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# ABSTRACT <br> SOFT DECISION ADAPTIVE MULTIUSER CDMA DETECTOR FOR ASYNCHRONOUS AWGN CHANNELS 

by<br>Frank Viehofer

A multiuser detector in an asynchronous, additive white Gaussian noise, codedivision multiple-access (CDMA) channel is proposed and analyzed. It employs a combination of a decorrelator and a nonlinear multiuser interference canceler that utilizes soft tentative decisions. The weights of the canceler are adaptively adjusted, in a manner that renders the knowledge of received signal energies and the use of training sequences unnecessary. The steady state error performance of the detector is obtained and found to be superior to the performance of the same detector that utilizes hard tentative decisions.

# SOFT DECISION ADAPTIVE MULTIUSER CDMA DETECTOR FOR ASYNCHRONOUS AWGN CHANNELS 

by<br>Frank Viehofer

A Thesis<br>Submitted to the Faculty of New Jersey Institute of Technology in Partial Fulfillment of the Requirements for the Degree of Master of Science in Electrical Engineering

Department of Electrical and Computer Engineering


## APPROVAL PAGE

# SOFT DECISION ADAPTIVE MULTIUSER CDMA DETECTOR FOR ASYNCHRONOUS AWGN CHANNELS 

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This thesis is dedicated to Afra, Henry, and Ellen

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## CHAPTER 1

## INTRODUCTION

Several techniques are used to transmit digital information simultaneously over a common channel. They differ among each other in the way they make use of the available resources; time, frequency, and code.


Figure 1.1 Communication cube

In Time-Division Multiple-Access (TDMA) the channel is partitioned into independent time slots to which a single user is assigned. Within his time slot, each user has access to the entire frequency band of the channel. In Frequency-Division Multiple-Access (FDMA), on the other hand, the channel is subdivided into independent frequency bands. Each user is allowed to transmit data consecutively but only within the assigned frequency band. Both techniques allow a certain maximum number of users to access the channel simultaneously, hence the preassigment of the channel tends to be wasteful in applications where most users send information during only a small percentage of the allotted time.

Contrary to those strategies, Code-Division Multiple-Access (CDMA) allows users to access the entire frequency band and to transmit their information simultaneously at the all time. CDMA does not impose a hard limit on the number of users that simultaneously access the channel; adding more users only gradually deteriorates the system performance.

In this thesis, we consider a CDMA system where a number of users simultaneously transmit information over a common channel by assigning different signature sequences to their information. A receiver observes a superimposed version of all asynchronously transmitted waveforms in additive white Gaussian noise (AWGN). To obtain desired information from a specific user, the receiver correlates the received signal with the same signature sequence that was originally assigned to that user in the transmitter. Since the cross-correlations between signature sequences are nonzero, the correlated signal consists of three parts:

- the desired signal,
- the multi-access interference (MAI), due to non-zero cross-correlations,
- and white Gaussian noise.

The goal of the CDMA detector is to separate the desired signal from the interference and to decide which information was actually sent. Several approaches have been taken in the past.

A conventional detector consists of a bank of matched filters followed by decision devices. It requires the knowledge of received amplitudes and signature sequences of all users. It demodulates each user's signal as if it were the only one present and thus does not take care of the multi-access interference. The detector has acceptable error performance as long as the interference component is not too strong compared to the desired signal; that is, the cross-correlations are low and the energies of the received signals are similar. The opposite of this is known as
the "near-far" problem. The advantage of the conventional detector is obviously its simplicity.

The optimum multiuser detector, based on the maximum likelihood principle, was proposed by Verdú in 1986 [1]. Besides the bank of matched filters, the detector consists of a decision system whose complexity is exponential with the number of users. The significance is that under the given conditions you can't do better than this detector regarding the output error probability. On the other hand, it requires the knowledge of all signature sequences as well as all received amplitudes. The main drawback is its complexity that makes it unsuitable for applications with a large number of users.

Several suboptimum detectors have been proposed which are much less complex, solve the "near-far" problem, and have error performance that comes fairly close to that of the optimum detector especially when the energies of the interferers increase. Suboptimum detectors can be divided into two categories, non-adaptive and adaptive detectors.

To the former category belongs Lupas and Verdú's [2,3] linear decorrelating detector. The decorrelator acts on the matched filter outputs by multiplying them with the inverse cross-correlation matrix resulting in interference-free outputs; however, those outputs do not result in optimum decisions since their noise components have larger variance compared to the matched filter outputs. It does not require the knowledge of the received amplitudes, but it requires the computation of the decorrelating coefficients from the cross-correlations.

Another approach is Varanasi and Aazhang's multistage detector, [4, 5], up on which our detector is based. It consists of the above mentioned decorrelator followed by hard tentative decisions on its first stage and a canceler with fixed weights on its second. Assuming that the decisions obtained from the first stage are correct, the second stage simply cancels the corresponding signals from the received waveform
resulting in a single-user performance if the previous decisions are indeed correct. It shows considerable improvement over the linear detector particularly in "near-far" situations; that is, those in the presence of relatively strong interfering signals. The knowledge of the received amplitudes is required here as well as in the next three cases.

Reference [6] analyses several classes of detectors that are similar to the preceding detector, except for using soft-decision tentative statistics instead.

The decorrelating decision-feedback detector proposed by Duel-Hallen, [7], is suitable for synchronous CDMA channels only, since it utilizes the difference in users' energies. A decorrelator estimates interference provided that the feedback data are correct. The feedback-structure first forms a decision from the strongest user since this is the most reliable one. Decisions of all other users are made in the order of decreasing received energies, for example, prior to the decision on the $k$ th user, the decisions of the $(k-1)$ stronger users are subtracted from the corresponding decorrelator output of user $k$. As in the former case, the feedback detector shows a performance gain with respect to the decorrelating detector especially for relatively weak users. It also has lower complexity than two-stage detectors with comparable performance.

Xie, Short, and Rushforth's, [8], detector consists of a linear transformation that operates on the matched filter outputs, followed by a set of threshold devices. They basically investigate two different versions of linear transformations using two different performance criteria:

- minimum mean square error MMSE and
- weighted least squares WLS.

The complexity is linear with the number of users, and under typical operating conditions these detectors perform nearly as good as the optimum detector.

In adaptive detectors, parameters are self-adjusted from observed received signals. The detector of Chen and Roy, [9], implements Verdú's decorrelating detector; however, its coefficients are adapted by decision feedback thus sidestepping the need to perform computations with cross-correlations. Here, the coefficients are obtained as the solution of a Least Square criterion that require the knowledge of all signature sequences but renders the computation of the coefficients from cross-correlations unnecessary.

Kohno et al., [10], consider a CDMA channel with limited bandwidth for which they design an adaptive MMSE detector that uses decision-feedback to remove multiaccess interference. The first stage performs preliminary decisions which are then used in the adaptive stage.

Rapajic and Vucetic, [11], investigate a similar adaptive MMSE detector, but their detector has no knowledge of the signature sequences and timing of other users. However there is a shortcoming: it has to be trained by a known sequence prior to data transmission.

A multistage detector based on [4], but with a canceler using adaptive weights on its second stage, was presented in [12] for the synchronous case and in [13] for the asynchronous case. It operates without the knowledge of received signals' amplitudes, and without the need to perform their prior estimation in order to set the values of the weights of the canceler. Instead, the weights are determined adaptively, without the requirement of a training sequence. The detector proposed here is also based on [4], and uses an adaptive canceler as [13] does, but we investigate the influence of soft tentative decisions, rather than hard decisions on the performance of a multiuser detector. Chapter 2 of this thesis will give a thorough description of the soft decision adaptive multiuser detector. Chapter 3 , we present the output error performance in different scenarios and compare it to other detector schemes.

## CHAPTER 2

## ADAPTIVE MULTIUSER DETECTOR

### 2.1 System Model



Figure 2.1 CDMA system model

A multi access AWGN channel is shown in Figure 2.1. A set of $K$ transmitters (users) desire to communicate with a base station over a common channel. Each transmitter produces its own data sequence $b_{k}(i) \in\{ \pm 1\}, k=1,2 \ldots, K$, for symbol interval $i$ of duration $T$. The narrow-band data sequence is multiplied by the unit energy signature sequence $s_{k}(t)$ of the same duration $T$, thus spreading the spectrum of the data sequence to that of the much wider signature sequence. The so encoded signals superimpose asynchronously in the channel. The baseband equivalent waveform $r(t)$ at the input of the detector is expressed as:

$$
\begin{equation*}
r(t)=\sum_{k=1}^{K} \sum_{i} b_{k}(i) \sqrt{a_{k}} s_{k}\left(t-i T-\tau_{k}\right)+n(t) \tag{2.1}
\end{equation*}
$$

where $n(t)$ is a zero-mean, white Gaussian noise with the two-sided power spectral density $N_{0} / 2$, and $a_{k}$ and $\tau_{k}$ are the received energy and relative delay for user $k$, respectively. While it is assumed that precise relative delay estimates are available for all users, their amplitudes are considered to be unknown to the detector.

### 2.2 Matched Filters

The detector has a bank of matched filters at its front end. To obtain desired information from a specific user, a matched filter correlates the received signal with the same signature sequence that was originally assigned to that user in the transmitter. Undesired signals of other users are affected by the signature sequence in the same way the original data sequence was affected at the transmitter, they are spread. However, since the cross-correlations between signature sequences are non-zero, not only the desired information is received but also the interference of all other users. Without loss of generality, the attention will be on the detection of bit 0 of user 1 , and it will be assumed that $0=\tau_{1}<\tau_{2}<\ldots<\tau_{K}<T$. The sampled output of the matched filter for user 1 is then:

$$
\begin{equation*}
x_{1}(0)=\sqrt{a_{1}} b_{1}(0)+\sum_{k=2}^{K} \sqrt{a_{k}}\left[\rho_{k 1} b_{k}(-1)+\rho_{1 k} b_{k}(0)\right]+n_{1}(0) \tag{2.2}
\end{equation*}
$$

The normalized partial cross-correlations for $k=2, \ldots, K$ are:

$$
\begin{equation*}
\rho_{k 1}=\int_{0}^{T} s_{1}(t) s_{k}\left(t+T-\tau_{k}\right) d t \quad \text { and } \quad \rho_{1 k}=\int_{0}^{T} s_{1}(t) s_{k}\left(t-\tau_{k}\right) d t \tag{2.3}
\end{equation*}
$$

Also, $n_{1}(0)=\int_{0}^{T} n(t) s_{1}(t) d t$ is a zero-mean Gaussian random variable with variance $N_{0} / 2$. Using the vector notations where $\rho_{1}=\left[\rho_{21}, \ldots, \rho_{K 1}, \rho_{12}, \ldots, \rho_{1 K}\right]^{T}$, $\boldsymbol{b}_{1}(0)=\left[b_{2}(-1), \ldots, b_{K}(-1), b_{2}(0), \ldots, b_{K}(0)\right]^{T}, \boldsymbol{A}_{1}=\operatorname{diag}\left[\sqrt{a_{2}}, \ldots, \sqrt{a_{K}}\right]$, and $\boldsymbol{A}=\left[\begin{array}{cc}\boldsymbol{A}_{\mathbf{1}} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{A}_{1}\end{array}\right]$, the matched filter output is:

$$
\begin{equation*}
x_{1}(0)=\sqrt{a_{1}} b_{1}(0)+\boldsymbol{\rho}_{1}^{T} \boldsymbol{A} \boldsymbol{b}_{1}(0)+n_{1}(0) \tag{2.4}
\end{equation*}
$$

The interference can be alleviated by using signature sequences with lower crosscorrelations or by using power control. Lower cross-correlations are usually achieved by employing Pseudo-noise sequences of long periodicity. However, this results in a lower bandwidth efficiency. The adaptive adjustment of transmitter power results in reductions in the transmitted power of stronger users and is thus self-defeating to the capability of a CDMA system.

### 2.3 Decorrelator

The goal of the first stage of our detector is to estimate the interference so that it can be eliminated from the matched filter outputs by a second stage. A 1 -shot decorrelator is employed, meaning that it considers one bit interval of the received signal at a time. It separates the users from each other so that each user's contribution to the interference can be seen. This decorrelator is the same as the one proposed by [14] for the two-user case here, extended to $K$ users.

We now implement a mathematical model that allows us to compute the statistics of the detector for user 1 , and by simple reordering of indices, for all other users.


Figure 2.2 Received asynchronous signal; general case

We make use of an idea that views an asynchronous channel as a synchronous channel. As can be seen in Figure 2.2, bit 0 of user 1 overlaps with bit -1 of user $k$, where $k=2,3, \ldots, K$, over the interval $\left[0, \tau_{k}\right]$ and with bit 0 of the same user over the interval $\left[\tau_{k}, T\right]$. We can view this situation as a $(2 K-1)$ - user synchronous channel with unit-energy signature waveforms $\bar{s}_{1}(t)=s_{1}(t), \bar{s}_{2 k-2}(t)=s_{k}^{L}(t) / \sqrt{e_{k}}$, and $\bar{s}_{2 k-1}(t)=s_{k}^{R}(t) / \sqrt{1-e_{k}}$. Thus every asynchronous user (except user 1 ) is assigned two synchronous users which results in $(2 K-1)$ synchronous users, where

$$
s_{k}^{L}(t)=\left\{\begin{array}{ll}
s_{k}\left(t+T-\tau_{k}\right) & 0 \leq t \leq \tau_{k} \\
0 & \tau_{k} \leq t \leq T,
\end{array} \quad s_{k}^{R}(t)= \begin{cases}0 & 0 \leq t \leq \tau_{k} \\
s_{k}\left(t-\tau_{k}\right) & \tau_{k} \leq t \leq T\end{cases}\right.
$$

and

$$
e_{k}=\int_{0}^{\tau_{k}} s_{k}^{2}\left(t+T-\tau_{k}\right) d t
$$

is the partial energy of the interfering signal over the left overlapping interval.
The cross-correlation matrix between the synchronous users is given as follows:

$$
\boldsymbol{H}=\left[\begin{array}{ccccc}
1 & \boldsymbol{\varphi}_{2} & \varphi_{3} & \cdots & \boldsymbol{\varphi}_{K}  \tag{2.5}\\
\boldsymbol{\varphi}_{2}^{T} & \boldsymbol{I} & \boldsymbol{\Phi}_{3}^{(2)} & \cdots & \boldsymbol{\Phi}_{K}^{(2)} \\
\boldsymbol{\varphi}_{3}^{T} & \boldsymbol{\Phi}_{3}^{(2)^{T}} & \boldsymbol{I} & \cdots & \boldsymbol{\Phi}_{K}^{(3)} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\boldsymbol{\varphi}_{K}^{T} & \boldsymbol{\Phi}_{K}^{(2)^{T}} & \boldsymbol{\Phi}_{K}^{(3)^{T}} & \cdots & \boldsymbol{I}
\end{array}\right]
$$

where

$$
\boldsymbol{\varphi}_{i}=\left[\frac{\rho_{1 i}}{\sqrt{e_{i}}}, \frac{\rho_{i 1}}{\sqrt{1-e_{i}}}\right], \quad i=2,3, \ldots, K
$$

and

$$
\boldsymbol{\Phi}_{i}^{(j)}=\left[\begin{array}{cc}
\theta_{j i}^{L} & 0 \\
\frac{\rho_{i j}}{\sqrt{e_{i}\left(1-e_{j}\right)}} & \theta_{j i}^{R}
\end{array}\right], \quad j=2,3, \ldots, K-1 \text { and } i=j+1, \ldots, K
$$

as well as

$$
\rho_{i 1}=\int_{0}^{T} s_{1}(t) s_{k}^{L}(t) d t \quad \text { and } \quad \rho_{1 i}=\int_{0}^{T} s_{1}(t) s_{k}^{R}(t) d t
$$

which is a restatement of Eq. (2.3), and:

$$
\rho_{i j}=\int_{\tau_{j}}^{\tau_{i}} s_{i}^{L}(t) s_{j}^{R}(t) d t
$$

$$
\theta_{j i}^{L}=\int_{0}^{\tau_{j}} \bar{s}_{2 j-2}(t) \bar{s}_{2 i-2}(t) d t \quad \text { and } \quad \theta_{j i}^{R}=\int_{\tau_{i}}^{T} \bar{s}_{2 j-1}(t) \bar{s}_{2 i-1}(t) d t
$$

For each additional user another $\varphi_{i}, \boldsymbol{\Phi}_{i}^{(j)}$ has to be added to the cross-correlation matrix.

We thus get the output vector of the matched filter bank as:

$$
\begin{equation*}
\boldsymbol{x}_{s}=\boldsymbol{H} \boldsymbol{A}_{s} \boldsymbol{b}_{\boldsymbol{s}}+\boldsymbol{n}_{\boldsymbol{s}} \tag{2.6}
\end{equation*}
$$

where $\boldsymbol{x}_{\mathrm{s}}=\left[x_{1}, x_{2}, \ldots, x_{2 K-1}\right]^{T}, \boldsymbol{A}_{s}=\operatorname{diag}\left[\sqrt{a_{1}}, \sqrt{a_{2}}, \sqrt{a_{2}}, \sqrt{a_{3}}, \sqrt{a_{3}}, \ldots, \sqrt{a_{K}}, \sqrt{a_{K}}\right]$ $\boldsymbol{b}_{s}=\left[b_{1}(0), b_{2}(-1), b_{2}(0), \ldots, b_{K}(-1), b_{K}(0)\right]^{T}$, and $\boldsymbol{n}_{s}=\left[\bar{n}_{1}, \bar{n}_{2}, \ldots, \bar{n}_{2 K-1}\right]^{T}$ with $\bar{n}_{m}=\int_{0}^{T} n(t) \bar{s}_{m}(t) d t, m=1,2, \ldots, 2 K-1$, are zero-mean Gaussian random variables with a variance of $N_{0} / 2$.

