ABSTRACT

Title of Thesis:	The Theory of Bootstrapped Algorithms and Their Applications to Cross Polarization Interference Cancelation
Abdulkadir Dinç	Doctor of Philosophy, 1991
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Dual-polarized transmission has become an important method for frequency re-use, particularly in satellite and microwave radio communication. Nevertheless, cross-polarization interference, which is inherent to this method, may cause degradation in system performance.

Different canceler structures have been proposed to mitigate the effect of cross-polarization. Among these are the diagonalizer, the least mean square (LMS) canceler and the bootstrapped cancelers. Bootstrapped canceler schemes have been proposed and implemented in different applications, such as satellites, tactical communications, and quadrature amplitude madulation (QAM) dual polarized microwave radio. Nevertheless, no attempt was made in the past to quantify the probability of error of dual polarized transmission systems when such cancelers are used, nor were important issues such as stability and the dynamic behavior of algorithms controlling such cancelers studied.

In this thesis, the error probability performance of dual polarized QAM transmission, for nondispersive fading channels and different configurations of bootstrapped cross-pol cancelers, is derived and compared to the performance for other cancelers. Stability analyses of different canceler configurations are investigated, and an application of orthogonal perturbation sequences in controlling the bootstrapped cancelers is considered.

It is shown that the error probability performance of the bootstrapped canceler is always better than that of other cancelers. such as the LMS canceler. It is also shown that. when the bootstrapped canceler is designed to meet certain conditions. it is asymptotically stable in converging to the calculated optimal points. Controlling the cancelers with adaptive algorithms using orthogonal dithering sequences is shown to be satisfactory; the canceler converges in the mean to the optimal condition.

The results indicate that bootstrapped algorithms are faster than other algorithms. Considering the fact such cancelers do not require decision feedback for their operation. we can conclude that bootstrapped algorithms are not only advantageous for cross polarization cancelation. but perhaps suitable for other adaptive signal processing applications, as well.

The Theory of Bootstrapped Algorithms and Their Applications to Cross Polarization Interference Cancelation

by

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Sevgili annem ve babama,

To my Parents

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Chapter 1

INTRODUCTION

In recent years, multi-level modulation and dual-polarization techniques have been applied to radio communication networks to increase the transmission capacity of limited bandwidth channels. Although multi-level modulation techniques have led to a considerable increase in spectral efficiency, in some applications, such as satellite communication and microwave radio, frequency re-use utilizing dually polarized transmission has become an indispensible technique that doubles the channel's capacity.

In a dual polarized transmission system, the available bandwidth is doubled by modulating the same carrier frequency with two independent information signals. The modulated signals are then transmitted through the channel, with one having vertical polarization (denoted by **V**) and the other having horizontal polarization (denoted by **H**). Because of antenna imperfections and/or non ideal transmission channel conditions (caused, for example, by fading), the received signals are not perfectly orthogonal. Therefore, cross-polarization coupling of each information signal into the other is created causing, in some cases, severe degradation in performance.

Channel models for dual polarized systems have been proposed by many authors [1-6]. It has been observed that in digital radio communication, the channel responses for both main polarization (co-pol) and cross-polarization (cross-pol) are slowly time-variant, and sometimes even dispersive.

Clearly, cross-polarization interference, like co-channel inband interference from other sources, causes errors in information extraction. Many approaches have been suggested to mitigate the effect of cross-pol interference. All involve adaptive canceling algorithms which can track slowly varying parameters of the channel and hence utilize the dually polarized channels more efficiently. Some of these methods assume nondispersive channels [1,3,4,7,8,9,10], as in satellite communications, while the others include dispersion in the channel's response, as in point-to-point microwave radio [5,6,11].

Nichols et. al. [7] proposed two adaptive cancelation algorithms; one is based on maximum likelihood detection (MLD), while the other is the least mean square (LMS) algorithm. QPSK modulation transmission through nondispersive fading channels is assumed. It is shown that implementing these adaptive algorithms at baseband avoids the the need for beacon signals, as in [1], or training sequences to find the parameters of the channel. It is also shown that the performance obtained by using these algorithms to eliminate cross-channel interference is significantly better than that obtained without cancelers. The performances of the two cancelers are compared. The authors also consider implementation problems and conclude that, while MLD can be implemented either at IF or baseband, the LMS canceler is implementable only at baseband. Steinberger [3] later suggested a recursive equalization algorithm that operates at RF and is used with dual polarized 8-level phase shift keying (8-PSK) signals.

Amitay [4] proposed an interference cancellation model which diagonalizes the overall channel matrix. Such a cross polarization interference canceler, termed a diagonalizer, will be discussed in more detail in chapter 2. Amitay considered 8-PSK modulation through a nondispersive fading channel.

Kavehrad [9] studied the performance of Amitay's diagonalizer [4] and the least mean square canceler (LMS) of [7] at baseband, for the case of dual-polarized M-ary quadrature amplitude modulation (M-QAM) signals. Performance results for these systems were obtained by deriving an average probability of error as a function of signal-to-noise ratio. The analysis is confined to a nondispersive fading channel. Kavehrad obtained Chernoff bounds for dual-polarized 16 QAM signals. He first considers the system performance without cancelation, then adding the LMS canceler or the diagonalizer suggested by Amitay, he shows the amount of improvement in performance and concludes that the LMS cross-polarization canceler outperforms the diagonalizer. In his analysis, Kavehrad completely ignores the effect of noise on the optimal weights for the diagonalizer, whereas he includes these effects in the LMS canceler case. One may note that the Chernoff bounds for these two baseband interference cancelers for 16 QAM signals might not be tight.

Brandwood [8] proposed an adaptive cross-pol interference canceler, for a dual polarized satellite communications system, that operates directly at RF. He assumes FM modulation and a nondispersive fading channel. He also presents experimental results.

Bar-Ness et. al. [12] suggested a group of adaptive cross-pol interference cancelers that are controlled by what are termed " bootstrapped algorithms ". They also present measured data which depicts the improvement in performance when using a cross pol canceler on a COMSTAR satellite link that transmits dual-pol QPSK. They emphasize the fact that the three different configurations of bootstrapping adaptive cross-pol cancelers proposed result in power separation, rather than interference cancelation [13]. Their cancelers, as well as Brandwoods's, have a distinct advantage over conventional interference cancelers, such as the LMS canceler and the diagonalizer. Initial carrier and timing acquisition are not a must for these cancelers' satisfactory operations, or at least they become much easier tasks [14]. Therefore, with bootstrapped cancelers, training sequences might not be needed. Nevertheless, as emphasized in [13], there is a need for "discriminators" which loosely distinguish between the two channel information contents.

In this work, the three configurations of bootstrapping cross-pol interference cancelers proposed in [13] are analysed. The performance measure derived is the symbol error probability as a function of receiver input signal-to-noise ratio. Dually polarized M-ary QAM signals are assumed and the channel is taken to be a nondispersive fading channel. Performance evaluations show that the bootstrapping algorithms outperforms the LMS and diagonalizer cancelers when they operate under the same conditions and with the same kind of information.

In chapter 2, the dually polarized M-ary signals are introduced. We also include, some relevant material regarding the LMS canceler and the diagonalizer previously mentioned. The chapter concludes with the presentation of the three bootstrapped cross-pol canceler schemes and their principles of operation.

In chapter 3 to chapter 5, we analyse the three bootstrap canceler configurations: the power-power, correlator-correlator and power-correlator algorithms , respectively. Their performances are evaluated and the results of numerical calculations are shown.

The question of stability is addressed in chapter 6, separately for each canceler configuration. The equilibrium points and the stability conditions for the weights are found.

In chapter 7, we study canceler convergence using simulation. Perturbation (dither) with orthogonal sequences, is used in the weight control process, and convergence in the mean of these weights to their predicted optimal value is shown. The computer experiments are performed for a nondispersive fading channel, and the residue interference power is observed as the weights are controlled adaptively by the perturbation algorithm.

Finally, in chapter 8, the performances of three bootstrapped cancelers are compared. Also shown in this chapter are performance comparisons between the bootstrapped canceler, on one hand, and the LMS canceler and the diagonalizer on the other. We conclude the chapter with a listing of suggested future work.

Chapter 2

DUAL POLARIZED CHANNEL AND CROSS–POLARIZATION CANCELERS

2.1 Introduction

Many transmission systems achieve frequency re-use by transmitting on two orthogonal polarizations. However, the isolation between these polarizations degraded by rain depolarization, channel impairments or antenna imperfections. Some of this depolarization are time varying and can only be eliminated by the use of adaptive interference cancelers. Many methods have been proposed in [1,2,3,4,5] to eliminate the degradation due to cross polarization interference. In this chapter, we first introduce a model for the dual-channel M-ary transmission in section (2.2) and then calculate the error probability caused by cross-polarization interference as well as the noise (section 2.3). Two interference cancelers, namely the diagonalizer and the LMS cancelers are introduced and their error probabilities are estimated in section (2.4). These results are mostly based on previously published material and presented in this work for completeness and convenience of the reader. The bootstrapped cancelers namely, the power-power the correlator-correlator and the power-correlator are introduced in section (2.5), their block diagram are presented and their principle of operation are discussed.

2.2 Dual-Channel M-ary QAM Transmission Model

The model for such channel have been well presented in the literature [1,2,3,4]. M-ary QAM bandpass signals with the same bandwidth and the same center frequency transmitted on two orthogonal channels can be presented as

$$s_i(t) = Re\{\tilde{s}_i(t)\exp(j2\pi f_c t)\} \tag{2.1}$$

where $Re\{\cdot\}$ stands for the real part, f_c denotes the carrier frequency and $\tilde{s}_i(t), i = 1,2$ is the complex envelopes of each of the orthogonal signals, respectively. These complex envelope can be expressed as

$$\tilde{s}_i(t) = \sum_{k=0}^{\infty} I_{ki} h(t - kT) \tag{2.2}$$

where I_{ki} $i = 1,2$ is a complex information symbol which takes on one of M different complex values, where $I_{ki} = I_{kR} + jI_{kI}$, and I_{kR} and I_{kI} (the in-phase and the quadrature component of the carrier), are independent M-ary symbols from the set $\{\pm c, \pm 3c, \ldots, \pm(\sqrt{M} - 1)c\}$. I_{kR} and I_{kI} can each take values equal to $2l - 1 - \sqrt{M}, l = 1, 2, \ldots, \sqrt{M}$. I_{kR} and I_{kI} are assumed to be statistically independent and equiprobable. Also, $h(t)$ is a complex low-pass equivalent of the overall system impulse response, M is the number of signal level of in-phase and quadrature component and c is a constant which determines the distance to the decision boundary from each signal location.

The channel is assumed to be slowly time varying and nondispersive. It accepts two orthogonally independent random data streams $I_1(n)$ and $I_2(n)$. It causes distortion; a fraction of one stream of data is added to the other [4].

In matrix notation the received signal is given by

$$\mathbf{x}(t) = \mathbf{A}\mathbf{s}(t) + \mathbf{n}(t) \tag{2.3}$$

where $\mathbf{A}$ is the dual-channel cross coupling matrix

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \tag{2.4}$$

a_{12} and a_{21} are complex valued constants that denote the channels cross-polarization (interchannel interference) responses. The factors a_{11} and a_{22} denote co-polarization (direct path attenuation) channel constants taken as real valued [3].

In (2.3) $\mathbf{s}(t) = [s_1(t), s_2(t)]^T$ and the noise $\mathbf{n(t)} = [n_1(t), n_2(t)]^T$ with $n_i(t) = n_{iR}(t) + jn_{iI}(t), i = 1, 2$

The received signals, sampled after matched filters, are denoted by;(see Fig 2.1)

$$\begin{aligned} x_1(n) &= a_{11}I_1(n) + a_{12}I_2(n) + n_1(n) \\ x_2(n) &= a_{21}I_1(n) + a_{22}I_2(n) + n_2(n) \end{aligned} \tag{2.5}$$

where $x_1(n)$ and $x_2(n)$ are the sampled received signals at the first and second channels respectively. $I_i(n)$ and $n_i(n)$ are the corresponding signals and noises at these outputs. Also $n_1(n)$ and $n_2(n)$ are independent samples of Gaussian zero mean random process.

The channel coefficients a_{ij} $i = 1, 2,$ $j = 1, 2$ are assumed to vary slowly with respect to the signal rate. These slow variations can be tracked by the adaptive algorithms.

We define the normalized cross-polarization coefficients,

$$\frac{a_{12}}{a_{22}} = r_1 e^{j\phi_1}, \quad \frac{a_{21}}{a_{11}} = r_2 e^{j\phi_2}, \tag{2.6}$$

where r_1, r_2 denote the magnitude of the normalized cross polarization constants and ϕ_1, ϕ_2 are the phases of these constants assumed to be independent and uniformly distributed over $[-\pi, \pi]$.

2.3 Performance of Dual Polarized M-ary QAM System

To estimate the performance of such system and realize the effect of cross-polarization, we will find an estimate for the probability of errors that each output will suffer. As a standard procedure, and based on the kind of signal processing performed at the outputs of receiver, we define decision parameter. In this chapter we will take as decision parameter $\hat{I}_1(n) = \frac{y_1(n)}{a_{11}}$ and $y_1(n) \neq x_1(n)^1$. In next chapter this normalization will be termed "amplitude" only compensation. Define,

$$Z_1(n) \triangleq \hat{I}_1(n) - I_1(n) \tag{2.7}$$

From (2.5) and (2.7), we write

$$Z_1(n) = \frac{a_{12}}{a_{11}} I_2(n) + \frac{n_1(n)}{a_{11}} \tag{2.8}$$

Next, we write $Z_1(n)$ in terms of its real , Z_{1R}, and imaginary,Z_{1I}, parts. For this, assuming $a_{11} = a_{22}$ we use (2.6) in (2.8) and present $I_2(n)$ and $n_1(n)$ in terms of their real and imaginary parts;

$$\begin{aligned} Z_{1R} &= r_1 I_{2R} cos\phi_1 - r_1 I_{2I} sin\phi_1 + \frac{n_{1R}}{a_{11}} \\ Z_{1I} &= r_1 I_{2R} sin\phi_1 + r_1 I_{2I} cos\phi_1 + \frac{n_{1I}}{a_{11}} \end{aligned} \tag{2.9}$$

For a matter of convenience, we dropped in 2.9 the dependence on the sampling time n.

Based on the decision parameters in (2.9), Kavehrad [9] finds the Chernoff bound on the probability of error at the output. He also uses the Gauss quadrature rule [14] to obtain an approximate value of the probability of error.

For the convenience of the reader, we will summarize Kavehrad's GQR procedure;

Define,

$$\begin{aligned} X_I &= r_1 (I_{2R} cos\phi_1 - I_{2I} sin\phi_1) \\ Y &= \frac{n_{1R}}{a_{11}} \end{aligned} \tag{2.10}$$

then

$$Z_{1R} = X_I + Y \tag{2.11}$$

[1] when we add canceler; $y_i(n)$ will be the output of the canceler.

where Y is zero mean Gaussian random variable with variance $\frac{\sigma_n^2}{a_{11}^2}$. Therefore, Z_{1R} is Gaussian with mean X_I and variance $\frac{\sigma_n^2}{a_{11}^2}$. It is possible to show that,

$$P_1(|Z_{1R}| > c|\phi_1, I_{2R}, I_{2I}) = 2Q\Big(\frac{c - X_I}{\sigma}\Big) \tag{2.12}$$

where

$$Q(x) \;=\; \frac{1}{\sqrt{2\pi}} \int_x^\infty exp(\frac{-t^2}{2})\, dt$$

and $\sigma^2 = \frac{\sigma_n^2}{a_{11}^2}$

Using the relation between the probability of error $P_1(e)$ and $P_1(|Z_{1R}| > c)$, we have

$$P_1(e|\phi_1, I_{2R}, I_{2I}) = 2(1 - \frac{1}{\sqrt{M}})Q\Big(a_{11}(c - X_I)\frac{1}{c}\sqrt{\frac{3\;SNR}{M-1}}\Big) \tag{2.13}$$

where we used the well known relation between SNR and M-QAM signal parameter;

$$SNR = \frac{M-1}{3}\frac{c^2}{\sigma_n^2} \tag{2.14}$$

Defining the random variable,

$$\mathrm{x} = \frac{X_I}{c} \tag{2.15}$$

then, (2.13) becomes,

$$P_1(e|x) = 2(1 - \frac{1}{\sqrt{M}})Q\Big(a_{11}(1 - \mathrm{x})\sqrt{\frac{3\;SNR}{M-1}}\Big) \tag{2.16}$$

The average probability can be approximated, using the GQR (see appendix B) by;

$$P_1(e) \approx 2(1 - \frac{1}{\sqrt{M}})\sum_{i=1}^{N} w_i Q\Big(a_{11}\sqrt{\frac{3\;SNR}{M-1}}(1 - x_i)\Big) \tag{2.17}$$

where x_i nd w_i are the nodes and weights of the GQR which can be obtained from the moments of the random variable x defined in (2.15) together with (2.10).

Based on (2.17,) we calculate and present in Fig. 2.2, the probability of error at the output of a channel for 16 QAM dually polarized signals as a function of the transmitter signal to noise ratio. The cross-coupling assumed to be -15 dB. 32 moments were used when applying the GQR in (2.17).

2.4 Cross-Polarization Cancelers

We notice from (2.5) that the output of the channel contains beside the noise term, an interference (cross polarization) through the cross coupling a_{12} and a_{21}. These interference undoubtly causes degradation of performance. Several different canceler structures were proposed to mitigate the effect of cross-polarization. Among these are the diagonalizer [4] and the LMS [7] and the bootstrapped canceler [12].

For the convenient of the reader, we will summarize Kavehrad's approach to estimate the performances of the diagonalizer and the LMS cross-pol cancelers. We will introduce the decision parameters of these cancelers as they have been derived for M-ary QAM system by Kavehrad; follow his derivation for the Chernoff bound on the probability of error and compares the results of these two algorithms with each other, for different cross-couplings. It should be noticed that the decision parameters for the outputs were derived in [9] under the assumption that only amplitude compensation is used at the output. Also, it should be emphasized that unlike the LMS canceler, Kavehrad completely neglects the effect of noise on optimal weight when he deals with diagonalizer.

2.4.1 The Diagonalizer and Its Performance

The structure of this canceler is well presented in [4], [9], see Fig 2.3. The output of the diagonalizer is given by

$$\begin{aligned} y_1(n) &= w_{11}x_1(n) + w_{12}x_2(n) \\ y_2(n) &= w_{21}x_1(n) + w_{22}x_2(n) \end{aligned} \tag{2.18}$$

and substituting (2.5) in (2.18), we get

$$\begin{aligned} y_1(n) &= w_{11}[a_{11}I_1(n) + a_{12}I_2(n) + n_1(n)] + w_{12}[a_{21}I_1(n) + a_{22}I_2(n) + n_2(n)] \\ y_2(n) &= w_{21}[a_{11}I_1(n) + a_{12}I_2(n) + n_1(n)] + w_{22}[a_{21}I_1(n) + a_{22}I_2(n) + n_2(n)] \end{aligned} \tag{2.19}$$

The canceler weights are found by forcing the coefficients of the interference signal to zero on each channel. Therefore, from (2.19) we must choose the weights to satisfy,

$$\begin{aligned} w_{11}a_{12} + w_{12}a_{22} &= 0 \\ w_{21}a_{11} + w_{22}a_{21} &= 0 \end{aligned} \tag{2.20}$$

By substituting the constraint of (2.20) in (2.19), we get after using (2.5),

$$\begin{aligned} y_1(n) &= a_{11}[1 - r_1r_2e^{j(\phi_1+\phi_2)}][I_{1R}(n) + jI_{1I}(n)] + n_{1R}(n) + jn_{1I}(n) \\ &\quad -(n_{2R} + jn_{2I})r_1e^{j\phi_1} \end{aligned}$$

$$\begin{aligned} y_2(n) &= a_{22}[1 - r_1r_2e^{j(\phi_1+\phi_2)}][I_{2R}(n) + jI_{2I}(n)] + n_{2R}(n) + jn_{2I}(n) \\ &\quad -(n_{1R} + jn_{1I})r_1e^{j\phi_2} \end{aligned} \tag{2.21}$$

Following Kavehrad, we define, $\hat{I}_1(n) = \frac{y_1(n)}{a_{11}}$ as an estimate of the transmitted signal $I_1(n)$ and hence definition of decision parameter follows

$$Z_1(n) = \hat{I}_1(n) - I_1(n) \tag{2.22}$$

Using (2.22) in (2.21), we write the decision parameter for the output of channel 1, in terms of its real and imaginary parts;

$$Z_{1R} = -I_{1R}(n)r_1r_2cos(\phi_1 + \phi_2) + I_{1I}(n)r_1r_2sin(\phi_1 + \phi_2)$$

$$+\frac{n_{1R}(n)}{a_{11}} - \frac{n_{2R}}{a_{11}} - \frac{n_{2R}}{a_{11}} r_1 cos\phi_1 + \frac{n_{2I}}{a_{11}} r_1 sin\phi_1$$

$$\begin{aligned} Z_{1I} &= -I_{1R}(n) r_1 r_2 sin(\phi_1+\phi_2) - I_{1I}(n) r_1 r_2 cos(\phi_1+\phi_2) \\ &\quad + \frac{n_{1I}(n)}{a_{11}} - \frac{n_{2R}}{a_{11}} - \frac{n_{2R}}{a_{11}} r_1 sin\phi_1 - \frac{n_{2I}}{a_{11}} r_1 cos\phi_1 \end{aligned} \tag{2.23}$$

The Chernoff bound on $P_1(|Z_{1R}| > c)$ is derived by [9] and some of the steps can be found detailed in section (3.3.1) From these analysis, we find that, the probability of error is bounded as follows;

$$P_i(e) \le (1 - \frac{1}{\sqrt{M}}) \exp\left[\frac{-3}{2(M-1)} \frac{SNR\ a_{ii}}{1 + SNR\ r_1^2 r_2^2 a_{ii}^2 + r_i^2}\right]\ i = 1,2 \tag{2.24}$$

where we again use the relation (2.14);

Kavehrad in different paper [14] uses another form of compensation;

From (2.21), he first finds the real and imaginary part of the canceler output,

$$\begin{aligned} y_{1R} &= a_{11}[1 - r_1 r_2 cos(\phi_1+\phi_2)] I_{1R}(n) + a_{11} I_{1I}(n) r_1 r_2 sin(\phi_1+\phi_2) + n_{1R}(n) \\ &\quad - n_{2R}(n) r_1 cos\phi_1 + n_{2I}(n) r_1 sin\phi_1, \end{aligned}$$

$$\begin{aligned} y_{1I} &= a_{11}[1 - r_1 r_2 cos(\phi_1+\phi_2)] I_{1I}(n) - a_{11} I_{1R}(n) r_1 r_2 sin(\phi_1+\phi_2) + n_{1I}(n) \\ &\quad - n_{2R}(n) r_1 sin\phi_1 - n_{2I}(n) r_1 cos\phi_1, \end{aligned} \tag{2.25}$$

then defines an estimate of the real part of $I_1(n)$,

$$\hat{I}_{1R}(n) = \frac{y_{1R}(n)}{a_{11}[1 - r_1 r_2 cos(\phi_1+\phi_2)]}, \tag{2.26}$$

and similar estimate for the imaginary part of $I_1(n)$. For the decision parameter Z_{1R}, then he uses

$$Z_{1R}(n) \triangleq \hat{I}_{1R}(n) - I_{1R}(n) \tag{2.27}$$

Therefore,

$$Z_{1R}(n) = \frac{1}{\Delta}[a_{11}I_{1I}(n)r_1r_2sin(\phi_1+\phi_2) + n_{1R}(n) - n_{2R}(n)r_1cos\phi_1 + n_{2I}(n)sin\phi_1] \tag{2.28}$$

where

$$\Delta \triangleq a_{11}[1 - r_1r_2cos(\phi_1+\phi_2)]. \tag{2.29}$$

This kind of compensation might be considered as "both amplitude and phase compensation" of the co-pol channel response. In next chapter, we will use, for the bootstrapped cancelers, a slightly different approach to this Kavehrad's compensation; we will apply compensation first on the complex output and then take a decision. Obviously, there will be difference in hardware needed to implement these approaches.

