ABSTRACT

Planning of Optimal Transit System Design

by

Yongning Liu

This study presents a methodological approach for finding an optimal design of bus transit service in an urban transportation corridor with elastic demand and under various objective functions. Service design decision variables to be optimized include the route length, route density, headway, number of stops and stop spacing pattern. The objective functions that optimize the transit system design include: minimizing operator cost, maximizing profit, minimizing sum of operator and user costs and maximizing the users' net benefits.

The analysis uses an elastic demand function wherein the number of trips are sensitive to the quality of service provided and price charged by the transit system.

Several analytical models of transit service design are developed and presented in a form of case studies. Computer-aided optimization methods are used to derive the optimal solutions in each of the case studies.

PLANNING OF OPTIMAL TRANSIT SYSTEM DESIGN

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LIST OF VARIABLES AND PARAMETERS

Description	Symbol	Units	Value
Headway	H	Minutes	*
Route Spacing	M	Miles/Route	*
Route Length	L	Miles	*
Number of Stop	N		*
Stop Spacing	S	Miles/stop	*
Length of The Served Area	E	Miles	5
Width of The Served Area	Y	Miles	3
Transit Speed	V	Miles/minute	0.167
Access Speed	G	Miles/minute	0.05
Supply Operating Cost	C	Cents/minute	50
Average Travel Time Lost Per Stop	d	Minutes	
Capacity of Bus	Cap	Seats	45
Bus Fare for Local Service	f	Cents	125
Ratio of Expected User Wait Time to Headway	k		0.4
Value of Passenger In-vehicle Time	Viv	Cents	5
Value of Passenger Waiting Time	Vw	Cents	15
Value of Passenger Access Time	Va	Cents	15

Demand Mode
Choice Coefficients:

Transit Constant	a1	0.38
Wait and walk time	a2	-0.0081
Invehicle Travel Time	a3	-0.0033
Fare	a4	-0.0014
Auto Time and Cost	a5	0.0328

* To Be Optimized

CHAPTER 1

INTRODUCTION

1.1 Problem Statement

Determining an optimal service design of public transportation systems in an urban area is one of the major research fields in urban transportation. The main problem faced by transit planners in designing a transit system is to determine the system parameters such as route length, route density, headway, number of stops and consequently stop spacing patterns. Since these elements have significant impacts on transit operating cost as well as user travel time and thus costs, their values should be carefully determined. Thus, the main problem studied here is finding an optimal transit system service area coverage , usually the optimal combination of route length, route spacing, headway and stop spacing pattern for a particular transit network so that various design objectives are satisfied. The system design objectives include maximization of user benefits, maximization of operator profit, minimization of operator cost and minimization of total operator and user time costs.

1.2 Objective of the Research

The objective of this research is to develop a methodological framework for optimizing transit service design when demand is elastic (i.e., varies with the fares charged and service provided). In most past studies, demand has been assumed to be constant. This may be appropriate for some general studies, but, it cannot reflect the attraction of users to the transit service under certain level of service quality and fare charged. The demand used in this thesis is a function of headway, route length, route spacing, stop spacing and fare. It is elastic in the sense that it is sensitive to changes in service quality parameters and fare paid for travel. Within this framework both theoretical and practical mathematical models were developed for determining the optimal combination values of route length, route spacing, headway and stop spacing for a bus transit system serving a rectangular urban transportation corridor. Initially, theoretical analytical models were formulated on an idealized basis but they were unable to provide a closed form solution for design variables. A more realistic model can be formulated after some of the initial assumptions are relaxed. This model is incorporated within an efficient optimization algorithm and used to compute the optimal service design for particular case studies. Each case study reflected a particular service design objective function. While all of the case studies focus on the same urban corridor, and have the same input data for costs and user value of time parameters, they differ in the objective the service design variables must meet. As stated earlier, the objectives considered in this research are: minimizing operator cost, maximizing operator profit, maximizing user benefit and minimizing total operator and user costs.

1.3 Scope

This research focuses on the operation of a bus transit system along a rectangular shaped urban corridor. The underlying street network is our urban grid network (also referred to as iron or Manhattan grid). The demand is given as a linear mode share model which yields the probabilities of travelers in the service corridor choosing transit. The travel pattern is "many to one", wherein all passengers have their origin or destination at a common central point. This pattern is common for commuting from residential areas to the central business district (CBD). All transit vehicles are assumed to be uniform in type and capacity. The operating speed is the same for all transit vehicles at all locations. Walking was assumed to be the only access mode.

1.4 Overview of Thesis

This thesis is divided in six chapters. Chapter 2 reviews the literature in the area of transit service modeling. Chapter 3 shows the model development and objective design function formulation. Four different cases of transit design in an urban area are examined in Chapter 4. Chapter 5 presents the sensitivity analysis of the four cases Chapter 6 summarizes the results and recommendations for future research.

CHAPTER 2
LITERATURE SURVEY

In the past 20 years, a number of attempts have been made to optimize various variables of transit system design. Most approaches have studied certain idealized problems by analytical methods. Their objective functions dealt generally with minimizing the transportation (sum of user and operator) costs and maximization of the transit system's profit.

Byrne and Vuchic (1972) studied bus route locations and frequency of parallel bus routes serving a rectangular area from which passengers travel to and from the CBD. Their objective function minimized the total operator and user costs. Their assumptions were that the passengers would be taking the lowest cost route to the CBD and that the bus operating speeds on each route were equal. They found that each bus route should be positioned in such a manner so that the populations using it from each side are equal.

Byrne (1976) extended the previous work by relaxing the assumption of equal bus operating speed to take into account different bus speeds on different routes. He concluded that when the high and low speed routes run in parallel in the same rectangular area, the lower speed routes will have to terminate at some distance from the CBD, otherwise all passengers would prefer to take the high speed routes.

S.C. Wirasinghe and N.S. Ghoneim (1981) used continuum approximations and methods of calculus to optimize bus stop locations along a local bus-route with non-uniform many-to-many travel demand. In their analysis daily demand was assumed to vary slowly with distance between stops.

Kocur and Hendrickson (1982) analyzed bus service design for a transit system with three different objective functions, namely profit maximization, maximization of net user benefits and operator profit, and maximization of net user benefits subject to a deficit constraint. The decision variables were defined as the spacing between parallel routes, the route headway, and the fare. The service area had a rectangular grid street network. The difference between their research and previous work is that their demand was elastic (i.e, it was sensitive to the transit levels of service). They found that the optimal line spacing is proportional to headway for all their objective functions and constraints.

Wirasinghe and Seneviratne (1986) also studied the rail line length in an urban transit corridor that would minimize the (user and operator) cost. In this analysis the "many to one" passenger demand travel pattern was assumed. The objective function included the cost of rail fleet, rail operating cost, bus operating cost and passenger time cost. The optimal value of line length, which minimizes the sum of user and supplier cost, was obtained by taking the partial derivative of the objective function with respect to line length and setting it equal to zero. They determined that, when the line cost is nonuniform, there could be several line lengths at which the total transit system cost is minimized or even maximized locally. When the cost per unit is uniform an optimal length exists only if the net gain in travel time and operating cost of transporting the total demand a unit distance by rail when compared to bus exceeds the marginal line and fleet cost per unit length. Closed form solutions for the line length were obtained for the cases of sectorial and rectangular corridor areas with uniformly distributed demand per unit area. They showed that the threshold demand and optimum line length are in fact related to the fleet cost.

Spasovic (1986) developed a model of service coverage for an urban transportation corridor. He minimized the total cost of a transit system where the various operator and user cost components are formulated as functions of the decision variables. The important findings of his study were that: the optimal route length varies directly with corridor length, operating headway, transit speed, value of access time, route spacing and passenger density and that it varies inversely with supplier operating cost and access speed. The optimal headway varies inversely with supplier operating cost, length of transit route, number of stops and time lost per stop. It varies inversely with the square root of passenger density, value of waiting time, transit speed and route spacing. He also confirmed results obtained by several researches that for the total operator and user cost objective function, at optimum, the costs of operating the service, and waiting costs are equalized.

A classification of the previously mentioned and other pertinent analytic models according to the type of service design elements to be optimized, and the objective function was made and is shown in Table 2.1.

In conclusion, the majority of the reviewed papers optimized the transit system design by finding an optimal headway (or frequency) and route spacing. All papers but one assumed that demand for service was constant. In marked contrast, this thesis will derive not only the headway and route spacing, but also the optimal route length and stop spacing pattern assuming variable (elastic) demand for service for four different objective functions.

Table 2.1 Summary of Pertinent Transit Design Analytic Models

Decision Variable	Objective Function	Transit Mode	Network Geometry	Passenger Demand	Authors
location headways	min. user & operator cost	bus	rectangular grid	uniform inelastic	Byrne & Vuchic 1972
route length headway	min. user & operator cost	bus & rail	rectangular grid	uniform inelastic	Byrne 1976
number of zones and headway	min. user & operator cost	bus	linear	uniform inelastic	Tsao & Schonfeld 1983
route length headway	min. user & operator cost	bus	rectangular grid	uniform inelastic	Tsao & Schonfeld 1983
vehicle size	min. user & operator cost	bus	rectangular grid	uniform inelastic	Jansson 1980
route space headway fare	max. profit max. benefit max. profit & benefit	bus	rectangular grid	elastic	Kocur & Hendrickson 1982
headway	min user time	bus	linear	uniform	Wirasinghe 1982
headway stops	min. user time	bus	linear	uniform inelastic	Kikuchi 1985
route length headway	min user & operator cost	bus	rectangular grid	uniform	Spasovic 1986
line length	min. user & operator cost	rail	rectangular grid	inelastic	Wirasinghe & seneviratne 1986
stops	min. user cost	bus	linear	uniform	Kuah & Perl 1988
headway	min. system cost	rail	linear	uniform	Keaton 1989
headway	min. user cost	bus	linear	uniform	Abkowitz 1990

CHAPTER 3

METHODOLOGICAL FRAMEWORK FOR BUS TRANSIT SERVICE PLANNING AND DESIGN

In the transportation services planning and designing in general, a supplier of the transportation service must select the level of service (LOS) characteristics that to be offered to the users (passengers or shippers of freight) and the rate (price) to be charged. Usually, the service characteristics are chosen in such a fashion so that they satisfy certain pre-specified service design objectives. The design objectives could range from maximizing service market penetration (e.g., market share), maximizing profit (or minimizing cost), and maximizing public benefit, to name a few.

In the public transit service planning process in particular, the supplier of the service must choose various LOS characteristics. Namely, how far to extend the transit route in a service area, how frequently to operate the vehicles, where, and how apart, the routes should be locate, how many stops (stations) will be on the route, and how far apart the stops will be located. In public transit, often the supplier of services is a local authority that provides equipment (i.e., buses) and grants operating rights to an operator who actually provides the service. Often the authority sets service characteristics externally, because it is actually the customer who "pays" for the service, rather than by the service operator. The authority usually specifies values of LOS characteristics that must be provided, thus ensuring that the riders are offered a minimum service.

The bus transit service planning process is presented schematically in Figure 3.1. This process relates the resources and cost required to provide the service to its operating characteristics.

In planning bus service planning, one must choose values for the following service characteristics, i.e., design variables:

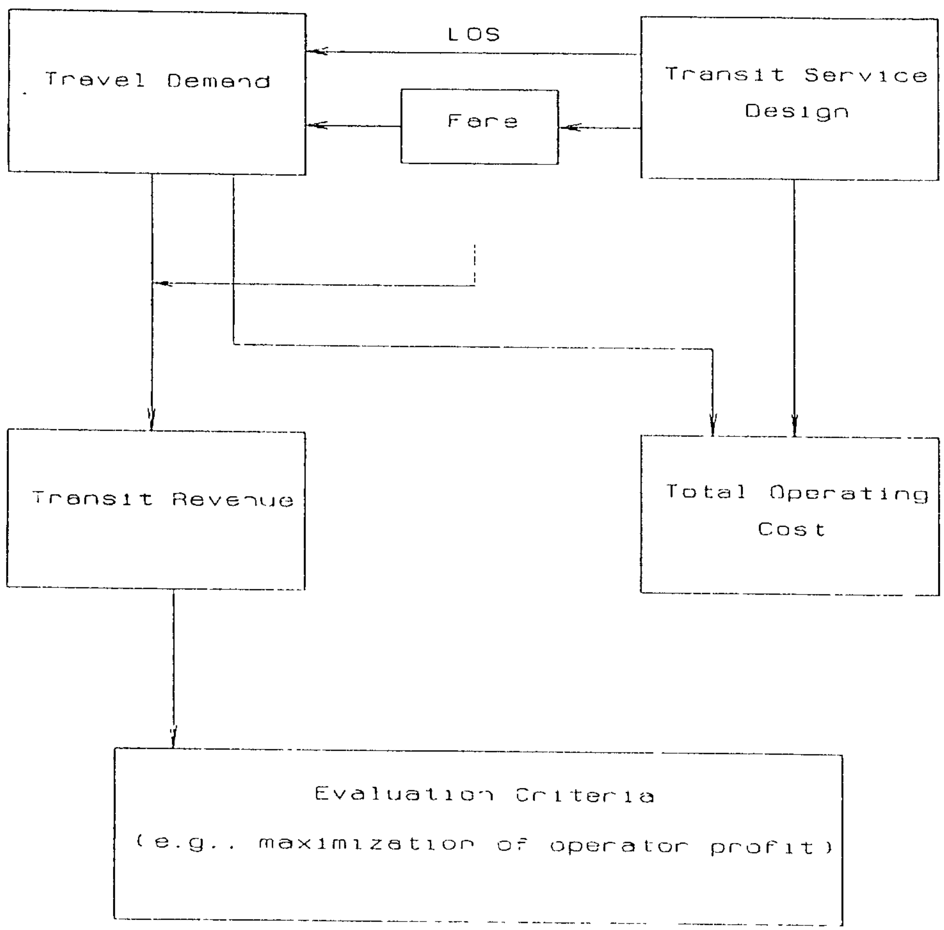

Figure 3.1 Methodological Framework For Bus Service

the length of route

headway (or its inverse, the frequency)

route spacing (i.e., route density)

number of stops on the route

stop spacing pattern must be chosen.

In providing the service, the operator must ensure that the regulatory measures that designate the minimum acceptable values imposed by the supplier are satisfied. The values of these variables will impact the cost of operation.

Passengers are sensitive to the values of service design variables. For example, the more frequent service is provided, the more passengers will use it. Therefore, the service variables impact the demand for service. The demand will in turn impact the service characteristics because certain frequency of buses will have to be provided on a route in order for the route capacity to meet demand. This in turn will impact the operating cost. Also, each passenger pays a fare for using the service. The total revenue obtained is equal to the number of passengers using the service multiplied by the individual fare per passenger. The total revenue collected from fares is called fare box revenue.

The service that is provided could be evaluated using several evaluation criteria. For example, the operating profit defined as a difference between the fare box revenue and operating cost -- could be one of the criteria. In reality, the majority of transit systems do not recover the operating cost from fare box moneys and need to be subsidized from additional external revenue sources. In this case, minimization of operating cost subject to the minimum LOS to be provided on the route becomes the relevant design criterion. The number of passengers attracted and transported via bus could be another criterion.

It should be mentioned that often there is a conflict between the operator's and users' objectives. The users would prefer to have short access to the route (e.g., frequent stops and longer routes) and little waiting time (i.e., frequent service). On the other hand, the operator would prefer to have very long headway and shorter, sparsely located routes with few stops so that cost is minimized. In order to take into account this conflict in operator's and users' objectives, the sum of operator and user cost is often used as a suitable criterion in designing the service. The operator cost is self-explanatory, while the user cost is defined as a time cost the user spends using the service (i.e., access, wait, in-vehicle riding, and egress times) multiplied by the perceived value of user time.

In summary, the service planning problem could be formulate as an optimization problem wherein the levels of various service characteristics must be chosen in such a fashion so as to improve (e.g., minimize or maximize) certain criteria in the form of service performance or design functions. Therefore, the service planning framework will be used to determine the optimal service design of a simplified bus transit system under various criteria or objective functions. The results of these case studies will be compared in order to evaluate the trade-offs between the values of design variables under various criteria.

3.1 Introduction to Analytic Approach

Optimal transit service design variables are derived in this thesis for four design functions presented in a form of case studies. The four design or objective functions are: minimization of operator cost, maximization of operator profit, minimization of total operator and user cost, and maximization of user benefits. The objective functions are formulated as a continuous function of design variables. The design variables are line length, line spacing, headway and stop spacing pattern. The optimal value of the design variables are found by solving the partial derivatives of the objective function with respect to the particular decision variables. Vehicle size constraints and certain minimum level of transit service quality (e.g., maximum acceptable walking distance) are considered in this study.

3.2 Model Assumptions

The following assumptions are made for all cases and models that are developed and analyzed in this thesis:

1. The service area is an urban transportation corridor of length E and width Y.
2. There is a many-to-one trip travel pattern. All trips either originate or terminate at the central business district (CBD).
3. The transit users (i.e., their trip origins) are uniformly distributed over the service area.
4. The transit demand is elastic, (i.e., sensitive to the change in price and quality of service).
5. A rectangular grid pattern street network (e.g., Manhattan grid) exists in the service area.
6. The area is served by a set of transit routes of length (L), the optimal number of which is to be determined. The routes extend from the CBD radially outbound.
7. Walking is assumed to be the only access mode.
8. Transit users always choose the shortest path to access the route. If the distance between routes is M, and stop spacing is S, passengers originating along the route walk an average distance of (M+S)/4. Passengers originating from the area beyond the last stop on the route, the average access distance is (E-L)/2+M/4. Average walking distance are shown in Figure 3.2.
9. The average transit speed (V) is assumed to be constant. Therefore, vehicles on the routes are operating with uniform headways.
10. Average waiting time is assumed to equal to one half of the headway. The headway is uniform along the route, as well as for all parallel routes.
11. The passenger access speed (G) is constant.

Figure 3.2 shows the corridor and the route configuration.