The decorrelator is simply the inverse matrix of the cross-correlation matrix $\boldsymbol{H}$, thus resulting in uncorrelated signals at its output:

$$
\begin{equation*}
z_{s}=H^{-1} x_{s}=A_{s} b_{s}+H^{-1} n_{s}=A_{s} b_{s}+\xi_{s} \tag{2.7}
\end{equation*}
$$

The noise vector $\boldsymbol{\xi}_{s}$ can be written more explicitly by expressing the inverse crosscorrelation matrix as follows:

$$
\boldsymbol{H}^{-1}=\frac{1}{\operatorname{det}(\boldsymbol{H})}\left[\begin{array}{c}
\boldsymbol{d}_{1} \\
\vdots \\
\boldsymbol{d}_{2 K-1}
\end{array}\right]
$$

where $\boldsymbol{d}_{m}$ is a $1 \times(2 K-1)$ vector. A specific entry of $\boldsymbol{\xi}_{s}$ is therefore:

$$
\xi_{s m}=\frac{1}{\operatorname{det}(\boldsymbol{H})} \boldsymbol{d}_{m} \boldsymbol{n}_{s}
$$

Since the $n_{m}$ 's are zero-mean Gaussian random variables and the performed mathematical operation is linear, $\xi_{s m}$ is also zero-mean Gaussian. We thus get for the variance of $\xi_{s m}$ :

$$
\sigma_{\xi_{s m}}^{2}=E\left\{\xi_{s m}^{2}\right\}=E\left\{\left(\boldsymbol{d}_{m} \boldsymbol{n}_{s}\right)^{2}\right\}
$$

Solving for bit 0 of user 1 :

$$
\begin{equation*}
z_{1}(0)=\sqrt{a_{1}} b_{1}(0)+\xi_{1}(0) \tag{2.8}
\end{equation*}
$$

The argument " 0 " indicates the corresponding time interval.
So far, the described model only allows us to compute the statistics of user 1. Other users can be computed simply by reordering the indices. Let $k$ denote the original order of the asynchronous users and $k^{\prime}$ the new order, with $k, k^{\prime}=1,2, \ldots, K$. Furthermore, let $l$ be the desired user for which the output of the decorrelator ought to be computed. The reordering must results in $k^{\prime}=1$ when $k=l$. The corresponding relationship between $k$ and $k^{\prime}$ is:

$$
\begin{equation*}
k^{\prime}=\bmod _{K}(K+k-l)+1 \tag{2.9}
\end{equation*}
$$

The above described method can now be applied to the remaining ( $K-1$ ) users.
We thus get the decorrelator outputs affecting bit 0 of user 1 :

$$
\begin{equation*}
\boldsymbol{z}_{1}(0)=\boldsymbol{A} \boldsymbol{b}_{1}(0)+\boldsymbol{\xi}_{1}(0) \tag{2.10}
\end{equation*}
$$

with $\boldsymbol{\xi}_{1}(0)=\left[\xi_{2}(-1), \ldots, \xi_{K}(-1), \xi_{2}(0), \ldots, \xi_{K}(0)\right]^{T}$ a zero-mean Gaussian vector having the covariance matrix $\boldsymbol{\Xi}_{1}$, whose diagonal elements are denoted by $\sigma_{\xi_{k}}^{2}$.

### 2.4 Soft Tentative Decisions

As mentioned earlier, we want to investigate the influence of tentative decisions on the performance of a multiuser detector in general and in particular the influence of soft tentative decision nonlinearities.

Taking into account that the task of the canceler on the second stage is to remove each user's interference from the output of the matched filters to obtain the desired signal without interference, the interference first has to be estimated. This is done by making decisions based on the decorrelator outputs. The vector of corresponding tentative decisions affecting bit 0 of user $1 \tilde{\boldsymbol{b}}_{1}(0)$ is:

$$
\begin{equation*}
\tilde{b}_{1}(0)=\boldsymbol{g}\left(z_{1}(0)\right)=\boldsymbol{g}\left(\boldsymbol{A} b_{1}(0)+\boldsymbol{\xi}_{1}(0)\right), \tag{2.11}
\end{equation*}
$$

where $\boldsymbol{g}(\cdot)$ denotes the performed tentative decisions.

Figure 2.3 justifies the implementation of decision devices in the multiuser detector in general. It shows the output error probability of the investigated detector without decision devices (linear detector) and with hard limiters versus the relative interference energy for fixed $S N R_{1}$.


Figure 2.3 Error probability $S N R_{1}=8 d B, \rho_{12}=0.2, \rho_{21}=0.6, e_{1}=0.4$

We can see from this figure that the overall performance of the hard limiter detector is better than that of the linear detector. Although the linear detector performs better than the hard limiter detector over a region of weak interference, its error probability gradually increases for increasing interference, whereas the hard limiter detector converges to the single-user bound.

The behavior of the hard limiter detector is intuitive. A weak interferer results in a bad or unreliable estimate. It is more likely to be faulty since the signal which the decision is based on contains a dominant noise part. If such a faulty estimate is used by a second stage for interference cancellation, the overall performance even deteriorates.

On the other hand, a strong interferer results in a good estimate, since most likely we estimated it correctly. A subtraction of this estimate from the matched filter outputs is desired since it results in interference cancellation.

To obtain better performance than the hard limiter detector, especially for weak interference, different tentative decision nonlinearities are implemented instead. The idea is to omit or attenuate weak interferers and thus bad estimates before subtracting them from the matched filter outputs, whereas strong interferers and thus good estimates should be subtracted without attenuation, thus they are hard limited. This is exactly what a soft limiter does. We consider two different tentative decision nonlinearities, a dead-zone limiter and a linear clipper.

Let us first consider how many decision devices are needed in the detector. Let user 1 be the user we wish to detect.


Figure 2.4 Interferers and their cross-correlations

According to Figure 2.4, interval -1 and 0 of user 2 partially overlap with interval 0 of user 1. Since each of those interferers ought to be subtracted we end up with two decision devices between the desired user and an interferer and consequently $2(K-1) K$ decision devices for the general $K$-user case.

The dead-zone limiter is shown in Figure 2.5 with respect to user 1. If the magnitude of a decorrelator output is smaller than the threshold, the corresponding estimate is omitted. If it is larger, however, the dead-zone limiter operates like a hard limiter.


Figure 2.5 Dead-zone limiter

The entries of $\tilde{\boldsymbol{b}}_{1}(0)$ for a dead-zone limiter are defined as follows:

$$
\tilde{b}_{k}(i)= \begin{cases}0 & \left|z_{k}(i)\right|<\lambda_{k 1}^{(i)}  \tag{2.12}\\ \operatorname{sgn}\left(z_{k}(i)\right) & \text { otherwise }\end{cases}
$$

where $i=-1,0$ and $k=2,3, \ldots, K . \lambda_{k 1}^{(i)}$ represents the threshold of the dead-zone limiter for the $k$ th interferer in time interval $i$.


Figure 2.6 Linear clipper

The same holds in the case of a linear clipper, Figure 2.6, when the magnitude of the decorrelator output is larger than the threshold. If it is smaller, though, the output of the linear clipper is a linear function of the input ranging between $\pm 1$, thus it attenuates weak interferers. The output of a linear clipper is thus given as:

$$
\tilde{b}_{k}(i)= \begin{cases}z_{k}(i) / \lambda_{k 1}^{(i)} & \left|z_{k}(i)\right|<\lambda_{k 1}^{(i)}  \tag{2.13}\\ \operatorname{sgn}\left(z_{k}(i)\right) & \text { otherwise } .\end{cases}
$$

The question is how the threshold should be determined now that the above mentioned is realized. The following has to be taken into account:

- For relatively weak interference, the threshold should be large to omit or attenuate the corresponding estimate whereas for relatively strong interference it should approach zero. The amount of interference can be inferred from the ratio between the energy of the desired and the interfering user. Those energies are best obtained from the decorrelator output where the users are uncorrelated.
- Other indicators for interference are the cross-correlation coefficients.

The best heuristic values of the thresholds can be obtained from the following investigation of the output error probability, shown here vicariously for the detector using dead-zone limiters. Figures 2.7 and 2.8 show its error performance versus variable strength of interference for fixed $S N R_{1}$. The thresholds for both dead-zone limiter and linear clipper between user 1, the desired signal, and user 2, the interferer, are chosen as follows:

$$
\lambda_{k 1}^{(-1)}=a \frac{\left[E\left\{\left|z_{1}(0)\right|\right\}\right]^{2}}{E\left\{\left|z_{k}(-1)\right|\right\}} \quad \text { and } \quad \lambda_{k 1}^{(0)}=b \frac{\left[E\left\{\left|z_{1}(0)\right|\right\}\right]^{2}}{E\left\{\left|z_{k}(0)\right|\right\}}
$$

where $k=2,3, \ldots, K . a$ and $b$ are functions of the cross-correlation coefficients.
Obviously, the solution with $a=\rho_{12}^{2}$ and $b=\rho_{21}^{2}$ provide the best results. This brings us to the general values of the thresholds between user 1 and user $k$ obtained with the above reasoning:

$$
\begin{equation*}
\lambda_{k 1}^{(-1)}=\frac{\rho_{1 k}^{2}\left[E\left\{\left|z_{1}(-1)\right|\right\}\right]^{2}}{E\left\{\left|z_{k}(-1)\right|\right\}} \quad \text { and } \quad \lambda_{k 1}^{(0)}=\frac{\rho_{k 1}^{2}\left[E\left\{\left|z_{1}(0)\right|\right\}\right]^{2}}{E\left\{\left|z_{k}(0)\right|\right\}} \tag{2.14}
\end{equation*}
$$

where the above expectations are evaluated as:

$$
\begin{equation*}
E\left\{\left|z_{k}(i)\right|\right\}=\sqrt{a_{k}}\left(1-2 Q\left(\frac{\sqrt{a_{k}}}{\sigma_{\xi_{k}(i)}}\right)\right)+\frac{2 \sigma_{\xi_{k}(i)}}{\sqrt{2 \pi}} \exp \left(-\frac{a_{k}}{2 \sigma_{\xi_{k}(i)}^{2}}\right), \tag{2.15}
\end{equation*}
$$

with:

$$
Q(x)=\frac{1}{\sqrt{2 \pi}} \int_{x}^{\infty} e^{-t^{2} / 2} d t
$$

For the elaborate computation of Eq. (2.15) see Appendix C.


Figure 2.7: Comparison of heuristic thresholds, error probability of user 1 for $K=2$, $S N R_{1}=8 d B, \rho_{12}=0.2, \rho_{21}=0.6, e_{1}=0.4$


Figure 2.8: Comparison of heuristic thresholds, error probability of user 1 for $K=2$, $S N R_{1}=12 d B, \rho_{12}=0.2, \rho_{21}=0.6, e_{1}=0.4$

Figures 2.9, 2.10, and 2.11 show that under the given conditions the so obtained output error probability is very close to the best possible output error probability for the given structure, obtained by search until the output error probability is minimized.


Figure 2.9: Comparison between best and heuristic threshold, error probability of user 1 for $K=2, S N R_{1}=4 d B, \rho_{12}=0.2, \rho_{21}=0.6, e_{1}=0.4$


Figure 2.10: Comparison between best and heuristic threshold, error probability of user 1 for $K=2, S N R_{1}=8 d B, \rho_{12}=0.2, \rho_{21}=0.6, e_{1}=0.4$


Figure 2.11: Comparison between best and heuristic threshold, error probability of user 1 for $K=2, S N R_{1}=12 d B, \rho_{12}=0.2, \rho_{21}=0.6, e_{1}=0.4$

### 2.5 Adaptive Canceler

The adaptive canceler takes the soft tentative decisions of the interfering users, multiplies each of them with a weight and subtracts them from the matched filter outputs. Ideally all interference is removed. The final decision statistics $y_{1}(0)$ and the corresponding final decision for bit 0 of user 1 are:

$$
\begin{equation*}
y_{1}(0)=x_{1}(0)-\boldsymbol{w}_{1}^{T}(0) \tilde{\boldsymbol{b}}_{1}(0) \quad \text { and } \quad \hat{b}_{1}(0)=\operatorname{sgn}\left(y_{1}(0)\right) \tag{2.16}
\end{equation*}
$$

where $\boldsymbol{w}_{1}(0)=\left[w_{21}^{(-1)}, \ldots, w_{K 1}^{(-1)}, w_{21}^{(0)}, \ldots, w_{K 1}^{(0)}\right]^{T}$ are the corresponding weights.
For controlling the weights, a steepest descent algorithm is used. This algorithm belongs to the group of Least-Mean-Square algorithms. It converges slowly but has good stability properties; its convergence is proven in [15].

The steepest descent algorithm minimizes the output signal energy $E\left\{y_{1}^{2}(0)\right\}$. The weights that achieve this are obtained by iterative search:

$$
\begin{equation*}
\boldsymbol{w}_{1}(i+1) \leftarrow \boldsymbol{w}_{1}(i)-\frac{\mu}{2} \frac{\partial}{\partial \boldsymbol{w}_{1}(i)} E\left\{y_{1}^{2}(i)\right\}=\boldsymbol{w}_{1}(i)+\mu E\left\{y_{1}(i) \tilde{\boldsymbol{b}}_{1}(i)\right\} \tag{2.17}
\end{equation*}
$$

The second term in Eq. (2.17) is forced to zero when the steady state values are reached. In this case the old value $\boldsymbol{w}_{1}(i)$ and the new value $\boldsymbol{w}_{1}(i+1)$ are identical. Two important facts can be deduced from this:

- The output energy $E\left\{y_{1}^{2}(0)\right\}$ of user 1 is minimized by this algorithm
- It forces the correlation between the output signal of user 1 and the vector of tentative decisions $\tilde{\boldsymbol{b}}_{1}(0)$ of interfering signals to zero. That means ideally weights are chosen such that the output of user 1 no longer contains any interference.

Thus we get for the steady state values:

$$
\begin{align*}
-\frac{1}{2} \frac{\partial}{\partial \boldsymbol{w}_{1}(0)} E\left\{y_{1}^{2}(0)\right\}=0 & =E\left\{y_{1}(0) \tilde{b}_{1}(0)\right\} \\
& =E\left\{x_{1}(0) \tilde{b}_{1}(0)-\boldsymbol{w}_{1}(0) \tilde{\boldsymbol{b}}_{1}(0) \tilde{\boldsymbol{b}}_{1}(0)^{T}\right\} \tag{2.18}
\end{align*}
$$

with:

$$
\begin{equation*}
E\left\{x_{1}(0) \tilde{\boldsymbol{b}}_{1}(0)\right\}=E\left\{\sqrt{a_{1}} b_{1}(0) \tilde{\boldsymbol{b}}_{1}(0)+\boldsymbol{\rho}_{1}^{T} \boldsymbol{A} \boldsymbol{b}_{1}(0) \tilde{\boldsymbol{b}}_{1}(0)+n_{1}(0) \tilde{\boldsymbol{b}}_{1}(0)\right\} \tag{2.19}
\end{equation*}
$$

The expected value of the first term in (2.19) is zero since the independence between a user and the estimates of all other users is a feature of the decorrelator. The expected value of the third term results in $E\left\{n_{1}(0) \boldsymbol{\xi}_{1}(0)\right\}$. The independence of $n_{1}(0)$ and the noise of interfering users at the output of the decorrelator is proven in [15]. Thus we get:

$$
\begin{equation*}
E\left\{x_{1}(0) \tilde{\boldsymbol{b}}_{1}(0)\right\}=\boldsymbol{A} E\left\{\boldsymbol{b}_{1}(0) \tilde{\boldsymbol{b}}_{1}^{T}(0)\right\} \boldsymbol{\rho}_{1} \tag{2.20}
\end{equation*}
$$

Clearly $E\left\{\boldsymbol{b}_{1}(0) \tilde{\boldsymbol{b}}_{1}^{T}(0)\right\}$ is diagonal; therefore, the system of $(2 K-1)$ linear equations (2.18), together with (2.20), gives the steady state values of the weights affecting the first output as:

$$
\begin{equation*}
\boldsymbol{w}_{1}(0)=\left[E\left\{\tilde{\boldsymbol{b}}_{1}(0) \tilde{\boldsymbol{b}}_{1}^{T}(0)\right\}\right]^{-1} \boldsymbol{A} E\left\{\boldsymbol{b}_{1}(0) \tilde{\boldsymbol{b}}_{1}^{T}(0)\right\} \boldsymbol{\rho}_{1} \tag{2.21}
\end{equation*}
$$

The expectations in the above expression are given in Appendix D.

