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ABSTRACT

DECENTRALIZED OPTIMAL CONTROL IN DESCRIPTOR SYSTEMS

by
Hag-Yeon Park

Application of Matrix Minimum Principle to a linear decentralized optimal control in descriptor systems is studied in this thesis. Linear-quadratic index of performance with Gaussian initial state is considered. The necessary and sufficient conditions for optimality are derived.

An additional constraint is imposed such that the controllers are linear function of output $y(t)$ rather than of the state vector $x(t)$. The optimal gain matrix G_i^* is then specified by the necessary conditions.

Two examples are developed to illustrate the concept.

**DECENTRALIZED OPTIMAL CONTROL
IN DESCRIPTOR SYSTEMS**

by
Hag-Yeon Park

**A Thesis
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in Partial Fulfillment of the Requirements for the Degree of
Master of Science in Electrical Engineering**

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APPROVAL PAGE

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IN DESCRIPTOR SYSTEMS

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This thesis is dedicated to
my family

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CHAPTER 1

INTRODUCTION

Descriptor systems occur frequently in system theory, often as natural representations of physical or economic systems, or as the necessary conditions representing optimal control, optimal estimation, or dynamic economic equilibrium. Therefore, it is important to understand the structure of such systems and develop efficient methods for solving them. Recently, the optimal control problem for singular or descriptor systems has been of interest in the field of control systems. Larson [1] and others considered the discrete case by first applying Luenberger's double sweep method to get a state system and then derived several recursive matrix equations which needed to be solved. Pandolfi [2] considered the continuous case and derived a feedback control which stabilizes the system. He used an augmented state system which is equivalent to the descriptor system, but the impulse elimination problem was not considered. In [3], Cobb also considered the continuous case by applying several preliminary feedbacks and then solved the optimal control problem for a state space system. His closed-loop system was both stable and impulse free. Campbell [4] used the Drazin inverse to analyze the cheap optimal control problem for state variables and the approach generalizes singular systems, but he did not give the control in terms of a feedback. Bender [5] solved the continuous-time linear-quadratic regulator problem for descriptor systems by using a singular value decomposition of E . To solve a finite-horizon problem or to compute the Riccati equation solution $P(t)$ required one transformation of the descriptor system in order to isolate the

dynamic portion. The dynamic portion of the system was the orthogonal complement of the part of the descriptor space contained in the kernel of E . He isolated it by performing a singular value decomposition of E , numerically quite robust way to determine the rank of a matrix. This approach yield with no undue difficulty the solution of the finite as well as infinite horizon problem. In this thesis approach, we use his method for computing the Riccati equation solution of $P(t)$.

In large-scale systems such as power systems, chemical processes, large space structures, and computer communication networks, a centralized control system or a single controller has access to all sensor measurements and generates all control commands for the entire system. However, as systems become more and more complex, it has been found that they cannot be handled by the centralized control method. As a result, decentralized control often arises as an important option in the design of strategies for controlling such systems, and the study of the stabilization of decentralized control systems has attracted much attention over the past few years[6] [7] [8]. These men motivated to do research in decentralized control because conventional modern control theory was not able to deal with certain issues of concern in large-scale systems. State feedback is a central idea in modern control theory. By combining the linear-quadratic optimal control technique and state feedback, it is possible to achieve improved system behavior. However, it is often impossible to design a system to the extent required for full state feedback. Therefore, many techniques including linear-quadratic Gaussian control were developed to overcome this difficulty. However, a central characteristic of all these techniques is that they result in a design in which every sensor output affects every actuator input. This situation is called centralized control. In large-scale systems, it is

impossible to put many feedback loops into the design [9]. Thus, decentralized feedback control has been applied to solve this problem. The basic characteristic of decentralized control is that there are restrictions of information transfer between certain groups of sensors and actuators. In addition, we will apply the Matrix Minimum Principle [10] to the design of optimal descriptor systems. As our model, we use a linear decentralized system with a quadratic performance and output-variable feedback. The goal is to determine an optimal set of feedback gains.

Before we discuss the decentralized optimal control problem in descriptor systems which is stated in detail in chapter 2, let us look at the overall content of the thesis. The system model is first formulated in the framework of decentralized optimal control theory in descriptor systems. Then, we transform the problem into the framework required by the Matrix Minimum Principle. In Chapter 3, we derive the necessary conditions for optimality by using the Matrix Minimum Principle. We also prove that the necessary conditions for optimality are sufficient. In Chapter 4, we summarize the main result of this thesis. In addition, two examples are shown to support this thesis in chapter 5. In Chapter 6, we conclude this thesis.

In this thesis approach, we consider that a general linear, descriptor system whose state vector $x(t)$, control vector $u_i(t)$, and output vector $y_i(t)$ are related by the following vector differential equations:

$$E \dot{x}(t) = A(t)x(t) + \sum_{i=1}^N B_i(t)u_i(t) \quad (1.1)$$

$$y_i(t) = C_i(t)x(t), \quad (1.2)$$

$$i = 1, 2, \dots, N$$

We look for a linear feedback law of the form

$$u_i(t) = -G_i(t)y_i(t) \quad (1.3)$$

and substituting (1.2) into (1.3) gives

$$u_i(t) = -G_i(t)C_i(t)x(t) \quad (1.4)$$

$$i = 1, 2, \dots, N$$

where $G_i(t)$ are gain matrices such that the quadratic cost functional

$$J = E \left\{ \int_{t_0}^T \left[x^T(t)Q(t)x(t) + \sum_{i=1}^N u_i^T(t)R_i(t)u_i(t) \right] dt \right\} \quad (1.5)$$

$$i = 1, 2, \dots, N$$

is minimized.

It is assumed that $x(t_0)$ is a Gaussian random vector with known mean and covariance.

Substituting (1.4) into (1.1) gives

$$E \dot{x}(t) = \left[A(t) - \sum_{i=1}^N B_i(t)G_i(t)C_i(t) \right] x(t) \quad (1.6)$$

where the solution of $x(t)$ is given as

$$x(t) = \Phi(t, t_0)x(t_0)$$

where $\Phi(t, t_0)$ is defined by (2.8). Hence, $u_i(t)$ is also a Gaussian process. Therefore, the problem reduces to finding the gain matrices $G_i(t)$, such that J in (1.5) is minimized with constraints (1.1), (1.2), and (1.3).

Our approach is similar to that of Levine and Athans [11] which was used for solving a centralized linear quadratic problem. We begin by transforming the original performance, a function of both initial states and feedback controls (gain matrix), into a

new performance criterion. The problem is therefore converted into a parameter optimization problem and then the necessary conditions for optimality are derived with the Matrix Minimum Principle.

CHAPTER 2

PROBLEM FORMULATION

2.1 System Dynamics

Consider a descriptor system with state vector $x(t)$ and control vector $u_i(t)$ $i = 1, 2, \dots, N$ related by the following vector differential equation:

$$E \dot{x}(t) = A(t)x(t) + \sum_{i=1}^N B_i(t)u_i(t), \quad (2.1)$$

$$y_i(t) = C_i(t)x(t), \quad (2.2)$$

$$i = 1, 2, \dots, N$$

with state vector $x(t) \in R^n$, control vectors $u_i \in R^{m_i}$, and output vectors $y_i \in R^{r_i}$. $A(t)$ is an $n \times n$ matrix, $B_i(t)$ are $n \times m_i$ real matrices, $C_i(t)$ are $r_i \times n$ real matrices of full rank and E^{-1} does not exist.

