ABSTRACT

Three-Dimensional Beam-Columns

by
Mohammad H. Shams

This thesis explores several special cases of three-dimensional beam-columns and suggests that the method of finite differences can be used when a solution of general case is required. The thesis begins with a review of the three-dimensional beam-column equations and then shows how these equations can be used to generate the member stiffness matrix for the nonlinear analysis of three-dimensional frames. Examples are included which discuss the differences between the analytical and numerical solutions.

THREE-DIMENSIONAL BEAM-COLUMNS

by

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This thesis is dedicated
to my parents.

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CHAPTER 1

INTRODUCTION

Typically, the term beam-column implies a two-dimensional beam in which there is bending about a single axis. This thesis is concerned with a straight three-dimensional prismatic beam in which there is bending about two axes and torsion about the longitudinal axis. In beam-column problems the member stiffness matrix becomes a function of axial load (1), and becomes singular at the flexural buckling load.

Three-dimensional beams have the potential of many complex beam-column like interactions. It is the intention here to show how all possible interactions can be included in the member stiffness matrix for a beam element to be used in a general computer program for three-dimensional frame analysis.

Of various interactions which can occur in three-dimensional beam problems, the best known is probably the phenomenon of torsional (lateral) buckling (2). In the most simple case, the presence of a bending moment about the strong axis of a beam weakens the effective stiffness about the weak axis (stress softening), leading eventually to lateral buckling. The presence of axial compression makes matters worse. Biot (3) in his classic book discusses another case of interaction in which axial tension increases torsional stiffness in a prismatic bar.

In two-dimensional beam-columns, the softening effect of axial compression upon member stiffness is obvious, while the softening effect of bending moment in lateral buckling problems in three-dimensional beam-columns is less so. It will be shown here that the entire initial state of stress in a member can interact with the member stiffness matrix in a rather complex manner. Since any incremental analysis has its initial state of stress, which is not simply axial load, there is every reason to include the appropriate "beam-column" effects.

In general, in three-dimensional beam-column problems, computing the member stiffness matrix is so complicated that an analytical solution is impossible. This thesis describes a general approach which subsumes all the effects cited above. A procedure for computing the member stiffness matrix for three-dimensional beams is presented in which four coupled differential equations are solved numerically using the method of finite differences. In order to find the 6 x 6 member stiffness matrix, this system of equations must be solved six times. Several special cases are also discussed.

CHAPTER 2

THE EQUATIONS OF THREE-DIMENSIONAL BEAM-COLUMNS

A perturbation method is used (4) to describe beam response. That is, a given straight beam in equilibrium under given forces is subjected to a small load perturbation. This approach produces the tangent stiffness now common in nonlinear analysis and describes buckling as the response to the load perturbation becomes singular.

The interest here is primarily three-dimensional beams. While there is a good understanding of two-dimensional problems and even three-dimensional cable nets, three-dimensional beams are still the subject of some discussion (5). The analysis below is straightforward, if cumbersome. The starting point is undeformed, but prestressed equilibrium in the sense that the initial configuration has internal stress-resultants associated with it (6). A small deformation is then superimposed upon the initial configuration and zero- and first-order solutions constructed with products of small terms neglected. Assuming that the initial configuration is straight and undeformed, the first-order solutions then return the expected theories for beam-column and torsional buckling (5) as special cases.

Small parameter analysis: Given an arbitrary beam, its equilibrium in its initial (fig. 1) and perturbed configurations can be described as (4):

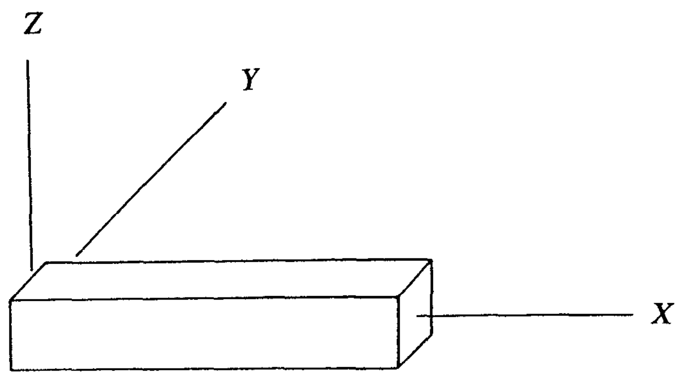

Figure 1 Typical three-dimensional beam segment.

$$P^{0\prime} + p^{0} = 0, \qquad P^{1\prime} + p^{1} = 0, \tag{1}$$
$$M^{0\prime} + m^{0} + t^{0} \times P^{0} = 0, \qquad M^{1\prime} + m^{1} + t^{1} \times P^{1} = 0.$$

Initial Configuration Perturbed Configuration

Here **P** and **M** are the usual force and moment stress-resultants, with **p** and **m** the applied forces and moments, the superscripts refer to the configuration, and the prime symbol refers to the differentiation with respect to arc length. The vector **t** is of course the unit tangent vector, which will also be referred to as **i** in the applications cited below. In component form

$$P = P_x i + P_y j + P_z k,$$
$$M = M_x i + M_y j + M_z k, \tag{2}$$

and the base vectors $i\ j\ k$ selected so that P_x and M_x represent thrust and torque, respectively. Going from the initial to the perturbed configuration, changes are indicated as

$$\boldsymbol{P}^0 \to \boldsymbol{P}^1 = \boldsymbol{P}^0 + \varepsilon\overline{\boldsymbol{P}},$$
$$\boldsymbol{M}^0 \to \boldsymbol{M}^1 = \boldsymbol{M}^0 + \varepsilon\overline{\boldsymbol{M}}, \tag{3}$$
$$\boldsymbol{i}^0 \to \boldsymbol{i}^1 = \boldsymbol{i}^0 + \varepsilon\overline{\boldsymbol{i}},$$

where ϵ is the anticipated small parameter and the bar is used to distinguish the perturbation term. The perturbed terms of Eq. (2) are now inserted into the equilibrium equations of the perturbed configuration given in Eq. (1). In component form these equations are

$$\begin{aligned} P_x^{1\prime}\boldsymbol{i}^1 + P_x^1\boldsymbol{i}^{1\prime} + P_y^{1\prime}\boldsymbol{j}^1 + P_y^1\boldsymbol{j}^{1\prime} + P_z^{1\prime}\boldsymbol{k}^1 \\ + P_z^1\boldsymbol{k}^{1\prime} + p_x^1\boldsymbol{i}^1 + p_y^1\boldsymbol{j}^1 + p_z^1\boldsymbol{k}^1 = 0, \end{aligned} \tag{4}$$

$$\begin{aligned} M_x^{1\prime}\boldsymbol{i}^1 + M_x^{1\prime}\boldsymbol{i} + M_y^{1\prime}\boldsymbol{j}^1 + M_y^1\boldsymbol{j}^{1\prime} + M_z^{1\prime}\boldsymbol{k}^1 \\ + M_z^1\boldsymbol{k}^{1\prime} + m_x^1\boldsymbol{i}^1 + m_y^1\boldsymbol{j}^1 + m_z^1\boldsymbol{k}^1 + \boldsymbol{t}^1 \times \boldsymbol{P}^1 = 0. \end{aligned} \tag{5}$$

