# ABSTRACT

## PROPERTIES AND DETERMINATION OF OPTIMAL DECENTRALIZED FEEDBACK STRUCTURE

by

Kangsong Han

In this thesis, the decentralized feedback structure for large scale, linear time invariant systems is studied. The internal differences and the relationship between decentralized feedback structure and centralized feedback structure are discussed. The conventional diagonal feedback structure, corresponding to the classical single loop design strategy, is first analyzed. This is followed by an arbitrary decentralized information flow constraint which is dependent upon the actual plant characteristics. Although signal flow graphs have limited use in describing decentralized control systems, the concept of a control cycle unit based on signal flow is introduced as a supplementary tool to characterize some fixed modes and decentralized feedback structures. For the decentralized feedback structure, the Jordan normal form method and essential control tuple space method are presented. The later method can be readily applied in a computer-aided design environment.

From the theory a set of relationships of eigenvalues and eigenvectors between the plant system and the synthesis system are deduced. Based upon such eigenstructures, conditions have been found to determine the optimal decentralized feedback structure, that is, one with the least number of non-zero gain elements. The notion of a feedback gain lattice is introduced for both the diagonal and Jordan form representation of the plant state matrices. This lattice structure is then utilized algorithmically to generate the optimal decentralized feedback structure. These algorithms can be used to reduce hardware implementation and system complexity for the control of large scale systems.

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by
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## Properties and Determination of Optimal Decentralized Feedback Structure

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| | |
|---|---|
| $\forall$ | for all |
| $\exists$ | there exists |
| $\Rightarrow$ | implies |
| $(\Rightarrow)$ | sufficient condition |
| $(\Leftarrow)$ | necessary condition |
| $\triangleq$ | equal by definition |
| $\phi$ | empty set |
| $\{x_i\}$ | the set of whose elements are $x_i$ |
| $x$ | elements of set or vector |
| $\in$ | belonging to |
| $\subset$ | contained in |
| $\subseteq$ | contained in or equal to |
| $\backslash 0$ | exclude 0 |
| $\cap$ | intersection |
| $\mathbf{L}$ | lattice |
| $\mathbf{C}$ | complex number |
| $\mathbf{R}$ | real number |
| $\mathbf{R}^{n \times m}$ | $n \times m$ matrix |
| $\mathbf{K}$ | decentralized feedback structure |
| $\mathbf{T}$ | control tuple space |
| $\square$ | end of proof |
| $\sigma(.)$ | spectrum |
| $\rho(.)$ | counting function |
| $G$ | combination operator |

# CHAPTER 1
# INTRODUCTION

In the control of large scale industrial process, implementing and maintaining the feedback links constitute a major hardware cost. For example, consider a linear system with $m$ inputs and $r$ outputs. In conventional centralized feedback control, the feedback matrix has $rm$ non-zero elements (interconnection). If $r = 100$, $m = 100$, then $r \times m = 10,000$. This may be an unrealistically large number from both design and hardware point of view. Therefore, a decentralized feedback structure with the fewest non-zero elements should be used, provided the resultant synthesis system remaining stablizable. The notion of decentralized fixed mode [4],[6] has long been used as the primary criterion in accessing the feasibility of a certain decentralized feedback structure.

A decentralized feedback structure is said to be admissible if the corresponding synthesis system has no decentralized fixed modes. In this thesis, we focus on the first step of the decentralized control system design: how to obtain an admissible decentralized feedback structure with a least number of non-zero interconnection elements, such a least number of structure is said to be optimal. In this case, the stabilization technique described in [4] may be used to control the plant.

It should be noted again that the word "optimal" in a decentralized feedback structure refers to the minimizing if feedback interconnections rather than performance of the closed loop systems. The selection of a decentralized feedback structure that balances structural and performance optimizing is a topic for future research. Here we introduce a decentralized feedback structure used to compensate the plant modes and to explicitly distinguish the decentralized from the usual centralized feedback. It can be thought as a part of feedback structure of whole control system because the other part will be designed to stabilize control system. Comparing with the centralized feedback structure (matrix), the decentralized feedback structure (matrix) has several internal properties: (1) there exist much more zero elements; (2) each zero element is determined before controller design; (3) each non-zero element value can be changed in larger range; (4) the selection of each element is limited by decentralized information flow constraint [4], and (5) each feedback structure is dependent on the actual plant models.

Before discussing the decentralized feedback structure, let us review the relevant literatures. In the decentralized and centralized control system field, the notion of fixed mode was introduced and researched by Davison [4],[5],[6],[17]. A mode is said to be a decentralized fixed mode (DFM) if it is an eigenvalue of the system matrix which can not be altered by some linear feedback components. This is a generalization of the uncontrollable or unobservable mode of centralized control problem. Traditional treatment of decentralized control structure is to pre-impose a certain decentralized information flow constraint, for example, one corresponding to a block diagonal feedback structure. Analysis is then applied to this decentralized structure to determine if the resultant system possesses any decentralized fixed modes. Such approach is useful in standardizing the analysis of decentralized fixed modes but may not be as convenient in dealing with the synthesis aspects of the system. Therefore, in this thesis, all decentralized structure are considered. It should be noted that any general decentralized information flow constraint (DIFC) may be converted into the traditional framework by means of non-singular input-output transformation. Finding the fixed modes from the aspect of transmission zero of a plant was addressed by Davison and Chang [4]. The advantage of this method not only the developed algorithms, but also introduced the certain square subsystems which can effect on fixed modes, and easily be applied in computer. Vaz and Davison [16] presented a method to find approximate decentralized fixed modes.

In the theoretical control area, many new mathematical tools were introduced to design control system. For example, the singular value decomposition and principal component analysis were given by Klema [11] and Moore [13]. They made use of singular value analysis and studied its application for controllability, observability, and minimal realization . The model order reduction of a plant can be completed by singular value analysis and then the feedback matrix dimension can be further reduced to the synthesis system control. In fact, the objective of model reduction is the same as that of decentralized feedback structure design, i.e. in order to significantly decrease the complexity of the controllers. Brockett [2] developed the linear central control theorem based on the transition matrix and the Gramian formula in time domain. Although many theorems were developed, but due to the limitation of transition matrix in time field, these could not be extended in application of practical plants. These theorems are limited in control system theory and analysis. Basile [1] developed the linear control theory by the tools of geometric theory and linear algebra. Based on the concept of an invariant space, many useful concepts, theorems, and algorithms were presented. Some of the results can be used to decentralized control systems and feedback structures to analyze qualitatively.

This thesis is based on decentralized control theory, linear time-invariant (LTI) system theory, linear algebra, geometric theory, and signal flow graph. The objective of thesis is to develop theories and algorithms for searching for an optimal decentralized feedback structure. Especially, I hope that a few of new idea in this thesis can be considered and developed in future. Because the decentralized feedback structure can be determined based on many different criteria, the objectives of this work are: 1) retain only those gain elements in the feedback matrix that can be used to shift all modes and 2) use the minimum number of non-zero elements in the feedback matrix to shift all modes. Since the Gramian formula is inconvenient in analyzing decentralized structure, two additional tools, signal flow graph and geometric approach will be introduced. Because the decentralized feedback matrix is determined before a system control design on a plant model,the value of each gain element in the decentralized feedback matrix often may be independent or partial dependent on the plant model. Generally speaking, non-unique solution will often occur. For a synthesis system, its structure is not changed, but the some element values in the synthesis model may be varied in a certain interval.

The organization of this thesis is as follows: Chapter 2 determines the problem and outline the relevant existing results. From a decentralized control aspect, the notion of fixed mode is defined and classified. A few well-known theorems are cited. The main result on fixed mode classification is the relationship between fixed modes and transmission zeros. Chapter 3 deals with the general feedback structure description in terms of matrices and signal flow graphs and determination of the minimal number of non-zero elements in the feedback structure necessary for shifting all modes in a plant. Finally, Chapter 4 deals with the theoretical development in decentralized feedback structure. The use of eigenstructure analysis provides a useful way in characterizing decentralized fixed mode and in determining the admissible and optimal feedback structure.

It should be noted that only the selection of admissible decentralized feedback structure is considered in this thesis. The stabilizing control system for the synthesis system may be developed in future.

The linear time-invariant system description has many forms. The strictly proper state-space description is given by

$$\begin{cases} \dot{x} &= Ax + Bu \\ y &= Cx \end{cases} \tag{1.1}$$

is referred to as a $(C, A, B)$ triple, while general proper dynamic system is given by

$$\begin{cases} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{cases} \tag{1.2}$$

is referred to as a $(C, A, B, D)$ quadruple. For the sake of simplicity, most of the analyses in this thesis will be referred to triple system, the method which is used to extend the triple system to the quadruple system awaits future research effect.

# CHAPTER 2
# DECENTRALIZED FEEDBACK STRUCTURE

## 2.1 Decentralized or Centralized Information Flow Constraint

Decentralized control approach is an extension to centralized control approach. Many papers [4] [5] [6] deal with the existence of decentralized fixed modes (DFM) in a large scale system under certain decentralized information flow constraint. Assume that the following output feedback controller

$$u = Ky \qquad \text{for } K \in \mathbf{K}_{\triangle} \tag{2.1}$$

is applied to (1.1) or (1.2), where $x \in \mathbf{R}^{n\times 1}$, $u \in \mathbf{R}^{r\times 1}$, $y \in \mathbf{R}^{m\times 1}$, $A \in \mathbf{R}^{n\times n}$, $B \in \mathbf{R}^{n\times m}$, $C \in \mathbf{R}^{r\times n}$, $D \in \mathbf{R}^{r\times m}$. The standard decentralized feedback structure is block diagonal defined as follow:

**Definition 2.1** (Decentralized Information Flow Constraint)[4]

The output feedback is said to have a decentralized information flow constraint $\mathbf{K}_{\triangle}$ imposed on linear time-invariant system (1.1) or (1.2) if $K \in \mathbf{K}_{\triangle}$ where

$$\mathbf{K}_{\triangle} \triangleq \{K \in \mathbf{R}^{m\times r} | \ K = \text{ block diag } (K_1, K_2, \cdots, K_{v_s}), K_i \in \mathbf{R}^{m_i\times r_i}, i = [1, v_s] \ \} \tag{2.2}$$

The linear time-invariant system (1.2) can be rewritten to explicitly show the dependency on the $v_s$ control agents, i.e.

$$\begin{cases} \dot{x} &= Ax + \sum_{i=1}^{v_s} B_i u_i \\ y_j &= C_j x + \sum_{i=1}^{v_s} D_{ji} u_i \end{cases} \qquad j = 1, 2, ..., v_s$$

where $u_i \in \mathbf{R}^{m_i}$, $y_j \in \mathbf{R}^{r_i}$.

Frequently, the control agents $v_s$ and $\mathbf{K}_\triangle$ structure constraint are determined in advance and independent on any element distribution in the $A$, $B$, $C$, $D$ matrices of a practical plant model. However, some plant modes may not be shifted by any one of $K_i$, $i = [1, v_s]$. In this case, the $m_i$ and $r_i$ in each $K_i$ need to be increased in order to shift all modes. The increase of $m_i$ and $r_i$ forces $\mathbf{K}_\triangle$ to approach $\mathbf{R}^{m\times r}$, central control structure.

**Definition 2.2** (Decentralized Fixed Modes)[4]

Assume that $u = Ky$, $K \in \mathbf{K}_\triangle$ controller is applied to (1.1) or (1.2). There exists a decentralized fixed mode (DFM) $\lambda_c \in \mathbf{C}$ with respect to $\mathbf{K}_\triangle$ if

$$\lambda_c \in \bigcap_{\forall K \in \mathbf{K}_\triangle} \sigma(A + BKC) \quad \text{or} \quad \lambda_c \in \bigcap_{\forall K \in \mathbf{K}_\triangle} \sigma(A + BK(I - DK)^{-1}C) \qquad (2.3)$$

where $\sigma$ denotes the set of eigenvalues of (.).

From Definition 2.2, $\lambda_c$ is exactly called fixed mode of output feedback control. Certainly, there is the fixed mode for state feedback control. For the sake of simplicity, the fixed modes mentioned in this thesis correspond to those of output feedback.

It should be further noted that an open loop system is often called plant (system) and a closed loop system is often called synthesis system. For the sake of explicitness, the $\lambda_o$, $\lambda_c$ are denoted as the open loop eigenvalue and the closed loop eigenvalue respectively. If $K = 0$ in (2.3), the fixed mode $\lambda_c$ of a synthesis system is equal to the eigenvalue $\lambda_o$ of a plant system. Definition 2.2 implies that any fixed mode is brought by any $K \in \mathbf{K}_\triangle$ in which the element does not shift the eigenvalue of a plant system.

**Definition 2.3** (Centralized Fixed Modes)[4]

Assume that $u = Ky$, $K \in \mathbf{R}^{m \times r}$ controller is applied to (1.1) or (1.2). There exist a centralized fixed mode (CFM) $\lambda_c \in \mathbf{C}$ if

$$\lambda_c \in \bigcap_{\forall K \in \mathbf{R}^{m \times r}} \sigma(A + BKC) \quad \text{or} \quad \lambda_c \in \bigcap_{\forall K \in \mathbf{R}^{m \times r}} \sigma(A + BK(I - DK)^{-1}C)$$

Comparing with Definition 2.2, $K$ in Definition 2.3 is not constrained. As we know, if a plant system is a minimal realization then the set of centralized fixed modes in its synthesis system is empty. Therefore, the centralized fixed modes correspond to the uncontrollable eigenvalues and/or the unobservable eigenvalue in a plant system.

**Theorem 2.1** Given a completely controllable and observable quadruple $(C, A, B, D)$ system. Assume a centralized output feedback of the form $u = Ky$, $K \in \mathbf{R}^{m \times r}$ is applied to it, then there does not exist any fixed mode in the synthesis system $(C, A + BK(I - DK)^{-1}C, B, D)$, i.e.

$$\bigcap_{\forall K \in \mathbf{R}^{m \times r}} \sigma(A + BK(I - DK)^{-1}C) = \phi$$

**Proof.** Because the plant system is controllable and observable, it implies that $\mathrm{Im} B \neq \phi$, $\ker C = \phi$ and almost any eigenvalues of synthesis system can be placed with a suitable $K$ [1]. Therefore any eigenvalues can be shifted by $K \in \mathbf{R}^{m \times r}$. □

Theorem 2.1 implies that if $K \in \mathbf{K}_{\triangle}$ is constrained, then the situation without any fixed mode can not be guaranteed even for a controllable and observable plant system. In this thesis, we mainly emphasize on how to find the special decentralized feedback structure, denoted by $\mathbf{K}$ so that no fixed modes occur for the synthesis system when the plant is controllable and observable. Because the number of control agents $v_s$ is limited for a decentralized control system, the choice $K \in \mathbf{R}^{m \times r}$ is not suitable and may not be admitted to be used for a decentralized control system. On the other hand, if using the fixed-limited control agents $v_s$ like (2.2) we may squander many agents which do not shift on some fixed modes. Therefore, a realistic decentralized information flow constraint (RDIFC) which depends on the realistic plant model is very important. A RDIFC not only should satisfy with the requirement that the $v_a$ control agents be limited in $K \in \mathbf{K}_\triangle$ (DIFC), but also should shift all modes (like $K \in \mathbf{R}^{m\times r}$) for a controllable and observable system. The RDIFC takes advantage of both virtues of $\mathbf{K}_\triangle$ (DIFC) and $\mathbf{R}^{m\times r}$.

**Definition 2.4** (Realistic Decentralized Information Flow Constraint (RDIFC))

The output feedback(2.1) is said to have a realistic decentralized information flow constraint $\mathbf{K}$ imposed on the linear time-invariant system (1.1) or (1.2) if $K \in \mathbf{K}$ and

$$\mathbf{K} \triangleq \{K = [k_{ij}] \in \mathbf{R}^{m\times r} | \ k_{ij} \equiv 0, \quad \forall (i,j) \notin S, \ i = [1,m], j = [1,r] \ \} \tag{2.4}$$

where $S$ is a set which depends on a realistic plant system and a algorithm, which will be detailed in later chapters.

Comparing with (2.2), if $S$ is replaced by $K =$ block diag $(K_1, K_2, \cdots, K_{v_s})$ then (2.4) is equal to (2.2). Hence the Definition 2.4 contains Definition 2.1 if $S$ is independent on the actual plant system and the algorithm. For the decentralized fixed mode. The Definition 2.2 is still valid if the $\mathbf{K}$ instead of $\mathbf{K}_\triangle$. .

## 2.2 Fixed Model Feature and Classification

A centralized fixed mode must also be a decentralized fixed mode, due to the decentralized feedbacks are contained in the centralized feedback. Although there is meaning for the invariant zero in both plant system and synthesis system, for the sake of simplicity, the invariant zero[7],[15] mentioned in this thesis often indicates that of plant system. According to Theorem 2.1, this result inversely implies that a decoupling zero[7],[15] may introduce some fixed modes. Because the decoupling zero will produce an non-minimal realization for a plant system, this implies that there exists the internal relationship between invariant zero and fixed mode.

**Definition 2.5** (Transmission Zero)[7][15]

A complex number $\lambda$ is a transmission zero of a quadruple $(C, A, B, D)$ if

$$\text{rank} \begin{bmatrix} A - \lambda I & B \\ C & D \end{bmatrix} < n + \min(r, m) \tag{2.5}$$

A complex number $\lambda$ is a input decoupling zero if rank $\begin{bmatrix} A - \lambda I & B \end{bmatrix} < n$. A complex number $\lambda$ is a output decoupling zero if rank $\begin{bmatrix} A - \lambda I \\ C \end{bmatrix} < n$. The transmission zeros and decoupling zeros are contained in the set of invariant zero.

**Lemma 2.1** [4] Consider a quadruple $(C, A, B, D)$ system. the synthesis system eigenvalue $\lambda_c \in \sigma(A)$ is a decentralized fixed mode with respect to $\mathbf{K}$ if and only if $\lambda_c$ is a transmission zero of all the following square subsystem:

1. $\{c_{j_1}, A, b_{i_1}\}$ $\forall (i_1, j_1) \in S$.
2. $\left\{ \begin{bmatrix} c_{j_1} \\ c_{j_2} \end{bmatrix}, A, \begin{bmatrix} b_{i_1} & b_{i_2} \end{bmatrix}, \begin{bmatrix} 0 & d_{j_1 i_2} \\ d_{j_2 i_1} & 0 \end{bmatrix} \right\}$ $\forall (i_1, j_1), (i_2, j_2) \in S$.
3. $\left\{ \begin{bmatrix} c_{j_1} \\ c_{j_2} \\ \vdots \\ c_{j_s} \end{bmatrix}, A, \begin{bmatrix} b_{i_1} & b_{i_2} & \cdots & b_{i_s} \end{bmatrix}, \begin{bmatrix} 0 & d_{j_1 i_2} & \cdots & d_{j_1 i_s} \\ d_{j_2 i_1} & 0 & \cdots & d_{j_2 i_s} \\ \vdots & & \ddots & \\ d_{j_s i_1} & \cdots & d_{j_s i_{s-1}} & 0 \end{bmatrix} \right\}$
   $\forall (i_1, j_1), (i_2, j_2), \ldots, (i_s, j_s) \in S$.

**Corollary 2.1** Consider a quadruple $(C, A, B, D)$ system. $\lambda_c \in \sigma(A)$ is a centralized fixed mode with respect to $\mathbf{R}^{m \times r}$ if and only if $\lambda_c$ is a transmission zero of all the square subsystems of $(C, A, B, D)$.

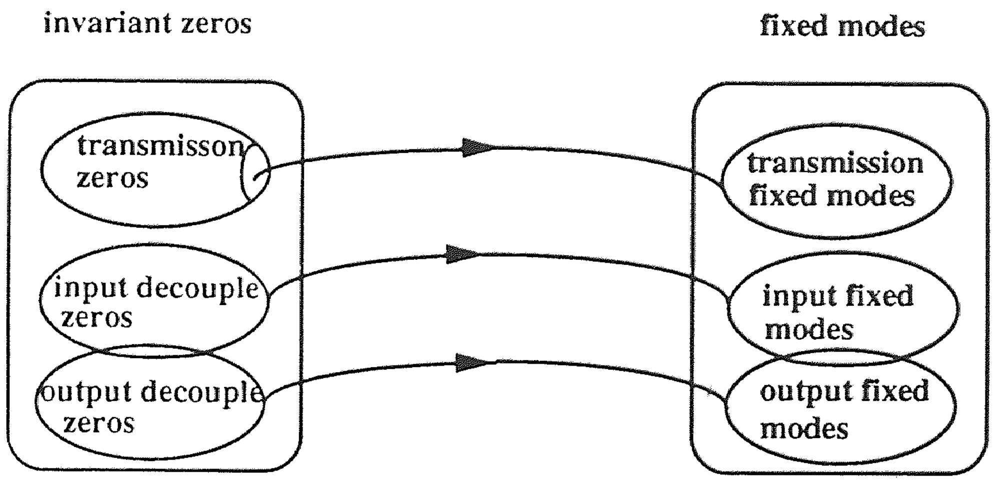


Figure 2.1: Relationship between Invariant Zeros and Fixed Modes

**Proof.** The proof is completed by equating $\mathbf{K}$ to $\mathbf{R}^{m \times r}$. The details are omitted. □

The fixed modes may be caused by invariant zeros. For the sake of clarity, fixed modes can be classified under the invariant zero classification [16]: invariant fixed modes and transmission fixed modes. This is because the fixed mode of synthesis system are caused by invariant zeros. The relationship can be described like Figure 2.1. In the case of $r = m$, all centralized fixed modes (CFM) are contained in the transmission zero (TZ) set, i.e. $CFM \subset TZ$. In the case of $r \neq m$, the centralized fixed modes are not contained in the transmission zero set, i.e. $CFM \not\subset TZ$[5]. The fixed modes which are caused by input decoupling zeros and output decoupling zeros are called input fixed mode and output fixed mode respectively. The fixed modes which are caused by transmission zeros are called transmission fixed mode. In fact, due to the cancellation between a transmission zero and a pole, the transmission zero may produce a fixed mode. The invariant fixed modes contain both input fixed modes and output fixed modes.

**Property 2.1** Consider a quadruple $(C, A, B, D)$ system. Let $u = Ky$, $K \in \mathbf{K}$ then the output fixed modes with $\mathbf{K}$, remain invariant i.e.RDIFC, can not be shifted by $K \in \mathbf{R}^{m \times r}$. They can be shifted only if $u = Kx$ by $K \in \mathbf{R}^{m \times n}$. According to duality, if $u = Ky$, $K \in \mathbf{K}$ then any input fixed modes with $\mathbf{K}$ can not be shifted by $K \in \mathbf{R}^{n \times m}$. They can be shifted only if $\dot{x} = Ku$ by $K \in \mathbf{R}^{n \times r}$

**Proof.** Because the output fixed modes are generated by a output decoupling zero, the output decoupling zero implies that there exist a disconnection between a state and an output. Therefore no output feedback can be imposed on these output fixed modes except for a state feedback. In duality, because the input fixed modes are generated by input decoupling zero, the input decoupling zeros implies that there exist a disconnection between an input and a deferential state. Therefore no output feedback can be imposed on these input fixed modes except for an additional input.
□

## 2.3 Relationship between Signal Flow Graph and Feedback Structure

**Definition 2.6** If a feedback matrix $K \in \mathbf{R}^{m \times r}$ has no zero elements, then the matrix is called full matrix or centralized feedback structure. Otherwise it is called a non-full matrix or decentralized feedback structure.

**Definition 2.7** Let $\rho$ be a counting function imposed on a K matrix or its structure described by

$$\rho(K) = \text{the number of non} - \text{zero elements in } K$$

**Example 2.1** Assume $K = \begin{bmatrix} \times & 0 \\ \times & 0 \\ 0 & \times \end{bmatrix}$. The $K$ is called non-full or decentralized feedback structure and $\rho(K) = 3$.