The performance index for all the control vectors is assumed to be

$$J = E \left\{ \int_{t_0}^T \left[x^T(t)Q(t)x(t) + \sum_{i=1}^N u_i^T(t)R_i(t)u_i(t) \right] dt \right\}, \quad (2.3)$$

$$i = 1, 2, \dots, N$$

where $t \in [t_0, T]$ and E is the expectation. We assume $Q(t)$ is an $n \times n$ symmetric semi-positive definite real matrix and $R_i(t)$ are $m_i \times m_i$ symmetric positive definite real matrices.

At this time, we introduce the constraint that the controls $u_i(t)$ be generated via output linear feedback, i.e.,

$$\begin{aligned}
u_i(t) &= -G_i(t)y_i(t), \\
&= -G_i(t)C_i(t)x(t)
\end{aligned} \tag{2.4}$$

where the gains $G_i(t)$ are $m_i \times r_i$ matrices.

We assume $x(t_0)$ is a Gaussian random process with known

$$\text{mean} \quad E\{x(t_0)\} = \bar{x}(t_0) = m$$

$$E\{x(t_0)x^T(t_0)\} = \Sigma_0 \quad \text{and}$$

$$\begin{aligned}
\text{covariance} \quad E\left\{ [x(t_0) - \bar{x}(t_0)][x(t_0) - \bar{x}(t_0)]^T \right\} &= E\left\{ [x(t_0)x^T(t_0)] - \bar{x}(t_0)\bar{x}^T(t_0) \right\} \\
&= \Sigma_0 - (m)(m)^T.
\end{aligned}$$

So, the system satisfies the closed-loop equation

$$E \dot{x}(t) = \left[A(t) - \sum_{i=1}^N B_i(t)G_i(t)C_i(t) \right] x(t) \tag{2.5}$$

and the cost functional J reduces to

$$J = E \left\{ \int_{t_0}^T x^T(t) \left[Q(t) + \sum_{i=1}^N C_i^T(t)G_i^T(t)R_i(t)G_i(t)C_i(t) \right] x(t) dt \right\}. \tag{2.6}$$

Thus, the choice of the gain matrices $G_i(t)$ obviously governs the closed-loop dynamics of the system. In fact, the response of the closed-loop system is given as:

$$x(t) = \Phi(t, t_0)x(t_0) \tag{2.7}$$

where $\Phi(t, t_0)$ is the $n \times n$ fundamental transition matrix associated with

$$\left[A(t) - \sum_{i=1}^N B_i(t)G_i(t)C_i(t) \right] \text{ and defined by:}$$

$$E \dot{\Phi}(t, t_0) = \left[A(t) - \sum_{i=1}^N B_i(t)G_i(t)C_i(t) \right] \Phi(t, t_0)$$

$$\Phi(t_0, t_0) = I. \quad (2.8)$$

Substituting (2.7) into (2.4) gives

$$u_i(t) = -G_i(t)C_i(t)\Phi(t, t_0)x(t_0), \quad (2.9)$$

$$i = 1, 2, \dots, N.$$

Hence, $u_i(t)$ is also a Gaussian process so the problem reduces to finding the gain matrices $G_i(t)$ which minimizes (2.3) subject to the constraints given by (2.1) and (2.2).

In order to find the gain matrices $G_i(t)$, the vector differential equations must be transformed into matrix differential equations which can be readily solved by the Matrix Minimum Principle.

2.2 Matrix Minimum Principle

The most common form of the minimum principle pertains to the optimal control of systems described like the following vector differential equation:

$$\dot{x}(t) = f[x(t), u(t), t] \quad (2.10)$$

where $x(t)$ is a column n -vector, $u(t)$ is a column r -vector and $f[t]$ is a vector-valued function. Plants described by equation (2.10) are very common. However, there are problems in which the evolution-in-time of their variables is most naturally described by means of matrix differential equations. To make this more precise, consider a system whose state variables are x_{ij} , with $i = 1, 2, \dots, n$ and $j = 1, 2, \dots, m$, and whose control variables are $u_{\alpha\beta}$, with $\alpha = 1, 2, \dots, r$ and $\beta = 1, 2, \dots, q$. In such problems, we may think of the "state matrix" $X(t)$, whose elements are the state variables $x_{ij}(t)$, and of the

“control matrix” $U(t)$, whose elements are the control variables $u_{\alpha\beta}(t)$; these are assumed to be related by the matrix differential equation

$$\dot{X}(t) = F[X(t), U(t), t] \quad (2.11)$$

where $F[t]$ is a matrix-valued function of its arguments.

As an example of a system with this type of description, consider a linear system

$$\dot{x}(t) = A(t)x(t) + v(t) \quad (2.12)$$

where $v(t)$ is a white-noise process with zero mean and covariance

$$E\{v(t)v^T(\tau)\} = \delta(t - \tau)Q(t). \quad (2.13)$$

If we denote the covariance matrix of the state vector $x(t)$ by $\Sigma(t)$, i.e.,

$$\Sigma(t) = E\{x(t)x^T(t)\}, \quad (2.14)$$

then it can be shown that $\Sigma(t)$ satisfies the linear matrix differential equation

$$\dot{\Sigma}(t) = A(t)\Sigma(t) + \Sigma(t)A^T(t) + Q(t) \quad (2.15)$$

which is in the form of equation (2.11). Indeed, the Matrix Minimum Principle has been applied to problems of filtering, control and signal design. In these types of problems, we are interested in minimizing a scalar-valued function of the covariance matrix $\Sigma(t)$ while the “control variables” are some of the elements of the matrix $A(t)$ or $Q(t)$.

If the system equations are naturally given by (2.11), it is easy to visualize an optimization problem. For example, consider a fixed-terminal time-optimization problem with a cost functional

$$J(U) = K[X(T)] + \int_{t_0}^T L[X(t), U(t), t] dt \quad (2.16)$$

where $K[T]$ and $L[t]$ are scalar-valued functions of their argument. Now we seek the optimal control matrix $U^*(t)$, which is constrained by

$$U^*(t) \in \Omega \quad (2.17)$$

which minimizes the cost function $J(U)$.

It should be clear that the tools are available to tackle this optimization problem. After all, equation (2.11) in component form can be written as

$$\dot{x}_y(t) = f_y[X(t), U(t), t] \quad (2.18)$$

and then we can proceed to apply the familiar minimum principle. However, an excessive number of equations result and it may become almost impossible to determine any structure and property of the solution. It is this complication which has provided the motivation for dealing with problems involving the time-evolution of matrices by constructing a systematic notational approach.

The first step towards this goal is to realize that the set of all, say, $n \times m$ real matrices forms a linear vector space with well-defined operations of addition and multiplication. We denote this vector space by S_{nm} . Then, it is possible to define an inner product in this space. Thus, if A and B are $n \times m$ matrices, i.e., $A \in S_{nm}$ and $B \in S_{nm}$, their inner product is defined by the trace operation

$$(A, B) = \text{tr}[AB^T] = \sum_{i=1}^n \sum_{j=1}^m a_{ij} b_{ij} . \quad (2.19)$$

Using this notation, we can form the Hamiltonian function for the optimization problem.

First, note that if $p_y(t)$ is the costate variable associated with $x_y(t)$, then the Hamiltonian must take the form

$$H = L[X(t), U(t), t] + \sum_{i=1}^n \sum_{j=1}^m \dot{x}_y(t) p_y(t) . \quad (2.20)$$

Using (2.20), it follows that the Hamiltonian can be written as

$$H = L[X(t), U(t), t] + \text{tr}[\dot{X}(t)P^T(t)], \quad (2.21)$$

where $P(t)$ is the costate matrix associated with the state matrix $X(t)$, in the sense that the costate variable $p_{ij}(t)$ is the ij th element of $P(t)$.

Using the notation of Athans and Falb [12], it is known that the costate variables satisfy the differential equations

$$\dot{p}_{ij}(t) = -\frac{\partial H}{\partial x_{ij}(t)}. \quad (2.22)$$

This type of equation leads to the definition of the so-called gradient matrix (see appendix A).