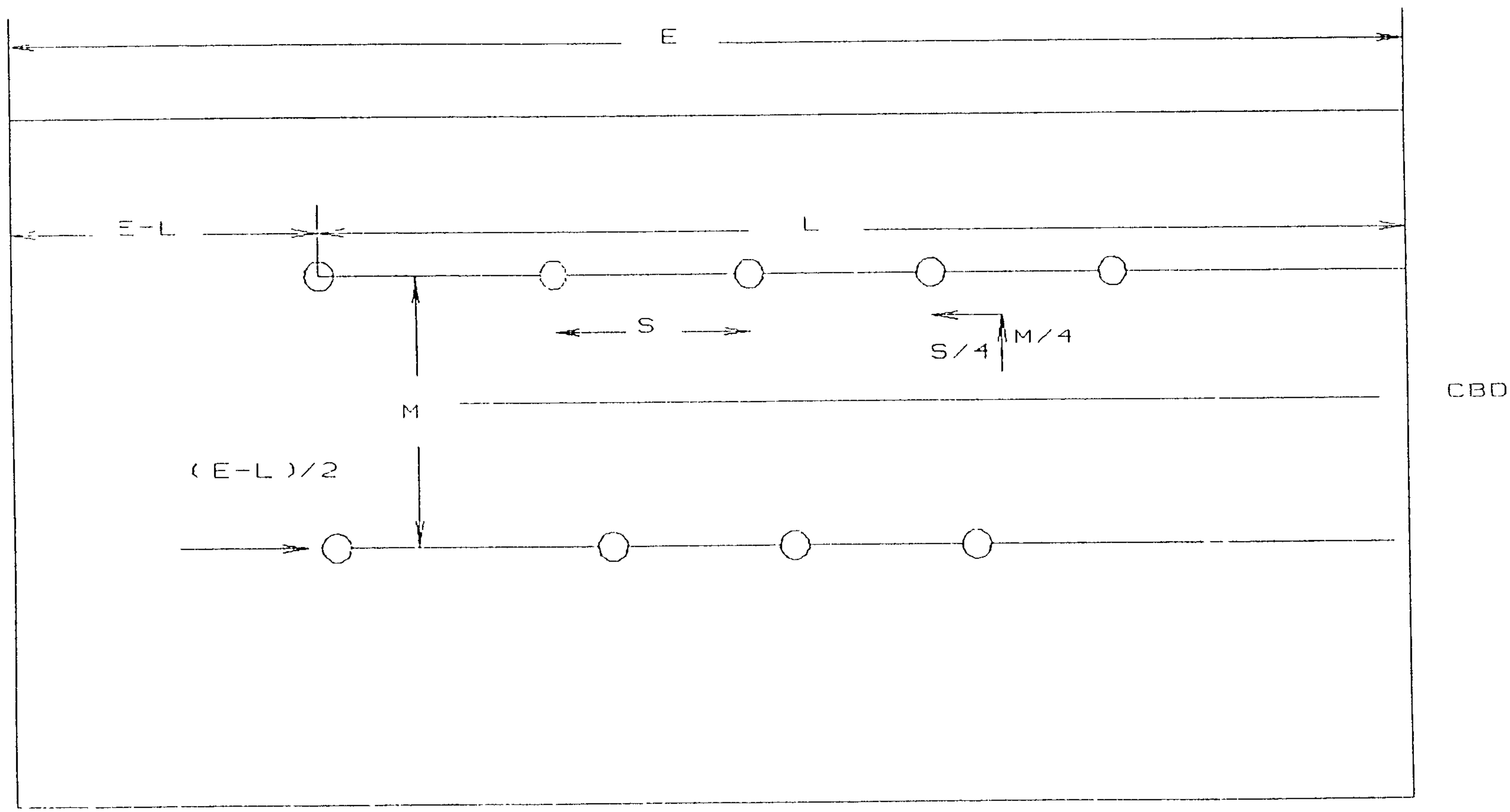

Figure 3.2 Transit Corridor Configuration

3.3 Model Formulation

3.3.1 Operator Cost

The operator or supplier cost usually includes labor expenses, fuel, maintenance cost and vehicle depreciation cost. In this study, we assume that all these expenses are included in the operator hourly cost (C). Thus, the total operator cost is equal to the product of the hourly transit cost (C) and fleet size:

Operator Cost = Vehicle Cost x Fleet Size

[\$/hr] = [\$/veh-hr] [vehicles]

The fleet size is equal to the number of transit vehicles required to provide the service on one route. That is:

Fleet Size = (Round Trip Time)/Headway.

In addition to the fleet size, the operator cost depends on the number of stations along the route. The vehicle stops to pick up or discharge passengers thus incurring additional cost due to the lost time and additional wear and tear of braking equipment. The time lost per stop (d) is considered in deriving operator cost. It is a combination of the time spent during deceleration and acceleration (given as a constant, a) and the time loss due to passengers' boarding the vehicle. The second component represents the time loss due to passenger volume at the stop. It is the product of the number of passengers waiting to board the vehicle Ps, multiplied by the boarding time per passenger. Using the notation introduced in List of Variables, the operator's cost can be written as:

$$SC = 2C\left(\frac{L\,Y}{M\,V} + \frac{L\,Y\,d}{M\,S}\right) / H$$

or

$$SC = \frac{2\,C\,L\,Y}{M\,H}\left(\frac{1}{V} + \frac{d}{S}\right) \tag{3.1}$$

3.3.2 User Cost

The hourly user cost (Cu) consists of user access (Ca), wait (Cw) and in-vehicle (riding) cost (Civ). The user cost is equal to the value of time multiplied by the total time spent on each portion of the trip. The access cost Ca consist of three parts:

access parallel to the route for passenger beyond the route, Ca1

access perpendicular to the route, Ca2

access in the area between the CBD and the route terminus, Ca3,

which is given by (an access speed G and the value of access time Va):

$$Ca1=2P(a1+a2(kH+(M+S)/4G)+(a3/2G+a5/2)L+a5f)YVa(E\text{-}L)/2G(E\text{-}L).$$

$$Ca2=2P(a1+a2(kH+(M+S)/4G)+(a3/2G+a5/2)L+a5f)YVaE(M/4G)$$

$$Ca3=2P(a1+a2(kH+(M+S)/4G)+(a3/2G+a5/2)L+a5f)YVaL(S/4G) \qquad (3.2)$$

Given the average wait time is one half of the headway (Assumption 10), we can derive the total waiting time as:

$$2P(a1+a2(kH+(M+S)/4G)+(a3/2G+a5/2)L+a5f)(H/2)EY$$

Then the total waiting cost equals the value of waiting time (Vw), times total waiting time:

$$Cw=Vw(H/2)2P(a1+a2(kH+(M+S)/4G)+(a3/2G+a5/2)L+a5f)EY. \qquad (3.3)$$

The average travel time to the CBD for people originating in the area beyond the end of the transit line equals the average distance traveled divided by the operating speed of the transit vehicle: $t_1=L/2V$. For passengers originating from a zone beyond the end of the transit line the distance traveled is L, so the average riding time is: $t_2=L/V$. The total user in-vehicle cost is then:

$$Civ=2P(a1+a2(kH+(M+S)/4G)+(a3/2G+a5/2)L+a5f)LYViv(L/2V+dL/2S)$$
$$+2P(a1+a2(kH+(M+S)/4G)+(a3/2G+a5/2)L+a5f)(E\text{-}L)YVivL/V+dL/S.$$

(3.4)

3.3.3 Service Constraints

The service constraints applied in this thesis ensure that the operator provides certain level of service (e.g., route spacing should be constrained by a maximum passenger access distance). In a case when passengers are distributed uniformly over the service area, an average passenger walks a distance of M/4 perpendicularly to the routes, and a distance of S/4 along the route he/she reaches a station. The passengers who are originating from the area beyond the last station walk an average distance of (E-L)/2. Assuming a maximum acceptable walking distance of 1 mile, the following service quality constraints can be derived:

$$\frac{E - L}{2} + \frac{M}{4} <= W ;$$

$$\frac{M + S}{4} <= W ; \tag{3.5}$$

In addition, there is a constraint that the capacity of the bus satisfies the demand (number of passengers) on the route during the headway.

3.3.4 Demand Function

The demand function used in this thesis is that of Kocur and Hendrickson (1982). It is elastic because it varies with the values of route length, route density (route spacing), transit headway, station spacing and fare. This demand function is actually a mode choice function and is defined as the sum of the probability of individuals in the service area choosing the transit service. The measure of the perceived difference in "quality" between the two alternatives namely auto and transit is given as:

$$a1+a2[kH+(M+S)/4G]+a3(L/2v)+a4f+a5(L/2).$$

where:

a1 = A constant reflecting variables excluded from the model and biases in preferences for auto and transit.

a2 = The coefficient for the waiting and walking time.

a3 = The mode choice coefficient for the utility of transit travel time.

a4 = The mode choice coefficient of transit fare.

a5 = The mode choice coefficient of the time and cost characteristics of the competing auto.

kH = the expected waiting time for bus

(M+S)/4 = the expected walking distance to reach a stop

L/2 = the average travel length for passenger to the destination

L/2v = the average travel time of passengers.

All individual passenger trip origins are assumed to be uniformly distributed over the service area.

CHAPTER 4

CASE STUDIES

4.1 Case 1: Minimizing Transit Operating Cost

The purpose of this case study is to develop an optimal transit network service design that will minimize the cost of operating the service. It is obvious that the transit operator will minimize its cost (i.e., have zero cost) by providing no service. Clearly this is not acceptable because the people in the service area demand service of certain quality. Therefore, certain minimum level of service quality must be set externally, usually by an authority that authorizes the operator to provide the service. This minimum service quality may specify minimum headway, the length of passengers access to the route, etc. This minimum level of service will ensure that the capacity of the route will meet demand as well as that the location of route and stations do not exceed the maximum allowable access distance. Constraints on the capacity of transit vehicles and on access distance are included in this Case. Therefore, this case investigates the optimal value of headway, transit length, space between transit route and stop spacing (actually the number of stops) that should be chosen to minimize operator cost. This problem is structured in the form of a constrained optimization problem, where the operator's cost is minimized subject to the service quality constraints. The problem is written as:

Minimize : Total operating cost

subject to: Route capacity meets demand, average longitudinal and perpendicular access distance to route does not exceed maximum acceptable distance.

That is:

Minimize $2CLY(1/V+d/S)/MH$

subject to:

$EMPH[a1+a2(kH+(M+S)/4G)+a3(L/2v)+a4f+a5(L/2) <= cap;$

$$\frac{E - L}{2} + \frac{M}{4} <= W ;$$

$$\frac{M + S}{4} <= W :$$

$$H,M,S,L >= 0 ; \qquad (4.1)$$

This problem is a non-linear programming problem with a non-linear objective function and linear constraints. In order to solve this problem, we will use the penalty function method. The penalty function method transforms a constrained problem into an unconstrained problem by pricing the inequality out of the constraint set and placing them into the objective function with a penalty. This penalty penalizes the violation of the constraints. The objective function can be rewritten as:

$$\text{Minimize:}Q=2CLY(1/v+d/S)/HM+u1\{[EMPvH(a1+a2(kH+(M+S)/4G)$$
$$+a3(L/2v)+a4f+a5(L/2)]\text{-cap}\}^2+u2[M+S\text{-}4W]^2+u3[2(E\text{-}L)+M\text{-}4W]^2$$

where:

ui=penalty attached with i-th constraints, for i=1,2,3. All u's must be non-negative.

From looking at the penalty function, we see that the optimal solution to this problem must have the second, third, and fourth term close to zero, otherwise a large penalty will be incurred. An optimization algorithm, shown in Figure 4.1, was developed to solve this problem. The algorithm searches feasible values of design variables L,H,M,S for certain "u"s, which have varying ranges of value. The iterations are repeated until the total operating cost has no obvious difference compared to the value of the penalty function Q. The resulting values of L,H,M,S are the optimal for certain "u"s. The Appendix A explains in detail the penalty function method and contains the code that was used to solve this case.

Due to the unavailability of numerical optimization software, calculus was used to compute the partial derivatives of the objective function with respect to L,H,M & S in order to speed up the process of computing the optimal values of the decision variables.

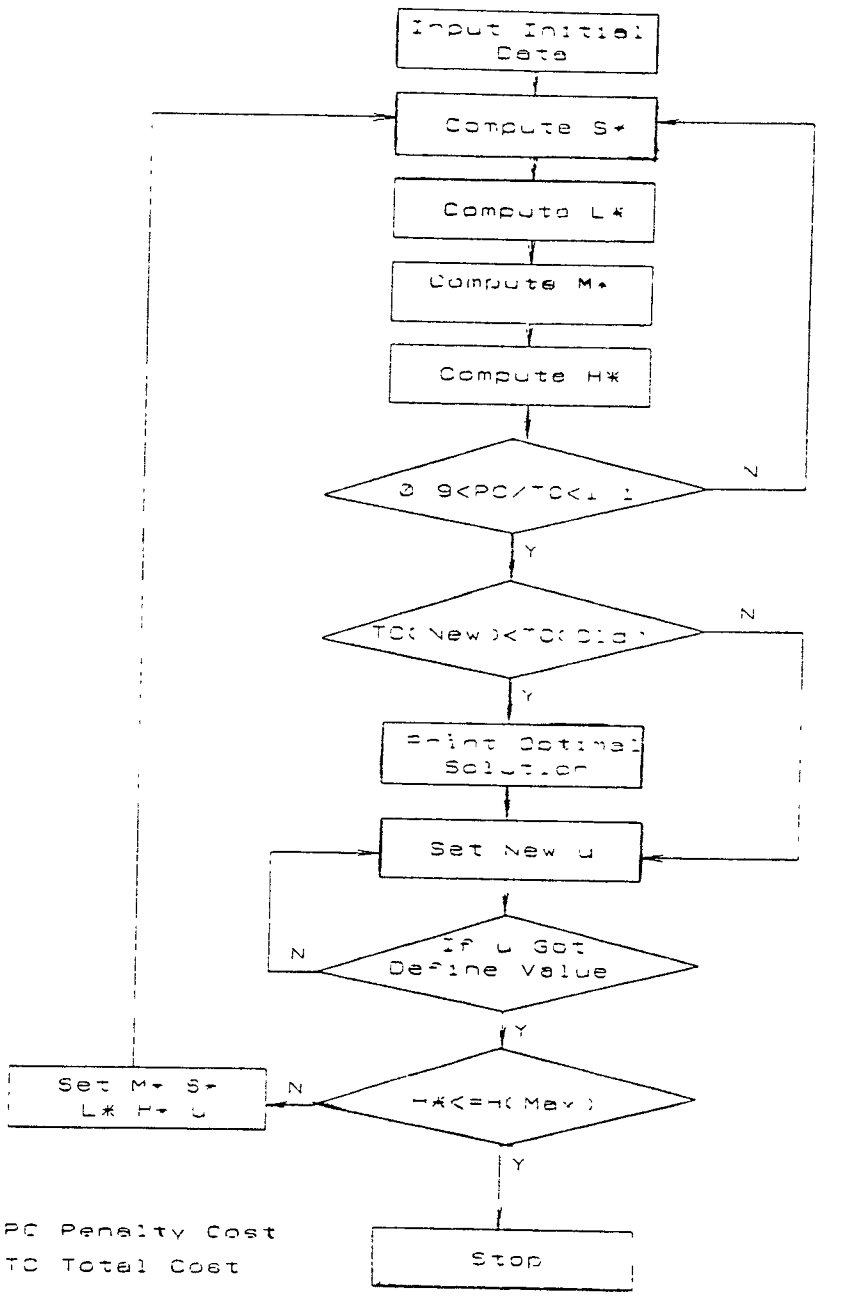

Figure 4.1 Optimization Algorithm

The partial derivatives of the penalty function with respect to L,H,M,S, are shown below:

$$dQ/dL=2CY(1/v+d/S)/HM+2u1\{EMPH[a1+a2(kH+(M+S)/4G)+a3(L/2v)$$
$$+a4f+a5(L/2)]-cap\}*EMPH(a3/2v+a5/2)-4u3[2(E-L)/2+M-4W]; \qquad (4.2)$$

$$dQ/dH=-2CLY(1/v+d/S)/H^2M+2u1\{EMHP[a1+a2(kH+(M+S)/4G)$$
$$+a3(L/2v)+a4f+a5(L/2)]-cap\}*EMP[a1+a2(2kH+(M+S)/4G)$$
$$+a3(L/2v)+a4f+a5(L/2)] \quad ; \qquad (4.3)$$

$$dQ/dM=-2CLY(1/v+d/S)/HM^2+2u1\{EMHP[a1+a2(kH+(M+S)/4G)$$
$$+a3(l/2v)+a4f+a5(L/2)]-cap\}*EHP[a1+a2(kH+(2M+S)/4G)+a3(L/2v)$$
$$+a4f+a5(L/2)]+2u2[(M+S)-4W]+2u3[2(E-L)+M-4W] \quad ; \qquad (4.4)$$

$$dQ/dS=-2CLYd/S^2HM+2u\{EMPH[a1+a2(kH+(M+S)/4G)+a3(L/2v)$$
$$+a4f+a5(L/2)]-cap\}*EMPH(a2/4G)+2u2[(M+S)-4W] \; ; \qquad (4.5)$$

The optimal expression for the line length is obtained by setting the partial derivative of the cost function with respect to L (Eq. 4.2) equal to zero:

$$L^*=\{2CY(1/v+d/S)/HM+2u1[EMPH(a1+a2(kH+(M+S)/4G)+a4f)-$$
$$cap]*EMPH(a3/2v+a5/2)-4u3(E+M-4W)\}/[-2u1(EMPH(a3/2v$$
$$+a3/2))^2-8u3]. \qquad (4.6)$$

Given the initial conditions for the optimal M^*, H^*, S^*, Eq.(4.6) is used within the algorithm to compute the optimal route length. Once the optimal route length is obtained, it is used in Eqs(4.3-4.5) to compute the optimal values of M, H, S, using the search methods to obtain numerical solutions.

4.1.1 Number of Stops and Stop Location

In order to determine the stop location, the corridor was partitioned into a finite number of small areas and scaned from its end toward the CBD. At any point along the route at X distance away from the CBD, and for a small increment x (e.g., 0.1 mile) the number of people in the increment computed as well as the cumulative mile) the number of people in the increment computed as well as the cumulative number of people (from the end of the corridor to X) aboard the vehicle. The trade-off used was between the operator and passenger delay cost of stopping the vehicle at the stop, and the access cost along the route to compute the increments of stop locations within the increments. The components of the total cost function that actually affect the number of stops and their location are the operator delay cost, user access cost (only longitudinal Component), and user in-vehicle delay cost. The partial cost function at any point X along the service area is:

$$TC(Si) = \frac{2cdY}{HM} \int_{x-dx}^{x} \frac{1}{Si}\, dx + \frac{VaY}{4G} \int_{x-dx}^{x} SiPdx + VivddxY \int_{x-dx}^{x} \frac{1}{Si}\, Pdx \qquad (4.7);$$

The optimal stop spacing is:

$$Si^{*} = \left[\frac{4Gd(c+VivHP(E-X)M}{VaPHM} \right]^{1/2} \qquad (4.8);$$

Eq.(4.8) is used to compute the stop increment in an area i. i.e., Ni*

$$N = \sum_{i=1}^{E/x} Ni, \quad \text{where } Ni = x/Si \qquad (4.9)$$

The number of stops is determined for each increment. For small increments the number is considerably less than 1.0. The stops are summed up for successive increments. When an integer number of stops is reached a "true" stop is established.