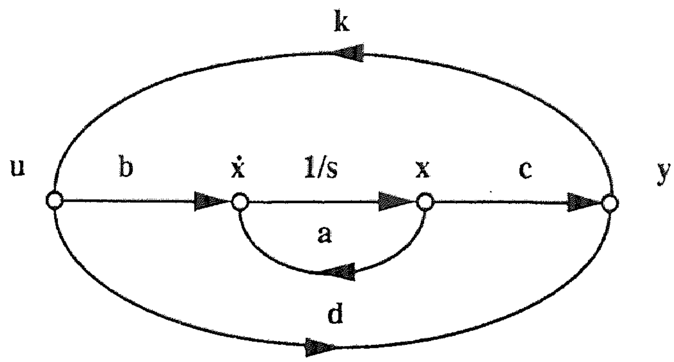


Figure 2.2: The Single-Input and Single Output System Description in Signal Flow Graph

Let the minimal and the maximal number of non-zero elements in $K$ be given by $\rho(\mathbf{K}^*)$ and $\rho(\mathbf{K})$ if $K \in [\mathbf{K}^*, \mathbf{K}]$, where $\mathbf{K}^* \subset \mathbf{K}$. Generally, the minimization of the number of non-zero feedback elements is appealing in the control of large scale systems. Hardware complexity can be significantly reduced if a suitable feedback structure is chosen so that the resultant system has no fixed modes.

For a single-input/single-output closed loop control system, the system consists of four signal nodes $(u, y, \dot{x}, x)$ and six oriented branches containing six gains $(a, b, c, d, \frac{1}{s}, k)$, which can be described as a signal flow graph in Figure 2.2.

Decomposing a complex system into interconnected unit systems, for example $(C_j, A, B_i)$ for a triple $(C, A, B)$, will facilitate the analysis of the system. The decomposing method is useful because many properties of a whole system are often determined by analyzing the corresponding properties of the subsystems. A complex system consisting of numerous interconnected parts can be presented by drawing a signal-flow graph. Its application range is restricted to show the internal structure. The major advantage of the signal-flow graph is that the input and output decoupling zero can be found directly. From them, the invariant fixed mode can be derived directly. The disadvantage is that signal-flow graphs does not show the transmission zeros of large scale systems and the cancellation transmission fixed mode directly, but it can show some transmission fixed modes and be used in computer for large scale system control. Please note that a fixed mode exists for $\forall K \in \mathbf{K}$ or $\mathbf{R}^{m \times r}$. Therefore the transmission fixed modes may be generated by a signal flow stuck, or a stuck around, or no signal flow pass through these modes which will be mentioned more detail in Chapter 3.

A signal-flow graph is composed of the brunches and the nodes. A brunch has oriented characteristics by an arrow. The nodes are classified as independent nodes and dependent nodes. Clearly, every dependent node represents a linear equation, so that the graph is equivalent to as many linear equations with many unknown variables, i.e. dependent nodes. When the feedback structure is synthesized for a plant system, the signal flow graph has an additional branch for every non-zero element in feedback matrix. The zero elements do not map any one corresponding branch. Some nodes may change from independent node into dependent node due to the additional interconnection by non-zero feedback gains.

**Definition 2.8** An oriented cyclic loop with respect to $u$ node is called an active loop. The corresponding node on an active loop is called a life node. Any node in a non-active loop is called fixed or dead node.

**Property 2.2** Given a triple $(A, B, C)$ and assume the number of distinct eigenvalues $v$ $(\leq n)$ and $A$ matrix can be transformed into a Jordan form with $v$ $(\leq n)$ distinct and elementary Jordan blocks, i.e. every algebraic multiplicity is equal to the corresponding geometric multiplicity. In this condition, a $i^{th}$ mode can be shifted if and only if the corresponding $\dot{x}_i$ is life node.

**Proof.** Because any an $i^{th}$ mode exists in $\dot{x}_i \rightarrow x_i$, the signal flow in a active loop for shifting the $i^{th}$ mode must enter $\dot{x}_i$ node. Because the fixed mode is based on any

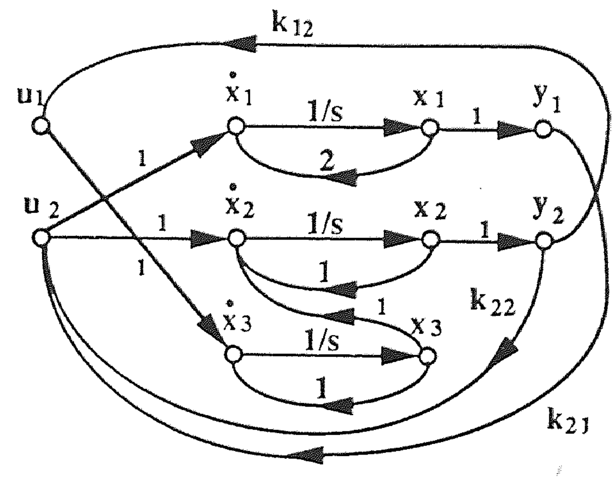


Figure 2.3: The Signal Flow Graph of A Synthesis System

$K$ and $v$ distinct eigenvalues condition, therefore the active loop can not generate a constant transmission zero for any $K$ if the cancellation[16] between zero and pole and the decoupling zero in transfer function does not occur. □

**Example 2.2** Consider a plant

$$\begin{cases} \dot{x} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} u \\ y = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} x \end{cases}$$

and a feedback structure $\mathbf{K} = \begin{bmatrix} 0 & k_{12} \\ k_{21} & 0 \end{bmatrix}$. In Figure 2.3, loops $u_1 \rightarrow \dot{x}_3(x_3) \rightarrow \dot{x}_2(x_2) \rightarrow y_2 \rightarrow u_1$ and $u_2 \rightarrow \dot{x}_1(x_1) \rightarrow y_1 \rightarrow u_2$ are active loop. All nodes are life nodes. There are no decentralized modes in this synthesis system.

**Example 2.3** Given a triple $(C, A, B)$ as $A = I_2$, $B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 1 \end{bmatrix}$. We can find that $\dot{x}_3(x_3)$, $\dot{x}_2(x_2)$ are live nodes, but the eigenvalue of this plant $\lambda_o = 1$ is non-distinct in two elementary Jordan blocks. Therefore the eigenvalue of the synthesis system $\lambda_c$ may a fixed mode. We can verify this result by obtaining the transfer

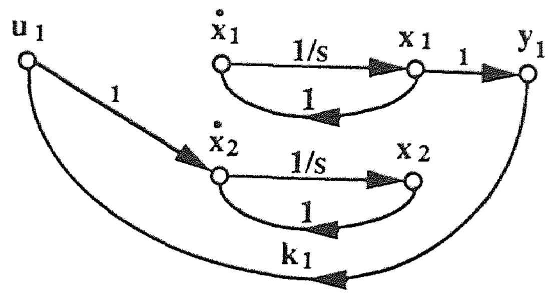


Figure 2.4: Invariant Fixed Mode Description

functions of the plant system $(C, A, B)$ and the synthesis system $(C, A + BKC, B)$, denoted by $H_o(s)$ and $H_c(s)$ respectively, are given

$$H_o(s) = \frac{1}{2(s-1)} = \frac{(s-1)}{2(s-1)^2}$$

$$H_c(s) = \frac{2}{(s-1-2K)} = \frac{2(s-1)}{(s-1)(s-1-2K)}$$

The one of two eigenvalues $\lambda_c = \lambda_o = 1$ is a fixed mode caused by the cancellation between a transmission zero and a pole in the both system.

**Corollary 2.2** An $i^{th}$ mode must be fixed mode if and only if the corresponding $\dot{x}_i$ is a dead node for a minimal realization system.

**Example 2.4** In Figure 2.3, if $\mathbf{K} = \begin{bmatrix} 0 & 0 \\ k_{21} & 0 \end{bmatrix}$ i.e. cross out $k_{12}$, then $\lambda_c = 1$ is fixed mode, because the corresponding $\dot{x}_2$, $\dot{x}_3$ are not in any oriented cycle loop.

**Example 2.5** In Figure 2.4, $\dot{x}_1$, $\dot{x}_2$ are dead nodes. So, $\lambda_c = 1$ is either input invariant mode or output invariant fixed mode.

## 2.4 Decentralized Feedback Structure Analysis

The invariant fixed modes are generated by decoupling zeros. This type of fixed modes can not be shifted directly by output feedback structure. In this thesis, we mainly consider transmission fixed modes which are not caused by cancellation zero with certain **K** .

General feedback structure can be described by a matrix and a signal flow graph. For Example 2.2, we can take $\mathbf{K} = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}$ too. In Example 2.5, if add only $k_{22}$, the "fixed" mode $\lambda_c = 1$ can now be shifted by either $k_{12}$ or $k_{22}$ because the loops is active, i.e.

$$\begin{array}{ll} \text{for } k_{12}: & u_1 \rightarrow \dot{x}_3(x_3) \rightarrow \dot{x}_2(x_2) \rightarrow y_2 \rightarrow u_1 \\ \text{for } k_{22}: & u_2 \rightarrow \dot{x}_2(x_2) \rightarrow y_2 \rightarrow u_2 \end{array}$$

In general, several different feedback elements may shift the same mode. It is of interest to come up with an "minimal" feedback structure in which the number of non-zero feedback elements $\rho(K)$ is minimized.

**Definition 2.9** A feedback structure, denoted by **K**, is called an admissible decentralized feedback structure if the resultant synthesis has no DFMs.

**Definition 2.10** A feedback structure, denoted by $\mathbf{K}^*$, is called minimal decentralized feedback structure if it is an admissible feedback structure with the least number of non-zero elements.

**Example 2.6** In Figure 2.3 if add $k_{11}$ and $k_{22}$, we can conclude that

$$\begin{array}{lll} k_{11} & \text{can shift} & \lambda_o = \{1,2\} \text{ based on } k_{22} \\ k_{12} & \text{can shift} & \lambda_o = \{1,1\} \\ k_{21} & \text{can shift} & \lambda_o = \{2\} \\ k_{22} & \text{can shift} & \lambda_o = \{1\} \end{array}$$

The admissible feedback structure is generally non-unique. This example give us the follow admissible feedback structure denoted by **K** and minimal feedback structure denoted by $\mathbf{K}^*$ in matrix description:

$$\mathrm{K} = \begin{bmatrix} k_{11} & 0 \\ 0 & k_{22} \end{bmatrix} \text{ or } \begin{bmatrix} 0 & k_{12} \\ k_{21} & 0 \end{bmatrix} \text{ or } \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \text{ or } \begin{bmatrix} 0 & k_{12} \\ k_{21} & k_{22} \end{bmatrix}$$

$$\mathrm{K}^* = \begin{bmatrix} k_{11} & 0 \\ 0 & k_{22} \end{bmatrix} \text{ or } \begin{bmatrix} 0 & k_{12} \\ k_{21} & 0 \end{bmatrix}$$

The corresponding feedback structures described by the signal flow graphs are in Figure 2.5

We can find that $\mathbf{K}^* \subseteq \mathbf{K}$. As the plant complexity increase, the signal flow graph description of the feedback structures becomes less intuitive. Now let us look at the fixed mode from invariant zero for Example 2.2.

$$\text{rank}\begin{bmatrix} A-\lambda I & B \\ C & D \end{bmatrix} = \text{rank}\begin{bmatrix} 2-\lambda & 0 & 0 & 0 & 1 \\ 0 & 1-\lambda & 1 & 0 & 1 \\ 0 & 0 & 1-\lambda & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} \equiv 5$$

This means that in this plant no invariant fixed mode exists for a suitable feedback structure. If $A$, $B$ change into $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}$ then

$$\text{rank}\begin{bmatrix} A-\lambda I & B \\ C & D \end{bmatrix} = \text{rank}\begin{bmatrix} 2-\lambda & 0 & 0 & 0 & 1 \\ 0 & 1-\lambda & 0 & 0 & 0 \\ 0 & 0 & 1-\lambda & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} < 5 \qquad \text{for } \lambda = 1$$

$$\text{rank}\begin{bmatrix} A-\lambda I & B \end{bmatrix} = \text{rank}\begin{bmatrix} 2-\lambda & 0 & 0 & 0 & 1 \\ 0 & 1-\lambda & 0 & 0 & 0 \\ 0 & 0 & 1-\lambda & 1 & 0 \end{bmatrix} < 3 \qquad \text{for } \lambda = 1$$

$$\text{rank}\begin{bmatrix} A-\lambda I \\ C \end{bmatrix} = \text{rank}\begin{bmatrix} 2-\lambda & 0 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} < 3 \qquad \text{for } \lambda = 1$$

we can find that $\lambda = 1$ is either input and output decoupling zeros. Therefore $\lambda = 1$ is also an invariant fixed mode. Generally, the invariant fixed mode is in no way to be moved by the output feedback controller. If there exists a decoupling zero, the system must exist on inherent fixed mode, because of non-minimal realization.

Based on the Gauss Elimination theory[14], the $D$ term in (2.5) may alter the system matrix rank, i.e. changing the possibility of the transmission zero emergence.

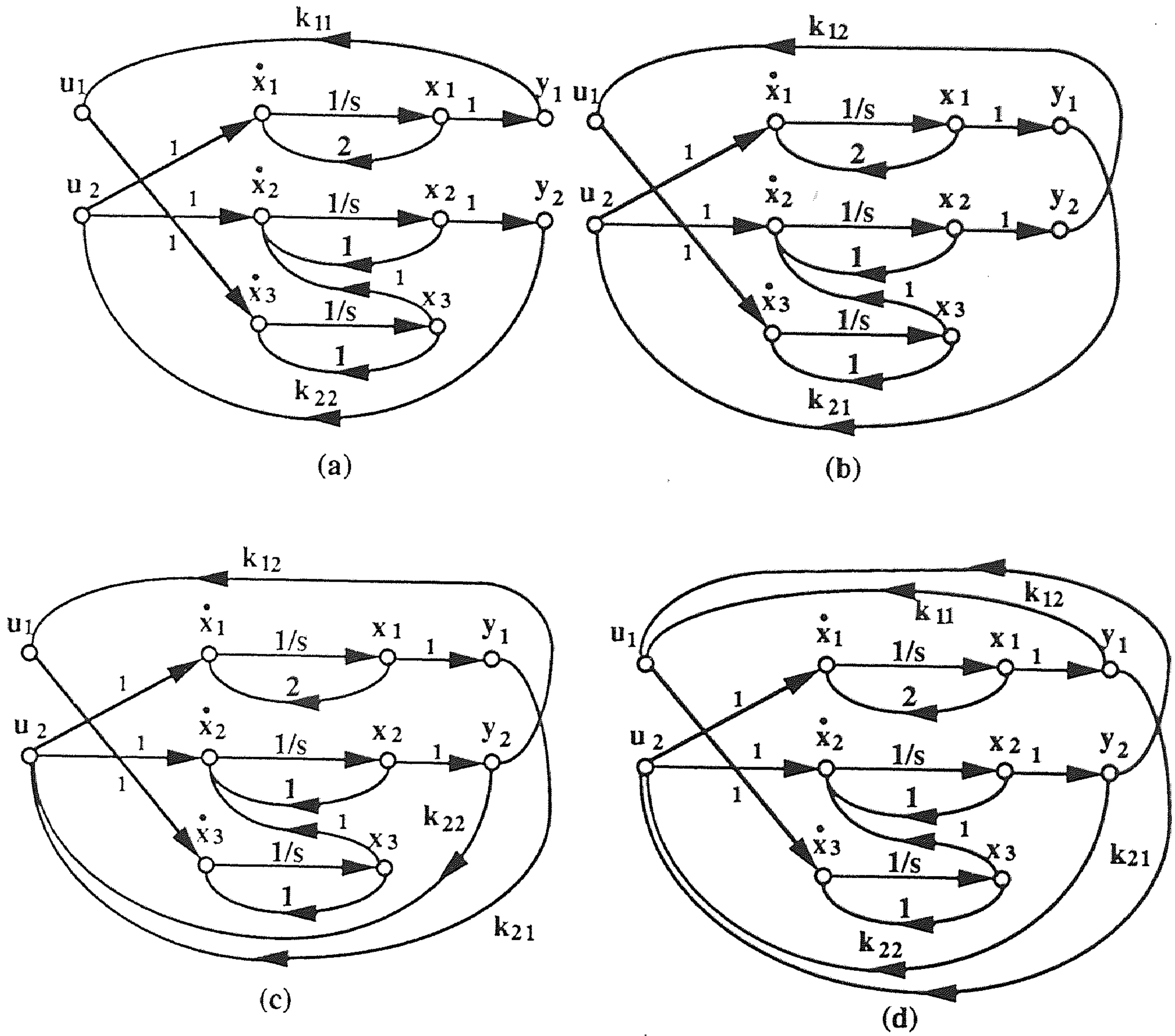


Figure 2.5: Feedback Structure Description in Signal Flow Graph

Therefore the full rank of $D$ matrix may compensate the non-full rank of $C$ or $B$ matrices and decrease the possibility of the fixed mode emergence in the decentralized control.

The minimal feedback structure is desirable for reduction of hardware complexity. It is also appealing because it saves many feedback components without reducing the function of mode shifting and generating any new fixed mode. In later Chapters, the search procedure for obtaining the minimal and admissible feedback structures will be addressed.

# CHAPTER 3

# DECENTRALIZED FEEDBACK STRUCTURE SEARCH

In Chapter 2, the realistic decentralized information flow constraint definition and decentralized feedback structure are defined. In this Chapter, two methods to search decentralized feedback structure are introduced: 1) feedback structure search based on the Jordan normal form and; 2) the signal flow graph.

## 3.1 The Jordan Normal Form Method

Since the Jordan normal form provides good information about the structure of a linear dynamic system, it can be easily to analyze the complete controllability and observability with respect to this form[1].

The linear time-invariant system (1.1) can be transformed into

$$\begin{cases} \dot{z} = & J_o z + \bar{B}u \\ y = & \bar{C}z \end{cases} \tag{3.1}$$

where $J_o \in \mathbf{C}^{n\times n}, \bar{B} \in \mathbf{C}^{n\times m}, \bar{C} \in \mathbf{C}^{r\times n}$. $J$ is Jordan form with $v$ ($\leq n$) elementary Jordan blocks. This means that $A$ may have at most $v$ distinct eigenvalues. The notations are

$$\begin{aligned} z = & \ T^{-1}x \\ J_o = & \ T^{-1}AT \quad = \text{block diag } (J(\lambda_1), J(\lambda_2), \cdots, J(\lambda_k),) \\ \bar{B} = & \ T^{-1}B \\ \bar{C} = & \ CT \end{aligned}$$

where

$$J(\lambda_j) = \begin{bmatrix} J_1(\lambda_j) & & \\ & \ddots & \\ & & J_s(\lambda_j) \end{bmatrix} \quad \text{and} \quad J_i(\lambda_j) = \begin{bmatrix} \lambda_j & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_j \end{bmatrix}$$

are the $j^{th}$ Jordan block and the $i^{th}$ elementary Jordan block respectively. Let

$$\bar{B} = \begin{bmatrix} \bar{B}_1 \\ \bar{B}_2 \\ \vdots \\ \bar{B}_v \end{bmatrix} \quad \text{and} \quad \bar{C} = \begin{bmatrix} \bar{C}_1 & \bar{C}_2 & \cdots & \bar{C}_v \end{bmatrix} \tag{3.2}$$

where $J_i(\lambda_j) \in \mathbf{C}^{n_i \times n_i}$, $\bar{B}_i \in \mathbf{C}^{n_i \times m}$, $\bar{C}_i \in \mathbf{C}^{r \times n_i}$, $i = 1, 2, ...v$. Denote the last row of every $\bar{B}_i$ as

$$\bar{b}_i = \begin{bmatrix} \bar{b}_{i_1} & \bar{b}_{i_2} & \cdots & \bar{b}_{i_m} \end{bmatrix} \qquad \text{for } i = 1, 2, ..., v, \ \bar{b}_i \in \mathbf{C}^{1 \times m}$$

and the first column of every $\bar{C}_i$ as

$$\bar{c}_i = \begin{bmatrix} \bar{c}_{i_1} \\ \bar{c}_{i_2} \\ \vdots \\ \bar{c}_{i_r} \end{bmatrix} \qquad \text{for } i = 1, 2, ..., v, \ \bar{c}_i \in \mathbf{C}^{r \times 1}$$

corresponding to every elementary Jordan block.

Many methods exist to determine the transform matrix $T$ [14],[3], i.e. the right generalized modal matrix.

**Lemma 3.1** [1] Given a triple $(C, A, B)$. A suitable transformation in the complex field yields the equivalent system $(\bar{C}, J_o, \bar{B})(3.1)$. The pair $(A, B)$ is controllable if and only if the every $\bar{b}_i$ corresponding to the last row of every elementary Jordan block is not zero. The pair $(C, A)$ is observable if and only if every $\bar{c}_i$ corresponding to the first column of every elementary Jordan block is not zero.

The set $S$ of Definition 2.4 and its RDIFC based on the Jordan normal form method can now be defined as follows:

**Definition 3.1** The output feedback, $u = Ky$, is said to have a realistic decentralized information flow constraint $\mathbf{K}$ imposed on the linear time-invariant system (1.1) if $K \in \mathbf{K}$ where

$$\mathbf{K} \triangleq \{K = [k_{ij}] \in \mathbf{R}^{m \times r} | \ k_{ij} \equiv 0, \forall (i,j) \notin S, \ i = [1, m], j = [1, r] \ \}$$

The set

$$S = \{(i_1, j_1), (i_2, j_2), \cdots, (i_s, j_s)\}$$

contains pair tuples which are dependent on the actual plant mode and describes the realistic internal connection information.

**Theorem 3.1** ($\mathbf{K}$, $\mathbf{K}^u$ Existence for minimal realization for a plant model)

Given a triple $(C, A, B)$ system which is controllable and observable, there exists at least one $K$ such that plant $(C, A, B)$ has no fixed mode with respect to $K \in \mathbf{K}$. In particular, $\mathbf{K}$ can be reduced into the upper bound structure $\mathbf{K}^u$ which contains $\min(v, r \times m)$ non-zero elements for an optimal structure $\mathbf{K}^*$.

**Proof.** Because (1.1), (1.2) can be transformed into $v$ subsystems $(\bar{C}_i, J_o, \bar{B}_i)$ (3.1),(3.2) corresponding to $v$ elementary Jordan block which are equivalent to (1.1), (1.2). every subsystem defined by one elementary Jordan block is controllable and observable based on Lemma 3.1. Every subsystem with its feedback has no fixed mode. Therefore, all subsystem feedbacks synthesize a $K \in \mathbf{K}$ such that the synthesis system has no fixed mode, because $\mathbf{K}$ depends on the actual model $(C, A, B)$, according to Theorem 2.1 and $\mathbf{K} \subseteq \mathbf{R}^{m \times r}$.

Every controllable and observable subsystem needs at least one feedback to shift its mode distinctly. The number of direct feedback components which shift modes is less than or equal to the any number of indirect feedback components which shift the same modes. Therefore the sum of every subsystem feedback element in $(C_i, A, B_i)$ is just $v$ non-zero elements which belongs to direct feedback elements in the feedback matrix. On the other hand, many subsystems use a common input node and/or a common output node. Therefore the number of non-zero element in $\mathbf{K}^u$ is $\min(v, rm)$.

□

It should be noted that the relationship among the feedback structures is $\mathbf{K}^* \subseteq \mathbf{K}^u \subseteq \mathbf{K} \subseteq \mathbf{R}^{m\times r}$. Based on Theorem 3.1, an algorithm is derived to determine $\mathbf{K}$.