Both Chernoff bound and the moment GQR can be used with (2.28). For the second approach one can find

$$P_1(e|\phi_1,\phi_2,I_{1I}) = 2(1-\frac{1}{\sqrt{M}})Q(\frac{c-X_I}{\sigma}) \tag{2.30}$$

where

$$\begin{aligned} X_I &= \frac{r_1r_2I_{1I}sin(\phi_1+\phi_2)}{\Delta(\phi_1,\phi_2)} \\ Y &= \frac{1}{\Delta(\phi_1,\phi_2)}[n_{1R} - n_{2R}r_1cos\phi_1 + n_{2I}r_1sin\phi_1] \end{aligned} \tag{2.31}$$

and a variance

$$\sigma^2 = \frac{\sigma_n^2(1+r_1^2)}{[1-r_1r_2cos(\phi_1+\phi_2)]^2} \tag{2.32}$$

Let x be the random variable

$$\mathrm{x} = r_1r_2[cos(\phi_1+\phi_2) - I_{1I}sin(\phi_1+\phi_2)] \tag{2.33}$$

then

$$P_1(e|\phi_1,\phi_2,I_{1I}) = 2(1-\frac{1}{\sqrt{M}})Q\Big(\sqrt{\frac{3\ SNR}{(M-1)(1+r_1^2)}}(1-\mathrm{x})\Big) \tag{2.34}$$

Equation (2.34) can be evaluated numerically, (see appendix B). Some results on the performance of the diagonalizer of Kavehrad are given in the following figures. In Fig. 2.4, we use 16 QAM and compare the error probability obtained with the moment method to the Chernoff upper bound. The cross polarization used was r=-15 dB. Fig. 2.5 depicts the same except for using 64 QAM instead. Comparing the performance when r=-10 dB to that when r=-15 dB is done in Fig. 2.6 using moment method.

2.4.2 LMS Canceler

The structure of this canceler (Fig 2.7), for dually polarized nondispersive channels is given in [7].

For the output of this canceler as in (2.18), Kavehrad performs an amplitude normalization and obtain an estimate of the transmitted signal $\hat{I}_i(n) = \dfrac{y_i(n)}{a_{ii}}$ $i = 1,2$. The optimal LMS weights are found by minimizing the sum of the squares of the errors. $E\{|e_1(n)|^2 + |e_2(n)|^2\}$, where

$$e_i(n) \triangleq \hat{I}_1(n) - I_1(n) \quad i = 1,2 \tag{2.35}$$

corresponding to the i-th output of the canceler.

These optimum weights are found by solving the matrix equation

$$\mathbf{R}\mathbf{w}_{\text{opt}} = \mathbf{S}^* \tag{2.36}$$

where

$$\mathbf{R} = \begin{bmatrix} |x_1(n)|^2 & x_1(n)x_2^*(n) \\ x_1(n)^* x_2(n) & |x_2(n)|^2 \end{bmatrix} \tag{2.37}$$

$$\mathbf{w}_{\text{opt}} = \begin{bmatrix} w_{11\text{opt}} & w_{12\text{opt}} \\ w_{21\text{opt}} & w_{22\text{opt}} \end{bmatrix}, \tag{2.38}$$

and

$$\mathbf{S} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}. \tag{2.39}$$

For $E\{I_1(n)\} = E\{I_2(n)\}$ and for a complex Gaussian noise, the optimum weights from (2.36) can be used in (2.18) to find the optimal output $y_1(n)$. (see appendix B of [9] for detail).

From the optimal output one can derive the decision parameter, and using it to find an approximation to, or upper bound on the probability of error.

Some results of these calculation are given in Fig. 2.8 to 2.10.

2.5 Bootstrapped Adaptive Algorithms

In this section, we present three different bootstrapped cross-pol interference cancelers. These cross-pol cancelers [13] differ from the other interference canceler systems [7,9] in that it is a power separator rather than interference canceler. That is each of the two input signals interferes with the other and the function of the canceler is to remove the interference from both input signals rather than just one.

To obtain a high signal to interference ratio at both outputs of these cancelers, we suggest to use bootstrapping technique. With this approach, two cancelation paths and two summation are used to obtain the two system outputs. An adaptive algorithm is employed to optimize the signal-to-interference power ratio at the two output ports simultaneously.

The three configurations of the bootstrapping algorithms differ in the criterion is set to obtain the optimal complex weight in the cancelation paths. The criterion used to minimize either the interfering signal power at the two output ports, the correlation between the two signals at the two output ports, or simultaneously the interfering signal power at one port and the correlation between the two output signals at ports. Correspondingly they will be termed ;power-power, power-correlation and correlation-correlation cross pol cancelers. The three configurations reported in the literature [3], [8], [13] differ in the topology of their two cancelation paths, the adaptive feedback information and hence in their hardware complexity. The use of either of these criteria lead power-inversion result. Each of the arrangement of bootstrapped algorithm results in power separation through the use of discrimination techniques.

2.5.1 Optimization Criteria

The following three configurations of bootstrapping algorithms employ different optimization criterion, namely power-power, correlator-correlator and power-correlator.

The first configuration has been studied by [12] , the second proposed by [8] and the third by [3].

Power-Power Canceler

The system in Fig. 2.11 consists of two distinct control loops: $Q - w_{21}$ loop and the $P - w_{12}$ loop. Let the power ratio of the two signals at point 3, be such that $I_1(n) > I_2(n)$. This is being the input to the weight w_{12} results in power-inversion in $I_2(n) > I_1(n)$ at point 4. However, point 4, being the input to the weight w_{21} result in $I_1(n)$ greater than $I_2(n)$ at point 3. This process continue resulting in a very high $\frac{I_1(n)}{I_2(n)}$ at one output and $\frac{I_2(n)}{I_1(n)}$ in the other. As a result the power-power canceler acts as a high quality power separator. The adaptive control algorithm varies the cancelation coefficient w_{12}, w_{21} so as to minimize the power P and Q at the output of the canceler. The blocks labeled "discrimination" performs functions which make the power detection more sensitive to the undesired ($a_{12}I_2$ at port P) signal than to the desired signal "$a_{11}I_1$". The effect of these blocks which will be discussed in later chapters and proven to be essential for bootstrapping operation.

Correlation-Correlation Canceler

From Fig. 2.12, we notice that the adaptive control algorithm is set up to control the cancelation weight w_{12} so as to minimize the magnitude square of the correlation

P_1 at one output and to control w_{21} so as to minimize the magnitude square of the correlation Q_1 simultaneously. The w_{21} control process can operate with sample of the single $I_2(n)$ at point 3, which is corrupted by the signal $I_1(n)$, but needs clean sample of the signal $I_1(n)$ at point 4 to generate its feedback. Similarly, for the w_{12} control processor. Since initially neither points 3, or 4, contains clean sample of $I_1(n)$ and $I_2(n)$, respectively, neither processor performs properly unless the other one does. However, if one processor starts its cancelation, it results in a cleaner sample of the proper signal needed by the other processor and vice versa. This bootstrapping behaviour results in the desired power separation at the output ports.

Power-Correlator Canceler

With the cross-pole canceler of Fig. 2.13, the cancelation weight w_{12} is controlled via a power criterion which minimizes the power P_2 and the weight w_{21} is controlled to minimize the magnitude square of the correlation at the second output. To obtain a perfect cancelation of the signal $I_2(n)$ at output port 1, it is required that the processor w_{12} has a clean sample of $I_2(n)$ at point 3 . The correlation process which controls w_{21} can operate with a sample of the signal $I_2(n)$ at point 3, which is corrupted by the signal $I_1(n)$, but needs a clean sample of the signal $I_1(n)$ at the other correlator input to generate its feedback signal. Since one of the two processor can in effect defer its need for a clean sample of $I_2(n)$, it makes sense to let that processor perform its cancelation first, making a clean sample of $I_2(n)$ available to the other processor (power processor) which in turn provides a clean sample of the sample of the signal $I_1(n)$ at point 4, to the first processor (correlation processor) to generate a feedback signal. Thus, although neither processor can function properly unless the other one does, both (in bootstrap operation) can operate properly together. Consequently the power-correlator canceler in Fig. 2.13

can perform as a power separator.

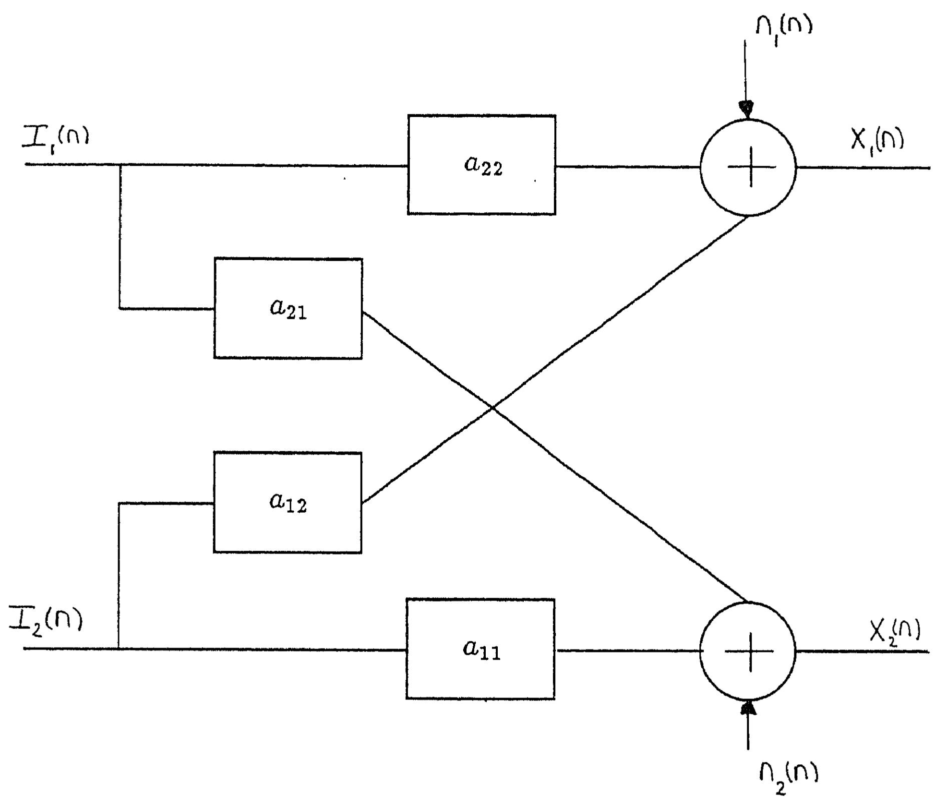

Figure 2.1: Dually Polarized Channel Model

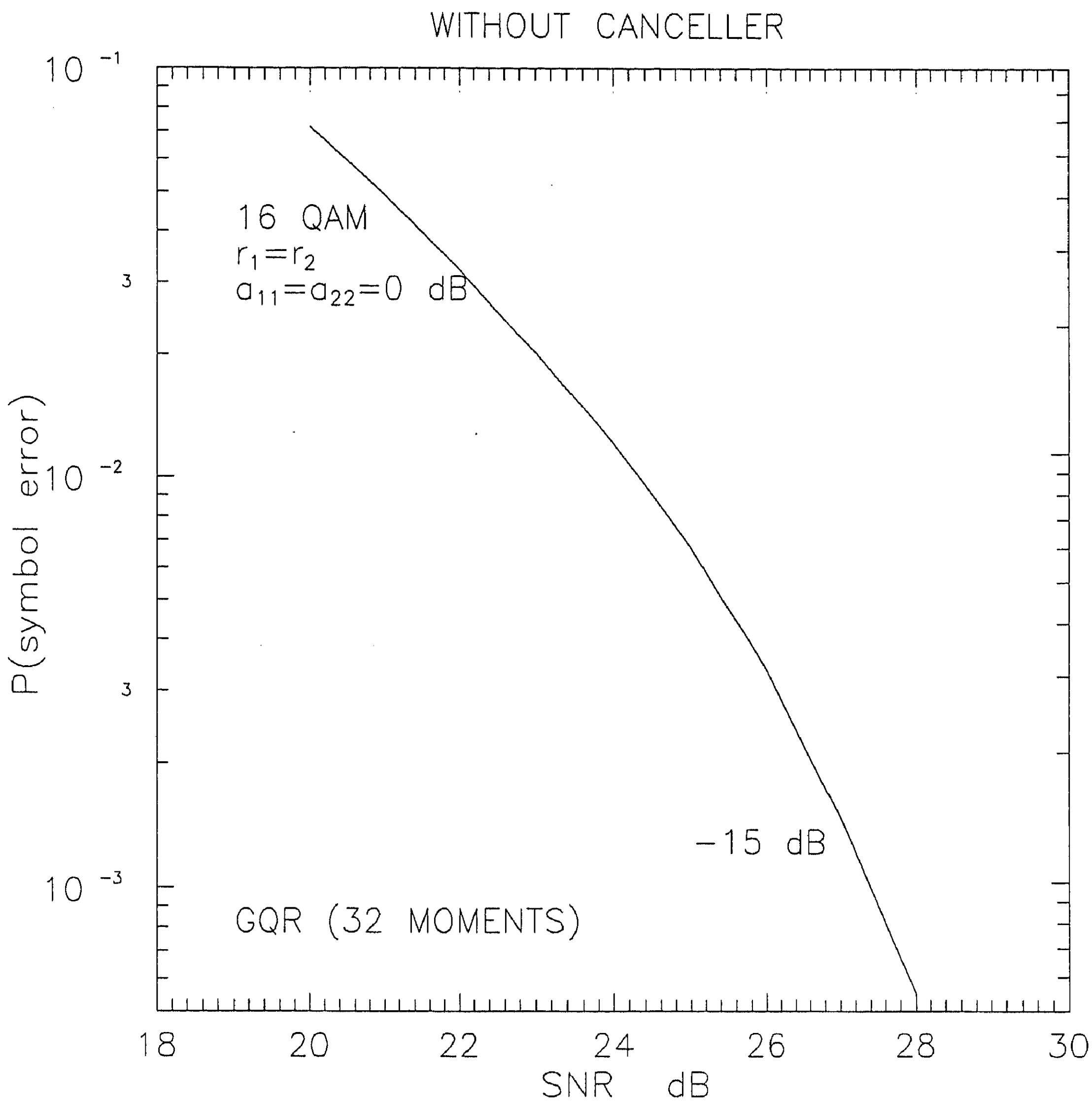

Figure 2.2: Performance of Dually Polarized 16 QAM System, without Cross-Pol Interference Canceler

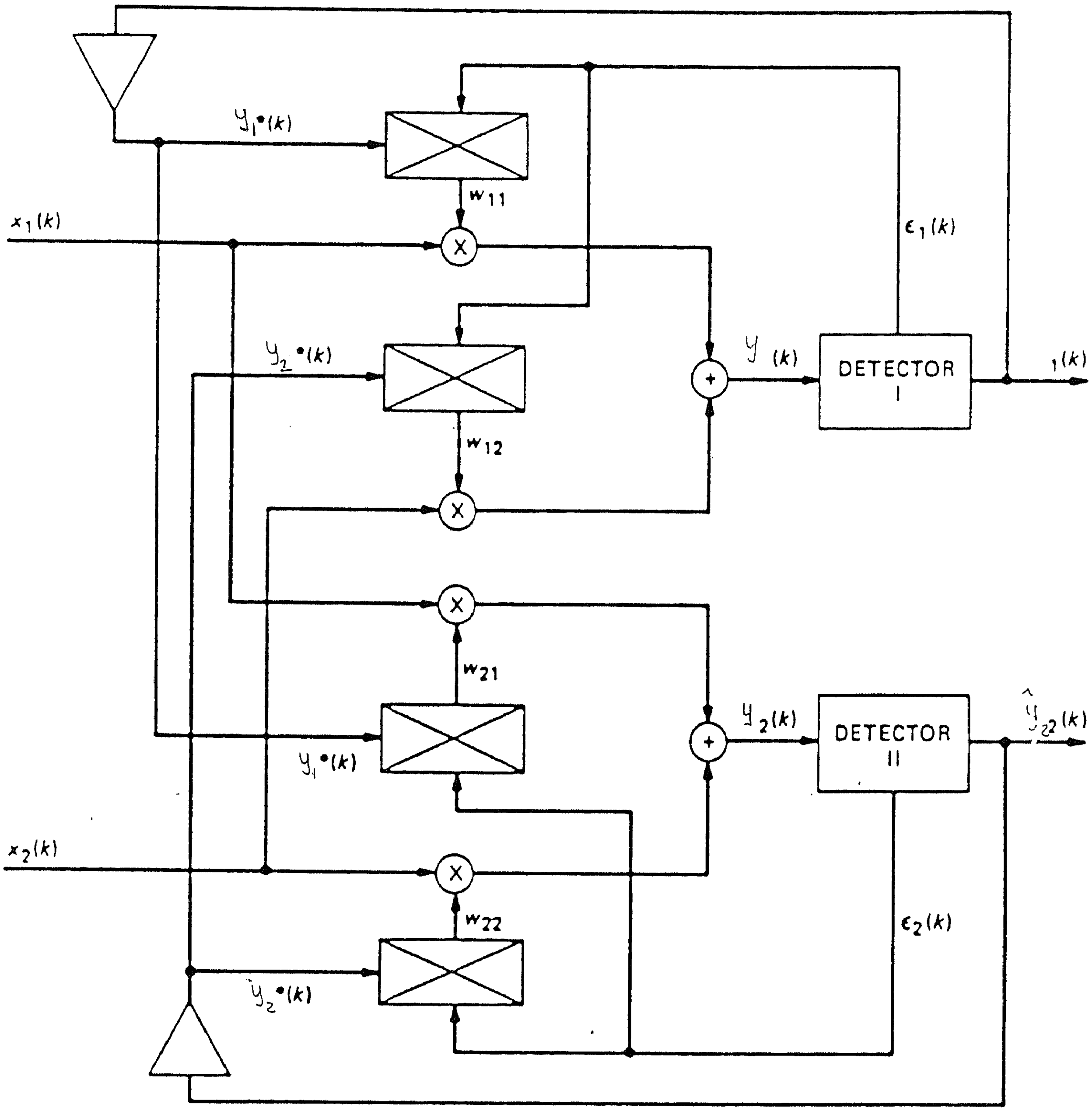

Figure 2.3: Diagonalizer Cross-Pol Interference Canceler

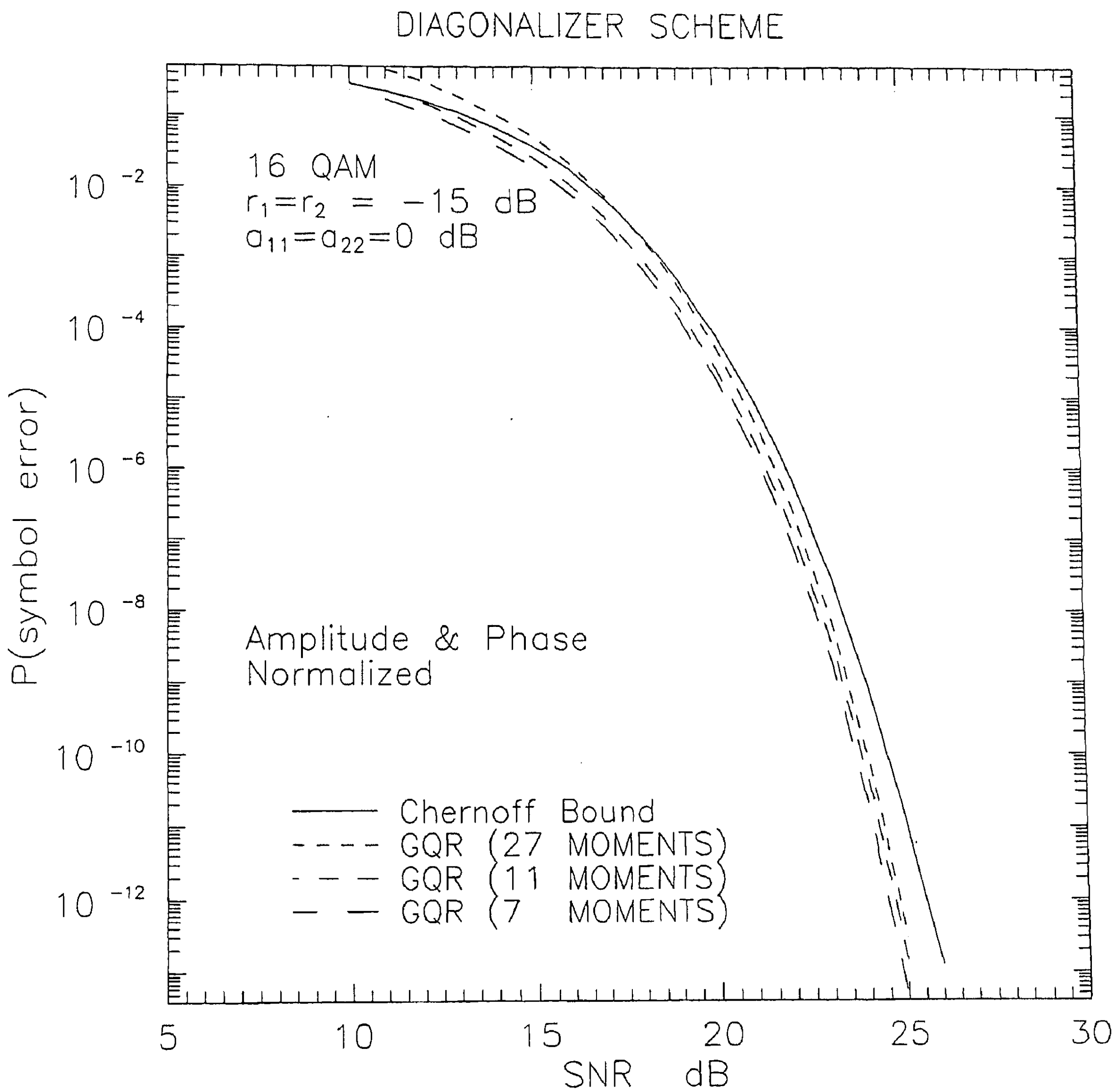

Figure 2.4: Diagonalizer Cross-Pol Interference Canceler, Chernoff bound and GQR calculation, 16 QAM with amplitude and phase compensation, cross coupling -15 dB