Delay is considered in the objective function. The time lost per stop is a function of demand.

$$\text{delay} = a + \text{demand}^{*}\ x(\text{passenger boarding time})^{*}H; \qquad (4.10)$$

that is:

$$d = 10 + P[a1 + a2[kH + (M+S)/4G] + a3(L/2v) + a4f + a5(L/2)]^{*}2^{*}H. \qquad (4.11)$$

The passenger boarding time here is set to be 2 seconds. The parameter "a" is a constant which represents the time spent during deceleration and acceleration. It is set to be 10 seconds. Now that optimal number of stops (or stop spacing) and route length are obtained they are input into Eqs.4.3 and 4.4 to colculate optimal M and H.

As an example the approach outlined above is used to optimize transit service in an urban transportation corridor characterized by the following:

The length of a corridor E is 5 miles. The width of the corridor Y is 3 miles. The passenger density is 3.59 passengers/mile^2 per minute per direction. The vehicle operating speed is assumed to be 0.167 mile/min., and the bus fare is assumed to be 125 cents. The value of operator cost is 50 cents/min. For access speed, normal human walk of 0.05 mile/min was assumed. The maximum acceptable access distance W is 1 mile. The capacity of the transit vehicle was assumed to be 45 seats.

Table 4.1 shows the optimal route length, spacing, headway, stop spacing and minimized operator cost obtained after run time of 45 minutes. Table 4.1 shows that the cost is minimized when the route length is 3.78 miles, route spacing 1.78 miles with 7 stations along the route and vehicle operating headway of 22 minutes. The optimal stop location and spacing pattern is shown in Figure 4.2.

4.2 Case 2: Maximizing Operator Profit

In this case, we will determine a bus transit network service area coverage, namely optimal route length, headway, route spacing and number of stops and their location (i.e., stop spacing pattern) that maximize operator profit. The operator's profit is defined as the difference between the total fare box revenue and the cost of operating service. The fare box revenue equals the total number of passengers multiplied by the fare each passenger pays for the service. The operating cost is expressed as the total number of hours of operation multiplied by the cost per vehicle hour. In addition to the line cost, the operating cost includes vehicle delay costs due to the stopping of vehicles to pick up and drop off passengers.

Table 4.1 Results of Case 1

	I	II	III	IV	V
Route Length(mile)	3.93	3.85	3.78	3.81	3.81
Route Spacing(mile)	1.78	1.78	1.78	1.74	1.74
Route Stops	6.58	6.42	6.01	5.84	5.73
Transit Headway(min)	22.2	22.0	21.8	21.7	21.8
Operating Cost(cent/min	193.6	191.7	189.9	196.6	195.8
Transit Demand(trips/hr	203.3	205.7	207.4	214.3	213.2
Profit(cent/min)	229.9	235.8	242.4	249.2	248.3
User In-vehicle Cost	263.6	263.0	264.3	273.9	272.7
User Access Cost	744.5	751.2	758.7	771.2	767.3
User Waiting Cost	565.9	565.7	565.5	581.6	581.5

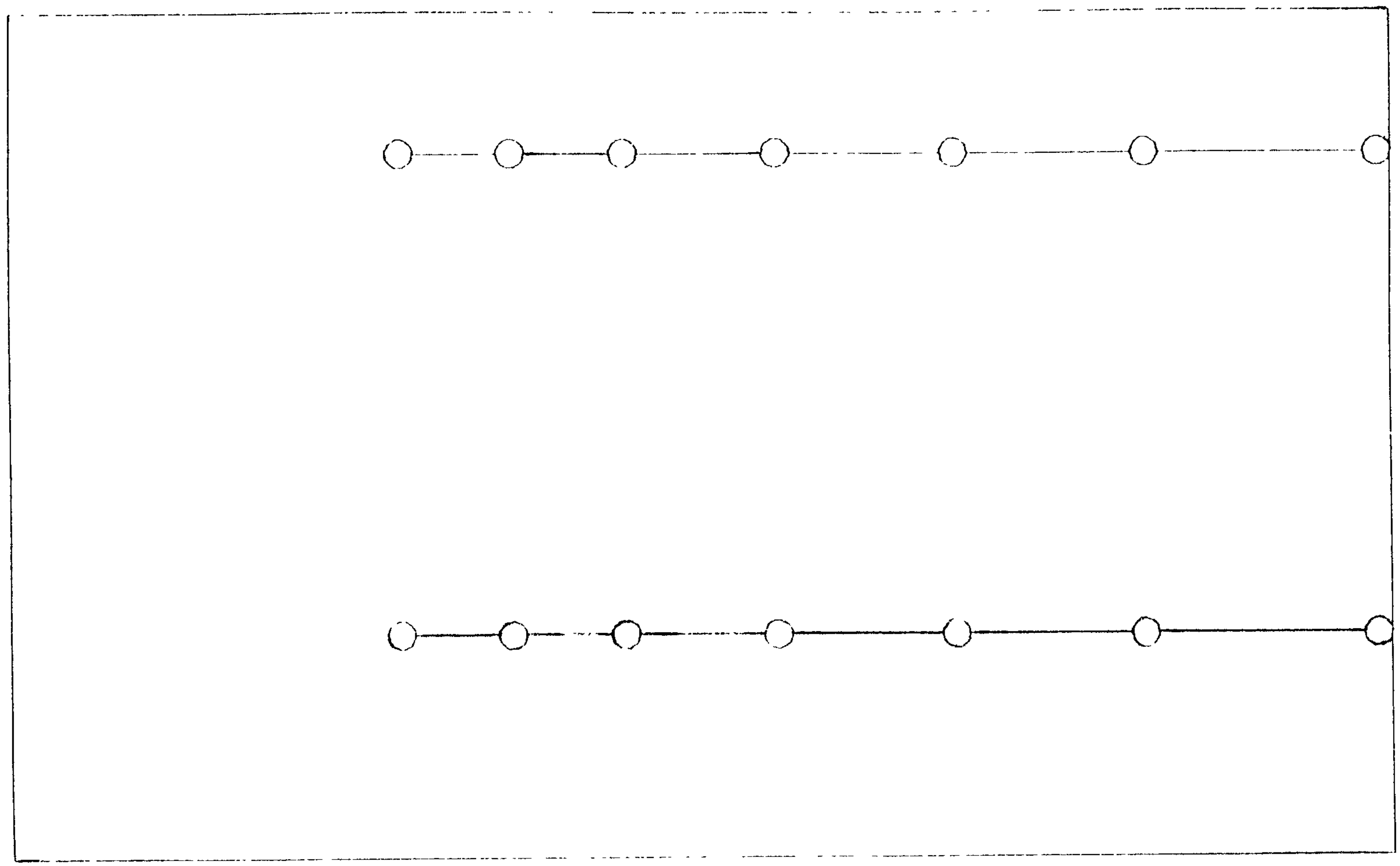

Figure 4.2 Transit Route Configuration And Stop Location for Case 1

From the service design point of view, there are several constraints imposed on the operation. These constraints ensure that a certain minimum level of service is provided to the public. They ensure that the service design parameters such as route length, stop spacing and route spacing are selected in such a way that there is a minimum access distance for the average passenger. In addition, the analysis considers two situations. First, it is assumed that there is no constraint imposed on vehicle size. This means that the capacity on routes might not meet demand, (i.e., there might not be a sufficient capacity to accommodate all passengers). In the second case, a vehicle size constraint is imposed. Hence this transit service design problem can be structured as an optimization problem in the following form:

Maximize Operator Profit

subject to :

maximum access distance to the route

The constraint represents a convenience to passengers accessing the route from their origins to the stations along the route.

$$\text{Max} : [a1+a2(kH+(M+S)/4G)+a3(L/2v)+a4f+a5(L/2)]EYP*f$$
$$-2CLY(1/V+d/S)/HM \;;$$

$$\text{s.t.} \quad \frac{E-L}{2} + \frac{M}{4} <= W \;;$$

$$\frac{M+S}{4} <= W \;;$$

$$S,M,L,H > 0 \;; \qquad (4.13)$$

This problem does not consider the vehicle size constraint. The problem with the vehicle size constraints will be discussed later.

The demand used here is the same function as in the previous case:

$[a1+a2(kH+(M+S)/4G)+a3(L/2v)+a4f+a5(L/2)]EMPH.$

The problem is a constrained non-liner programming problem and is solved using the penalty function method (explained in detail in the Appendix). The constraints are priced out of the constraint set via a penalty parameter and put into the objective function. The whole objective function is then transformed to:

$$Q = [a1+a2(kH+(M+S)/4G)+a3(L/2v)+a4f+a5(L/2)]EYPf$$
$$-2CLY(1/v+d/S)/HM+u1[4W-2(E-L)+M]^2+u2[4W-(M+S)]^2 \qquad (4.14)$$

The partial derivatives of the penalty objective function with respect to L,H,M,S are as follows:

$$dQ/dL=(a3/2v+a5/2)EYPf-2CY/HM(1/v+d/S)+4u1[4W-2(E-L)-M] \quad (4.15)$$

$$dQ/dS=a2EMPHf/4G+(2CLYd/HMS^2)-2u2[4W-(M+S)] \quad (4.16);$$

$$dQ/dM=EYPFa2/4G+2CLY(1/v+d/S)/HM^2+2u1[4W-2(E-L)-M] + u2[4W-(M+S)] \quad (4.17);$$

$$dQ/dH=EYPfa2+2CLY(1/v+d/S)/H^2M \quad (4.18);$$

The optimal value of L^* is obtained by setting the partial derivative of the penalty function dQ/dL equal to zero, and solving the equation with respect to L.

$$L^*=-[(a3/2v+a5/2)EYPf-2CY(1/v+d/S)/HM+4u1(4W-2E-M)]/(8u1) \quad (4.19);$$

The algorithm used in Case 1 is used to determine the optimal route length L^*, route density M^*, headway H^*, and the optimal average stop spacing S^*. A "scanning" technique is used to derive the optimal number of stops and their actual location along the route

$$N^* = \sum_{i=1}^{E/dx} Ni, \text{ where } Ni = dx/si \quad (4.20);$$

The input data are the same as for Case 1. The results of the optimization procedure are shown in Table 4.2. The optimal solution that maximizes the operator profit has route length of 4.31 miles, route density of 1.36 miles per route, headway of 16.6 minutes and 6 stations along the route. The transit route configuration and station spacing pattern are shown in Figure 4.3.

As mentioned earlier the problem was solved with no vehicle size constraint. Using the data from Table 4.2, the demand (expressed as number of passengers) per vehicle can be compute as followes:

$$369.1/1.36/(60/19.2) = 48 \text{ passengers.}$$

Since we assumed that the bus can only accommodate 45 passengers, it is obvious that the demand exceeds capacity.

Next, the problem that includes the vehicle size constraint is considered. To accomplish this, the following constraint is added to the previous problem statement:

$$[a1+a2(kH+(M+S)/4G)+a3(L/2v)+a4f+a5(l/2)]EMPH-cap<=0 ;$$

The problem is then written as:

Table 4.2 Results of Case 2 (Without Vehicle Size Constraint)

	I	II	III	IV	V
Route Length(Mile)	4.58	4.64	4.65	4.31	4.27
Route Space(Mile/Route)	1.63	1.62	1.61	1.36	1.30
Route Stops	6.2	6.2	6.4	5.7	5.8
Transit Headway(min)	26.3	26.2	26.3	16.6	17.9
Transit Demand(trip/Hr)	233.8	238.8	238.8	369.9	367.8
Operating Cost(Cent/Min	229.9	234.1	235.3	404.3	399.7
Profit(cents/Min)	260.3	262.2	262.4	366.1	365.6
User In-vehicle Cost	347.3	353.0	354.0	534.1	365.7
User Access Cost	772.2	780.9	785.7	767.5	745.3
User Waiting Cost	634.2	639.9	638.2	864.6	762.5

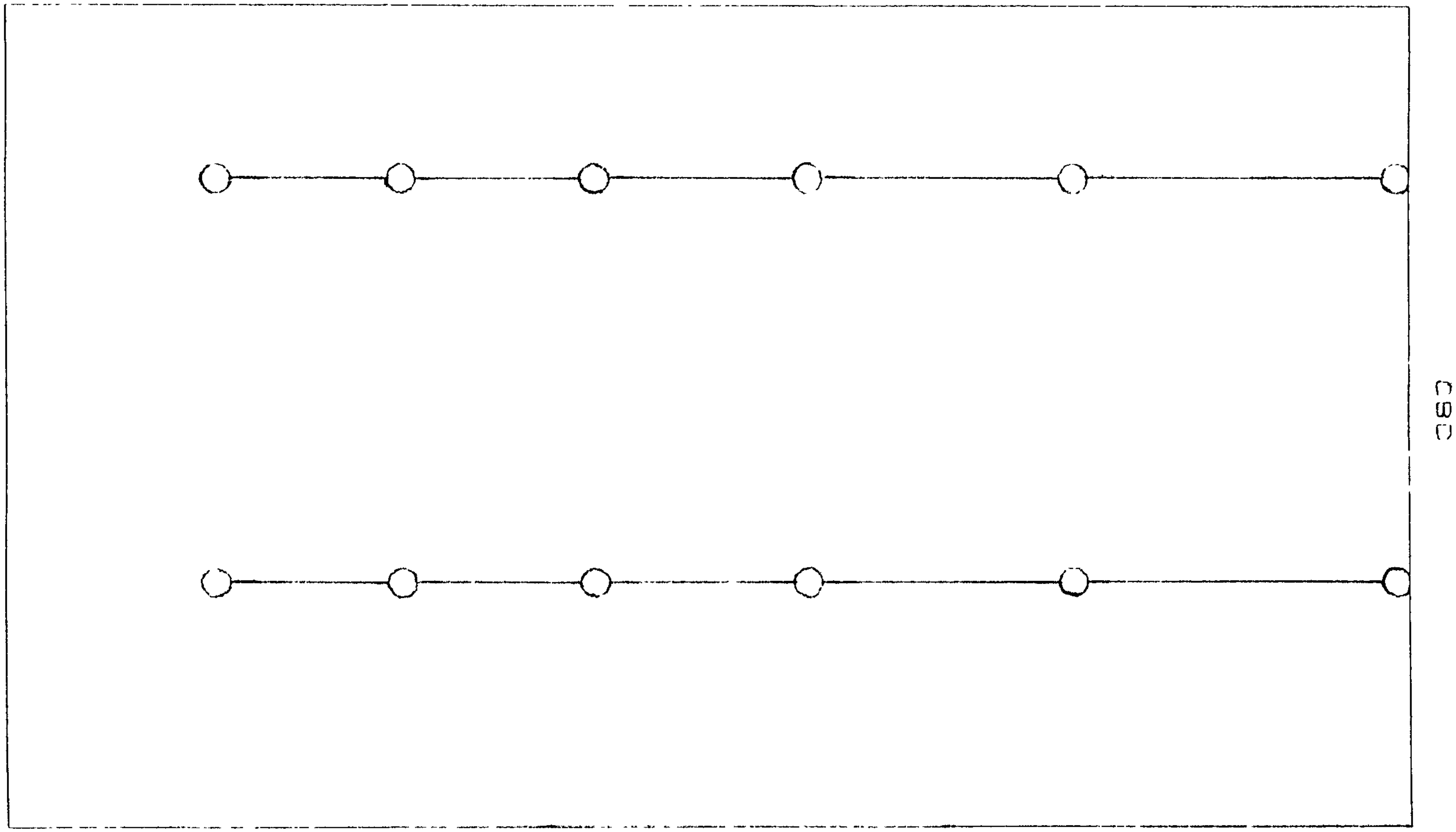

Figure 4.3 Transit Route Configuration And Stop Location for Case 2
(Without The Vehicle Size Constraint)

$$\text{Max}: [a1+a2(kH+(m+s)/4G)+a3(l/2v)+a4f+a5(l/2)]EYP^{*}f - 2CLY(1/V+d/S)/HM \;;$$

$$\text{s.t.} \quad \frac{E-L}{2} + \frac{M}{4} <= W \;;$$

$$\frac{M+S}{4} <= W \;;$$

$$[a1+a2(kH+(M+S)/4G)+a3(L/2v)+a4f+a5(l/2)]EMPH-cap <= 0$$

$$S,M,H,L > 0 \qquad (4.21);$$

The problem is solved using the penalty function method, with the following penalty function:

$$\text{Max } Q = [a1+a2(kH+(M+S)/4G)+a3(L/2v)+a4f+a5(L/2)]EYPf -2CLY(1/v+d/S)/HM+u1\{cap-[a1+a2(kH+(M+S)/4G)+a3(L/2v) +a4f+a5(L/2)]EMPH\}^2+u2(4W-2(E-L)-M)^2+u3[4W-(M+S)]^2;$$

$$(4.22);$$

The partial derivatives of the objective function with respect to L,H,M,S are:

$$dQ/dL=(a3/2v+a5/2)EYPf-2CY/HM(1/v+d/S)-2u1\{cap-[a1+a2(kH+ (M+S)/4G)+a3(L/2v)+a4f+a5(L/2)]EMPH\}[EMPH(a3/2v+a5/2)] +4u2(4W-2(E-L)-M) \qquad (4.23);$$

$$dQ/dS=a2EYPf/4G+(2CLYd/HMS^2)-2u1\{cap-[a1+a2(kH+(M+S)/4G) +a3(L/2v)+a4f+a5(L/2)]EMHP\}EMPHa2/4G-2u3[4W-(M+S)]$$

(4.24);

$$dQ/dM=a2EYPf/4G+2CLYd(1/V+d/S)/M^2H-2u1\{cap-[a1+a2(kH +(M+S)/4G)+a3(L/2v)+a4f+a5(L/2)]EMPH\}EPH[a1+a2(kH +(2M+S)/4G)+a4f+a3(L/2V)+a5L/2]-2u2(4W-2(E-L)-M) -2u3[W-(M+S)] \qquad (4.25);$$

$$dQ/dH=a2kEYPf+2CLY(1/v+d/S)/MH^2-2u1\{cap-[a1+a2(kH+(M+S)) +a3(L/2v)+a4f+a5(L/2)]EMPH\}[a1+a2(2kH+(M+S)/4G +a3(L/2V)+a5L/2+a4f]EMP \qquad (4.26);$$

The optimal value of the L^* is obtained by setting dQ/dL equal to zero.

$$L^*=\{(a3/2v+a5/2)EYPf-2CY(1/v+d/S)/HM-2u1[cap-(a1+a2(kH+(M+S)/4G) +a4f)EMPH][EMPH(a3/2v+a5/2)]+4u2(4W-2E-M)\}/\{-2u1(a3/2v$$

$+a5/2)^2(EMPH)^2-8u2\}$ (4.27);

The optimal values of decision variables that maximize operator profit for the case with the vehicle size constraint are shown in Table 4.3. The optimal solution has a route length of 4.15 miles, 6 stops, operating headway of 16.4 minutes, and spacing between routes of 1.30 miles. The resulting total demand is 351 trips per hour and the operating cost is \$3.38 per minute. The demand (number of passengers) per headway is 41.6 passengers which is less than the vehicle capacity. This implies that vehicles operate at maximum allowable capacity for the induced demand. The profit earned by the operator of the transit system is \$3.94 per minute. The stop location and spacing pattern is shown in Figure 4.4.