**Algorithm 3.1** ( Determine $\mathbf{K}$ and $\mathbf{K}^u$ )

1. Transform a triple $(C, A, B)$ system into its Jordan normal form $(\bar{C}, J_o, \bar{B})$.

2. Find a pair $(i_k, i_l)$ such that column in $\bar{B}$ and row in $\bar{C}$ respectively are

$$\bar{B}_{i_k} \triangleq \{ \begin{bmatrix} \times \\ \times \\ \bar{b}_{i_k} \\ \times \\ \times \end{bmatrix} \mid \bar{b}_{i_k} \in \bar{b}_i \backslash 0, \quad k = 1, 2, ..., m \}$$

   and

$$\bar{C}_{i_l} \triangleq \{ \begin{bmatrix} \times & \times & \bar{c}_{i_l} & \times & \times \end{bmatrix} \mid \bar{c}_{i_l} \in \bar{c}_i \backslash 0, \quad l = 1, 2, ..., r \}$$

   which is satisfied with rank $\begin{bmatrix} A - \lambda I & B_{i_k} \\ C_{i_l} & 0 \end{bmatrix} = n + 1$ for $i = 1, 2, ..., v$, where $\backslash \{.\}$ is an operator of excluding$\{.\}$. For the $i^{th}$ elementary Jordan block, obtain

$$S_i = (i_k, i_l) \tag{3.3}$$

   For $v$ elementary Jordan blocks, we can get

$$S = \{S_1, S_2, ..., S_v\} \tag{3.4}$$

3. The decentralized feedback structure is given by

$$\mathbf{K} = \{K = [k_{ij}] \in \mathbf{R}^{m\times r} \mid k_{ij} \equiv 0, \forall (i, j) \notin S\}$$

   where $S$ is given by (3.3),(3.4).

4. The optimal decentralized feedback structure is given by

$$\mathbf{K}^u = \{K = [k_{ij}] \in \mathbf{R}^{m\times r} \mid k_{ij} \equiv 0, \forall (i, j) \notin S^*\}$$

   where $S^* = \{S_1^*, S_2^*, ..., S_{v^*}^*\}$. $S^*$ is given by the following algorithm

(a) $S_1^* = \bigcap_{j_1 \in [1,v], \max G_v^{j_1}} S_{j_1} \neq \phi.$

(b) $S_2^* = \bigcap_{j_2 \in [1,v] \backslash j_1, \max G_v^{j_2}} S_{j_2} \neq \phi.$

(c) $S_3^* = \bigcap_{j_3 \in [1,v] \backslash \{j_1, j_2\}, \max G_v^{j_3}} S_{j_3} \neq \phi.$

$\vdots$

Until $S_{v^*+1}^* = \bigcap_{j_{v^*} \in [1,v] \backslash \{j_1, j_2, \ldots, j_{v^*}\}, \max G_v^{j_{v^*}}} S_{j_{v^*}} = \phi.$

where $G$ is an operator of combination.

**Remark 3.1** The feedback structure relationship is $\mathbf{K}^u \subseteq \mathbf{K} \subseteq \mathbf{R}^{m \times r}$. Generally, the K is unique. but $\mathbf{K}^u$ may not be unique. $\mathbf{K}^u$ may not be the best feedback structure for shifting a fixed mode because the elimination of the feedback signal flows mutually may be such that the region of mode variability is small.

For an illustration of the algorithm, take the plant model in Example 2.2 again.

**Example 3.1** Consider a plant

$$\begin{cases} \dot{x} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} u \\ \\ y = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} x \end{cases}$$

The plant is already in Jordan normal form with two elementary Jordan blocks ($v = 2$) and is controllable and observable. According to the previous algorithm, we can find that

$$\bar{B}_1 = \begin{bmatrix} 0 & 1 \end{bmatrix} \quad \bar{B}_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \bar{C}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \bar{C}_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$\bar{b}_1 = \begin{bmatrix} 0 & 1 \end{bmatrix} \quad \bar{b}_2 = \begin{bmatrix} 1 & 0 \end{bmatrix} \quad \bar{c}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \bar{c}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

For $\lambda_{o1} = 2$, because $1_k = 2$, $1_l = 1$, $\bar{B}_{1_2} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\bar{C}_{1_1} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$, and

$$\text{rank} \begin{bmatrix} A - \lambda I & B_{1_k} \\ C_{1_l} & 0 \end{bmatrix} = \text{rank} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = 4$$

then $S_1 = (2, 1)$.

For $\lambda_{o2} = 1$, $v = 1$, because $2_k = 1$, $2_l = 2$, $\bar{B}_{2_1} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $\bar{C}_{2_2} = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$, and

$$\text{rank} \begin{bmatrix} A - \lambda I & B_{2_k} \\ C_{2_l} & 0 \end{bmatrix} = \text{rank} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} = 4$$

then $S_1 = (1, 2)$. Therefore $S = \{S_1, S_2\} = \{(2,1), (1,2)\}$ and $\mathbf{K} = \begin{bmatrix} 0 & k_{12} \\ k_{21} & 0 \end{bmatrix}$. Because $S_1 \cap S_2 = \phi$, so $S_1^* = S_1 \neq \phi$ and $S_2^* = S_2 \neq \phi$, but $S_3^* = \phi$. Hence $S^* = \{S_1^*, S_2^*\} = S$.

The $K$ in which all feedback elements can directly shift some modes is called direct feedback structure. In fact, the $\mathbf{K}$ obtained from Jordan normal form method is a direct feedback structure. Because the non-zero elements of $\mathbf{K}$ is derived from each elementary Jordan block subsystem by direct feedback connection to shift corresponding modes, this method does not consider the indirect feedback connections to shift corresponding and other modes. In previous example, $S$ loses two indirect feedback structure, i.e.

$$S = \{S_1, S_2\} = \{\{\{(1,1), (2,2)\}, (2,1)\},\ \{\{(1,1), (2,2)\}, (1,2)\}\}$$

From this set, we can find that $\{(1,1), (2,2)\}$ belongs to the pair set of $(i_k, i_l)$. This means that the pair set existing appears in pair form. We can conclude that the triple or quadruple or more may exist and that the $k$ number indices of the set can shift at least $k$ modes. In the next section, we will see a graphic method which can find the indirect structures.

## 3.2 The Signal Flow Graph Method

A multivariable control system is composed of many single variable states. Although multivariable state nodes may contain many single state nodes like $\dot{x}$, and $x$. but from the $u, y$ nodes aspect, we can simply consider that there are many subsystems. In an open loop system the $u$ is called an independent node. In a closed loop system all nodes are called dependent nodes, but a $u$ node can still be considered as independent node. called start node, because it contains a signal flow path.

Because the signal flow graph tool may be difficult to analyze a large scale system, the signal flow definition, classification, and decomposition must be required for a large scale system analysis. Specifically, because a signal flow graph tool is easily realized in computer system, the standard definition of signal flow is needed to operate a complex signal flow graph.

**Definition 3.2** ( Control Cycle Unit)

The signal flow cycle form which starts from $u_i$ node, goes through at least a state node $x_k$, and ends to $u_i$ node in Figure 3.1 (a) is defined as a unit of the control cycle with $x_k$. Every control cycle has common features: 1) the start node and the end node of a signal flow are the same input node; 2) a signal flow completely goes through $\dot{x}_k$ to $x_k$.The types of control cycle can be classified as independent like Figure 3.1 (a) and dependent. Dependent control cycle has:

1. $y$-**dependent control cycle** with $x_k$ in Figure 3.1 (b). It has other feature: a signal flow goes through a $y_m$ node ($m \neq k$) before it comes down $x_k$ node.

2. $x$-**dependent control cycle** with $x_k$ in Figure 3.1 (c),(d). It has other feature: a signal flow goes through a $\dot{x}_m$ or $x_m$ node ($m \neq k$) before it comes down $x_k$ node.

3. $u$-**dependent control cycle** with $x_k$ in Figure 3.1 (e). It has other feature: a signal flow goes through a $u_m$ node ($m \neq i$) before it comes down $x_k$ node.

From Figure 3.1 and features of control cycle, we can conclude that the $y$-dependent control cycle must contain a $u$-dependent control cycle, and vice versa. The any $x$-dependent control cycle must contain a control cycle with $x_m$ besides with $x_k$. A simple single loop is the independent control cycle with $x_k$. The feedback connectional feature in independent and dependent control cycles are the same as those of direct and indirect connections, respectively.

**Definition 3.3** ( Deadlock Unit)

The non-cyclic signal flow form which starts from node $u_i$, goes through at least a state node $x_k$, and ends to any node except $u_i$ in Figure 3.2 (a) is defined as a unit of deadlock with $x_k$. Every deadlock unit has common feature: signal flow is stuck by the last node except $u_i$. The types of deadlock with $x_k$ can be classified as self-node deadlock in Figure 3.2 (a) and the other node deadlock. The other deadlock has are:

1. $y$-**deadlock** with $x_k$ in Figure 3.2 (b).

2. $x$-**deadlock** with $x_k$ in Figure 3.2 (c).

3. $u$-**deadlock** with $x_k$ in Figure 3.2 (d).

From Figure 3.2 and features of deadlock unit, we can conclude that two u-deadlock unit series connection may be a control cycle unit if the end node in a

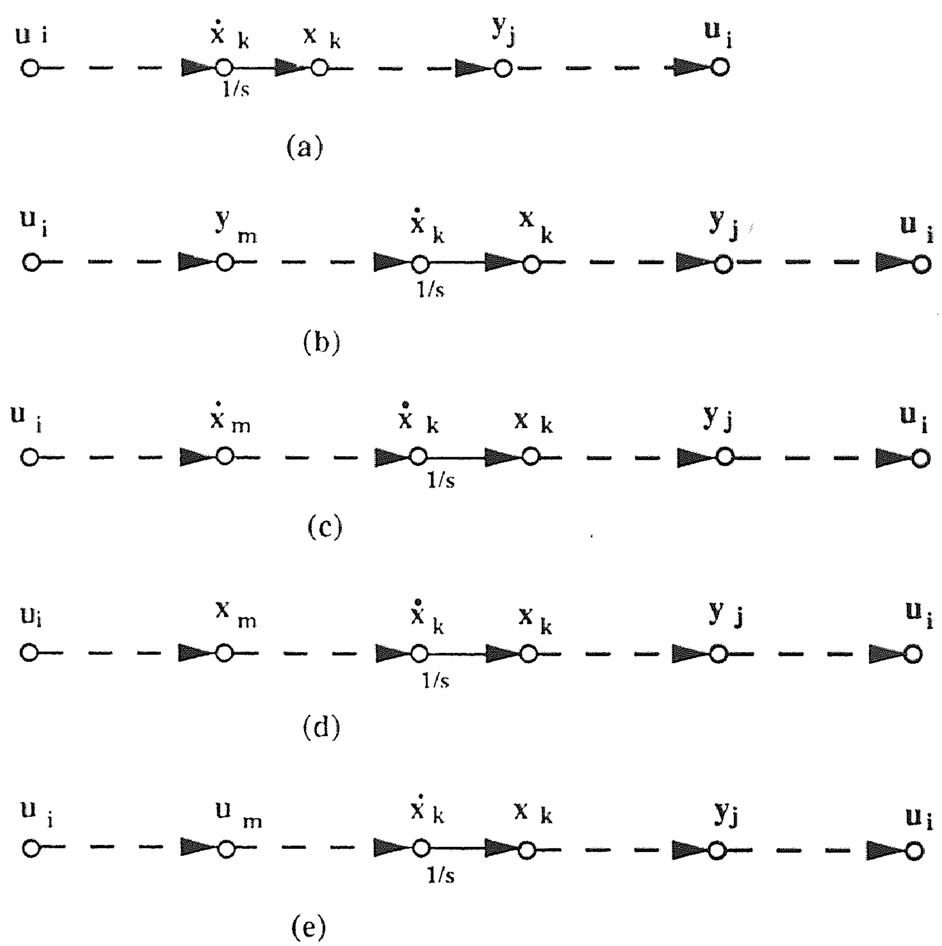


Figure 3.1: Signal Flow Type I

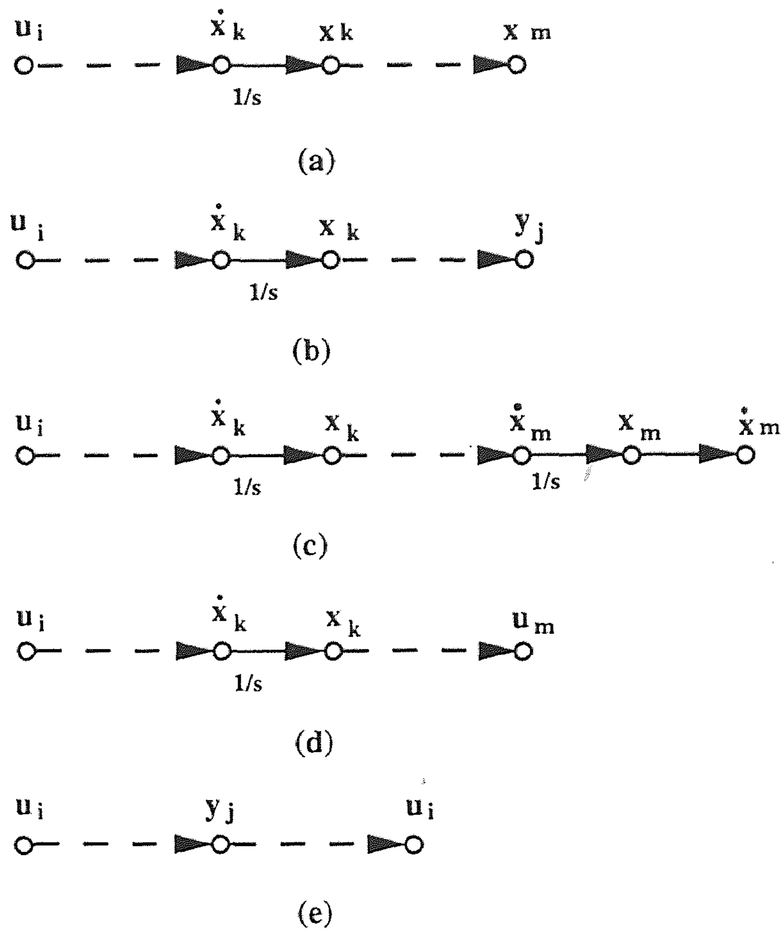


Figure 3.2: Signal Flow Types II

deadlock unit is equal to the start node in another deadlock unit. A control cycle unit may contain one or more $u$-deadlock unit.

**Definition 3.4** ( Shortcut Unit)

The signal flow form which starts from a node $u_i$, does not go through any one $x_k$ node and ends at a $u_i$ node in Figure 3.2 (e) is defined as an unit of shortcut with $u_i$. Every shortcut unit has a common feature: a signal flow shortcut is caused by $d$ (feedforward) or $k$ (feedback) gains between $u$ node and $y$ node without other gains in this signal flow.

### 3.2.1 Control System Tree Unit

Consider a quadruple $(C, A, B, D)$ system, then every vector and matrix in this system can be described as:

$$\dot{x} = \begin{bmatrix} \dot{x}_1 & \cdots & \dot{x}_k & \cdots & \dot{x}_n \end{bmatrix}^T \quad x = \begin{bmatrix} x_1 & \cdots & x_l & \cdots & x_n \end{bmatrix}^T$$

$$u = \begin{bmatrix} u_1 & \cdots & u_i & \cdots & u_m \end{bmatrix}^T \quad y = \begin{bmatrix} y_1 & \cdots & y_j & \cdots & y_r \end{bmatrix}^T$$

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{k1} & a_{kl} & a_{kn} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \qquad B = \begin{bmatrix} b_{11} & \cdots & b_{1m} \\ \vdots & & \vdots \\ b_{k1} & b_{ki} & b_{km} \\ \vdots & & \vdots \\ b_{n1} & \cdots & b_{nm} \end{bmatrix}$$

$$C = \begin{bmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & & \vdots \\ c_{j1} & c_{jl} & c_{jn} \\ \vdots & & \vdots \\ c_{r1} & \cdots & c_{rn} \end{bmatrix} \qquad D = \begin{bmatrix} d_{11} & \cdots & d_{1m} \\ \vdots & & \vdots \\ d_{j1} & d_{ji} & d_{jm} \\ \vdots & & \vdots \\ d_{r1} & \cdots & d_{rm} \end{bmatrix}$$

$$K = \begin{bmatrix} k_{11} & \cdots & k_{1r} \\ \vdots & & \vdots \\ k_{i1} & k_{ij} & k_{ir} \\ \vdots & & \vdots \\ k_{m1} & \cdots & k_{mr} \end{bmatrix}$$

For the sake of simplicity, an elementary unit tree containing two nodes and one gain is defined. Any elementary unit tree can be composed of one start node and many end nodes with same type. For example, $(u, x)$ can be described as $m$ elementary unit trees in 3.3(d). According to the above vectors and matrices, we can draw a set of control cycles with $x_k$ for each $u_i$ based on elementary unit types. Firstly, we draw a set of elementary pair nodes into elementary unit trees in Figure 3.3 which have six types in the linear system: $(x, \dot{x}), (x, y), (\dot{x}, x), (u, \dot{x}), (u, y)$, and $(y, u)$.

We can find that the total search steps of single control cycles are $rm$, $nrm$,

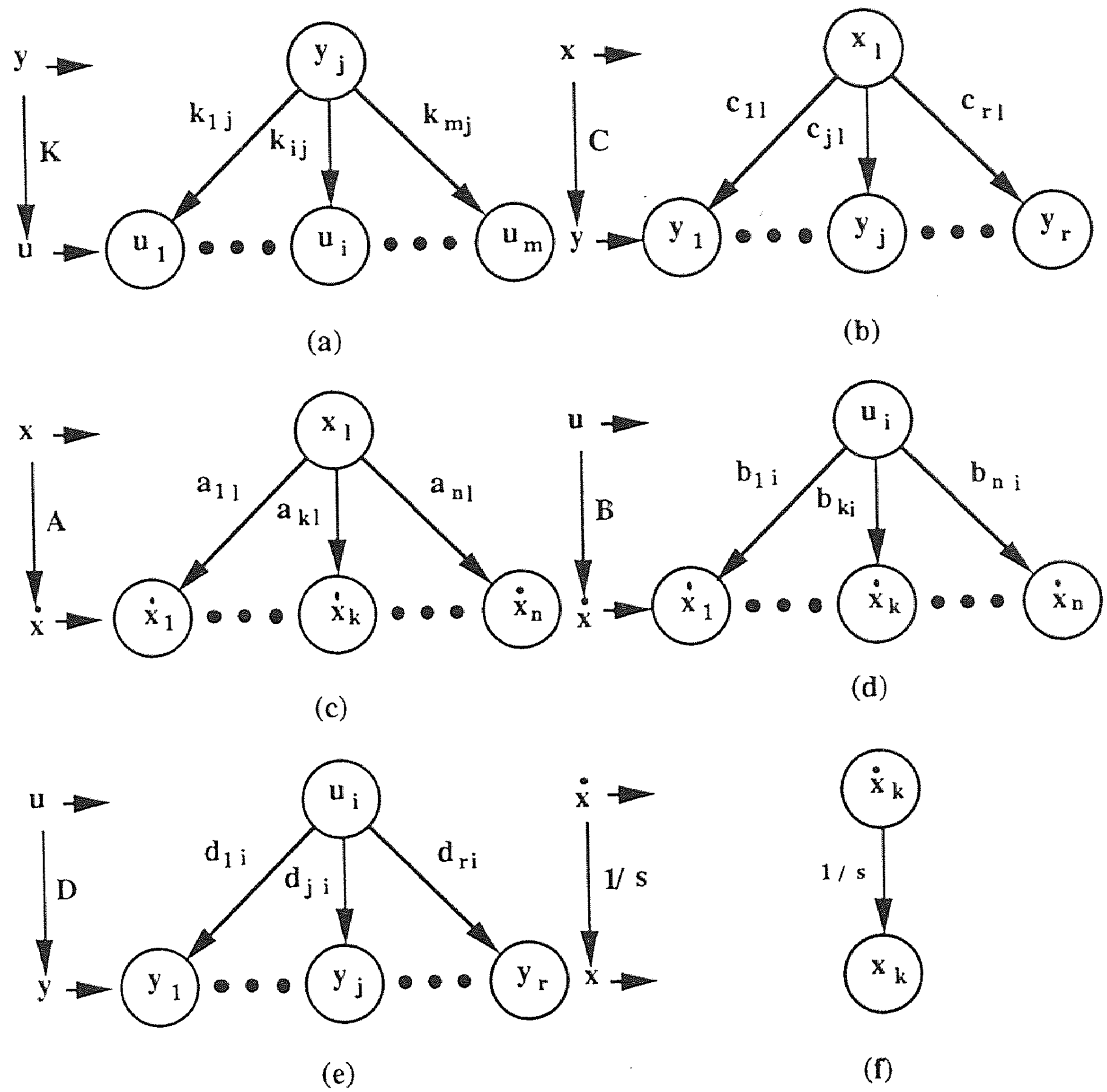


Figure 3.3: Elementary Unit Trees

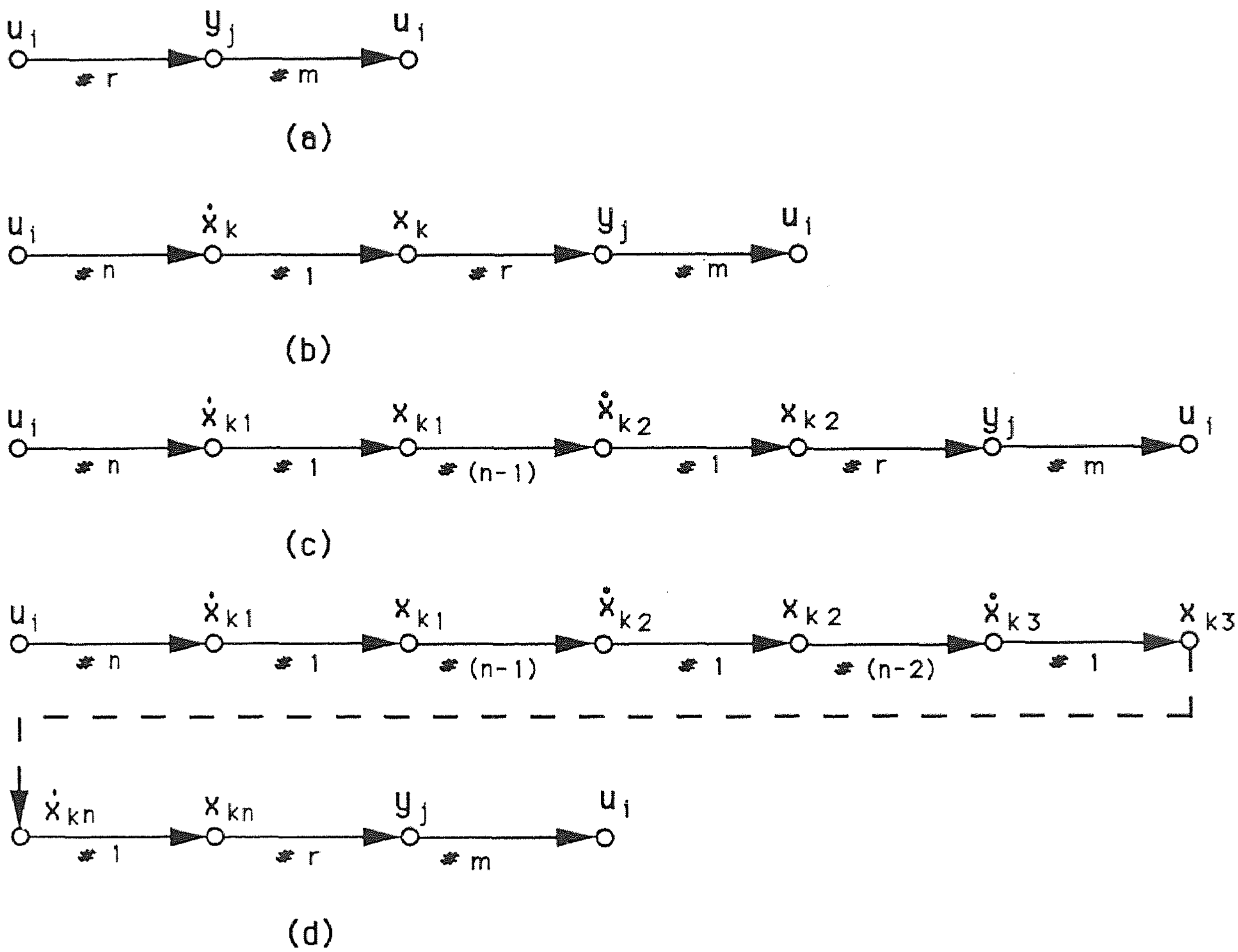


Figure 3.4: Search Steps in Single Control Cycles

$n(n-1)rm$, and $n!rm$ for the forms in Figure 3.4 (a), (b), (c), and (d) respectively. These search feature is one of the essential searches which can directly find the $u_i$ node and does not continually make a deep search for $u_i$. Hence the independent control cycle is safe for a control system due to the fact that it only corresponds to a single feedback gain $k_{ij}$, i.e. the signal flow transfer does not need any other feedback components as the bridges to connect $u$ nodes and $y$ nodes. A $u$ or $y$ dependent control cycle is seemly unsafe for a control system relatively, because it consists of many different gain $k_{ij}$s in series form. If one $k_{ij}$ in the $u$- or $y$- dependent control cycle is broken then signal flow is stuck and whole modes in this cycle unit can not move to their oriented places in a left-half S plan. The essential graphic path for searching a control cycle unit or $u$-deadlock unit tree and their search steps are shown in Figure 3.5.