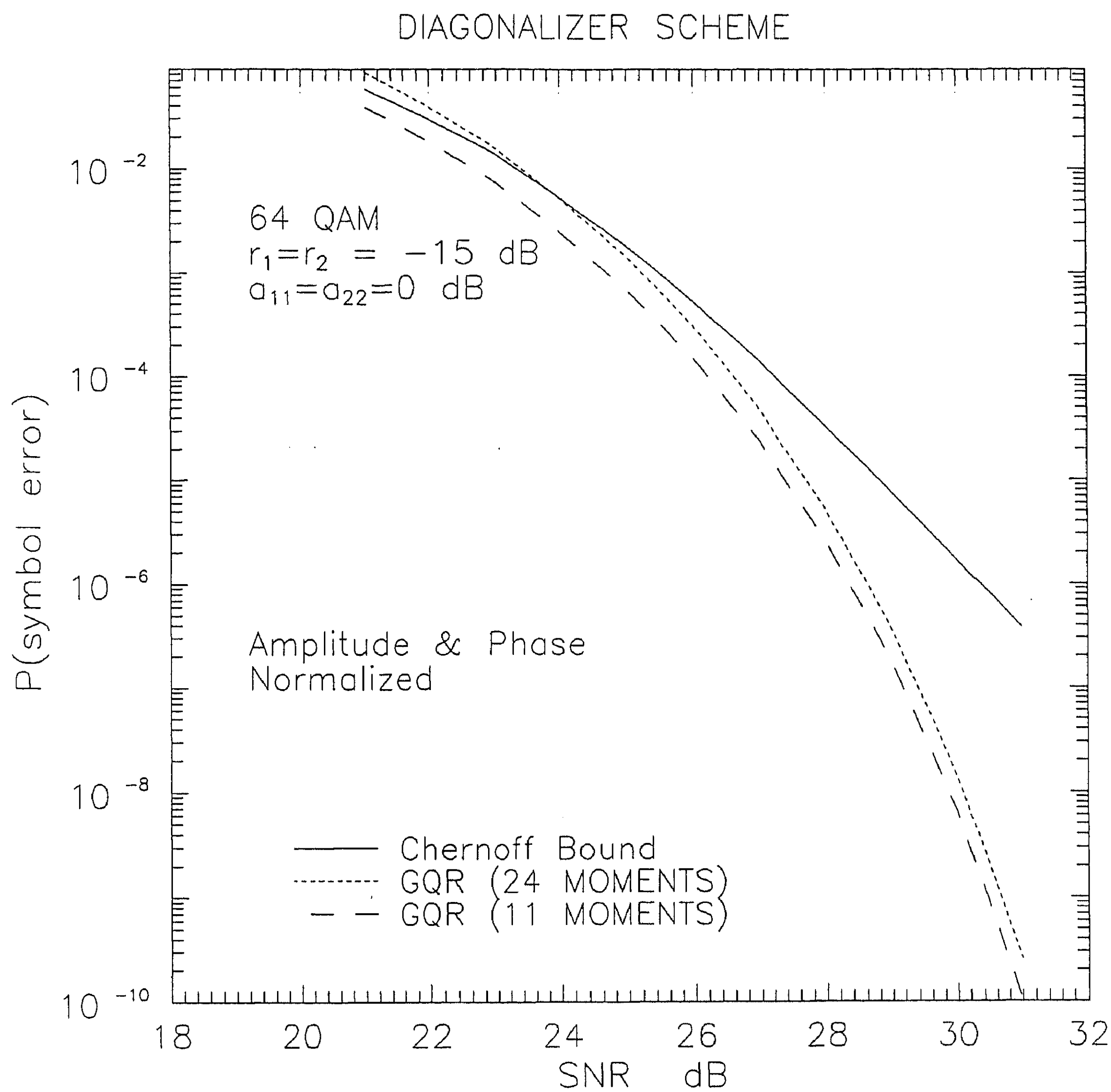

Figure 2.5: Diagonalizer Cross-Pol Interference Canceler, Chernoff bound and GQR calculation, 64 QAM with amplitude and phase compensation, cross coupling -15 dB

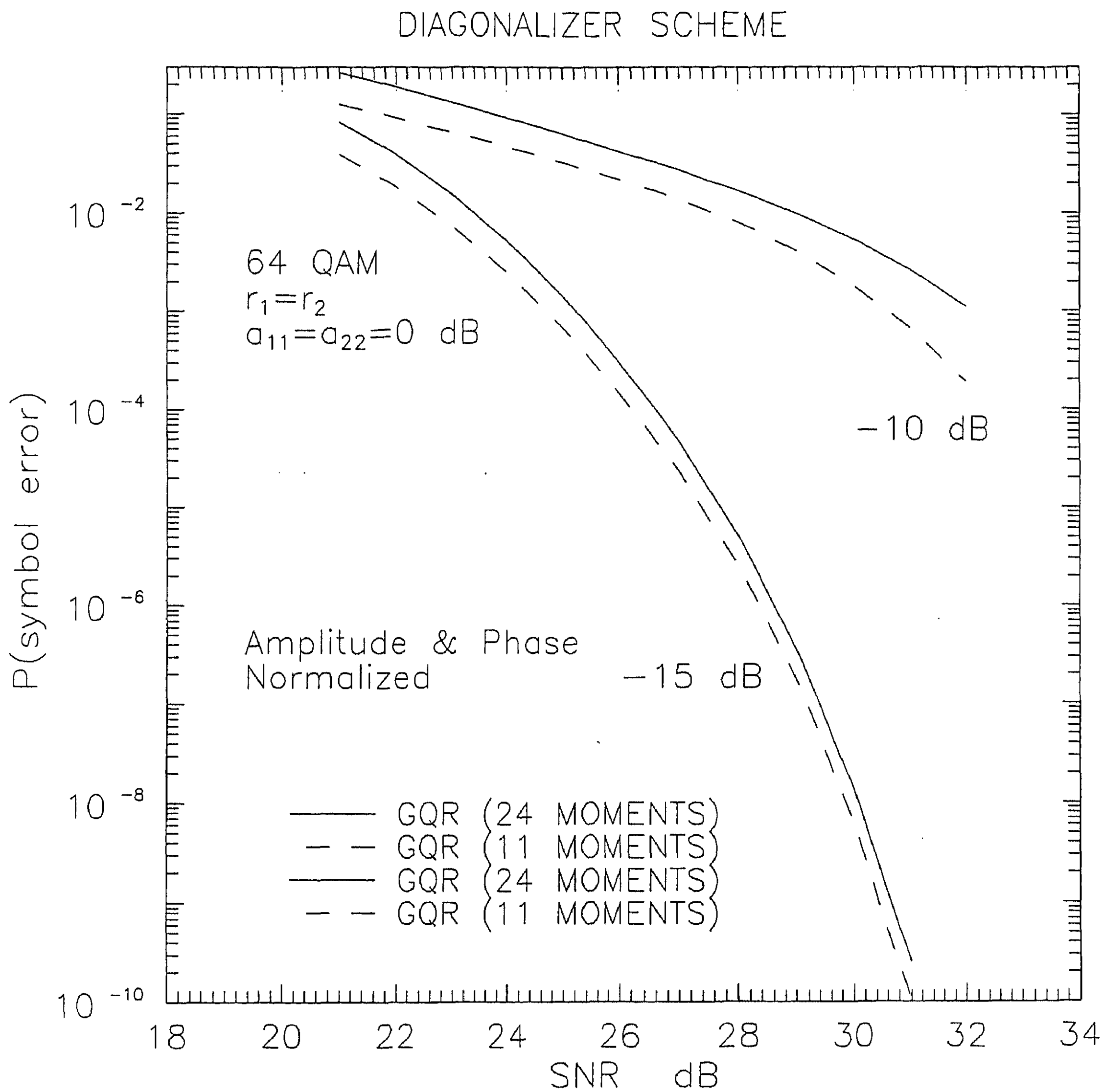

Figure 2.6: Diagonalizer Cross-Pol Interference Canceler, GQR calculation, 16 QAM vs. 64 QAM with amplitude and phase compensation, cross coupling -15 dB, -10 DB

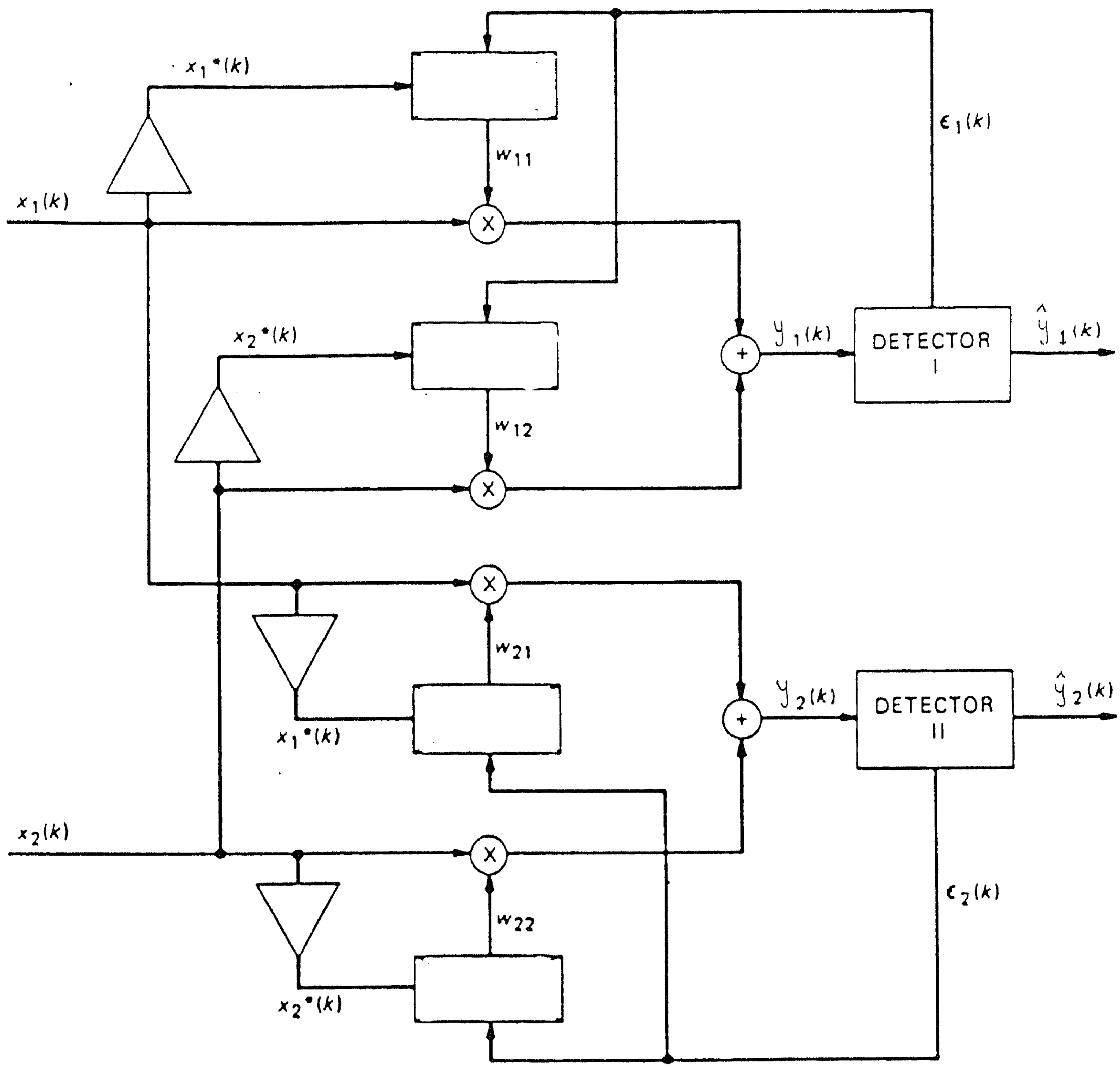

Figure 2.7: LMS Cross-Pol Interference Canceler

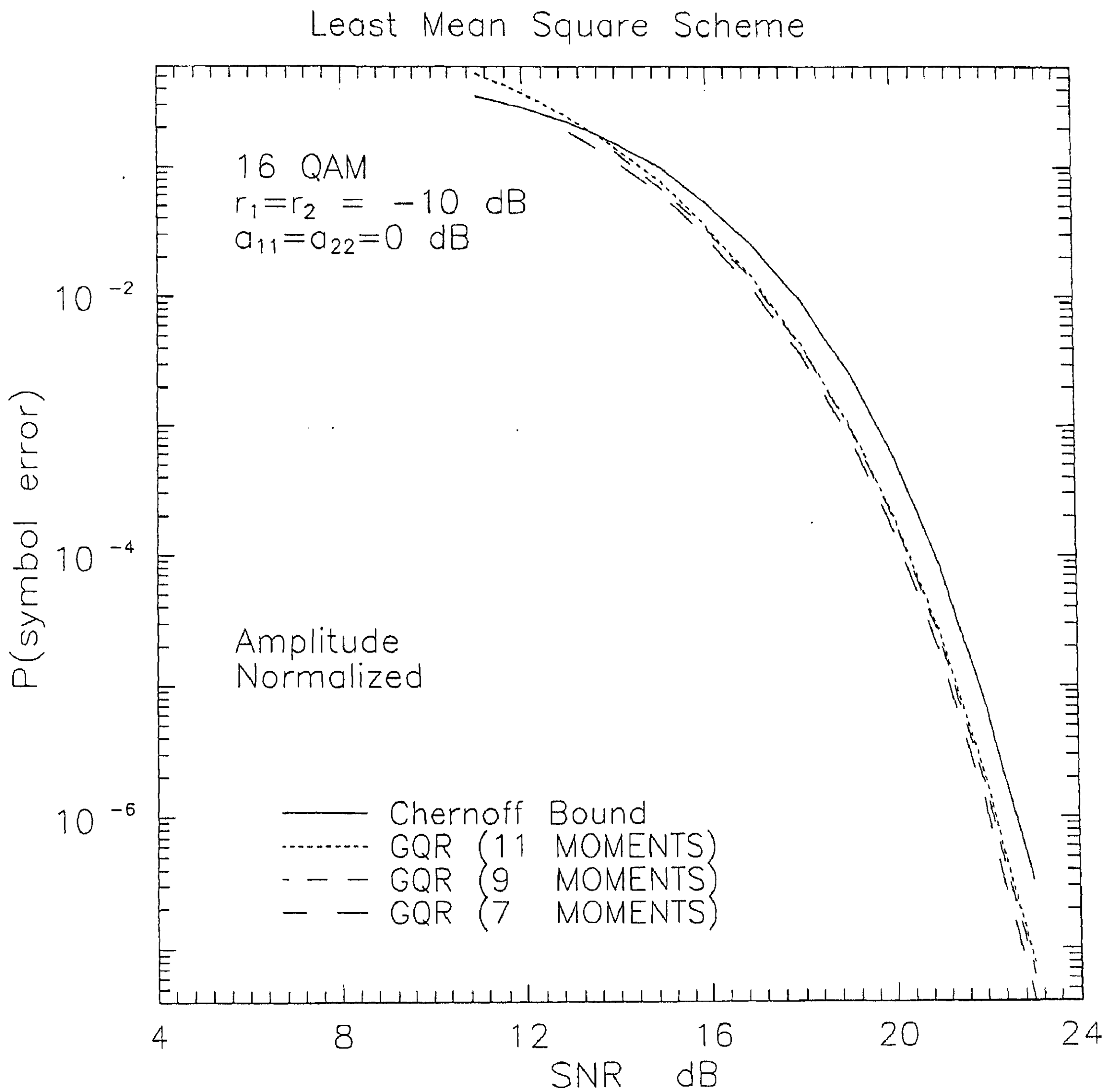

Figure 2.8: LMS Cross-Pol Interference Canceler, Chernoff bound and GQR calculation, 16 QAM with amplitude and phase compensation, cross coupling -10 dB

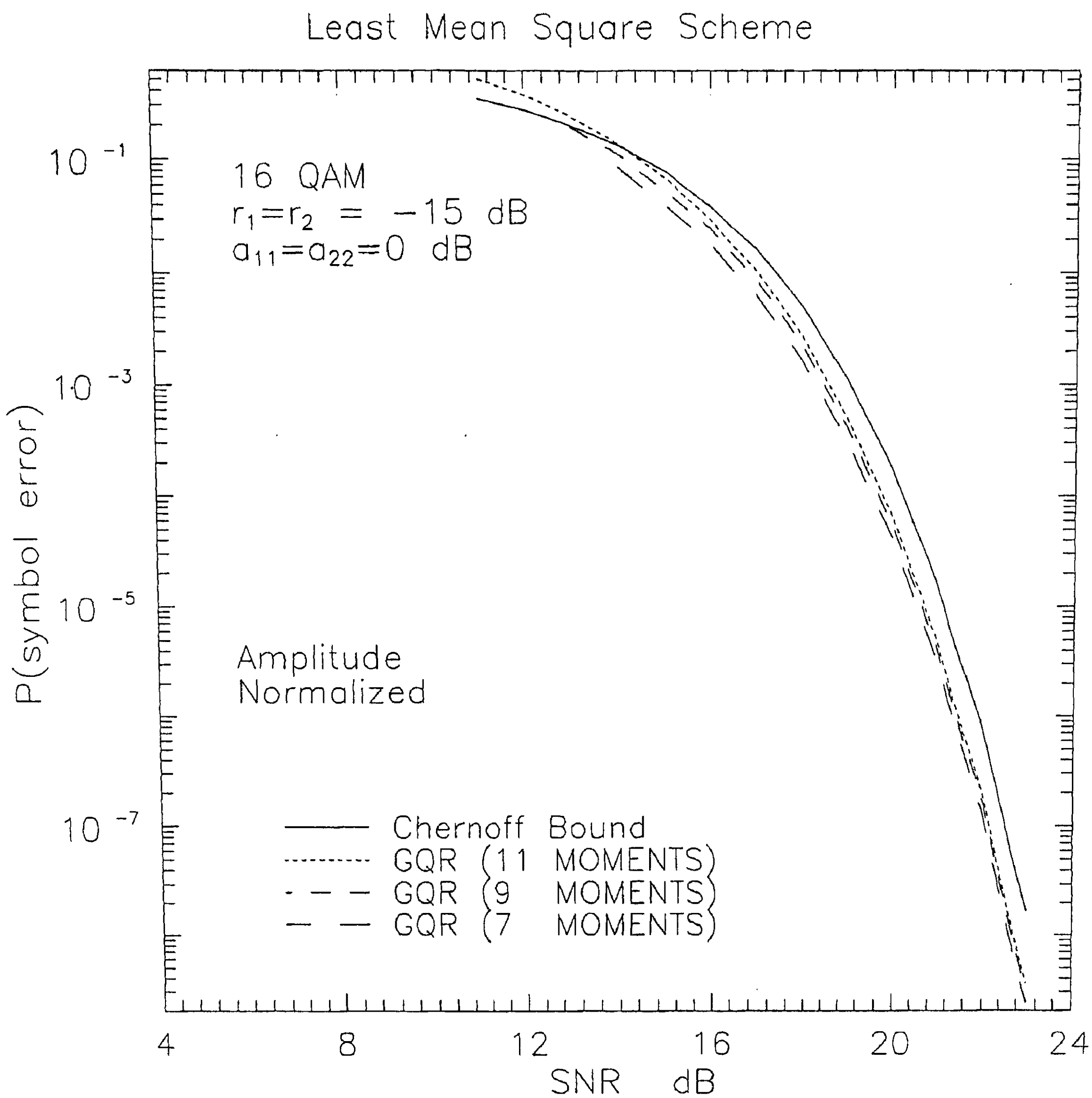

Figure 2.9: LMS Cross-Pol Interference Canceler, Chernoff bound and GQR calculation, 16 QAM with amplitude and phase compensation, cross coupling -15 dB

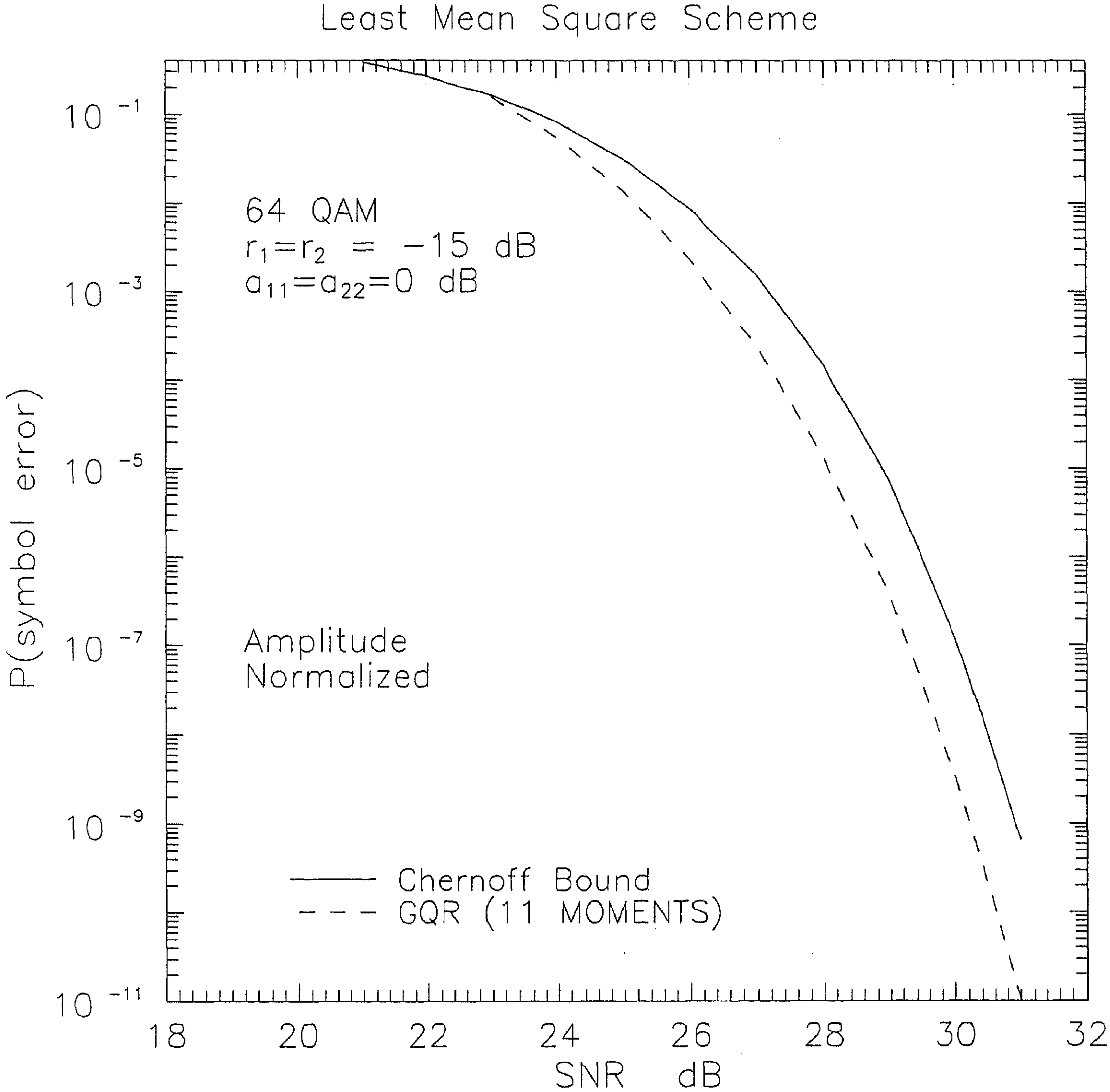

Figure 2.10: LMS Cross-Pol Interference Canceler, Chernoff bound and GQR calculation, 64 QAM with amplitude and phase compensation, cross coupling -15 dB