In order to complete the analysis, it is simply a matter of inserting the definition of the perturbations (Eq. (3)) into Eqs. (4-5) and collecting terms in the coordinate directions. Before doing so, it is convenient to introduce displacements in the following manner. Let ω represent the rotation vector associated with any beam element. If in the initial configuration the beam is straight and lies along the x axis, for the case of small rotations ω can be written as

$$\omega = \theta_x \boldsymbol{i} - \delta_{z,x}\boldsymbol{j} + \delta_{y,x}\boldsymbol{k} \tag{6}$$

where θ is the torsional rotation, δ_z is the z component of the beam displacement vector, δ_y is the y component of the beam displacement vector, and the comma is used to indicate differentiation. The rotation vector can, of course, be used to determine the changes of the base vectors as

$$
\begin{aligned}
\boldsymbol{i}^0 \rightarrow \boldsymbol{i}^1 &= \boldsymbol{i}^0 + \varepsilon\bar{\boldsymbol{i}} = \boldsymbol{i}^0 + \omega \times \boldsymbol{i}^0, \\
\boldsymbol{j}^0 \rightarrow \boldsymbol{j}^1 &= \boldsymbol{j}^0 + \varepsilon\bar{\boldsymbol{j}} = \boldsymbol{j}^0 + \omega \times \boldsymbol{j}^0, \\
\boldsymbol{k}^0 \rightarrow \boldsymbol{k}^1 &= \boldsymbol{k}^0 + \varepsilon\bar{\boldsymbol{k}} = \boldsymbol{k}^0 + \omega \times \boldsymbol{k}^0.
\end{aligned} \tag{7}
$$

Clearly

$$
\begin{aligned}
\omega \times \boldsymbol{i}^0 &= \delta_{y,x}\boldsymbol{j}^0 + \delta_{z,x}\boldsymbol{k}^0, \\
\omega \times \boldsymbol{j}^0 &= -\delta_{y,x}\boldsymbol{i}^0 + \theta_x\boldsymbol{k}^0, \\
\omega \times \boldsymbol{k}^0 &= -\delta_{z,x}\boldsymbol{i}^0 - \theta_x\boldsymbol{j}^0.
\end{aligned} \tag{8}
$$

$$
\boldsymbol{t} \times \boldsymbol{P} = (\boldsymbol{i}^0 + \omega \times \boldsymbol{i}^0)\,(\boldsymbol{P}^0 + \varepsilon\bar{\boldsymbol{P}}) \tag{9}
$$

After inserting Eqs. (7-8-9) in Eqs. (4-5), they can be written as

$$
\begin{aligned}
&(P_x^{0\prime} + \varepsilon\bar{P}_x{}')(\boldsymbol{i}^0 + \delta_{y,x}\boldsymbol{j}^0 + \delta_{z,x}\boldsymbol{k}^0) + (P_x^0 + \varepsilon\bar{P}_x)(\delta_{y,xx}\boldsymbol{j}^0 + \delta_{z,xx}\boldsymbol{k}^0) \\
&+(P_y^{0\prime} + \varepsilon\bar{P}_y{}')(\boldsymbol{j}^0 - \delta_{y,x}\boldsymbol{i}^0 + \theta_x\boldsymbol{k}^0) + (P_y^0 + \varepsilon\bar{P}_y)(-\delta_{y,xx}\boldsymbol{i}^0 + \theta_{x,x}\boldsymbol{k}^0) \\
&+(P_z^{0\prime} + \varepsilon\bar{P}_z{}')(\boldsymbol{k}^0 - \delta_{z,x}\boldsymbol{i}^0 - \theta_x\boldsymbol{j}^0) + (P_z^0 + \varepsilon\bar{P}_z)(-\delta_{z,xx}\boldsymbol{i}^0 - \theta_{x,x}\boldsymbol{j}^0) \\
&+(P_x^0 + \varepsilon\bar{P}_x)(\boldsymbol{i}^0 + \delta_{y,x}\boldsymbol{j}^0 + \delta_{z,x}\boldsymbol{k}^0) + (P_y^0 + \varepsilon\bar{P}_y)(\boldsymbol{j}^0 - \delta_{y,x}\boldsymbol{i}^0 + \theta_x\boldsymbol{k}^0) \\
&+(P_z^0 + \varepsilon\bar{P}_z)(\boldsymbol{k}^0 - \delta_{z,x}\boldsymbol{i}^0 - \theta_x\boldsymbol{j}^0) = 0
\end{aligned} \tag{10}
$$

$$
\begin{aligned}
&(M_x^{0\prime} + \varepsilon\bar{M}_x{}')(\boldsymbol{i}^0 + \delta_{y,x}\boldsymbol{j}^0 + \delta_{z,x}\boldsymbol{k}^0) + (M_x^0 + \varepsilon\bar{M}_x)(\delta_{y,xx}\boldsymbol{j}^0 + \delta_{z,xx}\boldsymbol{k}^0) + \\
&(M_y^{0\prime} + \varepsilon\bar{M}_y{}')(\boldsymbol{j}^0 - \delta_{y,x}\boldsymbol{i}^0 + \theta_x\boldsymbol{k}^0) + (M_y^0 + \varepsilon\bar{M}_y)(-\delta_{y,xx}\boldsymbol{i}^0 + \theta_{x,x}\boldsymbol{k}^0) + \\
&(M_z^{0\prime} + \varepsilon\bar{M}_z{}')(\boldsymbol{k}^0 - \delta_{z,x}\boldsymbol{i}^0 - \theta_x\boldsymbol{j}^0) + (M_z^0 + \varepsilon\bar{M}_z)(-\delta_{z,xx}\boldsymbol{i}^0 - \theta_{x,x}\boldsymbol{j}^0) + \\
&(M_x^0 + \varepsilon\bar{M}_x)(\boldsymbol{i}^0 + \delta_{y,x}\boldsymbol{j}^0 + \delta_{z,x}\boldsymbol{k}^0) + (M_y^0 + \varepsilon\bar{M}_y)(\boldsymbol{j}^0 - \delta_{y,x}\boldsymbol{i}^0 + \theta_x\boldsymbol{k}^0) + \\
&(M_z^0 + \varepsilon\bar{M}_z)(\boldsymbol{k}^0 - \delta_{z,x}\boldsymbol{i}^0 - \theta_x\boldsymbol{j}^0) + (\boldsymbol{i}^0 + \delta_{y,x}\boldsymbol{j}^0 + \delta_{z,x}\boldsymbol{k}^0)[(P_x^0 + \varepsilon\bar{P}_x) + \\
&(P_y^0 + \varepsilon\bar{P}_y) + (P_z^0 + \varepsilon\bar{P}_z)] = 0
\end{aligned} \tag{11}
$$

The first order equations are collected in Table 1 and will be discussed in the next chapter. The zero-order equations are assumed to be satisfied by equilibrium of the initial configuration and will not be discussed here. The equations of Table 1 are obtained by writing the equilibrium equations in the perturbed configuration and by keeping terms which are linear in the small parameter ε. For convenience, this parameter can be set to be one in which case the terms indicated by bars Eq. (3) then represent the full perturbation (5).

It should be noted that the assumptions of small displacement theory have been invoked above to allow arc length differentiation to be replaced by differentiation with respect to the space variable x (5).

Table 1. First-order equations.

