W(5,1,LL))+2D0*BB*F+CONW3R3)

*2)

D0)**2)-W(5,1,LL))-2D0*R-BB*F**3-(G*(F**4))*((1D0-(Y+DY/2D0)**2)-W(5,1,LL)))

Y)**2)-W(5,1,LL))

5,1,LL))+2D0*BB*F+CONW3R4)

*2)-W(5,1,LL))-2D0*R-BB*F**3-(G*(F**4))*((1D0-(Y+DY)**2)-W(5,1,LL)))

MS-DOS Editor  &lt;F1=Help&gt; Press ALT to activate menus        N 00633:119
```

REFERENCES

[1] Schlichting," Boundary Layer Theory".7th ed.McGraw Hill,1979,p.456

[2] Metzner,A.B.,"Polymer Solution and Fiber Suspension Rheology and Their Relationship to Turbulent Drag Reduction,".*Physics of Fluids,* Vol.20, No.10, Pt.11,1977, pp.S145-S149

[3] Harnoy A.,1976,"Stress Relaxation in Elastico-viscous Lubricants in Gears and Rollers".*J.of Fluid Mech*.vol.76.pp. 501-517.

[4] Harnoy A.,1989,"Normal Stresses and Relaxation Effects in Viscoelastic Boundary Layer Flow Past Submerged Bodies".*Journal of Rheology* Vol.33,pp.93-117.

[5] Tichy, J.A.and Modest,M.F.,1980, "A Simple Low Deborah Number Model for Unsteady Hydrodynamics Lubrication, Including Fluid Inertia, "*Journal of Rheology* Vol.24.pp.829-845.

[6] Sus, S, 1984,"Relaxations-und Normalspannungseffckte in der Questschstromung,"*Rheol Acta,* Vol.23,1984,pp.489-486.

[7] Mukherjee, A.Bhattacharyya,R, and Rao Dasary,A.M.,1985, "A Theoretical Study of Stability of Rigid Rotor Under Influence of Dilute Viscoelastic Lubricants,"*Journal of Tribology* ,Trans.ASME Vol.107,pp.75-81.

[8] Harnoy A.,"Investigation of Instability in Rheological equations of the differential type."*Proc. of Third European Rheology Conference* pp.208-210.

[9] Squire,H.B.," On the Stability of Three Dimensional Distribution of Viscous Fluid Between Parallel Walls",*Proc.Roy.Soc.of London* A142,621-628 (1933)

[10] Drazin and Reid " Hydrodynamic stability" 5th ed. chpt.4

[11] Burden and Faires "Numerical Analysis", 5th ed.