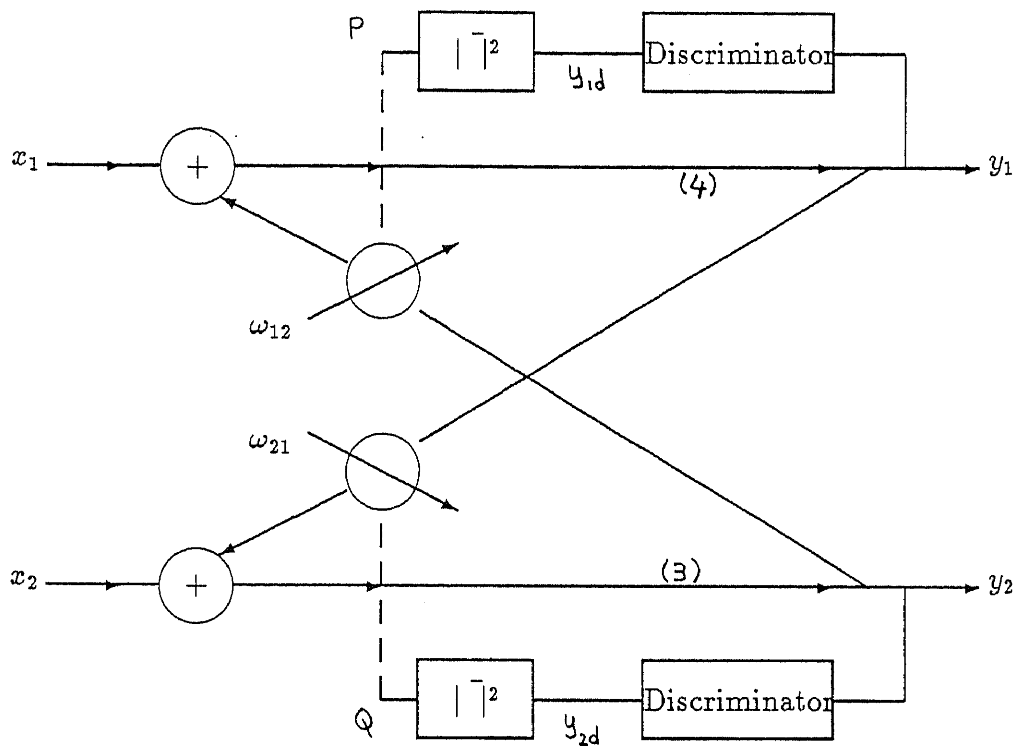

Figure 2.11: Power-Power Cross-Pol Interference Canceler

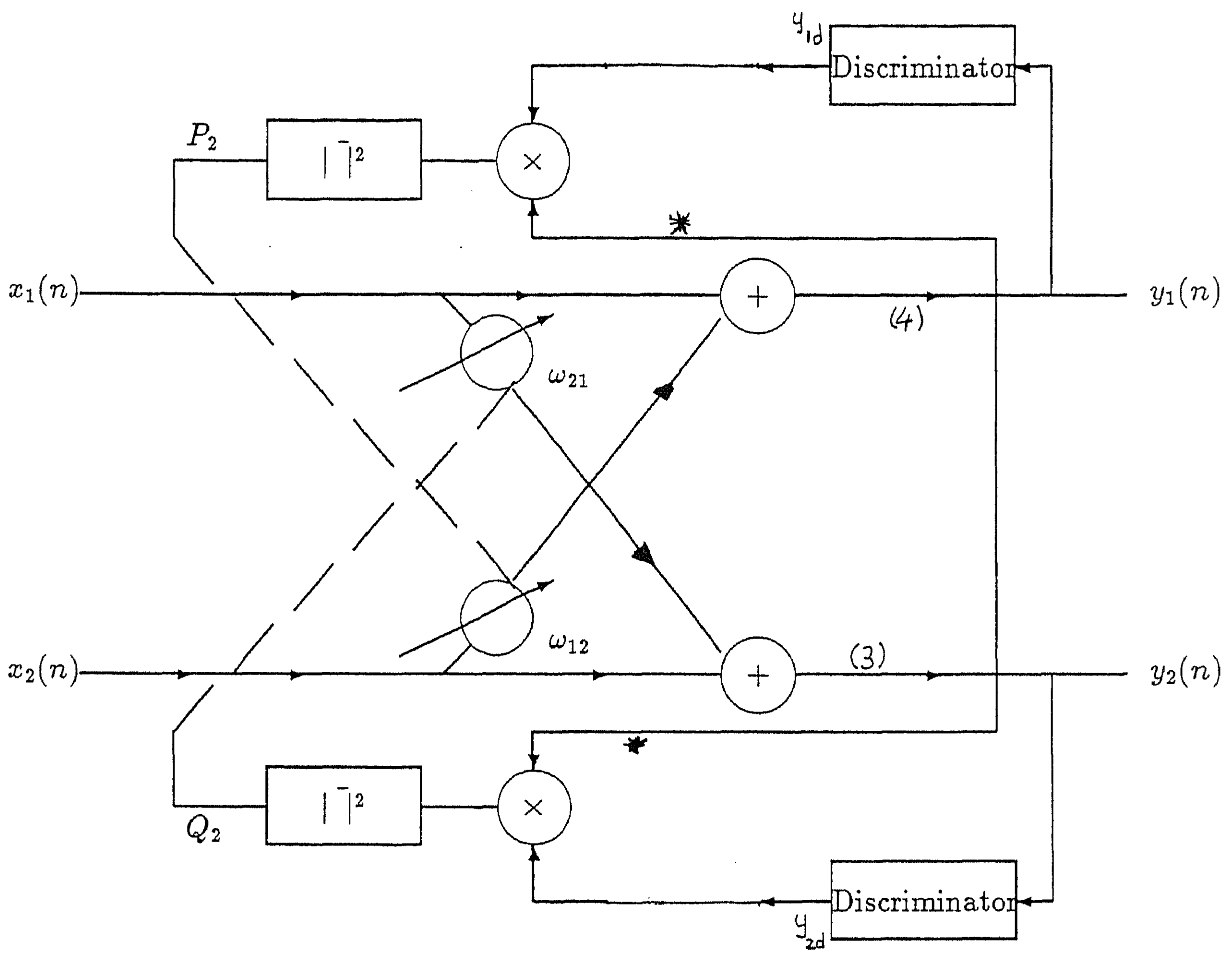

Figure 2.12: Correlator-Correlator Cross-Pol Interference Canceler

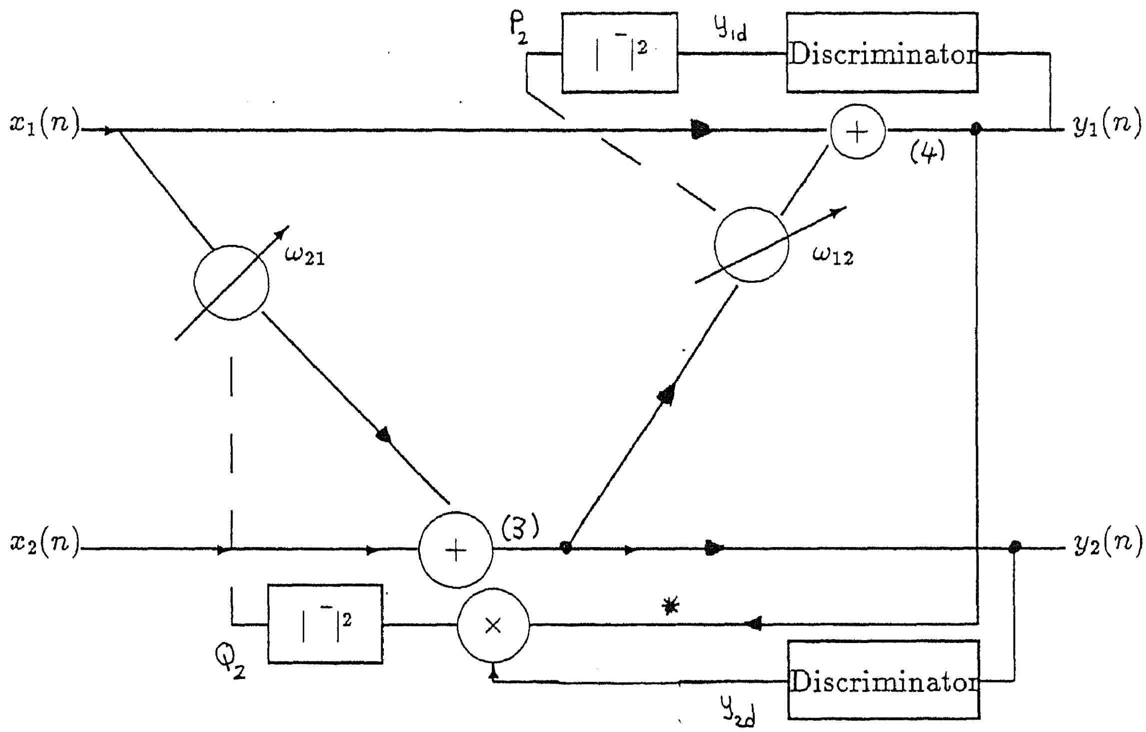

Figure 2.13: Power-Correlator Cross-Pol Interference Canceler

Chapter 3

PERFORMANCE ANALYSIS OF POWER-POWER SCHEME

3.1 Introduction

In this chapter, we present the performance of the power-power scheme of bootstrapped cross-polarization canceller for a dual M-QAM system over non-dispersive fading channels.

In this study, the performance is evaluated in two ways; by deriving an upper bound and by calculating the average error probability with the moment-generating method.

After deriving the canceler's parameter, such as optimal weights with and without noise effect, we find in the next section the canceler's optimal outputs. Assuming amplitude compensation alone, and together with phase compensation of the canceler's outputs, we derive decision parameters for deriving the corresponding error terms. In section 3.3, we derive the Chernoff bound on the average probability of error on one hand and define an expression to be used in calculating an approximation to the probability of error by the method of moments, on the other. The method of moments that is based on Gauss quadrature rule is discussed in an appendix to this chapter. Finally, in section 3.4, we present results on the performance of the power-power canceler showing Chernoff bounds and actual approximations to error probability based on the moments, for different cases and with different parameter. These results are also compared, in order to draw conclusion in section 3.5.

3.2 Canceler Scheme and Parameters

The canceler scheme is depicted in Fig. 2.11 and its operation was detailed in chapter 2. As it was discussed in this chapter the received signals which are sampled after matched filters, are given by;

$$\begin{aligned} x_1(n) &= a_{11}I_1(n) + a_{12}I_2(n) + n_1(n) \\ x_2(n) &= a_{21}I_1(n) + a_{22}I_2(n) + n_2(n) \end{aligned} \tag{3.1}$$

where $x_1(n)$ and $x_2(n)$ are the sampled received signals at the first and second channels respectively. $I_i(n)$ and $n_i(n)$ are the corresponding signals and noises at these outputs.

3.2.1 Canceler Outputs

The outputs $y_1(n)$ and $y_2(n)$ from Fig. 2.11 are as follows

$$y_1(n) = x_1(n) + y_2(n)w_{12}$$

$$y_2(n) = x_2(n) + y_1(n)w_{21} \tag{3.2}$$

Solving the system of equation (3.2) leads to

$$\begin{aligned} y_1(n) &= \frac{x_1(n) + x_2(n)w_{12}}{1 - w_{12}w_{21}} \\ y_2(n) &= \frac{x_2(n) + x_1(n)w_{21}}{1 - w_{12}w_{21}} \end{aligned} \tag{3.3}$$

Substituting for $x_1(n)$ and $x_2(n)$ from (3.1) we get

$$y_1(n) = \frac{I_1(n)(a_{11}+w_{12}a_{21}) + I_2(n)(a_{12}+w_{12}a_{22}) + n_1(n) + n_2(n)w_{12}}{1 - w_{12}w_{21}} \quad (3.4)$$

$$y_2(n) = \frac{I_1(n)(a_{21}+w_{21}a_{11}) + I_2(n)(a_{22}+w_{21}a_{12}) + n_1(n)w_{21} + n_2(n)}{1 - w_{12}w_{21}} \quad (3.5)$$

3.2.2 Optimal Weights

The control algorithm simultaneously minimizes the output powers

$$P(w_{12}^i, w_{21}^i) = E\{|y_{1d}^i(n)|^2\} \quad (3.6)$$

$$Q(w_{12}^i, w_{21}^i) = E\{|y_{2d}^i(n)|^2\} \quad (3.7)$$

where $y_{1d}(n)$ and $y_{2d}(n)$ are the samples of the corresponding output after the discriminations. In fact it simultaneously searches for $\partial E\{|y_{1d}(n)|^2\}/\partial w_{12} = 0$ and $\partial E\{|y_{2d}(n)|^2\}/\partial w_{21} = 0$, where $E\{\cdot\}$ and $|\cdot|$ denote the expected and magnitude respectively. The search for optimum weights can be performed by successive use of the following recursive equations, provided that $1 - w_{12}w_{21} \neq 0$.

$$w_{12}^{i+1} = w_{12}^i - \mu_1 \frac{\partial}{\partial w_{12}^i} P(w_{12}^i, w_{21}^i) \quad (3.8)$$

$$w_{21}^{i+1} = w_{21}^i - \mu_2 \frac{\partial}{\partial w_{21}^i} Q(w_{12}^i, w_{21}^i) \quad (3.9)$$

where μ_1 and μ_2 are the constants which determine the stability of convergence.

The optimum weights that minimize the powers are the steady state weights obtained from

$$\frac{\partial P(w_{12}^i, w_{21}^i)}{\partial w_{12}^i} = 0 \quad (3.10)$$

$$\frac{\partial Q(w_{12}^i, w_{21}^i)}{\partial w_{21}^i} = 0 \quad (3.11)$$

From (3.4) and (3.5), we first find the powers at the output of the discriminators, as

$$P(w_{12}, w_{21}) = \frac{1}{|1 - w_{12}w_{21}|^2}\Big[\delta_{11}E\{|I_1(n)|^2\}|a_{11} + w_{12}a_{21}|^2$$

$$+\delta_{12}E\{|I_2(n)|^2\}|a_{12} + w_{12}a_{22}|^2 + E\{|n_1(n)|^2\}$$

$$+E\{|n_2(n)|^2\}|w_{12}|^2\Big], \qquad (3.12)$$

$$Q(w_{12}, w_{21}) = \frac{1}{|1 - w_{12}w_{21}|^2}\Big[\delta_{21}E\{|I_1(n)|^2\}|a_{21} + w_{21}a_{11}|^2$$

$$+\delta_{22}E\{|I_2(n)|^2\}|a_{22} + w_{21}a_{12}|^2 + E\{|n_1(n)|^2\}|w_{21}|^2$$

$$+E\{|n_2(n)|^2\}\Big] \qquad (3.13)$$

where $\delta_{i,j}$ i,j=1,2 denotes the effect of the i th discriminator on the different signal ($I_1(n)$ or $I_2(n)$) powers.

Notice that in calculating the power, we assumed $I_1(n)$ and $I_2(n)$ are uncorrelated and zero mean. We will take for the derivative of any real function with respect to a complex variable [16].

$$\frac{\partial}{\partial w} = \frac{\partial}{\partial w_r} + j\frac{\partial}{\partial w_i} \qquad (3.14)$$

where $w = w_r + jw_i$. Hence, for the functions P in (3.12), we get,

$$\frac{\partial P}{\partial w_{12}} = \frac{2}{|1 - w_{12}w_{21}|^4}\Big[\Big(\delta_{11}E\{|I_1(n)|^2\}(a_{11} + w_{12}a_{21})a_{21}^*$$

$$+\delta_{12}E\{|I_2(n)|^2\}(a_{12}+w_{12}a_{22})a_{22}+E\{|n_2(n)|^2\}w_{12}\Big)|1-w_{12}w_{21}|^2$$

$$+(1-w_{12}w_{21})w_{21}^*\Big(\delta_{11}E\{|I_1(n)|^2\}|a_{11}+w_{12}a_{21}|^2$$

$$+\delta_{12}E\{|I_2(n)|^2\}|a_{12}+w_{12}a_{22}|^2+E\{|n_1(n)|^2\}+E\{|n_2(n)|^2\}|w_{12}|^2\Big)\Big] \tag{3.15}$$

Further simplification yields,

$$\frac{\partial P}{\partial w_{12}} = \frac{2(1-w_{12}w_{21})}{|1-w_{12}w_{21}|^4}\Big[(a_{11}+w_{12}a_{21})\Big(a_{21}^*(1-w_{12}w_{21})^*$$

$$+w_{21}^*(a_{11}+w_{12}a_{21})^*\Big)\delta_{11}E\{|I_1(n)|^2\}+(a_{12}+w_{12}a_{22})\Big(a_{22}(1-w_{12}w_{21})^*$$

$$+w_{21}^*(a_{12}+w_{12}a_{22})^*\Big)\delta_{12}E\{|I_2(n)|^2\}+E\{|n_1(n)|^2\}w_{21}^*$$

$$+E\{|n_2(n)|^2\}\Big(w_{12}(1-w_{12}w_{21})^*+w_{21}^*|w_{12}|^2\Big)\Big], \tag{3.16}$$

and finally we can write,

$$\frac{\partial P}{\partial w_{12}} = \frac{2(1-w_{12}w_{21})}{|1-w_{12}w_{21}|^4}\Big[(a_{11}+w_{12}a_{21})(a_{21}+w_{21}a_{11})^*\delta_{11}E\{|I_1(n)|^2\}$$

$$+(a_{12}+w_{12}a_{22})(a_{22}+w_{21}a_{12})^*\delta_{12}E\{|I_2(n)|^2\}$$

$$+E\{|n_1(n)|^2\}w_{21}^*+E\{|n_2(n)|^2\}w_{12}\Big]. \tag{3.17}$$

Similarly the derivative of the power Q in (3.13) can be written as,

$$\frac{\partial Q}{\partial w_{21}} = \frac{2}{|1-w_{12}w_{21}|^4}\Big[\Big(\delta_{21}E\{|I_1(n)|^2\}(a_{21}+w_{21}a_{11})a_{11}$$

$$+\delta_{22}E\{|I_2(n)|^2\}(a_{22}+w_{21}a_{12})a_{12}^* + E\{|n_1(n)|^2\}w_{21}\Big)|1-w_{12}w_{21}|^2$$

$$+(1-w_{12}w_{21})w_{12}^*\Big(\delta_{21}E\{|I_1(n)|^2\}|a_{21}+w_{21}a_{11}|^2$$

$$+\delta_{22}E\{|I_2(n)|^2\}|a_{22}+w_{21}a_{12}|^2 + E\{|n_2(n)|^2\} + E\{|n_1(n)|^2\}|w_{21}|^2\Big)\Big] \tag{3.18}$$

Further simplification yields to,

$$\frac{\partial Q}{\partial w_{21}} = \frac{2(1-w_{12}w_{21})}{|1-w_{12}w_{21}|^4}\Big[(a_{21}+w_{21}a_{11})(a_{11}+w_{12}a_{21})^*\delta_{21}E\{|I_1(n)|^2\}$$

$$+(a_{12}+w_{12}a_{22})^*(a_{22}+w_{21}a_{12})\delta_{22}E\{|I_2(n)|^2\}$$

$$+E\{|n_1(n)|^2\}w_{21} + E\{|n_2(n)|^2\}w_{12}^*\Big] \tag{3.19}$$

Provided $|1-w_{12}w_{21}| \neq 0$, equating (3.17) and (3.19) simultaneously to zero will result in $w_{12\mathrm{opt}}$ and $w_{21\mathrm{opt}}$.

$$w_{12\mathrm{opt}} = \frac{-1}{Dw_{12\mathrm{opt}}}\Big[a_{11}(a_{21}+w_{21\mathrm{opt}}a_{11})^*E\{|I_1(n)|^2\}\delta_{11}$$

$$+a_{12}(a_{22}+w_{21\mathrm{opt}}a_{12})^*E\{|I_2(n)|^2\}\delta_{12} + w_{21\mathrm{opt}}^*E\{|n_1(n)|^2\}\Big] \tag{3.20}$$

where,

$$
\begin{aligned}
Dw_{12\text{opt}} &= a_{21}(a_{21} + w_{21\text{opt}}a_{11})^* E\{|I_1(n)|^2\}\delta_{11} \\
&\quad + a_{22}(a_{22} + w_{21\text{opt}}a_{12})^* E\{|I_2(n)|^2\}\delta_{12} + E\{|n_2(n)|^2\}, \quad (3.21)
\end{aligned}
$$

and,

$$
\begin{aligned}
w_{21\text{opt}} &= \frac{-1}{Dw_{21\text{opt}}}\Big[a_{21}(a_{11} + w_{12\text{opt}}a_{21})^* E\{|I_1(n)|^2\}\delta_{21} \\
&\quad + a_{22}(a_{12} + w_{12\text{opt}}a_{22})^* E\{|I_2(n)|^2\}\delta_{22} + w_{12\text{opt}}^* E\{|n_2(n)|^2\}\Big] \quad (3.22)
\end{aligned}
$$

where

$$
\begin{aligned}
Dw_{21\text{opt}} &= a_{11}(a_{11} + w_{12\text{opt}}a_{21})^* E\{|I_1(n)|^2\}\delta_{21} \\
&\quad + a_{12}(a_{12} + w_{12\text{opt}}a_{22})^* E\{|I_2(n)|^2\}\delta_{22} + E\{|n_1(n)|^2\} \quad (3.23)
\end{aligned}
$$

The effect of the discriminators are presented by $\delta_{i,j} \quad i,j = 1,2$ which are real valued satisfying $\delta_{11}\delta_{22} < \delta_{12}\delta_{21}$. Note that, the first and second terms in (3.17) are complex conjugates of the terms in (3.19). Therefore, to find a unique solution for w_{12} and w_{21} using these equations, discriminators which enforce the constant $\delta_{i,j} \quad i,j = 1,2$ satisfying the above condition, are essential. The simultaneous solution of these non-linear equations give two equilibrium points; $[w_{12\text{opt}1}, w_{21\text{opt}1}]$ and $[w_{12\text{opt}2}, w_{21\text{opt}2}]$. One is a stable equilibrium which provide a solution to our problem.

3.2.3 Effect of Noise On Optimal Weights

In the absence of noise, that is when $E\{|n_1(n)|^2\} = E\{|n_2(n)|^2\} = 0$ the stable equilibrium points can easily found to be;

$$
w_{12\text{opt}} = -\frac{a_{12}}{a_{22}}, \qquad w_{21\text{opt}} = -\frac{a_{21}}{a_{11}}. \qquad (3.24)
$$

With noise, we will write

$$w_{12\text{opt}} = -\frac{a_{12}}{a_{22}} + \epsilon_1, \qquad w_{21\text{opt}} = -\frac{a_{21}}{a_{11}} + \epsilon_2. \tag{3.25}$$

where ϵ_1 and ϵ_2 are perturbations, due to noise on the optimal weights that we intend to find.