A gradient matrix is defined as follows: suppose that $f(X)$ is a scalar-valued function of the elements x_{ij} of X . Then the gradient matrix of $f(X)$ is denoted by

$$\frac{\partial f(X)}{\partial X}$$

and it is a matrix whose ij th element is simply given by

$$\left[\frac{\partial f(X)}{\partial X} \right]_{ij} = \frac{\partial f(X)}{\partial x_{ij}}. \quad (2.23)$$

Using the notation of the gradient matrix, it is readily seen that (2.22) can be written as

$$\dot{P}^*(t) = -\frac{\partial H}{\partial X(t)} \quad (2.24)$$

since the Hamiltonian H is a scalar-valued function.

Consider a system with “state matrix” $X(t)$ and “control matrix” $U(t) \in \Omega$ described by the matrix differential equation

$$\dot{X}(t) = F[X(t), U(t), t]; \quad X(t_0) = X_0 \quad (2.25)$$

and the cost functional

$$J(U) = K[X(T)] + \int_{t_0}^T L[X(t), U(t), t] dt; \quad T \text{ fixed} \quad (2.26)$$

where $K[T]$ and $L[t]$ are scalar-valued functions of their argument satisfying the usual differentiability conditions.

Let $P(t)$ denote the costate matrix. Define the scalar Hamiltonian function H by

$$H[X(t), P(t), t, U(t)] = L[X(t), U(t), t] + \text{tr}[F(X(t), U(t), t)P^T(t)]. \quad (2.27)$$

If $U^*(t)$ is the optimal control in the sense that it minimizes J , and if $X^*(t)$ is the corresponding state, then there exists a costate matrix $P^*(t)$ such that the following conditions hold:

(i) *Canonical Equations*

$$\dot{X}^*(t) = \left. \frac{\partial H}{\partial P(t)} \right|_* = F[X^*(t), U^*(t), t] \quad (2.28)$$

$$\begin{aligned} \dot{P}^*(t) = \left. -\frac{\partial H}{\partial X(t)} \right|_* &= -\frac{\partial}{\partial X^*(t)} L[X^*(t), U^*(t), t] \\ &\quad - \frac{\partial}{\partial X^*(t)} \text{tr}[F(X^*(t), U^*(t), t)P^{*T}(t)] \end{aligned} \quad (2.29)$$

(ii) *Boundary Conditions*

At the initial time

$$X^*(t_0) = X_0 \quad (2.30)$$

At the terminal time

$$P^*(t) = \frac{\partial}{\partial X^*(T)} K[X^*(T)] \quad (2.31)$$

(iii) *Minimization of the Hamiltonian*

$$H[X^*(t), P^*(t), t, U^*(t)] \leq H[X^*(t), P^*(t), t, U] \quad (2.32)$$

for every $U \in \Omega$ and for each $t \in [t_0, T]$.

Note that if $U(t)$ is unconstrained, then (2.32) implies the necessary condition

$$\left. \frac{\partial \mathcal{H}}{\partial U(t)} \right|_* = 0; \quad (2.33)$$

i.e., the gradient matrix of the Hamiltonian with respect to the control matrix U must vanish.

2.3 System Transformation

To complete the transformation of the problem into the framework required by the Matrix Minimum Principle, we define the $n \times n$ "state matrix" $X(t)$ as the outer vector product of the state vector $x(t)$ with itself, i.e.,

$$X(t) \triangleq x(t)x^T(t) \quad (2.34)$$

multiplying both sides by E and E^T gives

$$EX(t)E^T \triangleq Ex(t)x^T(t)E^T \quad (2.35)$$

noting that

$$x^T(t)x(t) = \text{tr}[X(t)] \quad (2.36)$$

$$\bar{X}(t) = E\{X(t)\} = E\{x(t)x^T(t)\} \quad (2.37)$$

$$E\{x^T(t)F(t)x(t)\} = \text{tr}[F(t)\bar{X}(t)] = \text{tr}[\bar{X}(t)F(t)]. \quad (2.38)$$

It follows from (2.5) and (2.35) that:

$$\begin{aligned} E\dot{X}(t)E^T &= E\dot{x}(t)x^T(t)E^T + Ex(t)\dot{x}^T(t)E^T \\ &= \left[A(t) - \sum_{i=1}^N B_i(t)G_i(t)C_i(t) \right] x(t)x^T(t)E^T \end{aligned}$$

$$+ E x(t)x^T(t) \left[A(t) - \sum_{i=1}^N B_i(t)G_i(t)C_i(t) \right]^T \quad (2.39)$$

so that the state matrix $X(t)$ satisfies the linear matrix differential equation.

Taking expectation on both sides, using (2.37), the dynamic constraint is transformed to:

$$\begin{aligned} E \dot{\bar{X}}(t)E^T &= \left[A(t) - \sum_{i=1}^N B_i(t)G_i(t)C_i(t) \right] \bar{X}(t)E^T \\ &+ E \bar{X}(t) \left[A(t) - \sum_{i=1}^N B_i(t)G_i(t)C_i(t) \right]^T \end{aligned} \quad (2.40)$$

with the initial condition

$$\bar{X}(t_0) = E \{ x(t_0)x^T(t_0) \} = \Sigma_0. \quad (2.41)$$

Therefore, the state transition matrix follows:

$$E \bar{X}(t)E^T = E \Phi(t, t_0) \bar{X}(t_0) \Phi^T(t, t_0) E^T, \quad (2.42)$$

and differentiating (2.42) with respect to t gives

$$\begin{aligned} E \dot{\bar{X}}(t)E^T &= E \dot{\Phi}(t, t_0) \bar{X}(t_0) \Phi^T(t, t_0) E^T + E \Phi(t, t_0) \bar{X}(t_0) \dot{\Phi}^T(t, t_0) E^T \\ &= E \dot{\Phi}(t, t_0) \bar{X}(t_0) \Phi^T(t, t_0) E^T + E \Phi(t, t_0) \bar{X}(t_0) \left[E \dot{\Phi}(t, t_0) \right]^T \\ &= \left[A(t) - \sum_{i=1}^N B_i(t)G_i(t)C_i(t) \right] \Phi(t, t_0) \bar{X}(t_0) \Phi^T(t, t_0) E^T \\ &+ E \Phi(t, t_0) \bar{X}(t_0) \Phi^T(t, t_0) \left[A(t) - \sum_{i=1}^N B_i(t)G_i(t)C_i(t) \right]^T \end{aligned} \quad (2.43)$$

where $\Phi(t, t_0)$ is the $n \times n$ transition matrix satisfying

$$\begin{aligned} E \dot{\Phi}(t, t_0) &= \left[A(t) - \sum_{i=1}^N B_i(t)G_i(t)C_i(t) \right] \Phi(t, t_0) \\ \Phi(t_0, t_0) &= I. \end{aligned}$$

The cost functional J reduces to

$$J = \int_{t_0}^T \text{tr} \left\{ \left[Q(t) + \sum_{i=1}^N C_i^T(t) G_i^T(t) R_i(t) G_i(t) C_i(t) \right] \bar{X}(t) \right\} dt. \quad (2.44)$$

The system (2.40) and cost functional (2.44) are in the form required to use the Matrix Minimum Principle. Therefore, given the dynamic constraint, the original problem can be restated as follows:

$$\begin{aligned} E \dot{\bar{X}}(t) E^T = & \left[A(t) - \sum_{i=1}^N B_i(t) G_i(t) C_i(t) \right] \bar{X}(t) E^T \\ & + E \bar{X}(t) \left[A(t) - \sum_{i=1}^N B_i(t) G_i(t) C_i(t) \right]^T \end{aligned} \quad (2.45)$$

with the initial condition $\bar{X}(t_0) = E \{ x(t_0) x^T(t_0) \} = \Sigma_0$

and the cost functional

$$J[\bar{X}(t_0), G_i(t), t_0] = \int_{t_0}^T \text{tr} \left\{ \left[Q(t) + \sum_{i=1}^N C_i^T(t) G_i^T(t) R_i(t) G_i(t) C_i(t) \right] \bar{X}(t) \right\} dt. \quad (2.46)$$

Find the gain matrices $G_i(t)$ such that J in (2.46) is minimized.