4.3 Case 3: Maximizing User Benefit

The objective of this case is to maximize the net user's benefit subject to the acceptable level of services and a deficit constraint in the objective function. The concept of consumer surplus (net willingness-to-pay) is used to measure net user benefit. The consumer surplus represents the difference between what passengers are willing to pay and what are actually paying. The expression for consumers surplus or net benefit is given in Kocur and Hendrickson (1982):

$$-YPE[a1+a2(kH+(M+S)/4G)+(a3/2V+a5/2)L+a4f]^2/2a4 \qquad (4.28);$$

This expression is derived by summing the total amount users would be willing to pay (including fare and the value of travel time) for a service with given L, M, H and S less the total cost they actually pay. Equation(4.28) is derived by multiplying one-half(average) of the number of passengers in the area:

$$0.5YPE[a1+a2(kH+(M+S)/4G)+(a3/2V+a5/2)L+a4f].$$

times the utility of the transit system relative to auto, and dividing it by the coefficient of the transit fare(a4) to convert it into money units:

$$-(1/a4)[a1+a2(kH+(M+S)/4G)+(a3/2V+a5/2)L+a4f].$$

So, the problem of maximizing user benefit could be stated as:

$$\text{Max} \quad -YPE[a1+a2(kH+(M+S)/4G)+(a3/2V+a5/2)L+a4f]^2/2a4$$

Table 4.3 Results of Case 2 (With Vehicle Size Constraint)

	I	II	III	IV	V
Route Length(Mile)	4.10	4.15	4.09	3.96	4.10
Route Space(Mile/Route)	1.66	1.32	1.65	1.30	1.65
Route Stops	6.7	5.7	7.5	6.3	7.5
Transit headway(Minute)	21.8	22.0	21.8	16.4	21.8
Transit Demand(trip/Hr)	223.3	277.3	224.5	351.8	224.0
Transit Operating Cost	220.5	280.5	223.8	338.8	221.2
Profit(cents/Min)	246.4	297.7	247.5	394.1	247.4
User In-vehicle Cost	290.8	366.8	292.3	462.2	292.5
User Access Cost	788.2	812.9	789.9	725.2	790.0
User Waiting Cost	608.7	763.9	612.1	950.6	612.3

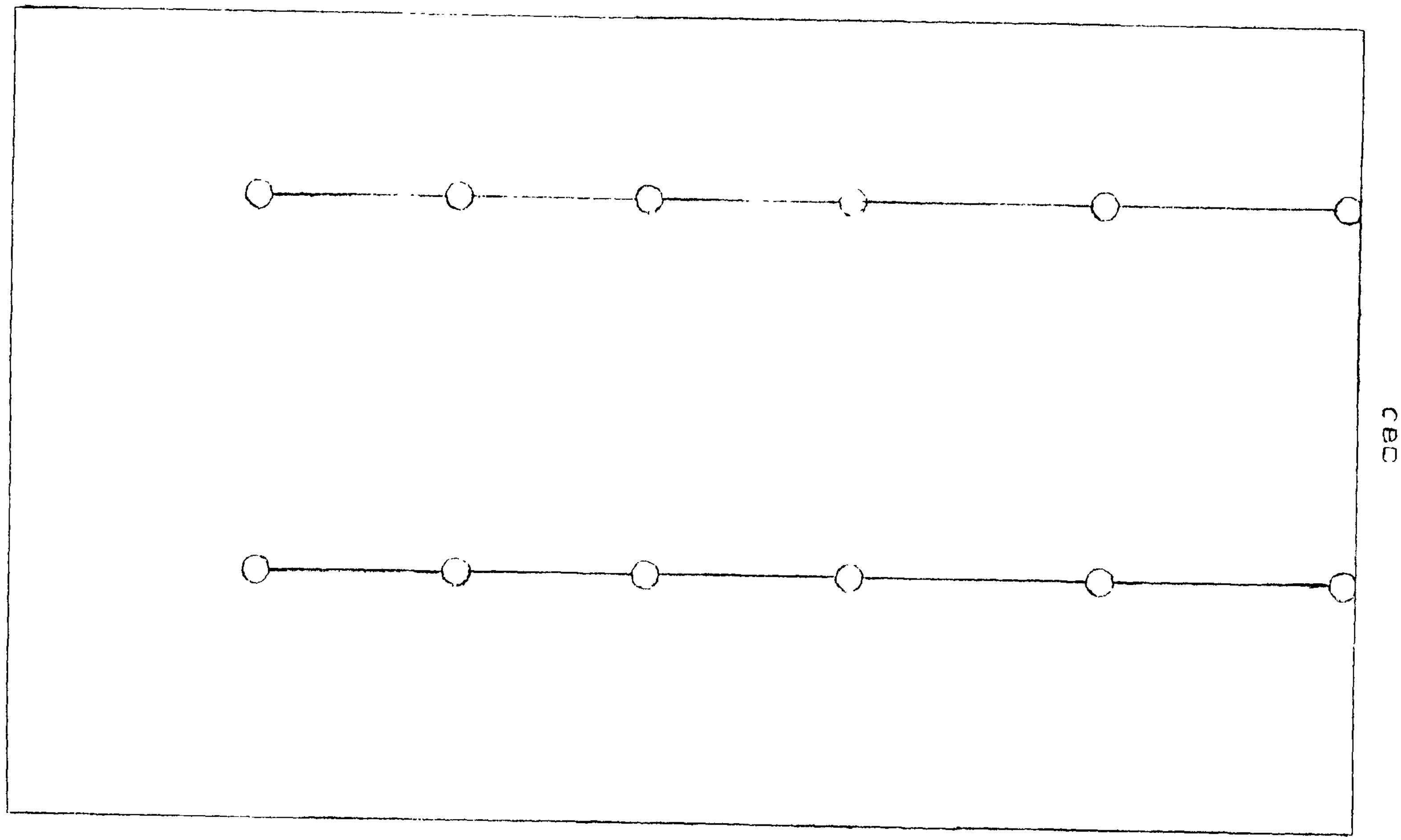

Figure 4.4 Transit Route Configuration And Stop Location for Case 2
(With The Vehicle Size Constraint)

$$\frac{2CYL(1/V+d/S)/MH}{YPEf[a1+a2(kH+(M+S)/4G)+(a3/2V+a5/2)L+a4f]} < 2$$

$$\frac{E-L}{2} + \frac{M}{4} <= W\ ;$$

$$\frac{M+S}{4} <= W\ ;$$

$$S,M,L,H > 0 \qquad (4.29);$$

The deficit constraint:

$$\frac{2CYL(1/V+d/S)/MH}{YPEf[a1+a2(kH+(M+S)/4G)+(a3/2V+a5/2)L+a4f]} < 2$$

implies that as a matter of the policy the operating cost can not exceed one half of the total revenue In order to compare the results of this case with other cases, the rest of the constraints are the same. The problem is also solved by the penalty function method. When the constraints are priced out of the constraint set and introduced into the objective function the following formulation is obtained:

$$Q= -YPE[a1+a2(kH+(M+S)/4G)+(a3/2V+a5/2)L+a4f]^2/2a4+u1\{cap-EMPH[a1+a2(kH+(M+S)/4G)+(a3/2V+a5/2)L+a4f]\}^2+u2\{2*[a1+a2(kH+(M+S)/4G)+(a3/2V+a5/2)L+a4f]*EYPf-2CLY/HM(1/V+d/S)\}^2+u3[4W-2(E-L)-M]^2+u4[4W-(M+S)]^2\ . \qquad (4.30);$$

The partial derivatives with respect to L, M, H, S are:

$$dQ/dL=-YPE[a1+a2(kH+(M+S)/4G)+(a3/2V+a5/2)L+a4f](a3/2V+a5/2)/a4 +2u1\{cap-EMPH[a1+a2(kH+(M+S)/4G)+(a3/2V+a5/2)L+a4f]\} [-EMPH(a3/2V+a5/2)]+2u2\{2[a1+a2(kH+(M+S)/4G)+(a3/2V +a5/2)L+a4f]*EYPF-2CLY/HM(1/V+d/S)\}[2(a3/2V+a5/2)EYPf -2CY(1/V+d/S)/HM]+4u3(4W-2(E-L)-M) \qquad (4.31);$$

$$dQ/dS=-YPE[a1+a2(kH+(M+S)/4G)+(a3/2V+a5/2)L+a4f](a2/4G)/a4 +2u1\{cap-EMPH[a1+a2(kH+(M+S)/4G)+(a3/2V+a5/2)L+a4f]\}$$

$$[-EMPH(a2/4G)]+2u2\{2[a1+a2(kH+(M+S)/4G)+(a3/2V+a5/2)L+a4f]*EYPF-2CLY/HM(1/V+d/S)\}[(2a2EYPf/4G)-2CLYd/HMS^2]-2u4[W-(M+S)] \quad (4.32):$$

$$dQ/dH=-YPE[a1+a2(kH+(M+S)/4G)+(a3/2V+a5/2)L+a4f](a2k)/a4 -2u1\{cap-EMPH[a1+a2(2kH+(M+S)/4G)+(a3/2V+a5/2)L+a4f]\} [EMP(a1+a2(kH+(2M+S)/4G)+(a3/2V+a5/2)L+a4f)]+2u2\{2 *[a1+a2(kH+(M+S)/4G)+(a3/2V+a5/2)L+a4f]*EYPF-2CLY/HM (1/V+d/S)\}[2a2kEYPf+2CLY/H^2M(1/V+d/S)] \quad (4.33).$$

$$dQ/dM=-YPE[a1+a2(kH+(M+S)/4G)+(a3/2V+a5/2)L+a4f](a2k)/a4 -2u1\{cap-EMPH[a1+a2(kH+(2M+S)/4G)+(a3/2V+a5/2)L+a4f]\} [EHP(a1+a2(kH+(M+S)/4G)+(a3/2V+a5/2)L+a4f)]+2u2\{2* [a1+a2(kH+(M+S)/4G)+(a3/2V+a5/2)L+a4f]*EYPF-2CLY/HM (1/V+d/S)\}[2a2EYPf/4G-2CLY/M^2H(1/V+d/S)]-2u3[4W-2(E-L)-M]-2u4[4W-(M+S)\} . \quad (4.34):$$

Setting dQ/dL, dQ/dS, dQ/dH and dQ/dM equal to zero and using the same algorithm and input data used for the previous cases, an optimal solution was obtained which is presented in Table 4.4. The transit system that maximizes user benefits has a route length of 4.27 miles, route spacing of 1.26 miles 15.9 minute headways and 5 stops along the route. The optimal stop spacing pattern is shown on Figure 4.5.

4.4 Case 4: Minimizing Total operator And User Cost

The objective function of this case is minimizing the sum of user and operator cost. Supplier cost represents the cost of resources used by the supplier to provide the transit service. The formulation of the supplier cost is the same as the one encountered in Case 1. That is:

$$SC= \quad 2LCY(1/V+d/S)/HM.$$

The users' cost is given as:

Table 4.4 Results of Case 3

	I	II	III	IV	V
Route Length(mile)	4.32	4.21	4.13	4.11	4.27
Route Spacing(mile)	1.36	1.40	1.86	1.35	1.26
Route Stops	4.1	5.5	4.9	4.4	5.2
Transit Headway(min)	18.6	17.1	13.3	17.0	15.9
Transit Demand(trip/hr)	254.6	245	267.2	260.7	367.3
User's Benefit(c/min)	198.6	210.2	189.7	218.8	248.3
Transit Operating Cost	341.1	351.3	334.3	358.6	382.2
Profit(cents/min)	189.7	159.1	222.4	184.5	383.2
User's In-vehicle cost	447.6	456.9	432.2	464.7	448.5
User's Access Cost	879.6	921.2	938.7	918.8	940.0
User's Waiting Cost	763.6	722.3	533.3	732.6	730.1

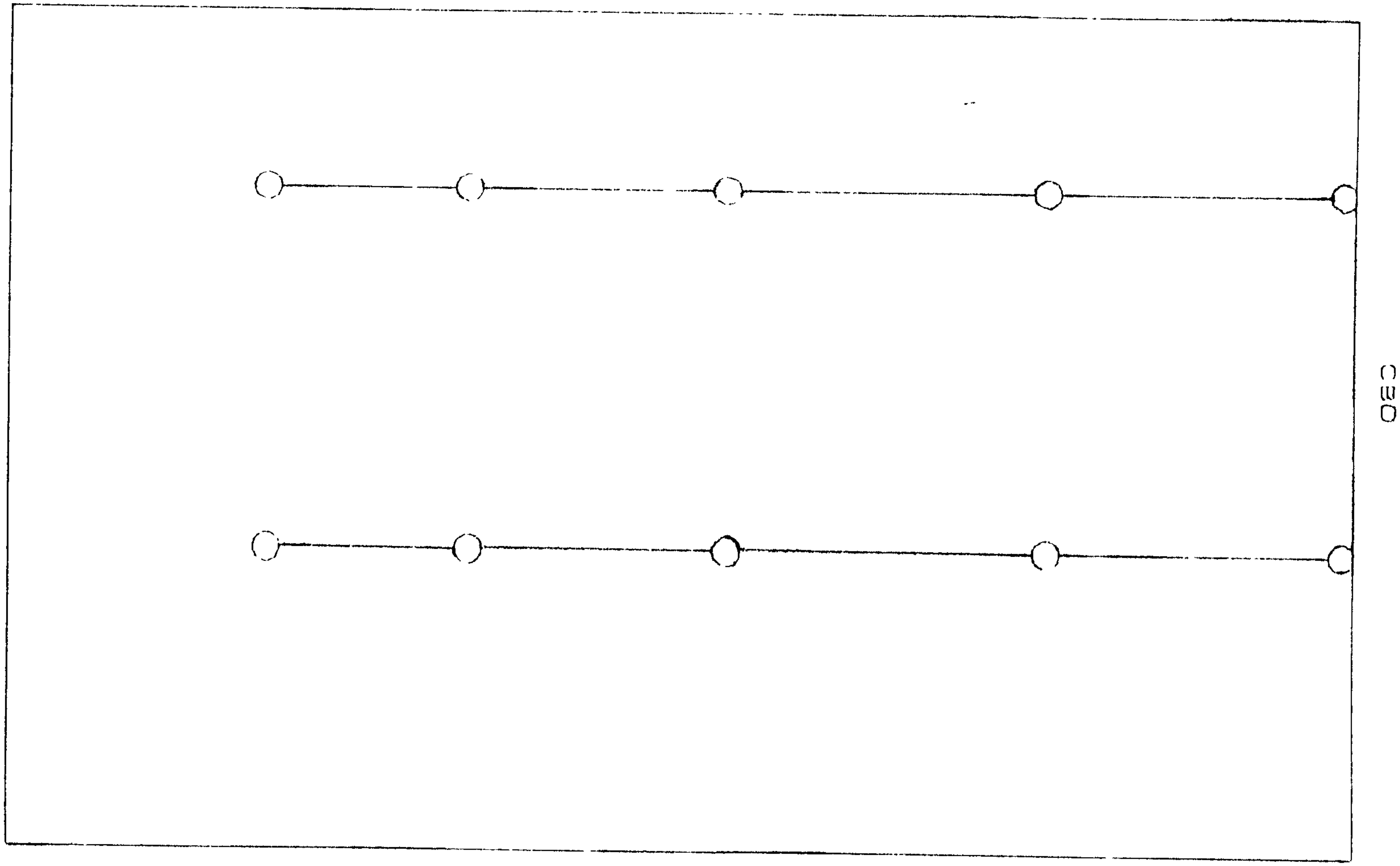

Figure 4.5 Transit Route Configuration And Stop Location for Case 3

$$UC = Ca + Cw + Civ \qquad (4.35);$$

where:

$$Ca = Ca1 + Ca2 + Ca3$$

$$Ca1 = P(a1+a2(kH+(M+S)/4G)+(a3/2V+a5/2)L+a4f)YEVaM/4G$$

$$Ca2 = P(a1+a2(kH+(M+S)/4G)+(a3/2V+a5/2)L+a4f)YLVaS/4G$$

$$Ca3 = P(a1+a2(kH+(M+S)/4G)+(a3/2V+a5/2)L+a4f)YVa(E\text{-}L)^2/2G \qquad (4.36);$$

$$Cw = Vw(H/2)2P(a1+a2(kH+(M+S)/4G)+(a3/2V+a5/2)L+a4f)EY \qquad (4.37);$$

$$Civ = P(a1+a2(kH+(M+S)/4G)+(a3/2V+a5/2)L+a4f)LYViv(L/2V+Ld/2S)$$
$$+2P(a1+a2(kH+(M+S)/4G)+(a3/2V+a5/2)L+a4f)(E\text{-}L)YViv(L/V+Ld/S).$$
$$(4.38);$$

The total cost is operator and user cost is:

$$TC = SC + UC$$

where: $UC = Ca + Cw + Civ$.