### 2.6 Error Probability

The output probability of error $P_{e_{1}}$ of user 1 is evaluated as:

$$
\begin{aligned}
P_{e_{1}} & =E_{b_{1}(0), \boldsymbol{b}_{1}(0), \tilde{\boldsymbol{b}}_{1}(0)} \operatorname{Pr}\left\{\hat{b}_{1} \neq b_{1}(0) \mid b_{1}(0), \boldsymbol{b}_{1}(0), \tilde{\boldsymbol{b}}_{1}(0)\right\} \\
& =\frac{1}{2} \sum_{\boldsymbol{b}_{\mathbf{1}}(0)} E_{\tilde{\boldsymbol{b}}_{1}(0)}\left[\operatorname{Pr}\left\{n_{1}(0)>\sqrt{a_{1}}-\boldsymbol{\rho}_{1}^{T} \boldsymbol{A} \boldsymbol{b}_{1}(0)+\boldsymbol{w}_{1}^{T}(0) \tilde{\boldsymbol{b}}_{\mathbf{1}}(0) \mid \tilde{\boldsymbol{b}}_{1}(0)\right\}\right. \\
& \left.+\operatorname{Pr}\left\{n_{1}(0)<-\sqrt{a_{1}}-\boldsymbol{\rho}_{1}^{T} \boldsymbol{A} \boldsymbol{b}_{1}(0)+\boldsymbol{w}_{1}^{T}(0) \tilde{\boldsymbol{b}}_{1}(0) \mid \tilde{\boldsymbol{b}}_{1}(0)\right\}\right] \operatorname{Pr}\left\{\boldsymbol{b}_{1}(0)\right\}(2.22)
\end{aligned}
$$

Since $\operatorname{Pr}\left\{\boldsymbol{b}_{1}(0)\right\}=2^{-(2 K-2)}$, and the above expression contains pairwise identical terms, it can be written as:

$$
\begin{equation*}
P_{e_{1}}=\frac{1}{2^{2 K-2}} \sum_{\boldsymbol{b}_{1}(0)} E_{\tilde{\boldsymbol{b}}_{1}(0)} \operatorname{Pr}\left\{n_{1}(0)>\sqrt{a_{\mathbf{1}}}-\boldsymbol{\rho}_{1}^{T} \boldsymbol{A} \boldsymbol{b}_{1}(0)+\boldsymbol{w}_{1}^{T}(0) \tilde{\boldsymbol{b}}_{1}(0) \mid \tilde{\boldsymbol{b}}_{1}(0)\right\} \tag{2.23}
\end{equation*}
$$

Focusing on the dead-zone soft tentative decision, the vector $\boldsymbol{\xi}_{1}$ spans a ( $2 K-$ 2)-dimensional space. Each dimension is partitioned as:

$$
\begin{gathered}
-\lambda_{k 1}^{(i)}-\sqrt{a_{k}} b_{k}(i)<\xi_{k}(i)<\lambda_{k 1}^{(i)}-\sqrt{a_{k}} b_{k}(i) \\
\text { or } \quad \xi_{k}(i)>\lambda_{k 1}^{(i)}-\sqrt{a_{k}} b_{k}(i), \quad \xi_{k}(i)<-\lambda_{k 1}^{(i)}-\sqrt{a_{k}} b_{k}(i),
\end{gathered}
$$

with $k=2,3, \ldots, K, \quad i=-1,0$. The created $(2 K-2)$-dimensional subregions $D_{m}$, $m=1,2, \ldots, 2^{2 K-2}$, have a corresponding vector $\tilde{\boldsymbol{b}}_{1}(0)=\tilde{\boldsymbol{b}}_{1 m}(0)$ whose elements, according to (2.12), take values of $\pm 1$ and 0 . Thus (2.23) can be rewritten as:

$$
\begin{equation*}
P_{e_{1}}=\frac{1}{2^{2 K-2}} \sum_{\boldsymbol{b}_{1}(0)} \sum_{m=1}^{2^{2 K-2}} \operatorname{Pr}\left\{n_{1}(0)>\sqrt{a_{1}}-\boldsymbol{\rho}_{1}^{T} \boldsymbol{A} \boldsymbol{b}_{1}(0)+\boldsymbol{w}_{1}^{T}(0) \tilde{\boldsymbol{b}}_{1 m}(0)\right\} \operatorname{Pr}\left\{\boldsymbol{\xi}_{1}(0) \in \boldsymbol{D}_{m}\right\} \tag{2.24}
\end{equation*}
$$

Since $n_{1}(0)$ and $\boldsymbol{z}_{1}(0)$ are uncorrelated, we finally get:

$$
\begin{equation*}
P_{e_{1}}=\frac{1}{2^{2 K-2}} \sum_{\boldsymbol{b}_{1}(0)} \sum_{m=1}^{2^{2 K-2}} Q\left(\frac{\sqrt{a_{1}}-\boldsymbol{\rho}_{1}^{T} \boldsymbol{A} \boldsymbol{b}_{1}(0)+\boldsymbol{w}_{1}^{T}(0) \tilde{\boldsymbol{b}}_{1_{m}}(0)}{\sqrt{N_{0} / 2}}\right) \int_{\boldsymbol{D}_{m}} \boldsymbol{f}_{\boldsymbol{\xi}_{1}(0)} d \boldsymbol{\xi}_{1}(0) \tag{2.25}
\end{equation*}
$$

where $f_{\boldsymbol{\xi}_{1}(0)}$ is a $(2 K-2)$-variate Gaussian density function.
In the case of a linear clipper, each entry of $\tilde{b}_{1}(0)$ in (2.23) can, according to (2.13), be $z_{k}(i) / \lambda_{k 1}^{(i)}$ or $\operatorname{sgn}\left(z_{k}(i)\right)$. A transformed Gaussian random variable $\psi_{1}$ is
defined as:

$$
\psi_{1}=n_{1}(0)-\sum_{k=2}^{K} c_{k}^{(0)} \frac{w_{k 1}^{(0)}}{\lambda_{k 1}^{(0)}} \xi_{k}(0)-\sum_{k=2}^{K} c_{k}^{(-1)} \frac{w_{k 1}^{(-1)}}{\lambda_{k 1}^{(-1)}} \xi_{k}(-1)
$$

where

$$
c_{k}^{(i)}=\left\{\begin{array}{ll}
1 & \left|z_{k}(0)\right|<\lambda_{k 1}^{(i)} \\
0 & \text { otherwise }
\end{array} \quad k=2,3, \ldots, K, \quad i=-1,0 .\right.
$$

The error probability (2.23) becomes:

$$
P_{e_{1}}=\frac{1}{2^{2 K-2}} \sum_{\boldsymbol{b}_{1}(0)} E_{\tilde{\boldsymbol{b}}_{1}(0)} \operatorname{Pr}\left\{\psi_{1}>\sqrt{a_{1}}-\boldsymbol{\rho}_{1}^{T} \boldsymbol{A} \boldsymbol{b}_{1}(0)+\boldsymbol{w}_{\mathbf{1}}^{T}(0) \boldsymbol{h}_{1}(0) \mid \tilde{\boldsymbol{b}}_{1}(0)\right\},
$$

where $\boldsymbol{h}_{1}(0)=\left[h_{2}(-1), \ldots, h_{K}(-1), h_{2}(0), \ldots, h_{K}(0)\right]^{T}$ and each entry is defined as:

$$
h_{k}(i)=\left\{\begin{array}{ll}
\sqrt{a_{k}} b_{k}(i) / \lambda_{k 1}^{(i)} & \left|z_{k}(i)\right|<\lambda_{k 1}^{(i)} \\
\operatorname{sgn}\left(z_{k}(i)\right) & \text { otherwise }
\end{array} \quad k=2,3, \ldots, K, \quad i=-1,0 .\right.
$$

Defining the vector $\boldsymbol{\zeta}$ as:

$$
\boldsymbol{\zeta}=\left[\zeta_{1}, \zeta_{2}, \ldots, \zeta_{2 K-1}\right]^{T}=\left[\psi_{1}, \xi_{2}(-1), \xi_{2}(0), \ldots, \ldots, \xi_{K}(-1), \xi_{K}(0)\right]^{T}
$$

that spans a ( $2 K-1$ )-dimensional space, the final expression for the error probability becomes

$$
\begin{equation*}
P_{e 1}=\frac{1}{2^{2 K-2}} \sum_{\boldsymbol{b}_{1}(0)} \sum_{m=1}^{2 K-2} \int_{\boldsymbol{D}_{m}} \boldsymbol{f}_{\boldsymbol{\zeta}} d \boldsymbol{\zeta} \tag{2.26}
\end{equation*}
$$

where $\boldsymbol{f}_{\boldsymbol{\zeta}}$ is a $(2 K-1)$-variate Gaussian density function. Each dimension is partitioned as $\sqrt{a_{1}}-\boldsymbol{b}_{\mathbf{1}}(0)^{T} \boldsymbol{A} \boldsymbol{\rho}_{1}+\boldsymbol{w}_{1}^{T}(0) \boldsymbol{h}_{1}(0)<\zeta_{1}$, and for $j=2,3, \ldots, 2 K-1$, each $\zeta_{j}$ falls in either of the following two regions:

$$
\begin{array}{r}
\quad-\lambda_{k 1}^{(i)}-\sqrt{a_{k}} b_{k}(i)<\zeta_{j}<\lambda_{k 1}^{(i)}-\sqrt{a_{k}} b_{k}(i) \\
\text { or } \quad \zeta_{j}>\lambda_{k 1}^{(i)}-\sqrt{a_{k}} b_{k}(i), \zeta_{j}<\lambda_{k 1}^{(i)}-\sqrt{a_{k}} b_{k}(i),
\end{array}
$$

again with $k=2,3, \ldots, K, \quad i=-1,0$.

### 2.7 Two-User Case

To illustrate the above general case, the derivations of the statistics for the two-user asynchronous detector are given here in detail.

| 0 |  | T |  |  |  |  |
| :--- | :--- | :--- | :--- | :--- | :---: | :---: |
|  | $\mathrm{~s}_{1}(t)$ |  |  |  |  |  |
|  | $\mathrm{s}_{2}^{\mathrm{I}}(\mathrm{t})$ | $\mathrm{s}_{2}^{\mathrm{R}}(\mathrm{t})$ |  |  |  |  |
| $\tau_{2}$ |  |  |  |  |  |  |

Figure 2.12 Received asynchronous signal; two-user case

From the view of bit 0 of user 1, Figure 2.12, it overlaps with bit -1 of user 2 over the interval $\left[0, \tau_{2}\right]$, assuming without loss of generality that $\tau_{1}=0$, and bit 0 of user 2 over the interval $\left[\tau_{2}, T\right]$. This situation can be viewed as a three-user synchronous channel. The unit-energy signature waveforms of the synchronous users can be derived from the waveforms of the original asynchronous users; $\bar{s}_{1}(t)=s_{1}(t)$, $\bar{s}_{2}(t)=s_{2}^{L}(t) / \sqrt{e_{2}}$, and $\bar{s}_{3}(t)=s_{2}^{R}(t) / \sqrt{1-e_{2}}$, where

$$
s_{2}^{L}(t)=\left\{\begin{array}{ll}
s_{k}\left(t+T-\tau_{2}\right) & 0 \leq t \leq \tau_{2} \\
0 & \tau_{2} \leq t \leq T,
\end{array} \quad s_{2}^{R}(t)= \begin{cases}0 & 0 \leq t \leq \tau_{2} \\
s_{k}\left(t-\tau_{2}\right) & \tau_{2} \leq t \leq T\end{cases}\right.
$$

and

$$
e_{2}=\int_{0}^{\tau_{2}} s_{k}^{2}\left(t+T-\tau_{2}\right) d t
$$

The received signal $r(t)$ in the synchronous channel is:

$$
r(t)=\sqrt{a_{1}} b_{1}(0) s_{1}(t)+\sqrt{a_{2}} b_{2}(-1) s_{2}^{L}(t)+\sqrt{a_{2}} b_{2}(0) s_{2}^{R}(t)+n(t) .
$$

The cross-correlation between the signature waveforms are given as follows:

$$
\rho_{21}=\int_{0}^{T} s_{1}(t) s_{2}^{L}(t) d t \quad \text { and } \quad \rho_{12}=\int_{0}^{T} s_{1}(t) s_{2}^{R}(t) d t
$$

The cross-correlation matrix between the three synchronous users is given as:

$$
\boldsymbol{H}=\left[\begin{array}{ccc}
1 & \frac{\rho_{21}}{\sqrt{e_{2}}} & \frac{\rho_{12}}{\sqrt{1-e_{2}}} \\
\frac{\rho_{21}}{\sqrt{e_{2}}} & 1 & 0 \\
\frac{\rho_{12}}{\sqrt{1-e_{2}}} & 0 & 1
\end{array}\right]
$$

The sampled output the matched filters is:

$$
\left[\begin{array}{l}
x_{1} \\
x_{2} \\
x_{3}
\end{array}\right]=\boldsymbol{H} \boldsymbol{A}_{s}\left[\begin{array}{c}
b_{1}(0) \\
b_{2}(-1) \\
b_{2}(0)
\end{array}\right]+\left[\begin{array}{l}
\bar{n}_{1} \\
\bar{n}_{2} \\
\bar{n}_{3}
\end{array}\right]
$$

where $\boldsymbol{A}_{s}=\operatorname{diag}\left[\sqrt{a_{1}}, \sqrt{a_{2}}, \sqrt{a_{2}}\right]$, and $\bar{n}_{m}=\int_{0}^{T} n(t) \bar{s}_{m}(t) d t, m=1,2,3$, are zeromean Gaussian random variables with a variance of $N_{0} / 2$.

In order to get interference-free signals, the decorrelator matrix can be chosen as the inverse of the cross-correlation matrix $\boldsymbol{H}$ :

$$
\boldsymbol{H}^{-1}=\frac{1}{1-\frac{\rho_{21}^{2}}{e_{2}}-\frac{\rho_{12}^{2}}{1-e_{2}}}\left[\begin{array}{ccc}
1 & -\frac{\rho_{21}}{\sqrt{e_{2}}} & -\frac{\rho_{12}}{\sqrt{1-e_{2}}} \\
-\frac{\rho_{21}}{\sqrt{e_{2}}} & 1-\frac{\rho_{12}^{2}}{1-e_{2}} & \frac{\rho_{12} \rho_{21}}{\sqrt{e_{2}\left(1-e_{2}\right)}} \\
-\frac{\rho_{12}}{\sqrt{1-e_{2}}} & \frac{\rho_{12} \rho_{21}}{\sqrt{e_{2}\left(1-e_{2}\right)}} & 1-\frac{\rho_{21}^{2}}{e_{2}}
\end{array}\right] .
$$

Thus, the interference-free outputs of the decorrelator are:

$$
\left[\begin{array}{l}
z_{1} \\
z_{2} \\
z_{3}
\end{array}\right]=\boldsymbol{A}_{s}\left[\begin{array}{c}
b_{1}(0) \\
b_{2}(-1) \\
b_{2}(0)
\end{array}\right]+\left[\begin{array}{c}
\xi_{s 1} \\
\xi_{s 2} \\
\xi_{s 3}
\end{array}\right] .
$$

The resulting noise at the output of the decorrelator for user 1 in time interval 0 is:

$$
\begin{aligned}
\xi_{s 1}=\xi_{1}(0) & =\frac{1}{\operatorname{det}(\boldsymbol{H})}\left[\begin{array}{lll}
1 & -\frac{\rho_{21}}{\sqrt{e_{2}}} & -\frac{\rho_{12}}{\sqrt{1-e_{2}}}
\end{array}\right]\left[\begin{array}{c}
\bar{n}_{1} \\
\bar{n}_{2} \\
\bar{n}_{3}
\end{array}\right] \\
& =\frac{1}{\operatorname{det}(\boldsymbol{H})}\left(\bar{n}_{1}-\frac{\rho_{21}}{\sqrt{e_{2}}} \bar{n}_{2}-\frac{\rho_{12}}{\sqrt{1-e_{2}}} \bar{n}_{3}\right) \\
& =\frac{1}{\operatorname{det}(\boldsymbol{H})}\left(n_{1}(0)-\frac{\rho_{21}}{e_{2}} n_{2}^{L}-\frac{\rho_{12}}{1-e_{2}} n_{2}^{R}\right) .
\end{aligned}
$$

$\xi_{1}(0)$ is a zero-mean Gaussian random variable with the variance:

$$
\begin{aligned}
\sigma_{\xi_{1}(0)}^{2} & =E\left\{\xi_{1}(0)^{2}\right\} \\
& =\frac{1}{\operatorname{det}(\boldsymbol{H})^{2}} E\left\{\left(n_{1}(0)-\frac{\rho_{21}}{e_{2}} n_{2}^{L}-\frac{\rho_{12}}{1-e_{2}} n_{2}^{R}\right)^{2}\right\} \\
& =\frac{1}{\operatorname{det}(\boldsymbol{H})^{2}}\left[E\left\{n_{1}^{2}(0)\right\}-2 \frac{\rho_{21}}{e_{2}} E\left\{n_{1}(0) n_{2}^{L}\right\}-2 \frac{\rho_{12}}{1-e_{2}} E\left\{n_{1}(0) n_{2}^{R}\right\}\right.
\end{aligned}
$$

$$
\begin{aligned}
& \left.+2 \frac{\rho_{12} \rho_{21}}{e_{2}\left(1-e_{2}\right)} E\left\{n_{2}^{L} n_{2}^{R}\right\}+\frac{\rho_{21}^{2}}{e_{2}} E\left\{n_{2}^{L^{2}}\right\}+\frac{\rho_{12}^{2}}{1-e_{2}} E\left\{n_{2}^{R^{2}}\right\}\right] \\
& =\frac{1}{\operatorname{det}(\boldsymbol{H})^{2}}\left[1-2 \frac{\rho_{21}^{2}}{e_{2}}-2 \frac{\rho_{12}^{2}}{1-e_{2}}+\frac{\rho_{21}^{2}}{e_{2}}+\frac{\rho_{12}^{2}}{1-e_{2}}\right] \frac{N_{0}}{2} \\
& =\frac{\frac{N_{0}}{2}}{1-\frac{\rho_{21}^{2}}{e_{2}}-\frac{\rho_{12}^{2}}{1-e_{2}}}
\end{aligned}
$$

and thus the output of the decorrelator for user 1 in the time interval $i=0$ :

$$
z_{1}(0)=\sqrt{a_{1}} b_{1}(0)+\xi_{1}(0)
$$

Likewise, we get the decorrelator output of user 2:

$$
z_{2}(0)=\sqrt{a_{2}} b_{2}(0)+\xi_{2}(0)
$$

with

$$
\xi_{2}(0)=\frac{1}{\operatorname{det}(\boldsymbol{H})}\left(n_{2}(0)-\frac{\rho_{12}}{e_{1}} n_{1}^{L}-\frac{\rho_{21}}{1-e_{1}} n_{1}^{R}\right)
$$

and

$$
\sigma_{\xi_{1}(0)}^{2}=\frac{\frac{N_{0}}{2}}{1-\frac{\rho_{12}^{2}}{e_{1}}-\frac{\rho_{21}^{2}}{1-e_{1}}}
$$