Perturbation On Optimal Weight, $w_{12\text{opt}}$

Using (3.20) and (3.24) in (3.25), we can find ϵ_1.

$$\begin{aligned}\epsilon_1 \;&=\; \frac{-1}{Dw_{12\text{opt}}}\Big[a_{11}(a_{21} + w_{21\text{opt}}a_{11})^*\delta_{11}E\{|I_1(n)|^2\} + a_{12}(a_{22} + w_{21\text{opt}}a_{12})^*\cdot \\ &\quad \delta_{12}E\{|I_2(n)|^2\} + E\{|n_1(n)|^2\}w^*_{21\text{opt}}\Big] + \frac{a_{12}}{a_{22}}\end{aligned} \tag{3.26}$$

where $Dw_{12\text{opt}}$ is defined in (3.21). Substituting for $w_{21\text{opt}}$ from (3.25) in (3.26), we get after some manipulations,

$$\begin{aligned}\epsilon_1 \;&=\; \frac{1}{\Delta_{\epsilon_1}}\Big[-a_{11}a_{22}(a_{21} + (\frac{-a_{21}}{a_{11}} + \epsilon_2)a_{11})^*\delta_{11}E\{|I_1(n)|^2\} \\ &\quad -a_{12}a_{22}(a_{22} + (\frac{-a_{21}}{a_{11}} + \epsilon_2)a_{12})^*\delta_{12}E\{|I_2(n)|^2\} - a_{22}(\frac{-a_{21}}{a_{11}} + \epsilon_2)^*E\{|n_1(n)|^2\} \\ &\quad +a_{21}a_{12}(a_{21} + (\frac{-a_{21}}{a_{11}} + \epsilon_2)a_{11})^*\delta_{11}E\{|I_1(n)|^2\} \\ &\quad +a_{12}a_{22}(a_{22} + (\frac{-a_{21}}{a_{11}} + \epsilon_2)a_{12})^*\delta_{12}E\{|I_2(n)|^2\} + E\{|n_2(n)|^2\}a_{12}\Big],\end{aligned} \tag{3.27}$$

where,

$$\Delta_{\epsilon_1} = a_{22}\Big[a_{21}(a_{21} + (\frac{-a_{21}}{a_{11}} + \epsilon_2)a_{11})^*\delta_{11}E\{|I_1(n)|^2\}$$

$$+a_{22}(a_{22} + (\frac{-a_{21}}{a_{11}} + \epsilon_2)a_{12})^*\delta_{12}E\{|I_2(n)|^2\} + E\{|n_2(n)|^2\}\Big] \quad (3.28)$$

It is easy to notice that the second and the fifth terms in (3.27) cancel one another resulting in

$$\epsilon_1 = \frac{1}{\Delta_{\epsilon_1}}\Big[-a_{11}^2 a_{22}\epsilon_2^*\delta_{11}E\{|I_1(n)|^2\} - a_{22}(\frac{-a_{21}}{a_{11}} + \epsilon_2)^*E\{|n_1(n)|^2\}$$

$$+a_{12}a_{21}\epsilon_2^*a_{11}\delta_{11}E\{|I_1(n)|^2\} + E\{|n_2(n)|^2\}a_{12}\Big], \quad (3.29)$$

or after collecting terms,

$$\epsilon_1 = \frac{a_{22}}{\Delta_{\epsilon_1}}\Big[\Big(a_{11}^2(\frac{a_{12}}{a_{22}}\frac{a_{21}}{a_{11}} - 1)\delta_{11}E\{|I_1(n)|^2\} - E\{|n_1(n)|^2\}\Big)\epsilon_2^*$$

$$+(\frac{a_{21}}{a_{11}})^*E\{|n_1(n)|^2\} + \frac{a_{12}}{a_{22}}E\{|n_2(n)|^2\}\Big]. \quad (3.30)$$

Δ_{ϵ_1} after some simplification becomes,

$$\Delta_{\epsilon_1} = a_{22}\Big[\Big(a_{21}a_{11}\delta_{11}E\{|I_1(n)|^2\} + a_{22}a_{12}^*\delta_{12}E\{|I_2(n)|^2\}\Big)\epsilon_2^*$$

$$+a_{22}(a_{22} - \frac{a_{21}}{a_{11}}a_{12})^*\delta_{12}E\{|I_2(n)|^2\} + E\{|n_2(n)|^2\}\Big]. \quad (3.31)$$

Now write (3.30) together with (3.31) as follows

$$\epsilon_1 = \frac{X_1\epsilon_2^* + A}{X_2\epsilon_2^* + X_3}, \tag{3.32}$$

or,

$$\epsilon_1\epsilon_2^* X_2 + \epsilon_1 X_3 = X_1\epsilon_2^* + A, \tag{3.33}$$

where

$$X_1 = -\Big[a_{11}^2(1 - \frac{a_{12}a_{21}}{a_{22}a_{11}})\delta_{11}E\{|I_1(n)|^2\} + E\{|n_1(n)|^2\}\Big], \tag{3.34}$$

$$A = (\frac{a_{21}}{a_{11}})^* E\{|n_1(n)|^2\} + \frac{a_{12}}{a_{22}}E\{|n_2(n)|^2\}, \tag{3.35}$$

$$X_2 = a_{21}a_{11}\delta_{11}E\{|I_1(n)|^2\} + a_{22}a_{12}^*\delta_{12}E\{|I_2(n)|^2\}, \tag{3.36}$$

$$X_3 = a_{22}^2[1 - \frac{a_{21}a_{12}}{a_{11}a_{22}}]^*\delta_{12}E\{|I_2(n)|^2\} + E\{|n_2(n)|^2\}, \tag{3.37}$$

We claim that in (3.33) $X_2\epsilon_1\epsilon_2^* << \epsilon_1 X_3$ with ϵ_1 and ϵ_2; the perturbation on the optimum weight are considered to be small enough at steady state. This could be justified as follows; since $a_{21} << a_{11}$ and $a_{12} << a_{22}$, hence for high signal to noise ratio; $(\frac{E\{|I_2(n)|^2\}}{E\{|n_2(n)|^2\}})$:

$$X_3 \approx a_{22}^2 E\{|I_2(n)|^2\}\delta_{12}, \tag{3.38}$$

$$X_2 = \frac{a_{21}}{a_{11}}a_{11}^2\delta_{11}E\{|I_1(n)|^2\} + (\frac{a_{12}}{a_{22}})^* a_{22}^2\delta_{12}E\{|I_2(n)|^2\}, \tag{3.39}$$

so that X_2 and X_3 are in the same order and our claim follows. Therefore we can write (3.33) as

$$\epsilon_1 X_3 - \epsilon_2^* X_1 \approx A \tag{3.40}$$

Perturbation on Optimal Weight, $w_{21\text{opt}}$

Similar step can be followed to determine ϵ_2 from (3.25) together with (3.22). In fact,

$$\epsilon_2 = w_{21\text{opt}} + \frac{a_{21}}{a_{11}}, \tag{3.41}$$

and using (3.22) we have,

$$\begin{aligned}
\epsilon_2 \;=\; & \frac{-1}{Dw_{21\text{opt}}}\Big[a_{21}(a_{11} + w_{12\text{opt}}a_{21})^*\delta_{21}E\{|I_1(n)|^2\} + a_{22}(a_{12} + w_{12\text{opt}}a_{22})^* \cdot \\
& \delta_{22}E\{|I_2(n)|^2\} + E\{|n_2(n)|^2\}w^*_{12\text{opt}}\Big] + \frac{a_{21}}{a_{11}},
\end{aligned} \tag{3.42}$$

where $Dw_{21\text{opt}}$ is given in (3.23).

Simplifying step by step we can get,

$$\begin{aligned}
\epsilon_2 \;=\; & \frac{1}{\Delta_{\epsilon_2}}\Big[-\delta_{21}E\{|I_1(n)|^2\}a_{11}a_{21}(a_{11} + (\frac{-a_{12}}{a_{22}} + \epsilon_1)a_{21})^* \\
& -\delta_{22}E\{|I_2(n)|^2\}a_{11}a_{22}(a_{12} + (\frac{-a_{12}}{a_{22}} + \epsilon_1)a_{22})^* - a_{11}E\{|n_2(n)|^2\}(\frac{-a_{12}}{a_{22}} + \epsilon_1)^* \\
& +\delta_{21}E\{|I_1(n)|^2\}a_{21}a_{11}(a_{11} + (\frac{-a_{12}}{a_{22}} + \epsilon_1)a_{21})^* + \delta_{22}E\{|I_2(n)|^2\}a_{12}a_{21} \cdot \\
& (a_{12} + (\frac{-a_{12}}{a_{22}} + \epsilon_1)a_{22})^* + E\{|n_1(n)|^2\}a_{21}\Big],
\end{aligned} \tag{3.43}$$

where,

$$\begin{aligned}
\Delta_{\epsilon_2} \;=\; & a_{11}\Big[a_{11}(a_{11} + (\frac{-a_{12}}{a_{22}} + \epsilon_1)a_{21})^*\delta_{21}E\{|I_1(n)|^2\} \\
& +a_{12}(a_{12} + (\frac{-a_{12}}{a_{22}} + \epsilon_1)a_{22})^*\delta_{22}E\{|I_2(n)|^2\} + E\{|n_1(n)|^2\}\Big],
\end{aligned} \tag{3.44}$$

or after collecting terms;

$$\epsilon_2 = \frac{1}{\Delta_{\epsilon_2}} a_{11}\Big[\Big(a_{22}^2(\frac{a_{12}}{a_{22}}\frac{a_{21}}{a_{11}} - 1)\delta_{22}E\{|I_2(n)|^2\} - E\{|n_2(n)|^2\}\Big)\epsilon_1^*$$

$$+(\frac{a_{12}}{a_{22}})^* E\{|n_2(n)|^2\} + \frac{a_{21}}{a_{11}}E\{|n_1(n)|^2\}\Big]. \quad (3.45)$$

Δ_{ϵ_2} after some algebraic simplification becomes,

$$\Delta_{\epsilon_2} = a_{11}\Big[\Big(a_{21}^* a_{11}\delta_{21}E\{|I_1(n)|^2\} + a_{12}a_{22}\delta_{22}E\{|I_2(n)|^2\}\Big)\epsilon_1^*$$

$$+a_{11}^2(1 - \frac{a_{21}}{a_{11}}\frac{a_{11}}{a_{12}})^*\delta_{21}E\{|I_1(n)|^2\} + E\{|n_1(n)|^2\}\Big]. \quad (3.46)$$

We can now write (3.45) together with (3.46) as follows;

$$\epsilon_2 = \frac{X_4\epsilon_1^* + B}{X_5\epsilon_1^* + X_6}, \quad (3.47)$$

or,

$$\epsilon_2\epsilon_1^* X_5 + \epsilon_2 X_6 = X_4\epsilon_1^* + B, \quad (3.48)$$

where,

$$X_4 = -\Big[a_{22}^2[1 - \frac{a_{21}a_{12}}{a_{22}a_{11}}]\delta_{22}E\{|I_2(n)|^2\} + E\{|n_2(n)|^2\}\Big], \quad (3.49)$$

$$B = (\frac{a_{12}}{a_{22}})^* E\{|n_2(n)|^2\} + \frac{a_{21}}{a_{11}}E\{|n_1(n)|^2\}, \quad (3.50)$$

$$X_5 = a_{11}a_{21}^*\delta_{21}E\{|I_1(n)|^2\} + a_{12}a_{22}\delta_{22}E\{|I_2(n)|^2\}, \quad (3.51)$$

$$X_6 = a_{11}^2[1 - \frac{a_{12}a_{21}}{a_{11}a_{22}}]^*\delta_{21}E\{|I_1(n)|^2\} + E\{|n_1(n)|^2\}. \quad (3.52)$$

Again by using the same argument as in (3.40), we get

$$\epsilon_2 X_6 - \epsilon_1^* X_4 \approx B. \tag{3.53}$$

Determination of Perturbations Final Expressions

Simultaneously solving (3.40) and (3.53), we get

$$\epsilon_1 = \frac{X_6^* A + X_1 B^*}{X_3 X_6^* - X_1 X_4^*}, \tag{3.54}$$

$$\epsilon_2 = \frac{X_4 A^* + X_3^* B}{X_3^* X_6 - X_1^* X_4}. \tag{3.55}$$

Using (3.34), (3.35), (3.37) together with (3.49), (3.50) and (3.52) in (3.54), we get,

$$\begin{aligned}\epsilon_1 = \frac{1}{\Delta}\Bigg(&\Big[a_{11}^2(1-\frac{a_{12}}{a_{11}}\frac{a_{21}}{a_{22}})E\{|I_1(n)|^2\}\delta_{21} + E\{|n_1(n)|^2\}\Big]\Big[(\frac{a_{21}}{a_{11}})^* E\{|n_1(n)|^2\} \\ &+\frac{a_{12}}{a_{22}}E\{|n_2(n)|^2\}\Big] - \Big[a_{11}^2(1-\frac{a_{12}}{a_{11}}\frac{a_{21}}{a_{22}})E\{|I_1(n)|^2\}\delta_{11} + E\{|n_1(n)|^2\}\Big]. \\ &\Big[(\frac{a_{21}}{a_{11}})^* E\{|n_1(n)|^2\} + \frac{a_{12}}{a_{22}}E\{|n_2(n)|^2\}\Big]\Bigg)\end{aligned} \tag{3.56}$$

Without loss of generality, we will assume that noise variance in V and H polarized channels are equal; $E\{|n_1(n)|^2\} = E\{|n_2(n)|^2\} = E\{|n_n(n)|^2\}$. This assumption is particularly useful when calculating the probability of error. In fact, this is the most difficult case that puts the worst requirement on the discriminators. Using this assumption and the definition of the channel parameter, we write (3.56) as

$$\epsilon_1 = \frac{1}{\Delta}a_{11}^2[1 - r_1 r_2 e^{j(\phi_1+\phi_2)}][r_1 e^{j\phi_1} + r_2 e^{-j\phi_2}]E\{|n_n(n)|^2\}E\{|I_1(n)|^2\}(\delta_{21}-\delta_{11})$$

$$(3.57)$$

where

$$\Delta = X_3 X_6^* - X_1 X_4^* \tag{3.58}$$

Δ in (3.58) can be simplified as follows: By using (3.34), (3.37), (3.49) and (3.52), we get,

$$\begin{aligned}\Delta \;=\; & \left[a_{22}^2(1-\frac{a_{12}}{a_{11}}\frac{a_{21}}{a_{22}})^* E\{|I_2(n)|^2\}\delta_{12} + E\{|n_n(n)|^2\}\right]\left[a_{11}^2(1-\frac{a_{12}}{a_{11}}\frac{a_{21}}{a_{22}}).\right.\\ & \left. E\{|I_1(n)|^2\}\delta_{21} + E\{|n_n(n)|^2\}\right] - \left[a_{11}^2(1-\frac{a_{12}}{a_{11}}\frac{a_{21}}{a_{22}})E\{|I_1(n)|^2\}\delta_{11}\right.\\ & \left. + E\{|n_n(n)|^2\}\right].\left[a_{22}^2(1-\frac{a_{12}}{a_{11}}\frac{a_{21}}{a_{22}})^* E\{|I_2(n)|^2\}\delta_{22} + E\{|n_n(n)|^2\}\right].\end{aligned} \tag{3.59}$$

Let

$$k = 1 - \frac{a_{21}}{a_{11}}\frac{a_{12}}{a_{22}} = 1 - r_1 r_2 e^{j(\phi_1+\phi_2)}, \tag{3.60}$$

then we can write

$$\begin{aligned}\Delta \;=\; & a_{22}^2 a_{11}^2 |k|^2 E\{|I_1(n)|^2\}E\{|I_2(n)|^2\}\delta_{12}\delta_{21} + a_{22}^2 k^* E\{|I_2(n)|^2\}\delta_{12}E\{|n_n(n)|^2\}\\ & + a_{11}^2 k E\{|I_1(n)|^2\}\delta_{21}E\{|n_n(n)|^2\} - a_{11}^2 a_{22}^2 |k|^2 E\{|I_1(n)|^2\}E\{|I_2(n)|^2\}\delta_{11}\delta_{22}\\ & - a_{11}^2 k E\{|I_1(n)|^2\}\delta_{11}E\{|n_n(n)|^2\} - a_{22}^2 k^* E\{|I_2(n)|^2\}E\{|n_n(n)|^2\}\delta_{22}\end{aligned} \tag{3.61}$$

Further simplification yields,

$$\Delta \;=\; a_{22}^2 a_{11}^2 |k|^2 E\{|I_1(n)|^2\}E\{|I_2(n)|^2\}(\delta_{12}\delta_{21} - \delta_{11}\delta_{22})$$

$$+a_{22}^2 k^* E\{|I_2(n)|^2\}E\{|n_n(n)|^2\}(\delta_{12}-\delta_{22})$$

$$+a_{11}^2 k E\{|I_1(n)|^2\}E\{|n_n(n)|^2\}(\delta_{21}-\delta_{11}), \tag{3.62}$$

and using (3.60) we get,

$$\begin{aligned}\Delta \; = \; & a_{22}^2 a_{11}^2 E\{|I_1(n)|^2\}E\{|I_2(n)|^2\}(1-2r_1r_2cos(\phi_1+\phi_2)+r_1^2r_2^2). \\ & (\delta_{12}\delta_{21}-\delta_{11}\delta_{22}) + a_{22}^2 E\{|I_2(n)|^2\}E\{|n_n(n)|^2\}(\delta_{12}-\delta_{22})(1-r_1r_2e^{-j(\phi_1+\phi_2)}) \\ & +a_{11}^2 E\{|I_1(n)|^2\}E\{|n_n(n)|^2\}(\delta_{21}-\delta_{11})(1-r_1r_2e^{j(\phi_1+\phi_2)}).\end{aligned} \tag{3.63}$$

Let Δ_R and Δ_I be the real and imaginary part of Δ, that is;

$$\Delta = \Delta_R + j\Delta_I, \tag{3.64}$$

then,

$$\begin{aligned}\Delta_R \; = \; & a_{11}^2 a_{22}^2 (1-2r_1r_2cos(\phi_1+\phi_2)+r_1^2r_2^2).E\{|I_1(n)|^2\}E\{|I_2(n)|^2\} \\ & (\delta_{12}\delta_{21}-\delta_{11}\delta_{22}) + \Big[|a_{22}|^2 E\{|I_2(n)|^2\}E\{|n_n(n)|^2\}(\delta_{12}-\delta_{22}) + a_{11}^2 E\{|I_1(n)|^2\}. \\ & E\{|n_n(n)|^2\}(\delta_{21}-\delta_{11})\Big].(1-r_1r_2cos(\phi_1+\phi_2)), \\ \Delta_I \; = \; & \Big[a_{22}^2 E\{|I_2(n)|^2\}(\delta_{12}-\delta_{22}) - a_{11}^2 E\{|I_1(n)|^2\}(\delta_{21}-\delta_{11})\Big].\end{aligned}$$

$$r_1 r_2 sin(\phi_1 + \phi_2) E\{|n_n(n)|^2\}. \tag{3.65}$$

Finally from (3.57)

$$\epsilon_1 = \frac{1}{\Delta_R + j\Delta_I} a_{11}^2 \Big[r_1 e^{j\phi_1} + r_2 e^{-j\phi_2} - r_1^2 r_2 e^{j(2\phi_1 + \phi_2)} - r_1 r_2^2 e^{j\phi_1} \Big].$$

$$E\{|n_n(n)|^2\} E\{|I_1(n)|^2\} (\delta_{21} - \delta_{11}) \tag{3.66}$$

Defining ϵ_1 as;

$$\epsilon_1 = \frac{\epsilon_{1AR} + j\epsilon_{1AI}}{\Delta_R + j\Delta_I}, \tag{3.67}$$

then,

$$\begin{aligned} \epsilon_1 &= \epsilon_{1R} + j\epsilon_{1I} \\ &= \frac{1}{\Delta_R^2 + \Delta_I^2} \Big[(\epsilon_{1AR}\Delta_R + \epsilon_{1AI}\Delta_I) + j(\epsilon_{1AI}\Delta_R - \epsilon_{1AR}\Delta_I) \Big] \end{aligned} \tag{3.68}$$

with

$$\epsilon_{1R} = \frac{\epsilon_{1AR}\Delta_R + \epsilon_{1AI}\Delta_I}{|\Delta|^2}, \tag{3.69}$$

$$\epsilon_{1I} = \frac{\epsilon_{1AI}\Delta_R - \epsilon_{1AR}\Delta_I}{|\Delta|^2}, \tag{3.70}$$

and

$$|\Delta|^2 = \Delta_R^2 + \Delta_I^2. \tag{3.71}$$

From (3.66) we have,

$$\epsilon_{1AR} = a_{11}^2\Big[r_1cos\phi_1 + r_2cos\phi_2 - r_1^2r_2cos(2\phi_1+\phi_2) - r_1r_2^2cos\phi_1\Big].$$

$$E\{|n_n(n)|^2\}E\{|I_1(n)|^2\}(\delta_{21}-\delta_{11}), \tag{3.72}$$

$$\epsilon_{1AI} = a_{11}^2\Big[r_1sin\phi_1 - r_2sin\phi_2 - r_1^2r_2sin(2\phi_1+\phi_2) - r_1r_2^2sin\phi_1\Big].$$

$$E\{|n_n(n)|^2\}E\{|I_1(n)|^2\}(\delta_{21}-\delta_{11}). \tag{3.73}$$

Similar calculation can be performed for ϵ_2: Using (3.34), (3.35) and (3.37) together with (3.49), (3.50) and (3.52) in (3.55) we get,

$$\epsilon_2 = \frac{-1}{\Delta^*}\Big[\Big[a_{22}^2(1-\frac{a_{12}}{a_{11}}\frac{a_{21}}{a_{22}})E\{|I_2(n)|^2\}\delta_{22} + E\{|n_n(n)|^2\}\Big].\Big[\frac{a_{21}}{a_{11}}E\{|n_n(n)|^2\}$$

$$+(\frac{a_{12}}{a_{22}})^*E\{|n_n(n)|^2\}\Big] + \Big[a_{22}^2(1-\frac{a_{12}}{a_{11}}\frac{a_{21}}{a_{22}})E\{|I_2(n)|^2\}\delta_{12} + E\{|n_n(n)|^2\}\Big]$$

$$\Big[\frac{a_{21}}{a_{11}}E\{|n_n(n)|^2\} + (\frac{a_{12}}{a_{22}})^*E\{|n_n(n)|^2\}\Big]\Big], \tag{3.74}$$

with Δ as in (3.58) and the fact that $E\{|n_1(n)|^2\} = E\{|n_2(n)|^2\} = E\{|n_n(n)|^2\}$. Using the definition of the channel parameter we write (3.74),

$$\epsilon_2 = \frac{a_{22}^2}{\Delta^*}[1-r_1r_2e^{j(\phi_1+\phi_2)}][r_1e^{-j\phi_1}+r_2e^{j\phi_2}]E\{|n_n(n)|^2\}E\{|I_2(n)|^2\}(\delta_{12}-\delta_{22}). \tag{3.75}$$