The problem of minimizing the total system cost can be written in the following form:

$$\text{Min}: 2LCY(1/V+d/S)/HM+(a1+a2(kH+(M+S)/4G)+(a3/2V+a5/2)L$$
$$+a4f)YVa(E\text{-}L)^2/2G+P(a1+a2(kH+(M+S)/4G)+(a3/2V+a5/2)L$$
$$+a4f)YEVaM/4G+P(a1+a2(kH+(M+S)/4G)+(a3/2V+a5/2)L$$
$$+a4f)YLVaS/4G+P(a1+a2(kH+(M+S)/4G)+(a3/2V+a5/2)L$$
$$+a4f)YVa(E\text{-}L)^2/2G+Vw(H/2)2P(a1+a2(kH+(M+S)/4G)+(a3/2V$$
$$+a4/2)L+a5f)EY+2P(a1+a2(kH+(M+S)/4G)+(a3/2V+a5/2)L$$
$$+a4f)LYViv(L/2V+Ld/2S)+2P(a1+a2(kH+(M+S)/4G)+(a3/2V$$
$$+a5/2)L+a4f)(E\text{-}L)YViv(L/V+dL/S).$$

subject to:

$$(a1+a2(kH+(M+S)/4G)+(a3/2G+a5/2)L+a4f)EMPH < cap$$

$$\frac{E - L}{2} + \frac{M}{4} <= W ;$$

$$\frac{M + S}{4} <= W ;$$

$$S,M,L,H > 0 \qquad (4.39):$$

The penalty function is written in the following form.

$$Q=2LCY(1/V+d/S)/HM+(a1+a2(kH+(M+S)/4G)+(a3/2V+a5/2)L$$
$$+a4f)YVa(E-L)^2/2G+P(a1+a2(kH+(M+S)/4G)+(a3/2V+a5/2)L$$
$$+a4f)YEVaM/4G+P(a1+a2(kH+(M+S)/4G)+(a3/2V+a5/2)L$$
$$+a4f)YLVaS/4G+P(a1+a2(kH+(M+S)/4G)+(a3/2V+a5/2)L$$
$$+a4f)YVa(E-L)^2/2G+Vw(H/2)P(a1+a2(kH+(M+S)/4G)+(a3/2V$$
$$+a4/2)L+a4f)EY+P(a1+a2(kH+(M+S)/4G)+(a3/2V+a5/2)L$$
$$+a4f)LYViv(L/2V+Ld/2S)+P(a1+a2(kH+(M+S)/4G)+(a3/2V$$
$$+a5/2)L+a4f)(E-L)YViv(L/V+dL/S)+u1(a1+a2(kH+(M+S)/4G)$$
$$+(a3/2V+a5/2)L+a4f-cap)^2EMPH+u2(2(E-L)+M-4W)^2$$
$$+u3(M+S-4W)^2 \qquad (4.40):$$

The partial derivatives with respect to L, M, H, S are:

$$dQ/dL=2CY(1/V+d/S)/HM+PVivY\{(a3/2V+a5/2)[(L^2/2V+L^2d/2S)$$
$$+(E-L)(L/V+Ld/S)]+(a1+a2(kH+(M+S)/4G)+(a3/2V+a5/2)L$$
$$+a4f)[(L/V+Ld/S)-L(1/V+d/S)^2(E-L)]\}+VwH/2PEY(a3/2V$$
$$+a5/2)+PYVa\{(a3/2V+a5/2)[EM/4G+LS/4G+(E-L)^2/2G)]$$
$$+(a1+a2(kH+(M+S)/4G)+(a3/2V+a5/2)L+a4f)[S/4G-(E-$$
$$L)/G]\}+2u1(a1+a2(kH+(M+S)/4G)+(a3/2V+a5/2)L+a4f)EMPH$$
$$-cap)EMPH(a3/2V+a5/2)-4u2(2(E-L)+M-4W) \qquad (4.41);$$

$$dQ/dS=-2CLYd/(S^2HM)+[PYVivL(L/2V+Ld/2S)+P(E-L)YViv(L/V$$
$$+Ld/S)]a2/4G-[PL^2YVivd/2S^2+P(E-L)YVivd/S^2](a1+a2(kH$$
$$+(M+S)/4G)+(a3/2V+a5/2)L+a4f)+VwH/2PEYa2/4G+PYVa$$
$$*[a2/4G(EM/4G+LS/4G+(E-L)^2/2G)+L/4G(a1+a2(kH+(M+S)/4G)$$
$$+(a3/2V+a5/2)L+a4f)]+2u1[(a1+a2(kH+(M+S)/4G)+(a3/2V$$

$$+a5/2)L+a4f)EMPH\text{-}cap]a2EMPH/4G+2u3[(M+S)\text{-}4W] \qquad (4\ 42):$$

$$dQ/dH=-2CLY(1/V+d/S)/(H^2M)+[PLYViv(L/2V+Ld/2S)+P(E\text{-}L)YViV$$
$$*(L/V+Ld/s)]a2k+VwPEY/2(a1+a2(2kH+(M+S)/4G)+(a3/2V$$
$$+a5/2)L+a4f)+PYVaka2[EM/4G+LS/4G+(E\text{-}L)^2/2G]+u1[(a1$$
$$+a2(kH+(M+S)/4G)+(a3/2V+a5/2)L+a4f)EMPH\text{-}cap](a1$$
$$+a2(2kH+(M+S)/4G)+(a3/2V+a5/2)L+a4f)EMP \qquad (4.43):$$

$$dQ/dM=-2CLY(1/V+d/S)/(M^2H)+PYVivL[(L/2V+Ld/2S)+(E\text{-}L)(1/V$$
$$+d/S)]a2/4G+VwH/2PEYa2/4G+PYVa[E/4G(a1+a2(kH$$
$$+(M+S)/4G)+(a3/2V+a5/2)L+a4f)+(EM/4G+LS/4G+(E\text{-}L)^2)$$
$$*a2/2G]+u1[(a1+a2(kH+(M+S)/4G)+(a3/2V+a5/2)L+a4f)$$
$$*EMPH\text{-}cap](a1+a2(kH+(2M+S)/4G)+(a3/2V+a5/2)L+a4f)$$
$$*EHP+2u2[2(E\text{-}L)+M\text{-}4W]+2u3(M+S\text{-}4W) \qquad (4.44).$$

Using the same algorithm as for the previous cases, the results shown in Table 4.5 was obtained. The transit service design that minimizes the total system cost has a route length of 4.84 miles, route spacing of 2.11 miles, 7 stops along the route and operating headway of 21.0 minutes. The route configuration, stop pattern location and spacing pattern is shown on Figure 4.6.

Table 4.5 Results of Case 4

	I	II	III	IV	V
Route Length(mile)	4.72	4.81	4.84	5.00	4.99
Route Spacing(mile)	1.67	1.67	2.11	1.86	1.87
Route Stops	6.49	6.95	6.39	5.73	5.92
Transit headway(minute)	23.9	21.0	21.0	23.9	23.8
Transit Demand(trip/hr)	185.1	183.1	181.5	157.5	150.0
User's Cost (cent/min)	1532.3	1531.5	1492.7	1516	1520
Transit Operating Cost	210.6	210.9	210.7	216.9	215.7
Total Cost(cents/min)	1742.9	1742.5	1703.8	1732.9	1735
Profit(cent/min)	175.0	171.8	167.5	111.2	96.8
User's In-vehicle cost	254.6	249.8	243.2	244.7	250.1
User's Access Cost	690.5	677.2	771.7	658.8	664.8
User's Waiting Cost	586.1	604.6	477.3	612.3	604.4

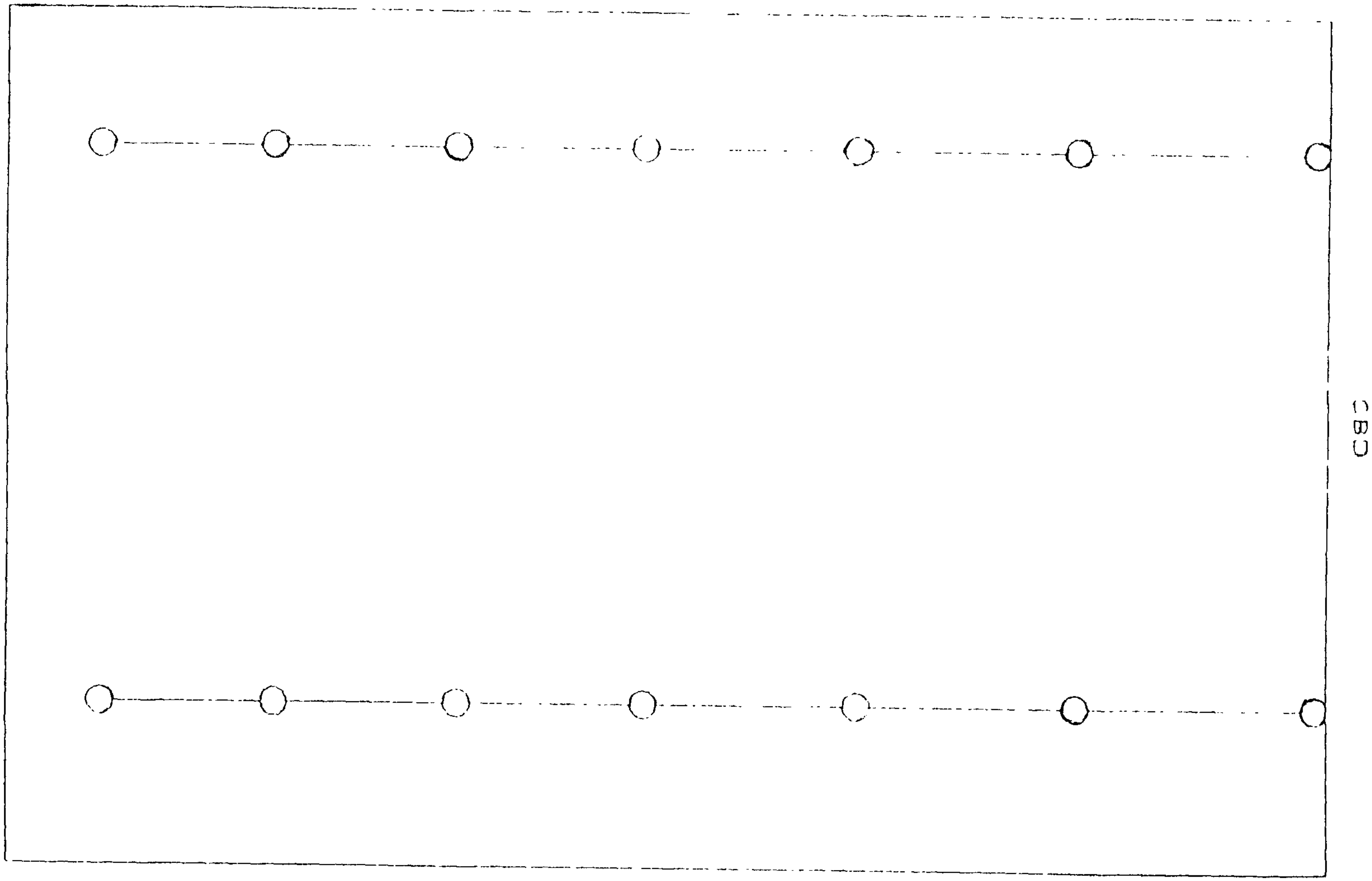

Figure 4.6 Transit Route Configuration And Stop Location for Case 4

CHAPTER 5

SENSITIVITY ANALYSIS

5.1 Introduction

Sensitivity analysis is performed to show the relations between design variables and various important exogenous parameters. Elasticities are used as measures of sensitivity. The elasticity of a variable Y with respect to a parameter X, Ex(Y), is defined as a percentage change in Y for a one percent change in X. For example, the elasticity of the route length (L) with respect to the corridor length (E) is then:

$$E_E(L) = \frac{E}{L}\,\frac{dL}{dE} \qquad (5.1)$$

When the absolute value $|Ex(Y)| < 1$, Y is said to be inelastic with respect to X. When $|Ex(Y)| > 1$, Y is said to be elastic with respect to X.

The graphical representation of elasticity is shown in Figure 5.1. The figure shows the arc elasticity of a transit line of length (L) with respect to changes in the corridor length. As corridor length changes for E0 to Ea, route length changes for L0 to La, the elasticity can be obtained as Equation 5.1. Sensitivity analysis is conducted for the four case studies, and the results are described in the following sections.

5.2 Analysis Approach

5.2.1 Case 1: Minimizing operator cost

Table 5.1 shows the elasticities of route length, headway, route spacing and number of stops with respect to ten parameters, namely corridor length, passenger density, transit speed, access speed, supplier cost, value of riding time, value of access time, vehicle capacity, value of waiting time and transit fare.

When the supplier cost or passenger density increase by 10%, the route length decreases by 4.2%. If the access speed increases by 10%, the route length will increase by 8.1%. The route length decreases by 7.3% if the bus fare is increased by 10%.

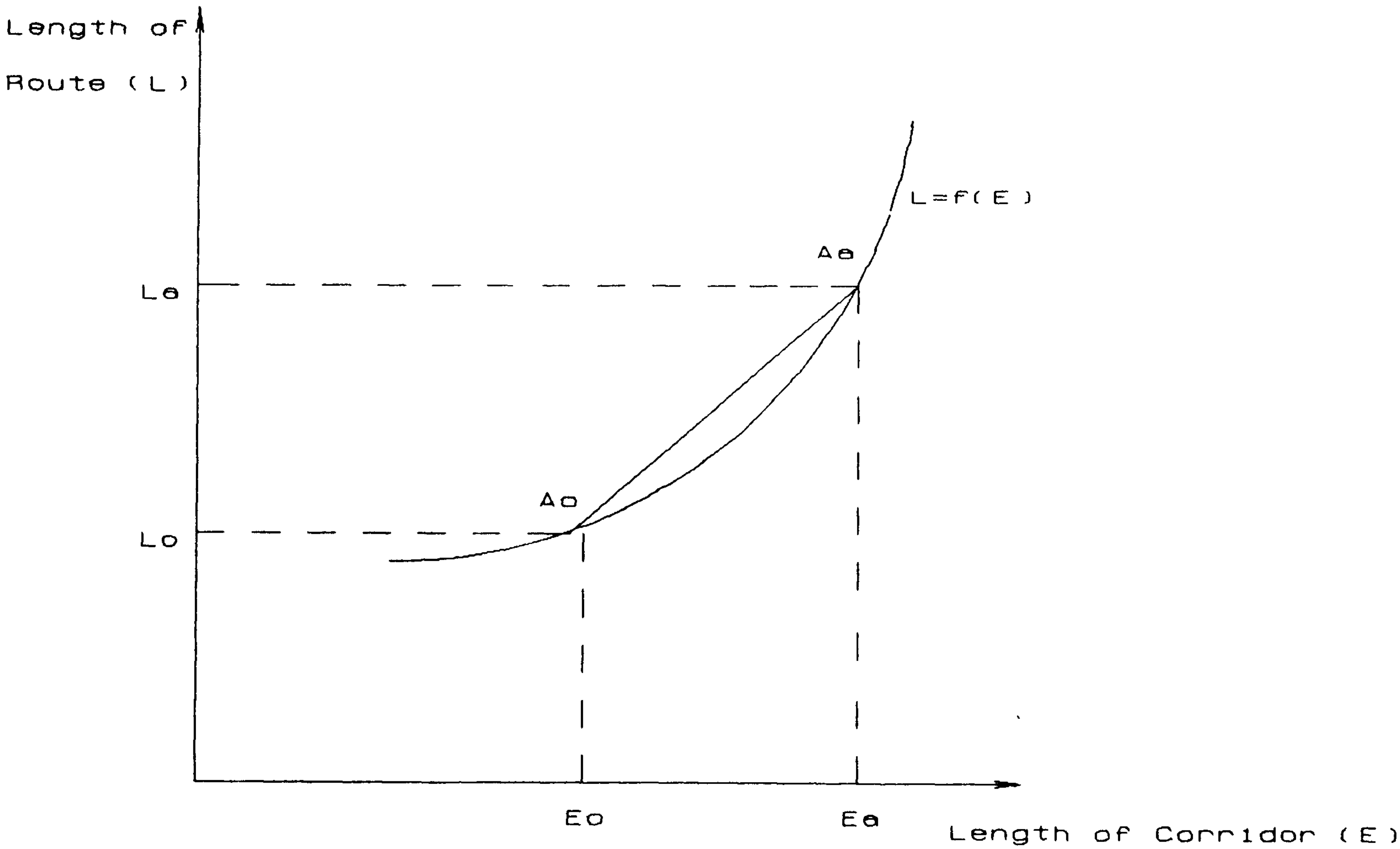

Figure 5.1 Graphical Representation of the Elasticity Concept

Table 5.1 Elasticities of Case 1.

	L	H	M	N
Corridor Length	1.5835	1.1617	-1.1183	0.8258
Passenger Density	0.4245	0.0047	-0.3600	1.6480
Transit Speed	1.3292	-0.6650	1.7320	0.7050
Access Speed	0.8148	0.3851	0.0980	-2.2298
Supplier Cost	0.4582	0.3717	0.0045	-1.1885
Value of Riding Time	0.0000	0.0000	0.0000	1.5931
Value of Access Time	0.0000	0.0000	0.0000	-1.7925
Vehicle Capacity	-0.1900	-2.8726	0.0000	-1.8844
Value of Waiting	0.0000	0.0000	0.0000	0.0000
Transit Fare	-0.7308	-2.2958	2.3692	2.7599

When the vehicle capacity increases by 10%, the route length will decrease by 1.9%. The change in waiting time cost and riding time will leave the route length unaltered, because these parameters do not appear in the objective function that minimizes only transit operating cost. The route length is elastic with respect to corridor length and the transit speed.

A 10% increase in transit speed decreases the headway by 6.6%. When the passenger access speed is increased by 10%, the headway will increase 3.8%. The headway is elastic with respect to transit fare, corridor length and vehicle capacity (i.e., The absolute value of the elasticity exceeds 1.0). If operator costs are increased by 10%, the value of headway will increase by 3.7%. Values of riding time, waiting time and access time leave the headway unaltered, because these parameters do not appear in the objective function.

The corridor length, transit speed and transit fare are elastic with respect to the route spacing. Route spacing will increase by 0.98% if the access speed is increased by 10%. If the operator cost is increased by 10%, the route spacing will decrease by 0.04%. If the passenger density is increased by 10%, route spacing will decrease by 3.6%. The route is insensitive to change in the rest of the parameters.

The number of stops is elastic with respect to passenger density, access speed, supplier cost, vehicle capacity, transit fare, value of riding time and value of access time. If the corridor length is increased by 10 percent, the number of stops will increase by 8% and if the transit speed is increased by 10%, the number of stops will decrease 7%. The number of stops will increase by 7.5% when the transit fare is increased by 10%. As the vehicle capacity increase 10% the number of stops will decrease by 18%. The number of stops is insensitive to changes in the cost of waiting time.