Having computed the statistics for all users involved, we can now rewrite the equation for the output of the matched filter for user 1 using the vector notations where $\boldsymbol{\rho}_{1}=\left[\rho_{21}, \rho_{12}\right]^{T}, \boldsymbol{b}_{1}(0)=\left[b_{2}(-1), b_{2}(0)\right]^{T}$, and $\boldsymbol{A}=\left[\begin{array}{cc}\sqrt{a_{2}} & 0 \\ 0 & \sqrt{a_{2}}\end{array}\right]$ :

$$
x_{1}(0)=\sqrt{a_{1}} b_{1}(0)+\boldsymbol{\rho}_{1}^{T} \boldsymbol{A} \boldsymbol{b}_{1}(0)+n_{1}(0)
$$

and the decorrelator output affecting bit 0 of user 1 :

$$
\boldsymbol{z}_{1}(0)=\boldsymbol{A} \boldsymbol{b}_{1}(0)+\boldsymbol{\xi}_{1}(0)
$$

with $\boldsymbol{\xi}_{1}(0)=\left[\xi_{2}(-1), \xi_{2}(0)\right]^{T}$.
The decorrelator follows the tentative decision part of the detector. Its task is it to provide estimates on the amount of interference which is then subtracted from the desired user using an adaptive network of weights. The definitions for the
tentative decisions $\tilde{\boldsymbol{b}}_{1}(0)$ and the thresholds $\lambda_{k 1}^{(i)}$, respectively, are straightforward, see Eq. (2.12) to Eq. (2.15). We thus obtain the final output $y_{1}(0)$ of the interference canceler and the corresponding final decision:

$$
y_{1}(0)=x_{1}(0)-\boldsymbol{w}_{1}^{T}(0) \tilde{\boldsymbol{b}}_{1}(0) \quad \text { and } \quad \hat{b}_{1}(0)=\operatorname{sgn}\left(y_{1}(0)\right)
$$

where $\boldsymbol{w}_{1}(0)=\left[w_{21}^{(-1)}, w_{21}^{(0)}\right]^{T}$. It follows a detailed computation of the steady state values of the weights for the dead-zone limiter and the linear clipper.

The steady state values of the weights can be inferred from Eq. (2.21) by substituting the matrices for the general case with their two-user case equivalent; $\boldsymbol{b}_{\mathbf{1}}(0)=\left[b_{2}(-1), b_{2}(0)\right]^{T}, \tilde{\boldsymbol{b}}_{1}(0)=\left[\tilde{b}_{1}(-1), \tilde{b}_{1}(0)\right]^{T}, \boldsymbol{A}=\operatorname{diag}\left[\sqrt{a_{2}}, \sqrt{a_{2}}\right]$, and $\boldsymbol{\rho}_{1}=$ $\left[\rho_{21}, \rho_{12}\right]^{T}$.

$$
\begin{equation*}
w_{21}(0)=\rho_{12} \sqrt{a_{2}} \frac{E\left\{b_{2}(0) \tilde{b}_{2}(0)\right\}}{E\left\{\tilde{b}_{2}(0)^{2}\right\}} \tag{2.27}
\end{equation*}
$$

is obtained when considering time interval $i=0$ only, which can be done without loss of generality.

### 2.7.1 Steady State Values of Weights for Dead-Zone Limiter

The numerator of Eq. (2.27) can be written as follows:
$E\left\{b_{2}(0) \tilde{b}_{2}(0)\right\}=E\left\{b_{2}(0) \cdot 0| | z_{2}(0) \mid<\lambda_{21}^{(0)}\right\}+E\left\{b_{2}(0) \operatorname{sgn}\left(z_{2}(0)\right)| | z_{2}(0) \mid>\lambda_{21}^{(0)}\right\}$.
Only the expected value for the interval $\left|z_{2}(0)\right|>\lambda_{21}^{(0)}$ is non-zero.

$$
\begin{aligned}
E\left\{b_{2}(0) \tilde{b}_{2}(0)\right\} & =\left[\operatorname{Pr}\left\{\tilde{b}_{2}(0)=1 \mid b_{2}(0)=1\right\} \operatorname{Pr}\left\{b_{2}(0)=1\right\}\right. \\
& -\operatorname{Pr}\left\{\tilde{b}_{2}(0)=-1 \mid b_{2}(0)=1\right\} \operatorname{Pr}\left\{b_{2}(0)=1\right\} \\
& -\operatorname{Pr}\left\{\tilde{b}_{2}(0)=1 \mid b_{2}(0)=-1\right\} \operatorname{Pr}\left\{b_{2}(0)=-1\right\} \\
& \left.+\operatorname{Pr}\left\{\tilde{b}_{2}(0)=-1 \mid b_{2}(0)=-1\right\} \operatorname{Pr}\left\{b_{2}(0)=-1\right\}\right]
\end{aligned}
$$

With $\operatorname{Pr}\left\{b_{2}(0)=1\right\}=\frac{1}{2}$ and the fact that the above expression contains pairwise identical terms, it can be written as:

$$
E\left\{b_{2}(0) \tilde{b}_{2}(0)\right\}=\left[\operatorname{Pr}\left\{\xi_{2}(0)>\lambda_{21}^{(0)}-\sqrt{a_{2}}\right\}\right.
$$

$$
\begin{aligned}
& \left.-\operatorname{Pr}\left\{\xi_{2}(0)>\lambda_{21}^{(0)}+\sqrt{a_{2}}\right\}\right] \\
& =\left[Q\left(\frac{\lambda_{21}^{(0)}-\sqrt{a_{2}}}{\sigma_{\xi_{2}(0)}}\right)-Q\left(\frac{\lambda_{21}^{(0)}+\sqrt{a_{2}}}{\sigma_{\xi_{2}(0)}}\right)\right],
\end{aligned}
$$

where:

$$
Q(x)=\frac{1}{\sqrt{2 \pi}} \int_{x}^{\infty} e^{-t^{2} / 2} d t
$$

The denominator of Eq. (2.27) can be subdivided equivalently:

$$
\begin{aligned}
E\left\{\tilde{b}_{2}(0)^{2}\right\} & =E\left\{\tilde{b}_{2}(0)^{2}| | z_{2}(0) \mid>\lambda_{21}^{(0)}\right\} \\
& =E\left\{\int_{\left|z_{2}(0)\right|>\lambda_{21}^{(0)}} \tilde{b}_{2}(0)^{2} f\left(z_{2}(0)\right) d z_{2}(0)\right\} \\
& =E\left\{\int_{z_{2}(0)<\lambda_{21}^{(0)}} f\left(z_{2}(0)\right) d z_{2}(0)+\int_{z_{2}(0)>\lambda_{21}^{(0)}} f\left(z_{2}(0)\right) d z_{2}(0)\right\} \\
& =E\left\{\int_{-\infty}^{-\infty} \lambda_{21}^{(0)}-\sqrt{a_{2} b_{2}(0)} f\left(\xi_{2}(0)\right) d \xi_{2}(0)+\int_{\lambda_{21}^{(0)}-\sqrt{a_{2}} b_{2}(0)}^{\infty} f\left(\xi_{2}(0)\right) d \xi_{2}(0)\right\} \\
& =\frac{1}{2}\left[\int_{-\infty}^{-\lambda_{21}^{(0)}+\sqrt{a_{2}}} f\left(\xi_{2}(0)\right) d \xi_{2}(0)+\int_{-\infty}^{-\lambda_{21}^{(0)}-\sqrt{a_{2}}} f\left(\xi_{2}(0)\right) d \xi_{2}(0)\right. \\
& \left.+\int_{\lambda_{21}^{(0)}+\sqrt{a_{2}}}^{\infty} f\left(\xi_{2}(0)\right) d \xi_{2}(0)+\int_{\lambda_{21}^{(0)}-\sqrt{a_{2}}}^{\infty} f\left(\xi_{2}(0)\right) d \xi_{2}(0)\right] \\
& =\int_{-\infty}^{\lambda_{21}^{(0)}+\sqrt{a_{2}}} f\left(\xi_{2}(0)\right) d \xi_{2}(0)+\int_{\lambda_{21}^{(0)}-\sqrt{a_{2}}}^{\infty} f\left(\xi_{2}(0)\right) d \xi_{2}(0) \\
& =\left[Q\left(\frac{\lambda_{21}^{(0)}+\sqrt{a_{2}}}{\sigma_{\xi_{2}(0)}}\right)+Q\left(\frac{\lambda_{21}^{(0)}-\sqrt{a_{2}}}{\sigma_{\xi_{2}(0)}}\right)\right] .
\end{aligned}
$$

The weights for time interval $i=-1$ results similarly only by substituting $i=0$ by $i=-1$ and $\rho_{12}$ by $\rho_{21}$.

### 2.7.2 Steady State Values of Weights for Linear Clipper

Again, the numerator of Eq. (2.27) can be written as follows:

$$
\begin{aligned}
E\left\{b_{2}(0) \tilde{b}_{2}(0)\right\} & =E\left\{\left.b_{2}(0) \frac{z_{2}(0)}{\lambda_{21}^{(0)}}| | z_{2}(0) \right\rvert\,<\lambda_{21}^{(0)}\right\} \\
& +E\left\{b_{2}(0) \operatorname{sgn}\left(z_{2}(0)\right)| | z_{2}(0) \mid>\lambda_{21}^{(0)}\right\}
\end{aligned}
$$

For $\left|z_{2}(0)\right|>\lambda_{21}^{(0)}$ we get:

$$
E\left\{b_{2}(0) \operatorname{sgn}\left(z_{2}(0)\right)| | z_{2}(0) \mid>\lambda_{21}^{(0)}\right\}
$$

$$
\begin{aligned}
& =\left[\operatorname{Pr}\left\{\tilde{b}_{2}(0)=1 \mid b_{2}(0)=1\right\} \operatorname{Pr}\left\{b_{2}(0)=1\right\}\right. \\
& -\operatorname{Pr}\left\{\tilde{b}_{2}(0)=-1 \mid b_{2}(0)=1\right\} \operatorname{Pr}\left\{b_{2}(0)=1\right\} \\
& -\operatorname{Pr}\left\{\tilde{b}_{2}(0)=1 \mid b_{2}(0)=-1\right\} \operatorname{Pr}\left\{b_{2}(0)=-1\right\} \\
& \left.+\operatorname{Pr}\left\{\tilde{b}_{2}(0)=-1 \mid b_{2}(0)=-1\right\} \operatorname{Pr}\left\{b_{2}(0)=-1\right\}\right]
\end{aligned}
$$

With $\operatorname{Pr}\left\{b_{2}(0)=1\right\}=\frac{1}{2}$ and the fact that the above expression contains pairwise identical terms it can be written as:

$$
\begin{aligned}
E\left\{b_{2}(0) \operatorname{sgn}\left(z_{2}(0)\right)| | z_{2}(0) \mid>\lambda_{21}^{(0)}\right\} & =\left[\operatorname{Pr}\left\{\xi_{2}(0)>\lambda_{21}^{(0)}-\sqrt{a_{2}}\right\}\right. \\
& \left.-\operatorname{Pr}\left\{\xi_{2}(0)>\lambda_{21}^{(0)}+\sqrt{a_{2}}\right\}\right] \\
& =\left[Q\left(\frac{\lambda_{21}^{(0)}-\sqrt{a_{2}}}{\sigma_{\xi_{2}(0)}}\right)-Q\left(\frac{\lambda_{21}^{(0)}+\sqrt{a_{2}}}{\sigma_{\xi_{2}(0)}}\right)\right] .
\end{aligned}
$$

For $\left|z_{2}(0)\right|<\lambda_{21}^{(0)}$ we get:

$$
\begin{align*}
E\left\{\left.b_{2}(0) \frac{z_{2}(0)}{\lambda_{21}^{(0)}}| | z_{2}(0) \right\rvert\,<\lambda_{21}^{(0)}\right\} & =\frac{1}{\lambda_{21}^{(0)}} E\left\{b_{2}(0)\left(\sqrt{a_{2}} b_{2}(0)+\xi_{2}(0)\right)| | z_{2}(0) \mid<\lambda_{21}^{(0)}\right\} \\
& =\frac{1}{\lambda_{21}^{(0)}}\left[E\left\{\sqrt{a_{2}} b_{2}(0)^{2}| | z_{2}(0) \mid<\lambda_{21}^{(0)}\right\}\right. \\
& \left.+E\left\{\xi_{2}(0) b_{2}(0)| | z_{2}(0) \mid<\lambda_{21}^{(0)}\right\}\right] . \tag{2.28}
\end{align*}
$$

The first term of Eq. (2.28) is:

$$
\begin{aligned}
& E\left\{\sqrt{a_{2}} b_{2}(0)^{2}| | z_{2}(0) \mid<\lambda_{21}^{(0)}\right\} \\
= & \sqrt{a_{2}} E\left\{\int_{\left|z_{2}(0)\right|<\lambda_{21}^{(0)}} f\left(z_{2}(0)\right) d z_{2}(0)\right\} \\
= & \sqrt{a_{2}} E\left\{\int_{-\lambda_{21}^{(0)}-\sqrt{a_{2} b_{2}(0)}}^{\lambda_{(0)}^{(0)}-\sqrt{a_{2}} b_{2}(0)} f\left(z_{2}(0)\right) d z_{2}(0)\right\} \\
= & \frac{1}{2} \sqrt{a_{2}}\left[\int_{-\lambda_{21}^{(0)}-\sqrt{a_{2}}}^{\lambda_{21}^{(0)}-\sqrt{a_{2}}} f\left(\xi_{2}(0)\right) d \xi_{2}(0)+\int_{-\lambda_{21}^{(0)}+\sqrt{a_{2}}}^{\lambda_{12}^{(0)}+\sqrt{a_{2}}} f\left(\xi_{2}(0)\right) d \xi_{2}(0)\right] \\
= & \sqrt{a_{2}}\left[Q\left(\frac{-\lambda_{21}^{(0)}+\sqrt{a_{2}}}{\sigma_{\xi_{2}(0)}^{(0)}}\right)-Q\left(\frac{\lambda_{21}^{(0)}+\sqrt{a_{2}}}{\sigma_{\xi_{2}(0)}}\right)\right] .
\end{aligned}
$$

The second term of Eq. (2.28) can be written as:

$$
E\left\{\xi_{2}(0) b_{2}(0)| | z_{2}(0) \mid<\lambda_{21}^{(0)}\right\}
$$

$$
\begin{aligned}
& =E\left\{\int_{-\lambda_{21}(0)}^{\lambda_{21}^{(0)}-\sqrt{a_{2}} b_{2}(0)} b_{2}(0)\right. \\
& =\frac{1}{2}\left[\int_{-\lambda_{21}^{(0)}-\sqrt{a_{2}}}^{\lambda_{21}^{(0)}-\sqrt{a_{2}}} \xi_{2}(0) \xi_{2}(0) f\left(\xi_{2}(0)\right) d \xi_{2}(0)\right\} \\
& \left.-\int_{-\lambda_{21}}^{\lambda_{21}^{(0)}+\sqrt{a_{2}}} \xi_{2}(0) f\left(\xi_{2}(0)\right) d \xi_{2}(0)\right] \\
& =\frac{1}{2 \sqrt{2 \pi} \sigma_{\xi_{2}(0)}}\left[\int_{-\lambda_{21}^{(0)}-\sqrt{a_{2}}}^{\lambda_{21}^{(0)}-\sqrt{a_{2}}} \xi_{2}(0) \exp \left(-\frac{\xi_{2}^{2}(0)}{2 \sigma_{\xi_{2}(0)}^{2}(0)}\right) d \xi_{2}(0)\right. \\
& \left.-\int_{-\lambda_{21}^{(0)}+\sqrt{a_{2}}}^{\lambda_{21}^{(0)}+\sqrt{a_{2}}} \xi_{2}(0) \exp \left(-\frac{\xi_{2}^{2}(0)}{2 \sigma_{\xi_{2}(0)}^{2}}\right) d \xi_{2}(0)\right]
\end{aligned}
$$

where $b_{2}(0) \in\{ \pm 1\}$ and $z_{2}(0)$ within the given limits. Thus:

$$
E\left\{\xi_{2}(0) b_{2}(0)\right\}=\frac{\sigma_{\xi_{2}(0)}}{\sqrt{2 \pi}}\left[\exp \left(-\frac{\left(\lambda_{21}^{(0)}+\sqrt{a_{2}}\right)^{2}}{2 \sigma_{\xi_{2}(0)}^{2}}\right)-\exp \left(-\frac{\left(\lambda_{21}^{(0)}-\sqrt{a_{2}}\right)^{2}}{2 \sigma_{\xi_{2}(0)}^{2}}\right)\right]
$$