Defining ϵ_2 as ,

$$\epsilon_2 = \frac{\epsilon_{2AR}+j\,\epsilon_{2AI}}{\Delta_R - j\,\Delta_I} \tag{3.76}$$

then from (3.75) we have,

$$\epsilon_{2AR} = a_{22}^2\Big[r_1cos\phi_1 + r_2cos\phi_2 - r_1^2r_2cos\phi_2 - r_1r_2^2cos(\phi_1+2\phi_2)\Big].$$

$$E\{|n_n(n)|^2\}E\{|I_2(n)|^2\}(\delta_{12}-\delta_{22}), \tag{3.77}$$

$$\epsilon_{2AI} = a_{22}^2\Big[-r_1sin\phi_1 + r_2sin\phi_2 - r_1^2r_2sin\phi_2 - r_1r_2^2sin(\phi_1+2\phi_2)\Big].$$

$$E\{|n_n(n)|^2\}E\{|I_2(n)|^2\}(\delta_{12}-\delta_{22}). \tag{3.78}$$

Clearly,

$$\epsilon_2 = \epsilon_{2R} + j\epsilon_{2I} \tag{3.79}$$

$$= \frac{1}{\Delta_R^2+\Delta_I^2}[(\epsilon_{2AR}\Delta_R - \epsilon_{2AI}\Delta_I) + j(\epsilon_{2AR}\Delta_I + \epsilon_{2AI}\Delta_R)], \tag{3.80}$$

then,

$$\epsilon_{2R} = \frac{\epsilon_{2AR}\Delta_R - \epsilon_{2AI}\Delta_I}{|\Delta|^2} \tag{3.81}$$

$$\epsilon_{2I} = \frac{\epsilon_{2AR}\Delta_I + \epsilon_{2AI}\Delta_R}{|\Delta|^2} \tag{3.82}$$

3.2.4 Canceler Optimal Outputs

Next using (3.25) in (3.4) with $w_{12} = w_{12\text{opt}}$ and $w_{21} = w_{21\text{opt}}$, we obtain $y_1(n)$

$$y_1(n) = \frac{1}{\Big(1-(\frac{-a_{12}}{a_{22}}+\epsilon_1)(\frac{-a_{21}}{a_{11}}+\epsilon_2)\Big)}\Big[I_1(n)[a_{11}+(\frac{-a_{12}}{a_{22}}+\epsilon_1)a_{21}]$$

$$+I_2(n)[a_{12}+(-\frac{a_{12}}{a_{22}}+\epsilon_1)a_{22}] + n_1(n) + n_2(n)[\frac{-a_{12}}{a_{22}}+\epsilon_1]\Big], \tag{3.83}$$

and after combining terms,

$$y_1(n) = \frac{a_{11}I_1(n)[1 - \frac{a_{12}a_{21}}{a_{11}a_{22}} + \epsilon_1 \frac{a_{21}}{a_{11}}] + I_2(n)\epsilon_1 a_{22} + n_1(n) - n_2(n)\frac{a_{12}}{a_{22}} + n_2(n)\epsilon_1}{1 - \frac{a_{12}a_{21}}{a_{11}a_{22}} + \frac{a_{12}}{a_{22}}\epsilon_2 + \frac{a_{21}}{a_{11}}\epsilon_1 - \epsilon_1\epsilon_2} \tag{3.84}$$

3.2.5 Decision Parameters with Amplitude Compensation at the Canceler Output

The co-pole horizontally polarized signal at the output of the channel is $a_{11}I_1(n)$ and hence it is reasonable to take $y_1(n)a_{11}$ as estimate of this signal by compensating for the attenuation in the co-pol by a_{11}. Therefore, we will take $\hat{I}_1(n) = \frac{y_1(n)}{a_{11}}$ as estimate of the transmitted signal $I_1(n)$. We will also assume the $\epsilon_1\epsilon_2$ in (3.84) is negligible with respect to the other terms in the denominator of this equation. Hence from (3.84) we can write,

$$\begin{aligned}
\hat{I}_1(n)[1 - \frac{a_{12}a_{21}}{a_{11}a_{22}} + \frac{a_{12}}{a_{22}}\epsilon_2 + \frac{a_{21}}{a_{11}}\epsilon_1] &= I_1(n)[1 - \frac{a_{12}a_{21}}{a_{11}a_{22}} + \frac{a_{21}}{a_{11}}\epsilon_1 + \frac{a_{12}}{a_{22}}\epsilon_2 - \frac{a_{12}}{a_{22}}\epsilon_2] \\
&\quad + I_2(n)\epsilon_1\frac{a_{22}}{a_{11}} + \frac{n_1(n)}{a_{11}} + n_2(n)(\frac{-a_{12}}{a_{11}a_{22}} + \frac{\epsilon_1}{a_{11}})
\end{aligned} \tag{3.85}$$

Define,

$$Z_1(n) \triangleq \hat{I}_1(n) - I_1(n), \tag{3.86}$$

with $Z_1(n)$ is taken as the decision parameter. That is the probability of error is given by $P_1(e) = P\{|Z_1(n)| > c\}$ with c is the half of the distance between two signals in the corresponding signal space. From (3.85) together with (3.86), we have

$$Z_1(n) = \frac{-I_1(n)\epsilon_2\frac{a_{12}}{a_{22}} + I_2(n)\epsilon_1\frac{a_{22}}{a_{11}} + \frac{n_1(n)}{a_{11}} + \frac{n_2(n)}{a_{11}}(\frac{-a_{12}}{a_{22}} + \epsilon_1)}{1 - \frac{a_{12}a_{21}}{a_{11}a_{22}} + \frac{a_{12}}{a_{22}}\epsilon_2 + \frac{a_{21}}{a_{11}}\epsilon_1} \tag{3.87}$$

Further simplification leads to,

$$Z_1(n) = \frac{-I_1(n)K + I_2(n)\epsilon_1 \frac{a_{22}}{a_{11}} + \frac{n_1(n)}{a_{11}} - \frac{n_2(n)}{a_{11}} r_1 e^{j\phi_1} + \frac{n_2(n)}{a_{11}} \epsilon_1}{1 - r_1 r_2 e^{j(\phi_1+\phi_2)} + K + V} \quad (3.88)$$

where

$$K = \epsilon_2 \frac{a_{12}}{a_{22}}, \quad (3.89)$$

$$V = \epsilon_1 \frac{a_{21}}{a_{11}}. \quad (3.90)$$

In order to be able to calculate the probability of error, we must find the real and imaginary part of $Z_1(n)$. For this, we first find K_R, V_R ,K_I and V_I, the real and imaginary parts of K and V respectively.

Define $K = \frac{K_A}{\Delta^*}$, then from (2.6) and (3.75), we have

$$K = \frac{K_A}{\Delta^*} = \frac{a_{22}^2}{\Delta_R - j\Delta_I} E\{|n_n(n)|^2\} E\{|I_2(n)|^2\}(\delta_{12} - \delta_{22}).$$

$$\left[r_1^2 + r_1 r_2 e^{j(\phi_1+\phi_2)} - r_1^3 r_2 e^{j(\phi_1+\phi_2)} - r_1^2 r_2^2 e^{j2(\phi_1+\phi_2)}\right]. \quad (3.91)$$

From (3.91), the real and imaginary parts of K_A;

$$K_{AR} = a_{22}^2 E\{|n_n(n)|^2\} E\{|I_2(n)|^2\}(\delta_{12} - \delta_{22}).$$

$$\left[r_1^2 + r_1 r_2 cos(\phi_1 + \phi_2) - r_1^3 r_2 cos(\phi_1 + \phi_2) - r_1^2 r_2^2 cos(2(\phi_1 + \phi_2))\right] \quad (3.92)$$

$$K_{AI} = a_{22}^2 E\{|n_n(n)|^2\} E\{|I_2(n)|^2\}(\delta_{12} - \delta_{22}).$$
$$\left[r_1 r_2 sin(\phi_1 + \phi_2) - r_1^3 r_2 sin(\phi_1 + \phi_2) - r_1^2 r_2^2 sin(2(\phi_1 + \phi_2))\right] \quad (3.93)$$

But,

$$K = K_R + jK_I = \frac{K_{AR} + jK_{AI}}{\Delta_R - j\Delta_I}, \tag{3.94}$$

therefore,

$$K_R = \frac{K_{AR}\Delta_R - K_{AI}\Delta_I}{\Delta_R^2 + \Delta_I^2} \tag{3.95}$$

$$K_I = \frac{K_{AR}\Delta_I + K_{AI}\Delta_R}{\Delta_R^2 + \Delta_I^2} \tag{3.96}$$

with K_{AR} and K_{AI} as in (3.92) and (3.93), while Δ_R and Δ_I as in (3.65) . Note that K_R and K_I are functions of the random variables ϕ_1 and ϕ_2.

Similarly, define $V = \frac{V_A}{\Delta}$, then from (2.6) and (3.66),

$$V = \frac{V_A}{\Delta} = \frac{a_{11}^2 E\{|n_n(n)|^2\}E\{|I_1(n)|^2\}(\delta_{21} - \delta_{11})}{\Delta_R + j\Delta_I}\Big[r_1 r_2 e^{j(\phi_1+\phi_2)} + r_2^2 - r_1^2 r_2^2 e^{j2(\phi_1+\phi_2)} - r_1 r_2^3 e^{j(\phi_1+\phi_2)}\Big] \tag{3.97}$$

From (3.97), the real and imaginary of V_A are ,

$$V_{AR} = a_{11}^2 E\{|n_n(n)|^2\}E\{|I_1(n)|^2\}(\delta_{21} - \delta_{11})\Big[r_1 r_2 cos(\phi_1 + \phi_2) + r_2^2 - r_1 r_2^3 cos(\phi_1 + \phi_2) - r_1^2 r_2^2 cos(2(\phi_1 + \phi_2))\Big] \tag{3.98}$$

$$V_{AI} = a_{11}^2 E\{|n_n(n)|^2\}E\{|I_1(n)|^2\}(\delta_{21} - \delta_{11})\Big[r_1 r_2 sin(\phi_1 + \phi_2)$$

$$-r_1 r_2^3 sin(\phi_1+\phi_2) - r_1^2 r_2^2 sin(2(\phi_1+\phi_2))\Big] \quad (3.99)$$

But,

$$V = V_R + jV_I = \frac{V_{AR} + jV_{AI}}{\Delta_R + j\Delta_I}, \quad (3.100)$$

therefore,

$$V_R = \frac{V_{AR}\Delta_R + V_{AI}\Delta_I}{\Delta_R^2 + \Delta_I^2} \quad (3.101)$$

$$V_I = \frac{V_{AI}\Delta_R - V_{AR}\Delta_I}{\Delta_R^2 + \Delta_I^2} \quad (3.102)$$

with V_{AR} and V_{AI} as in (3.98) and (3.99) while Δ_R and Δ_I as in (3.65). Similar to the real and imaginary part of K, V_R and V_I are functions of the random variables, ϕ_1 and ϕ_2.

Using (3.95), (3.96), (3.101), (3.102) in (3.88), we get;

$$\begin{aligned} Z_1 &= \frac{1}{Z_D}\Big[-(I_{1R}+jI_{1I})(K_R+jK_I) + (I_{2R}+jI_{2I})(\epsilon_{1R}+j\epsilon_{1I})\frac{a_{22}}{a_{11}} \\ &+(n_{1R}+jn_{1I})\frac{1}{a_{11}} - \frac{r_1}{a_{11}}(n_{2R}+jn_{2I})(cos\phi_1 + jsin\phi_1) \\ &+\frac{1}{a_{11}}(n_{2R}+jn_{2I})(\epsilon_{1R}+j\epsilon_{1I})\Big] \end{aligned} \quad (3.103)$$

where we used $I_i = I_{iR} + jI_{iI}$ and $n_i = n_{iR} + jn_{iI}, \quad i = 1,2$ and in our notation, we also drop the dependence of terms on the sampling time n.

Also from (3.88) the real and imaginary part of the denominator Z_D are respectively given by,

$$Z_{DR} = 1 - r_1 r_2 cos(\phi_1+\phi_2) + K_R + V_R \quad (3.104)$$

$$Z_{DI} = -r_1 r_2 sin(\phi_1 + \phi_2) + K_I + V_I. \qquad (3.105)$$

Clearly Z_{DR} and Z_{DI} are functions of the random variables ϕ_1 and ϕ_2.

The real and imaginary part of numerator of (3.103), Z_N are given by,

$$Z_{NR} = -(I_{1R}K_R - I_{1I}K_I) + (I_{2R}\epsilon_{1R} - I_{2I}\epsilon_{1I})\frac{a_{22}}{a_{11}}$$

$$+\frac{n_{1R}}{a_{11}} - \frac{r_1}{a_{11}}(n_{2R}cos\phi_1 - n_{2I}sin\phi_1) + \frac{1}{a_{11}}[n_{2R}\epsilon_{1R} - n_{2I}\epsilon_{1I}] \qquad (3.106)$$

$$Z_{NI} = -(I_{1R}K_I + I_{1I}K_R) + (I_{2R}\epsilon_{1I} + I_{2I}\epsilon_{1R})\frac{a_{22}}{a_{11}}$$

$$+\frac{n_{1I}}{a_{11}} - \frac{r_1}{a_{11}}(n_{2R}sin\phi_1 + n_{2I}cos\phi_1) + \frac{1}{a_{11}}[n_{2R}\epsilon_{1I} + n_{2I}\epsilon_{1R}] \qquad (3.107)$$

Z_{NR} and Z_{NI} beside being function of the random variables ϕ_1 and ϕ_2, they are also function of the signals and noises random variables.

Finally, we can write (3.103) as

$$Z_1 = Z_{1R} + jZ_{1I} = \frac{Z_{NR} + jZ_{NI}}{Z_{DR} + jZ_{DI}} \qquad (3.108)$$

with

$$Z_{1R} = \frac{Z_{NR}Z_{DR} + Z_{NI}Z_{DI}}{Z_{DR}^2 + Z_{DI}^2} \qquad (3.109)$$

$$Z_{1I} = \frac{Z_{NI}Z_{DR} - Z_{NR}Z_{DI}}{Z_{DR}^2 + Z_{DI}^2} \qquad (3.110)$$

Using (3.104), (3.105), (3.106) and (3.107) in (3.108), we can get after some simplification, which emphasizes the dependency of the different terms on the different random variables, the real part of the decision variable:

$$
\begin{aligned}
Z_{1R} &= \frac{1}{|\Delta_Z|^2}\Big[I_{1R}(K_R Z_{DR} + K_I Z_{DI}) + I_{1I}(K_I Z_{DR} - K_R Z_{DI}) \\
&+ I_{2R}(\epsilon_{1R} Z_{DR} + \epsilon_{1I} Z_{DI})\frac{a_{22}}{a_{11}} + I_{2I}(\epsilon_{1R} Z_{DI} - \epsilon_{1I} Z_{DR})\frac{a_{22}}{a_{11}} \\
&+ n_{1R}\frac{Z_{DR}}{a_{11}} + n_{1I}\frac{Z_{DI}}{a_{11}} \\
&+ n_{2R}[(\frac{\epsilon_{1R}}{a_{11}} - \frac{r_1 cos\phi_1}{a_{11}})Z_{DR} + (\frac{\epsilon_{1I}}{a_{11}} - \frac{r_1 sin\phi_1}{a_{11}})Z_{DI}] \\
&+ n_{2I}[(\frac{-\epsilon_{1I}}{a_{11}} + \frac{r_1 sin\phi_1}{a_{11}})Z_{DR} + (\frac{\epsilon_{1R}}{a_{11}} - \frac{r_1 cos\phi_1}{a_{11}})Z_{DI}]\Big]. \qquad (3.111)
\end{aligned}
$$

Similar expression can be found for Z_{1I}.

$$
\begin{aligned}
Z_{1I} &= \frac{1}{|\Delta_Z|^2}\Big[-I_{1R}(K_I Z_{DR} - K_R Z_{DI}) - I_{1I}(K_R Z_{DR} + K_I Z_{DI}) \\
&- I_{2R}(\epsilon_{1R} Z_{DI} - \epsilon_{1I} Z_{DR})\frac{a_{22}}{a_{11}} + I_{2I}(\epsilon_{1R} Z_{DR} + \epsilon_{1I} Z_{DI})\frac{a_{22}}{a_{11}} \\
&- n_{1R}\frac{Z_{DI}}{a_{11}} + n_{1I}\frac{Z_{DR}}{a_{11}} \\
&- n_{2R}[(\frac{-\epsilon_{1I}}{a_{11}} + \frac{r_1 sin\phi_1}{a_{11}})Z_{DR} + (\frac{\epsilon_{1R}}{a_{11}} - \frac{r_1 cos\phi_1}{a_{11}})Z_{DI}] \\
&+ n_{2I}[(\frac{\epsilon_{1R}}{a_{11}} - \frac{r_1 cos\phi_1}{a_{11}})Z_{DR} + (\frac{\epsilon_{1I}}{a_{11}} - \frac{r_1 sin\phi_1}{a_{11}})Z_{DI}]\Big] \qquad (3.112)
\end{aligned}
$$

The Decision Parameters Final Expressions

Finally we write the real and imaginary parts of $Z_1(n)$ in terms of the random variable representing the real and imaginary part of signal and noises of channel 1;

$$Z_{1R} = I_{1R}Y_1 + I_{1I}Y_2 + I_{2R}Y_3 + I_{2I}Y_4 + n_{1R}Y_5 + n_{1I}Y_6 + n_{2R}Y_7 + n_{2I}Y_8 \quad (3.113)$$

$$Z_{1I} = -I_{1R}Y_2 + I_{1I}Y_1 - I_{2R}Y_4 + I_{2I}Y_3 - n_{1R}Y_6 + n_{1I}Y_5 - n_{2R}Y_8 + n_{2I}Y_7 \quad (3.114)$$

where

$$Y_1 = -\frac{K_R Z_{DR} + K_I Z_{DI}}{|\Delta_Z|^2} \quad (3.115)$$

$$Y_2 = \frac{K_I Z_{DR} - K_R Z_{DI}}{|\Delta_Z|^2} \quad (3.116)$$

$$Y_3 = \frac{a_{22}}{a_{11}} \frac{\epsilon_{1R} Z_{DR} + \epsilon_{1I} Z_{DI}}{|\Delta_Z|^2} \quad (3.117)$$

$$Y_4 = \frac{a_{22}}{a_{11}} \frac{\epsilon_{1R} Z_{DI} - \epsilon_{1I} Z_{DR}}{|\Delta_Z|^2} \quad (3.118)$$

$$Y_5 = \frac{Z_{DR}}{a_{11}|\Delta_Z|^2} \quad (3.119)$$

$$Y_6 = \frac{Z_{DI}}{a_{11}|\Delta_Z|^2} \quad (3.120)$$

$$Y_7 = \frac{(\epsilon_{1R} - r_1 cos\phi_1) Z_{DR} + (\epsilon_{1I} - r_1 sin\phi_1) Z_{DI}}{a_{11}|\Delta_Z|^2} \quad (3.121)$$

$$Y_8 = \frac{(r_1 sin\phi_1 - \epsilon_{1I}) Z_{DR} + (\epsilon_{1R} - r_1 cos\phi_1) Z_{DI}}{a_{11}|\Delta_Z|^2} \quad (3.122)$$

$$|\Delta_Z|^2 = Z_{DR}^2 + Z_{DI}^2 \quad (3.123)$$

Notice that $Y_i, \quad i = 1, 2..8$ depend only on the random variables ϕ_1 and ϕ_2.

3.2.6 Decision Parameter with Both Amplitude and Phase Compensation at the Canceler Output

Instead of the amplitude compensation used in section 3.2.5, in this section we use compensation on both amplitude and phase of the co-pol signal. That is at the output of the canceler, the co-pol signal is the same as that send by the transmitter and the error will be caused only by the cross coupling and the noise processes.

From (3.84) we can write

$$y_1(n) = \frac{1}{\Delta_I}\Big[I_1(n)\Delta_y + I_2(n)\epsilon_1 a_{22} + n_1(n) + n_2(n)[-\frac{a_{12}}{a_{22}} + \epsilon_1]\Big] \quad (3.124)$$

where

$$\Delta_I = 1 - \frac{a_{12}a_{21}}{a_{11}a_{22}} + \epsilon_2\frac{a_{12}}{a_{22}} + \epsilon_1\frac{a_{21}}{a_{11}} - \epsilon_1\epsilon_2$$

$$\Delta_y = a_{11}[1 - \frac{a_{12}a_{21}}{a_{11}a_{22}} + \epsilon_1\frac{a_{21}}{a_{11}}], \quad (3.125)$$

and with amplitude and phase compensation, we take $\hat{I}_1(n) = y_1(n)\frac{\Delta_I}{\Delta_y}$ as estimate of the transmitted signal $I_1(n)$.

Define,

$$Z_1(n) \triangleq \hat{I}_1(n) - I_1(n) \quad (3.126)$$

with Z_1 denotes the amplitude and phase compensated output decision variable at channel 1.

We will perform similar analysis as in previous section to find the real and imaginary parts of the decision variable $Z_1(n)$.

From (3.124) and (3.126) we get,

$$Z_1(n) = \frac{1}{\Delta_y}\Big[I_2(n)\epsilon_1 a_{22} + n_1(n) + n_2(n)[-\frac{a_{12}}{a_{22}} + \epsilon_1]\Big] \quad (3.127)$$

Using (2.6) in (3.127), we get the decision variable,

$$Z_1 = \frac{1}{\Delta_y}\Big[(I_{2R} + jI_{2I})(\epsilon_{1R} + j\epsilon_{1I})a_{22}$$

$$+n_{1R} + jn_{1I} - r_1(n_{2R} + jn_{2I})(cos\phi_1 + jsin\phi_1)$$

$$+(n_{2R} + jn_{2I})(\epsilon_{1R} + j\epsilon_{1I})\Big], \tag{3.128}$$

where we drop the dependence of terms on the sampling time n.

Also from (3.125) the real and imaginary part of the denominator Δ_y are respectively given by,

$$\Delta_{yR} = [1 - r_1 r_2 cos(\phi_1 + \phi_2) + V_R]a_{11} \tag{3.129}$$

$$\Delta_{yI} = [-r_1 r_2 sin(\phi_1 + \phi_2) + V_I]a_{11} \tag{3.130}$$

where V_R and V_I are defined in the previous section. Clearly Δ_{yR} and Δ_{yI} are functions of the random variables ϕ_1 and ϕ_2.