Force Equilibrium

$$i\ comp\text{: } \bar{P}_x{}' + \bar{p}_x - P_y^{0\prime}\delta_{y,x} - P_z^{0\prime}\delta_{z,x} - P_y^0\delta_{y,xx} - P_z^0\delta_{z,xx} - p_y^0\delta_{y,x} - p_z^0\delta_{z,x} = 0$$

$$j\ comp\text{: } \bar{P}_y{}' + \bar{p}_y + P_x^{0\prime}\delta_{y,x} - P_z^{0\prime}\theta_x + P_x^0\delta_{y,xx} - P_z^0\theta_{x,x} + p_x^0\delta_{y,x} - p_z^0\theta_x = 0$$

$$k\ comp\text{: } \bar{P}_z{}' + \bar{p}_z + P_x^{0\prime}\delta_{z,x} + P_y^{0\prime}\theta_x + P_x^0\delta_{z,xx} + P_y^0\theta_{x,x} + p_x^0\delta_{z,x} + p_y^0\theta_x = 0$$

Moment Equilibrium

$$i\ comp\text{: } \bar{M}_x{}' + \bar{m}_x - M_y^{0\prime}\delta_{y,x} - M_z^{0\prime}\delta_{z,x} - M_y^0\delta_{y,xx} - M_z^0\delta_{z,xx}$$

$$-m_y^0\delta_{y,x} - m_z^0\delta_{z,x} + P_z^0\delta_{y,x} - P_y^0\delta_{z,x} = 0$$

$$j\ comp\text{: } \bar{M}_y{}' + \bar{m}_y + M_x^{0\prime}\delta_{y,x} - M_z^{0\prime}\theta x + M_x^0\delta_{y,xx} - M_z^0\theta_{x,x}$$

$$+m_x^0\delta_{y,x} - m_z^0\theta x_x - \bar{P}_z - P_y^0\theta_x = 0$$

$$k\ comp\text{: } \bar{M}_z{}' + \bar{m}_z + M_x^{0\prime}\delta_{z,x} + M_y^{0\prime}\theta x + M_x^0\delta_{z,xx} + M_y^0\theta_{x,x}$$

$$+m_x^0\delta_{z,x} + m_y^0\theta_x + \bar{P}_y - P_z^0\theta_x = 0$$

CHAPTER 3

THE MEMBER STIFFNESS MATRIX

The interest here is how the equations of Table 1 might be used in a computer program for the nonlinear analysis of three-dimensional beams. From consideration of equilibrium it can be argued that the member stiffness matrix is a 6 x 6 matrix. Using Newton's method for nonlinear structural analysis, each step (iteration) of the nonlinear analysis becomes simply a linear analysis which uses the local tangent stiffness. Therefore the terms in the member stiffness matrix are "forces" due to unit "displacements." If the member forces are chosen properly the terms in the member stiffness matrix may be computed by introducing sequentially 6 discontinuities into the boundary conditions of the system of equations in Table 1:

- A unit axial discontinuity
- A unit torsional discontinuity
- Four flexural discontinuities

Note that the four flexural discontinuities are those which are used in moment distribution: a unit rotation about an axis of flexure is applied at one end of a beam while the other end is held fixed. Two beam ends and two axes of flexure then imply four flexural discontinuities.

The next step will combine some of the equations in Table 1 to produce a system of four equations in the four displacements $\delta_x, \delta_y, \delta_z$ *and* θ_x.

Furthermore, at this point all the member loads can be eliminated as not of interest.

$$\begin{aligned} m_x^0 &= m_y^0 = m_z^0 = \overline{m}_x = \overline{m}_y = \overline{m}_t = 0 \\ P_x^0 &= P_y^0 = P_z^0 = \overline{P}_x = \overline{P}_y = \overline{P}_z = 0 \end{aligned} \tag{12}$$

This implies that the initial axial thrust and torque, P_x^0, M_x^0 must be constant and that the initial bending moment, M_y^0, M_z^0 can at most be linear functions.

Inserting Eq. (12) in the equations in Table 1, the equations can be written as

Force Eqs.:

$$i\ comp\text{:}\ \overline{P}_x{}' - P_y^{0\prime}\delta_{y,x} - P_z^{0\prime}\delta_{z,x} - P_y^0\delta_{y,xx} - P_z^0\delta_{z,xx} = 0 \tag{13-a}$$

$$j\ comp\text{:}\ \overline{P}_y{}' + P_x^{0\prime}\delta_{y,x} - P_z^{0\prime}\theta_x + P_x^0\delta_{y,xx} - P_z^0\theta_{x,x} = 0 \tag{13-b}$$

$$k\ comp\text{:}\ \overline{P}_z{}' + P_x^{0\prime}\delta_{z,x} + P_y^{0\prime}\theta_x + P_x^0\delta_{z,xx} + P_y^0\theta_{x,x} = 0 \tag{13-c}$$

Moment Equilibrium:

$$\begin{aligned} i\ comp\text{:}\ \overline{M}_x{}' &- M_y^{0\prime}\delta_{y,x} - M_z^{0\prime}\delta_{z,x} - M_y^0\delta_{y,xx} \\ &- M_z^0\delta_{z,xx} + P_z^0\delta_{y,x} - P_y^0\delta_{z,x} = 0 \end{aligned} \tag{13-d}$$

$$\begin{aligned} j\ comp\text{:}\ \overline{M}_y{}' &+ M_x^{0\prime}\delta_{y,x} - M_z^{0\prime}\theta_x + M_x^0\delta_{y,xx} \\ &- M_z^0\theta_{x,x} - \overline{P}_z - P_y^0\theta_x = 0 \end{aligned} \tag{13-e}$$

$$\begin{aligned} k\ comp\text{:}\ \overline{M}_z{}' &+ M_x^{0\prime}\delta_{z,x} + M_y^{0\prime}\theta_x + M_x^0\delta_{z,xx} \\ &+ M_y^0\theta_{x,x} + \overline{P}_y - P_z^0\theta_x = 0 \end{aligned} \tag{13-f}$$

Equation (13-b) can be combined with Eq. (13-f) eliminating shear term $\overline{P}_y$; similarly Eq. (13-c) can be combined with Eq. (13-e) eliminating shear term $\overline{P}_z$. For eliminating $\overline{P}_y$, the differentiation of Eq. (13-f) will subtract from Eq.

(13-b), and the same procedure will be used for eliminating $\overline{P}_z$ in Eqs. (13-c, 13-e). Then:

$$\overline{M}_z'' + 2M_y^{0'}\theta_{x,x} + M_x^0\delta_{z,xxx} + M_y^0\theta_{x,xx} - P_x^0\delta_{y,xx} = 0$$
$$\overline{M}_y'' - 2M_z^{0'}\theta_{x,x} + M_x^0\delta_{y,xxx} - M_z^0\theta_{x,xx} + P_x^0\delta_{z,xx} = 0 \tag{14}$$

To complete the formulation, four constitutive equations are appended:

$$\overline{P}_x = k_x\delta_{x,x}\ ,\ \overline{M}_x = k_T\theta_{x,x}\ ,\ \overline{M}_y = -k_y\delta_{z,xx}\ ,\ \overline{M}_z = k_z\delta_{y,xx} \tag{15}$$

Here the k's are usual spring constants from considerations of strength of materials.

The six equations (13-a to 13-f) then reduce to the following four equations.

$$k_x\delta_{x,xx} - P_y^0\delta_{y,xx} - P_z^0\delta_{z,xx} = 0 \tag{16-a}$$

$$k_T\theta_{x,xx} - M_y^{0'}\delta_{y,x} - M_z^{0'}\delta_{z,x} - M_y^0\delta_{y,xx} - M_z\delta_{z,xx} + P_z^0\delta_{y,x} - P_y^0\delta_{z,x} = 0 \tag{16-b}$$

$$-k_y\delta_{z,xxxx} - 2M_z^{0'}\theta_{x,x} + M_x^0\delta_{y,xxx} - M_z^0\theta_{x,xx} + P_x^0\delta_{z,xx} = 0 \tag{16-c}$$

$$k_z\delta_{y,xxxx} + 2M_y^{0'}\theta_{x,x} + M_x^0\delta_{z,xxx} - M_y^0\theta_{x,xx} - P_x^0\delta_{y,xx} = 0 \tag{16-d}$$

Some general comments on this system of equations can now be made:

- The last three equations are coupled and must be solved simultaneously; then the first equation can be integrated to complete the solution.
- The last two equations are fourth order in the beam displacements

(like the linear elastic beam equations); the other two equations are second order.