Generally, a LTI system can be equivalently transformed into the Jordan normal form. Note that the unit tree $(x, \dot{x})$ is constrained only by the multiplicity of eigenvalues and its branch number is only one for single distinct eigenvalues in the Jordan normal form.

**Example 3.2** Consider Example 3.1. The corresponding elementary unit trees are in Figure 3.6. According to the oriented connection of elementary unit tree as

$$(u, \dot{x})(\dot{x}, x)(x, \dot{x})(\dot{x}, x)(x, y)(y, u)$$

and

$$(u, \dot{x})(\dot{x}, x)(x, y)(y, u)$$

direction, the corresponding control cycle unit and $u$-deadlock unit are constructed in Figure 3.7. Figure 3.7 shows that $\dot{x}_2$, $\dot{x}_3$ are live nodes and their states, $x_2$ and $x_3$, can be adjusted by signal flow between $(u_1, u_1)$ if $k_{12}$ changes or between $(u_1, u_2)(u_2, u_1)$

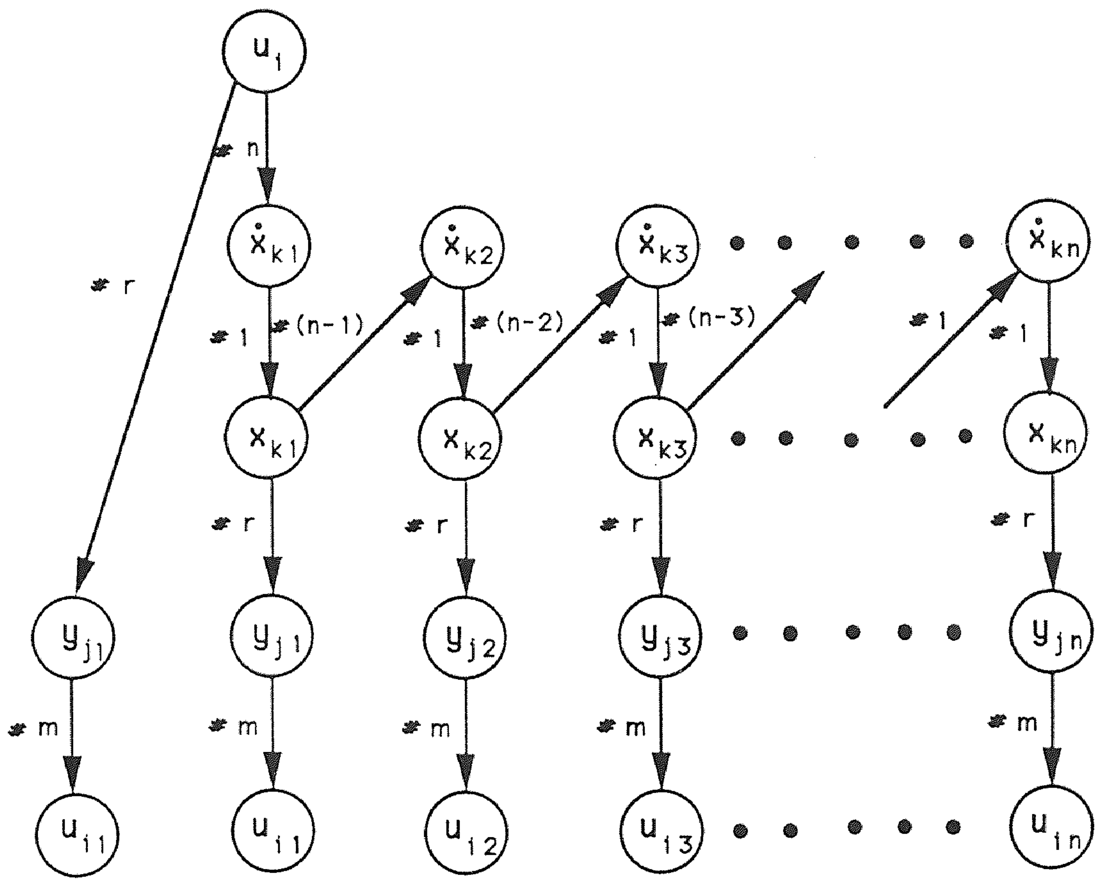


Figure 3.5: Essential Search Trees

if $k_{11}$, and/or $k_{22}$ change, where (.)(.) is denoted as unit oriented connection. The $\dot{x}_1$, live node, $\dot{x}_2$ can be adjusted by signal flow between $(u_2, u_2)$ if $k_{21}, k_{22}$ change respectively or between $(u_2, u_1)(u_1, u_2)$ if $k_{11}$ and/or $k_{22}$ change. The eigenvalues corresponding to live state nodes can be shifted. For the sake of explicitness, the relationship in which a $x$ can be adjusted by a $k$ is denoted by $x(k)$.

We can define a tuple to describe the unit of a $u$-deadlock or a $u$-control cycle with $x_k$ as

$$t(u_{i_1}, u_{i_2}) \triangleq t(u_{i_1}, x_l, k_{ij}, u_{i_2})$$

where $u_{i_1}$ is a starting independent input, $x_k$ a controlled state, $k_{ij}$ a feedback gain, and $u_{i_2}$ is a ending dependent input . Each variable is a single variable for an elementary unit $(u_{i_1}, u_{i_2})$. $x_l$ is used as a symbol for $(\dot{x}_l, x_l)$ with a signal flow to avoid deadlock and short cut states. Therefore in this example we can find that elementary units can be described by four cases: $(u_{i_1}, u_{i_2})$ for $i_1, i_2 = 1, 2$.

### 3.2.1 Control Tuple Space Method

**Definition 3.5** A tuple of control is a set of nodes and branches, which can describe a control signal flow and is denoted by

$$t(u_i, u_j) \triangleq t(u_i, x_k, k_{lj}, u_j) \triangleq t(u_i, x_k, y_l, k_{lj}, u_j)$$

where $u_i, x_k, y_l$ are nodes in the graph, and $k_{lj}$ is the corresponding output feedback gain. $x_k$, $y_l$, $k_{lj}$ can be a set of $x_k$'s, $y_l$'s, $k_{lj}$'s respectively, for example $x_k = (x_{k_1}, x_{k_2}, ..., x_{k_k})$. $u_i$, $u_j$ are a starting element and a ending element, respectively. $x_k \triangleq (\dot{x}_k, x_k)$ is a unique oriented set.

**Definition 3.6** ( Control Tuple Space $\mathbf{T}$)

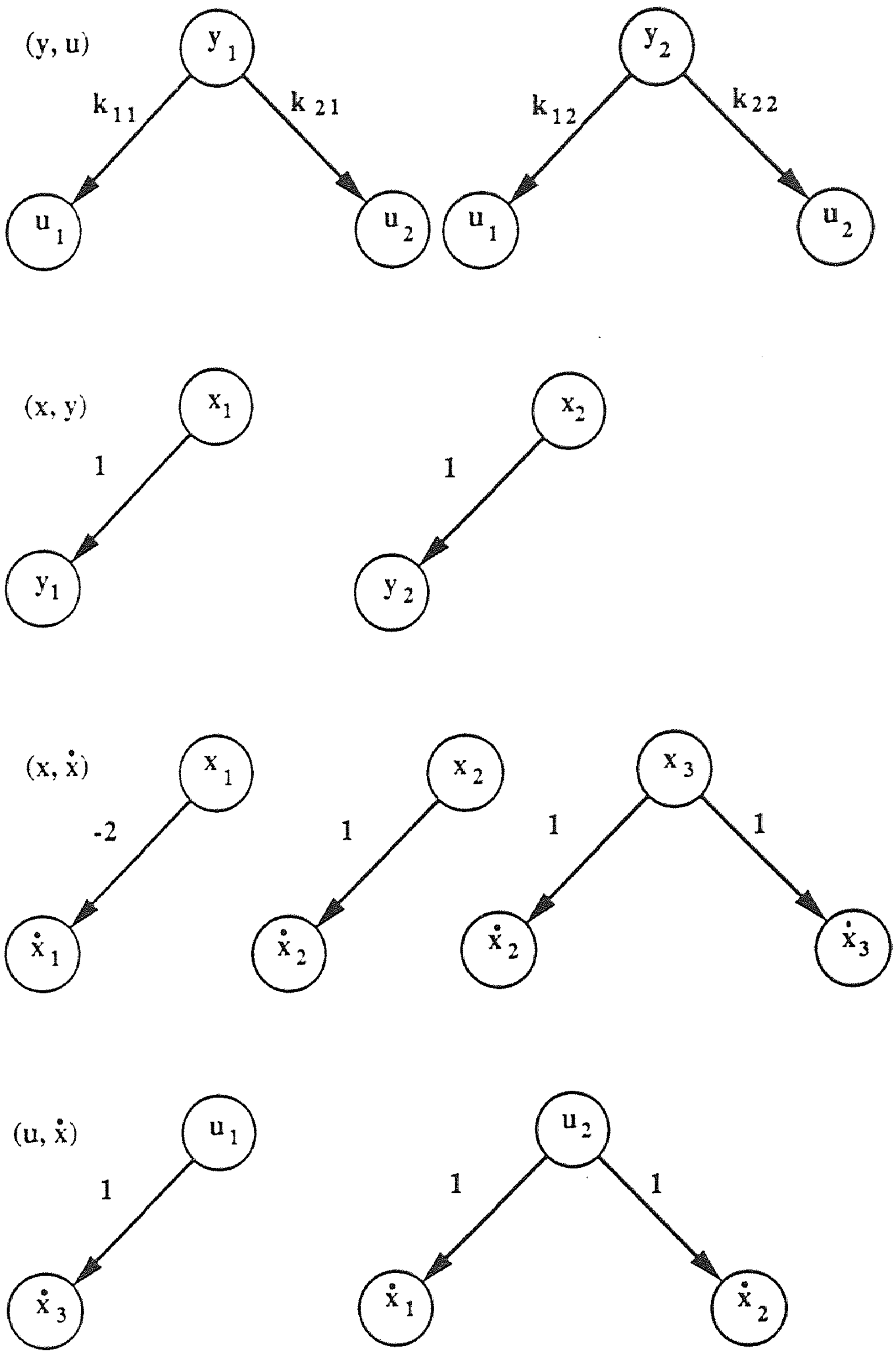


Figure 3.6: Elementary Unit Tree for an Example

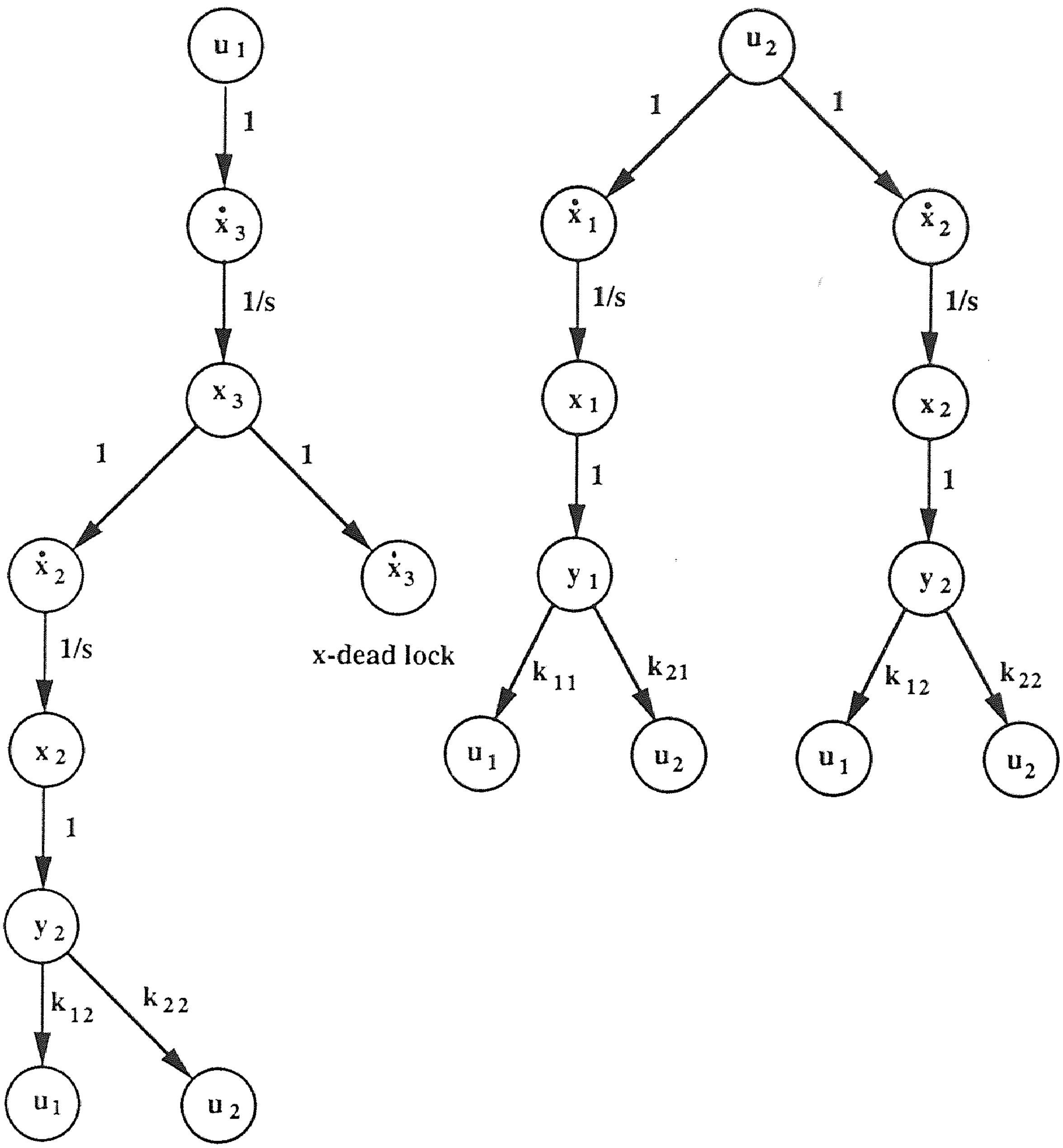


Figure 3.7: Essential Control Search Tree for An Example

A set of control tuples organizes a space of a control system, denoted by $\mathbf{T}$, which satisfies the following conditions with + oriented operation:

1. Let $t_1(u_1, u_2)$, $t_2(u_2, u_1)$, $t_3(u_2, u_3) \in \mathbf{T}$, then

   $t_1 + t_2 = t_{12}(u_1, u_2, u_1) = t_{12}(u_1, u_1)$.

   $t_2 + t_1 = t_{21}(u_2, u_1, u_2) = t_{21}(u_2, u_2)$.

   $t_1 + t_2 \neq t_2 + t_1 \qquad \text{if } t_1 \neq t_2$.

2. $t_1 + t_3 = t_{13}(u_1, u_3)$.

   $t_3 + t_1 = t_{31}(u_2, u_3, u_1, u_2) = \phi$.

**Property 3.1** The operational result of any two control tuples is contained in tuple space $\mathbf{T}$. i.e. $\mathbf{T}$ is closed.

**Example 3.3** Consider Example 3.1. We can find the four control tuples in Figure 3.7:

$$\begin{aligned}
t_1(u_1, u_1) &= t_1(u_1, (x_3, x_2), k_{12}, u_1) = t_1(u_1, (x_3, x_2), y_2, k_{12}, u_1) \\
t_2(u_1, u_2) &= t_2(u_1, (x_3, x_2), k_{22}, u_2) = t_2(u_1, (x_3, x_2), y_2, k_{22}, u_2) \\
t_3(u_2, u_1) &= t_3(u_2, x_1(k_{11}), x_2(k_{12}), u_1) = t_3(u_2, x_1(k_{11}), x_2(k_{12}), (y_1, y_2), u_1) \\
t_4(u_2, u_2) &= t_4(u_2, x_1(k_{21}), x_2(k_{22}), u_2) = t_3(u_2, x_1(k_{21}), x_2(k_{22}), (y_1, y_2), u_2)
\end{aligned} \tag{3.5}$$

where $t_1$, $t_4$, are control cycles, $t_2$, $t_3$ are $u$-deadlock based on the tuple description in (3.5). Please note that the description $x(k)$ in $t_3$, $t_4$ which reflects the relationship between $x$ and $k$. In fact, the $t_3$, $t_4$ are not unique. They contain two $u$-deadlock units and two control cycle units, respectively as follows.

$$\begin{aligned}
t_3^1(u_2, u_1) &= t_3^1(u_2, x_1(k_{11}), u_1) \\
t_3^2(u_2, u_1) &= t_3^2(u_2, x_2(k_{12}), u_1) \\
t_4^1(u_2, u_2) &= t_4^1(u_2, x_1(k_{21}), u_2) \\
t_4^2(u_2, u_2) &= t_4^2(u_2, x_2(k_{22}), u_2)
\end{aligned}$$

It should be noted that the $u$-deadlock unit can be changed into control cycle if and only if there exists at least an intermediate tuple such that these two unit can be connected by it.

**Example 3.4** Consider Example 3.3 where $t_2(u_1, u_2)$, $t_3(u_2, u_1) \in \mathbf{T}$. Then

$$\begin{aligned} t_2 + t_3 = \quad t_{23}(u_1, u_1) = \quad & t_{23}(u_1, x_1(k_{11}, k_{22}), x_2(k_{12}, k_{22}), x_3(k_{11}, k_{22}), x_3(k_{12}, k_{22}), u_1) \\ t_3 + t_2 = \quad t_{32}(u_2, u_2) = \quad & t_{32}(u_2, x_1(k_{11}, k_{22}), x_2(k_{12}, k_{22}), x_3(k_{11}, k_{22}), x_3(k_{12}, k_{22}), u_2) \end{aligned}$$

Therefore the total control cycles are $t_1$, $t_2 + t_3$, $t_3 + t_2$, and $t_4$ and

$$\text{In } t_1 \text{ cycle:} \qquad k_{12} \qquad \text{can shift} \qquad x_2, x_3$$

$$\text{In } t_2 + t_3 \text{ cycle:} \quad \begin{cases} (k_{11}, k_{22}) & \qquad\qquad x_1, x_2, x_3 \\ & \text{can shift} \\ (k_{12}, k_{22}) & \qquad\qquad x_2, x_3 \end{cases}$$

$$\text{In } t_4 \text{ cycle:} \quad \begin{cases} k_{21} & \qquad\qquad x_1 \\ & \text{can shift} \\ k_{22} & \qquad\qquad x_2 \end{cases}$$

$$\text{In } t_3 + t_2 \text{ cycle:} \quad \begin{cases} (k_{11}, k_{22}) & \qquad\qquad x_1, x_2, x_3 \\ & \text{can shift} \\ (k_{12}, k_{22}) & \qquad\qquad x_2, x_3 \end{cases}$$

Based on the above four cases, we can easily conclude that

$$\mathbf{K} = \begin{bmatrix} k_{11} & 0 \\ 0 & k_{22} \end{bmatrix} \quad \text{or} \quad \mathbf{K} = \begin{bmatrix} k_{11} & k_{12} \\ 0 & k_{22} \end{bmatrix} \quad \text{or} \quad \mathbf{K} = \begin{bmatrix} 0 & k_{12} \\ k_{21} & 0 \end{bmatrix}$$

**Definition 3.7** The any type of unit is called elementary unit if it contains only one single flow. The relationship between $x$ and $k$ is unique.

**Theorem 3.2** The $x(k)$ in the new tuple $t_{12}$ or $t_{21}$, which is produced by any two elementary $u$-deadlock unit tuples $t_1$ and $t_2$ operating with $+$, is unique if $t_1 + t_2 \neq \phi$ and $t_2 + t_1 \neq \phi$. And the corresponding new tuples are a control cycle unit.

**Proof.** $\forall t_1(u_1, x_1, k_{11}, u_2)$, $t_2(u_2, x_2, k_{21}, u_1) \in \mathbf{T}$ are elementary unit, then

$$\begin{aligned} t_1 + t_2 &= t_{12}(u_1, (x_1(k_{11}), x_2(k_{21})), u_1) \\ t_2 + t_1 &= t_{21}(u_2, (x_1(k_{11}), x_2(k_{21})), u_2) \end{aligned}$$

therefore The $x(k)$ in the new tuple $t_{12}$ or $t_{21}$ are unique, i.e. $x_1(k_{11}), x_2(k_{21})$. □

Clearly any $x(k)$ in elementary unit is unique. If $x(k)$ is unique, then the corresponding tuple is an elementary unit. Therefore the elementary unit may be cascaded by many other elementary units.

**Corollary 3.1** If finite elementary unit tuples operate with + and produce at least two new non empty tuples, then $x(k)$ relationship in the new tuples are unique and the corresponding new tuples are a control cycle unit. If any finite elementary unit tuples operate with + and produce the new tuples with non unique $x(k)$, then the new tuples are deadlock unit.