CHAPTER 3

CONDITION FOR OPTIMALITY

3.1 Necessary Conditions for Optimality

We shall derive the necessary conditions for optimality by using the Matrix Minimum Principle. Let $P(t)$ be the $n \times n$ costate matrix associated with $X(t)$. The Hamiltonian function H for this problem is:

$$H = \text{tr} \left\{ L[\bar{X}(t), U(t), t] + \left[E \dot{\bar{X}}(t) E^T P^T(t) \right] \right\} \quad (3.1)$$

$$\text{where } L[\bar{X}(t), U(t), t] = \left[Q(t) + \sum_{i=1}^N C_i^T(t) G_i^T(t) R_i(t) G_i(t) C_i(t) \right] \bar{X}(t).$$

The Hamiltonian can be written as

$$\begin{aligned} H = & \text{tr} \left[Q(t) \bar{X}(t) \right] + \text{tr} \left[\sum_{i=1}^N C_i^T(t) G_i^T(t) R_i(t) G_i(t) C_i(t) \bar{X}(t) \right] \\ & + \text{tr} \left[A(t) \bar{X}(t) E^T P^T(t) \right] - \text{tr} \left[\sum_{i=1}^N B_i(t) G_i(t) C_i(t) \bar{X}(t) E^T P^T(t) \right] \\ & + \text{tr} \left[E \bar{X}(t) A^T(t) P^T(t) \right] - \text{tr} \left\{ E \bar{X}(t) \left[\sum_{i=1}^N B_i(t) G_i(t) C_i(t) \right]^T P^T(t) \right\}. \end{aligned} \quad (3.2)$$

Now consider the functional equation,

$$J = \text{tr} \int_{t_0}^T L[\bar{X}(t), U(t), t] dt, \quad (3.3)$$

then substituting (3.1) into (3.3) gives

$$J = \int_{t_0}^T \left\{ H - \text{tr} \left[E \dot{\bar{X}}(t) E^T P^T(t) \right] \right\} dt \quad (3.4)$$

where H is the Hamiltonian.

Variation of the function of J gives

$$\delta J = \int_{t_0}^T \left\{ \delta H - \delta \text{tr} \left[E \dot{\bar{X}}(t) E^T P^T(t) \right] \right\} dt \quad (3.5)$$

where $\delta H = \text{tr} \left\{ \frac{\partial H}{\partial \bar{X}} \delta \bar{X}^T + \frac{\partial H}{\partial P} \delta P^T + \frac{\partial H}{\partial G} \delta G^T \right\}$ and

$$\delta \left\{ \text{tr} \left[E \dot{\bar{X}}(t) E^T P^T(t) \right] \right\} = \text{tr} \left[E \dot{\bar{X}}(t) E^T \delta P^T(t) \right] + \text{tr} \left[E^T P(t) E \delta \dot{\bar{X}}^T(t) \right].$$

So, (3.5) gives

$$\begin{aligned} \delta J = \text{tr} \int_{t_0}^T \left\{ \frac{\partial H}{\partial \bar{X}} \delta \bar{X}^T + \frac{\partial H}{\partial P} \delta P^T + \frac{\partial H}{\partial G} \delta G^T \right\} dt \\ - \text{tr} \int_{t_0}^T \left\{ E \dot{\bar{X}}(t) E^T \delta P^T(t) \right\} dt - \text{tr} \int_{t_0}^T \left\{ E^T P(t) E \delta \dot{\bar{X}}^T(t) \right\} dt. \end{aligned} \quad (3.6)$$

Integration by parts of the last term of (3.6) gives

$$\text{tr} \int_{t_0}^T \left\{ E^T P(t) E \delta \dot{\bar{X}}^T(t) \right\} dt = \text{tr} \left[E^T P(t) E \delta \bar{X}^T(t) \right]_{t_0}^T - \text{tr} \int_{t_0}^T \left\{ E^T \dot{P}(t) E \delta \bar{X}^T(t) \right\} dt. \quad (3.7)$$

Substituting (3.7) into (3.6) gives

$$\begin{aligned} \delta J = \text{tr} \int_{t_0}^T \left\{ \left[\frac{\partial H}{\partial \bar{X}} + E^T \dot{P}(t) E \right] \delta \bar{X}^T + \left[\frac{\partial H}{\partial P} - E \dot{\bar{X}}(t) E^T \right] \delta P^T + \frac{\partial H}{\partial G} \delta G^T \right\} dt \\ - \text{tr} \left[E^T P(t) E \delta \bar{X}^T(T) \right] \end{aligned} \quad (3.8)$$

where $\delta \bar{X}(t_0) = 0$.

The necessary conditions for the optimality of (3.8) requires $\delta J = 0$ and gives

$$E \dot{\bar{X}}(t) E^T = \frac{\partial H}{\partial P} \quad (3.9)$$

$$E^T \dot{P}(t) E = - \frac{\partial H}{\partial \bar{X}} \quad (3.10)$$

$$\frac{\partial H}{\partial G} = 0 \quad (3.11)$$

$$E^T P(T) E = 0. \quad (3.12)$$

The canonical equations using (3.9), (3.10) and the gradient matrix formulae of

Appendix A yields:

$$E \dot{\bar{X}}^*(t) E^T = \frac{\partial H^*}{\partial P(t)} = \left[A(t) - \sum_{i=1}^N B_i(t) G_i^*(t) C_i(t) \right] \bar{X}^*(t) E^T + E \bar{X}^*(t) \left[A(t) - \sum_{i=1}^N B_i(t) G_i^*(t) C_i(t) \right]^T; \quad (3.13)$$

$$E^T \dot{P}^*(t) E = -\frac{\partial H^*}{\partial \bar{X}(t)} = -Q(t) - \sum_{i=1}^N C_i^T(t) G_i^{*T}(t) R_i(t) G_i^*(t) C_i(t) - \left[A(t) - \sum_{i=1}^N B_i(t) G_i^*(t) C_i(t) \right]^T P^*(t) E - E^T P^*(t) \left[A(t) - \sum_{i=1}^N B_i(t) G_i^*(t) C_i(t) \right] \quad (3.14)$$

with boundary conditions:

$$\bar{X}(t_0) = \Sigma_0; \quad E^T P(T) E = 0 \quad (3.15)$$

and (3.11) yields:

$$0 = \frac{\partial H^*}{\partial G_i(t)} = R_i(t) G_i^*(t) C_i(t) \bar{X}^{*T}(t) C_i^T(t) + R_i^T(t) G_i^{*T}(t) C_i(t) \bar{X}^*(t) C_i^T(t) - B_i^T(t) P^*(t) E \bar{X}^{*T}(t) C_i^T(t) - B_i^T(t) P^{*T}(t) E \bar{X}^*(t) C_i^T(t). \quad (3.16)$$