- The equations themselves are linear in x since the initial moment diagrams M_y^0, M_z^0 may be linear in x. (Timoshenko (2) remarks that equations of this type may be solved using Bessel functions.)

For computing the member stiffness matrix, this system of equations must be solved six times.

CHAPTER 4

FINITE DIFFERENCE SOLUTIONS

In general, there is not an analytical solution for Eqs. (16). This set of equations can be solved using the method of finite difference. As mentioned before, the last three equations (Eqs. (16-b, c, d)) are coupled and will be solved simultaneously. In this approach each beam is divided into a number of segments. The Eq. (16-c) and Eq. (16-d) are fourth order in displacement, then two fictitious points at each end are required. For any typical point "n", the displacements and their differentiations can be written as:

$$\begin{aligned}
\theta_{x_n} &= \theta_n \\
\theta_{x_n}' &= \frac{1}{2h}(\theta_{n+1} - \theta_{n-1}) \\
\theta_{x_n}'' &= \frac{1}{h^2}(\theta_{n+1} - 2\theta_n + \theta_{n-1})
\end{aligned} \tag{17}$$

$$\begin{aligned}
\delta_{y_n} &= y_n \\
\delta_{y_n}'' &= \frac{1}{h^2}(y_{n+1}-2y_n+y_{n-1}) \\
\delta_{y_n}' &= \frac{1}{2h}(y_{n+1}-y_{n-1}) \\
\delta_{y_n}''' &= \frac{1}{2h^3}(y_{n+2}-2y_{n+1}+2y_{n-1}-y_{n-2}) \\
\delta_{y_n}^{IV} &= \frac{1}{h^4}(y_{n+2}-4y_{n+1}+6y_n-4y_{n-1}+y_{n-2})
\end{aligned} \tag{18}$$

$$\delta_{z_n} = z_n \quad ; \quad \delta_{z_n}'' = \frac{1}{h^2}(z_{n+1}-2z_n+z_{n-1})$$

$$\delta_{z_n}' = \frac{1}{2h}(z_{n+1}-z_{n-1}) \quad ; \quad \delta_{z_n}''' = \frac{1}{2h^3}(z_{n+2}-2z_{n+1}+2z_{n-1}-z_{n-2})$$

$$\delta_{z_n}^{IV} = \frac{1}{h^4}(z_{n+2}-4z_{n+1}+6z_n-4z_{n-1}+z_{n-2})$$

Using Eq. (17) to Eq. (18), the set of Eq. (16) can be written as:

$$(\frac{k_T}{h})\theta_{n+1} - (\frac{2k_T}{h})\theta_n + (\frac{k_T}{h})\theta_{n-1} + (\frac{P_z^0}{2} - \frac{M_y^0}{h})y_{n+1} +$$
$$(-\frac{P_z^0}{2} - \frac{M_y^0}{h})y_{n-1} + (-\frac{P_y^0}{2} - \frac{M_z^0}{h})z_{n+1} - (\frac{2M_z^0}{h})z_n + (\frac{P_y^0}{2} - \frac{M_z^0}{h})z_{n-1} = 0$$

$$(-\frac{M_z^0}{h})\theta_{n+1} + (\frac{2M_z^0}{h})\theta_n + (-\frac{M_z^0}{h})\theta_{n-1} + (\frac{M_x^0}{2h^2})y_{n+2} + (-\frac{M_x^0}{h^2})y_{n+1} +$$
$$(\frac{M_x^0}{h^2})y_{n-1} + (-\frac{M_x^0}{2h^2})y_{n-2} + (\frac{k_y}{h^3})z_{n+2} + (-\frac{4k_y}{h^3} + \frac{P_x^0}{h})z_{n+1} +$$
$$(\frac{6k_y}{h^3} - \frac{2P_x^0}{h})z_n + (-\frac{4k_y}{h^3} + \frac{P_x^0}{h})z_{n-1} + (\frac{k_y}{h^3})z_{n-2} = 0$$

$$(\frac{M_y^0}{h})\theta_{n+1} - (\frac{2M_y^0}{h})\theta_n + (\frac{M_y^0}{h})\theta_{n-1} + (\frac{M_y^0}{2h^2})z_{n+2} + (-\frac{M_y^0}{h^2})z_{n+1} +$$
$$(\frac{M_y^0}{h^2})z_{n-1} - (\frac{M_y^0}{2h^2})z_{n-2} + (\frac{k_z}{h^3})y_{n+2} + (-\frac{4k_z}{h^3} - \frac{P_x^0}{h})y_{n+1} +$$
$$(\frac{6k_z}{h^3} + \frac{2P_x^0}{h})y_n + (-\frac{4k_z}{h^3} - \frac{P_x^0}{h})y_{n-1} + (\frac{k_z}{h^3})y_{n-2} = 0$$

In the computer program which is described in the next chapter, each beam is divided into 20 segments. With two fictitious points at each end, there exist 25 nodes on each beam. Then 75 simultaneous equations are to be solved on a PC for every case.

CHAPTER 5

A COMPUTER PROGRAM FOR THREE-DIMENSIONAL BEAM COLUMNS ANALYSIS

This chapter lists the code of FORTRAN subroutine which will compute the 6 x 6 member stiffness matrix for a three-dimensional beam given the member force matrix and the appropriate stiffness coefficients.

This subroutine divides each beam into 20 segments and then uses the method of central differences to compute the displacements at the resulting nodes. As described in Chapter 2, the displacements θ_x, δ_y and δ_z are coupled, then the set of Eq. (20) is first solved to obtain θ_x, δ_y and δ_z at each node; then Eq. (16-a) is solved for δ_x. In order to generate 6 x 6 member stiffness matrix, it is necessary to solve Eq. (20) six times.

The input to the subroutine STIFF is the member force matrix and some stiffness parameter:

ALEN - member length

AKL - axial stiffness

AKT - torsional stiffness

AKY - y-axis bending stiffness

AKZ - z-axis bending stiffness

FORCE (1) - axial load

FORCE (2) - torsion

FORCE (3) - y-axis bending moment at +end of the member

FORCE (4) - z-axis bending moment at +end of the member

FORCE (5) - y-axis bending moment at -end of the member

FORCE (6) - z-axis bending moment at -end of the member

The subroutines return the 6 x 6 member stiffness matrix AK (I,J).

The first part of the subroutine solves for θ_x, δ_z and δ_y at each node point using Eq. (20) in the form of a linear system AX=b. The linear equation solver SIMQ form the IBM Scientific Subroutine Package (7) is being used for solving the above linear system. The order of the variables in the matrix is that just given, i.e.,

$$x = \begin{bmatrix} \theta_{x_1} \\ \delta_{z_1} \\ \delta_{y_1} \\ \theta_{x_2} \\ \delta_{z_2} \\ \delta_{y_2} \\ . \\ . \\ \vdots \end{bmatrix} \qquad (21\text{-a})$$

with 20 spaces (21 nodes) and 2 fictitious nodes at each end for boundary conditions, the system matrix A is 75 by 75. This implies an unused component θ_{x_i} at each beam and since the torsional response is second order while the bending response is fourth order.