**Algorithm 3.2** ( Construct the Elementary Tuple and Control Cycle Unit)

1. Transform the state-space equation into an elementary tree set as in Figure 3.3.
2. Make the oriented connecting operation + in related elementary set as following direction.

$$u \rightarrow \dot{x} \rightarrow x \rightarrow \cdots \rightarrow \dot{x} \rightarrow x \rightarrow y \rightarrow u \tag{3.6}$$

3. Find all elementary tuples
4. Carry out the operation + for all elementary tuples, find all control cycles in **T**.

**Example 3.5** Consider Example 3.1, then

$$A = \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \end{array} \begin{array}{c} \\ \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{array} \Rightarrow \begin{bmatrix} (x_1, \dot{x}_1) & & \\ & (x_2, \dot{x}_2) & (x_3, \dot{x}_2) \\ & & (x_3, \dot{x}_3) \end{bmatrix}$$

$$B = \begin{array}{c} \begin{array}{cc} u_1 & u_2 \end{array} \\ \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \end{array} \begin{array}{c} \\ \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{array} \Rightarrow \begin{bmatrix} & (u_2, \dot{x}_1) \\ & (u_2, \dot{x}_2) \\ (u_1, \dot{x}_3) & \end{bmatrix}$$

$$C = \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \end{array} \begin{array}{c} \\ y_1 \\ y_2 \end{array} \Rightarrow \begin{bmatrix} (x_1, y_1) & & \\ & (x_2, y_2) & \end{bmatrix}$$

$$K = \begin{array}{c} \begin{array}{cc} y_1 & y_2 \end{array} \\ \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \end{array} \begin{array}{c} \\ u_1 \\ u_2 \end{array} \Rightarrow \begin{bmatrix} (y_1, k_{11}, u_1) & (y_2, k_{12}, u_1) \\ (y_1, k_{21}, u_2) & (y_2, k_{22}, u_2) \end{bmatrix}$$

According to (3.6), the corresponding direction is .

$$B \rightarrow \frac{1}{s} \rightarrow A \rightarrow \frac{1}{s} \rightarrow A \rightarrow \cdots \rightarrow \frac{1}{s} \rightarrow C \rightarrow K$$

therefore the all elementary tuples are

$$\begin{aligned}
&t_1(u_1, \dot{x}_3, x_3, \dot{x}_2, x_2, y_2, k_{12}, u_1) = && t_1(u_1, (x_3, x_2), k_{12}, u_1) \\
&t_2(u_1, \dot{x}_3, x_3, \dot{x}_2, x_2, y_2, k_{22}, u_2) = && t_2(u_1, (x_3, x_2), k_{22}, u_2) \\
&t_3(u_1, \dot{x}_3, x_3) && \\
&t_4(u_2, \dot{x}_1, x_1, y_1, k_{11}, u_1) = && t_4(u_2, x_1, k_{11}, u_1) \\
&t_5(u_2, \dot{x}_1, x_1, y_1, k_{21}, u_2) = && t_5(u_2, x_1, k_{21}, u_2) \\
&t_6(u_2, \dot{x}_2, x_2, y_2, k_{12}, u_1) = && t_6(u_2, x_2, k_{12}, u_1) \\
&t_7(u_2, \dot{x}_2, x_2, y_2, k_{22}, u_2) = && t_7(u_2, x_2, k_{22}, u_2)
\end{aligned}$$

where $t_1$, $t_5$, $t_7$ are control cycles; $t_2$, $t_4$, $t_6$ are $u$-deadlocks; and $t_3$ is an $x$-deadlock. The control cycles are given by $t_2 + t_4$, $t_4 + t_2$, $t_2 + t_6$, $t_6 + t_2$, $t_1$, $t_5$, and $t_7$. The total number of control cycles in $\mathbf{T}$ is seven for this example.

**Algorithm 3.3** ( Minimize **K** Structure Elements ).

1. Categorize the control cycles in $\mathbf{T}$ into $(u_i, u_i)$ type, $i = 1, 2, ..., m$.

2. Choose just one control cycle in each type and then make combinations in tuples for all states.

3. Find the minimal number of **K** elements with all live state nodes $x$ in all combination set, through the standard sorting algorithms.

**Example 3.6** Continue Example 3.5. Categorize the control cycle into two type:$(u_1, u_1)$ and $(u_2, u_2)$, i.e.

$$\begin{array}{ll} t_1,\ t_2 + t_4,\ t_2 + t_6, & \in t(u_1, u_1) \\ t_5,\ t_7,\ t_4 + t_2,\ t_6 + t_2, & \in t(u_2, u_2) \end{array}$$

According to Corollary 3.1, $t_2 + t_4 \sim t_4 + t_2$ and $t_2 + t_6 \sim t_6 + t_2$, therefore we can find a $3 \times 4$ combination. The minimal number of elements of combination for K are $t_2 + t_4\ and\ (t_5, t_1)$, then the corresponding **K** are $\begin{bmatrix} k_{11} & 0 \\ 0 & k_{22} \end{bmatrix}$ and $\begin{bmatrix} 0 & k_{12} \\ k_{21} & 0 \end{bmatrix}$.

Algorithm 3.2, can be realized in a computer through a certain stream operation by a hashing function search[9]. Construct a function to search **K** structure with input $(C, A, B, D)$ system and output the optimal structure **K**.

# CHAPTER 4
# THEORETICAL DEVELOPMENT ON FEEDBACK STRUCTURE

In Chapter 3, the essential decentralized feedback structures are obtained by the Jordan normal form method and the signal flow graph methods. In this Chapter, the closed loop eigenstructure under RDIFC is analyzed to minimize the feedback structure for a synthesis system. Firstly, the general relationships of eignevectors and eigenvalues of an open loop and the corresponding closed loop systems are discussed. Secondly, the relationships are considered for the system in which the $A$ is the diagonal matrix. Thirdly, the relationship are further developed to the system in which the $A$ contains only one elementary Jordan block and then to the system in which $A$ contains many elementary Jordan blocks. Finally, the relationships between a plant and corresponding synthesis system under right and left eigenvectors are developed.

## 4.1 General Theory for Eigenvalue and Eigenvector

Consider a triple $(C, A, B)$ and assume that there exists a $K \in \mathbf{K}$ such that corresponding synthesis system is $(C, A + BKC, B)$. The $i^{th}$ open loop system eigenvalue $\lambda_{oi} \in \sigma(A)$ and the eigenvector $V_{oi}$ for $\lambda_{oi}$ may be changed into $\lambda_{ci} \in \sigma(A + BKC)$ and/or $V_{ci}$ by $K$ imposed or may be invariant. To classify the variation of eigenvalues and eigenvectors, a definition of characteristics of them is given as follow:

**Definition 4.1** Given a triple $(C, A, B)$, a RDIFC $\mathbf{K}$, and the corresponding synthesis system $(C, A + BKC, B)$ for $K \in \mathbf{K}$. The $i^{th}$ synthesis system eigenvalue $\lambda_{ci} \in \mathrm{C}$ is said to be fixed if $\lambda_{ci} \in \sigma(A)$, i.e. $\lambda_{ci} = \lambda_{oi}$, for $\forall K \in \mathbf{K}$. Similarly, the $i^{th}$ right eigenvector $V_{ci}$ of the corresponding synthesis system is said to be fixed if

$V_{ci} \in \ker(A - \lambda_{oi}I)$ $\forall K \in \mathbf{K}$, i.e. $V_{ci} = \alpha V_{oi}$, for all $\alpha \in C\backslash 0$. As the duality of eigenvectors, the $i^{th}$ left eigenvector $W_{ci}$ of the corresponding synthesis system is said to be fixed if $W_{ci}^T \in \ker(A^T - \lambda_{oi}I)$ $\forall K \in \mathbf{K}$, i.e. $V_{ci} = \alpha V_{oi}$, for all $\alpha \in C\backslash 0$.

**Theorem 4.1** Consider a triple $(C, A, B)$ and the corresponding synthesis system $(C, A+BKC, B)$ systems. Assume that $(\lambda_{oi}, V_{oi})$ and $(\lambda_{ci}, V_{ci})$ are the corresponding eigenvalue and eigenvector pairs for two systems, respectively. Then

$$(A + BKC - \lambda_{oi}I)V_{ci} \neq 0 \tag{4.1}$$

if and only if $\lambda_{oi} \notin \sigma(A + BKC)$

**Proof.** $(\Leftarrow)$ Since $(A + BKC)V_{ci} = \lambda_{ci}V_{ci}$ and $(A + BKC)V_{ci} \neq V_{ci}\lambda_{oi}$, hence $\lambda_{oi} \neq \lambda_{ci}$.

$(\Rightarrow)$ $\lambda_{oi} \notin \sigma(A + BKC)$ implies $(A + BKC)V_{ci} \neq V_{ci}\lambda_{oi}$ □

Please note that $V_{ci}$ variation is the vector direction change in a geometric space. A free $V_{ci}$ means that $V_{ci}$ is linearly independent of the corresponding $V_{oi}$. $\lambda_{ci}$ change is scale variation in complex field. Generally, the $\lambda_{ci}$ free does not preserve the $V_{ci}$ free, vice versa. Because the $BKCV_{oi}$ item includes the information of both systems, i.e. the plant system and the synthesis system, it can be used to reflect the properties of their relationship.

According to the Definition 4.1 and an assumption of $A - \lambda_{oi}I \neq 0$, the states of eigenvalues and right eigenvectors changed from a pair ($\lambda_{oi}$, $V_{oi}$) to a pair ($\lambda_{ci}$, $V_{ci}$) with $K \in \mathbf{K}$ have four case: 1) $\lambda_{ci} \neq \lambda_{oi}$, $V_{ci} = \alpha V_{oi}$ for $\forall\alpha \in C\backslash 0$; 2) $\lambda_{ci} \neq \lambda_{oi}$, $V_{ci} \neq \alpha V_{oi}$ for $\forall\alpha \in C\backslash 0$; 3) $\lambda_{ci} = \lambda_{oi}$, $V_{ci} \neq \alpha V_{oi}$ for $\forall\alpha \in C\backslash 0$; and 4) $\lambda_{ci} = \lambda_{oi}$, $V_{ci} = \alpha V_{oi}$ for $\forall\alpha \in C\backslash 0$. The four cases can reflect on the $BKCV_{oi}$ term change as following properties:

**Property 4.1** If $V_{ci} = \alpha V_{oi}$, $\forall \alpha \in \mathbf{C} \backslash 0$ and $\lambda_{ci} \neq \lambda_{oi}$ then

$$BKCV_{oi} = (\lambda_{ci} - \lambda_{oi})V_{oi} \neq 0 \tag{4.2}$$

**Proof.** $(\Rightarrow)$ $(A+BKC)V_{ci} = \lambda_{ci}V_{ci}$ implies $(A+BKC)\alpha V_{oi} = \lambda_{ci}\alpha V_{oi}$ by assumption. Cancelling $\alpha \neq 0$, we obtain

$$(A + BKC)V_{oi} = \lambda_{ci}V_{oi} = AV_{oi} + BKCV_{oi} = \lambda_{oi}V_{oi} + BKCV_{oi}$$

Hence

$$BKCV_{oi} = V_{oi}(\lambda_{ci} - \lambda_{oi}) \neq 0. \ \square$$

This shows that $V_{oi}$ is not only $A$ invariant, but also $BKC$ invariant when $V_{ci} = \alpha V_{oi}$ and $\lambda_{ci} \neq \lambda_{oi}$.

**Property 4.2** If $V_{ci} \neq \alpha V_{oi}$, $\forall \alpha \in \mathbf{C} \backslash 0$ and $\lambda_{ci} \neq \lambda_{oi}$ then

$$BKCV_{oi} \neq V_{oi}(\lambda_{ci} - \lambda_{oi}) \neq 0$$

and

$$BKCV_{oi} \neq 0$$

**Proof.** $(\Rightarrow)$

$$(A + BKC)\alpha V_{oi} \neq \alpha V_{oi}\lambda_{ci} \tag{4.3}$$

Substituting $AV_{oi} = V_{oi}\lambda_{oi}$ into (4.3) and cancelling $\alpha \neq 0$ it becomes

$$BKCV_{oi} \neq V_{oi}(\lambda_{ci} - \lambda_{oi}) \neq 0 \tag{4.4}$$

To establish the second condition, assume $BKCV_{oi} = 0$. However, a result deduced from (4.4) is

$$0 \neq V_{oi}(\lambda_{ci} - \lambda_{oi}) \neq 0 \tag{4.5}$$

Based on the assumption, $(A+BKC)V_{oi} = AV_{oi} = V_{oi}\lambda_{oi}$. This means $\lambda_{oi} \in \sigma(A+BKC)$ and the $i^{th}$ eigenvalue $\lambda_{ci} = \lambda_{oi}$. Substituting $\lambda_{ci} = \lambda_{oi}$ into (4.5), we obtaining

$$0 \neq V_{oi}(\lambda_{ci} - \lambda_{oi}) = 0$$

Therefore, this equation does not exist and a contradiction to (4.4). □

**Property 4.3** If $V_{ci} \neq \alpha V_{oi}, \alpha \in C\backslash 0$ and $\lambda_{ci} = \lambda_{oi}$ then $BKCV_{oi} \neq 0$.

**Proof.** (⇒) Since $(A+BKC)V_{oi} \neq V_{oi}\lambda_{oi}$ and $AV_{oi} = V_{oi}\lambda_{oi}$ then $BKCV_{oi} \neq 0$.
□

**Property 4.4** If $V_{ci} = \alpha V_{oi}, \alpha \in C\backslash 0$ and $\lambda_{ci} = \lambda_{oi}$ then $BKCV_{oi} = 0$.

**Proof.** (⇒) Since $(A+BKC)V_{oi} = V_{oi}\lambda_{oi}$ then $BKCV_{oi} = 0$. □

**Remark 4.1** Based on Properties 4.1-4.4, it can be deduced that

1. Both $V_{oi}$ and $\lambda_{oi}$ are fixed $\Rightarrow BKCV_{oi} = 0$.

2. Either $V_{oi}$ or $\lambda_{oi}$ is free $\Rightarrow BKCV_{oi} \neq 0$.

**Example 4.1** Consider a plant system as

$$\begin{cases} \dot{x} = & Ax + Bu \\ y = & Cx \end{cases}$$

where $A = \begin{bmatrix} a_{11} & 0 \\ 0 & a_{22} \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, and $a_{11} \neq a_{22}$. The eigenvalues and right eigenvectors of the plant are given by

$$\lambda_o = \{\lambda_{o1}, \lambda_{o2}\} = \{a_{11}, a_{22}\}, \qquad V_o = \begin{bmatrix} V_{o1} & V_{o2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Consider the four elementary output feedback cases with $u = Ky$:

1. If $K = \begin{bmatrix} k_{11} & 0 \\ 0 & 0 \end{bmatrix}$, the corresponding eigenvalues, eigenvectors are given by

$$\lambda_c = \{\lambda_{c1}, \lambda_{c2}\} = \{a_{11}, a_{22} + k_{11}\}, \quad Vc = \begin{bmatrix} V_{c1} & V_{c2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = V_o$$

Furthermore,

$$BKC = \begin{bmatrix} 0 & 0 \\ 0 & k_{11} \end{bmatrix}, \ BKCV_{o1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \ BKCV_{o2} = \begin{bmatrix} 0 \\ k_{11} \end{bmatrix}$$

$$V_{o2}(\lambda_{c2} - \lambda_{o2}) = \begin{bmatrix} 0 \\ k_{11} \end{bmatrix}$$

Hence, $BKCV_{o1}$ satisfies Property 4.4 and $BKCV_{o1} = V_{o2}(\lambda_{c2} - \lambda_{o2})$ satisfies Property 4.1.

2. If $K = \begin{bmatrix} 0 & k_{12} \\ 0 & 0 \end{bmatrix}$, then

$$\lambda_c = \{\lambda_{c1}, \lambda_{c2}\} = \{a_{11}, a_{22}\} = \lambda_o, \qquad Vc = \begin{bmatrix} V_{c1} & V_{c2} \end{bmatrix} = \begin{bmatrix} a_{22} - a_{11} & 0 \\ -k_{12} & 1 \end{bmatrix}$$

$$BKC = \begin{bmatrix} 0 & 0 \\ k_{12} & 0 \end{bmatrix}, \ BKCV_{o1} = \begin{bmatrix} 0 \\ k_{12} \end{bmatrix}, \ BKCV_{o2} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Hence, $BKCV_{o1}$ and $BKCV_{o2}$ satisfy Property 4.3 and Property 4.4 respectively.

3. If $K = \begin{bmatrix} 0 & 0 \\ k_{21} & 0 \end{bmatrix}$, then

$$\lambda_c = \{\lambda_{c1}, \lambda_{c2}\} = \{a_{11}, a_{22}\} = \lambda_o, \ Vc = \begin{bmatrix} V_{c1} & V_{c2} \end{bmatrix} = \begin{bmatrix} 1 & -k_{21} \\ 0 & a_{22} - a_{11} \end{bmatrix}$$

$$BKC = \begin{bmatrix} 0 & k_{21} \\ 0 & 0 \end{bmatrix}, \ BKCV_{o1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \ BKCV_{o2} = \begin{bmatrix} k_{21} \\ 0 \end{bmatrix}$$

Hence, $BKCV_{o1}$ and $BKCV_{o2}$ satisfy Property 4.4 and Property 4.3 respectively.

4. If $K = \begin{bmatrix} 0 & 0 \\ 0 & k_{22} \end{bmatrix}$, then

$$\lambda_c = \{\lambda_{c1}, \lambda_{c2}\} = \{a_{11} + k_{22}, a_{22}\},\ Vc = \begin{bmatrix} V_{c1} & V_{c2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ o & 1 \end{bmatrix} = V_o$$

$$BKC = \begin{bmatrix} k_{22} & 0 \\ 0 & 0 \end{bmatrix},\ BKCV_{o1} = \begin{bmatrix} k_{22} \\ 0 \end{bmatrix},\ BKCV_{o2} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Hence, $BKCV_{o1}$ and $BKCV_{o2}$ satisfy Property 4.1 and Property 4.4 respectively.

**Corollary 4.1** If rank$B = m$, then $KCV_{oi} \neq 0$ if and only if $BKCV_{oi} \neq 0$.

**Proof.** Consider $BKCV_{oi} = 0$ with rank$B = m$. Pre-multiply $B$ by its left pseudo-inverse $(B^TB)^{-1}B^T$and yields

$$(B^TB)^{-1}(B^TB)KCV_{oi} = KCV_{oi} \neq 0$$

for $BKCV_{oi} \neq 0$. the proof is similar and is omitted.□

**Example 4.2** Consider Example 4.1. Assume the feedback of controllers is

$$K = \begin{bmatrix} 0 & k_{12} \\ k_{21} & 0 \end{bmatrix}$$

then

$$A + BKC = \begin{bmatrix} a_{11} & k_{21} \\ k_{12} & a_{22} \end{bmatrix} \quad \text{and} \quad |\lambda I - A - BKC| = \lambda^2 - b\lambda + c$$

$$\lambda_c = \{\lambda_{c1}, \lambda_{c2}\} = \{\frac{b + \sqrt{b^2 - 4c}}{2}, \frac{b + \sqrt{b^2 - 4c}}{2}\} \neq \{\lambda_{o1}, \lambda_{o2}\}$$

where $b = (a_{11} + a_{22})$, $c = a_{11}a_{22} - k_{12}k_{21}$. Let $V_{ci} = \begin{bmatrix} v_{ci}^1 \\ v_{ci}^2 \end{bmatrix}$, $i = 1, 2$, then

$$(A + BKC - \lambda_{ci}I)V_{ci} = \begin{bmatrix} (\lambda_{ci} - a_{11})v_{ci}^1 - k_{21}v_{ci}^2 \\ -k_{12}v_{ci}^1 + (a_{11} - a_{22})v_{ci}^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Assume $k_{21} \neq 0$, from the first row of the previous matrix, we can get

$$v_{ci}^2 = \frac{(\lambda_{ci} - a_{11})}{k_{21}} v_{ci}^1 = \frac{1}{2k_{21}}(a_{22} - a_{11} \pm \sqrt{(a_{11} - a_{22})^2 + 4k_{12}k_{21}})v_{ci}^1 \qquad (4.6)$$

substitute it into the second row of same matrix, the result is given by

$$-k_{12}v_{ci}^1 + \frac{1}{k_{21}}(\lambda_{ci} - a_{22})(\lambda_{ci} - a_{11})v_{ci}^1 = 0$$

$v_{ci}^1, v_{ci}^2 = 0$ are trivial solution. This implies that two corresponding equations are mutually dependent. This result can be verified as follows:

Assume $\lambda_{c1} \neq a_{11}$, from the second row of matrix , the corresponding equation is

$$v_{ci}^2 = \frac{1}{\lambda_{ci} - a_{22}} v_{ci}^1 = \frac{1}{2k_{21}}(a_{22} - a_{11} \pm \sqrt{(a_{11} - a_{22})^2 + 4k_{12}k_{21}})v_{ci}^1$$

too. i.e. same as (4.6). So

$$\text{rank}(A + BKC) = \text{rank}\begin{bmatrix} \lambda_{ci} - a_{11} & -k_{21} \\ -k_{12} & \lambda_{ci} - a_{22} \end{bmatrix} = 1 \qquad \text{for } i = 1, 2$$

$$V_c = \begin{bmatrix} v_{c1}^1 & v_{c2}^1 \\ v_{c1}^2 & v_{c2}^2 \end{bmatrix} = \begin{bmatrix} v_{c1} & v_{c2}^1 \\ \frac{1}{\lambda_{c1}-a_{22}}v_{c1}^1 & \frac{1}{\lambda_{c2}-a_{22}}v_{c2}^1 \end{bmatrix} \neq V_0 \qquad \text{for } \forall v_{ci}^1, v_{c2}^1 \in \mathbf{C}\backslash 0, \lambda_{ci} \neq a_{22}$$

$$BKCV_{o1} = \begin{bmatrix} k_{21} & v_{o1}^2 \\ k_{12} & v_{o1}^1 \end{bmatrix} \neq v_{o1}\,(\lambda_{c1} - \lambda_{o1}) = \begin{bmatrix} v_{01}^1(\lambda_{c1} - \lambda_{o1}) \\ v_{01}^2(\lambda_{c1} - \lambda_{o1}) \end{bmatrix} = \begin{bmatrix} 0 \\ (\lambda_{c1} - \lambda_{o1}) \end{bmatrix}$$

satisfying with Property 4.2.

For the $i^{th}$ left eigenvector $W_{oi}$, the states of eigenvalues and left eigenvectors from a pair $(W_{oi}, \lambda_{oi})$ to a pair $(W_{ci}, \lambda_{ci})$ with respect to $K \in \mathbf{K}$ have four cases as those of the $i^{th}$ right eigenvector. The dual properties of Properties 4.1-4.4 are given by

**Property 4.5**

1. If $W_{ci} = \alpha W_{oi}$, $\forall \alpha \in \mathbf{C}\backslash 0$, and $\lambda_{ci} \neq \lambda_{oi}$, then $W_{oi}BKC = (\lambda_{ci} - \lambda_{oi})W_{oi} \neq 0$.

2. If $W_{ci} \neq \alpha W_{oi}$, $\forall \alpha \in \mathbf{C}\backslash 0$, and $\lambda_{ci} \neq \lambda_{oi}$, then $W_{oi}BKC \neq (\lambda_{ci} - \lambda_{oi})W_{oi} \neq 0$ and $W_{oi}BKC \neq 0$.

3. If $W_{ci} \neq \alpha W_{oi}$, $\forall \alpha \in \mathbf{C}\backslash 0$, and $\lambda_{ci} = \lambda_{oi}$, then $W_{oi}BKC \neq 0$.

4. If $W_{ci} = \alpha W_{oi}$, $\forall \alpha \in \mathbf{C}\backslash 0$, and $\lambda_{ci} = \lambda_{oi}$, then $W_{oi}BKC = 0$.