The denominator of Eq. (2.27) is given as:

$$
E\left\{\tilde{b}_{2}(0)^{2}\right\}=\frac{1}{\lambda_{21}^{(0)^{2}}} E\left\{z_{2}(0)^{2}| | z_{2}(0) \mid<\lambda_{21}^{(0)}\right\}+E\left\{\tilde{b}_{2}(0)^{2}| | z_{2}(0) \mid>\lambda_{21}^{(0)}\right\}
$$

For $\left|z_{2}(0)\right|>\lambda_{21}^{(0)}$ we get:

$$
\begin{aligned}
& E\left\{\tilde{b}_{2}(0)^{2}| | z_{2}(0) \mid>\lambda_{21}^{(0)}\right\} \\
= & E\left\{\int_{\left|z_{2}(0)\right|>\lambda_{21}^{(0)}} 1 \cdot f\left(z_{2}(0)\right) d z_{2}(0)\right\} \\
= & E\left\{\int_{z_{2}(0)<-\lambda_{21}^{(0)}} f\left(z_{2}(0)\right) d z_{2}(0)+\int_{z_{2}(0)>\lambda_{21}^{(0)}} f\left(z_{2}(0)\right) d z_{2}(0)\right\} \\
= & E\left\{\int_{-\infty}^{-\lambda_{21}^{(0)}-\sqrt{a_{2}} b_{2}(0)} f\left(\xi_{2}(0)\right) d \xi_{2}(0)+\int_{\lambda_{21}^{(0)}-\sqrt{a_{2} b_{2}}(0)}^{\infty} f\left(\xi_{2}(0)\right) d \xi_{2}(0)\right\} \\
= & \frac{1}{2}\left[\int_{-\infty}^{-\lambda_{21}^{(0)}+\sqrt{a_{2}}} f\left(\xi_{2}(0)\right) d \xi_{2}(0)+\int_{-\infty}^{-\lambda_{21}^{(0)}-\sqrt{a_{2}}} f\left(\xi_{2}(0)\right) d \xi_{2}(0)\right. \\
+ & \left.\int_{\lambda_{21}^{(0)}+\sqrt{a_{2}}} f\left(\xi_{2}(0)\right) d \xi_{2}(0)+\int_{\lambda_{21}^{(0)}-\sqrt{a_{2}}} f\left(\xi_{2}(0)\right) d \xi_{2}(0)\right] \\
= & \int_{-\infty}^{-\lambda_{21}^{(0)}} f\left(\xi_{2}(0)\right) d \xi_{2}(0)+\int_{\lambda_{21}^{(0)}}^{\infty} f\left(\xi_{2}(0)\right) d \xi_{2}(0) \\
= & {\left[Q\left(\frac{\lambda_{21}^{(0)}+\sqrt{a_{2}}}{\sigma_{\xi_{2}(0)}}\right)+Q\left(\frac{\lambda_{21}^{(0)}-\sqrt{a_{2}}}{\sigma_{\xi_{2}(0)}}\right)\right] . }
\end{aligned}
$$

For $\left|z_{2}(0)\right|<\lambda_{21}^{(0)}$ we get:

$$
\begin{aligned}
E\left\{z_{2}(0)^{2}| | z_{2}(0) \mid<\lambda_{21}^{(0)}\right\} & =E\left\{a_{2} b_{2}(0)^{2}+2 \sqrt{a_{2}} \xi_{2}(0) b_{2}(0)+\xi_{2}(0)^{2}| | z_{2}(0) \mid<\lambda_{21}^{(0)}\right\} \\
& =a_{2} E\left\{b_{2}(0)^{2}| | z_{2}(0) \mid<\lambda_{21}^{(0)}\right\} \\
& +2 \sqrt{a_{2}} E\left\{\xi_{2}(0) b_{2}(0)| | z_{2}(0) \mid<\lambda_{21}^{(0)}\right\} \\
& +E\left\{\xi_{2}(0)^{2}| | z_{2}(0) \mid<\lambda_{21}^{(0)}\right\}
\end{aligned}
$$

with:

$$
E\left\{b_{2}(0)^{2}| | z_{2}(0) \mid<\lambda_{21}^{(0)}\right\}=\left[Q\left(\frac{-\lambda_{21}^{(0)}+\sqrt{a_{2}}}{\sigma_{\xi_{2}(0)}}\right)-Q\left(\frac{\lambda_{21}^{(0)}-\sqrt{a_{2}}}{\sigma_{\xi_{2}(0)}}\right)\right]
$$

as derived before, and:

$$
\begin{aligned}
& E\left\{\xi_{2}(0) b_{2}(0)| | z_{2}(0) \mid<\lambda_{21}^{(0)}\right\} \\
= & \frac{\sigma_{\xi_{2}(0)}}{\sqrt{2 \pi}}\left[\exp \left(-\frac{\left(\lambda_{21}^{(0)}+\sqrt{a_{2}}\right)^{2}}{2 \sigma_{\xi_{2}(0)}^{2}}\right)-\exp \left(-\frac{\left(\lambda_{21}^{(0)}-\sqrt{a_{2}}\right)^{2}}{2 \sigma_{\xi_{2}(0)}^{2}}\right)\right],
\end{aligned}
$$

furthermore:

$$
\begin{aligned}
E\left\{\xi_{2}(0)^{2}\right\} & =E\left\{\int_{-\lambda_{21}^{(0)}-\sqrt{a_{2}} b_{2}(0)}^{\lambda_{0}^{(0)}-\sqrt{a_{2}} b_{2}(0)} \xi_{2}(0)^{2} f\left(\xi_{2}(0)\right) d \xi_{2}(0)\right\} \\
& =\frac{1}{2}\left[\int_{-\lambda_{21}^{(0)}-\sqrt{a_{2}}}^{\lambda_{21}^{(0)}-\sqrt{a_{2}}} \xi_{2}(0)^{2} f\left(\xi_{2}(0)\right) d \xi_{2}(0)\right. \\
& \left.+\int_{-\lambda_{21}^{(0)}+\sqrt{a_{2}}}^{\lambda_{21}^{(0)}+\sqrt{a_{2}}} \xi_{2}(0)^{2} f\left(\xi_{2}(0)\right) d \xi_{2}(0)\right] \\
& =\int_{-\lambda_{21}^{(0)}+\sqrt{a_{2}}}^{\lambda_{21}^{(0)}+\sqrt{a_{2}}} \xi_{2}(0)^{2} f\left(\xi_{2}(0)\right) d \xi_{2}(0) \\
& =\frac{1}{\sqrt{2 \pi} \sigma_{\xi_{2}(0)}} \int_{-\lambda_{21}^{(0)}+\sqrt{a_{2}}}^{\lambda_{2}^{(0)}+\sqrt{a_{2}}} \xi_{2}(0)^{2} \exp \left(-\frac{\xi_{2}^{2}(0)}{2 \sigma_{\xi_{2}(0)}^{2}}\right) d \xi_{2}(0) \\
& =-\frac{\sigma_{\xi_{2}(0)}}{\sqrt{2 \pi}}\left[\left(\lambda_{21}^{(0)}+\sqrt{a_{2}}\right) \exp \left(-\frac{\left(\lambda_{21}^{(0)}+\sqrt{a_{2}}\right)^{2}}{2 \sigma_{\xi_{2}(0)}^{2}}\right)\right. \\
& \left.-\left(-\lambda_{21}^{(0)}+\sqrt{a_{2}}\right) \exp \left(-\frac{\left(-\lambda_{21}^{(0)}+\sqrt{a_{2}}\right)^{2}}{2 \sigma_{\xi_{2}(0)}^{2}}\right)\right] \\
& +\sigma_{\xi_{2}(0)}^{2}\left[Q\left(\frac{-\lambda_{21}^{(0)}+\sqrt{a_{2}}}{\sigma_{\xi_{2}(0)}}\right)-Q\left(\frac{\lambda_{21}^{(0)}+\sqrt{a_{2}}}{\sigma_{\xi_{2}(0)}}\right)\right]
\end{aligned}
$$

The weights for time interval $i=-1$ result similarly only by substituting $i=0$ by $i=-1$ and $\rho_{12}$ by $\rho_{21}$.

### 2.7.3 Error Probability

An elaborate computation of the output error probability of the multiuser detector using dead-zone limiters and linear clippers is given in Appendix A.

## CHAPTER 3

## PERFORMANCE ANALYSIS

We now want to investigate the performance of the proposed detector, especially its output error probability compared to other detector schemes.

### 3.1 Preliminaries

The determination of the output error probability involved simulations and computations carried out on a SUN SPARC workstation. The simulations assume an already demodulated stream of information bits at the input of the detector. It is obtained from a random generator that provides $\pm 1$ uniformly distributed.


Figure 3.1 Received asynchronous signal; two-user case

To generate the noise at the output of the matched filters and especially the correlated noise at the output of the decorrelator, the partitioning depicted in Figure 3.1 is used. It shows the received waveform for the two-user case with respect to bit 0 of user 1. Those signals can be subdivided into four intervals. For each interval, a random generator generates the corresponding independent white Gaussian noise components $n_{1 R}, n_{1 L}, n_{2 R}$, and $n_{2 L}$.

The signal at the output of the matched filters is obtained by Eq. (2.2), the noise $n_{1}(0)$ is simply the addition of $n_{1 R}$ and $n_{1 L}$.

The reason for partitioning the noise becomes apparent when the noise at the decorrelator output and the corresponding output, Eq. (2.10), have to be simulated. Because of the partitioning, the correlated noise can be obtained by performing the matrix multiplication of $\boldsymbol{H}^{-1}$, the inverse cross-correlation matrix, and the noise components, see Eq. (2.7).

In order to obtain the values for the threshold, the expected values of the decorrelator outputs have to be determined. This is done by time averaging.

This leads us to the outputs of the decision devices according to Eq. (2.11) and the final decision corresponding to Eq. (2.16). An error event is determined by comparing the bit at the detector output with the originally generated bit in each interval i. Appendix E shows the flowchart of the simulation program.

The computation of the error probability is realized according to the equations in Appendix A. In general MATLAB was used. However, for correlated Gaussian random variables we used FORTRAN since we have a FORTRAN subroutine that can handle those cases.

### 3.2 Probability of Error Curves

Since the numerical analysis of the error probability for the $K$-user case (where $K$ is a considerably large number) is computational very intensive, we restrict ourselves to the two-user case. For the purpose of comparison, the decorrelating and the detector using hard tentative decisions are also shown in most cases. In the examples the signal-to-noise ratio for user $k$ is defined as $S N R_{k}=a_{k} / N_{0}$. The first examples, depicted in Figure 3.2 to 3.4 , are related to our discussion in Chapter 2.4, where we introduced soft tentative decisions with the goal of improving the output error probability for low interference compared to the detector using hard tentative decisions only (see Figure 2.3).


Figure 3.2: Error probability of user 1 for $K=2, S N R_{1}=4 d B, \rho_{12}=0.2$, $\rho_{21}=0.6, e_{1}=0.4$


Figure 3.3: Error probability of user 1 for $K=2, S N R_{1}=8 d B, \rho_{12}=0.2$, $\rho_{21}=0.6, e_{1}=0.4$


Figure 3.4: Error probability of user 1 for $K=2, S N R_{1}=12 d B, \rho_{12}=0.2$, $\rho_{21}=0.6, e_{1}=0.4$

The figures show the output error probability of user 1 for several detector schemes versus the relative interference energy for fixed $S N R_{1}$. The decorrelator alone is a straight line since the users are uncorrelated at its output. The detectors using soft limiters bring significant improvement over the one using hard limiters for weak interference, especially as $S N R_{1}$ increases. In general, the multistage detectors clearly outperform the decorrelator and achieve the single-user bound for a relative interference level above $5 d B$.

The next figure, 3.5 , shows the output error probability of user 1 versus its SNR for the two-user case. The given ratio between the energy of users 1 and 2 indicates weak interference. Weak interference means that the conventional detector doesn't fail completely, but it also means that the detector using soft tentative decisions performs better than the one using hard decisions since it diminishes the influence of bad estimates that would otherwise deteriorate the error performance as described in Chapter 2.4. The linear clipper slightly outperforms the dead-zone limiter. Also,
given the single-user bound as a reference, it represents the best possible error performance.


Figure 3.5: Error probability of user 1 for $K=2, \rho_{12}=0.2, \rho_{21}=0.6, e_{1}=0.4$, $a_{2} / a_{1}=0.6$

In the next figure, 3.6, all quantities remain the same except that the ratio between users 1 and 2 has changed. Strong interference is now being considered. Obviously, in this case the conventional detector exhibits its lack of coping with the "near-far" problem. It can also be seen that the detector using hard limiters performs slightly better than that using dead-zone limiters. For strong interference we actually wish to have a hard limiter anyway, since most of our estimations are correct. In fact, the dead-zone limiter seems to omit some of the good estimates. Nevertheless I would still say that the dead-zone limiter is superior because its error performance varies in a smaller range for the changing strength of interference compared to the hard limiter. For an error probability of $10^{-6}$ for example, the hard limiter varies by $4 d B$ from $17 d B$ to $13 d B$ whereas the dead-zone limiter varies only by about $1 d B$. And still the linear clipper performs better than the hard limiter.


Figure 3.6: Error probability of user 1 for $K=2, \rho_{12}=0.2, \rho_{21}=0.6, e_{1}=0.4$, $a_{1} / a_{2}=0.6$

Figures 3.8 and 3.9 give insight into the performance of the detector in a more practical environment where the delay $\tau$ between asynchronous users is subject to changes. They show the average and worst error probability of users 1 versus the SNR of user 1 and 2. The signature sequences are Gold-codes, shown in Figure 3.7. The delay between them is gradually increased from 0 to $T$, the period of the sequences. For each delay the output error probability is computed, and at the end the average and worst error probabilities are determined. In the scenario of Figure 3.8, where the signal energies of users 1 and 2 are equal, dead-zone limiter and linear clipper virtually perform identically, which is why only one curve is drawn for them. Again, the performance of the conventional detector is rather poor, whereas the average error probability of the soft limiters is very close to the single-user bound. Compared to the hard limiter, the curves are not too dissimilar since we didn't choose a scenario with weak interference.



Figure 3.7 Gold-code signature waveforms


Figure 3.8 Error probability of user 1 for $K=2, a_{1}=a_{2}$

This difference becomes more apparent when we consider a scenario with weak interference, as in Figure 3.9. The linear clipper now outperforms the dead-zone limiter above $S N R_{1}=12 d B$ and the gap between soft and hard limiters increased.


Figure 3.9 Error probability of user 1 for $K=2, a_{2} / a_{1}=0.6$

## CHAPTER 4

## CONCLUSION

A multiuser detector in an additive white Gaussian noise, code-division multipleaccess (CDMA) channel was proposed and analyzed. It employed a combination of a decorrelator and a nonlinear multiuser interference canceler utilizing soft tentative decisions and adaptively adjusted weights. Because of the adaptability of the canceler, neither the knowledge of received signals' amplitudes nor training sequences are necessary, while it is assumed that precise relative delay estimates are available for all users.

Emphasis was placed on the influence of soft tentative decisions on the output error performance. The thresholds of those tentative decision nonlinearities, in particular dead-zone limiters and linear clippers, are adjusted by heuristically obtained equations that were shown to be very close to the best threshold under the prospect of minimizing the output error probability. The statistics of the detector were calculated analytically for the general $K$-user case. The superiority of this detector versus the detector using hard tentative decisions was illustrated in figures for the two-user case.

Future effort should investigate other tentative decision nonlinearities such as the multilevel quantizer, look into other possibilities to determine the threshold of the nonlinearities, and attempt to apply other algorithms to adjust the weights of the canceler.