Note that both $\bar{X}^*(t)$ and $P^*(t)$ are symmetrical. The symmetry of both $\bar{X}(t)$ and $\bar{X}(t_0)$ follows from equation (2.43). A similar argument can be used to establish the symmetry of $P(t)$. These symmetrical properties and (3.16) yield:

$$\left[R_i(t) G_i^*(t) C_i(t) - B_i^T(t) P^*(t) E \right] \bar{X}^*(t) C_i^T(t) = 0; \quad (3.17)$$

if we assume C_i^{-1} exists, equation (3.17) reduces to

$$G_i^*(t) = R_i^{-1}(t) B_i^T(t) P^*(t) E C_i^{-1}(t). \quad (3.18)$$

To completely specify the gain matrix $G_i^*(t)$, we must determine the costate matrix $P(t)$ by substituting (3.18) into (3.14). We find that the costate matrix $P(t)$ is similar to the solution of the familiar Riccati matrix differential equation

$$\begin{aligned} E^T \dot{P}^*(t)E &= -E^T P^*(t)A(t) - A^T(t) P^*(t)E \\ &+ \sum_{i=1}^N E^T P^*(t)B_i(t)R_i^{-1}(t)B_i^T(t)P^*(t)E - Q(t) \quad (3.19) \\ &i = 1, 2, \dots, N \end{aligned}$$

with the boundary condition

$$E^T P^*(T)E = 0. \quad (3.20)$$

Thus, the optimal open loop control for the system (2.1) and (2.2) with the performance index defined in (2.3) is given by

$$\begin{aligned} u_i^*(t) &= -G_i^*(t)C_i(t)\Phi(t, t_0)x(t_0) \quad (3.21) \\ &i = 1, 2, \dots, N \end{aligned}$$

where $G_i^*(t)$ is defined in (3.18) with $P(t)$ satisfying the Riccati matrix differential equation expressed by (3.19).

The mean and covariance of the Gaussian process for $u_i^*(t)$ can then be determined by taking expectation on both sides of (3.21) to give the mean of $\bar{u}_i^*(t)$:

$$\begin{aligned} E\{u_i^*(t)\} &= -G_i^*(t)C_i(t)\Phi(t, t_0)E\{x(t_0)\} \\ \text{or } \bar{u}_i^*(t) &= -G_i^*(t)C_i(t)\Phi(t, t_0)m. \quad (3.22) \end{aligned}$$

The covariance of $u_i^*(t)$ is expressed by

$$\begin{aligned} &E\left\{\left[u_i^*(t) - \bar{u}_i^*(t)\right]\left[u_i^*(t) - \bar{u}_i^*(t)\right]^T\right\} \\ &= E\left\{u_i^*(t) u_i^{*T}(t)\right\} - \bar{u}_i^*(t) \bar{u}_i^{*T}(t) \end{aligned}$$

$$\begin{aligned}
&= \mathbb{E}\left\{G_i^*(t)C_i(t)\Phi(t, t_0)x(t_0)x^T(t_0)\Phi^T(t, t_0)C_i^T(t)G_i^{*T}(t)\right\} - m_i m_i^T \\
&= G_i^*(t)C_i(t)\Phi(t, t_0)\Sigma_0\Phi^T(t, t_0)C_i^T(t)G_i^{*T}(t) - m_i m_i^T. \tag{3.23}
\end{aligned}$$

3.2 Sufficient Conditions for Optimality

In order for $G_i(t)$ at $t \in [t_0, T]$ to be an optimal gain matrix, it is sufficient that the following conditions hold:

$$(i) \quad EP(t)E^T = \frac{\partial J}{\partial \bar{X}}(\bar{X}^*(t), t) \tag{3.24}$$

$$(ii) \quad \frac{\partial J}{\partial a}(\bar{X}^*(t), t) + H[\bar{X}^*(t), P^*(t), G_i^*(t), t] = 0 \tag{3.25}$$

$$(iii) \quad H[\bar{X}^*(t), P^*(t), G_i(t), t] \geq H[\bar{X}^*(t), P^*(t), G_i^*(t), t] \tag{3.26}$$

for every $G_i^* \in \Omega$,

$$(iv) \quad J(\bar{X}^*(t), t) \text{ is a function of } C^{(1)} \text{ in } (\bar{X}^*(t), t) \in \Omega_{\bar{X}} \times \{T\} \tag{3.27}$$

$$(v) \quad J(\bar{X}^*(T), T) = 0. \tag{3.28}$$

Sufficient conditions are proved as follows:

$G_i^*(t)$ calculated from (3.18) are functions of $P^*(t)$. We may denote it by

$$G_i^*(t) = G_i^*[P^*(t), t] = G_i^*\left[\frac{\partial J^*}{\partial \bar{X}}(t), t\right] \text{ in view of (3.24) and define a function}$$

$g[\bar{X}^*(t), G_i(t), t]$ such that

$$H\left[\bar{X}^*(t), \frac{\partial J(\bar{X}^*(t), t)}{\partial \bar{X}}, G_i(t), t\right] + \frac{\partial J(\bar{X}^*(t), t)}{\partial a} = g[\bar{X}^*(t), G_i(t), t]. \tag{3.29}$$

At $G_i(t) = G_i^*(t)$ from (3.25) we have

$$\frac{\partial J}{\partial a}(\bar{X}^*(t), t) + H[\bar{X}^*(t), P^*(t), G_i^*(t), t] = g[\bar{X}^*(t), G_i^*(t), t] = 0. \tag{3.30}$$

It follows from (3.24) that (3.1) may be written as:

$$\begin{aligned} H &= \text{tr} \left\{ L[\bar{X}(t), G_i(t), t] + \left[E^T P(t) E \dot{\bar{X}}(t) \right] \right\} \\ &= \text{tr} \left\{ L[\bar{X}(t), G_i(t), t] + \left(\frac{\partial J}{\partial \bar{X}} \right)^T \dot{\bar{X}} \right\} \end{aligned} \quad (3.31)$$

and substituting (3.31) into (3.30) gives:

$$\text{tr} \left\{ L[\bar{X}^*(t), G_i^*(t), t] + \left(\frac{\partial J}{\partial \bar{X}} \right)^T \frac{d\bar{X}^*}{dt} \right\} + \frac{\partial J(\bar{X}^*(t), t)}{\partial t} = g[\bar{X}^*(t), G_i^*(t), t] \quad (3.32)$$

$$\text{where } L[\bar{X}^*(t), G_i^*(t), t] = \left[Q(t) + \sum_{i=1}^N C_i^T(t) G_i^{*T}(t) R_i(t) G_i^*(t) C_i(t) \right] \bar{X}^*(t).$$

Since

$$\frac{dJ(\bar{X}^*(t), t)}{dt} = \text{tr} \left\{ \left[\frac{\partial J(\bar{X}^*(t), t)}{\partial \bar{X}} \right]^T \frac{d\bar{X}^*}{dt} \right\} + \frac{\partial J(\bar{X}^*(t), t)}{\partial t}, \quad (3.33)$$

(3.32) reduces to

$$\frac{dJ(\bar{X}^*(t), t)}{dt} + \text{tr} \left\{ L[\bar{X}^*(t), G_i^*(t), t] \right\} = g[\bar{X}^*(t), G_i^*(t), t]. \quad (3.34)$$

Integrating (3.34) from t_0 to T gives

$$\begin{aligned} \int_{t_0}^T g[\bar{X}^*(t), G_i^*(t), t] dt &= \int_{t_0}^T \text{tr} \left\{ L[\bar{X}^*(t), G_i^*(t), t] \right\} dt + \\ &\quad J(\bar{X}^*(T), T) - J(\bar{X}^*(t_0), t_0) \\ &= J(\bar{X}^*(t_0), G_i^*(t), t_0) - J(\bar{X}^*(t_0), t_0). \end{aligned} \quad (3.35)$$

In view of (3.28) and (2.46), similarly it can be shown that

$$\int_{t_0}^T g[\bar{X}^*(t), G_i(t), t] dt = J(\bar{X}^*(t_0), G_i(t), t_0) - J(\bar{X}^*(t_0), t_0). \quad (3.36)$$

Equation (3.26), (3.29) and (3.30) implies that

$$g[\bar{X}^*(t), G_i(t), t] \geq g[\bar{X}^*(t), G_i^*(t), t]; \quad (3.37)$$

and conditions (3.35), (3.36) and (3.37) gives

$$J[\bar{X}^*(t_0), G_i(t), t_0] \geq J[\bar{X}^*(t_0), G_i^*(t), t_0]$$

or

$$J[\bar{X}(t), G_i(t), t] \geq J[\bar{X}(t), G_i^*(t), t]$$

for all initial conditions $(\bar{X}^*(t), t) \in \Omega_x \times \{T\}$ and every $G_i^* \in \Omega$, where $\bar{X}^*(t_0) = \bar{X}(t_0)$.