**Proof.** The proof is similar to that of Properties 4.1-4.4 and is omitted.

## 4.2 Eigenvector and Eigenvalue Analysis for A Diagonal Matrix

**Property 4.6** Given the triple $(C, A, B)$, where $A$ is assumed to have $n$ distinct eigenvalues. Let $V_o = \begin{bmatrix} V_{o1} & V_{o2} & \cdots & V_{on} \end{bmatrix}$ be the right modal matrix of $A$ and $D_o$ =diag $\begin{bmatrix} \lambda_{o1} & \lambda_{o2} & \cdots & \lambda_{on} \end{bmatrix}$ eigenvalue matrix of $A$. Let $V_c = \begin{bmatrix} V_{c1} & V_{c2} & \cdots & V_{cn} \end{bmatrix}$ and $D_c$ =diag $\begin{bmatrix} \lambda_{c1} & \lambda_{c2} & \cdots & \lambda_{cn} \end{bmatrix}$ are corresponding right modal matrix and eigenvalue matrix for the synthesis system $(C, A+BKC, B)$ with respect to $K \in \mathbf{K}$, respectively. Assume that the synthesis system has a fixed mode $\lambda_{ci} \in \sigma(A+BKC) \cap \sigma(A)$ for all $K$. Then

$$(A + BKC - \lambda_{oi}I)V_{oi} = 0 \quad \text{if } \lambda_{ci} = \lambda_{oi} \text{ and } V_{ci} = \alpha V_{oi},\ \alpha \in \mathbf{C}\backslash 0$$

or

$$(A + BKC - \lambda_{oi}I)V_{oi} \neq 0 \quad \text{if } \lambda_{ci} = \lambda_{oi} \text{ but } V_{ci} \neq \alpha V_{oi},\ \alpha \in \mathbf{C}\backslash 0$$

**Example 4.3** Refer to Example 4.1, in which $D_o$ =diag $[a_{11,}a_{22}]$ and $V_o = I_2$. Take $K = \begin{bmatrix} 0 & k_{12} \\ 0 & 0 \end{bmatrix}$ then $D_c = D_o, V_c = \begin{bmatrix} a_{22} - a_{11} & 0 \\ -k_{12} & 1 \end{bmatrix}$

for $\lambda_{c1} = \lambda_{o1}$ but $V_{c1} \neq \alpha V_{o1}$ then $(A + BKC - \lambda_{o1}I)V_{o1} = \begin{bmatrix} 0 \\ k_{12} \end{bmatrix} \neq 0$

for $\lambda_{c1} = \lambda_{o1}$ and $V_{c1} = \alpha V_{o1}$ then $(A + BKC - \lambda_{o1}I)V_{o1} = 0$ .

Consider a triple $(C, A, B)$. Let $V_o$, $D_o$ be the right modal matrix and eigenvalue matrix of A, respectively. It is well known that if the eigenvalues are distinct, then

$$AV_o = V_oD_o \quad \Leftrightarrow \quad A^TW_o^T = W_o^TD_o \tag{4.7}$$

and

$$AV_{oi} = V_{oi}\lambda_{oi} \text{ is dual of } A^T W_{oi}^T = W_{oi}^T \lambda_{oi} \quad \text{for } i = 1, 2, ..., n \tag{4.8}$$

where $V_o = \begin{bmatrix} V_{o1} & V_{o2} & \cdots & V_{on} \end{bmatrix}$ and $W_o = V_o^{-1}$, is known as the left modal matrix

$$W_o = \begin{bmatrix} W_{o1} \\ W_{o2} \\ \vdots \\ W_{on} \end{bmatrix}$$

Since $V_o W_o = \sum_{i=1}^{n} V_{oi} W_{oi} = I_n$ and

$$W_o V_o = \text{diag} \begin{bmatrix} W_{o1}V_{o1} & W_{o2}V_{o2} & \cdots & W_{on}V_{on} \end{bmatrix} = I_n$$

$$W_{oi}V_{oj} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

Pre-multiply and post-multiply $W_o$ in the left equation of (4.7), then take transpose on it. The result is the right equation of (4.7). Because $W_o^T = \begin{bmatrix} W_{o1}^T & W_{o2}^T & \cdots & W_{on}^T \end{bmatrix}$, $D_o = D_o^T$, and

$$A^T W_o^T = \begin{bmatrix} A^T W_{o1}^T & A^T W_{o2}^T & \cdots & A^T W_{on}^T \end{bmatrix} = \begin{bmatrix} W_{o1}^T \lambda_{o1} & W_{o2}^T \lambda_{o2} & \cdots & W_{on}^T \lambda_{on} \end{bmatrix}$$

$$AV_o = \begin{bmatrix} AV_{o1} & AV_{o2} & \cdots & AV_{on} \end{bmatrix} = \begin{bmatrix} V_{o1}\lambda_{o1} & V_{o2}\lambda_{o2} & \cdots & V_{on}\lambda_{on} \end{bmatrix}$$

Therefore $AV_{oi} = V_{oi}\lambda_{oi}$ and $A^T W_{oi}^T = W_{oi}^T \lambda_o$ are dual.

Consider a triple$(C, A + BKC, B)$, let $V_c$, $D_c$ be the right modal matrix and the diagonal eigenvalues matrix, respectively. Assume that all the eigenvalues are distinct. Then

$$(A + BKC)V_{ci} = V_{ci}\lambda_{ci} \text{ is the dual of } (A + BKC)^T W_{ci}^T = W_{ci}^T \lambda_{ci} \text{ for } i = 1, 2, ..., n$$

$$(A + BKC)V_c = V_c D_c \quad \Leftrightarrow \quad (A + BKC)^T W_c^T = W_c^T D_c$$

**Theorem 4.2** Consider a triple $(C, A, B)$ system and assume $\lambda_{oi} \in \sigma(A)$ with $V_{oi}$. There exists $\forall\, K \in \mathbf{K}$ such that synthesis system $(C, A + BKC, B)$ has $\lambda_{ci} \in \sigma(A + BKC)$ with $V_{ci}$. Then $V_{ci}$ is a fixed eigenvector with respect to $\mathbf{K}$ if and only if

*1.* $V_{oi} \in \ker(BKC)$, for $\lambda_{ci}$ fixed. or

*2.* $V_{oi}$ is $BKC$ invariant with $BKCV_{oi} = V_{oi}(\lambda_{ci} - \lambda_{oi})$, for $\lambda_{ci}$ free.

**Proof.**

1. ($\Rightarrow$)Because $\lambda_{ci} = \lambda_{oi}$, $V_{oi} \in \ker(BKC)$, i.e. $BKCV_{oi} = 0$, so

$$AV_{oi} + BKCV_{oi} = AV_{oi} = V_{oi}\lambda_{oi} = V_{oi}\lambda_{ci}$$

Comparing with $(A + BKC)V_{ci} = V_{ci}\lambda_{ci}$, we can find a $\alpha$ such that $V_{ci} = \alpha V_{oi}$, $\alpha \in C\backslash 0$ and

$$(A + BKC)\alpha V_{oi} = \alpha V_{oi}\lambda_{ci} = V_{ci}\lambda_{ci}$$

($\Leftarrow$) $V_{ci}$ fixed means that $\exists\alpha \neq 0$, $V_{ci} = \alpha V_{oi}$. Together with the condition $\lambda_{oi} = \lambda_{ci}$, we can find that the invariant equation of $(C, A + BKC, B)$ change into

$$(A + BKC)\alpha V_{oi} = \alpha V_{oi}\lambda_{ci} = \alpha V_{oi}\lambda_{oi}$$

Because $AV_{oi} = V_{oi}\lambda_{oi}$, cancelling $\alpha \neq 0$, then $BKCV_{oi} = 0$, i.e. $V_{oi} \in \ker(BKC)$.

2. ($\Rightarrow$) Since

$$BKCV_{oi} = V_{oi}(\lambda_{ci} - \lambda_{oi}) = V_{oi}\lambda_{ci} - AV_{oi}$$

Multiply $\alpha \neq 0$ and add $AV_{oi}$ in both sides of the previous equation

$$(A + BKC)\alpha V_{oi} = \alpha V_{oi}\lambda_{ci}$$

Comparing with $(A+BKC)V_{ci} = V_{ci}\lambda_{ci}$, we can find that $V_{ci} = \alpha V_{oi}$, $\forall\alpha \in \mathbf{C}\backslash 0$.

($\Leftarrow$) If $V_{ci}$ fixed then $\exists\alpha \neq 0$, $V_{ci} = \alpha V_{oi}$ and

$$(A + BKC)\alpha V_{oi} = \alpha V_{oi}\lambda_{ci}$$

Due to $\lambda_{oi} \neq \lambda_{ci}$, and $AV_{oi} = V_{oi}\lambda_{oi}$, after cancelling $\alpha \neq 0$, then

$$(A + BKC)V_{oi} = V_{oi}\lambda_{oi} + BKCV_{oi} = V_{oi}\lambda_{ci}$$

This equation can be reduced to $BKCV_{oi} = V_{oi}(\lambda_{ci} - \lambda_{oi})$. □

**Example 4.4** Take the plant model in Example 4.1 with $K = \begin{bmatrix} 0 & 0 \\ k_{21} & 0 \end{bmatrix}$, $V_o = I$, and

$$D_o = \text{diag} \begin{bmatrix} a_{11} & a_{22} \end{bmatrix}.$$

then

$$Vc = \begin{bmatrix} 1 & -k_{21} \\ 0 & a_{22} - a_{11} \end{bmatrix}, \qquad D_c = D_o$$

$$BKCV_{o1} = 0, \Rightarrow V_{o1} \in \ker(BKC)$$

$$BKCV_{o2} = \begin{bmatrix} k_{21} \\ 0 \end{bmatrix}, \text{ but } V_{o2}(\lambda_{c2} - \lambda_{o2}) = 0 \Rightarrow V_{o2} \text{ is not } BKC \text{ invariant.}$$

Therefore we can conclude that $V_{c1}$ is fixed for $\lambda_{c1}$ fixed, $V_{c2}$ free for $\lambda_{c2}$ fixed in agreement with $Vc = \begin{bmatrix} 1 & -k_{21} \\ 0 & a_{22} - a_{11} \end{bmatrix}$.

**Theorem 4.3** Consider a triple $(C, A, B)$. Assume that $A$ is a diagonal matrix with distinct eigenvalues. The synthesis system $(C, A + BKC, B)$ has a decentralized fixed mode(DFM) $\lambda_{ci}$ with respect to $\forall K \in \mathbf{K}$ if and only if the $i^{th}$ row or the $i^{th}$ column of $BKC$ is identically zero.

**Proof.** Because A is diagonal matrix, $A = D_o$ and $V_o = W_o = I_n$. From Theorem 4.2, we can conclude that

$$V_{ci} = \alpha V_{oi} \Leftrightarrow BKCV_{oi} = 0 \quad \text{for } \lambda_{ci} \text{ fixed}$$

and

$$\lambda_{ci} = \lambda_{oi} \Leftrightarrow BKCV_{oi} = 0 \quad \text{for } V_{ci} \text{ fixed} \tag{4.9}$$

From (4.8), the dual of (4.9) and Property 4.5, we can conclude that

$$\lambda_{ci} = \lambda_{oi} \Leftrightarrow W_{oi}BKC = 0 \quad \text{for } W_{ci} \text{ fixed} \tag{4.10}$$

Due to $W_{oi}^T = V_{oi} = \begin{bmatrix} 0 & \cdots & 0 & \underset{i^{th}}{1} & 0 & \cdots & 0 \end{bmatrix}^T$, Therefore, together with the (4.9) and (4.10), we obtain

$$\lambda_{ci} = \lambda_{oi} \Leftrightarrow \text{the } i^{th} \text{ column and/or the } i^{th} \text{ row of } BKC \text{ are zero.} \ \Box$$

**Corollary 4.2** Consider a triple $(C, A, B)$ system. Assume $A$ is a diagonal matrix. The synthesis system $(C, A+BKC, B)$ has no DFM with respect to $K \in \mathbf{K}$ if and only if $BKC$ has no zero rows and no zero columns.

**Example 4.5** Consider $A = \begin{bmatrix} a_{11} & 0 \\ 0 & a_{22} \end{bmatrix}$, $B = C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, then

$$BKC = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} k_{22} & k_{21} \\ k_{12} & k_{11} \end{bmatrix}$$

Therefore, using Corollary 4.2 and inspection, the special $BKC$ structures, which may not produce any one DFM, are given by

$$BKC = \begin{bmatrix} k_{22} & 0 \\ 0 & k_{11} \end{bmatrix}, \text{ or } \begin{bmatrix} 0 & k_{21} \\ k_{12} & 0 \end{bmatrix}$$

the corresponding optimal feedback structures are

$$\mathbf{K} = \begin{bmatrix} k_{11} & 0 \\ 0 & k_{22} \end{bmatrix}, \text{ or } \begin{bmatrix} 0 & k_{12} \\ k_{21} & 0 \end{bmatrix}$$

**Example 4.6** Consider $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, then

$$BKC = B\begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}C = \begin{bmatrix} k_{11}+k_{21} & k_{12}+k_{22} & 0 \\ k_{21} & k_{22} & 0 \\ k_{21} & k_{22} & 0 \end{bmatrix}$$

Therefore, because there exists the $3^{rd}$ zero column in $BKC$, $\lambda_{c3} = 3$ is a fixed mode.

If $C$ is changed into $C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$,then

$$BKC = B\begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}C = \begin{bmatrix} k_{11}+k_{21} & k_{12}+k_{22} & k_{12}+k_{22} \\ k_{21} & k_{22} & k_{22} \\ k_{21} & k_{22} & k_{22} \end{bmatrix}$$

Therefore, using Corollary 4.2 by inspection, the special $BKC$ structures, which may not produce any one DFM, are given by

$$BKC = \begin{bmatrix} k_{21} & k_{22} & k_{22} \\ k_{21} & k_{22} & k_{22} \\ k_{21} & k_{22} & k_{22} \end{bmatrix}, \text{ or } \begin{bmatrix} k_{21} & k_{12} & k_{12} \\ k_{21} & 0 & 0 \\ k_{21} & 0 & 0 \end{bmatrix}, \text{ or } \begin{bmatrix} k_{11} & k_{22} & k_{22} \\ 0 & k_{22} & k_{22} \\ 0 & k_{22} & k_{22} \end{bmatrix}$$

the corresponding feedback structures are

$$\mathbf{K} = \begin{bmatrix} 0 & 0 \\ k_{21} & k_{22} \end{bmatrix}, \text{ or } \begin{bmatrix} 0 & k_{12} \\ k_{21} & 0 \end{bmatrix}, \text{ or } \begin{bmatrix} k_{11} & 0 \\ 0 & k_{22} \end{bmatrix}$$

In Chapter 2, $\rho$ is defined as a counting function on a $K$, It can be extended to any matrix $T$, $T \in \mathbf{C}^{m \times n}$.

**Example 4.7** If $T_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 2 & 3 \\ 0 & 0 & 4 \end{bmatrix}$, $T_2 = \begin{bmatrix} 0 & 1 & 2 \\ 3 & 2 & 3 \\ 0 & 0 & 4 \end{bmatrix}$, $T_3 = \begin{bmatrix} 0 & 1 & 0 \\ 7 & 0 & 5 \\ 0 & 0 & 4 \end{bmatrix}$ then

$$\rho(T_1) = \rho(T_3) = 4 \quad \text{and} \quad \rho(T_2) = 6$$

**Definition 4.2** (Lattice)[1] A lattice $\mathbf{L}$ is partially ordered set in which for any pair $x$, $y \in \mathbf{L}$ there exists a least upper bound, i.e. an $\eta \geq x$, $\eta \geq y$ and $z \geq \eta$ for all $z \in \mathbf{L}$ such that $z \geq x$, $z \geq y$, and greatest lower bound, i.e. an $\epsilon \in \mathbf{L}$ such that $x \geq \epsilon$, $y \geq \epsilon$, and $\epsilon \geq z$ for all $z \in \mathbf{L}$ such that $z \leq x$, $z \leq y$.

**Definition 4.3** For any a $T \in \mathbf{C}^{n \times n}$, we can decompose $T$ as row vectors, column vectors, and $R$ distinct symbol element matrices, i.e.

$$T = \begin{bmatrix} T_{r1} \\ T_{r2} \\ \vdots \\ T_{rn} \end{bmatrix} = \begin{bmatrix} T_{c1} & T_{c2} & \cdots & T_{cn} \end{bmatrix} = \sum_{k=1}^{R} T_k^{\circ} \tag{4.11}$$

where $R = \rho(K)$, $K = [k_{ij}]$, $K \in \mathbf{K}$

The $R$ distinct matrices $T_k^\circ$ $(k = 1, 2, ..., R)$ contain $R$ distinct symbol elements $k_{ij}$, respectively. Any two element set intersection of the distinct matrices is empty, i.e.

$$\{t_{ij}\}_{k1} \cap \{t_{ij}\}_{k2} = \phi \qquad \text{for } \forall \{t_{ij}\}_{k1} \in T_{k1}^\circ, \{t_{ij}\}_{k2} \in T_{k2}^\circ, \; k1 \neq k2$$

Based on the previous Theorems 4.3 and Corollary 4.2, an algorithm is devised to determine $\mathbf{K}$ as follows.

**Algorithm 4.1** (Determine $\mathbf{K}$ and $\mathbf{K}^*$)

1. Given a controllable and observable triple $(C, A, B)$ system in which the eigenvalues of $A$ are distinct. Transform $(C, A, B)$ into its diagonal normal form $(\bar{C}, D_o, \bar{B})$ through

$$D_o = V_o^{-1} A V_o, \; \bar{B} = V_o^{-1} B, \; \bar{C} = C V_o$$

2. Calculate $T \triangleq \bar{B} K \bar{C}$ by the symbol method for $K = [k_{ij}]$.

3. Find $\mathbf{K}$ is denoted in the Chapter 2 by

$$\mathbf{K} \; = \{ \; K \in \mathbf{R}^{m \times r} \mid k_{ij} \equiv 0 \;\; \text{if } k_{ij} \notin T, \; i = [1, m], j = [1, r] \}$$

4. Calculate $R = \rho(K)$, where $K \in \mathbf{K}$.

5. According to each $k_{ij} \neq 0$, decompose $T$ as $R$ distinct matrices $T_k^\circ, (k = 1, 2, ..., R)$ by (4.11).

6. Calculate $\rho(T_k^\circ)$ for every $T_k^\circ$, construct a lattice [1] as

$$\mathbf{L} = \{T_1^\circ, T_2^\circ, ..., T_R^\circ\}$$

with partially ordered set based on

$$\rho(T_1^\circ) \geq \rho(T_2^\circ) \geq \cdots \geq \rho(T_R^\circ)$$

7. Make combination for any distinct matrices in $\mathbf{L}$ from one-matrix combination to $\min(n, r \times m)$-matrix combination. Start from $T_1^\circ$ and end to $T_R^\circ$. $T^*$ is denoted by

$$T^* \triangleq T_{i_1}^\circ + T_{i_2}^\circ + \cdots + T_{i_h}^\circ \qquad 1 \leq i_h \leq R,\ T_k^\circ \in \mathbf{L},\ k = i_1, i_2, ..., i_h \tag{4.12}$$

such that $T^*$ has no zero rows and columns as (4.11), i.e.

$$T_{ri}^* \neq 0, \quad \text{and} \quad T_{cj}^* \neq 0 \quad \text{for } \forall T_{ri}^*, T_{cj}^* \in T^* \tag{4.13}$$

until at least one of $i_h$-matrix combination is satisfied with (4.12)-(4.13).

8. Find $\mathbf{K}^*$ is denoted by

$$\mathbf{K}^* = \{\ K \in \mathbf{R}^{m \times r}\ |\ k_{ij} \equiv 0 \ \text{ if } k_{ij} \notin T^*,\ i = [1, m],\ j = [1, r]\}$$

**Example 4.8** Consider Example 4.1, let

$$T = BKC = \begin{bmatrix} k_{22} & k_{21} \\ k_{12} & k_{11} \end{bmatrix}$$

then

$$\mathbf{K} = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \quad \text{and} \quad R = \rho(K) = 4 \ \text{ for } K \in \mathbf{K},$$

because

$$T=\sum_{k=1}^{4}T_k^{\circ}=\begin{bmatrix}k_{11}&0\\0&0\end{bmatrix}+\begin{bmatrix}0&k_{12}\\0&0\end{bmatrix}+\begin{bmatrix}0&0\\k_{21}&0\end{bmatrix}+\begin{bmatrix}0&0\\0&k_{22}\end{bmatrix}$$

and $\rho(T_k^{\circ})\equiv 1,\ k=1,2,3,4$, hence $\mathbf{L}=\{T_1^{\circ},T_2^{\circ},T_3^{\circ},T_4^{\circ}\}$.

Now start from one-matrix combination from $T_1^{\circ}$ and end to $\min(n, r\times m)=2$-matrix combination, in which

$$\mathrm{T}^{*}\neq \mathrm{T}_1^{\circ}+\mathrm{T}_2^{\circ}\ =\begin{bmatrix}k_{22}&k_{21}\\0&0\end{bmatrix}=\begin{bmatrix}T_{r1}^{\circ}\\T_{r2}^{\circ}\end{bmatrix}\qquad \text{because } \mathrm{T}_{r2}^{\circ}=0$$

$$\mathrm{T}^{*}\neq \mathrm{T}_1^{\circ}+\mathrm{T}_3^{\circ}\ =\begin{bmatrix}k_{22}&0\\k_{12}&0\end{bmatrix}.\qquad \text{because } \mathrm{T}_{c2}^{\circ}=0$$

$$\mathrm{T}^{*}= \mathrm{T}_1^{\circ}+\mathrm{T}_4^{\circ}\ =\begin{bmatrix}k_{22}&0\\0&k_{11}\end{bmatrix}\qquad \text{it is satistied with (4.13)}$$

as the same way. we can find other $T^{*}=T_2^{\circ}+T_3^{\circ}$ and stop more elementary matrices combination. Based on Algorithm 4.1 and $T^{*}$ the optimal feedback structures are given by $\mathbf{K}^{*}=\begin{bmatrix}k_{11}&0\\0&k_{22}\end{bmatrix}$, and $\begin{bmatrix}0&k_{12}\\k_{21}&0\end{bmatrix}$.

**Remark 4.2** From Algorithm 4.1 and Example 4.8, we can conclude that

1. $T$ is unique but $T^{*}$ may be non-unique for a plant system.

2. $\mathbf{K}$ is unique but $\mathbf{K}^{*}$ may be non-unique for a plant system.

**Example 4.9** From Example 4.6, the $T$ is given by

$$T=BKC=\begin{bmatrix}k_{11}+k_{21}&k_{12}+k_{22}&k_{12}+k_{22}\\k_{21}&k_{22}&k_{22}\\k_{21}&k_{22}&k_{22}\end{bmatrix}\quad \text{for } C=\begin{bmatrix}1&0&0\\0&1&1\end{bmatrix}$$

then $\mathbf{K}=\begin{bmatrix}k_{11}&k_{12}\\k_{21}&k_{22}\end{bmatrix}$ and $R=\rho(\mathbf{K})=4$, and

$$T=\sum_{k=1}^{4}T_k^{\circ}=\begin{bmatrix}0&k_{22}&k_{22}\\0&k_{22}&k_{22}\\0&k_{22}&k_{22}\end{bmatrix}+\begin{bmatrix}k_{21}&0&0\\k_{21}&0&0\\k_{21}&0&0\end{bmatrix}+\begin{bmatrix}0&k_{12}&k_{12}\\0&0&0\\0&0&0\end{bmatrix}+\begin{bmatrix}k_{11}&0&0\\0&0&0\\0&0&0\end{bmatrix}$$

$$\rho(T_1^\circ) > \rho(T_2^\circ) > \rho(T_3^\circ) > \rho(T_4^\circ) \, . \quad \mathbf{L} = \{T_1^\circ, T_2^\circ, T_3^\circ, T_4^\circ\}$$

Based on (4.12)-(4.13) in step 7 of Algorithm 4.1, we can find that

$$T^* = T_1^\circ + T_2^\circ = \begin{bmatrix} k_{21} & k_{22} & k_{22} \\ k_{21} & k_{22} & k_{22} \\ k_{21} & k_{22} & k_{22} \end{bmatrix}, \text{ or } \quad T^* = T_1^\circ + T_4^\circ = \begin{bmatrix} k_{11} & k_{22} & k_{22} \\ 0 & k_{22} & k_{22} \\ 0 & k_{22} & k_{22} \end{bmatrix}$$

$$\text{or} \quad T^* = T_2^\circ + T_3^\circ = \begin{bmatrix} k_{21} & k_{12} & k_{12} \\ k_{21} & 0 & 0 \\ k_{21} & 0 & 0 \end{bmatrix}$$

Therefore the corresponding minimal feedback structures are

$$\mathbf{K}^* = \begin{bmatrix} k_{11} & 0 \\ 0 & k_{22} \end{bmatrix}, \text{or} \begin{bmatrix} 0 & 0 \\ k_{21} & k_{22} \end{bmatrix}, \text{or} \begin{bmatrix} 0 & k_{12} \\ k_{21} & 0 \end{bmatrix}$$

## 4.3 Eigenvector and Eigenvalue Analysis for Jordan Form

For the case that $A$ can not be diagonalized, the Jordan form of $A$ may be used. The theorems and corollaries in the previous section remains valid for Jordan form. For the sake of explicitness in this section, firstly, the $A$ is assumed as an elementary Jordan block in the theoretical development, and then the $A$ is assumed as the Jordan form in which contains many elementary Jordan blocks.