Table 5.2 Elasticities of Case 2

	L	H	M	N
Corridor Length	0.4618	-2.0125	-1.8836	1.8693
Passenger Density	-0.0382	-0.3243	-0.2456	-1.0900
Transit Speed	0.2328	0.0795	0.0047	-2.2954
Access Speed	0.2362	-0.4903	0.9980	-0.2710
Supplier Cost	0.4208	-1.5976	-2.0565	-0.2710
Value of Riding Time	0.0000	0.0000	0.0000	0.4791
Value of Access Time	0.0000	0.0000	0.0000	-0.4596
Vehicle Capacity	0.3422	-1.8107	-2.5909	-1.5225
Value of Waiting	0.0000	0.0000	0.0000	0.0000
Transit Fare	0.3487	-2.3777	2.6351	-0.0266

5.2.2 Case 2: Maximizing operator profit

Table 5.2 presents the elasticities of route length, headway, route spacing and number of stops with respect to ten parameters, namely corridor length, passenger density, transit speed, access speed, supplier cost, value of riding time, value of access time, vehicle capacity, value of waiting time and transit fare for the case of maximizing operator profit.

The route length is increased by 4.6% when corridor length is increased by 10 percent. The route length will decrease 0.3% with an increase of 10 percent in the passenger density. When the transit speed and access speed increase by 10%, the route length will increase by 2.5%. If the supplier cost is increased by 10%, the route length will increase 0.3%. When transit fare is increased by 10%, the route length will increase by 3.4%. If vehicle capacity is increased by 10%, the route length will increase by 3.4%. A change in value of access time, value of riding time and value of waiting time leaves the route length unaltered because they do not appear in the objective function.

With a 10 percent increase in the corridor length and passenger density, the headway will decrease by 20%. The headway will increase 0.79% when the transit speed is increased by 10%. As the access speed is increased by 10%, the headway decreases by 4.9%. An increase of 10 percent in transit fare will cause a 23% decrease of the headway. If the supplier cost is increased by 10% the headway will decrease by 0.17% and if vehicle capacity increases by 10% the headway will decrease 18%. The headway is elastic with respect to the corridor length, operator cost, vehicle capacity and transit fare. The headway is insensitive to changes in the value of riding time, value of access time and value of waiting time.

The route spacing is elastic with respect to access speed, corridor length, operator cost, vehicle capacity and transit fare. When passenger density is increased by 10 percent, route spacing increases 2.45%. Route spacing will increase 0.47% when the transit speed is increased by 10%. As transit fare is increased by 10%, the headway is decreases 26%. If operator cost increases by 10% route spacing will decrease by 20%. If vehicle capacity increases by 10% route spacing will decrease 25%. The headway is insensitive to change in the value of riding time, value of access time, value of waiting time and vehicle capacity.

The number of stops is elastic with respect to corridor length, passenger density, transit speed, value of access time and vehicle capacity. A 10 percent increase in the value of riding time will cause a 4.7% increase in the number of stops. If the access speed or operator cost increases by 10%, the number of stops will decrease by 2.7%. Any change in the value of waiting time will leave the number of stops unaltered.

5.2.4 Case 3: Maximizing user benefit

Table 5.3 presents the elasticities of route length, headway, route spacing and number of stops with respect to ten parameters for the case of user benefit maximization.

If the corridor length is increased by 10 percent the route length will increase by 0.8%. If the passenger density increases by 10%, the route length will increase by 0.08%. When transit speed and access speed is increased by 10 percent, the route length will increase by 6.4%. A 10% increase of the transit fare will decrease route length by 7.4%. If vehicle capacity increases by 10%, route length will increase by 1.9%. The route length is elastic with respect to the operator cost. Changes in the value of riding time, value of access time or value of waiting time will leave the route length unaltered.

Headway is elastic with respect to change in corridor length, passenger density, transit speed, access speed, operator cost, vehicle capacity or transit fare. The headway will increase 12% with a 10% increase in transit speed. When corridor length and

Table 5.3 Elasticities of Case 3.

	L	H	M	N
Corridor Length	0.0799	2.0755	2.0769	2.6518
Passenger Density	0.0087	-1.9485	2.2931	-1.1123
Transit Speed	0.2403	1.2000	2.1928	1.7389
Access Speed	0.8400	2.2025	0.7266	1.7772
Supplier Cost	-1.5672	1.6569	1.1351	2.8455
Value of Riding Time	0.0000	0.0000	0.0000	-0.4695
Value of Access Time	0.0000	0.0000	0.0000	1.8948
Vehicle Capacity	0.1927	1.6400	1.0502	1.6188
Value of Waiting	0.0000	0.0000	0.0000	0.0000
Transit Fare	-0.7554	2.4882	2.4185	1.6471

passenger density increased by 10%, the headway will decrease by 20%. If vehicle capacity is increased by 10%, the headway will decrease by 16%. The headway will not change when the rest of the parameters change.

Route spacing is elastic with respect to change in access speed, corridor length, passenger density, operator cost vehicle capacity or transit fare. If the corridor length, passenger density and transit speed are increased by 10%, a 20% increase will occur in route spacing. If operator cost increases by 10%, route spacing will increase by 11%. Route spacing will remain the same if the value of access time, riding time or waiting time for these parameters do not appear in the objective function.

The number of stops is elastic with respect to corridor length, passenger density, access speed, supplier cost, value of access time, vehicle capacity and transit fare. The number of stops will decrease 5% if the transit speed and value of riding time are increased by 10%. A change in value of waiting time will leave the route length unaltered.

5.2.4 Case 4: Minimizing Total supplier cost and user cost

Table 5.4 shows the elasticities of route length, headway, route spacing and number of stops with respect to ten parameters for the case of minimizing the sum of operator cost and user costs.

The route length is elastic with respect to the corridor length or transit fare. It will increase by 2.6% if passenger density is increased by 10%. When transit speed or value of waiting time increases by 10%, the route length will increase by 4.4%. With a 10 percent increase in the value of access time, route length will decrease 2.1%. Route length will be reduced by 6.5% if the access speed is increased by 10%. When vehicle capacity is increased by 10%, the route length will increase by 7.4%. A 10% change in operator cost will cause the route length to be increased by 4.8%.

Table 5.4 Elasticities of Case 4

	L	H	M	N
Corridor Length	1.2779	-0.5966	-2.2500	-0.1533
Passenger Density	0.2687	-3.9485	2.2931	-1.1123
Transit Speed	0.4440	-0.7200	-2.805	-0.3326
Access Speed	-0.6548	2.5624	0.3987	1.7786
Supplier Cost	0.4873	-2.6087	-2.0851	-3.5737
Value of Riding Time	0.5305	-2.0322	3.5735	-1.5478
Value of Access Time	-0.2293	0.5645	1.2727	-0.1354
Vehicle Capacity	0.7437	-5.2000	3.8605	-1.8458
Value of Waiting	0.4440	4.1067	0.0474	-1.8458
Transit Fare	1.5457	-2.4066	-2.0500	1.1783

If the corridor length is increased by 10 percent the headway will be decreased by 5.9 percent. An increase of 10 percent in transit speed will decrease headway by 7.2%. When passenger access speed is increased by 10 percent, the headway will increase by 20%. When supplier cost is increased by 10%, the headway will decrease by 26%. If the value of access time is increased by 10%, the headway will increase by 5.6%. The headway is elastic with respect to changes in passenger density, access speed, operator cost, value of riding time, vehicle capacity or transit fare.

Route spacing is elastic with respect to corridor length, passenger density, transit speed, operator cost, value of riding time, value of access time, vehicle capacity or transit fare. With a 10% increase in passenger access speed, route spacing will increase by 3.9%, and it will increase by 0.47% with a 10% increase in the value of access time.

The number of stops is elastic with respect to passenger density, value of riding time, access speed, value of waiting time, vehicle capacity and transit fare. If the passenger density is increased by 10 percent, the number of stops will decrease 11%. With a 10 percent increase in transit speed, the number of stops will decrease by 3.3%. The number of stops will decrease 1.5% if the corridor length is increased by 10%. A 10% increase in the value of access time results in a 1.3% increase in stops.

CHAPTER 6

CONCLUSIONS AND SUGGESTIONS

6.1 Summary

This thesis developed a methodology for optimal transit service planning and design. The key elements of service design that were optimized are the length of the transit routes, the spacing between parallel transit routes, operating headway, and stop location and spacing pattern. Four different objective functions were developed; operator cost minimization, profit maximization, welfare (user benefit) maximization subject to a deficit constraint and minimization of total operator and user cost.

An efficient optimization algorithm was developed and used to generate the optimal solutions for each objective function. The transit design methodology and the efficient optimization algorithm are the main contributions of this thesis.

6.2 Conclusions

Table 6.1 shows the value of the optimal design variables and the objective function for all four design objectives. The column under the heading "Min. System Cost" contains the results of the model that minimizing the sum of user and operator costs. The comparison of optimal results reveals that for given input data and demand characteristics the results of minimization of operator cost with a minimum service quality constraint are almost superior to the results of minimization of operator and user cost. The operator cost minimization model generates more demand and profit with almost the same user costs as the minimization of system cost model. It appears that saving a negligible amount of waiting time for an average passenger, resulted in substantial reduction in passenger demand and profit. It should be noted that the system cost is sensitive to the value of passenger access time. It can be expected that if

Table 6.1 Optimal Transit Design Under Various Objectives

	Min Op. Cost	Max Op. Profit	Max Walfare	Min System Cost
Route Length(miles)	3.78	3.96	4.27	4.84
Route Spacing(miles)	1.78	1.30	1.26	2.11
Route Stops	6.01	6.3	5.2	6.39
Transit Headway(min)	21.8	16.4	15.9	21.0
Operating Cost(cent/min	189.9	351.8	367.3	210.7
Transit Demand(trips/hr	207.4	338.8	382.8	181.5
Profit(cent/min)	242.4	394.1	383.2	167.5
User In-vehicle Cost	264.3	462.2	448.5	243.2
User Access Cost	758.7	950.6	940.0	771.7
User Waiting Cost	565.5	725.C	730.1	477.3
Total Cost	1778.4	2476.€	2501.3	1703.8
User Benefit			248.1	

passenger access time increases the new trade-off between operator and user cost will result in a service design more advantagous for the passengers.

The maximization of the operator profit is clearly superior in terms of demand generation to the minimization of operator and user cost. From both policy and operating stand point the small decrease in wait time is not worth the substantial loss in demand and, therefore, profit.

In conclusion, the methodology presented in this thesis provides a good framework for deriving an optimal transit system design under various design objectives. It facilitates evaluation of the trade-offs in system design before the actual system design is implemented in real world applications. Also, it should be noted that the values of design variables are similar in some cases. It is clear that if similar service quality is provided to the public by two alternatives, the alternative with larger profit (or smaller operating deficit) should be preferred.

6.3 Suggestion for Further tudy

The validity of some assumptions should be re-exan ned. The transit corridor is assumed rectangular in this study. This could be rela ed to take into account more realistic shapes such as wedge areas. Demand that vari s with space and time may be also considered.

The objective function may be modified to include additional elements which influence the transit service such as right-of-way cost so that the operation will represent the real world more realistically.

APPENDIX

A Penalty Function

1.The Concept of Penalty Function

The penalty function method transforms a constrained optimization problem in a form of a mathematical program into a single unconstrained problem or into a sequence of unconstrained problems. The constraints are placed into the objective function via a penalty parameter in such a way that any violation of constraints is penalized. Consider the following problem in which the objective function, $f(\mathbf{x})$ is to be minimized subject to the constraints $p_i(\mathbf{x})=0$

Minimize $f(\mathbf{x})$

subject to $p_i(\mathbf{x}) = 0$; (1)

where: $i = 1,2,\ldots,m$

$\mathbf{x} = (x_1,x_2,\ldots x_m)$

The penalty function is then defined as:

Minimize $f(\mathbf{x}) + \mathbf{u}\ p_i^2(\mathbf{x})$ (2)

where **u** is a large positive constant

For sufficiently small "**u**" it can be reasoned that the solutions to problem (1) and (2) will be nearly equal. The term $\mathbf{u}\ p_i^2(\mathbf{x})$ is referred to as the penalty function since in effect it assigns a specific cost to violations of the constraints.

Now consider the following problem with the inequality constraints

Minimize $f(\mathbf{x})$

subject to $p_i(\mathbf{x}) <= 0$ (3)

It is clear that the form $f(\mathbf{x})+u\ p_i^2(\mathbf{x})$ is not appropriate, since a penalty will be incurred whether $p(\mathbf{x})<0$ or $p(\mathbf{x})>0$. Needless to say, a penalty is desired only if the point x is not feasible, that is, $p(\mathbf{x})>0$. A suitable unconstrained problem is therefore given by

Minimize $f(\mathbf{x}) + \mathbf{u}$ maximum $\{0, p_i(\mathbf{x})\}$

where $p(\mathbf{x})$ satisfy:

(1) $p(\mathbf{x})$ is continuous,

(2) for any $\mathbf{x}$ En, have $p(\mathbf{x}) >= 0$,

(3) if $p(\mathbf{x}) <= 0$ then maximum $\{0,P(\mathbf{x})\}=0$.

The of penalty function method is simple and effective, provided that suitable values of parameters **u** are chosen. The concept of a penalty function is demonstrated below via an example.

Example:

Consider the following problem:

Minimize $x_1^2 + x_2^2$

subject to $x_1 + x_2 - 1 = 0$ (4)

The optimal solution lies at the point (x1,x2)= (1/2,1/2), and has the objective value of 1/2. Now consider the following penalty problem, where $u > 0$ is a large number.

Minimize $x_1^2 + x_2^2 + u(x_1 + x_2 - 1)^2$

subject to $x_1, x_2 >= 0$ (5)

Note that for any $u >= 0$, the objective function is convex. Thus, a necessary and sufficient condition for optimality is that the gradient of $x_1^2 + x_2^2 + u(x_1+x_2-1)^2$ is equal to zero, yielding

$$x_1 + u(x_1+x_2-1)=0$$

$$x_2 + u(x_1+x_2-1)=0 \quad (6)$$

Solving these two equations simultaneously yie $x_1+x_2=u/(2u+1)$. Thus, the optimal solution of the penalty problem can be made itrarily close to the solution of the original problem by choosing sufficiently large u Often u and thus **x** are chosen according to the certain set of rules and the whole pr edure is presented in a form of an algorithm below.

2. Penalty Function Algorithm

The algorithm operates as follows:

Initialization Step:

Let >0 be a termination scalar. Choose an initial point $\mathbf{x}_k$, a penalty parameter $\mathbf{u}_k > 0$, and a scalar $B > 1$. Let $k=1$ and go to the main step.

Main Step

1 Starting with $\mathbf{x}_k$, solve the following problem:

Minimize $f(\mathbf{x}) + \mathbf{u}_k(\mathbf{x})$

Designate the resulting Xk to be an optimal solution, use it as x_{k+1}, and go to step 2.

2 If $u_k(x_{k+1}) < \epsilon$ stop; otherwise, let $u_{k+1} = u_k$, replace k by k+1, and go to step 1.

Example 1:

$$\text{Minimize} \quad f(x) = (x1 - 3)^2 + (x2 - 2)^2$$

$$\text{subject to} \quad h(x) = x1 + x2 - 4 = 0 \tag{7}$$

For a given penalty parameter u_k, the problem to be solved is:

$$\text{Minimize } Q = (x1-3)^2 + (x-2)^2 + u(x1+x2-4)^2$$

$$dQ/dx1 = 2(x1-3) + 2u(x1+x2-4),$$

$$dQ/dx2 = 2(x2-2) + 2u(x1+x2-4),$$

For $dQ/dx1 = 0$, $dQ/dx2 = 0$ we get:

$$x1 = (5u+3)/(2u+1), \quad x2 = (3u+2)/(2u+1),$$

If $u \to \epsilon$, $x^* = \lim x(u) = (5/2, 3/2)^T$, then the minimum value is $f(x) = 1/2$.

Example 2:

$$\text{Minimize} \quad f(x) = (x1-3)^2 + (x2-2)^2,$$

$$\text{subject to } h(x) = x1 + x2 - 4 <= 0.$$

Since it is a inequality constrained function, we have to consider

$$h(x) = \begin{cases} 0 & \text{for } h(x) <= 0; \\ h(x)^2 & \text{for } h(x) > 0; \end{cases}$$

The above function transform to:

$$\text{Min } Q = (x-3)^2 + (x-2)^2 + uh(x),$$

$$dQ/dx1 = \begin{cases} 2(x1-3) + 2u(x1+x2-4), & h(x) > 0 \\ 2(x1-3), & h(x) <= 0 \end{cases}$$

$$dQ/dx2 = \begin{cases} 2(x2-2) + 2u(x1+x2-4), & h(x) > 0 \\ 2(x2-2), & h(x) <= 0. \end{cases}$$

Consider when $h(x) > 0$:

$$\begin{cases} (x1-3) + u(x1+x2-4) = 0 \\ (x2-2) + u(x1+x2-4) = 0 \end{cases}$$

$$x1 = (5u+3)/(2u+1) \quad x2 = (3u+2)/(2u+1)$$

If $u \to \epsilon$, we have $x^* = \lim x(u) = (5/2, 3/2)$, $h(x^*) <= 0$. The resulting optimal function was minimum value in x^*, $f(x^*) = 1/2$.