The real and imaginary part of numerator of (3.128), Z_N are given by

$$Z_{NR} = (I_{2R}\epsilon_{1R} - I_{2I}\epsilon_{1I})a_{22}$$

$$+n_{1R} + n_{2R}(\epsilon_{1R} - r_1 cos\phi_1) - n_{2I}(\epsilon_{1I} - r_1 sin\phi_1) \tag{3.131}$$

$$Z_{NI} = (I_{2R}\epsilon_{1I} + I_{2I}\epsilon_{1R})a_{22}$$

$$+n_{1I} + n_{2R}(\epsilon_{1I} - r_1 sin\phi_1) + n_{2I}(\epsilon_{1R} - r_1 cos\phi_1) \tag{3.132}$$

Finally, we can write (3.128) as,

$$Z_1 = Z_{1R} + jZ_{1I} = \frac{Z_{NR} + jZ_{NI}}{\Delta_{yR} + j\Delta_{yI}} \tag{3.133}$$

with

$$Z_{1R} = \frac{Z_{NR}\Delta_{yR} + Z_{NI}\Delta_{yI}}{\Delta_{yR}^2 + \Delta_{yI}^2} \tag{3.134}$$

$$Z_{1I} = \frac{Z_{NI}\Delta_{yR} - Z_{NR}\Delta_{yI}}{\Delta_{yR}^2 + \Delta_{yI}^2} \tag{3.135}$$

Using (3.129), (3.130), (3.131) and (3.132) in (3.134) and (3.135), we can get after some simplification, which emphasizes the dependency of the different terms on the different random variables, the real part of the decision variable:

$$\begin{aligned} Z_{1R} = \frac{1}{|\Delta_y|^2}\Big[& I_{2R}(\epsilon_{1R}\Delta_{yR} + \epsilon_{1I}\Delta_{yI})a_{22} + I_{2I}(\epsilon_{1R}\Delta_{yI} - \epsilon_{1I}\Delta_{yR})a_{22} \\ & + n_{1R}\Delta_{yR} + n_{1I}\Delta_{yI} \\ & + n_{2R}[(\epsilon_{1R} - r_1 cos\phi_1)\Delta_{yR} + (\epsilon_{1I} - r_1 sin\phi_1)\Delta_{yI}] \\ & + n_{2I}[(-\epsilon_{1I} + r_1 sin\phi_1)\Delta_{yR} + (\epsilon_{1R} - r_1 cos\phi_1)\Delta_{yI}]\Big] \end{aligned} \tag{3.136}$$

Similar expression can be found for Z_{1I};

$$\begin{aligned} Z_{1I} = \frac{1}{|\Delta_y|^2}\Big[& -I_{2R}(\epsilon_{1R}\Delta_{yI} - \epsilon_{1I}\Delta_{yR})a_{22} + I_{2I}(\epsilon_{1R}\Delta_{yR} + \epsilon_{1I}\Delta_{yI})a_{22} \\ & - n_{1R}\Delta_{yI} + n_{1I}\Delta_{yR} \end{aligned}$$

$$-n_{2R}[(-\epsilon_{1I}+r_1 sin\phi_1)\Delta_{yR}+(\epsilon_{1R}-r_1 cos\phi_1)\Delta_{yI}$$

$$+n_{2I}[(\epsilon_{1R}-r_1 cos\phi_1)\Delta_{yR}+(\epsilon_{1I}-r_1 sin\phi_1)\Delta_{yI}]\Big] \quad (3.137)$$

with $|\Delta_y|^2=\Delta_{yR}^2+\Delta_{yI}^2$

The Decision Parameters Final Expressions

Finally we write the real and imaginary parts of $Z_1(n)$ in terms of the random variable representing the real and imaginary part of signal and noises of channel 1;

$$Z_{1R}=I_{2R}Y_{1AP}+I_{2I}Y_{2AP}+n_{1R}Y_{3AP}+n_{1I}Y_{4AP}+n_{2R}Y_{5AP}+n_{2I}Y_{6AP} \quad (3.138)$$

$$Z_{1I}=-I_{2R}Y_{2AP}+I_{2I}Y_{1AP}-n_{1R}Y_{4AP}+n_{1I}Y_{3AP}-n_{2R}Y_{6AP}+n_{2I}Y_{5AP} \quad (3.139)$$

where

$$Y_{1AP} = a_{22}\frac{\epsilon_{1R}\Delta_{yR}+\epsilon_{1I}\Delta_{yI}}{|\Delta_y|^2} \quad (3.140)$$

$$Y_{2AP} = a_{22}\frac{\epsilon_{1R}\Delta_{yI}-\epsilon_{1I}\Delta_{yR}}{|\Delta_y|^2} \quad (3.141)$$

$$Y_{3AP} = \frac{\Delta_{yR}}{|\Delta_y|^2} \quad (3.142)$$

$$Y_{4AP} = \frac{\Delta_{yI}}{|\Delta_y|^2} \quad (3.143)$$

$$Y_{5AP} = \frac{(\epsilon_{1R}-r_1 cos\phi_1)\Delta_{yR}+(\epsilon_{1I}-r_1 sin\phi_1)\Delta_{yI}}{|\Delta_y|^2} \quad (3.144)$$

$$Y_{6AP} = \frac{(r_1 sin\phi_1-\epsilon_{1I})\Delta_{yR}+(\epsilon_{1R}-r_1 cos\phi_1)\Delta_{yI}}{|\Delta_y|^2} \quad (3.145)$$

$$|\Delta_y|^2 = \Delta_{yR}^2 + \Delta_{yI}^2 \tag{3.146}$$

3.3 The Performance Analysis

3.3.1 Chernoff Bound

With Amplitude Compensation

Using the real and imaginary parts of the decision variable for channel 1 (3.113) and (3.114) obtained under the assumption of amplitude compensation, we calculate an upper bound for the average symbol error probability for power-power scheme of BXPC with dual-polarized M-ary QAM system.

An error is made on this channel if the decision variable $|Z_{1R}| > c$ or $|Z_{1I}| > c$. The probability of error on channel 1 can be written [9] as,

$$P_1(e) = \frac{1}{2}\Big[1 - \frac{1}{\sqrt{M}}\Big]\{P(|Z_{1R}| > c) + P(|Z_{1I}| > c)\} \tag{3.147}$$

For a bound on the probability, $P(|Z_{1R}| > c)$ or $P(|Z_{1I}| > c)$ we will use the Chernoff bound [25,26]. Such a bound is defined as follows; for any random variable Z and a constant c, one can find a $\lambda \geq 0$ such that

$$P(Z > c) \leq E\{e^{\lambda(Z-c)}\} \quad \lambda \geq 0 \tag{3.148}$$

Obviously λ that minimizes the right hand side of (3.148) establishes the least upper bound on $P(Z > c)$. Using (3.113) in (3.125) we find

$$P(|Z_{1R}| > c) \;\; \leq \;\; e^{-\lambda c} E_{\phi_1,\phi_2}\Big\{ E_{I_{1R}}[\exp(\lambda I_{1R} Y_1)].E_{I_{1I}}[\exp(\lambda I_{1I} Y_2)].$$

$$E_{I_{2R}}[\exp(\lambda I_{2R} Y_3)].E_{I_{2I}}[\exp(\lambda I_{2I} Y_4)].$$

$$E_{n_{1R}}[\exp(\lambda n_{1R}Y_5)].E_{n_{1I}}[\exp(\lambda n_{1I}Y_6)].$$

$$E_{n_{2R}}[\exp(\lambda n_{2R}Y_7)].E_{n_{2I}}[\exp(\lambda n_{2I}Y_8)]\Big\}, \qquad (3.149)$$

where we used the fact that I_{iR}, I_{iI}, n_{iR} and n_{iI}, $\quad i = 1,2$ are independent of each other. Also note that all the expected value operations inside the large parenthesis , are conditional on ϕ_1 and ϕ_2, and hence the random function $Y_i \quad i = 1,..8$ conditioned on ϕ_1 and ϕ_2 are constant with respect to these operations.

Following Kavehrad [9], we derive these expected values: The random variable I_{1R} is a discrete M-ary random variable which takes the values $\{\pm 1c, \pm 3c, .. \pm (\sqrt{(M)} - 1)c\}$ with equal probability. For such random variable, we derive in appendix A an upper bound on $E\{\exp(aI_{1R})\}$, with a given constant a. Noting that conditioned on ϕ_1 and ϕ_2, λY_1 is a constant, we obtain using (A.3) and (A.4),

$$\begin{aligned} E_{I_{1R}}[\exp(\lambda I_{1R}Y_1)] &= \frac{2}{\sqrt{M}} \sum_{i=1}^{\sqrt{M}/2} cosh[(\lambda Y_1)(2i-1)c] \\ &\leq \exp(\frac{\lambda^2}{2}c^2\frac{M-1}{3}Y_1^2) \end{aligned} \qquad (3.150)$$

Similar terms are in effect for the following expected values of (3.149);

$$E_{I_{1I}}[\exp(\lambda I_{1I}Y_2)] \leq \exp(\frac{\lambda^2}{2}c^2\frac{M-1}{3}Y_2^2), \qquad (3.151)$$

$$E_{I_{2R}}[\exp(\lambda I_{2R}Y_3)] \leq \exp(\frac{\lambda^2}{2}c^2\frac{M-1}{3}Y_3^2), \qquad (3.152)$$

$$E_{I_{2I}}[\exp(\lambda I_{2I}Y_4)] \leq \exp(\frac{\lambda^2}{2}c^2\frac{M-1}{3}Y_4^2). \qquad (3.153)$$

The additive noise $n_1(n)$ and $n_2(n)$ are assumed to be independent samples of zero mean complex Gaussian random variables with $E\{|n_i(n)|^2\} = 2\sigma_n^2 \quad i = 1,2$.

Therefore, n_{iR} and n_{iI} $i = 1,2$ are real, zero mean Gaussian random variable with variance= σ_n^2. For such random variable $\mathbf{n}$, we derive in appendix A the value of $E\{\exp(a\mathbf{n})\}$ with a as given constant. Again noting that conditioned on ϕ_1 and ϕ_2, λY_5 is a constant, therefore we obtain from (A.7),

$$E_{n_{1R}}[\exp(\lambda n_{1R} Y_5)] = \exp(\frac{\lambda^2}{2} c^2 \sigma_n^2 Y_5^2). \tag{3.154}$$

Similar terms for ;

$$E_{n_{1I}}[\exp(\lambda n_{1R} Y_6)] = \exp(\frac{\lambda^2}{2} c^2 \sigma_n^2 Y_6^2), \tag{3.155}$$

$$E_{n_{2R}}[\exp(\lambda n_{2R} Y_7)] = \exp(\frac{\lambda^2}{2} c^2 \sigma_n^2 Y_7^2), \tag{3.156}$$

$$E_{n_{2I}}[\exp(\lambda n_{2I} Y_8)] = \exp(\frac{\lambda^2}{2} c^2 \sigma_n^2 Y_8^2). \tag{3.157}$$

Collecting terms from (3.150) to (3.157) in (3.149), we rewrite (3.149);

$$\begin{aligned} P(|Z_{1R}| > c) \leq\ & E_{\phi_1,\phi_2}\Big\{\exp[-\lambda c + \frac{\lambda^2 c^2}{2}\frac{M-1}{3}(Y_1^2 + Y_2^2 + Y_3^2 + Y_4^2) \\ & + \frac{\lambda^2 c^2}{2}\sigma_n^2(Y_5^2 + Y_6^2 + Y_7^2 + Y_8^2)]\Big\} \end{aligned} \tag{3.158}$$

As a function of λ, we write (3.158);

$$P(|Z_{1R}| > c) \leq E_{\phi_1,\phi_2}\{\exp(-\lambda c + \lambda^2[U(\phi_1,\phi_2) + W(\phi_1,\phi_2)])\} \tag{3.159}$$

where

$$\begin{aligned} U(\phi_1,\phi_2) &= \frac{c^2}{2}\frac{M-1}{3}(Y_1^2 + Y_2^2 + Y_3^2 + Y_4^2) \\ W(\phi_1,\phi_2) &= \frac{\sigma_n^2 c^2}{2}(Y_5^2 + Y_6^2 + Y_7^2 + Y_8^2) \end{aligned} \tag{3.160}$$

Minimizing the exponent of (3.159) with respect to λ, we obtain the least upper bound;

$$P(|Z_{1R}| > c) \leq E_{\phi_1,\phi_2}\Big\{\exp\Big[\frac{-c^2}{4[U(\phi_1,\phi_2) + W(\phi_1,\phi_2)]}\Big]\Big\}, \tag{3.161}$$

or by using (3.160) we get,

$$P(|Z_{1R}| > c) \leq E_{\phi_1,\phi_2}\Big\{\exp\Big(\frac{-c^2}{4[\frac{c^2}{2}\frac{M-1}{3}U_1(\phi_1,\phi_2) + \frac{\sigma_n^2}{2}W_1(\phi_1,\phi_2)]}\Big)\Big\} \tag{3.162}$$

where,

$$\begin{aligned} U_1(\phi_1,\phi_2) &= Y_1^2 + Y_2^2 + Y_3^2 + Y_4^2 \\ W_1(\phi_1,\phi_2) &= Y_5^2 + Y_6^2 + Y_7^2 + Y_8^2 \end{aligned} \tag{3.163}$$

Rearranging terms, we get

$$\begin{aligned} P(|Z_{1R}| > c) &\leq E_{\phi_1,\phi_2}\Big\{\exp\Big[\frac{\frac{-c^2}{\sigma_n^2}}{2[c^2\frac{M-1}{3\sigma_n^2}U_1(\phi_1,\phi_2) + W_1(\phi_1,\phi_2)]}\Big]\Big\} \\ &= E_{\phi_1,\phi_2}\Big\{\exp\Big[\frac{-3(SNR)}{2(M-1)[(SNR)U_1(\phi_1,\phi_2) + W_1(\phi_1,\phi_2)]}\Big]\Big\}, \end{aligned} \tag{3.164}$$

where in the last step, we used (see [17])

$$SNR = \frac{S}{N} = \frac{M-1}{3}\frac{c^2}{\sigma_n^2} \tag{3.165}$$

Due to symmetry $P(|Z_{1R}| > c) = P(|Z_{1I}| > c)$. By using the minimum upper bound on $P(|Z_{1R}| > c)$ from (3.164) in (3.147), we can write the resulting least upper bound on the probability error ;

$$
\begin{aligned}
P_1(e) \;\; &\leq \;\; (1 - \frac{1}{\sqrt{M}})\frac{1}{4\pi^2}. \\
&\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \exp\Big[\frac{-3\,(SNR)}{2(M-1)}\frac{1}{(SNR)U_1(\phi_1,\phi_2)+W_1(\phi_1,\phi_2)}\Big]\, d\phi_1 d\phi_2
\end{aligned} \tag{3.166}
$$

where we used the fact that probability density function of $\phi_i \quad i = 1, 2$ are, $p(\phi_1) = p(\phi_2) = \dfrac{1}{2\pi}$.

With Amplitude and Phase Compensation

To calculate the Chernoff bound in the case where the decision variable is obtained with both amplitude and phase compensation, we follow the same steps as in previous section except with different value of $Y_i \;\; i = 1, 2.., 6$.

As in (3.158) we have here

$$
\begin{aligned}
P(|Z_{1R}| > c) \;\; &\leq \;\; E_{\phi_1,\phi_2}\Big\{\exp[-\lambda c + \frac{\lambda^2 c^2}{2}\frac{M-1}{3}(Y_{1AP}^2 + Y_{2AP}^2) \\
&+ \frac{\lambda^2 c^2}{2}\sigma_n^2(Y_{3AP}^2 + Y_{4AP}^2 + Y_{5AP}^2 + Y_{6AP}^2)]\Big\}
\end{aligned} \tag{3.167}
$$

where $Y_{iAP} \;\; i = 1, 2, ..6$ are defined by (3.140) to (3.146).

Minimizing the right hand side of (3.167) and taking the expected value over ϕ_1 and ϕ_2, we write error the bound for the amplitude and phase compensated channel 1 output,

$$
\begin{aligned}
P_{1AP}(e) \;\; &\leq \;\; (1 - \frac{1}{\sqrt{M}})\frac{1}{4\pi^2}. \\
&\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \exp\Big[\frac{-3\,(SNR)}{2(M-1)}\frac{1}{(SNR)U_{1AP}(\phi_1,\phi_2)+W_{1AP}(\phi_1,\phi_2)}\Big]\, d\phi_1 d\phi_2
\end{aligned} \tag{3.168}
$$

where

$$U_{1AP}(\phi_1, \phi_2) = Y_{1AP}^2 + Y_{2AP}^2$$

$$W_{1AP}(\phi_1, \phi_2) = Y_{3AP}^2 + Y_{4AP}^2 + Y_{5AP}^2 + Y_{6AP}^2 \tag{3.169}$$

and

$$SNR = \frac{S}{N} = \frac{M-1}{3}\frac{c^2}{\sigma_n^2} \tag{3.170}$$

3.3.2 Method of Moments for Probability of Error Calculation

In some cases, Chernoff bound might not be sufficiently tight [18]. Therefore to use it as a measure of performance in comparing different system might not be adequate. Hence, in this section, we will present another method which actually compute an approximate to the average probability of error for the power-power scheme of BXPC. The method based upon Gauss quadrature rules (GQR) which was shown to assure accurate and satisfactory results. We will first give a brief description of GQR and apply it to calculate the average probability of error of the power-power scheme. These calculations will be performed for both amplitude compensated, and amplitude and phase compensated received signals, respectively.

With Amplitude Compensation

Again the probability of error on channel 1 is given in (3.147);

$$P_1(e) = \frac{1}{2}\Big[1 - \frac{1}{\sqrt{M}}\Big]\{P(|Z_{1R}| > c) + P(|Z_{1I}| > c)\} \tag{3.171}$$

Using (3.113), we first find the conditional probability, $P(|Z_{1R}| > c|\phi_1, \phi_2, I_{1R}, I_{1I}, I_{2R}, I_{2I})$. That is, we integrate on the joint probability of the random variable Y;

$$Y = n_{1R}Y_5 + n_{1I}Y_6 + n_{2R}Y_7 + n_{2I}Y_8 \tag{3.172}$$

Define,

$$X_I = I_{1R}Y_1 + I_{1I}Y_2 + I_{2R}Y_3 + I_{2I}Y_4, \tag{3.173}$$

then from (3.113),

$$Z_{1R} = X_I + Y \tag{3.174}$$

The random variable Y is zero mean Gaussian and have variance ;

$$\sigma^2(\phi_1, \phi_2) = (Y_5^2 + Y_6^2 + Y_7^2 + Y_8^2)\sigma_n^2 \tag{3.175}$$

Also conditioned on ϕ_1,ϕ_2,I_{1R},I_{1I}, I_{2R} and I_{2I} , the random variable Z_{1R} is Gaussian with mean equals X_I and variance $\sigma^2(\phi_1, \phi_2)$. Therefore

$$\begin{aligned} P(|Z_{1R}| > c|\phi_1, \phi_2, I_{1R}, I_{1I}, I_{2R}, I_{2I}) &= \frac{2}{\sqrt{2\pi}} \int_c^\infty \exp\Big(\frac{-1}{2}\Big[\frac{z_{1R} - X_I}{\sigma}\Big]^2\Big) dz_{1R} \\ &= \frac{2}{\sqrt{2\pi}} \int_{\frac{c-X_I}{\sigma}}^\infty \exp(\frac{-t^2}{2})\ dt \end{aligned}$$

Hence,

$$P(|Z_{1R}| > c|\phi_1, \phi_2, I_{1R}, I_{1I}, I_{2R}, I_{2I}) = 2Q(\frac{c - X_I}{\sigma}) \tag{3.176}$$

where

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty \exp(\frac{-t^2}{2})\ dt \tag{3.177}$$

Again due to symmetry $P(|Z_{1R}| > c)$= $P(|Z_{1I}| > c)$, so that together with (3.171), we get

$$P_1(e|\phi_1, \phi_2, I_{1R}, I_{1I}, I_{2R}, I_{2I}) = 2(1 - \frac{1}{\sqrt{M}})Q\Big(\frac{c - X_I}{\sigma}\Big). \tag{3.178}$$

We can write

$$P_1(e|\mathbf{x}) = 2(1 - \frac{1}{\sqrt{M}})Q(\frac{c}{\sigma_n}\mathbf{x}), \tag{3.179}$$

where the random variable $\mathbf{x}$;

$$\mathrm{x} = \frac{c - X_I}{\sigma_o} \tag{3.180}$$

with

$$\sigma_o^2 = (Y_5^2 + Y_6^2 + Y_7^2 + Y_8^2)c^2, \tag{3.181}$$

is a function of the random variables $(\phi_1, \phi_2, I_{1R}, I_{1I}, I_{2R}, I_{2I})$.

Clearly, the average error probability on channel 1 can be evaluated from

$$P_1(e) = \int_x P_1(e|x) f_{\mathbf{x}}(x)\, dx \tag{3.182}$$

with $f_{\mathbf{x}}(x)$ as the pdf of the random variable $\mathbf{x}$. Using (B.4) of the appendix together with (3.170), we have

$$P_1(e) = 2(1 - \frac{1}{\sqrt{M}}) \sum_{i=1}^{N} w_i Q\Big(\sqrt{\frac{3(SNR)}{M-1}} x_i\Big) \tag{3.183}$$

The GQR nodes x_i and the weights w_i are determined from the moments of random variable $\mathbf{x}$, as it is described in appendix B.

Moments of x

Using the Gauss Quadrature integration, the average probability of error in (3.180) can be calculated numerically by evaluating the $2N+1$ moments of random variable x in (3.180).

Denoting the moments of x by μ_n $n = 0, 1, ..2N$,

$$\begin{aligned}\mu_n &= E\{\mathrm{x^n}\} = \int_a^b x^n f_{\mathbf{x}}(x)\, dx \\ &= E\{[\frac{c - X_I}{\sigma_o}]^n\} \end{aligned} \tag{3.184}$$

Substituting X_I and σ_o from (3.173) and (3.181) respectively, we get

$$\mu_n = E\Big\{\Big[\frac{c-(I_{1R}Y_1+I_{1I}Y_2+I_{2R}Y_3+I_{2I}Y_4)}{(\sqrt{Y_5^2+Y_6^2+Y_7^2+Y_8^2})c}\Big]^n\Big\} \tag{3.185}$$

Since, the random variable in (3.187) is a function of the random variables $\phi_1, \phi_2, I_{1R}, I_{1I}, I_{2R}$ and I_{2I} which are assumed independent, therefore

$$\begin{aligned} f_{\mathbf{x}}(x) &= f_{\mathbf{I_{1R}},\mathbf{I_{1I}},\mathbf{I_{2R}},\mathbf{I_{2I}},\phi_1,\phi_2}(I_{1R}, I_{1I}, I_{2R}, I_{2I}, \phi_1, \phi_2) \\ &= f_{\mathbf{I_{1R}}}(I_{1R}) f_{\mathbf{I_{1I}}}(I_{1I}) f_{\mathbf{I_{2R}}}(I_{2R}) f_{\mathbf{I_{2I}}}(I_{2I}) f_{\phi_1}(\phi_1) f_{\phi_2}(\phi_2) \end{aligned} \tag{3.186}$$

Using the simple binomial rule, one can write (3.187),

$$\begin{aligned} E\{\mathbf{x}^n\} &= E\{[\frac{c-(A+B)}{\sigma_o}]^n\} = E\Big\{\frac{\sum_{k=0}^{n}\binom{n}{k} c^k(-1)^{n-k}(A+B)^{n-k}}{\sigma_o^n}\Big\} \\ &= E\Big\{\frac{\sum_{k=0}^{n}\binom{n}{k} c^k(-1)^{n-k}\Big(\sum_{m=0}^{n-k}\binom{n-k}{m} A^m B^{n-k-m}\Big)}{\sigma_o^n}\Big\} \\ &= E\Big\{\frac{1}{\sigma_o^n}\sum_{k=0}^{n}\binom{n}{k} c^k(-1)^{n-k}. \\ &\quad \Big[\sum_{m=0}^{n-k}\binom{n-k}{m}(I_{1R}Y_1+I_{1I}Y_2)^m(I_{2R}Y_3+I_{2I}Y_4)^{n-k-m}\Big]\Big\} \end{aligned} \tag{3.187}$$

where

$$\begin{aligned} A &= I_{1R}Y_1 + I_{1I}Y_2 \\ B &= I_{2R}Y_3 + I_{2I}Y_4 \end{aligned} \tag{3.188}$$

The expectation in (3.184) are taken first with respect to I_{ij} $i = 1, 2$ $j = R, I$ conditioned on ϕ_1 and ϕ_2 and then expectation over ϕ_1 and ϕ_2 .