CHAPTER 4

THE MAIN RESULT

The major result of this research is summarized below.

It specifies the properties of the optimal gain matrices $G_i^*(t)$, $t \in [t_0, T]$.

To find the gain matrices $G_i(t)$ such that J in (2.46) is minimized,

we use the Hamiltonian function:

$$H = L[\bar{X}(t), U(t), t] + \text{tr}[E \dot{\bar{X}}(t) E^T P^*(t)] \quad (3.1)$$

where $L[\bar{X}(t), U(t), t] = \text{tr} \left\{ \left[Q(t) + \sum_{i=1}^N C_i^T(t) G_i^T(t) R_i(t) G_i(t) C_i(t) \right] \bar{X}(t) \right\}$

to give the gain matrices

$$G_i^*(t) = R_i^{-1}(t) B_i^T(t) P^*(t) E C_i^{-1}(t) \quad (3.18)$$

$$i = 1, 2, \dots, N$$

where we assume C_i^{-1} exist.

The costate matrix $P(t)$ is similar to the solution of the familiar Riccati matrix differential equation:

$$\begin{aligned} E^T \dot{P}^*(t) E &= -E^T P^*(t) A(t) - A^T(t) P^*(t) E \\ &+ \sum_{i=1}^N E^T P^*(t) B_i(t) R_i^{-1}(t) B_i^T(t) P^*(t) E - Q(t) \end{aligned} \quad (3.19)$$

with the boundary condition

$$E^T P^*(T) E = 0. \quad (3.20)$$

The mean of the Gaussian process for $u_i^*(t)$ has been determined as:

$$\bar{u}_i^*(t) = -G_i^*(t)C_i^T(t)\Phi(t, t_0)m. \quad (3.22)$$

The covariance of the Gaussian process for $u_i^*(t)$ is expressed by

$$\begin{aligned} E\left\{\left[u_i^*(t) - \bar{u}_i^*(t)\right]\left[u_i^*(t) - \bar{u}_i^*(t)\right]^T\right\} &= E\left\{u_i^*(t)u_i^{*T}(t)\right\} - \bar{u}_i^*(t)\bar{u}_i^{*T}(t) \\ &= E\left\{G_i^*(t)C_i^T(t)\Phi(t, t_0)x(t_0)x^T(t_0)\Phi^T(t, t_0)C_i^T(t)G_i^{*T}(t)\right\} - m_i m_i^T \\ &= G_i^*(t)C_i^T(t)\Phi(t, t_0)\Sigma_0\Phi^T(t, t_0)C_i^T(t)G_i^{*T}(t) - m_i m_i^T. \end{aligned} \quad (3.23)$$

Previously, the sufficient conditions for optimality were given:

$$(i) \quad E P(t) E^T = \frac{\partial J}{\partial \bar{X}}(\bar{X}^*(t), t) \quad (3.24)$$

$$(ii) \quad \frac{\partial J}{\partial \bar{X}}(\bar{X}^*(t), t) + H[\bar{X}^*(t), P^*(t), G_i^*(t), t] = 0 \quad (3.25)$$

$$(iii) \quad H[\bar{X}^*(t), P^*(t), G_i(t), t] \geq H[\bar{X}^*(t), P^*(t), G_i^*(t), t] \quad (3.26)$$

for every $G_i^* \in \Omega$,

$$(iv) \quad J(\bar{X}^*(t), t) \text{ is a function of } C^{(1)} \text{ in } (\bar{X}^*(t), t) \in \Omega_X \times \{T\} \quad (3.27)$$

$$(v) \quad J(\bar{X}^*(T), T) = 0. \quad (3.28)$$

CHAPTER 5

EXAMPLES

5.1 Example 1

Find the gain matrices such that J is minimized for the system

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} u_{11}(t) \\ u_{12}(t) \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u_{21}(t) \\ u_{22}(t) \end{bmatrix} \quad (5.1)$$

$$y_1(t) = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

$$y_2(t) = \begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \quad (5.2)$$

to minimize the performance measure

$$J = E \left\{ \int_0^{\infty} \left[\begin{pmatrix} x_1(t) & x_2(t) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} u_{11}(t) & u_{12}(t) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u_{11}(t) \\ u_{12}(t) \end{pmatrix} + \begin{pmatrix} u_{21}(t) & u_{22}(t) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u_{21}(t) \\ u_{22}(t) \end{pmatrix} \right] dt \right\} \quad (5.3)$$

To solve this problem, the cost functional J reduces to

$$J = \int_0^{\infty} \text{tr} \left\{ \left[Q(t) + \sum_{i=1}^N C_i^T(t) G_i^T(t) R_i(t) G_i(t) C_i(t) \right] \bar{X}(t) \right\} dt \quad (5.4)$$

and by using the necessary condition results in:

$$G_i^*(t) = R_i^{-1}(t) B_i^T(t) P^*(t) E C_i^{-1}(t), \quad (5.5)$$

$$E^T \dot{P}^*(t) E = -E^T P^*(t) A(t) - A^T(t) P^*(t) E + \sum_{i=1}^N E^T P^*(t) B_i(t) R_i^{-1}(t) B_i^T(t) P^*(t) E - Q(t) \quad (5.6)$$

and the boundary condition

$$E^T P^*(T) E = 0. \quad (5.7)$$

For this performance criterion, the weighting matrices are seen to be

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad R_1 = R_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Let the costate matrix $P^*(t)$ be:

$$P^* = \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}$$

So to solve the $P^*(t)$, we apply the singular value decomposition of E

$$U E V = \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} \quad (5.8)$$

where $\Sigma^2 = \text{diag}(\sigma_1, \sigma_2, \dots, 0)$ and U and V are unitary matrices [5].

In example 1, equation (5.6) has the following form

$$-\Sigma^2 \dot{P}(t) \Sigma^2 = \Sigma^2 P(t) A_{11} + A_{11}^T P(t) \Sigma^2 + Q_{11} - (\Sigma^2 P(t) \tilde{B}_1) \tilde{R}_1^{-1} (\tilde{B}_1^T(t) P(t) \Sigma^2) - (\Sigma^2 P(t) \tilde{B}_2) \tilde{R}_2^{-1} (\tilde{B}_2^T(t) P(t) \Sigma^2)$$

where

$$\tilde{R}_i \triangleq \begin{bmatrix} 0 & A_{22} & B_{12} \\ A_{22}^T & Q_{22} & S_2 \\ B_{12} & S_2^T & R_i \end{bmatrix}$$

$$\tilde{B}_i \triangleq [0 \quad A_{12} \quad B_{11}]$$

$$i = 1, 2.$$

$$A_{22} = 1, \quad A_{12} = 2, \quad Q_{22} = 1, \quad S_2 = [0 \quad 0]$$

$$B_1 = \begin{bmatrix} B_{11} \\ B_{12} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, \quad B_2 = \begin{bmatrix} B_{21} \\ B_{22} \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$$

Thus,

$$-1\dot{P}(t)1 = 1P(t)1 + 1P(t)1 + 1 - \left\{ 1P(t) \begin{bmatrix} 0 & 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 2 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 1 \\ 1 \end{bmatrix} P(t)1 \right\}$$

$$- \left\{ 1P(t) \begin{bmatrix} 0 & 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 2 \\ 1 \end{bmatrix} P(t)1 \right\} \quad (5.9)$$

$$-\begin{bmatrix} \dot{p}_1 & \dot{p}_2 \\ \dot{p}_2 & \dot{p}_3 \end{bmatrix} = \begin{bmatrix} -2.833p_1^2 + 2p_1 + 1 & -2.833p_2^2 + 2p_2 + 1 \\ -2.833p_2^2 + 2p_2 + 1 & -2.833p_3^2 + 2p_3 + 1 \end{bmatrix} \quad (5.10)$$

Now we can get $p_i(t)$ by solving the differential equation of $\dot{p}_i(t)$ with boundary condition of $p_i(T) = 0$.