When generating the system matrix A, each term in the differential Eq. (16) puts corresponding terms in A in the program that is done by the PUT subroutines. For example, in order to introduce a unit torsional discontinuity, it is necessary to specify that:

$$\begin{aligned} &\theta_{x_1} = \theta_{x_n} = 0 && \textit{(unused fictitious points)} \\ &\theta_{x_3} = 0 && (\theta_x = 0 \quad @ \quad x = 0) \\ &\theta_{x_{n-2}} = 1 && (\theta_x = 1 \quad @ \quad x = l) \end{aligned} \tag{20-b}$$

In terms of FORTRAN code, these conditions are given as

$$\begin{aligned} &\theta_{x_1} = 0 \rightarrow A(1,1) = 1.0 \quad , \quad B(1) = 0 \\ &\theta_{x_n} = 0 \rightarrow A(2,M-2) = 1.0, \quad B(2) = 0 \\ &\theta_{x_3} = 0 \rightarrow A(3,7) = 1.0 \quad , \quad B(3) = 0 \\ &\theta_{x_{n-2}} = 1 \rightarrow A(4,M-8) = 1.0; \quad B(4) = 1.0 \end{aligned} \tag{20-c}$$

The other boundary conditions follow in a similar fashion.

After the node displacements are computed by the subroutine SIMQ, the subroutines TORQ, BEMDY, BEMDZ are called to find the torque and the bending moments.

The axial response (Eq. (16-a)) is determined in a similar fashion. Finally, reactions are computed as elements of the member stiffness matrix. This, of course, requires corrections since the perturbation method gives results in the deformed coordinate system while the structural analysis program requires components in the member coordinate system.

Appendix B represents a listing of FORTRAN code and output of two special cases.

CHAPTER 6

ANALYTICAL SOLUTIONS FOR SPECIAL CASES

In this chapter, some special cases will be solved using Eqs. (16).

Two-dimensional beam-columns: In a classical example of a beam-column, a two-dimensional simply supported beam is subjected to an axial compression, Figure 2.

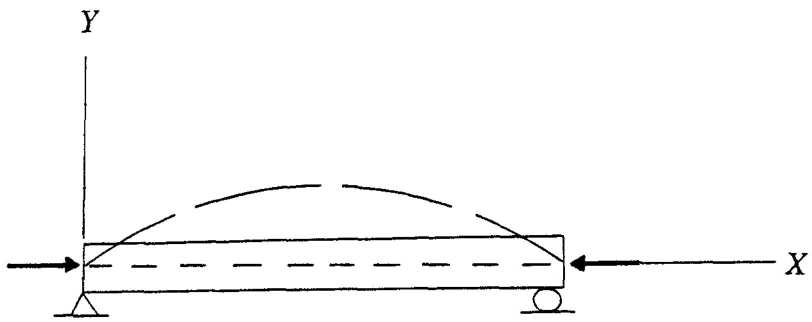

Figure 2 Simply supported beam subjected to axial compression.

In this case, $P_x^0 = -P$ and

$$M_y^0 = M_z^0 = P_y^0 = P_z^0 = 0 \quad ; \quad \delta_z = 0 \tag{21-a}$$

Then the rest of Eq. (16) can be written as:

$$\begin{aligned} &k_x \delta_{x,xx} = 0 \\ &k_T \theta_{x,xx} = 0 \quad ; \quad k_z \delta_y^{IV} + P \delta_{y,xx} = 0 \end{aligned} \tag{21-b}$$

This equation (Eq. (21)) is uncoupled and can be solved easily.

Three-dimensional beam-columns: In three-dimensional beam-column problems, when only an axial load, P_x^0, is present, Eq. (16) is again uncoupled. In this case, the y and z axis bending is controlled by the beam-column equations:

$$k_x\delta_x'' = 0 \quad ; \quad -k_y\delta_z^{IV} + P_x^0\delta_z'' = 0$$
$$k_T\theta_x'' = 0 \quad ; \quad k_z\delta_y^{IV} - P_x^0\delta_y'' = 0 \qquad (22)$$

This is the case which is most commonly used in the nonlinear analysis of three-dimensional frames.

Lateral buckling: If a constant initial moment, M_z^0, is added to the previous case, the member twist (θ_x) becomes coupled with δ_z while δ_x and δ_y remain uncoupled:

$$k_x\delta_x'' = 0 \quad ; \quad k_z\delta_y^{IV} - P_x^0\delta_y'' = 0$$
$$k_T\theta_x'' - M_z^0\delta_z'' = 0 \quad ; \quad -k_y\delta_z^{IV} + \left(P_x^0 - \frac{M_z^{0^2}}{k_T}\right)\delta_z'' = 0 \qquad (23)$$

The last equation is particularly interesting and can be thought of as a beam-column equation in which the constant $(\frac{P_0}{k_y})$ is replaced by $(P_x^0 - \frac{M_z^{0^2}}{k_T})/ky$. In these terms Gere's beam-column charts (1) can also be used in this case.

A more complex case: If an initial constant moment (M_y^0) is added to the problem, fig. (4), θ_x, δ_y and δ_z become fully coupled and it is no longer possible to invoke charts for well-known solutions. In this case the equations take on the form:

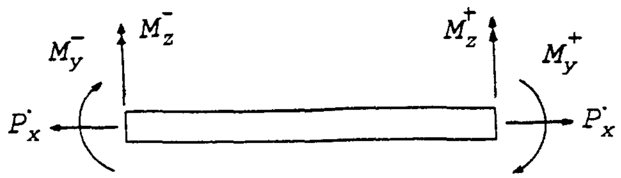

Figure 3 Three-dimensional beam-column subjected to biaxial bending.

$$
\begin{aligned}
&k_x \delta_x'' = 0 \\
&k_T \theta_x'' - M_y^0 \delta y'' - M_z^0 \delta_z'' = 0 \\
&k_y \delta_z^{IV} + M_z^0 \theta_x'' - P_x^0 \delta_z'' = 0 \\
&k_z \delta_y^{IV} + M_y^0 \theta_x'' - P_x^0 \delta_y'' = 0
\end{aligned}
\tag{24}
$$

The last three equations are coupled and must be solved simultaneously. These equations can be written as:

$$
D^2 \begin{bmatrix} k_T & -M_z^0 & -M_y^0 \\ M_z^0 & k_y D^2 - P_x^0 & 0 \\ M_y^0 & 0 & k_z D^2 - P_x^0 \end{bmatrix} \begin{Bmatrix} \theta_x \\ \delta_z \\ \delta_y \end{Bmatrix} = 0 \tag{25}
$$

For nontrivial solutions the determinant of Eq. (25) must be zero, then:

$$
D^6 \langle k_T (k_y D^2 - P_x^0)(k_z D^2 - P_x^0) + M_z^{0^2} (k_z D^2 - P_x^0) + M_y^{0^2} (k_y D^2 - P_x^0) \rangle = 0 \tag{26}
$$

Eq. (26) can be factorized as:

$$
D^6 = 0 \tag{27-a}
$$

$$
\begin{aligned}
&D^4 (k_T k_y k_z) + D^2 (-P_x^0 k_T k_y - P_x^0 k_T k_z + M_z^{0^2} k_z + M_y^{0^2} k_y) + \\
&(k_T P_x^{0^2} - P_x^0 M_z^{0^2} - P_x^0 M_y^{0^2}) = 0
\end{aligned}
\tag{27-b}
$$

If Eq. (27-b) is quadratic with respect to D, then the solution contains 30 coefficients which must be determined, an unlikely choice to be carried out by hand. Appendix A discusses this case in more detail and compares the results with the numerical solution using the method of finite difference.