B Computer Programs

Case 1 Minmizing Transit Operating Cost

```
PROGRAM findout Case1(input, output);
VAR
 a,k,l,s,h,m,c,y,v,p,d,e,sc3,g,cap : real ;
 sc1,sc2,pnty,u1,u2,u3,x,n,nn1,viv,va,a1,a2,a3,a4,f,a5 : real ;
 ha,hb,hc,hd,hy,hx,h1,h2,hdd,hdd1,mdd,mdd1,sdd,sdd1,ldd,ldd1 : real ;
 ma,mb,mc,md,my,mx,m1,m2 : real ;
 sa,sb,sc,sd,sp,sq,sr,detla,stt,stt1,sk:real;
 s1,l1,pp,lk,lu,lv,w: real;
 so,xx,ub,hbe,lb,mbe,nb,hk,tst,tst1,scb,sob,uv,ua,uw,vw : real ;
 j, jj,jjj,tt, kkkk, x1 : integer ;
 found : boolean ;
 outfile,outfile1 : text ;
BEGIN
 assign(outfile,'lgot3.dat');
 rewrite(outfile);
 viv:=5;
 va:=15;
 vw:=15;
 g:=0.05;
 y := 3 ;
 c := 50 ;
 p := 3.59 ;
 cap := 45 ;
 v := 0.167 ;
 k := 0.4 ;
 e := 5 ;
 f :=125;
 a1 := 0.38;
 a2 := -0.0081;
 a3 := -0.0033;
 a4 := -0.0014;
 a5 := 0.0328;
 sc2 := 10000 ;
 w :=1 ;
 h := 20 ;
 h1 :=100 ;
 s := 1 ;
 m :=1.0 ;
 l:= 4.5 ;
 tt := 0 ;
 tst1:=1.6;
 writeln(outfile,'  L    M    H    S    N    SC    SO    P   UV    UA
UW');
 for jj :=3 to 15 do
 begin
  u2 :=jj;
 for j :=2 to 3 do
  begin
   u1 :=exp(j*ln(10));
   for jjj :=6 to 10 do
```

```
 begin
 u3 :=jjj;
found := false ;
while not found do
 begin
 hdd1 :=100000;
 for kkkk :=1 to 600 do
  begin
   h := kkkk/10 ;
   a := 2*50*0.1;
   d := (1/360+a*h/60*2/3600)*60 ;
   hdd := -2*c*l*y/(m*h*h)*(1/v+d/s)+2*u1*((a1+a2*(k*h+(m+s)/(4*g))
          +(a3/(2*v)+a5/2)*l+a4*f)*e*m*p*h-cap)*(a1+a2*(2*k*h+(m+s)/(4*g))
          +(a3/(2*v)+a5/2)*l+a4*f)*(e*m*p) ;
   if abs(hdd1)>abs(hdd) then
     begin
      hdd1 := hdd ;
      h1 := h;
     end;
  end;
h := h1 ;
writeln('h ',h:9:4);
mdd1:=100000;
for kkkk := 1 to 300 do
  begin
   m := kkkk/100 ;
   a := 2*50*0.1;
   d := (1/360+a*h/60*2/3600)*60 ;
   mdd := -2*c*l*y/(h*m*m)*(1/v+d/s)+2*u1*((a1+a2*(k*h+(m+s)/(4*g))
         +(a3/(2*v)+a5/2)*l+a4*f)*e*m*p*h-cap)*(a1+a2*(k*h+(2*m+s)/(4*g))
         +(a3/(2*v)+a5/2)*l+a4*f)*(e*h*p)+2*u2*((m+s)-4*w)+2*u3*(2*(e-l)
         +m-4*w) ;
   if abs(mdd1)>abs(mdd) then
    begin
     mdd1 := mdd ;
     m1 := m
    end;
  end;
 m := m1 ;
 writeln('m ',m:9:4);
 a := 2*50*0.1;
 d := (1/360+a*h/60*2/3600)*60 ;
 sdd1 :=100000;
 for kkkk :=1 to 300 do
  begin
   s := kkkk/100;
   sdd :=-2*c*l*y*d/(s*s*h*m)+2*u1*((a1+a2*(k*h+(m+s)/(4*g))
        +(a3/(2*v)+a5/2)*l+a4*f)*e*m*h*p-cap)*a2*(e*m*h*p)/(4*g)
        +2*u2*((m+s)-4*w) ;
   if abs(sdd1)>abs(sdd) then
    begin
     sdd1 := sdd ;
     s1 := s
    end
```

```
 end;
 s := s1 ;

writeln('s ',s:8:3);
  l := (2*c*y*(1/v+d/s)/(h*m)+2*u1*((a1+a2*(k*h+(m+s)/(4*g))
   +a4*f)*e*m*h*p-cap)*(a3/(2*v)+a5/2)*(e*m*h*p)-4*u3*(e+m-4*W))
   /(-2*u1*sqr(e*m*p*h*(a3/(2*v)+a5/2))-8*u3) ;
writeln('l= ',l:8:3);
  lk:=2*u1*((a1+a2*(k*h+(m+s)/(4*g))+a4*f)*e*m*h*p-cap)*(a3/(2*v)+a5/2)
       *(e*m*h*p);
  lu :=2*c*y*(1/v+d/s)/(h*m);
  lv:=-2*u1*sqr(e*m*p*h*(a3/(2*v)+a5/2));
writeln('lkk=',lk:9:4,'luu=',lu:8:3,'lvv=',lv:8:3,u1:6:2,u2:6:2,u3:6:2);
sc1 := 2*c*l*y*(1/v+d/s)/(h*m)+u1*sqr((a1+a2*(k*h+(m+s)/(4*g))
       +(a3/(2*v)+a5/2)*l+a4*f)*e*m*h*p-cap)+u2*sqr(m+s-4*w)
       +u3*sqr(2*(e-l)+m-4*w);
so := 2*c*l*y*(1/v+d/s)/(h*m);
pp:=(a1+a2*(k*h+(m+s)/(4*g))+(a3/(2*v)+a5/2)*l+a4*f)*e*m*h*p/h*60;
 writeln('sc1 ',sc1:9:3,'so ',so:9:3);
 if (sc1/so<1.1) and (sc1/so>0.9) then
 begin
  found := true ;
 a := 2*50*0.1;
d := (1/360+a*h/60*2/3600)*60 ;
n:=0;
for x1:=50 downto 0 do
 begin
  x:=x1/10 ;
  stt1 :=1000000;
  for kkkk :=1 to 200 do
   begin
    sk := kkkk/100;
    stt :=-2*c*d/(sk*sk*h*m)+2*va*p*(a1+a2*(k*h-(m+2*sk)/(4*g))
         +(a3/(2*v)+a5/2)*l+a4*f)/(4*g)-2*viv*d*p*(e-x)/sqr(sk)*(a1
         +a2*(k*h+m/(4*g))+(a3/(2*v)+a5/2)*l+a4*f) ;
    if abs(stt1)>abs(stt) then
     begin
      stt1 := stt ;
      s1 := sk;
     end
   end;
   nn1:=0.1/s1;
   n:=n+nn1;
 end;
uv:=p*(a1+a2*(k*h+(m+s)/(4*g))+(a3/(2*v)
    +a5/2)*l+a4*f)*y*viv*l*((l/(2*v)+l*d/(2*s))+(e-l)*(1/v+d/s));

ua:=p*Y*va*(a1+a2*(k*h+(m+s)/(4*g))+(a3/(2*v)+a5/2)*l+a4*f)*(sqr(e-l)
   /(4*g)+m*e/(4*g)+s*l/(2*g));
uw:=vw*h/2*p*(a1+a2*(k*h+(m+s)/(4*g))+(a3/(2*v)+a5/2)*l+a4*f)*e*y;
  writeln(s:8:3,n:10:3);
  writeln(outfile,l:6:2,m:6:2,h:8:2,s:8:2,
        n:8:2,sc1:8:2,so:8:2,pp:7:2,uv:6:1,ua:6:1,uw:6:1);
 end
```

```
  end;
  end;
 end;
 end;
close(outfile)
end.
```

Case 2 Maximizing Operator profit

```
PROGRAM findout Case2(input, output);
VAR
  a,k,l,s,h,m,c,y,v,p,d,e,sc3,g,cap,w : real ;
  sc1,sc2,pnty,u1,u2,u3,u4,x,n,nn1,viv,va,a1,a2,a3,a4,f,a5 : real ;
  ha,hb,hc,hd,hy,hx,h1,h2,hdd,hdd1,mdd,mdd1,sdd,sdd1,ldd,ldd1 : real ;
  ma,mb,mc,md,my,mx,m1,m2 : real ;
  sa,sb,sc,sd,sp,sq,sr,detla,pt:real;
  s1,l1,pp,op,oo,opp,stt,stt1,sk: real;
  so,xx,ub,hbe,lb,mbe,nb,hk,tst,tst1,scb,sob,ua,uv,uw,vw : real ;
  j, jj,jjj,jjjj,tt, kkkk, x1 : integer ;
  found : boolean ;
  outfile,outfile1 : text ;
BEGIN
  assign(outfile,'lgot4.dat');
  assign(outfile1,'lgot5.dat');
  rewrite(outfile1);
  rewrite(outfile);
  viv:=5;
  va:=15;
  vw:=15
  g:=0.05;
  y := 3 ;
  c := 50 ;
  p := 3.59 ;
  op := 150 ;
  v := 0.167 ;
  k := 0.4 ;
  e := 5 ;
  f :=125;
  a1 := 0.38;
  a2 := -0.0081;
  a3 := -0.0033;
  a4 := -0.0014;
  a5 := 0.0328;
  sc2 := 10000 ;
  h := 20 ;
  w := 1 ;
  h1 :=100 ;
  s := 1 ;
  m :=1.0 ;
  cap:=45;
  l:= 4.5 ;
  tt := 0 ;
```

```
 tst1:=1.6;
 writeln(outfile,' u1  u2  u3    L    M     H    S    N     SC     SO     P     OP');
for j :=9 to 15 do
begin
 u1 :=j;
for jj :=1 to 2 do
 begin
  u2 :=exp(jj*ln(10));
for jjj:=0 to 1 do
 begin
  u3 :=exp(jjj*ln(10));
  found := false ;
  while not found do
   begin
   hdd1 :=100000;
   writeln(u1:6:0,u2:6:0,u3:6:1);
   for kkkk :=1 to 600 do
    begin
     h := kkkk/10 ;
     a := 2*50*0.1;
     d := (1/360+a*h/60*2/3600)*60 ;
     hdd := e*y*p*f*a2*k+2*c*y*l*(1/v+d/s)/(h*h*m)-2*u1*(cap-(a1
            +a2*(k*h+(m+s)/(4*g))+(a3/(2*v)+a5/2)*l+a4*f)*e*m*p*h)
            *(e*m*p*(a1+a2*(2*k*h+(m+s)/(4*g))+(a3/(2*v)+a5/2)*l+a4*f)) ;
     if abs(hdd1)>abs(hdd) then
       begin
        hdd1 := hdd ;
        h1 := h;
       end;
    end;
  h := h1 ;
  writeln('h ',h:9:4);
  mdd1:=100000;
  for kkkk := 1 to 500 do
    begin
     m := kkkk/100 ;
     a := 2*50*0.1;
     d := (1/360+a*h/60*2/3600)*60 ;
     mdd :=e*y*p*f*a2/(4*g)+2*c*y*l*(1/v+d/s)/(h*m*m)-2*u1*(cap-(a1
            +a2*(k*h+(m+s)/(4*g))+(a3/(2*v)+a5/2)*l+a4*f)*e*m*p*h)
            *(e*h*p*(a1+a2*(k*h+(2*m+s)/(4*g))+(a3/(2*v)+a5/2)*l+a4*f))
            -2*u2*(4*w-2*(e-l)-m)-2*u3*(4*w-(m+S)) ;
     if abs(mdd1)>abs(mdd) then
      begin
       mdd1 := mdd ;
       m1 := m
      end;
    end;
   m := m1 ;
   writeln('m ',m:9:4);
   a := 2*50*0.1;
   d := (1/360+a*h/60*2/3600)*60 ;
   sdd1 :=100000;
   for kkkk :=1 to 300 do
```

```
 begin
  s := kkkk/100;
  sdd :=a2*e*y*p*f/(4*g)+2*c*l*y*d/(h*m*s*s)-2*u1*(cap-(a1
        +a2*(k*h+(m+s)/(4*g))+(a3/(2*v)+a5/2)*l+a4*f)*e*m*p*h)
        *e*m*p*h*a2/(4*g)-2*u3*(4*w-(m+s)) ;
  if abs(sdd1)>abs(sdd) then
    begin
     sdd1 := sdd ;
     s1 := s
    end
 end;
 s := s1 ;

writeln('s ',s:8:3);
 l := (2*c*y*(1/v+d/s)/(h*m)-(a3/(2*v)+a5/2)*e*y*p*f+2*u1*(cap-(a1
        +a2*(k*h+(m+s)/(4*g))+a4*f)*e*m*p*h)*(a3/(2*v)+a5/2)*e*m*p*h
        -4*u2*(4*w-e*2-m))/(2*u1*sqr(a3/(2*v)+a5/2)*sqr(e*m*h*p)+u2*8) ;
writeln('l= ',l:8:3);
sc1 := ((a1+a2*(k*h+(m+s)/(4*g))+(a3/(2*v)+a5/2)*l+a4*f)*e*y*p*f
      -2*c*l*y/(h*m)*(1/v+d/s))+u1*sqr(cap-(a1+a2*(k*h+(m+s)/(4*g))
      +(a3/(2*v)+a5/2)*l+a4*f)*e*m*p*h)+u2*sqr(4*w-2*(e-l)-M)
      +u3*sqr(4*w-(m+s));
so := ((a1+a2*(k*h+(m+s)/(4*g))+(a3/(2*v)+a5/2)*l+a4*f)*e*y*p*f
      -2*c*l*y/(h*m)*(1/v+d/s));
oo :=a1+a2*(k*h+(m+s)/(4*g))+(a3/(2*v)+a5/2)*l+a4*f;
pp:=(a1+a2*(k*h+(m+s)/(4*g))+(a3/(2*v)+a5/2)*l+a4*f)*e*m*60*p;
opp:=2*c*l*y/(h*m)*(1/v+d/s);
 writeln('sc1 ',sc1:9:3,'so ',so:9:3,'oo=',oo:9:3);
 if (sc1/so<1.1) and (sc1/so>0.9) then
 begin
  found := true ;
a:= 2*50*0.1;
d := (1/360+a*h/60*2/3600)*60 ;
n:=0;
for x1:=50 downto 1 do
 begin
  x:=x1/10 ;
  stt1 :=1000000;
  for kkkk :=1 to 200 do
   begin
    sk := kkkk/100;
    stt :=-2*c*d/(sk*sk*h*m)+2*va*p*(a1+a2*(k*h+(m+2*sk)/(4*g))
         +(a3/(2*v)+a5/2)*l+a4*f)/(4*g)-2*viv*d*p*(e-x)/sqr(sk)*(a1
         +a2*(k*h+m/(4*g))+(a3/(2*v)+a5/2)*l+a4*f) ;
    if abs(stt1)>abs(stt) then
     begin
      stt1 := stt ;
      s1 := sk;
     end
   end;
   nn1:=0.1/s1;
   n:=n+nn1;
 end;
 uv:=p*(a1+a2*(k*h+(m+s)/(4*g))+(a3/(2*v)
```

```
      +a5/2)*l+a4*f)*y*viv*l*((l/(2*v)+l*d/(2*s))+(e-l)*(1/v+d/s));

   ua:=p*Y*va*(a1+a2*(k*h+(m+s)/(4*g))+(a3/(2*v)+a5/2)*l+a4*f)*(sqr(e-l)
      /(4*g)+m*e/(4*g)+s*l/(2*g));
   uw:=vw*h/2*p*(a1+a2*(k*h+(m+s)/(4*g))+(a3/(2*v)+a5/2)*l+a4*f)*e*y;
    writeln(s:8:3,n:10:3);
    writeln(outfile,l:6:1m:6:2,h:7:2,s:6:1,n:6:1,sc1:8:2,so:8:2,pp:7:2,
           opp:7:2,uv:6:1,ua:6:1,uw:6:1);
   end
 end;
 end;
end;
end;
close(outfile) ;
close(outfile1)
end.
```