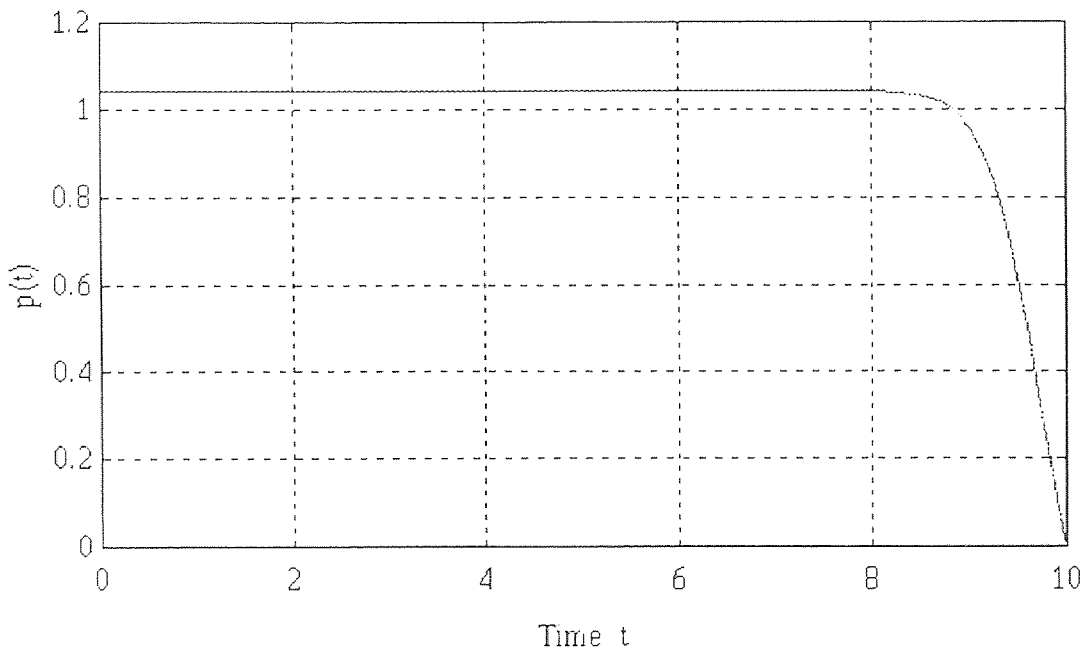


Figure 5.1 The solution of the Riccati equation for G_i^* .

From Figure 5.1, we can see that $p_i(t)$ are constants for $0 \leq t \leq 8.2$. Therefore, from a practical view point, it may be feasible to use the fixed gain matrix for processes of finite duration [13]. Therefore, the costate matrix $P_i^*(t)$ has elements

$$P_i^* = \begin{bmatrix} 1.04 & 1.04 \\ 1.04 & 1.04 \end{bmatrix} \quad (5.11)$$

Now we substitute all matrices into (5.5) which gives the solution of optimal gain

$$G_1^* = \begin{bmatrix} 2.08 & 2.08 \\ 3.12 & 3.12 \end{bmatrix} \quad (5.12)$$

$$G_2^* = \begin{bmatrix} 1.56 & 0.78 \\ 0.52 & 0.26 \end{bmatrix} \quad (5.13)$$

5.2 Example 2

Find the gain matrices such that J is minimized for the system

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} u_{11}(t) \\ u_{12}(t) \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} u_{21}(t) \\ u_{22}(t) \end{bmatrix} \quad (5.14)$$

$$y_1(t) = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

$$y_2(t) = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \quad (5.15)$$

to minimize the performance measure

$$J = E \left\{ \int_0^{10} \left[\begin{pmatrix} x_1(t) & x_2(t) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} u_{11}(t) & u_{12}(t) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u_{11}(t) \\ u_{12}(t) \end{pmatrix} + \begin{pmatrix} u_{21}(t) & u_{22}(t) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u_{21}(t) \\ u_{22}(t) \end{pmatrix} \right] dt \right\} \quad (5.16)$$

To solve this problem, the cost functional J reduces to

$$J = \int_0^{t_0} \text{tr} \left\{ \left[Q(t) + \sum_{i=1}^N C_i^T(t) G_i^T(t) R_i(t) G_i(t) C_i(t) \right] \bar{X}(t) \right\} dt \quad (5.17)$$

and by using the necessary condition results in:

$$G_i^*(t) = R_i^{-1}(t) B_i^T(t) P^*(t) E C_i^{-1}(t), \quad (5.18)$$

$$\begin{aligned} E^T \dot{P}^*(t) E = & -E^T P^*(t) A(t) - A^T(t) P^*(t) E \\ & + \sum_{i=1}^N E^T P^*(t) B_i(t) R_i^{-1}(t) B_i^T(t) P^*(t) E - Q(t) \end{aligned} \quad (5.19)$$

and the boundary condition

$$E^T P^*(T) E = 0. \quad (5.20)$$

For this performance criterion, the weighting matrices are seen to be

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad R_1 = R_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Let the costate matrix $P^*(t)$ be:

$$P^* = \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}$$

So to solve the $P^*(t)$, we apply the singular value decomposition of E

$$UEV^T = \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} \quad (5.21)$$

where $\Sigma^2 = \text{diag}(\sigma_1, \sigma_2, \dots, 0)$ and U and V are unitary matrices.

In example 1, equation (5.19) has the following form

$$\begin{aligned} -\Sigma^2 \dot{P}(t) \Sigma^2 = & \Sigma^2 P(t) A_{11} + A_{11}^T P(t) \Sigma^2 + Q_{11} - (\Sigma^2 P(t) \tilde{B}_1) \tilde{R}_1^{-1} (\tilde{B}_1^T(t) P(t) \Sigma^2) \\ & - (\Sigma^2 P(t) \tilde{B}_2) \tilde{R}_2^{-1} (\tilde{B}_2^T(t) P(t) \Sigma^2) \end{aligned}$$

where

$$\tilde{R}_i \triangleq \begin{bmatrix} 0 & A_{22} & B_{i2} \\ A_{22}^T & Q_{22} & S_2 \\ B_{i2} & S_2^T & R_i \end{bmatrix}$$

$$\tilde{B}_i \triangleq \begin{bmatrix} 0 & A_{i2} & B_{i1} \end{bmatrix}$$

$$i = 1, 2.$$

$$A_{22} = 3, \quad A_{i2} = 2, \quad Q_{22} = 1, \quad S_2 = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} B_{11} \\ B_{12} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, \quad B_2 = \begin{bmatrix} B_{21} \\ B_{22} \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

Thus,

$$\begin{aligned} -1\dot{P}(t)1 &= 1P(t)1 + 1P(t)1 + 1 - \left\{ 1P(t) \begin{bmatrix} 0 & 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 3 & 1 & 2 \\ 3 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 2 \\ 1 \\ 1 \end{bmatrix} P(t)1 \right\} \\ &\quad - \left\{ 1P(t) \begin{bmatrix} 0 & 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 3 & 1 & 1 \\ 3 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 2 \\ 2 \\ 1 \end{bmatrix} P(t)1 \right\} \end{aligned} \quad (5.22)$$

then gives

$$-\begin{bmatrix} \dot{p}_1 & \dot{p}_2 \\ \dot{p}_2 & \dot{p}_3 \end{bmatrix} = \begin{bmatrix} -1851p_1^2 + 2p_1 + 1 & -1851p_2^2 + 2p_2 + 1 \\ -1851p_2^2 + 2p_2 + 1 & -1851p_3^2 + 2p_3 + 1 \end{bmatrix}. \quad (5.23)$$

Now we can get $p_i(t)$ by solving the differential equation of $\dot{p}_i(t)$ with boundary condition of $p_i(T) = 0$.