The effect of axial torsion: The discussion of special cases has been motivated by practical applications and available solutions and therefore has been dominated by the effect of axial load. An alternative would have been to start with the case of initial constant torsion, M_x^0. In this case the set of Eq. (16) takes on the form:

$$
\begin{aligned}
k_x\delta_x'' &= 0\\
k_T\theta_x'' &= 0\\
-k_y\delta_z^{IV} + M_x^0\delta_y'' &= 0\\
k_z\delta_y^{IV} + M_x^0\delta_z'' &= 0
\end{aligned}
\tag{28}
$$

The last two equations are coupled. These two equations will be solved simultaneously, and it is possible to solve the system of equations by hand. This case will be discussed in more detail in appendix A.

CHAPTER 7

CONCLUSION

This thesis has explored several special cases of three-dimensional beam-columns and suggests that the method of finite differences can be used when a solution of the general case is required. While not elegant, the method of finite differences has the added advantage of not being concerned with the signs of coefficients in the equations and thus including both stress hardening and softening in a single algorithm.

The real driving force here is the need to construct a member stiffness matrix for the nonlinear analysis of three-dimensional frames. At the moment, many of the available computer programs and even theoretical discussions such as See and McConnel (8) use two-dimensional beam-column theory to construct such a member stiffness matrix. Clearly such an approach leaves out the effect of lateral buckling. As pointed out here, it also leaves out other possible three-dimensional couplings of initial stress effects.

The finite difference approach has the added advantage of being quite general and including all initial stress interactions. As the results show, the maximum error difference with analytical solutions is 3.4% for solved cases.

APPENDIX A

ANALYTICAL SOLUTION

This appendix details the solution of the special case in which the initial state of stress is one of pure torsion. In this case the displacement δ_y and δ_z are coupled:

$$D^3 \begin{bmatrix} -k_y D & M_x^0 \\ M_x^0 & k_z D \end{bmatrix} \begin{bmatrix} \delta_z \\ \delta_y \end{bmatrix} = 0 \tag{29}$$

Here D represents the derivative d/dx. If the determinant of the system is set to be zero, it follows that

$$D^6(k_y k_z D^2 + M_x^{0^2}) = 0 \tag{30}$$

and that δ_z and δ_y have the form

$$\begin{aligned} \delta &= A_1 + A_2 x + A_3 x^2 + A_4 x^3 + A_5 x^4 + A_6 x^5 + A_7 \sin kx + A_8 \cos kx \\ \delta &= B_1 + B_2 x + B_3 x^2 + B_4 x^3 + B_5 x^4 + B_6 x^5 + B_7 \sin kx + B_8 \cos kx \end{aligned} \tag{31}$$

with $k^2 = M_x^{0^2}/(k_y k_z)$. Coupling of their coefficients implies that

$$A_4 = A_5 = A_6 = B_4 = B_5 = B_6 = 0 \tag{32}$$

and that

$$\begin{aligned} -K_y A_7 k + B_8 M_x^0 &= 0 \\ K_y A_8 k + B_7 M_x^0 &= 0 \end{aligned} \tag{33}$$

Given the eight boundary conditions (flexure about two axes) of this system, the A's and B's can be computed explicitly by hand.

One special case of interest is the case in which a unit y-axis rotation is applies at one end while keeping the other boundary conditions homogeneous. (This case produces the moment distribution member stiffness.) For this case, the eight boundary conditions are as follows:

$$@\ x = 0\ ,\quad \delta_z = 0 \rightarrow\ A_1 + A_8 = 0$$
$$@\ x = l\ ,\quad \delta_2 = 0 \rightarrow\ A_1 + A_2 l + A_3 l^2 + A_7 \sin kl + A_8 \cos kl = 0$$
$$@\ x = 0\ ,\quad \delta_z' = 0 \rightarrow\ A_2 + A_7 = 0$$
$$@\ x = l\ ,\quad \delta_z = 0 \rightarrow\ A_2 + 2A_3 l + A_7 \cos kl - A_8 \sin kl = 0 \qquad (33\text{-a})$$
$$@\ x = 0\ ,\quad \delta_y' = 0 \rightarrow\ B_1 + B_8 = 0$$
$$@\ x = l\ ,\quad \delta_y = 0 \rightarrow\ B_1 + B_2 l + B_3 l^2 + B_7 \sin kl + B_8 \cos kl = 0$$
$$@\ x = 0\ ,\quad \delta_y' = 0 \rightarrow\ B_2 + B_7 = 0$$
$$@\ x = l\ ,\quad \delta_y' = 0 \rightarrow\ B_2 + 2B_3 l + B_7 \cos kl - B_8 \sin kl = 0$$

The ten coefficients can be determined using these eight equations and Eq. (33).

As a numerical example, let $l = 100$, $k_T = 10$, $k_y = k_z = 100$ and $M_x = 0.5$, then the values of A's and B's are as follows:

$$\begin{array}{llll} A_1 = -1189.711 & ; & B_1 = 4661.615 & \\ A_2 = 23.308707 & ; & B_3 = 5.94855 & \\ A_3 = 0.004971 & ; & B_3 = -0.059515 & (33\text{-b}) \\ A_7 = -4661.614 & ; & B_7 = -1189.711 & \\ A_8 = 1189.711 & ; & B_8 = -4661.614 & \end{array}$$

Then

$$\delta_z = -1189.711 + 23.30807x + 0.004971x^2 - 4661.614 \sin kx + 1189.711 \cos kx \qquad (34\text{-a})$$

$$\delta_y = 4661.615 + 5.94855\, x - 0.059515x^2 - 1189.711 \sin kx - 4661.614 \cos kx \qquad (34\text{-b})$$

This problem was also solved numerically by the computer using the method of finite differences. The results are compared in Table 2.

Table 2 Results I

x	δ_y		δ_z	
	Analytical	Numerical	Analytical	Numerical
0	0.00	0.00	0.00	0.00
20	0.31826	0.328771	3.17865	3.21370
40	0.718169	0.732784	9.55834	9.60447
60	0.719593	0.732786	14.35800	14.39558
80	0.32311	0.328772	12.77260	12.78634
100	0.00	0.00	0.00	0.00

As indicated in Table 2, the maximum difference between analytical results and numerical results is 3.19%.

The next case is a beam subjected to an initial state of constant axial load and constant biaxial bending, M_y^0 and M_z^0. The first of Eq. (24) is trivial. The remaining equations can be written in operator form as

$$D^2 \begin{bmatrix} k_T & -M_z^0 & -M_y^0 \\ M_z^0 & k_y D^2 - P_x^0 & 0 \\ M_y^0 & 0 & k_z D^2 - P_x^0 \end{bmatrix} \begin{Bmatrix} \theta_x \\ \delta_z \\ \delta_y \end{Bmatrix} = 0 \tag{35}$$

The determinantal equation for this system is

$$D^6 \langle D^4(k_T k_y k_z) + D^2(-P_x^0 k_T k_y - P_x^0 k_T k_z + M_z^{0^2} k_z + M_y^{0^2} k_y) + (k_T P_x^{0^2} - P_x^0 M_z^{0^2} - P_x^0 M_y^{0^2}) \rangle = 0 \tag{36}$$

which is basically quadratic in D^2 and thus should be regarded to be accessible. The solution of this system generally has the form

$$\begin{aligned} \theta_x &= A_1 + A_2 x + A_3 x^2 + A_4 x^3 + A_5 x^4 + A_6 x^5 + \\ &\quad A_7 \sin k_1 x + A_8 \cos k_1 x + A_9 \sin k_2 x + A_{10} \cos k_2 x \\ \delta_z &= B_1 + B_2 x + B_3 x^2 + B_4 x^3 + B_5 x^4 + B_6 x^5 + \\ &\quad B_7 \sin k_1 x + B_8 \cos k_1 x + B_9 \sin k_2 x + B_{10} \cos k_2 x \\ \delta_y &= C_1 + C_x + C_3 x^2 + C_4 x^3 + C_5 x^4 + C_6 x^5 + \\ &\quad C_7 \sin k_1 x + C_8 \cos k_1 x + C_9 \sin k_2 x + C_{10} \cos k_2 x \end{aligned} \tag{37}$$

where k_1 and k_2 are the root of the quadratic described above. Coupling of these coefficients requires that the polynomial terms of order quadratic and higher must vanish and also provides eight relationships between the coefficients of the trigonometric terms. The ten boundary conditions then complete the problem statement.