## APPENDIX A

## Error Probability of Detector Using Soft Tentative Decisions

## A. 1 Dead-Zone

The computation of the error probability for the two-user case clarifies the derivation. Without loss of generality, consider only user 1:

$$
\begin{aligned}
P_{e_{1}}= & E_{b_{1}(0), \boldsymbol{b}_{1}(0), \tilde{\boldsymbol{b}}_{1}(0)} \operatorname{Pr}\left\{\hat{b}_{1}(0) \neq b_{1}(0) \mid b_{1}(0), \boldsymbol{b}_{1}(0), \tilde{\boldsymbol{b}}_{1}(0)\right\} \\
= & \frac{1}{2} \sum_{\boldsymbol{b}_{1}(0), \tilde{\boldsymbol{b}}_{1}(0)}\left[\operatorname{Pr}\left\{n_{1}(0)>\sqrt{a_{1}}-\boldsymbol{\rho}_{1}^{T} \boldsymbol{A} \boldsymbol{b}_{1}(0)+\boldsymbol{w}_{1}^{T}(0) \tilde{\boldsymbol{b}}_{1}(0)\right\}\right. \\
+ & \left.\operatorname{Pr}\left\{n_{1}(0)<-\sqrt{a_{1}}-\boldsymbol{\rho}_{1}^{T} \boldsymbol{A} \boldsymbol{b}_{1}(0)+\boldsymbol{w}_{1}^{T}(0) \tilde{\boldsymbol{b}}_{1}(0)\right\}\right] . \\
& \operatorname{Pr}\left\{\tilde{b}_{2}(-1), b_{2}(-1), \tilde{b}_{2}(0), b_{2}(0)\right\} .
\end{aligned}
$$

Random variables in distinct time intervals are not correlated. The joint probability can thus be written as:

$$
\operatorname{Pr}\left\{\tilde{b}_{2}(-1), b_{2}(-1), \tilde{b}_{2}(0), b_{2}(0)\right\}=\operatorname{Pr}\left\{\tilde{b}_{2}(-1), b_{1}(-1)\right\} \operatorname{Pr}\left\{\tilde{b}_{2}(0), b_{2}(0)\right\}
$$

Since all random processes are wide sense stationary, only one joint probability has to be considered in the sequel. Additionally, the above expression contains pairwise identical terms, thus:

$$
\begin{aligned}
P_{e_{1}} & =\sum_{\boldsymbol{b}_{1}(0), \tilde{\boldsymbol{b}}_{1}(0)} \operatorname{Pr}\left\{n_{1}(0)>\sqrt{a_{1}}-\boldsymbol{\rho}_{1}^{T} \boldsymbol{A} \boldsymbol{b}_{1}(0)+\boldsymbol{w}_{1}^{T}(0) \tilde{\boldsymbol{b}}_{1}(0)\right\} \operatorname{Pr}\left\{\tilde{b}_{2}(0), b_{2}(0)\right\}^{2} \\
& =\sum_{\boldsymbol{b}_{1}(0), \tilde{b}_{1}(0)} Q\left(\frac{\sqrt{a_{1}}-\boldsymbol{\rho}_{1}^{T} \boldsymbol{A} \boldsymbol{b}_{1}(0)+\boldsymbol{w}_{1}^{T}(0) \tilde{\boldsymbol{b}}_{1}(0)}{\sqrt{N_{0} / 2}}\right) \operatorname{Pr}\left\{\tilde{b}_{2}(0), b_{2}(0)\right\}^{2}
\end{aligned}
$$

where:

$$
Q(x)=\frac{1}{\sqrt{2 \pi}} \int_{x}^{\infty} e^{-t^{2} / 2} d t
$$

The joint probability $\operatorname{Pr}\left\{\tilde{b}_{1}(0), b_{1}(0)\right\}$ has the following outcomes:
1.

$$
\begin{aligned}
\operatorname{Pr}\left\{\tilde{b}_{2}(0)=1 \mid b_{2}(0)=1\right\} \operatorname{Pr}\left\{b_{2}(0)=1\right\} & =\frac{1}{2} \operatorname{Pr}\left\{g\left(z_{2}(0)\right)=1 \mid b_{1}(0)=1\right\} \\
& =\frac{1}{2} \operatorname{Pr}\left\{\sqrt{a_{2}}+\xi_{2}(0)>\lambda_{21}^{(0)}\right\} \\
& =\frac{1}{2} \operatorname{Pr}\left\{\xi_{2}(0)>\lambda_{21}^{(0)}-\sqrt{a_{2}}\right. \\
& =\frac{1}{2} Q\left(\frac{\lambda_{21}^{(0)}-\sqrt{a_{2}}}{\sigma_{\xi_{2}(0)}}\right)
\end{aligned}
$$

where $g(\cdot)$ denotes the dead-zone limiter.
2.

$$
\operatorname{Pr}\left\{\tilde{b}_{2}(0)=1 \mid b_{2}(0)=-1\right\} \operatorname{Pr}\left\{b_{2}(0)=-1\right\}=\frac{1}{2} Q\left(\frac{\lambda_{21}^{(0)}+\sqrt{a_{2}}}{\sigma_{\xi_{2}(0)}}\right)
$$

3. 

$$
\begin{aligned}
& \operatorname{Pr}\left\{\tilde{b}_{2}(0)=0 \mid b_{2}(0)=1\right\} \operatorname{Pr}\left\{b_{2}(0)=1\right\} \\
= & \frac{1}{2} \operatorname{Pr}\left\{-\lambda_{21}^{(0)}<\sqrt{a_{2}}+\xi_{2}(0)<\lambda_{21}^{(0)}\right\} \\
= & \frac{1}{2}\left[Q\left(\frac{-\lambda_{21}^{(0)}-\sqrt{a_{2}}}{\sigma_{\xi_{2}(0)}}\right)-Q\left(\frac{\lambda_{21}^{(0)}-\sqrt{a_{2}}}{\sigma_{\xi_{2}(0)}}\right)\right] .
\end{aligned}
$$

4. 

$$
\begin{aligned}
& \operatorname{Pr}\left\{\tilde{b}_{2}(0)=0 \mid b_{2}(0)=-1\right\} \operatorname{Pr}\left\{b_{2}(0)=-1\right\} \\
= & \frac{1}{2} \operatorname{Pr}\left\{-\lambda_{21}^{(0)}<-\sqrt{a_{2}}+\xi_{2}(0)<\lambda_{21}^{(0)}\right\} \\
= & \frac{1}{2}\left[Q\left(\frac{-\lambda_{21}^{(0)}+\sqrt{a_{2}}}{\sigma_{\xi_{2}(0)}}\right)-Q\left(\frac{\lambda_{21}^{(0)}+\sqrt{a_{2}}}{\sigma_{\xi_{2}(0)}}\right)\right] .
\end{aligned}
$$

5. 

$$
\operatorname{Pr}\left\{\tilde{b}_{2}(0)=-1 \mid b_{2}(0)=1\right\} \operatorname{Pr}\left\{b_{2}(0)=1\right\}=\text { see } 2
$$

6. 

$$
\operatorname{Pr}\left\{\tilde{b}_{2}(0)=-1 \mid b_{2}(0)=-1\right\} \operatorname{Pr}\left\{b_{2}(0)=-1\right\}=\text { see } 1
$$

Thus, the error probability is:

$$
\begin{aligned}
& P_{e 1}=\left[Q\left(\frac{\sqrt{a_{1}}+\sqrt{a_{2}}\left(-\rho_{12}-\rho_{21}\right)+w_{21}^{(0)}+w_{21}^{(-1)}}{\sqrt{N_{0} / 2}}\right)\right. \\
& +Q\left(\frac{\sqrt{a_{1}}+\sqrt{a_{2}}\left(-\rho_{12}+\rho_{21}\right)-w_{21}^{(0)}+w_{21}^{(-1)}}{\sqrt{N_{0} / 2}}\right) \\
& +Q\left(\frac{\sqrt{a_{1}}+\sqrt{a_{2}}\left(\rho_{12}-\rho_{21}\right)+w_{21}^{(0)}-w_{21}^{(-1)}}{\sqrt{N_{0} / 2}}\right) \\
& \left.+Q\left(\frac{\sqrt{a_{1}}+\sqrt{a_{2}}\left(\rho_{12}+\rho_{21}\right)-w_{21}^{(0)}-w_{21}^{(-1)}}{\sqrt{N_{0} / 2}}\right)\right] \cdot \frac{1}{4} Q\left(\frac{\lambda_{21}^{(0)}-\sqrt{a_{2}}}{\sigma_{\xi_{2}(0)}}\right)^{2} \\
& +\left[Q\left(\frac{\sqrt{a_{1}}+\sqrt{a_{2}}\left(-\rho_{12}-\rho_{21}\right)+w_{21}^{(-1)}}{\sqrt{N_{0} / 2}}\right)\right. \\
& +Q\left(\frac{\sqrt{a_{1}}+\sqrt{a_{2}}\left(-\rho_{12}-\rho_{21}\right)+w_{21}^{(0)}}{\sqrt{N_{0} / 2}}\right) \\
& +Q\left(\frac{\sqrt{a_{1}}+\sqrt{a_{2}}\left(-\rho_{12}+\rho_{21}\right)-w_{21}^{(0)}}{\sqrt{N_{0} / 2}}\right) \\
& \left.+Q\left(\frac{\sqrt{a_{1}}+\sqrt{a_{2}}\left(\rho_{12}-\rho_{21}\right)-w_{21}^{(-1)}}{\sqrt{N_{0} / 2}}\right)\right] \\
& \cdot \frac{1}{4} Q\left(\frac{\lambda_{21}^{(0)}-\sqrt{a_{2}}}{\sigma_{\xi_{2}(0)}}\right)\left[Q\left(\frac{-\lambda_{21}^{(0)}-\sqrt{a_{2}}}{\sigma_{\xi_{2}(0)}}\right)-Q\left(\frac{\lambda_{21}^{(0)}-\sqrt{a_{2}}}{\sigma_{\xi_{2}(0)}}\right)\right] \\
& +\left[Q\left(\frac{\sqrt{a_{1}}+\sqrt{a_{2}}\left(-\rho_{12}-\rho_{21}\right)-w_{21}^{(0)}+w_{21}^{(-1)}}{\sqrt{N_{0} / 2}}\right)\right. \\
& +Q\left(\frac{\sqrt{a_{1}}+\sqrt{a_{2}}\left(-\rho_{12}+\rho_{21}\right)+w_{21}^{(0)}+w_{21}^{(-1)}}{\sqrt{N_{0} / 2}}\right) \\
& +Q\left(\frac{\sqrt{a_{1}}+\sqrt{a_{2}}\left(-\rho_{12}-\rho_{21}\right)+w_{21}^{(0)}-w_{21}^{(-1)}}{\sqrt{N_{0} / 2}}\right) \\
& +Q\left(\frac{\sqrt{a_{1}}+\sqrt{a_{2}}\left(-\rho_{12}+\rho_{21}\right)-w_{21}^{(0)}-w_{21}^{(-1)}}{\sqrt{N_{0} / 2}}\right) \\
& +Q\left(\frac{\sqrt{a_{1}}+\sqrt{a_{2}}\left(\rho_{12}-\rho_{21}\right)+w_{21}^{(0)}+w_{21}^{(-1)}}{\sqrt{N_{0} / 2}}\right) \\
& +Q\left(\frac{\sqrt{a_{1}}+\sqrt{a_{2}}\left(\rho_{12}+\rho_{21}\right)-w_{21}^{(0)}+w_{21}^{(-1)}}{\sqrt{N_{0} / 2}}\right)
\end{aligned}
$$

$$
\begin{aligned}
& +Q\left(\frac{\sqrt{a_{1}}+\sqrt{a_{2}}\left(\rho_{12}-\rho_{21}\right)-w_{21}^{(0)}-w_{21}^{(-1)}}{\sqrt{N_{0} / 2}}\right) \\
& \left.+Q\left(\frac{\sqrt{a_{1}}+\sqrt{a_{2}}\left(\rho_{12}+\rho_{21}\right)+w_{21}^{(0)}-w_{21}^{(-1)}}{\sqrt{N_{0} / 2}}\right)\right] \\
& \text { - } \frac{1}{4} Q\left(\frac{\lambda_{21}^{(0)}-\sqrt{a_{2}}}{\sigma_{\xi_{2}(0)}}\right) Q\left(\frac{\lambda_{21}^{(0)}+\sqrt{a_{2}}}{\sigma_{\xi_{2}(0)}}\right) \\
& +\left[Q\left(\frac{\sqrt{a_{1}}+\sqrt{a_{2}}\left(-\rho_{12}+\rho_{21}\right)+w_{21}^{(-1)}}{\sqrt{N_{0} / 2}}\right)\right. \\
& +Q\left(\frac{\sqrt{a_{1}}+\sqrt{a_{2}}\left(\rho_{12}-\rho_{21}\right)+w_{21}^{(0)}}{\sqrt{N_{0} / 2}}\right) \\
& +Q\left(\frac{\sqrt{a_{1}}+\sqrt{a_{2}}\left(\rho_{12}+\rho_{21}\right)-w_{21}^{(0)}}{\sqrt{N_{0} / 2}}\right) \\
& \left.+Q\left(\frac{\sqrt{a_{1}}+\sqrt{a_{2}}\left(\rho_{12}+\rho_{21}\right)-w_{21}^{(-1)}}{\sqrt{N_{0} / 2}}\right)\right] \\
& \cdot \frac{1}{4} Q\left(\frac{\lambda_{21}^{(0)}-\sqrt{a_{2}}}{\sigma_{\xi_{2}(0)}}\right)\left[Q\left(\frac{-\lambda_{21}^{(0)}+\sqrt{a_{2}}}{\sigma_{\xi_{2}(0)}}\right)-Q\left(\frac{\lambda_{21}^{(0)}+\sqrt{a_{2}}}{\sigma_{\xi_{2}(0)}}\right)\right] \\
& +Q\left(\frac{\sqrt{a_{1}}+\sqrt{a_{2}}\left(-\rho_{12}-\rho_{21}\right)}{\sqrt{N_{0} / 2}}\right) \\
& \text { - } \frac{1}{4}\left[Q\left(\frac{-\lambda_{21}^{(0)}-\sqrt{a_{2}}}{\sigma_{\xi_{2}(0)}}\right)-Q\left(\frac{\lambda_{21}^{(0)}-\sqrt{a_{2}}}{\sigma_{\xi_{2}(0)}}\right)\right]^{2} \\
& +\left[Q\left(\frac{\sqrt{a_{1}}+\sqrt{a_{2}}\left(-\rho_{12}-\rho_{21}\right)-w_{21}^{(0)}}{\sqrt{N_{0} / 2}}\right)\right. \\
& +Q\left(\frac{\sqrt{a_{1}}+\sqrt{a_{2}}\left(-\rho_{12}+\rho_{21}\right)+w_{21}^{(0)}}{\sqrt{N_{0} / 2}}\right) \\
& +Q\left(\frac{\sqrt{a_{1}}+\sqrt{a_{2}}\left(-\rho_{12}-\rho_{21}\right)-w_{21}^{(-1)}}{\sqrt{N_{0} / 2}}\right) \\
& \left.+Q\left(\frac{\sqrt{a_{1}}+\sqrt{a_{2}}\left(\rho_{12}-\rho_{21}\right)+w_{21}^{(-1)}}{\sqrt{N_{0} / 2}}\right)\right] \\
& \cdot \frac{1}{4} Q\left(\frac{\lambda_{21}^{(0)}+\sqrt{a_{2}}}{\sigma_{\xi_{2}(0)}}\right)\left[Q\left(\frac{-\lambda_{21}^{(0)}+\sqrt{a_{2}}}{\sigma_{\xi_{2}(0)}}\right)-Q\left(\frac{\lambda_{21}^{(0)}+\sqrt{a_{2}}}{\sigma_{\xi_{2}(0)}}\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
& +\left[Q\left(\frac{\sqrt{a_{1}}+\sqrt{a_{2}}\left(-\rho_{12}+\rho_{21}\right)}{\sqrt{N_{0} / 2}}\right)\right. \\
& \left.+Q\left(\frac{\sqrt{a_{1}}+\sqrt{a_{2}}\left(\rho_{12}-\rho_{21}\right)}{\sqrt{N_{0} / 2}}\right)\right] \\
& -\frac{1}{4}\left[Q\left(\frac{-\lambda_{21}^{(0)}+\sqrt{a_{2}}}{\sigma_{\xi_{2}(0)}}\right)-Q\left(\frac{\lambda_{21}^{(0)}+\sqrt{a_{2}}}{\sigma_{\xi_{2}(0)}}\right)\right] \\
& \cdot\left[Q\left(\frac{-\lambda_{21}^{(0)}+\sqrt{a_{2}}}{\sigma_{\xi_{2}(0)}}\right)-Q\left(\frac{\lambda_{21}^{(0)}+\sqrt{a_{2}}}{\sigma_{\xi_{2}(0)}}\right)\right] \\
& +\left[Q\left(\frac{\sqrt{a_{1}}+\sqrt{a_{2}}\left(-\rho_{12}-\rho_{21}\right)-w_{21}^{(0)}-w_{21}^{(-1)}}{\sqrt{N_{0} / 2}}\right)\right. \\
& +Q\left(\frac{\sqrt{a_{1}}+\sqrt{a_{2}}\left(-\rho_{12}+\rho_{21}\right)+w_{21}^{(0)}-w_{21}^{(-1)}}{\sqrt{N_{0} / 2}}\right) \\
& +Q\left(\frac{\sqrt{a_{1}}+\sqrt{a_{2}}\left(\rho_{12}-\rho_{21}\right)-w_{21}^{(0)}+w_{21}^{(-1)}}{\sqrt{N_{0} / 2}}\right) \\
& \left.+Q\left(\frac{\sqrt{a_{1}}+\sqrt{a_{2}}\left(\rho_{12}+\rho_{21}\right)+w_{21}^{(0)}+w_{21}^{(-1)}}{\sqrt{N_{0} / 2}}\right)\right] \frac{1}{4} Q\left(\frac{\lambda_{21}^{(0)}+\sqrt{a_{2}}}{\sigma_{\xi_{2}(0)}}\right)^{2} \\
& +\left[Q\left(\frac{\sqrt{a_{1}}+\sqrt{a_{2}}\left(\rho_{12}+\rho_{21}\right)+w_{21}^{(-1)}}{\sqrt{N_{0} / 2}}\right)\right. \\
& +Q\left(\frac{\sqrt{a_{1}}+\sqrt{a_{2}}\left(-\rho_{12}+\rho_{21}\right)-w_{21}^{(-1)}}{\sqrt{N_{0} / 2}}\right) \\
& +Q\left(\frac{\sqrt{a_{1}}+\sqrt{a_{2}}\left(\rho_{12}-\rho_{21}\right)-w_{21}^{(0)}}{\sqrt{N_{0} / 2}}\right) \\
& \left.+Q\left(\frac{\sqrt{a_{1}}+\sqrt{a_{2}}\left(\rho_{12}+\rho_{21}\right)+w_{21}^{(0)}}{\sqrt{N_{0} / 2}}\right)\right] \\
& \cdot \frac{1}{4} Q\left(\frac{\lambda_{21}^{(0)}+\sqrt{a_{2}}}{\sigma_{\xi_{2}(0)}}\right)\left[Q\left(\frac{-\lambda_{21}^{(0)}+\sqrt{a_{2}}}{\sigma_{\xi_{2}(0)}}\right)-Q\left(\frac{\lambda_{21}^{(0)}+\sqrt{a_{2}}}{\sigma_{\xi_{2}(0)}}\right)\right] \\
& +Q\left(\frac{\sqrt{a_{1}}+\sqrt{a_{2}}\left(\rho_{12}+\rho_{21}\right)}{\sqrt{N_{0} / 2}}\right) \\
& \cdot \frac{1}{4}\left[Q\left(\frac{-\lambda_{21}^{(0)}+\sqrt{a_{2}}}{\sigma_{\xi_{2}(0)}}\right)-Q\left(\frac{\lambda_{21}^{(0)}+\sqrt{a_{2}}}{\sigma_{\xi_{2}(0)}}\right)\right]^{2} .
\end{aligned}
$$