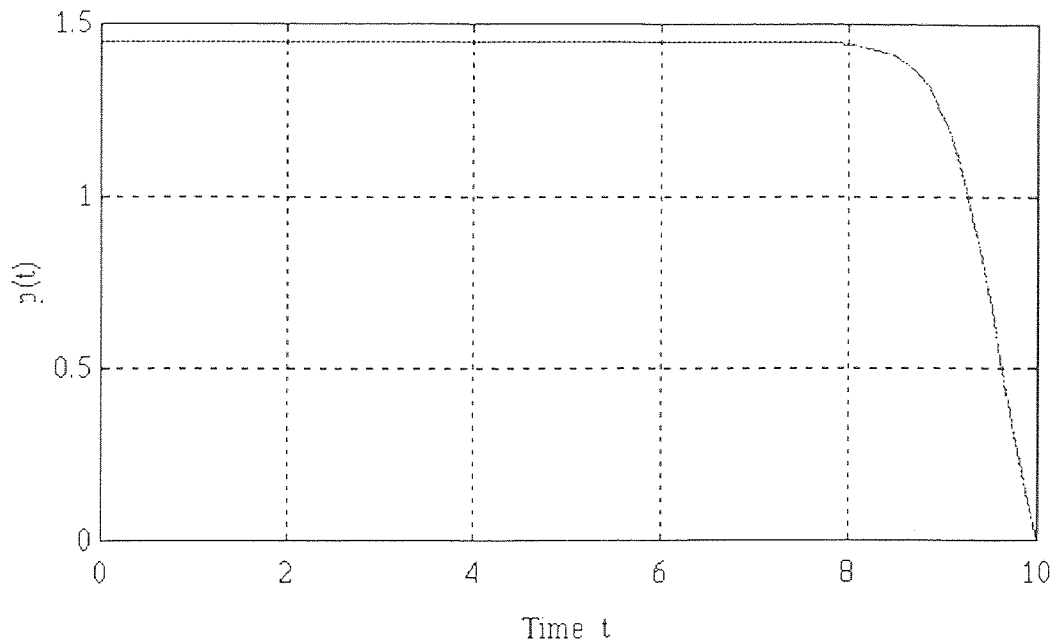


Figure 5.2 The solution of the Riccati equation for G_t^* .

From Figure 5.2, we can see that $p_i(t)$ are constants for $0 \leq t \leq 7.9$. Therefore, from a practical view point, it may be feasible to use the fixed gain matrix even for processes of finite duration. Therefore, the costate matrix $P_i^*(t)$ has elements

$$P_i^* = \begin{bmatrix} 1.45 & 1.45 \\ 1.45 & 1.45 \end{bmatrix}. \quad (5.23)$$

Now we substitute all matrices into (5.18) which gives the solution of optimal gain

$$G_1^* = \begin{bmatrix} 2.900 & 2.900 \\ 4.350 & 4.350 \end{bmatrix} \quad (5.24)$$

$$G_2^* = \begin{bmatrix} 4.350 & 2.175 \\ 2.900 & 1.450 \end{bmatrix}. \quad (5.25)$$

CHAPTER 6

CONCLUSION

In conclusion, the objective of this research paper was to consider the Decentralized Optimal Control in Descriptor systems using the Matrix Minimum Principle as stated in the introduction. In this thesis, we have presented a study of Decentralized Optimal Control in Descriptor systems which makes use of the linear-quadratic-Gaussian technique and output-variable feedback. Unlike the work done previously in Centralized Optimal Control Systems using output-variable feedback, our discussion is focused on a decentralized control approach.

In addition, Riccati equations for costate matrix $P(t)$ were derived which are analogous to the well-known Riccati equation of optimal control for state-space problems. However, the Riccati equation we derived was difficult to solve. So, in order to overcome this obstacle, we applied Bender's [5] method which uses the singular value decomposition of E . By applying this method, we solved the costate matrix $P(t)$ which provided us with the required optimal gain matrices $G_i^*(t)$. Thus, we found a complete feedback solution for our linear decentralized control problem in descriptor systems.

The contribution of this thesis was the application of the Matrix Minimum Principle to the design of the Decentralized Optimal Regulator in Descriptor Systems with the Gaussian random-vector initial state.

APPENDIX A

List of Gradient Matrices

$$\frac{\partial}{\partial X} \text{tr}[X] = I \quad (\text{A.1})$$

$$\frac{\partial}{\partial X} \text{tr}[AX] = A^T \quad (\text{A.2})$$

$$\frac{\partial}{\partial X} \text{tr}[AX^T] = A \quad (\text{A.3})$$

$$\frac{\partial}{\partial X} \text{tr}[AXB] = A^T B^T \quad (\text{A.4})$$

$$\frac{\partial}{\partial X} \text{tr}[AX^T B] = BA \quad (\text{A.5})$$

$$\frac{\partial}{\partial X^T} \text{tr}[AX] = A \quad (\text{A.6})$$

$$\frac{\partial}{\partial X^T} \text{tr}[AX^T] = A^T \quad (\text{A.7})$$

$$\frac{\partial}{\partial X^T} \text{tr}[AXB] = BA \quad (\text{A.8})$$

$$\frac{\partial}{\partial X^T} \text{tr}[AX^T B] = A^T B^T \quad (\text{A.9})$$

$$\frac{\partial}{\partial X} \text{tr}[XX] = 2X^T \quad (\text{A.10})$$

$$\frac{\partial}{\partial X} \text{tr}[XX^T] = 2X \quad (\text{A.11})$$

$$\frac{\partial}{\partial X} \text{tr}[X^n] = n(X^{n-1})^T \quad (\text{A.12})$$

$$\frac{\partial}{\partial X} \text{tr}[AX^n] = \left(\sum_{i=0}^{n-1} X^i A X^{n-1-i} \right)^T \quad (\text{A.13})$$

$$\frac{\partial}{\partial X} \text{tr}[AXBX] = A^T X^T B^T + B^T X^T A^T \quad (\text{A.14})$$

$$\frac{\partial}{\partial X} \text{tr}[AXBX^T] = A^T X^T B^T + AXB \quad (\text{A.15})$$

$$\frac{\partial}{\partial X} \text{tr}[e^X] = e^{X^T} \quad (\text{A.16})$$

$$\frac{\partial}{\partial X} \text{tr}[X^{-1}] = -(X^{-1} X^{-1})^T = -(X^{-2})^T \quad (\text{A.17})$$

$$\frac{\partial}{\partial X} \text{tr}[AX^{-1}B] = -(X^{-1} B A X^{-1})^T \quad (\text{A.18})$$

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