For one particular case where $l = 100$, $k_T = 10$, $k_z = k_y = 100$, $P_x^0 = -0.03$ and $M_z^0 = M_y^0 = 0.5$, the solution of this system has been completed. In this case there was at most a 3.4% difference with a numerical solution generated using the finite difference method (Table 2).

Table 3 Results II

x	θ_x		δ_z		δ_y	
	Analytical	Numerical	Analytical	Numerical	Analytical	Numerical
0	0.00	0.00	0.00	0.00	0.00	0.00
20	0.19529	0.19696	3.66379	3.69509	0.24090	0.24423
40	0.58708	0.58956	11.01302	11.05138	0.72620	0.73984
60	0.85278	0.85502	16.15949	16.18721	0.89259	0.91327
80	0.70911	0.71022	13.71975	13.72970	0.45764	0.47483
100	0.00	0.00	0.00	0.00	0.00	0.0

APPENDIX B

PROGRAM AND OUTPUT

```
C       Dummy main program to access subroutine
        DIMENSION FORCE(6),AK(6,6)
        OPEN(UNIT=6,FILE="BMGEN.OUT")
        READ(5,1) AKL,AKT,AKY,AKZ,ALEN,FORCE
        WRITE(6,1)AKL,AKT,AKY,AKY,ALEN,FORCE
      1 FORMAT(5F5.0/6F5.2)
        CALL STIFF(ALEN,AKL,AKT,AKY,AKZ,FORCE,AK)
        WRITE(6,75)((AK(I,J),J=1,6),I=1,6)
     75 FORMAT(6E18.8)
        STOP
        END
C
        SUBROUTINE STIFF(ALEN,AKL,AKT,AKY,AKZ,FORCE,AK)
C       GENERALIZED MEMBER STIFFNESS
        COMMON A,B,C,A1
        COMMON /PROP/ H
        DIMENSION A(75,75),B(75),C(25),A1(25,25),FORCE(6)
        1 ,AK(6,6),DISC(6)
        AMY0(X)=PZ0     *X-FORCE(5)
        AMZ0(X)=-PY0           *X-FORCE(6)
        PY0=-(FORCE(4)+FORCE(6))/ALEN
        PZ0=(FORCE(3)+FORCE(5))/ALEN
        P=FORCE(1)
        TOR=FORCE(2)
        NSPACES=24
        N=NSPACES+1
        M=3*N
        N2=N-2
        H=ALEN/FLOAT(NSPACES-4)
        DO 22 I=1,6
     22 DISC(I)=0.

        Solve for bending and torsion
        DO 999 ICOL=1,6
        DISC(ICOL)=1.
        DO 1 I=1,m
        B(I)=0.
        DO 1 J=1,M
      1 A(I,J)=0.
        DO 2 I=3,N2
        X=(I-3)
        X=X*H
        CALL PUT4(I,3,3,AKZ)
        CALL PUT1(I,3,1,2.*PZ0)
        CALL PUT3(I,3,2,TOR)
        CALL PUT2(I,3,1,AMY0(X))
        CALL PUT2(I,3,3,-P)
        CALL PUT4(I,2,2,-AKY)
        CALL PUT1(I,2,1,2.*PY0)
        CALL PUT3(I,2,3,TOR)
        CALL PUT2(I,2,1,-AMZ0(X))
        CALL PUT2(I,2,2,P)
        CALL PUT2(I,1,1,AKT)
        CALL PUT1(I,1,3,-PZ0)
        CALL PUT1(I,1,2,PY0)
        CALL PUT2(I,1,3,-AMY0(X))
        CALL PUT2(I,1,2,-AMZ0(X))
```

```
      CALL PUT1(I,1,3,PZ0)
      CALL PUT1(I,1,2,-PY0)
    2 CONTINUE
C     BC FOR THX
      A(1,1)=1.
      A(2,M-2)=1.
      A(3,7)=1.
      A(4,M-8)=1.
      B(4)=DISC(2)
C
C     BC FOR DZ
      A(5,8)=1.
      A(6,M-7)=1.
      A(M,11)=.5/H
      A(M,5)=-.5/H
      B(M)=-DISC(5)
      A(M-1,M-4)=.5/H
      A(M-1,M-10)=-.5/H
      B(M-1)=-DISC(3)
C
C     BC FOR DY
      A(M-2,M-6)=1.
      A(M-3,9  )=1.
      A(M-4,12 )=.5/H
      A(M-4,6  )=-.5/H
      B(M-4)=DISC(6)
      A(M-5,M-3)=.5/H
      a(m-5,M-9)=-.5/H
      B(M-5)=DISC(4)
      CALL SIMQ(A,B,M,KS)
      CALL TORQ(AKT,3,1,ATORM)
      CALL TORQ(AKT,N-2,1,ATORP)
      CALL BENDY(AKY,3,2,AMYM1)
      AMYM1=-AMYM1
      CALL BENDZ(AKZ,3,3,AMZM1)
      AMZM1=-AMZM1
      CALL BENDY(AKY,N-2,2,AMYP1)
      CALL BENDZ(AKZ,N-2,3,AMZP1)
      WRITE(6,66) (I,(B(3*I-3+J),J=1,3),I=1,N)
   66 FORMAT(I5,3E20.8)
C
C     Solve for axial compression
      WRITE(6,67)ATORM,ATORP,AMYM1,AMZM1,AMYP1,AMZP1
   67 FORMAT(6E20.8)
      DO 11 I=1,N
      C(I)=0.
      DO 11 J=1,N
   11 A1(I,J)=0.
      DO 4 I=3,N2
      CALL PUT22(I,1,1,AKL)
      CALL GET2(I,3,VAL)
      C(I)=C(I)+VAL*PY0
      CALL GET2(I,2,VAL)
    4 C(I)=C(I)+VAL*PZ0
      A1(1,1)=1.
      A1(N,N)=1.
      A1(2,3)=1.
```

```
      A1(N-1,N-2)=1.
      c(n-1)=DISC(1)
      CALL SIMQ(A1,C,N,KS)
      WRITE(6,69) (I,C(I),I=1,N)
   69 FORMAT(I5,E20.8)
      CALL THR(AKL,3,PM)
      CALL THR(AKL,N-2,PP)
      WRITE(6,68)PM,PP
   68 FORMAT(2E20.8)
C
C     Components for stiffness matrix
C     Corrections for deformed geometry
      IF(ICOL.NE.2) GO TO 555
      AMYP1= AMYP1-FORCE(4)
      AMZP1= AMZP1+Force(3)
  555 CONTINUE
      IF(ICOL.NE.3) GO TO 556
      AMZP1=-FORCE(2)+AMZP1
      PP=PM
      ATORP=ATORM
  556 CONTINUE
      IF(ICOL.NE.4) GO TO 557
      AMYP1=AMYP1+FORCE(2)
      PP=PM
      ATORP=ATORM
  557 CONTINUE
      IF(ICOL.NE.5) GO TO 558
      AMZM1=AMZM1+FORCE(2)
  558 CONTINUE
      IF(ICOL.NE.6) GO TO 559
      AMYM1=AMYM1-FORCE(2)
  559 CONTINUE
      AK(1,ICOL)=PP
      AK(2,ICOL)=ATORP