## A. 2 Linear Clipper

The computation of the error probability for the two-user case clarifies the derivation. Without loss of generality, consider only user 1 :

$$
\begin{aligned}
P_{e_{1}} & =E_{b_{1}(0), \boldsymbol{b}_{1}(0), \tilde{\boldsymbol{b}}_{1}(0)} \operatorname{Pr}\left\{\hat{b}_{1}(0) \neq b_{1}(0) \mid b_{1}(0), \boldsymbol{b}_{1}(0), \tilde{\boldsymbol{b}}_{1}(0)\right\} \\
& =\frac{1}{2} \sum_{\boldsymbol{b}_{1}(0), \tilde{\boldsymbol{b}}_{1}(0)}\left[\operatorname{Pr}\left\{n_{1}(0)>\sqrt{a_{1}}-\boldsymbol{\rho}_{1}^{T} \boldsymbol{A} \boldsymbol{b}_{1}(0)+\boldsymbol{w}_{1}^{T}(0) \tilde{\boldsymbol{b}}_{1}(0) \mid \tilde{\boldsymbol{b}}_{1}(0)\right\}\right. \\
& \left.+\operatorname{Pr}\left\{n_{1}(0)<-\sqrt{a_{\mathbf{1}}}-\boldsymbol{\rho}_{1}^{T} \boldsymbol{A} \boldsymbol{b}_{1}(0)+\boldsymbol{w}_{1}^{T}(0) \tilde{\boldsymbol{b}}_{1}(0) \mid \tilde{\boldsymbol{b}}_{1}(0)\right\}\right] \operatorname{Pr}\left\{\boldsymbol{b}_{1}(0)\right\} .
\end{aligned}
$$

Since $\operatorname{Pr}\left\{\boldsymbol{b}_{1}(0)\right\}=1 / 4$, and the above expression contains pairwise identical terms, it can be written as:

$$
\begin{aligned}
& P_{e_{1}} \\
= & \frac{1}{4} \sum_{\boldsymbol{b}_{1}(0)}\left[\operatorname { P r } \left\{n_{1}(0)>\sqrt{a_{1}}-\boldsymbol{\rho}_{1}^{T} \boldsymbol{A} \boldsymbol{b}_{1}(0)+\boldsymbol{w}_{1}^{T}(0) \tilde{\boldsymbol{b}}_{1}(0),\left|z_{2}(0)\right|<\lambda_{21}^{(0)},\right.\right. \\
& \left.\left|z_{2}(-1)\right|<\lambda_{21}^{(-1)}\right\} \\
+ & \operatorname{Pr}\left\{n_{1}(0)>\sqrt{a_{1}}-\boldsymbol{\rho}_{1}^{T} \boldsymbol{A} \boldsymbol{b}_{1}(0)+\boldsymbol{w}_{1}^{T}(0) \tilde{\boldsymbol{b}}_{1}(0),\left|z_{2}(0)\right|<\lambda_{21}^{(0)},\left|z_{2}(-1)\right|>\lambda_{21}^{(-1)}\right\} \\
+ & \operatorname{Pr}\left\{n_{1}(0)>\sqrt{a_{1}}-\boldsymbol{\rho}_{1}^{T} \boldsymbol{A} \boldsymbol{b}_{1}(0)+\boldsymbol{w}_{1}^{T}(0) \tilde{\boldsymbol{b}}_{1}(0),\left|z_{2}(0)\right|>\lambda_{21}^{(0)},\left|z_{2}(-1)\right|<\lambda_{21}^{(-1)}\right\} \\
+ & \left.\operatorname{Pr}\left\{n_{1}(0)>\sqrt{a_{1}}-\boldsymbol{\rho}_{1}^{T} \boldsymbol{A} \boldsymbol{b}_{1}(0)+\boldsymbol{w}_{1}^{T}(0) \tilde{\boldsymbol{b}}_{1}(0),\left|z_{2}(0)\right|>\lambda_{21}^{(0)},\left|z_{2}(-1)\right|>\lambda_{21}^{(-1)}\right\}\right],
\end{aligned}
$$

where:

$$
\begin{gathered}
z_{2}(0)=\sqrt{a_{2}} b_{2}(0)+\xi_{2}(0), \\
z_{2}(-1)=\sqrt{a_{2}} b_{2}(-1)+\xi_{2}(-1)
\end{gathered}
$$

and:

$$
\begin{gathered}
\sigma_{\xi_{2}(0)}^{2}=\frac{1}{1-\frac{\rho_{12}^{2}}{e_{1}}-\frac{\rho_{21}^{2}}{1-e_{1}}} . \\
P_{e_{1}}=\frac{1}{4} \sum_{\boldsymbol{b}_{1}(0)}\left[\operatorname { P r } \left\{\psi_{1}(0)>\sqrt{a_{1}}-\boldsymbol{\rho}_{1}^{T} \boldsymbol{A} \boldsymbol{b}_{1}(0)+\sqrt{a_{2}}\left[\frac{w_{21}^{(0)}}{\lambda_{21}^{(0)}} b_{2}(0)+\frac{w_{21}^{(-1)}}{\lambda_{21}^{(-1)}} b_{2}(-1)\right]\right.\right. \\
\left.\left|\sqrt{a_{2}} b_{2}(0)+\xi_{2}(0)\right|<\lambda_{21}^{(0)},\left|\sqrt{a_{2}} b_{2}(-1)+\xi_{2}(-1)\right|<\lambda_{21}^{(-1)}\right\}
\end{gathered}
$$

$$
\begin{aligned}
+ & \operatorname{Pr}\left\{\psi_{2}(0)>\sqrt{a_{1}}-\boldsymbol{\rho}_{1}^{T} \boldsymbol{A} \boldsymbol{b}_{1}(0)+\frac{w_{21}^{(0)}}{\lambda_{21}^{(0)}} \sqrt{a_{2}} b_{2}(0)+w_{21}^{(-1)},\right. \\
& \left.\left|\sqrt{a_{2}} b_{2}(0)+\xi_{2}(0)\right|<\lambda_{21}^{(0)},\left|\sqrt{a_{2}} b_{2}(-1)+\xi_{2}(-1)\right|>\lambda_{21}^{(-1)}\right\} \\
+\quad & \operatorname{Pr}\left\{\psi_{3}(0)>\sqrt{a_{1}}-\boldsymbol{\rho}_{1}^{T} \boldsymbol{A} \boldsymbol{b}_{1}(0)+w_{21}^{(0)}+\frac{w_{21}^{(-1)}}{\lambda_{21}^{(-1)}} \sqrt{a_{2}} b_{2}(-1),\right. \\
& \left.\left|\sqrt{a_{2}} b_{2}(0)+\xi_{2}(0)\right|>\lambda_{21}^{(0)},\left|\sqrt{a_{2}} b_{2}(-1)+\xi_{2}(-1)\right|<\lambda_{21}^{(-1)}\right\} \\
+ & \operatorname{Pr}\left\{\psi_{4}(0)>\sqrt{a_{1}}-\boldsymbol{\rho}_{1}^{T} \boldsymbol{A} \boldsymbol{b}_{1}(0)+w_{21}^{(0)}+w_{21}^{(-1)},\right. \\
& \left.\left.\left|\sqrt{a_{2}} b_{2}(0)+\xi_{2}(0)\right|>\lambda_{21}^{(0)},\left|\sqrt{a_{2}} b_{2}(-1)+\xi_{2}(-1)\right|>\lambda_{21}^{(-1)}\right\}\right] .
\end{aligned}
$$

Where the noise terms are the following zero mean Gaussian random variables:

$$
\begin{aligned}
& \psi_{1}=n_{1}(0)-\frac{w_{21}^{(0)}}{\lambda_{21}^{(0)}} \xi_{2}(0)-\frac{w_{21}^{(-1)}}{\lambda_{21}^{(-1)}} \xi_{2}(-1) \\
& \psi_{2}=n_{1}(0)-\frac{w_{21}^{(0)}}{\lambda_{21}^{(0)}} \xi_{2}(0) \\
& \psi_{3}=n_{1}(0)-\frac{w_{21}^{(-1)}}{\lambda_{21}^{(-1)}} \xi_{2}(-1) \\
& \psi_{4}=n_{1}(0), \\
& P_{e_{1}}=\frac{1}{4} \sum_{\boldsymbol{b}_{1}(0)}\left[\operatorname { P r } \left\{\psi_{1}(0)>\sqrt{a_{1}}-\boldsymbol{\rho}_{1}^{T} \boldsymbol{A} \boldsymbol{b}_{1}(0)+\sqrt{a_{2}}\left[\frac{w_{21}^{(0)}}{\lambda_{21}^{(0)}} b_{2}(0)+\frac{w_{21}^{(-1)}}{\lambda_{21}^{(-1)}} b_{2}(-1)\right],\right.\right. \\
& -\lambda_{21}^{(0)}-\sqrt{a_{2}} b_{2}(0)<\xi_{2}(0)<\lambda_{21}^{(0)}-\sqrt{a_{2}} b_{2}(0), \\
& \left.-\lambda_{21}^{(-1)}-\sqrt{a_{2}} b_{2}(-1)<\xi_{2}(-1)<\lambda_{21}^{(-1)}-\sqrt{a_{2}} b_{2}(-1)\right\} \\
& +\operatorname{Pr}\left\{\psi_{2}(0)>\sqrt{a_{1}}-\boldsymbol{\rho}_{1}^{T} \boldsymbol{A} \boldsymbol{b}_{1}(0)+\frac{w_{21}^{(0)}}{\lambda_{21}^{(0)}} \sqrt{a_{2}} b_{2}(0)+w_{21}^{(-1)},\right. \\
& -\lambda_{21}^{(0)}-\sqrt{a_{2}} b_{2}(0)<\xi_{2}(0)<\lambda_{21}^{(0)}-\sqrt{a_{2}} b_{2}(0), \\
& \left.\xi_{2}(-1)<-\lambda_{21}^{(-1)}-\sqrt{a_{2}} b_{2}(-1) \quad \text { or } \quad \xi_{2}(-1)>\lambda_{21}^{(-1)}-\sqrt{a_{2}} b_{2}(-1)\right\} \\
& +\operatorname{Pr}\left\{\psi_{3}(0)>\sqrt{a_{1}}-\boldsymbol{\rho}_{1}^{T} \boldsymbol{A} \boldsymbol{b}_{1}(0)+w_{21}^{(0)}+\frac{w_{21}^{(-1)}}{\lambda_{21}^{(-1)}} \sqrt{a_{2}} b_{2}(-1),\right. \\
& \xi_{2}\left(0<-\lambda_{21}^{(0)}-\sqrt{a_{2}} b_{2}(0) \quad \text { or } \quad \xi_{2}(0)>\lambda_{21}^{(0)}-\sqrt{a_{2}} b_{2}(0),\right. \\
& \left.-\lambda_{21}^{(-1)}-\sqrt{a_{2}} b_{2}(-1)<\xi_{2}(-1)<\lambda_{21}^{(-1)}-\sqrt{a_{2}} b_{2}(-1)\right\} \\
& +\operatorname{Pr}\left\{\psi_{4}(0)>\sqrt{a_{1}}-\boldsymbol{\rho}_{1}^{T} \boldsymbol{A} \boldsymbol{b}_{1}(0)+w_{21}^{(0)}+w_{21}^{(-1)},\right. \\
& \xi_{2}\left(0<-\lambda_{21}^{(0)}-\sqrt{a_{2}} b_{2}(0) \quad \text { or } \quad \xi_{2}(0)>\lambda_{21}^{(0)}-\sqrt{a_{2}} b_{2}(0),\right.
\end{aligned}
$$

$$
\left.\left.\xi_{2}(-1)<-\lambda_{21}^{(-1)}-\sqrt{a_{2}} b_{2}(-1) \quad \text { or } \quad \xi_{2}(-1)>\lambda_{21}^{(-1)}-\sqrt{a_{2}} b_{2}(-1)\right\}\right]
$$

Solving this equation for all possible $\boldsymbol{b}_{1}(0)=\left[b_{2}(-1), b_{2}(0)\right]^{T}$ and $b_{2}(i) \in\{ \pm 1\}$ leads to sixteen terms that are omitted here for brevity's sake.

$$
\begin{aligned}
P_{e_{1}} & =\frac{1}{4}\left[\sum_{i=1}^{4} \iiint_{D_{i}} f_{\psi_{1}, \xi_{2}(0), \xi_{2}(-1)}\left(\psi_{1}, \xi_{2}(0), \xi_{2}(-1)\right) d \psi_{1} d \xi_{2}(0) d \xi_{2}(-1)+\right. \\
& +\sum_{i=1}^{4} \iiint_{M_{i}} f_{\psi_{2}, \xi_{2}(0), \xi_{2}(-1)}\left(\psi_{2}, \xi_{2}(0), \xi_{2}(-1)\right) d \psi_{2} d \xi_{2}(0) d \xi_{2}(-1)+ \\
& +\sum_{i=1}^{4} \iiint_{S_{i}} f_{\psi_{3}, \xi_{2}(0), \xi_{2}(-1)}\left(\psi_{3}, \xi_{2}(0), \xi_{2}(-1)\right) d \psi_{3} d \xi_{2}(0) d \xi_{2}(-1)+ \\
& \left.+\sum_{i=1}^{4} \iiint_{R_{i}} f_{\psi_{4}, \xi_{2}(0), \xi_{2}(-1)}\left(\psi_{4}, \xi_{2}(0), \xi_{2}(-1)\right) d \psi_{4} d \xi_{2}(0) d \xi_{2}(-1)\right]
\end{aligned}
$$

where the three-dimensional zero-mean jointly Gaussian random variables can be computed as follows:

$$
f_{\psi_{i}, \xi_{2}(0), \xi_{2}(-1)}\left(\psi_{i}, \xi_{2}(0), \xi_{2}(-1)\right)=\frac{\exp \left(-\frac{1}{2} \boldsymbol{x}^{T} K^{-1} \boldsymbol{x}\right)}{\left.\left.(2 \pi)^{\frac{3}{2}} \right\rvert\, K\right)^{\frac{1}{2}}}
$$

and

$$
K=\left[\begin{array}{ccc}
1 & \rho_{\psi_{i}, \xi_{2}(0)} & \rho_{\psi_{i}, \xi_{2}(-1)} \\
\rho_{\psi_{i}, \xi_{2}(0)} & 1 & 0 \\
\rho_{\psi_{i}, \xi_{2}(-1)} & 0 & 1
\end{array}\right], \quad \boldsymbol{x}=\left[\begin{array}{c}
\psi_{i} \\
\xi_{2}(0) \\
\xi_{2}(-1)
\end{array}\right]
$$