**Theorem 4.4** Consider a triple $(C, A, B)$. Assume $\lambda_o \in \sigma(A)$ with geometric multiplicity $\mu = n$ and the corresponding right generalized modal matrix $V_o$. There exists an $K \in \mathbf{K}$ such that a synthesis system $(C, A + BKC, B)$ has $\lambda_c \in \sigma(A + BKC)$ with geometric multiplicity $n$ and the corresponding right generalized modal matrix $V_c$. Then

$$(A + BKC)V_{c1} \neq \lambda_o V_{c1} \tag{4.14}$$

$$(A + BKC)V_{ci} \neq \lambda_o V_{ci} + V_{ci-1} \qquad \text{for } i = 2, 3, ..., n \tag{4.15}$$

if and only if $\lambda_o \notin \sigma(A + BKC)$.

**Proof.** ($\Rightarrow$) Because $\lambda_c \neq \lambda_o$, then [14]

$$(A + BKC)V_{ci} = \lambda_c V_{ci} + V_{ci-1} \quad \text{for } i = 1, 2, 3, ..., n \tag{4.16}$$

$$(A + BKC)V_{ci} \neq \lambda_o V_{ci} + V_{ci-1} \quad \text{for } i = 1, 2, 3, ..., n$$

let $V_{c0} = 0$, so $(A + BKC)V_{c1} \neq \lambda_o V_{c1}$.

($\Leftarrow$) Because there exist (4.14), (4.15), (4.16), and $V_{c0} = 0$, then $\lambda_c \neq \lambda_o$, i.e. $\lambda_o \notin \sigma(A + BKC)$. □

In following properties, the Jordan form with only one Jordan block is considered i.e. $\mu = n$. The characteristics of $BKCV_{oi}$ with respect to the "fixedness" of the eigenvalue and right generalized eigenvector are described:

**Property 4.7** If $V_{ci} = \alpha V_{oi}$, $\forall \alpha \in \mathrm{C} \backslash 0$, and $\lambda_c \neq \lambda_o$, then $BKCV_{oi} = (\lambda_c - \lambda_o)V_{oi} \neq 0$, $i = 1, 2, ..., n$.

**Proof.** ($\Rightarrow$)Let $V_{o0} = 0$. $(A + BKC)V_{oi} = \lambda_c V_{oi} + V_{oi-1} \neq \lambda_o V_{oi} + V_{oi-1}$. Because $AV_{oi} = \lambda_o V_{oi} + V_{oi-1}$, substituting previous equation leads to $BKCV_{oi} \neq 0$, and $BKCV_{oi} = \lambda_c V_{oi} + V_{oi-1} - AV_{oi} = (\lambda_c - \lambda_o)V_{oi}$. □

**Property 4.8** If $V_{ci} \neq \alpha V_{oi}$, $\forall \alpha \in \mathrm{C} \backslash 0$, and $\lambda_c \neq \lambda_o$, then $BKCV_{oi} \neq (\lambda_c - \lambda_o)V_{oi} \neq 0$ and $BKCV_{oi} \neq 0$, $i = 1, 2, ..., n$.

**Proof.** ($\Rightarrow$)Let $V_{o0} = 0$. Because $(A + BKC)V_{oi} \neq V_{oi}\lambda_c + V_{oi-1}$ and $AV_{oi} = V_{oi}\lambda_o + V_{oi-1}$, combine two equation then $BKCV_{oi} \neq (\lambda_c - \lambda_o)V_{oi} \neq 0$.

Assume a contrary that $BKCV_{oi} = 0$. However, $BKCV_{oi} = 0 \neq V_{oi}(\lambda_c - \lambda_o)$ and $(A + BKC)V_{oi} = AV_{oi} = V_{oi}\lambda_o$. This means $\lambda_o \in \sigma(A + BKC)$, i.e. $\lambda_c = \lambda_o$. Substituting $\lambda_c = \lambda_o$ into previous equation, we obtain $0 \neq (\lambda_c - \lambda_o)V_{oi} = 0$. Therefore the equation does not exist and assumption is a contradiction to $(\lambda_c - \lambda_o)V_{oi} \neq 0$. □

**Property 4.9** If $V_{ci} \neq \alpha V_{oi}$, $\forall \alpha \in \mathbf{C}\backslash 0$, and $\lambda_c = \lambda_o$, then $BKCV_{oi} \neq 0$, $i = 1, 2, \ldots, n$.

**Proof.** $(\Rightarrow)$Let $V_{o0} = 0$. Because $(A + BKC)V_{oi} \neq V_{oi}\lambda_o + V_{oi-1}$ and $AV_{oi} = V_{oi}\lambda_{oi} + V_{oi-1}$ then $BKCV_{oi} \neq 0$. □

**Property 4.10** If $V_{ci} = \alpha V_{oi}$, $\forall \alpha \in \mathbf{C}\backslash 0$, and $\lambda_c = \lambda_o$, then $BKCV_{oi} = 0$, $i = 1, 2, \ldots, n$.

**Proof.** $(\Rightarrow)$Let $V_{o0} = 0$. Because $(A + BKC)V_{oi} = V_{oi}\lambda_o + V_{oi-1}$ then $BKCV_{oi} = 0$.□

**Corollary 4.3** Consider the elementary Jordan block for $A$ or $A + BKC$. Assume $\lambda_o \in \sigma(A)$ with $V_o$. $\lambda_c \in \sigma(A + BKC)$ with $V_c$. Let the geometric multiplicities for both system be $n$, then $(A + BKC)V_c \neq V_cJ_o$ if and only if $\lambda_o \notin \sigma(A + BKC)$.

**Proof.** $(\Rightarrow)$ $\lambda_o \notin \sigma(A + BKC)$ obviously means $(A + BKC)V_c \neq V_cJ_o$.

$(\Leftarrow)$ Because $(A + BKC)V_{c1} = \lambda_c V_{c1}$ is independent of other Jordan blocks if they exist and $(A + BKC)V_{ci} = \lambda_c V_{ci} - V_{ci-1}$, then

$$\begin{aligned}
(A + BKC)V_c &= \begin{bmatrix} (A + BKC)V_{c1} & (A + BKC)V_{c2} & \cdots & (A + BKC)V_{cn} \end{bmatrix} \\
&= V_cJ_c = \begin{bmatrix} V_{c1} & V_{c2} & \cdots & V_{cn} \end{bmatrix} \begin{bmatrix} \lambda_c & 1 & & & \\ & \lambda_c & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ & & & & \lambda_c \end{bmatrix} \\
&= \begin{bmatrix} V_{c1}\lambda_c & V_{c2}\lambda_c + V_{c2} & \cdots & V_{cn}\lambda_c + V_{c(n-1)} \end{bmatrix}
\end{aligned}$$

Therefore, if $(A + BKC)V_c \neq V_cJ_o$, then $J_c \neq J_o \Rightarrow \lambda_c \neq \lambda_o \Rightarrow \lambda_o \notin \sigma(A + BKC)$. □

**Corollary 4.4** Consider the case that both $A$ and $A + BKC$ consist of a single elementary Jordan block.

1. If $V_{ci} = \alpha V_{oi}, \forall \alpha \in \mathbf{C} \backslash 0$, and $\lambda_c \neq \lambda_o$, then $BKCV_o = V_o(J_c - J_o) \neq 0$

2. If $V_{ci} \neq \alpha V_{oi}, \forall \alpha \in \mathbf{C} \backslash 0$, and $\lambda_c \neq \lambda_o$, then $BKCV_o \neq V_o(J_c - J_o) \neq 0$ and $BKCV_o \neq 0$.

3. If $V_{ci} \neq \alpha V_{oi}, \forall \alpha \in \mathbf{C} \backslash 0$, and $\lambda_c = \lambda_o$, then $BKCV_o \neq 0$.

4. If $V_{ci} = \alpha V_{oi}, \forall \alpha \in \mathbf{C} \backslash 0$, and $\lambda_c = \lambda_o$, then $BKCV_o = 0$.

**Proof.** The proof is similar to that of Property 4.7-4.10 and is omitted.

For the $i^{th}$ left generalized eigenvector $W_{oi}$, the states of eigenvalues and left eigenvectors changed from a pair $(W_{oi}, \lambda_o)$ to a pair $(W_{ci}, \lambda_c)$ with respect to $K \in \mathbf{K}$ have four cases as those of the $i^{th}$ right generalized eigenvector. For a Jordan block, the dual properties and corollary of Properties 4.1-4.4 and Corollary 4.2, respectively, are given by:

**Property 4.11** Consider the case that both $A$ and $A + BKC$ consist of a single elementary Jordan block.

1. If $W_{ci} = \alpha W_{oi}$, $\forall \alpha \in \mathbf{C} \backslash 0$, and $\lambda_c \neq \lambda_o$, then $W_{oi}BKC = (\lambda_c - \lambda_o)W_{oi} \neq 0$.

2. If $W_{ci} \neq \alpha W_{oi}$, $\forall \alpha \in \mathbf{C} \backslash 0$, and $\lambda_c \neq \lambda_o$, then $W_{oi}BKC \neq (\lambda_c - \lambda_o)W_{oi} \neq 0$ and $W_{oi}BKC \neq 0$.

3. If $W_{ci} \neq \alpha W_{oi}$, $\forall \alpha \in \mathbf{C} \backslash 0$, and $\lambda_c = \lambda_o$, then $W_{oi}BKC \neq 0$.

4. If $W_{ci} = \alpha W_{oi}$, $\forall \alpha \in \mathbf{C} \backslash 0$, and $\lambda_c = \lambda_o$, then $W_{oi}BKC = 0$.

**Corollary 4.5** Consider the case that both $A$ and $A + BKC$ consist of a single elementary Jordan block.

1. If $W_{ci} = \alpha W_{oi}, \forall \alpha \in \mathbf{C}\backslash 0$, and $\lambda_c \neq \lambda_o$, then $W_o BKC = (J_c - J_o)W_o \neq 0$

2. If $W_{ci} \neq \alpha W_{oi}, \forall \alpha \in \mathbf{C}\backslash 0$, and $\lambda_c \neq \lambda_o$, then $W_o BKC \neq (J_c - J_o)W_o \neq 0$ and $W_o BKC \neq 0$.

3. If $W_{ci} \neq \alpha W_{oi}, \forall \alpha \in \mathbf{C}\backslash 0$, and $\lambda_c = \lambda_o$, then $W_o BKC \neq 0$.

4. If $W_{ci} = \alpha W_{oi}, \forall \alpha \in \mathbf{C}\backslash 0$, and $\lambda_c = \lambda_o$, then $W_o BKC = 0$.

**Corollary 4.6** Consider the case that both $A$ and $A + BKC$ consist of a single elementary Jordan block.

1. If $\lambda_c \neq \lambda_o$ then $BKCV_o \neq 0$ or $BKCV_{oi} \neq 0$, $i = 1, 2, ..., n$.

2. If $V_c \neq \alpha V_o$ then $BKCV_o \neq 0$ or $BKCV_{oi} \neq 0$, $i = 1, 2, ..., n$.

**Theorem 4.5** Consider a triple $(C, A, B)$. Assume $\lambda_o \in \sigma(A)$ with geometric multiplicity $n$ and $V_o$, then

1. $AV_o = V_o J_o \;\Leftrightarrow\; A^T W_o^T = W_o^T J_o^T$

2. $AV_{o1} = V_{o1}\lambda_o$ is dual of $A^T W_{on}^T = W_{on}^T \lambda_o$

3. $AV_{oi} = V_{oi}\lambda_o + V_{o(i-1)}$, $i = 2, 3, ..., n$ is dual of $A^T W_{oi}^T = W_{oi}^T \lambda_o + W_{o(i+1)}^T$, $i = n-1, n-2, ..., 2$, respectively.

where $W_o = V_o^{-1}$, $V_o = \begin{bmatrix} V_{o1} & V_{o2} & \cdots & V_{on} \end{bmatrix}$, $W_o = \begin{bmatrix} W_{o1} \\ W_{o2} \\ \vdots \\ W_{on} \end{bmatrix}$.

**Proof.** Because $W_oV_o = V_oW_o = I_n$ and

$$W_{oi}V_{oj} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

Decompose $AV_o = V_oJ_o$ as

$$\begin{bmatrix} AV_{o1} & AV_{o2} & \cdots & AV_{on} \end{bmatrix} = \begin{bmatrix} V_{o1}\lambda_o & V_{o2}\lambda_o + V_{o1} & \cdots & V_{on}\lambda_o + V_{o(n-1)} \end{bmatrix}$$

Pre-multiply and post-multiply $W_o$ in the left equation of the result 1 in Theorem 4.5. then $W_oAV_oW_o = W_oV_oJ_oW_o \Rightarrow W_oA = J_oW_o \Rightarrow A^TW_o^T = W_o^TJ_o^T$, vice versa. Decompose $A^TW_o^T = W_o^TJ_o^T$ as

$$\begin{bmatrix} A^TW_{o1}^T & A^TW_{o2}^T & \cdots & A^TW_{on}^T \end{bmatrix} = \begin{bmatrix} W_{o1}^T\lambda_o + W_{o2}^T & W_{o2}^T\lambda_o + W_{o3}^T & \cdots & W_{on}^T \end{bmatrix}$$

Because the leading independent vectors are $V_{o1}$ and $W_{on}^T$ corresponding to the other elementary Jordan blocks if they exist, therefore, the relationship is just an inverse order in the elementary Jordan block. □

**Theorem 4.6** Consider a triple $(C, A, B)$. Assume $\lambda_o \in \sigma(A)$ with geometric multiplicity $n$ and right generalized modal matrix $V_o$ corresponding to the elementary Jordan block. There exists $\forall\ K \in \mathbf{K}$ (RDIFC) such that synthesis system $(C, A + BKC, B)$ has $\lambda_c \in \sigma(A + BKC)$ with geometric multiplicity $n$ and right generalized modal matrix $V_c$. Then $V_c$ is fixed with $K \in \mathbf{K}$ if and only if

1. $V_o \in \ker\,(BKC)$ or $W_o^T \in \ker((BKC)^T)$ for $\lambda_c$ fixed.

2. $V_o$ is $BKC$ invariant with $BKCV_o = V_o(J_c - J_o)$ or $W_o^T$ is $(BKC)^T$ invariant with $(BKC)^TW_o^T = W_o^T(J_c^T - J_o^T)$ for $\lambda_c$ free.

**Proof.**

1. ($\Rightarrow$) For same geometric multiplicity, $\lambda_c = \lambda_o \Leftrightarrow J_c = J_o$. Because $V_o \in$ ker $(BKC) \Rightarrow BKCV_{oi} = 0$, $V_o = \begin{bmatrix} V_{o1} & V_{o2} & \cdots & V_{on} \end{bmatrix}$, then

$$AV_o + BKCV_o = AV_o = V_oJ_o = V_oJ_c \Rightarrow \alpha V_o = V_c \ , \ \forall \alpha \in \mathbf{C}\backslash 0$$

($\Leftarrow$) Due to $\alpha V_o = V_c$, $\forall \alpha \neq 0$, and $\lambda_c = \lambda_o \Leftrightarrow J_c = J_o$ then

$$(A + BKC)\frac{1}{\alpha}V_o = \frac{1}{\alpha}V_oJ_c = \frac{1}{\alpha}V_oJ_o = \frac{1}{\alpha}AV_o \Rightarrow BKCV_o = 0$$

Because

$$W_oBKCV_oW_o = 0 \Rightarrow W_oBKC = 0 \Rightarrow (BKC)^TW_o^T = 0$$

then $W^T \in \ker((BKC)^T)$.

2. ($\Rightarrow$) For $J_c \neq J_o$ , because $BKCV_o = V_o(J_c - J_o) = V_oJ_c - AV_o$, then

$$(A + BKC)V_o = V_oJ_c \Rightarrow (A + BKC)V_c = V_cJ_c \Rightarrow \exists\alpha, V_c = \alpha V_c$$

($\Leftarrow$) For $J_c \neq J_o$, because $V_c = \alpha V_o$ and $(A + BKC)V_c = V_cJ_c$ then

$$(A + BKC)V_o = V_oJ_c \neq V_oJ_o = AV_o \Rightarrow BKCV_o = V_o(J_c - J_o) \neq 0$$

From result 1 in Theorem 4.5, we can conclude that $(A + BKC)V_o = V_oJ_c \Leftrightarrow (A + BKC)^TW_o^T = W_o^T\lambda_c$. Therefore

$$BKCV_o = V_o(\lambda_c - \lambda_o) \Leftrightarrow (BKC)^TW_o^T = W_o^T(\lambda_c - \lambda_o). \ \square$$

**Example 4.10** Consider the plant model

$$\begin{cases} \dot{x} = \begin{bmatrix} a_{11} & 1 \\ 0 & a_{22} \end{bmatrix} x + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} u \\ y = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x \end{cases}$$

Because $J_o = A$. then $AV_o = V_oJ_o = V_oA$, and $V_o = I_2 = W_o$

If K= $\begin{bmatrix} 0 & 0 \\ k_{21} & 0 \end{bmatrix}$, then $BKCV_o = \begin{bmatrix} 0 & k_{21} \\ 0 & 0 \end{bmatrix} \neq 0$

Because $J_c = J_o$ and the leading independent vector $V_{c1} = V_{o1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, based on (4.16) we can find $V_{c2} = \begin{bmatrix} \beta \\ \frac{1}{k_{21}+1} \end{bmatrix}$, $k_{21} \neq -1$, $\beta \neq 0$. Since $BKCV_o \neq 0$ and $BKCV_o \neq V_o(J_c - J_o) = 0$, then $V_c$ is free. If $\beta$= 0 then $V_c = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{k_{21}+1} \end{bmatrix} \neq \alpha V_o$ and $V_c$ also free.

**Theorem 4.7** Consider a triple $(C, A, B)$. Assume that $A$ is the elementary Jordan block. The synthesis system $(C, A + BKC, B)$ has a DFM $\lambda_c$ with respect to $K$ $\in$**K** if and only if $BKC$ is a zero matrix for $V_c = \alpha V_o$, $\alpha \in$R\0.

**Proof.** Because $A = J_o$ then $V_o = W_o = I_n$. From Theorem 4.6, we can conclude that $\lambda_c$ is fixed if and only if $BKCV_o = 0$ for $V_c = \alpha V_o$. Therefore $BKCI_n = BKC = 0$. □

**Remark 4.3** Theorem 4.7 is almost same as Theorem 4.3 except $A$. One is diagonal matrix, other is Jordan block. Comparing the two theorems, we can conclude that every Jordan block in Jordan form satisfies the theorems based on Jordan block. Jordan form also is satisfied with the separation rule.

**Theorem 4.8** Consider a triple $(C, A, B)$. Assume $\lambda_o \in \sigma(A)$ with geometric multiplicity $n$ and right generalized modal matrix $V_o$ corresponding to the elementary Jordan block. There exists a $K$ $\in$**K** (RDIFC) such that synthesis system $(C, A + BKC, B)$ has $\lambda_c \in \sigma(A + BKC)$ with geometric multiplicity $n$ and right generalized modal matrix $V_c$ . Then $V_{c1} \in V_c$ is fixed with respect to $K$ $\in$**K** if and only if

1. $V_{o1} \in$ ker $(BKC)$ or $W_{on}^T \in$ ker$((BKC)^T)$ for $\lambda_c$ fixed.

2. $V_{o1}$ is $BKC$ invariant with $BKCV_{o1} = V_{o1}(\lambda_c - \lambda_o)$ or $W_{on}^T$ is $(BKC)^T$ invariant with $(BKC)^T W_{on}^T = W_{on}^T(\lambda_c - \lambda_o)$ for $\lambda_c$ free.

**Proof.**

1. ($\Rightarrow$) Because $\lambda_c = \lambda_o$ and $BKCV_{o1} = 0$, then

$$AV_{o1} + BKCV_{o1} = AV_{o1} = V_{o1}\lambda_o = V_{o1}\lambda_c \Rightarrow \alpha V_{o1} = V_{c1} \ , \ \forall\alpha \in \mathbf{C}\backslash 0$$

($\Leftarrow$) Due to $\alpha V_{o1} = V_{c1}$ $\forall\alpha \in \mathbf{C}\backslash 0$ and $\lambda_c = \lambda_o$, then

$$(A + BKC)V_{c1} = V_{c1}\lambda_c = \alpha V_{o1}\lambda_o \Rightarrow BKCV_{o1} = 0$$

From the result 2 in Theorem 4.5 $AV_{o1} = V_{o1}\lambda_o$ is the dual of $A^T W_{on}^T = W_{on}^T\lambda_o$, then

The dual of $V_{o1} \in \ker(BKC)$ is $W_{on}^T \in \ker((BKC)^T)$

2. ($\Rightarrow$) Because $BKCV_{o1} = V_{o1}(\lambda_c - \lambda_o) = V_{o1}\lambda_c - AV_{o1}$, then

$$(A + BKC)V_{o1} = V_{o1}\lambda_c \Rightarrow (A + BKC)V_{c1} = V_{c1}\lambda_c \Rightarrow \exists\alpha \neq 0, V_{c1} = \alpha V_{c1}$$

($\Leftarrow$) Because $V_{c1} = \alpha V_{o1}$ and $(A + BKC)V_{c1} = V_{c1}\lambda_c$ then

$$(A + BKC)V_{o1} = V_{o1}\lambda_c \neq V_{o1}\lambda_o = AV_{o1} \Rightarrow BKCV_{o1} = V_{o1}(\lambda_c - \lambda_o) \neq 0$$

From the result 2 in Theorem 4.5, we can conclude that

The dual of $(A + BKC)V_{o1} = V_{o1}\lambda_c$ is $(A + BKC)^T W_{on}^T = W_{on}^T\lambda_c$

Therefore

The dual of $BKCV_{o1} = V_{o1}(\lambda_c - \lambda_o)$ is $(BKC)^T W_{on}^T = W_{on}^T(\lambda_c - \lambda_o)$. □

**Remark 4.4** Theorem 4.8 explanation is more precise then Theorem 4.6 for the condition of a $V_{c1}$ fixed.

**Theorem 4.9** Consider a triple $(C, A, B)$. Assume $A$ as the elementary Jordan block. The synthesis system $(C, A + BKC, B)$ has a DFM with respect to $K \in \mathbf{K}$ if and only if $BKC$ has the first zero column or the last zero row.