Case 3 Maximizing User Benefit

```
PROGRAM findout Case3(input, output);
VAR
 a,k,l,s,h,m,c,y,v,p,d,e,sc3,g,cap,w : real ;
 sc1,sc2,pnty,u1,u2,u3,u4,x,n,nn1,viv,va,a1,a2,a3,a4,f,a5 : real ;
 ha,hb,hc,hd,hy,hx,h1,h2,hdd,hdd1,mdd,mdd1,sdd,sdd1,ldd,ldd1 : real ;
 ma,mb,mc,md,my,mx,m1,m2 : real ;
 sa,sb,sc,sd,sp,sq,sr,detla,pt,ff,oo1:real;
 s1,l1,pp,op,oo,opp,stt,stt1,sk: real;
 so,xx,ub,hbe,lb,mbe,nb,hk,tst,tst1,scb,sob : real ;
 j, jj,jjj,jjjj,tt, kkkk, x1 : integer ;
 found : boolean ;
 outfile,outfile1 : text ;
BEGIN
 assign(outfile,'lgot4.dat');
 assign(outfile1,'lgot5.dat');
 rewrite(outfile1);
 rewrite(outfile);
 viv:=5;
 va:=15;
 g:=0.05;
 y := 3 ;
 c := 50 ;
 p := 3.59 ;
 op := 150 ;
 v := 0.167 ;
 k := 0.4 ;
 e := 5 ;
 f :=125;
 a1 := 0.38;
 a2 := -0.0081;
 a3 := -0.0033;
 a4 := -0.0014;
 a5 := 0.0328;
```

```
 sc2 := 10000 ;
 h := 20 ;
 w := 1 ;
 h1 :=100 ;
 ff:=0.5;
 s := 1 ;
 m :=1.0 ;
 cap:=45;
 l:= 4.5 ;
 tt := 0 ;
 tst1:=1.6;
 writeln(outfile,' u1  u2  u3  u4  L    M    H    S    N    SC    SO    P    OP');
for j :=0 to 4 do
begin
 u1 :=exp(j*ln(10));
 for jj :=1 to 3 do
  begin
  u2 :=jj;
   for jjj:=6 to 6 do
    begin
     u3 :=jjj;
     for jjjj := 4 to 8 do
     begin
     u4:=jjjj;
  found := false ;
  while not found do
   begin
   hdd1 :=100000;
   writeln(u1:6:0,u2:6:0,u3:6:1,u4:6:1);
   for kkkk :=1 to 600 do
    begin
      h := kkkk/10 ;
      a := 2*50*0.1;
      d := (1/360+a*h/60*2/3600)*60 ;
      hdd :=-e*y*p/a4*(a1+a2*(k*h+(m+s)/(4*g))+(a3/(2*v)+a5/2)*l+a4*f)
             *a2*k-2*u1*(cap-(a1+a2*(k*h+(m+s)/(4*g))+(a3/(2*v)+a5/2)*l
             +a4*f)*e*m*p*h)*(e*m*p*(a1+a2*(2*k*h+(m+s)/(4*g))+(a3/(2*v)
             +a5/2)*l+a4*f))+2*u2*(ff*((a1+a2*(k*h+(m+s)/(4*g))+(a3/(2*v)
             +a5/2)*l+a4*f)*e*y*p*f)-2*c*y*l/(h*m)*(1/v+d/s))*(ff*a2*k*e
             *y*p*f+2*c*l*y/(h*h*m)*(1/v+d/s));
      if abs(hdd1)>abs(hdd) then
        begin
         hdd1 := hdd ;
         h1 := h;
        end;
    end;
  h := h1 ;
  writeln('h ',h:9:4);
  mdd1:=1000000000;
  for kkkk := 1 to 500 do
    begin
     m := kkkk/100 ;
     a := 2*50*0.1;
     d := (1/360+a*h/60*2/3600)*60 ;
```

```
  mdd :=-e*y*p/a4*(a1+a2*(k*h+(m+s)/(4*g))+(a3/(2*v)+a5/2)*l+a4*f)
        *a2/(4*g)-2*u1*(cap-(a1+a2*(k*h+(m+s)/(4*g))+(a3/(2*v)+a5/2)*l
        +a4*f)*e*m*p*h)*(e*h*p*(a1+a2*(k*h+(2*m+s)/(4*g))+(a3/(2*v)
        +a5/2)*l+a4*f))+2*u2*(ff*((a1+a2*(k*h+(m+s)/(4*g))+(a3/(2*v)
        +a5/2)*l+a4*f)*e*y*p*f)-2*c*y*l/(h*m)*(1/v+d/s))*(ff*a2*e*y*p
        *f/(4*g)-2*c*l*y/(m*m*h)*(1/v+d/s))-2*u3*(4*w-2*(e-l)-m)-2*u4
        *(4*w-(m+s));
  if abs(mdd1)>abs(mdd) then
   begin
    mdd1 := mdd ;
    m1 := m;

   end;
 end;
m := m1 ;
writeln('m ',m:9:4);
a := 2*50*0.1;
d := (1/360+a*h/60*2/3600)*60 ;
sdd1 :=1000000000;
for kkkk :=1 to 300 do
 begin
  s := kkkk/100;
  sdd :=-e*y*p/a4*(a1+a2*(k*h+(m+s)/(4*g))+(a3/(2*v)+a5/2)*l+a4*f)*a2
      /(4*g)-2*u1*(cap-(a1+a2*(k*h+(m+s)/(4*g))+(a3/(2*v)+a5/2)*l+a4*f)
      *e*m*p*h)*e*m*p*h*a2/(4*g)+2*u2*(ff*((a1+a2*(k*h+(m+s)/(4*g))
      +(a3/(2*v)+a5/2)*l+a4*f)*e*y*p*f)-2*c*y*l/(h*m)*(1/v+d/s))*(ff*
      a2*e*y*p*f/(4*g)+2*c*l*y*d/(h*m*s*s))-2*u4*(w-(m+s));
  if abs(sdd1)>abs(sdd) then
    begin
     sdd1 := sdd ;
     s1 := s
    end
 end;
 s := s1 ;
writeln('s ',s:8:3);
a := 2*50*0.1;
d := (1/360+a*h/60*2/3600)*60 ;
ldd1 :=1000000000;
for kkkk :=1 to 500 do
 begin
  l := kkkk/100;
  ldd :=-e*y*p/a4*(a1+a2*(k*h+(m+s)/(4*g))+(a3/(2*v)+a5/2)*l+a4*f)*(a3/
      (2*v)+a5/2)-2*u1*(cap-(a1+a2*(k*h+(m+s)/(4*g))+(a3/(2*v)+a5/2)
      *l+a4*f)*e*m*p*h)*e*m*p*h*(a3/(2*v)+a5/2)+2*u2*(ff*((a1+a2*(k*h
      +(m+s)/(4*g))+(a3/(2*v)+a5/2)*l+a4*f)*e*y*p*f)-2*c*y*l/(h*m)
      *(1/v+d/s))*(ff*(a3/(2*v)+a5/2)*e*y*p*f-2*c*y*(1/v+d/s)/(h*m))
      +4*u3*(4*w-2*(e-l)-m) ;
  if abs(ldd1)>abs(ldd) then
    begin
     ldd1 := ldd ;
     l1 := l
    end
 end;
 l:= l1 ;
```

```
writeln('l= ',l:8:3);
sc1 :=-y*p*e*sqr(a1+a2*(k*h+(m+s)/(4*g))+(a3/(2*v)+a5/2)*l+a4*f)/(2*a4)
     +u1*sqr(cap-(a1+a2*(k*h+(m+s)/(4*g))+(a3/(2*v)+a5/2)*l+a4*f)*e*m
     *p*h)+u2*sqr(ff*(a1+a2*(k*h+(m+s)/(4*g))+(a3/(2*v)+a5/2)*l+a4*f)*f
     *e*y*p
     -2*c*y*l/(h*m)*(1/v+d/s))+u3*sqr(4*w-2*(e-l)-m)+u4*sqr(4*w-(m+s)) ;
so := -y*p*e*sqr(a1+a2*(k*h+(m+s)/(4*g))+(a3/(2*v)+a5/2)*l+a4*f)/(2*a4);
oo :=(a1+a2*(k*h+(m+s)/(4*g))+(a3/(2*v)+a5/2)*l+a4*f);
oo1:=u2*sqr(ff*(a1+a2*(k*h+(m+s)/(4*g))+(a3/(2*v)+a5/2)*l+a4*f)
    -2*c*y*l/(h*m)*(1/v+d/s));
pp:=(a1+a2*(k*h+(m+s)/(4*g))+(a3/(2*v)+a5/2)*l+a4*f)*e*m*60*p;
opp:=2*c*l*y/(h*m)*(1/v+d/s);
 writeln('sc1 ',sc1:9:3,'so ',so:9:3,'oo=',oo:9:3,'oo1',opp:9:3);
 if (sc1/so<1.20) and (sc1/so>0.80) then
 begin
  found := true ;
a:= 2*50*0.1;
d := (1/360+a*h/60*2/3600)*60 ;
n:=0;
for x1:=50 downto 1 do
 begin
  x:=x1/10 ;
  stt1 :=1000000;
  for kkkk :=1 to 200 do
   begin
    sk := kkkk/100;
    stt :=-2*c*d/(sk*sk*h*m)+2*va*p*(a1+a2*(k*h+(m+2*sk)/(4*g))
         +(a3/(2*v)+a5/2)*l+a4*f)/(4*g)-2*viv*d*p*(e-x)/sqr(sk)*(a1
         +a2*(k*h+m/(4*g))+(a3/(2*v)+a5/2)*l+a4*f) ;
    if abs(stt1)>abs(stt) then
     begin
      stt1 := stt ;
      s1 := sk;
     end
   end;
   nn1:=0.1/s1;
   n:=n+nn1;
   writeln(outfile1,nn1:7:2);
 end;
  writeln(s:8:3,n:10:3);
  writeln(outfile1);

writeln(outfile,u1:4:0,u2:4:0,u3:3:0,u4:3:0,l:6:2,m:6:2,h:7:2,s:6:2,n:6:1,sc1:8:2,so:8
:2,pp:7:2,opp:7:2);
    end
  end;
  end;
  end;
 end;
 end;
 close(outfile) ;
 close(outfile1)
 end.
```

Case 4 Minimizing Total Supplier and User Cost

```
PROGRAM findout(input, output);
VAR
  a,k,l,s,h,m,c,y,v,p,d,e,sc3,g,cap : real ;
  sc1,sc2,pnty,u1,u2,u3,x,n,nn1,viv,va,vw,a1,a2,a3,a4,f,a5 : real ;
  ha,hb,hc,hd,hy,hx,h1,h2,hdd,hdd1,mdd,mdd1,sdd,sdd1,ldd,ldd1 : real ;
  ma,mb,mc,md,my,mx,m1,m2,ua,uw,uv : real ;
  sa,sb,sc,sd,sp,sq,sr,detla,stt,stt1,sk:real;
  s1,l1,pp,lk,lu,lv,w,op,uc: real;
  so,xx,ub,hbe,lb,mbe,nb,hk,tst,tst1,scb,sob : real ;
  j, jj,jjj,tt, kkkk, x1 : integer ;
  found : boolean ;
  outfile,outfile1 : text ;
BEGIN
  assign(outfile,'lgot3.dat');
  rewrite(outfile);
  assign(outfile1,'lgot5.dat');
  rewrite(outfile1);
  viv:=5;
  vw :=15 ;
  va:=15;
  g:=0.05;
  y := 3 ;
  c := 50 ;
  p := 3.59 ;
  cap := 45 ;
  v := 0.167 ;
  k := 0.4 ;
  e := 5 ;
  f :=125;
  a1 := 0.38;
  a2 := -0.0081;
  a3 := -0.0033;
  a4 := -0.0014;
  a5 := 0.0328;
  sc2 := 10000 ;
  w :=1 ;
  h := 20 ;
  h1 :=100 ;
  s := 1 ;
  m :=1.0 ;
  l:= 4.5 ;
  tt := 0 ;
  tst1:=1.6;
  writeln(outfile,' L    M    H    N     SC     SO    P    OP    UC    UA    UV
UW');
 for jj :=4 to 9 do
 begin
  u2 :=jj;
 for j :=2 to 3 do
  begin
```

```
u1 :=exp(j*ln(10));
for jjj :=0 to 2 do
 begin
 u3 :=exp(jjj*ln(10));
found := false ;
while not found do
 begin
 hdd1 :=100000;
 for kkkk :=1 to 600 do
  begin
   h := kkkk/10 ;
   a := 2*50*0.1;
   d := (1/360+a*h/60*2/3600)*60 ;
   hdd := -2*c*l*y/(m*h*h)*(1/v+d/s)+(p*y*viv*l*(l/(2*v)+l*d/(2*s))
          +p*(e-l)*y*viv*l/v+l*d/s)*a2*k+vw*p*e
          *y*(a1+a2*(2*k*h+(m+s)/(4*g))+(a3/(2*v)+a5/2)*l+a4*f)
          +p*y*va/g*a2*k*(sqr(e-l)/2+m*e/4+s*l/4)+2*u1*((a1
          +a2*(k*h+(m+s)/(4*g))+(a3/(2*v)+a5/2)*l+a4*f)*e*m*p*h-cap)
          *(a1+a2*(2*k*h+(m+s)/(4*g))+(a3/(2*v)+a5/2)*l+a4*f)*(e*m*p) ;
   if abs(hdd1)>abs(hdd) then
     begin
      hdd1 := hdd ;
      h1 := h;
     end;
  end;
h := h1 ;
writeln('h ',h:9:4);
mdd1:=100000;
for kkkk := 1 to 300 do
  begin
  m := kkkk/100 ;
  a := 2*50*0.1;
  d := (1/360+a*h/60*2/3600)*60 ;
  mdd := -2*c*l*y/(h*m*m)*(1/v+d/s)+p*y*viv*l*(l/(2*v)+l*d/(2*s)
         +(e-l)*(1/v+d/s))*a2/(4*g)+vw*h/2*p*e*y*a2/(4*g)+p*y*va*(e/(4*g)
         *(a1+a2*(k*h+(m+s)/(4*g))+(a3/(2*v)+a5/2)*l+a4*f)+(sqr(e-l)/(2*g)
         +m*e/(4*g)+s*l/(4*g))*a2/(4*g))+2*u1*((a1+a2*(k*h+(m+s)/(4*g))
         +(a3/(2*v)+a5/2)*l+a4*f)*e*m*p*h-cap)*(a1+a2*(k*h+(2*m+s)/(4*g))
         +(a3/(2*v)+a5/2)*l+a4*f)*(e*h*p)+2*u2*(2*(e-l)+m-4*w)+2*u3*(m+s-
4*w) ;
  if  abs(mdd1)>abs(mdd) then
   begin
    mdd1 := mdd ;
    m1 := m
   end;
  end;
 m := m1 ;
 writeln('m ',m:9:4);
 a := 2*50*0.1;
 d := (1/360+a*h/60*2/3600)*60 ;
 sdd1 :=100000;
 for kkkk :=1 to 300 do
  begin
  s := kkkk/100;
```

```
  sdd :=-2*c*l*y*d/(s*s*h*m)+(p*y*viv*l*(l/(2*v)+l*d/(2*s))+p*(e-l)*y*viv
       *(l/v+l*d/s))*a2/(4*g)-(p*l*l*y*viv/(2*s*s)+p*(e-l)*y*viv*d/(s*s))
       *(a1+a2*(k*h+(m+s)/(4*g))+(a3/(2*v)+a5/2)*l+a4*f)+vw*h/2*p*e*y*a2
       /(4*g)+p*y*va/g*(l/(4*G)*(a1+a2*(k*h+(m+s)/(4*g))+(a3/(2*v)+a5/2)
       *l+a4*f)+(sqr(e-l)/(2*g)+m*e/(4*g)+s*l/(4*g))*a2/(4*g))
       +2*u1*((a1+a2*(k*h+(m+s)/(4*g))+(a3/(2*v)+a5/2)*l+a4*f)
       *e*m*h*p-cap)*a2*(e*m*h*p)/(4*g)+2*u3*((m+s)-4*w) ;
  if abs(sdd1)>abs(sdd) then
    begin
     sdd1 := sdd ;
     s1 := s
    end
 end;
 s := s1 ;

writeln('s ',s:8:3);
ldd1 :=100000;
for kkkk :=1 to 500 do
 begin
  l := kkkk/100;
  ldd :=2*c*y/(h*m)*(1/v+d/s)+p*viv*y*((a3/(2*v)+a5/2)*((e-l)*(l/v
       +l*d/s)+(l*l/(2*v)+l*l*d/(2*s)))+(a1+a2*(k*h+(m+s)/(4*g))
       +(a3/(2*v)+a5/2)*l+a4*f)*((l/v+l*d/s)-l*(e-l)*sqr(1/v+d/s)))
       +vw*h/2*p*e*y*(a3/(2*v)+a5/2)+p*y*va*((a3/(2*v)+a5/2)
       *(sqr(e-l)/(2*g)+m*e/(4*g)+s*l/(4*g))+(a1+a2*(k*h+(m+s)/(4*g))
       +(a3/(2*v)+a5/2)*l+a4*f)*(s/(4*g)-(e-l)/g))
       +2*u1*((a1+a2*(k*h+(m+s)/(4*g))+(a3/(2*v)+a5/2)*l+a4*f)
       *e*m*p*h-cap)*e*m*p*h*(a3/(2*v)+a5/2)-4*u2*(2*(e-l)+m-4*w) ;
  if abs(ldd1)>abs(ldd) then
    begin
     ldd1 := ldd ;
     l1 := l
    end
 end;
 l := l1 ;
writeln('l= ',l:8:3);
writeln(u1:6:2,u2:6:2,u3:6:2);
sc1 := 2*c*l*y*(1/v+d/s)/(h*m)+p*(a1+a2*(k*h+(m+s)/(4*g))+(a3/(2*v)
      +a5/2)*l+a4*f)*y*viv*l*((l/(2*v)+l*d/(2*s))+(e-l)*(1/v+d/s))
      +vw*h/2*p*(a1+a2*(k*h+(m+s)/(4*g))+(a3/(2*v)+a5/2)*l+a4*f)*e*y
      +p*Y*va*(a1+a2*(k*h+(m+s)/(4*g))+(a3/(2*v)+a5/2)*l+a4*f)*(sqr(e-l)
      /(4*g)+m*e/(4*g)+s*l/(2*g))+u1*sqr((a1+a2*(k*h
      +(m+s)/(4*g))+(a3/(2*v)+a5/2)*l+a4*f)*e*m*h*p-cap)+u2*sqr(2*(e-l)
      +m-4*w)+u3*sqr(m+s-4*w);
so := 2*c*l*y*(1/v+d/s)/(h*m)+p*(a1+a2*(k*h+(m+s)/(4*g))+(a3/(2*v)
      +a5/2)*l+a4*f)*y*viv*l*((l/(2*v)+l*d/(2*s))+(e-l)*(1/v+d/s))
      +vw*h/2*p*(a1+a2*(k*h+(m+s)/(4*g))+(a3/(2*v)+a5/2)*l+a4*f)*e*y
      +p*Y*va*(a1+a2*(k*h+(m+s)/(4*g))+(a3/(2*v)+a5/2)*l+a4*f)*(sqr(e-l)
      /(4*g)+m*e/(4*g)+s*l/(2*g));
op:=2*c*l*y*(1/v+d/s)/(h*m);
uc:=p*(a1+a2*(k*h+(m+s)/(4*g))+(a3/(2*v)
   +a5/2)*l+a4*f)*y*viv*l*((l/(2*v)+l*d/(2*s))+(e-l)*(1/v+d/s))
   +vw*h/2*p*(a1+a2*(k*h+(m+s)/(4*g))+(a3/(2*v)+a5/2)*l+a4*f)*e*y
   +p*Y*va*(a1+a2*(k*h+(m+s)/(4*g))+(a3/(2*v)+a5/2)*l+a4*f)*(sqr(e-l)
```

```
        /(4*g)+m*e/(4*g)+s*l/(2*g));
    pp:=(a1+a2*(k*h+(m+s)/(4*g))+(a3/(2*v)+a5/2)*l+a4*f)*e*m*h*p/h*60;
    uv:=p*(a1+a2*(k*h+(m+s)/(4*g))+(a3/(2*v)
        +a5/2)*l+a4*f)*y*viv*l*((l/(2*v)+l*d/(2*s))+(e-l)*(1/v+d/s));

    ua:=p*Y*va*(a1+a2*(k*h+(m+s)/(4*g))+(a3/(2*v)+a5/2)*l+a4*f)*(sqr(e-l)
        /(4*g)+m*e/(4*g)+s*l/(2*g));
    uw:=vw*h/2*p*(a1+a2*(k*h+(m+s)/(4*g))+(a3/(2*v)+a5/2)*l+a4*f)*e*y;
     writeln('sc1 ',sc1:9:3,'so ',so:9:3, 'op',op:9:3);
     if (sc1/so<1.1) and (sc1/so>0.9) then
     begin
      found := true ;
    a := 2*50*0.1;
    d := (1/360+a*h/60*2/3600)*60 ;
    n:=0;
    for x1:=50 downto 0 do
     begin
      x:=x1/10 ;
      stt1 :=1000000;
      for kkkk :=1 to 200 do
       begin
        sk := kkkk/100;
        stt :=-2*c*d/(sk*sk*h*m)+2*va*p*(a1+a2*(k*h+(m+2*sk)/(4*g))
             +(a3/(2*v)+a5/2)*l+a4*f)/(4*g)-2*viv*d*p*(e-x)/sqr(sk)*(a1
             +a2*(k*h+m/(4*g))+(a3/(2*v)+a5/2)*l+a4*f) ;
        if abs(stt1)>abs(stt) then
         begin
          stt1 := stt ;
          s1 := sk;
         end
       end;
       nn1:=0.1/s1;

       n:=n+nn1;
     end;
      writeln(s:8:3,n:10:3);
      writeln(outfile,l:5:2,m:5:2,h:6:2,n:5:2,sc1:8:2,
             so:8:2,pp:7:2,op:7:2,uc:7:2,ua:6:2,uv:6:1,uw:6:1);
     end
  end;
  end;
 end;
 end;
 close(outfile)
 end.
```

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