Again under the constraint of brevity the regions of integration are:

$$
\begin{aligned}
D_{i}: & {\left[\psi_{1}>\sqrt{a_{1}}+\rho_{12} \sqrt{a_{2}}+\rho_{21} \sqrt{a_{2}}-\frac{w_{21}^{(0)}}{\lambda_{21}^{(0)}} \sqrt{a_{2}}-\frac{w_{21}^{(-1)}}{\lambda_{21}^{(-1)}} \sqrt{a_{2}},\right.} \\
& -\lambda_{21}^{(0)}-\sqrt{a_{2}} b_{2}(0)<\xi_{2}(0)<\lambda_{21}^{(0)}-\sqrt{a_{2}} b_{2}(0), \\
& -\lambda_{21}^{(-1)}-\sqrt{a_{2}} b_{2}(0)<\xi_{2}(-1)<\lambda_{21}^{(-1)}-\sqrt{\left.a_{2} b_{2}(0)\right]} \\
M_{i}: \quad & {\left[\psi_{2}>\sqrt{a_{1}}+\rho_{12} \sqrt{a_{2}}+\rho_{21} \sqrt{a_{2}}-\frac{w_{21}^{(0)}}{\lambda_{21}^{(0)}} \sqrt{a_{2}}-\frac{w_{21}^{(-1)}}{\lambda_{21}^{((1)}} \sqrt{a_{2}},\right.} \\
& -\lambda_{21}^{(0)}+\sqrt{a_{2} b_{2}(0)<\xi_{2}(0)<\lambda_{21}^{(0)}+\sqrt{a_{2}}, b_{2}(0)} \\
& \left.\xi_{2}(-1)<-\lambda_{21}^{(-1)}-\sqrt{a_{2}} b_{2}(0) \quad \text { or } \quad \xi_{2}(-1)>\lambda_{21}^{(-1)}-\sqrt{a_{2}} b_{2}(0)\right]
\end{aligned}
$$

$$
\begin{aligned}
& S_{i}: {\left[\psi_{3}>\sqrt{a_{1}}+\rho_{12} \sqrt{a_{2}}+\rho_{21} \sqrt{a_{2}}-\frac{w_{21}^{(0)}}{\lambda_{21}^{(0)}} \sqrt{a_{2}}-\frac{w_{21}^{(-1)}}{\lambda_{21}^{(-1)}} \sqrt{a_{2}},\right.} \\
& \xi_{2}(0)<-\lambda_{21}^{(0)}-\sqrt{a_{2}} b_{2}(0) \quad \text { or } \quad \xi_{2}(0)>\lambda_{21}^{(0)}-\sqrt{a_{2}} b_{2}(0), \\
&\left.-\lambda_{21}^{(-1)}-\sqrt{a_{2}} b_{2}(0)<\xi_{2}(-1)<\lambda_{21}^{(-1)}-\sqrt{a_{2}} b_{2}(0)\right] \\
& R_{i}: \quad\left[\psi_{4}>\sqrt{a_{1}}+\rho_{12} \sqrt{a_{2}}+\rho_{21} \sqrt{a_{2}}-\frac{w_{21}^{(0)}}{\lambda_{21}^{(0)}} \sqrt{a_{2}}-\frac{w_{21}^{(-1)}}{\lambda_{21}^{(-1)}} \sqrt{a_{2}},\right. \\
& \xi_{2}(0)<-\lambda_{21}^{(0)}-\sqrt{a_{2}} b_{2}(0) \quad \text { or } \quad \xi_{2}(0)>\lambda_{21}^{(0)}-\sqrt{a_{2}} b_{2}(0), \\
&\left.\xi_{2}(-1)<-\lambda_{21}^{(-1)}-\sqrt{a_{2}} b_{2}(0) \quad \text { or } \quad \xi_{2}(-1)>\lambda_{21}^{(-1)}-\sqrt{a_{2}} b_{2}(0)\right]
\end{aligned}
$$

## APPENDIX B

## Pairwise Identical Terms

$$
\begin{aligned}
& \operatorname{Pr}\left\{\tilde{\boldsymbol{b}}_{1}(0) \mid \boldsymbol{b}_{1}(0)\right\} \\
= & \operatorname{Pr}\left\{g_{2}^{(-1)}\left(z_{2}(-1)\right), \ldots, g_{K}^{(-1)}\left(z_{K}(-1)\right), g_{2}^{(0)}\left(z_{2}(0)\right), \ldots, g_{K}^{(0)}\left(z_{K}(0)\right)\right\} .
\end{aligned}
$$

Without loss of generality consider only one random variable:

$$
g_{k}^{(i)}\left(z_{k}(i)\right)=\left\{\begin{array}{ll}
z_{k}(i) / \lambda_{k l}^{(i)} & \left|z_{k}(i)\right|<\lambda_{k 1}^{(i)} \\
\operatorname{sgn}\left(z_{k}(i)\right) & \text { otherwise. }
\end{array}, \quad l, k=1,2, \ldots, K, \quad l \neq k .\right.
$$

1. $\left|z_{k}(i)\right|<\lambda_{k l}^{(i)}$

$$
\begin{aligned}
\operatorname{Pr}\left\{\tilde{\boldsymbol{b}}_{k}(i) \mid \boldsymbol{b}_{k}(i)\right\} & =\operatorname{Pr}\left\{\left|z_{k}(i)\right|<\lambda_{k l}^{(i)}\right\} \\
& =\operatorname{Pr}\left\{-\lambda_{k l}^{(i)}-\sqrt{a_{k}} b_{k}(i)<\xi_{k}(i)<\lambda_{k l}^{(i)}-\sqrt{a_{k}} b_{k}(i)\right\} \\
& =\operatorname{Pr}\left\{-\lambda_{k l}^{(i)}+\sqrt{a_{k}} b_{k}(i)<\xi_{k}(i)<\lambda_{k l}^{(i)}+\sqrt{a_{k}} b_{k}(i)\right\} \\
& =\operatorname{Pr}\left\{-\left[\lambda_{k l}^{(i)}-\sqrt{a_{k}} b_{k}(i)\right]<\xi_{k}(i)<\left[-\lambda_{k l}^{(i)}-\sqrt{a_{k}} b_{k}(i)\right]\right\} \\
& =\operatorname{Pr}\left\{-\tilde{\boldsymbol{b}}_{k}(i) \mid-\boldsymbol{b}_{k}(i)\right\} .
\end{aligned}
$$

2. $\left|z_{k}(i)\right|>\lambda_{k l}^{(i)}$

$$
\begin{aligned}
\operatorname{Pr}\left\{\tilde{\boldsymbol{b}}_{k}(i) \mid \boldsymbol{b}_{k}(i)\right\} & =\operatorname{Pr}\left\{\operatorname{sgn}\left(\sqrt{a_{k}} b_{k}(i)+\xi_{k}(i)\right)\right\} \\
& \left.=\operatorname{Pr}\left\{\tilde{\boldsymbol{b}}_{k}(i) \xi_{k}(i)\right)>-\sqrt{a_{k}} b_{k}(i) \tilde{\boldsymbol{b}}_{k}(i)\right\} \\
& \left.=\operatorname{Pr}\left\{\tilde{\boldsymbol{b}}_{k}(i) \xi_{k}(i)\right)<\sqrt{a_{k}} b_{k}(i) \tilde{\boldsymbol{b}}_{k}(i)\right\} \\
& =\operatorname{Pr}\left\{-\tilde{\boldsymbol{b}}_{k}(i) \mid-\boldsymbol{b}_{k}(i)\right\} .
\end{aligned}
$$

The same properties can be shown for all other random variables of the joint proba bility. Thus:

$$
\operatorname{Pr}\left\{\tilde{\boldsymbol{b}}_{k}(i) \mid \boldsymbol{b}_{k}(i)\right\}=\operatorname{Pr}\left\{-\tilde{\boldsymbol{b}}_{k}(i) \mid-\boldsymbol{b}_{k}(i)\right\} .
$$

In case of a dead-zone limiters, the first part of the proof becomes zero whereas the second part remains the same, making the proof also valid for the dead-zone limiter.

## APPENDIX C

## Thresholds of Tentative Decision Nonlinearities

The thresholds of the tentative decision nonlinearity are determined heuristically from the observed values of the decorrelator outputs as:

$$
\lambda_{21}^{(-1)}=\frac{\rho_{12}^{2}\left[E\left\{\left|z_{1}(0)\right|\right\}\right]^{2}}{E\left\{\left|z_{2}(-1)\right|\right\}} \quad \text { and } \quad \lambda_{21}^{(0)}=\frac{\rho_{21}^{2}\left[E\left\{\left|z_{1}(0)\right|\right\}\right]^{2}}{E\left\{\left|z_{2}(0)\right|\right\}}
$$

and

$$
\lambda_{12}^{(-1)}=\frac{\rho_{21}^{2}\left[E\left\{\left|z_{2}(0)\right|\right\}\right]^{2}}{E\left\{\left|z_{1}(-1)\right|\right\}} \quad \text { and } \quad \lambda_{12}^{(0)}=\frac{\rho_{12}^{2}\left[E\left\{\left|z_{2}(0)\right|\right\}\right]^{2}}{E\left\{\left|z_{1}(0)\right|\right\}}
$$

As shown in Chapter 3, these thresholds provide good results versus other heuristically determined thresholds.

Without loss of generality, only the threshold influencing the tentative decision of user 1 is considered and this only for time interval $i=0$. The other cases can be easily deduced from the following derivation of the above expectation.

$$
\begin{aligned}
E\left\{\left|z_{2}(0)\right|\right\} & =E\left\{\left|\sqrt{a_{2}} b_{2}(0)+\xi_{2}(0)\right|\right\} \\
& =\frac{1}{2} E\left\{\left|\sqrt{a_{2}} b_{2}(0)+\xi_{2}(0)\right| \mid b_{2}(0)=-1\right\} \\
& +\frac{1}{2} E\left\{\left|\sqrt{a_{2}} b_{2}(0)+\xi_{2}(0)\right| \mid b_{2}(0)=1\right\} \\
& =\frac{1}{2}\left[E\left\{\left|-\sqrt{a_{2}}+\xi_{2}(0)\right|\right\}+E\left\{\left|\sqrt{a_{2}}+\xi_{2}(0)\right|\right\}\right] \\
& =\frac{1}{2}\left[E\left\{\sqrt{a_{2}}-\xi_{2}(0) \mid-\sqrt{a_{2}}+\xi_{2}(0)<0\right\}\right. \\
& +E\left\{-\sqrt{a_{2}}+\xi_{2}(0) \mid-\sqrt{a_{2}}+\xi_{2}(0)>0\right\} \\
& +E\left\{-\sqrt{a_{2}}-\xi_{2}(0) \mid \sqrt{a_{2}}+\xi_{2}(0)<0\right\} \\
& \left.+E\left\{\sqrt{a_{2}}+\xi_{2}(0) \mid \sqrt{a_{2}}+\xi_{2}(0)>0\right\}\right]
\end{aligned}
$$

Since $\xi_{2}(0)$ is a zero-mean Gaussian random variable:

$$
E\left\{\left|z_{2}(0)\right|\right\}=\frac{1}{2}\left[\int_{-\infty}^{\sqrt{a_{2}}}\left(\sqrt{a_{2}}-\xi_{2}(0)\right) f\left(\xi_{2}(0)\right) d \xi_{2}(0)\right.
$$

$$
\begin{aligned}
& +\int_{\sqrt{a_{2}}}^{\infty}\left(-\sqrt{a_{2}}+\xi_{2}(0)\right) f\left(\xi_{2}(0)\right) d \xi_{2}(0) \\
& +\int_{-\infty}^{-\sqrt{a_{2}}}\left(-\sqrt{a_{2}}-\xi_{2}(0)\right) f\left(\xi_{2}(0)\right) d \xi_{2}(0) \\
& \left.+\int_{-\sqrt{a_{2}}}^{\infty}\left(\sqrt{a_{2}}+\xi_{2}(0)\right) f\left(\xi_{2}(0)\right) d \xi_{2}(0)\right] \\
& =\frac{1}{2}\left[\sqrt{a_{2}}\left(1-Q\left(\frac{\sqrt{a_{2}}}{\sigma_{\xi_{2}(0)}}\right)\right)+\frac{\sigma_{\xi_{2}(0)}}{\sqrt{2 \pi}} \exp \left(-\frac{a_{2}^{2}}{2 \sigma_{\xi_{2}(0)}^{2}}\right)\right. \\
& -\sqrt{a_{2} Q}\left(\frac{\sqrt{a_{2}}}{\sigma_{\xi_{2}(0)}}\right)+\frac{\sigma_{\xi_{2}(0)}}{\sqrt{2 \pi}} \exp \left(-\frac{a_{2}^{2}}{2 \sigma_{\xi_{2}(0)}^{2}}\right) \\
& -\sqrt{a_{2} Q}\left(\frac{\sqrt{a_{2}}}{\sigma_{\xi_{2}(0)}}\right)+\frac{\sigma_{\xi_{2}(0)}}{\sqrt{2 \pi}} \exp \left(-\frac{a_{2}^{2}}{2 \sigma_{\xi_{2}(0)}^{2}}\right) \\
& \left.+\sqrt{a_{2}}\left(1-Q\left(\frac{\sqrt{a_{2}}}{\sigma_{\xi_{2}(0)}}\right)\right)+\frac{\sigma_{\xi_{2}(0)}}{\sqrt{2 \pi}} \exp \left(-\frac{a_{2}^{2}}{2 \sigma_{\xi_{2}(0)}^{2}}\right)\right] \\
& =\sqrt{a_{2}}\left(1-2 Q\left(\frac{\sqrt{a_{2}}}{\sigma_{\xi_{2}(0)}}\right)\right)+\frac{2 \sigma_{\xi_{2}(0)}}{\sqrt{2 \pi}} \exp \left(-\frac{a_{2}^{2}}{2 \sigma_{\xi_{2}(0)}^{2}}\right) .
\end{aligned}
$$

## APPENDIX D

## Steady State Values of Weights

Let $f_{z_{k}}$ be the density function of $z_{k}(i)$, and $\boldsymbol{f}_{z_{k} z_{l}}$ the joint density function of $z_{k}(i)$ and $z_{l}(i)$. For the entries in the matrix $E\left\{\tilde{\boldsymbol{b}}_{1}(0) \tilde{\boldsymbol{b}}_{1}^{T}(0)\right\}$ in (2.21), the diagonal elements in the case of a linear clipper are:

$$
E\left\{\tilde{b}_{k}^{2}(i)\right\}=\frac{1}{2} \sum_{b_{k}(i)}\left[\int_{\boldsymbol{L}_{1}} f_{z_{k}} d z_{k}+\int_{\boldsymbol{L}_{2}} \frac{z_{k}^{2}(i)}{\lambda_{k 1}^{(i)^{2}}} f_{z_{k}} d z_{k}\right]
$$

while for the dead-zone limiter:

$$
E\left\{\tilde{b}_{k}^{2}(i)\right\}=\frac{1}{2} \sum_{b_{k}(i)} \int_{\boldsymbol{L}_{1}} f_{z_{k}} d z_{k}
$$

The off-diagonal entries for the linear clipper are:

$$
\begin{aligned}
E\left\{\tilde{b}_{k}(i) \tilde{b}_{l}(i)\right\} & =\frac{1}{4} \sum_{b_{k}(i), b_{l}(i)}\left[\iint_{\boldsymbol{Z}_{1}} \operatorname{sgn}\left(z_{k}(i)\right) \operatorname{sgn}\left(z_{l}(i)\right) \boldsymbol{f}_{z_{k} z_{l}} d z_{k} d z_{l}\right. \\
& +\iint_{\boldsymbol{Z}_{2}} \frac{z_{k}(i)}{\lambda_{k 1}^{(i)}} \operatorname{sgn}\left(z_{l}(i)\right) \boldsymbol{f}_{z_{k} z_{l}} d z_{k} d z_{l} \\
& +\iint_{\boldsymbol{Z}_{3}} \frac{z_{l}(i)}{\lambda_{l 1}^{(i)}} \operatorname{sgn}\left(z_{k}(i)\right) \boldsymbol{f}_{z_{k} z_{l}} d z_{k} d z_{l} \\
& \left.+\iint_{\boldsymbol{Z}_{4}} \frac{z_{k}(i) z_{l}(i)}{\lambda_{k 1}^{(i)} \lambda_{l 1}^{(i)}} \boldsymbol{f}_{z_{k} z_{l}} d z_{k} d z_{l}\right] \\
E\left\{b_{k}(i) \tilde{b}_{k}(i)\right\} & =\frac{1}{2} \sum_{b_{k}(i)}\left[\int_{\boldsymbol{L}_{1}} b_{k}(i) \operatorname{sgn}\left(z_{k}(i)\right) f_{z_{k}} d z_{k}\right. \\
& \left.+\int_{\boldsymbol{L}_{2}} b_{k} \frac{z_{k}(i)}{\lambda_{k 1}^{(i)}} f_{z_{k}} d z_{k}\right]
\end{aligned}
$$

and for the dead-zone:

$$
\begin{aligned}
E\left\{\tilde{b}_{k}(i) \tilde{b}_{l}(i)\right\} & =\frac{1}{4} \sum_{b_{k}(i), b_{l}(i)} \iint_{\boldsymbol{Z}_{1}} \operatorname{sgn}\left(z_{k}(i)\right) \operatorname{sgn}\left(z_{l}(i)\right) \boldsymbol{f}_{z_{k} z_{l}} d z_{k} d z_{l} \\
E\left\{b_{k}(i) \tilde{b}_{k}(i)\right\} & =\frac{1}{2} \sum_{b_{k}(i)} \int_{\boldsymbol{L}_{1}} b_{k}(i) \operatorname{sgn}\left(z_{k}(i)\right) f_{z_{k}} d z_{k}
\end{aligned}
$$

For the second expectation in $(2.21), E\left\{\boldsymbol{b}_{1}(0) \tilde{\boldsymbol{b}}_{1}^{T}(0)\right\}$, we have in the case of the linear clipper:

$$
E\left\{b_{k}(i) \tilde{b}_{k}(i)\right\}=\frac{1}{2} \sum_{b_{k}(i)}\left[\int_{\boldsymbol{L}_{1}} b_{k}(i) \operatorname{sgn}\left(z_{k}(i)\right) f_{z_{k}} d z_{k}+\int_{\boldsymbol{L}_{2}} b_{k} \frac{z_{k}(i)}{\lambda_{k 1}^{(i)}} f_{z_{k}} d z_{k}\right]
$$

and for the dead-zone:

$$
E\left\{b_{k}(i) \tilde{b}_{k}(i)\right\}=\frac{1}{2} \sum_{b_{k}(i)} \int_{\boldsymbol{L}_{1}} b_{k}(i) \operatorname{sgn}\left(z_{k}(i)\right) f_{z_{k}} d z_{k}
$$

where we assume $i, j=-1,0, \quad k, l=2, \ldots, K, \quad k \neq l$.
$\boldsymbol{L}_{j}, j=1,2$ corresponds to the appropriate intervals of $z_{k}$, and $\boldsymbol{Z}_{j}, j=1, \ldots, 4$ corresponds to the appropriate rectangular regions in the $\left(z_{k}, z_{l}\right)$ plane.

## APPENDIX E

## Flow Chart Simulation Program



Figure E. 1 Flow chart simulation program


Figure E. 2 Flow chart simulation program (continued)

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