      AK(3,ICOL)= AMYP1
      AK(4,ICOL)=AMZP1
      AK(5,ICOL)= AMYM1
      AK(6,ICOL)= AMZM1
      DISC(ICOL)=0.
  999 CONTINUE
      RETURN
      END
C
      SUBROUTINE PUT1(IROW,IEQ,IVAR,COEFF)
      COMMON A
      COMMON /PRO /H
      DIMENSION A(75,75)
      I=3*IROW-3+IEQ
      J=3*IROW-3+IVAR
      A(I,J+3)=A(I,J+3)+COEFF*.5/H
      A(I,J-3)=A(I,J-3)-COEFF*.5/H
      RETURN
      END
C
      SUBROUTINE PUT2(IROW,IEQ,IVAR,COEFF)
      COMMON A
      COMMON /PROP/ H
```

```
      DIMENSION A(75,75)
      I=3*IROW-3+IEQ
      J=3*IROW-3+IVAR
      A(I,J+3)=A(I,J+3)+COEFF/H**2
      A(I,J-3)=A(I,J-3)+COEFF/H**2
      A(I,J  )=A(I,J  )-COEFF*2./H**2
      RETURN
      END
C
      SUBROUTINE PUT22(IROW,IEQ,IVAR,COEFF)
      COMMON A,B,C,A1
      COMMON /PROP/ H
      DIMENSION A(75,75),B(75),C(25),A1(25,25)
      I=  IROW-1+IEQ
      J=  IROW-1+IVAR
      A1(I,J+1)=A1(I,J+1)+COEFF/H**2
      A1(I,J-1)=A1(I,J-1)+COEFF/H**2
      A1(I,J  )=A1(I,J  )-COEFF*2./H**2
      RETURN
      END
C
      SUBROUTINE PUT3(IROW,IEQ,IVAR,COEFF)
      COMMON A
      COMMON /PROP/ H
      DIMENSION A(75,75)
      I=3*IROW-3+IEQ
      J=3*IROW-3+IVAR
      A(I,J+3)=A(I,J+3)-COEFF/H**3
      A(I,J-3)=A(I,J-3)+COEFF/H**3
      A(I,J+6)=A(I,J+6)+COEFF*.5/H**3
      A(I,J-6)=A(I,J-6)-COEFF*.5/H**3
      RETURN
      END
C
      SUBROUTINE PUT4(IROW,IEQ,IVAR,COEFF)
      COMMON A
      COMMON /PROP/ H
      DIMENSION A(75,75)
      I=3*IROW-3+IEQ
      J=3*IROW-3+IVAR
      A(I,J+3)=A(I,J+3)-COEFF*4./H**4
      A(I,J-3)=A(I,J-3)-COEFF*4./H**4
      A(I,J+6)=A(I,J+6)+COEFF   /H**4
      A(I,J-6)=A(I,J-6)+COEFF   /H**4
      A(I,J  )=A(I,J  )+COEFF*6./H**4
      RETURN
      END

      SUBROUTINE TORQ(AKT,NODE,IVAR,TOR)
      COMMON /PROP/H
      COMMON A,B
      DIMENSION A(75,75),B(75)
      I=3*NODE-3+IVAR
      TOR=AKT*(B(I+3)-B(I-3))*.5/H
      RETURN
      END
```

```
      SUBROUTINE BENDY(AKY,NODE,IVAR,BEND)
      COMMON /PROP/H
      COMMON A,B
      DIMENSION A(75,75),B(75)
      I=3*NODE-3+IVAR
      BEND=-AKY*(B(I+3)+B(I-3)-2.*B(I))/H**2
      RETURN
      END
C
      SUBROUTINE BENDZ(AKZ,NODE,IVAR,BEND)
      COMMON /PROP/H
      COMMON A,B
      DIMENSION A(75,75),B(75)
      I=3*NODE-3+IVAR
      BEND=AKZ*(B(I+3)+B(I-3)-2.*B(I))/H**2
      RETURN
      END
C
      SUBROUTINE GET2(NODE,IVAR,VAL )
      COMMON /PROP/H
      COMMON A,B
      DIMENSION A(75,75),B(75)
      I=3*NODE-3+IVAR
      VAL = (B(I+3)+B(I-3)-2.*B(I))/H**2
      RETURN
      END
C
      SUBROUTINE THR(AKL,NODE,P)
      COMMON /PROP/H
      COMMON A,B,C
      DIMENSION A(75,75),B(75),C(25)
      I=NODE
      P    =AKL*(C(I+1)-C(I-1))*.5/H
      RETURN
      END
C
      SUBROUTINE SIMQ(A,B,N,KS)
      DIMENSION A(1),B(1)
C
C     FORWARD SOLUTION
C
      TOL=0.0
      KS=0
      JJ=-N
      DO 65 J=1,N
      JY=J+1
      JJ=JJ+N+1
      BIGA=0
      IT=JJ-J
      DO 30 I=J,N
C
C     SEARCH FOR MAXIMUM COEFFICIENT IN COLUMN
C
      IJ=IT+I
      IF(ABS(BIGA)-ABS(A(IJ))) 20,30,30
   20 BIGA=A(IJ)
      IMAX=I
   30 CONTINUE
```

```
C     TEST FOR PIVOT LESS THAN TOLERANCE (SINGULAR MATRIX)
C
      IF(ABS(BIGA)-TOL) 35,35,40
   35 KS=1
      RETURN
C
C     INTERCHANGE ROWS IF NECESSARY
C
   40 I1=J+N*(J-2)
      IT=IMAX-J
      DO 50 K=J,N
      I1=I1+N
      I2=I1+IT
      SAVE=A(I1)
      A(I1)=A(I2)
      A(I2)=SAVE
C
C     DIVIDE EQUATION BY LEADING COEFFICIENT
C
   50 A(I1)=A(I1)/BIGA
      SAVE=B(IMAX)
      B(IMAX)=B(J)
      B(J)=SAVE/BIGA
C
C     ELIMINATE NEXT VARIABLE
C
      IF(J-N) 55,70,55
   55 IQS=N*(J-1)
   DO 65 IX=JY,N
      IXJ=IQS+IX
      IT=J-IX
      DO 60 JX=JY,N
      IXJX=N*(JX-1)+IX
      JJX=IXJX+IT
   60 A(IXJX)=A(IXJX)-(A(IXJ)*A(JJX))
   65 B(IX)=B(IX)-(B(J)*A(IXJ))
C
C     BACK SOLUTION
C
   70 NY=N-1
      IT=N*N
      DO 80 J=1,NY
      IA=IT-J
      IB=N-J
      IC=N
      DO 80 K=1,J
      B(IB)=B(IB)-A(IA)*B(IC)
      IA=IA-N
   80 IC=IC-1
      RETURN
      END
```

L	Kt	Ka	Ky	Kz
100.	10.	100.	100.	100.

F1	F2	F3	F4	F5	F6
.00	.50	.00	.00	.00	.00

	TETA(x)	DALTA(z)	DALTA(y)
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2	.00000E+00	.00000E+00	.00000E+00
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6	.00000E+00	.00000E+00	.00000E+00
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MEMBER STIFFNESS MATRIX :

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