**Proof.** Because $A = J_o$ then $AV_o = V_oJ_o$ and $AV_{o1} = V_{o1}\lambda_o$. From Theorem 4.8, we know that $V_{c1} = \alpha V_{o1} \Leftrightarrow BKCV_{o1} = 0$ for $\lambda_c$ fixed. Since $V_{o1} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, therefore $BKCV_{o1} = 0$ if and only if the first column vector is zero column vector in $BKC$ corresponding to the elementary Jordan block. Because the dual of $BKCV_{o1} = 0$ is $(BKC)^T W_{on}^T = 0$ (see Theorem 4.5 and Property 4.11) for $\lambda_c$ fixed and $W_{on}^T = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$, Therefore $(BKC)^T W_{on}^T = 0$ if and only if the last row vector is zero in $BKC$ corresponding to the elementary Jordan block. □

**Theorem 4.10** Consider a triple $(C, A, B)$ with $\lambda_o \in \sigma(A)$, and assume $A$ as the Jordan block. The synthesis system $(C, A + BKC, B)$ has no DFM with respect to $K \in \mathrm{K}$ only if the $BKCV_{o1} \neq 0$ (please note that this is only a necessary condition).

**Proof.** (omit) see proof in Property 4.7, 4.8. □

The follow theoretic development is described under an assumption that $A$ is Jordan form containing more than one elementary Jordan block.

**Theorem 4.11** Consider a triple $(C, A, B)$. Assume $A$ as the Jordan form. The synthesis system $(C, A + BKC, B)$ has no DFM with respect to $K \in\mathrm{K}$ if and only if the first column vectors and the last row vectors in $BKC$ matrix, corresponding to each elementary Jordan block of $A$, are not zero vectors.

**Proof.** (omit) It is easily concluded from Theorem 4.9, 4.10 and the Jordan block separation principal [14]. □

**Example 4.11** Consider the plant model in Example 4.10. Then $T \triangleq BKC = \begin{bmatrix} k_{22} & k_{21} \\ k_{12} & k_{11} \end{bmatrix}$. According to Theorem 4.11, we can find that $T^* = \begin{bmatrix} 0 & 0 \\ k_{12} & 0 \end{bmatrix}$ mentioned in the pre-

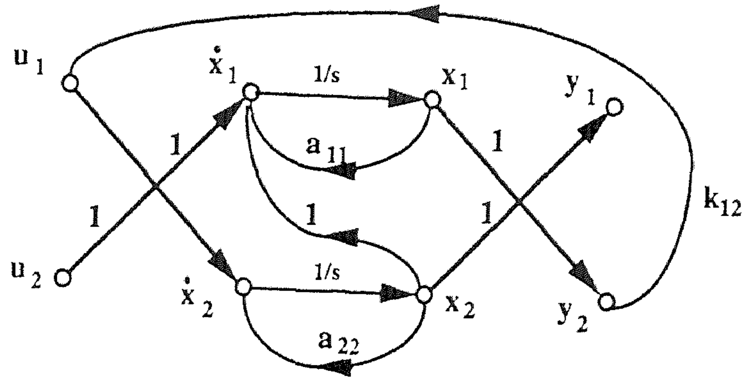


Figure 4.1: Signal Flow Graph for An Example

vious section. Therefore $\mathbf{K} = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}$ and $\mathbf{K}^* = \begin{bmatrix} 0 & k_{12} \\ 0 & 0 \end{bmatrix}$. We can verify that result by signal flow graph. Figure 4.1 shows that if add a controller between $u_2$ and $y_1$ with $\mathbf{K}^*$, then the signal flow from $u_2$ to $y_1$ is a cycle through two modes. No fixed mode exists.

**Example 4.12** Consider the plant model

$$\begin{cases} \dot{x} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} u \\ y = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} x \end{cases}$$

and $A = J = \begin{bmatrix} J_1 & \\ & J_2 \end{bmatrix}$, then

$$T \triangleq BKC = \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} k_{21} & k_{21}+k_{22} & 0 \\ k_{21} & k_{21}+k_{22} & 0 \\ k_{11} & k_{11}+k_{12} & 0 \end{bmatrix}$$

Because $J_o$ has two Jordan blocks, based on Theorem 4.11 and inspection, $T^*$ is selected by

$$T^* = \begin{bmatrix} k_{21} & 0 & 0 \\ k_{21} & 0 & 0 \\ 0 & k_{12} & 0 \end{bmatrix}, \qquad \text{or} \qquad T^* = \begin{bmatrix} 0 & k_{22} & 0 \\ 0 & 0 & 0 \\ k_{11} & k_{11} & 0 \end{bmatrix}$$

Both matrices of $T^*$ satisfy the conditions of Theorem 4.11 corresponding to each elementary Jordan blocks. Therefore, for preserving the elements, the $K^* = \begin{bmatrix} k_{11} & 0 \\ 0 & k_{22} \end{bmatrix}$ or $K^* = \begin{bmatrix} 0 & k_{12} \\ k_{21} & 0 \end{bmatrix}$.

**Algorithm 4.2** (Find **K** and **K**$^*$ for a $\lambda_{oi}$ with geometric multiplicity $\geq 1$)

1. Given a $(C, A, B)$ controllable and observable system. Transform $(C, A, B)$ into Jordan normal form $(\bar{C}, J_o, \bar{B})$ as $J_o = V_o^{-1}AV_o$, $\bar{B} = V_o^{-1}B$, $\bar{C} = CV_o$.

2. Calculate $T \triangleq \bar{B}K\bar{C}$ by the partial symbol method for $K = [k_{ij}]$.

3. Find decentralized feedback structure

$$\mathbf{K} = \{K \in \mathbf{R}^{m \times r} | \; k_{ij} \equiv 0 \quad \text{if } k_{ij} \in T, \; i = [1, m], j = [1, r]\}$$

4. Calculate $R = \rho(\mathbf{K})$.

5. According $k_{ij} \neq 0$, decompose $T$ as $R$ distinct matrices $T_k^*(k = 1, 2, ..., R)$, $k = 1, 2, ..., R$ as in Definition 4.3.

6. Calculate each $\rho(T_k^\circ)$ for $T_k^\circ$, then construct a lattice $\mathbf{L} = \{T_1^\circ, T_2^\circ, ..., T_R^\circ\}$ with partially ordered set based on $\rho(T_1^\circ) \geq \rho(T_2^\circ) \geq \cdots \geq \rho(T_R^\circ)$.

7. Make combination for $R$ distinct matrices $T_k^\circ$, $k = 1, 2, ..., R$, from one to $\min(v, rm)$. Start from $T_1^\circ$ .

   Denote $T^*$ by

$$T^* \triangleq T_{i_1}^\circ + T_{i_2}^\circ + \cdots + T_{i_h}^\circ \quad \text{for } 1 \leq i_h \leq r, \; T_{i_h}^\circ \in \mathbf{L} \tag{4.17}$$

   such that $T^*$ has no the first zero column and no the last zero row corresponding to each elementary Jordan block, i.e.

$$\forall T_{r_i}^* \neq 0 \quad \text{and} \quad \forall T_{c_j}^* \neq 0 \qquad \text{for } i, j = 1, 2, ..., v \tag{4.18}$$

where assume the corresponding $J_o$ as $v$ elementary Jordan blocks, $T^*_{r_i}$ and $T^*_{c_j}$, corresponding to the first column and the last row of each elementary Jordan block, are the first column vectors and the last row vectors in $T^*$.

End to until there exists at least one of $T^*$ of $i_h$-matrix set combination is satisfied with (4.17), (4.18).

8. Find the minimal feedback structure

$$\mathbf{K}^* = \{K \in \mathbf{R}^{m \times r} |\ k_{ij} \equiv 0 \quad \text{if } k_{ij} \in T^*,\ i = [1, m],\ j = [1, r]\}$$

**Remark 4.5** From Algorithm 4.2 for a distinct $\lambda_{oi}$ repeated at least one time, similar to Remark 4.2, it is possible conclude that

1. $T$ is unique but $T^*$ may be non-unique for a plant.
2. $\mathbf{K}$ is unique but $\mathbf{K}^*$ may be non-unique for a plant.

**Example 4.13** Assume that a plant model is given in Jordan normal form as

$$\begin{cases} \dot{x} = \begin{bmatrix} 1 & & & \\ & 2 & 1 & \\ & & 2 & 1 \\ & & & 2 \end{bmatrix} x + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} u \\ y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} x \end{cases}$$

and

$$A = J_o = \begin{bmatrix} J_1 & \\ & J_2 \end{bmatrix}$$

then

$$T \triangleq BKC = \begin{bmatrix} k_{11} & k_{13} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ k_{31} & k_{33} & 0 & 0 \\ k_{21} & k_{23} & 0 & 0 \end{bmatrix} \Rightarrow \mathbf{K} = \begin{bmatrix} k_{11} & 0 & k_{13} \\ k_{21} & 0 & k_{23} \\ k_{31} & 0 & k_{33} \end{bmatrix} \quad \text{and} \quad \rho(\mathbf{K}) = 6.$$

Now let us find $T^*$. Because $\rho(T_k^\circ) \equiv 1$ in

$$T = \begin{bmatrix} k_{11} & 0 & \cdots & 0 \\ 0 & \cdots & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & \cdots & \cdots & 0 \end{bmatrix} + \begin{bmatrix} 0 & k_{13} & 0 & 0 \\ 0 & \cdots & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & \cdots & \cdots & 0 \end{bmatrix} + \begin{bmatrix} 0 & \cdots & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & \cdots & \cdots & 0 \\ k_{21} & 0 & \cdots & 0 \end{bmatrix} +$$

$$+ \begin{bmatrix} 0 & \cdots & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & \cdots & \cdots & 0 \\ 0 & k_{23} & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & \cdots & \cdots & 0 \\ 0 & \cdots & \cdots & 0 \\ k_{31} & 0 & \cdots & 0 \\ 0 & \cdots & \cdots & 0 \end{bmatrix} + \begin{bmatrix} 0 & \cdots & \cdots & 0 \\ 0 & \cdots & \cdots & 0 \\ \vdots & k_{33} & 0 & \vdots \\ 0 & \cdots & \cdots & 0 \end{bmatrix}$$

and **L**= $\{T_1^\circ, T_2^\circ, ..., T_r^\circ\}$, therefore based on (4.18) $r_1 = 1 = r1$, $c_1 = 1 = c1$ for $J_1$ and $r_2 = 4 = r4$, $c_2 = 2 = c2$ for $J_2$, and the first, second columns and the first, fourth rows in $BKC$ can not be zero, i.e. $T^*_{r_1}$, $T^*_{c_1}$, $T^*_{r_2}$, $T^*_{c_2} \neq 0$. Please note that $T^*_{ri} \neq T^*_{r_i}$ and $T^*_{ci} \neq T^*_{c_i}$ in (4.11). Because of $\min(v, rm) = 2$, start to make one-matrix combination, then make two-matrix combination

$$\mathrm{T}^* \neq \mathrm{T}_1^\circ + \mathrm{T}_2^\circ = \begin{bmatrix} k_{11} & k_{13} & 0 & 0 \\ 0 & \cdots & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & \cdots & \cdots & 0 \end{bmatrix}$$

because $T^*_{r_2} = 0$. As the same way $T^* \neq T_1^\circ + T_3^\circ$, $T_1^\circ + T_5^\circ$, $T_1^\circ + T_6^\circ$, $T_2^\circ + T_4^\circ$, $T_2^\circ + T_5^\circ$, ..., $T_5^\circ + T_6^\circ$. But in two-matrix combination, we have found two $T^*$ satisfied (4.17)(4.18),i.e.

$$T^* = T_1^\circ + T_4^\circ = \begin{bmatrix} k_{11} & 0 & \cdots & 0 \\ 0 & \cdots & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & k_{23} & 0 & 0 \end{bmatrix} \quad \text{and} \quad T^* = T_2^\circ + T_3^\circ = \begin{bmatrix} 0 & k_{13} & 0 & 0 \\ \vdots & \cdots & \cdots & 0 \\ 0 & & & \vdots \\ k_{21} & 0 & \cdots & 0 \end{bmatrix}$$

Therefore the more matrix combination process is ended and corresponding optimal feedback structure is

$$\mathrm{K}^* = \begin{bmatrix} k_{11} & 0 & 0 \\ 0 & 0 & k_{23} \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \mathrm{K}^* = \begin{bmatrix} 0 & 0 & k_{13} \\ k_{21} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

## 4.4 Right and Left Eigenvectors Analysis For DFM

In the previous sections, we only discuss that right eigenvector or left eigenvector distinctively for a eigenvalue. In fact, there exist many relationship between these two eigenvectors. In this section, an alternative characterization of DFM in terms of the left and right modal matrices are given.

**Theorem 4.12** Given a triple $(C, A, B)$ and assume that $A$ can be diagonalized. Let $V_o, W_o^T$ be the corresponding right and left modal matrices for the $(C, A, B)$, where $V_o = \begin{bmatrix} V_{o1} & V_{o2} & \cdots & V_{on} \end{bmatrix}$ and $W_o^T = \begin{bmatrix} W_{o1}^T & W_{o2}^T & \cdots & W_{on}^T \end{bmatrix}$. Then a $\lambda_{ci} = \lambda_{oi}$ is fixed for the synthesis system $(C, A + BKC, B)$ if and only if the $W_o BKCV_o$ has the $i^{th}$ zero column and/or the $i^{th}$ zero row.

**Proof.** Since

$$W_o(A + BKC)V_o = W_o A V_o + W_o BKCV_o$$

$$= D_o + W_o BKCV_o = D_o + \bar{B}K\bar{C}$$

i.e.

$$(C, A, B) \iff (\bar{C}, D_o, \bar{B})$$

Use the result in Theorem 4.3. The synthesis system $(\bar{C}, D_o + \bar{B}K\bar{C}, \bar{B})$ has a $\lambda_{ci} = \lambda_{oi}$ DFM with respect to $K \in \mathbf{K}$ if and only if $\bar{B}K\bar{C} = W_o BKCV_o$ has zero rows and/or zero columns. □

**Example 4.14** Consider a plant system which is partially generated by a random function in the Matlab softpackage, i.e.

$$A = \begin{bmatrix} 0.7564 & 0 & 0 & 0 \\ 0 & 0.7227 & 0 & 0 \\ 1 & 0 & 0.8847 & 0 \\ 1 & 1 & 0 & 0.2378 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Using Matlab, we obtain

$$D_o = \begin{bmatrix} 0.8847 & 0 & 0 & 0 \\ 0 & 0.2378 & 0 & 0 \\ 0 & 0 & 0.7564 & 0 \\ 0 & 0 & 0 & 0.7727 \end{bmatrix}$$

$$V_o = \begin{bmatrix} 0 & 0 & 0.1236 & 0 \\ 0 & 0 & 0 & 0.4363 \\ 1 & 0 & -0.9633 & 0 \\ 0 & 1 & 0.2383 & 0.8998 \end{bmatrix},$$

$$W_o = \begin{bmatrix} 0.77942 & 0 & 1 & 0 \\ -1.9283 & -2.0623 & 0 & 1 \\ 8.9012 & 0 & 0 & 0 \\ 0 & 2.2919 & 0 & 0 \end{bmatrix}$$

If a centralized feedback structure is given as

$$\mathbf{K} = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} = \mathbf{R}^{2\times 2}, \qquad \text{and} \qquad K = \begin{bmatrix} 0.2749 & 0.1665 \\ 0.3593 & 0.4865 \end{bmatrix}$$

where $K$ is generated by the random function in the Matlab, then

$$W_oBKCV_o = \begin{bmatrix} 0 & 0 & 0.3461 & 1.6544 \\ 0 & 0 & -0.0072 & -0.1244 \\ 0 & 0 & 0.3593 & 1.7175 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

According to Theorem 4.12, because the $1^{th}$, the $2^{th}$ columns and the $4^{th}$ row are zero, we can conclude that $\lambda_{oi} = 0.8847,\ 0.2378,\ 0.7727$, $i = 1, 2, 4$, are fixed mode with respect to $\mathbf{K}$. Now let us verify the conclusion by the Matlab, then

$$D_c = \begin{bmatrix} 0.8847 & & & \\ & 0.2378 & & \\ & & 1.1157 & \\ & & & 0.7727 \end{bmatrix}$$

agree with the previous conclusion.

**Corollary 4.7** Given a triple $(C, A, B)$ and assume that $A$ can be diagonalized. Let $V_o$, $W_o^T$ be corresponding right and left modal matrices for the $(C, A, B)$. Then a $\lambda_{ci} \neq \lambda_{oi}$ is free for the synthesis system $(C, A+BKC, B)$ if and only if the $W_oBKCV_o$ has the $i^{th}$ non-zero column and the $i^{th}$ non-zero row.

**Proof.** Its proof is similar to that of Theorem 4.12 and omitted. □

**Theorem 4.13** Given a triple $(C, A, B)$ and assume that there is $v$ elementary Jordan blocks for $A$. Let $V_o$, $W_o^T$ be corresponding generalized right and left modal matrices for the $(C, A, B)$, where

$$V_o = \left[ \begin{array}{ccc|c|ccc} V_{o1_1} & \dots & V_{on_1} & \cdots & V_{o1_v} & \cdots & V_{on_v} \end{array} \right]$$

$$W_o^T = \left[ \begin{array}{ccc|c|ccc} W_{o1_1}^T & \dots & W_{on_1}^T & \cdots & W_{o1_v}^T & \cdots & W_{on_v}^T \end{array} \right]$$

Then a $\lambda_{ci} = \lambda_{oi}$ is fixed for the synthesis system $(C, A + BKC, B)$ if and only if the $W_oBKCV_o$ has the first zero column and/or the last zero row for the $i^{th}$ elementary Jordan block.

**Proof.** Since

$$W_o(A + BKC)V_o = W_oAV_o + W_oBKCV_o$$

$$= J_o + W_oBKCV_o = J_o + \bar{B}K\bar{C}$$

i.e.

$$(C, A, B) \iff (\bar{C}, J_o, \bar{B})$$

Based on the result in Theorem 4.11. The synthesis system $(\bar{C}, J_o + \bar{B}K\bar{C}, \bar{B})$ has a $\lambda_{ci} = \lambda_{oi}$ DFM with respect to $K \in \mathbf{K}$ if and only if $\bar{B}K\bar{C} = W_oBKCV_o$ has the last zero row and/or the first zero column corresponding to the $i^{th}$ elementary Jordan block. □

**Corollary 4.8** Given a triple $(C, A, B)$ and assume that there is $v(< n)$ elementary Jordan blocks for $A$. Let $V_o, W_o^T$ be corresponding right and left generalized modal matrices for the $(C, A, B)$. Then a $\lambda_{ci} \neq \lambda_{oi}$ is free for the synthesis system $(C, A + BKC, B)$ if and only if the $W_oBKCV_o$ has the first non-zero column and the last non-zero row for the $i^{th}$ elementary

**Proof.** Its proof is similar to that of Theorem 4.13 and omitted. □

**Example 4.15** Consider a triple $(C, A, B)$ as

$$A = \begin{bmatrix} 2 & & \\ & 1 & 1 \\ & & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Let $\mathbf{K} = \begin{bmatrix} 0 & k_{12} \\ k_{21} & 0 \end{bmatrix}$ and $K = \begin{bmatrix} 0 & 10 \\ 10 & 0 \end{bmatrix} \in \mathbf{K}$. we can find that $V_o = W_o = I_3$, $J_o = \begin{bmatrix} J_{o1} & \\ & J_{o2} \end{bmatrix}$ and

$$W_o BKCV_o = \begin{bmatrix} 10 & 0 & 0 \\ 10 & 0 & 0 \\ 0 & 10 & 0 \end{bmatrix}$$

satisfies the condition of Corollary 4.8. Therefore the **K** structure belongs to the RDIFC and can shifts all modes for this system.

**Corollary 4.9** Consider a triple $(C, A, B)$ and assume that there are $v$ elementary Jordan blocks for $A$. The synthesis system $(C, A+BKC, B)$ has no DFM with respect to $K \in \mathbf{K}$ if and only if the first column vectors and the last row vectors in $W_o BKCV_o$ matrix, corresponding to each Jordan block of $A$, are non-zero vectors.

**Corollary 4.10** Consider a triple $(C, A, B)$ with $n$ distinct eigenvalues of $A$, The synthesis $(C, A + BKC, B)$ has no fixed mode (DFM) with respect to $K \in \mathbf{K}$ if and only if the every column and row in$W_o BKCV_o$ are not zero vector.

**Example 4.16** Consider a triple system

$$A = \begin{bmatrix} 0.6868 & 0.5269 & 0.7012 & 0.0475 \\ 0 & 0.092 & 0.9103 & 0 \\ 0 & 0.6539 & 0.7622 & 0 \\ 0.8462 & 0 & 0 & 0.6326 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

which generated by a random function partially in the Matlab. Take $\mathbf{K} = \begin{bmatrix} 0 & 0 \\ k_{21} & 0 \end{bmatrix}$, and $K = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \in \mathbf{K}$. Using Matlab, we can obtain

$$D_o = \begin{bmatrix} 0.8619 & & & \\ & 0.4575 & & \\ & & 1.2683 & \\ & & & -0.4141 \end{bmatrix}$$

$$V_o = \begin{bmatrix} 0.2615 & 0.2027 & -0.5660 & 0.1116 \\ 0 & 0 & -0.2047 & -0.8650 \\ 0 & 0 & -0.2645 & 0.4809 \\ 0.9652 & -0.9792 & -0.7535 & -0.0902 \end{bmatrix}$$

$$W_o = \begin{bmatrix} 2.1676 & -2.1366 & -4.2624 & 0.4487 \\ 2.1365 & -0.9008 & -2.2251 & -0.5789 \\ 0 & -1.4694 & -2.6431 & 0 \\ 0 & -0.8083 & 0.6256 & 0 \end{bmatrix}$$

and

$$W_oBKCV_o = \begin{bmatrix} 0.5669 & 0.4393 & 1.2265 & 0.2418 \\ 0.5588 & 0.4331 & 1.2089 & 0.2384 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Therefore we can conclude that $\lambda_{o3} = 1.2683$ and $\lambda_{o4} = -0.4141$ are fixed mode in a synthesis system with $\mathbf{K}$. Now let us chick the result by Matlab. Because

$$D_o = \begin{bmatrix} \mathbf{01.7236} & & & \\ & \mathbf{0.5958} & & \\ & & 1.2683 & \\ & & & -0.4141 \end{bmatrix}$$

with respect to $K$. it agrees with this result.

# CHAPTER 5
# CONCLUSIONS

This thesis deals with the selection of an optimal the decentralized feedback structure in large scale control systems. From some aspects of mathematic theory and application, various definitions, theorems, corollaries, and algorithms are generated and developed. These theories can be extended into the general control theory. The main conclusions of this thesis are:

1. Presented a systematic method to determine if a decentralized control system possesses decentralized fixed modes.

2. Introduced the relationship between the system zeros of plant and the fixed modes of synthesis system. The fixed modes are clarified according to system zeros. Addressed that only the transmission fixed mode can be avoid with respect to decentralized feedback structure under output feedback.

3. Characterized the fixed modes and decentralized feedback structures by means of signal flow graphs. Defined decentralized feedback structure and set up algorithm to search this structure in plant system based on the control system tree unit.

4. Defined and set up the tuple control space. Transfer control system tree into a set of corresponding tuples. Constructed operation rules in tuple control space such that the feedback search can be carried aid algorithmically.

5. Introduced the fixed eigenvector concept and applied such concept in the determination of an admissible DIFC.

6. Developed the theorems and corollaries for the cases that $A$ can be diagonalized and transformed into the Jordan canonical form.

7. Designed a set of algorithms to find the realistic decentralized feedback structure (RDIFC) $\mathbf{K}$ and the optimal feedback structure $\mathbf{K}^*$.

8. Applied the right- and the left-generalized modal matrix to survey the fixed mode existing in the synthesis system.

As for the future development of the decentralized control system, the following suggestions are now made:

1. To extend the algorithms in this thesis into computer program for application.

2. To develop a set of realistic algorithm to find feedback structure and controller parameters.

3. To realize the second step of control strategy. Find reductional algorithm such that the multivariable system can be transferred into single variable control system successfully based on decentralized feedback compensator.

4. To develop results on how the geometric number variation of Jordan block affects the feedback structure design.

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