Equation (3.184) can be rewritten in the form,

$$E\{\mathbf{x^n}\} = \sum_{k=0}^{n}\binom{n}{k} c^k(-1)^{n-k}\sum_{m=0}^{n-k}\binom{n-k}{m} E\{A_x B_x\} \tag{3.189}$$

where

$$A_x = \sum_{l=0}^{m} \begin{pmatrix} m \\ l \end{pmatrix} I_{1R}^{l} \frac{Y_1^l}{\sigma_o^n} I_{1I}^{(m-l)} \frac{Y_2^{(m-l)}}{\sigma_o^n}$$

$$B_x = \sum_{u=0}^{n-k-m} \begin{pmatrix} n-k-m \\ u \end{pmatrix} I_{2R}^{u} \frac{Y_3^u}{\sigma_o^n} I_{2I}^{(n-k-m-u)} \frac{Y_4^{(n-k-m-u)}}{\sigma_o^n} \tag{3.190}$$

Recall that I_{ij}, $i = 1,2$, $j = R, I$ are all independent equally likely M-ary symbols from the set $\{\pm 1c, \pm 3c, \ldots.. \pm(\sqrt{M}-1)c\}$, and Y_k, $k = 1, 2..8$ are functions of ϕ_1 and ϕ_2 which are independent and uniformly distributed over $[-\pi, \pi]$. In the processes of evaluating (3.189), we note that the n th moment of equally likely M-ary symbol [17,19] is given by

$$E_{I_{ij}}\{I_{ij}^n\} = \frac{1}{\sqrt{M}} \sum_{m=0}^{\sqrt{M}-1} (2m + 1 - \sqrt{M})^n c^n, \tag{3.191}$$

and for the case of independent and zero mean M-ary symbols I_{ij}, we have

$$\begin{aligned} E_{\mathbf{I_{ij}},\mathbf{I_{kl}}}\{I_{ij}I_{kl}\} &= 0 \quad i \neq k, \ \ or \ \ j \neq l \\ E_{\mathbf{I_{ij}}}\{I_{ij}\} &= 0 \quad i = 1,2 \ \ j = I, R \end{aligned} \tag{3.192}$$

Furthermore for the n th moment of Y_i $i = 1, ..8$ which are function of ϕ_1 and ϕ_2, we use

$$\begin{aligned} E_{\phi_1,\phi_2}\{Y_i(\phi_1,\phi_2)^n\} &= \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} Y_i(\phi_1,\phi_2)^n f_{\phi_1,\phi_2}(\phi_1,\phi_2) \ d\phi_1, d\phi_2 \\ &= \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} Y_i(\phi_1,\phi_2)^n f_{\phi_1}(\phi_1) f_{\phi_2}(\phi_2) \ d\phi_1 d\phi_2 \quad i = 1, 2..8 \end{aligned} \tag{3.193}$$

Finally the moments obtained for the random variable x will be used to find the GQR nodes and weights $\{x_i, w_i\}_{i=1}^{N}$ which are needed for calculating approximate average error probability in (3.183). The evaluation of the nodes and weights are given in appendix B.

With Amplitude and Phase compensation

Similar analysis is used to find the average probability of error for the case when the decision variable is obtained with both amplitude and phase compensation.

In this case, using (3.138), we first calculate the conditional probability, $P(|Z_{1R}| > c|\phi_1, \phi_2, I_{2R}, I_{2I})$. That is, we integrate on the joint probability of the random variable Y_{AP},

Define

$$Y_{AP} = n_{1R}Y_{3AP} + n_{1I}Y_{4AP} + n_{2R}Y_{5AP} + n_{2I}Y_{6AP} \quad (3.194)$$

$$X_{IAP} = I_{2R}Y_{1AP} + I_{2I}Y_{2AP}, \quad (3.195)$$

then from (3.138),

$$Z_{1R} = X_{IAP} + Y_{AP} \quad (3.196)$$

The random variable Y_{AP} is zero mean Gaussian and have variance ;

$$\sigma_{oAP}^2(\phi_1, \phi_2) = (Y_{3AP}^2 + Y_{4AP}^2 + Y_{5AP}^2 + Y_{6AP}^2)\sigma_n^2 \quad (3.197)$$

Also conditioned on ϕ_1,ϕ_2, I_{2R} and I_{2I} , the random variable Z_{1R} is Gaussian with mean equals X_{IAP} and variance $\sigma_{oAP}^2(\phi_1, \phi_2)$. Similar to (3.178),

$$P_{1AP}(e|\phi_1, \phi_2, I_{2R}, I_{2I}) = 2(1 - \frac{1}{\sqrt{M}})Q\Big(\frac{c - X_{IAP}}{\sigma_{AP}}\Big), \quad (3.198)$$

or

$$P_{1AP}(e|\mathbf{x_{AP}}) = 2(1 - \frac{1}{\sqrt{M}})Q(\frac{c}{\sigma_n}\mathbf{x_{AP}}), \quad (3.199)$$

where the random variable $\mathbf{x_{AP}}$,

$$\mathbf{x_{AP}} = \frac{c - X_{IAP}}{\sigma_{oAP}}, \quad (3.200)$$

with

$$\sigma_{oAP}^2 = (Y_{3AP}^2 + Y_{4AP}^2 + Y_{5AP}^2 + Y_{6AP}^2)c^2, \tag{3.201}$$

is a function of the random variables $(\phi_1, \phi_2, I_{2R}, I_{2I})$. Because of independence assumption,

$$f_{\mathbf{x_{AP}}}(x) \;=\; f_{\mathbf{I_{2R}}}(I_{2R}) f_{\mathbf{I_{2I}}}(I_{2I}) f_{\phi_1}(\phi_1) f_{\phi_2}(\phi_2) \tag{3.202}$$

Similar to (3.182), we can use GQR to calculate the probability of error from the moments of the random variable $\mathbf{x_{AP}}$; viz,

$$P_{1AP}(e) = 2(1 - \frac{1}{\sqrt{M}}) \sum_{i}^{N} w_i Q\Big(\sqrt{\frac{3(SNR)}{M-1}} x_i\Big) \tag{3.203}$$

where again x_i and w_i are the nodes and the weights of the GQR.

Moments of $\mathbf{x_{AP}}$

By denoting the moments of $\mathbf{x_{AP}}$ by the sequence $\{\mu_n\}|_{n=0}^{2N}$, we can write,

$$\mu_n \;=\; E\{\mathbf{x_{AP}^n}\} = \int_a^b x_{AP}^n f_{\mathbf{x_{AP}}}(x)\, dx_{AP}$$

$$=\; E\{[\frac{c - X_{IAP}}{\sigma_{oAP}}]^n\}$$

Substituting X_{IAP} and σ_{oAP} from (3.195) and (3.201) respectively,

$$=\; E\Big\{\Big[\frac{c - (I_{2R}Y_{1AP} + I_{2I}Y_{2AP})}{(\sqrt{Y_{3AP}^2 + Y_{4AP}^2 + Y_{5AP}^2 + Y_{6AP}^2})c}\Big]^n\Big\} \tag{3.204}$$

Since, $\mathbf{x_{AP}}$ is a function of independent random variables; $(\phi_1, \phi_2, I_{2R}, I_{2I})$, then by using the simple binomial rule, one can write (3.204),

$$E\{\mathbf{x_{AP}^n}\} \;=\; E\{[\frac{c - (I_{2R}Y_{1AP} + I_{2I}Y_{2AP})}{\sigma_{oAP}}]^n\}$$

$$E\left\{\frac{\sum_{k=0}^{n}\begin{pmatrix} n \\ k \end{pmatrix} c^k(-1)^{n-k}(I_{2R}Y_{1AP}+I_{2I}Y_{2AP})^{n-k}}{\sigma_{oAP}^n}\right\} \tag{3.205}$$

We take the expected values of the inner terms in (3.205), then,

$$E\{\mathbf{x}_{\mathbf{AP}}^{\mathbf{n}}\} = \sum_{k=0}^{n}\begin{pmatrix} n \\ k \end{pmatrix} c^k(-1)^{n-k}E\{A_{xAP}\} \tag{3.206}$$

where

$$A_{xAP} = \sum_{l=0}^{n-k}\begin{pmatrix} n-k \\ l \end{pmatrix} I_{2R}^l\frac{Y_{1AP}^l}{\sigma_{oAP}^n}I_{2I}^{(n-k-l)}\frac{Y_{2AP}^{(n-k-l)}}{\sigma_{oAP}^n} \tag{3.207}$$

3.4 Results

The Chernoff upper bound on the average probability of error as a function of signal-to-noise (SNR) ratio is evaluated for various cross coupling constants and for 16 QAM and 64 QAM signals. The Gauss quadrature rule is also used to find approximations to these probability of errors.

In Fig. 3.1, the bounds on error probability with 16 QAM and with cross polarization coupling r= -15 dB, -10 dB and -5 dB are calculated and compared. Equation (3.166) is used in calculating these bounds when only amplitude compensation is employed, while (3.168) is used when both amplitude and phase compensation is employed. Notice that adding phase compensation improve the bound when the cross coupling is high (i.e., r=-5 dB). The effect of adding phase compensation is hardly noticeable with low cross coupling (r=-15 dB). Fig. 3.2 depicts the same for 64 QAM. Effect of compensation is similar. Nevertheless as expected the bounds are higher for 64 QAM displaying possibility of higher error rates with the same SNR. Comparison of these bounds for 16 QAM and 64 QAM are shown in Fig. 3.3.

In Fig. 3.4 and Fig. 3.5, we depict the probability of error as it is calculated using the Gauss quadrature rule, for 16 QAM and 64 QAM, respectively. These calculations were done with cross coupling of -15 dB, -10 dB and -5 dB, and in each a total of 9 moments were used. Only the case with amplitude compensation was shown since adding phase compensation did not change these results very much. In order to show how tight are the Chernoff bounds shown in in Fig. 3.1, we depict in Fig. 3.6 a comparison of the results obtained with GQR moments calculations to their corresponding Chernoff bounds for 16 QAM and cross coupling of -15 dB, -10 dB and -5 dB. Fig. 3.7 shows the same for the 64 QAM case. To show the effect of increasing the number of moments used in obtaining the GQR results, we show in the next two figures these results with 7, 9 and 11 moments. Chernoff bound was added to these curves for comparison. In Fig. 3.8, we present these comparisons for 16 QAM case, while Fig. 3.9 presents the same for the 64 QAM case. Finally, we listed in Tables 3.1 to 3.5 some of the results shown in these figures.

3.5 Conclusion

The power-power bootstrapped canceler was analysed and its performance was studied in this chapter. Particularly, the average probability of error was estimated, using the moment generating method or by finding the Chernoff bounds. Results of the analysis as well as computer calculation show, as expected, that with 16 QAM performance is well better than 64 QAM. Also shown that adding phase compensation to the canceler output adds very little to the performance when only amplitude compensation is included.

From comparing the results obtained with moment generating method to the corresponding Chernoff bound, we concluded that these bounds are sufficiently tight. Comparing the results when different number of moments are used, and concluded tightness of the Chernoff bound, we infer that approximately 10 moments are sufficient for deriving a good approximation for the average probability of error using the Gauss quadrature rule.

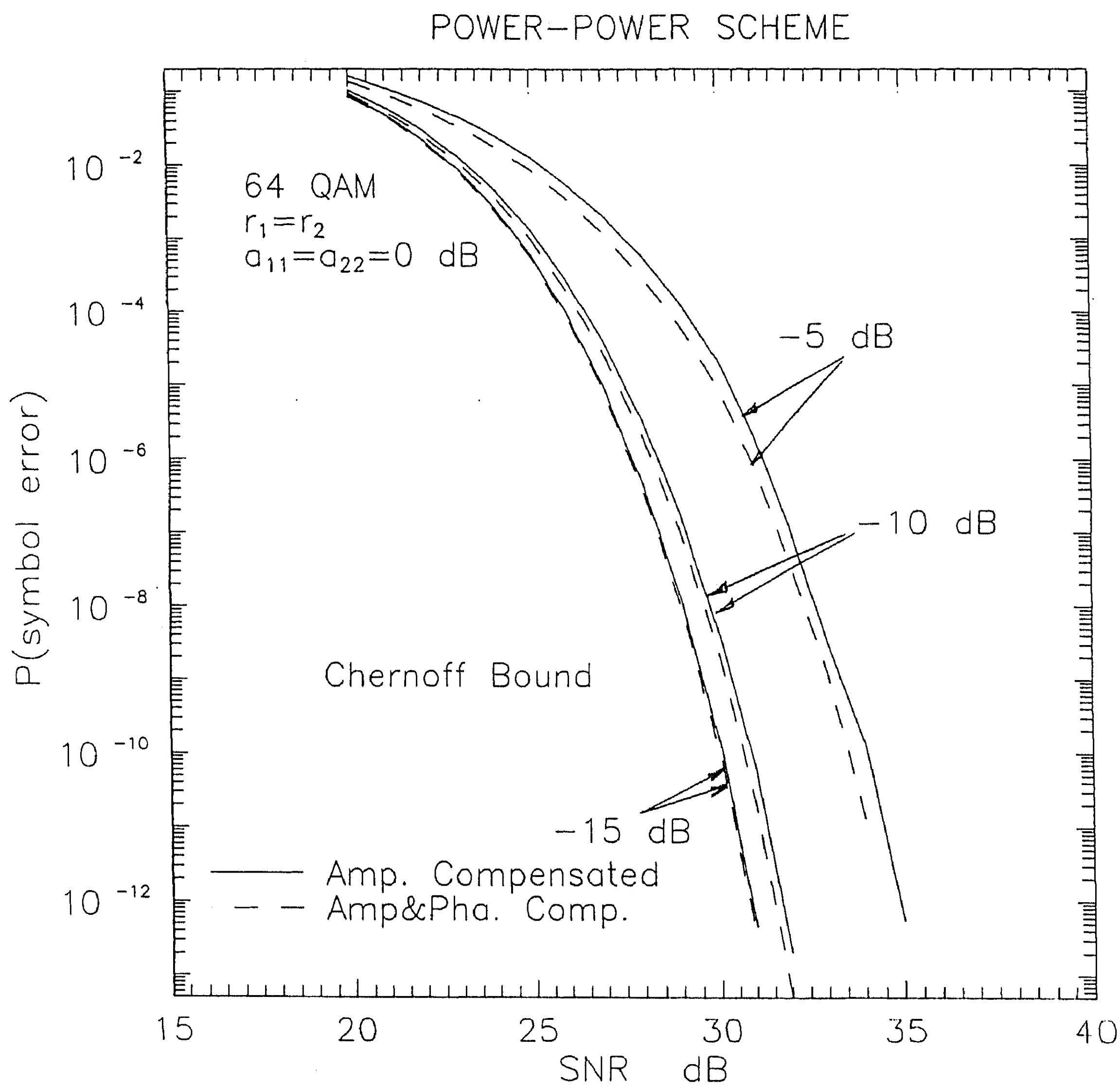

Figure 3.1: Power-Power Cross-Pol Canceler, Chernoff bound 16 QAM

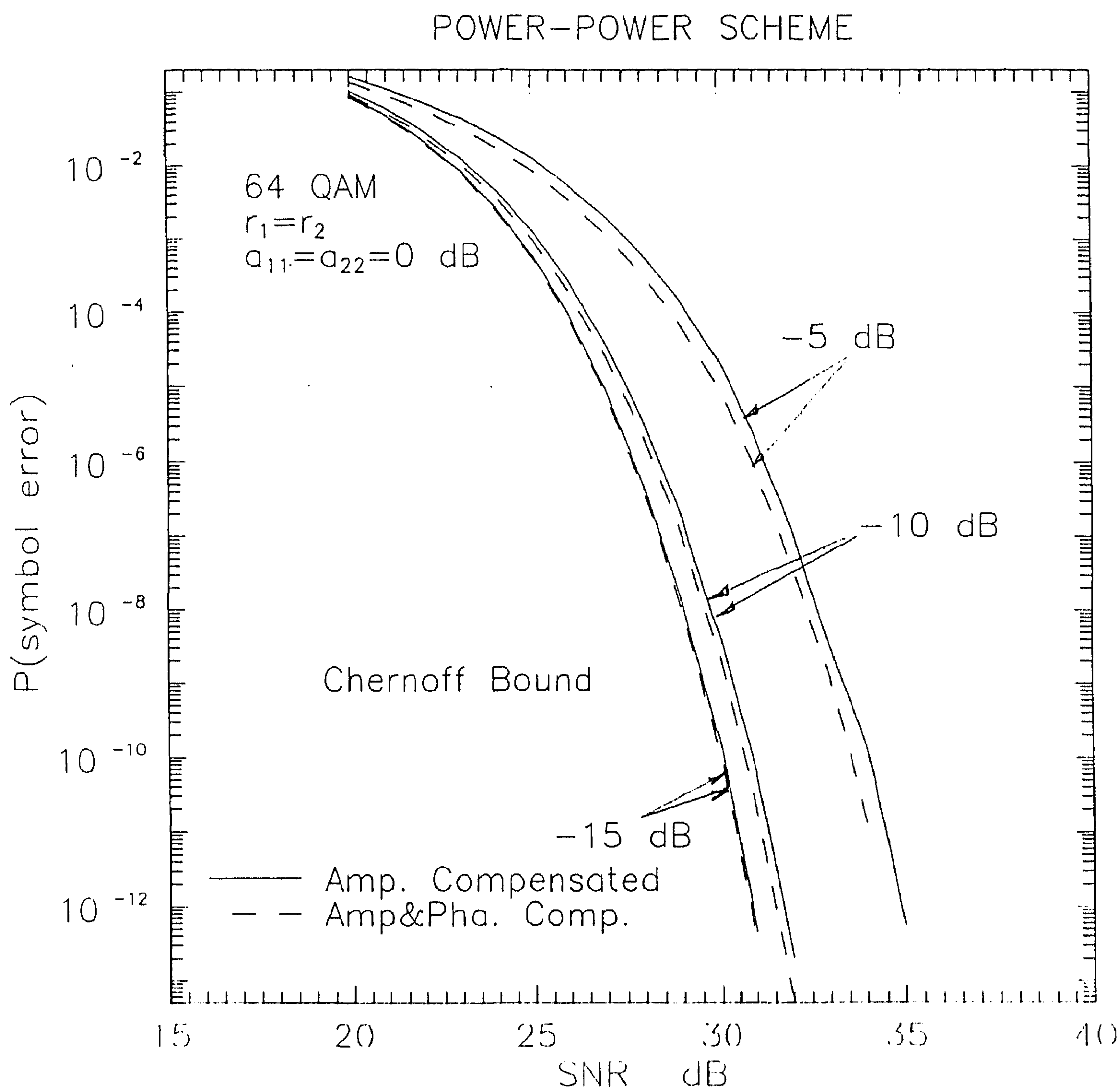

Figure 3.2: Power-Power Cross-Pol Canceler, Chernoff bound, 64 QAM

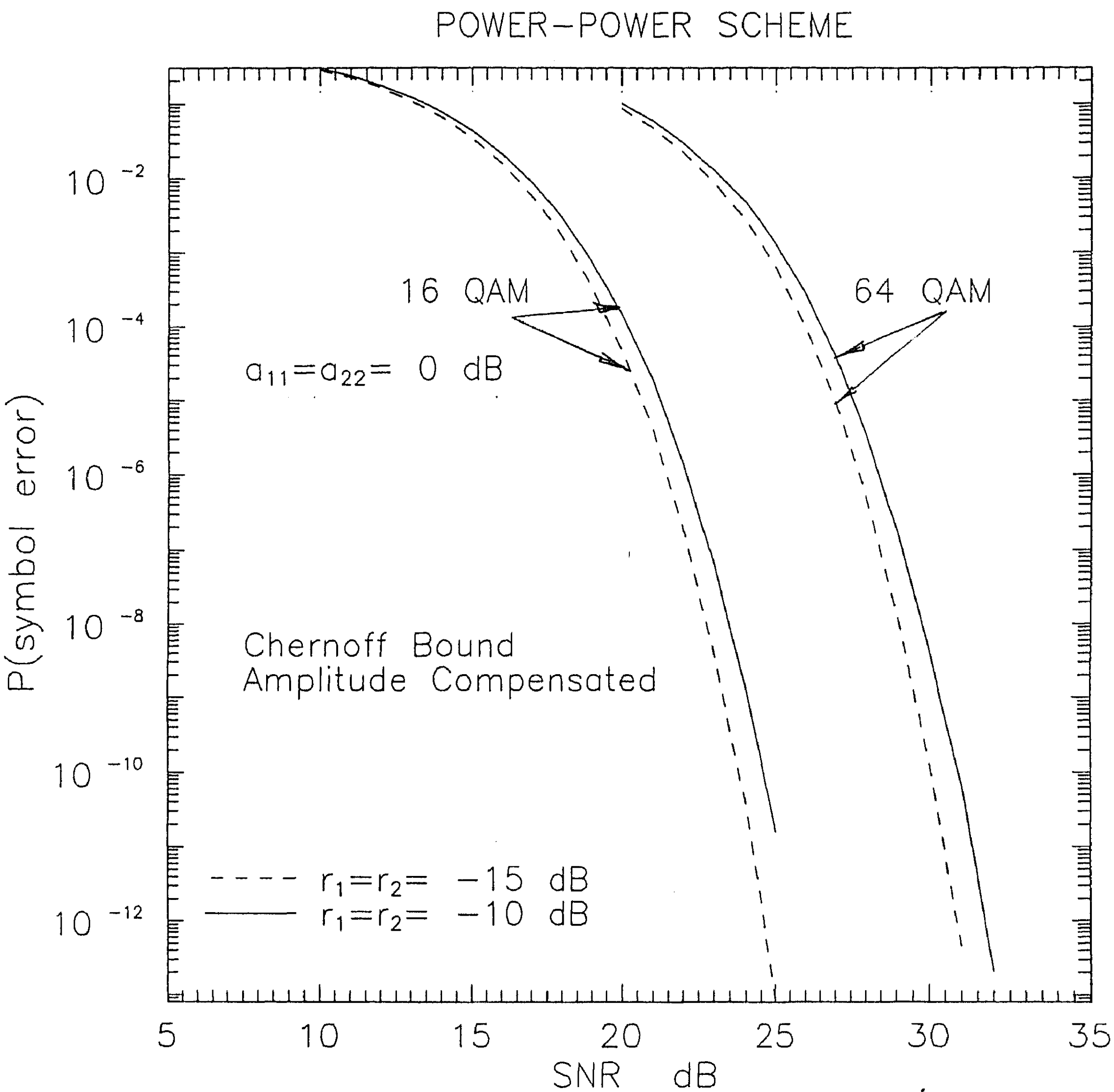

Figure 3.3: Power-Power Cross-Pol Canceler, Chernoff bound comparison 16 QAM v.s 64 QAM

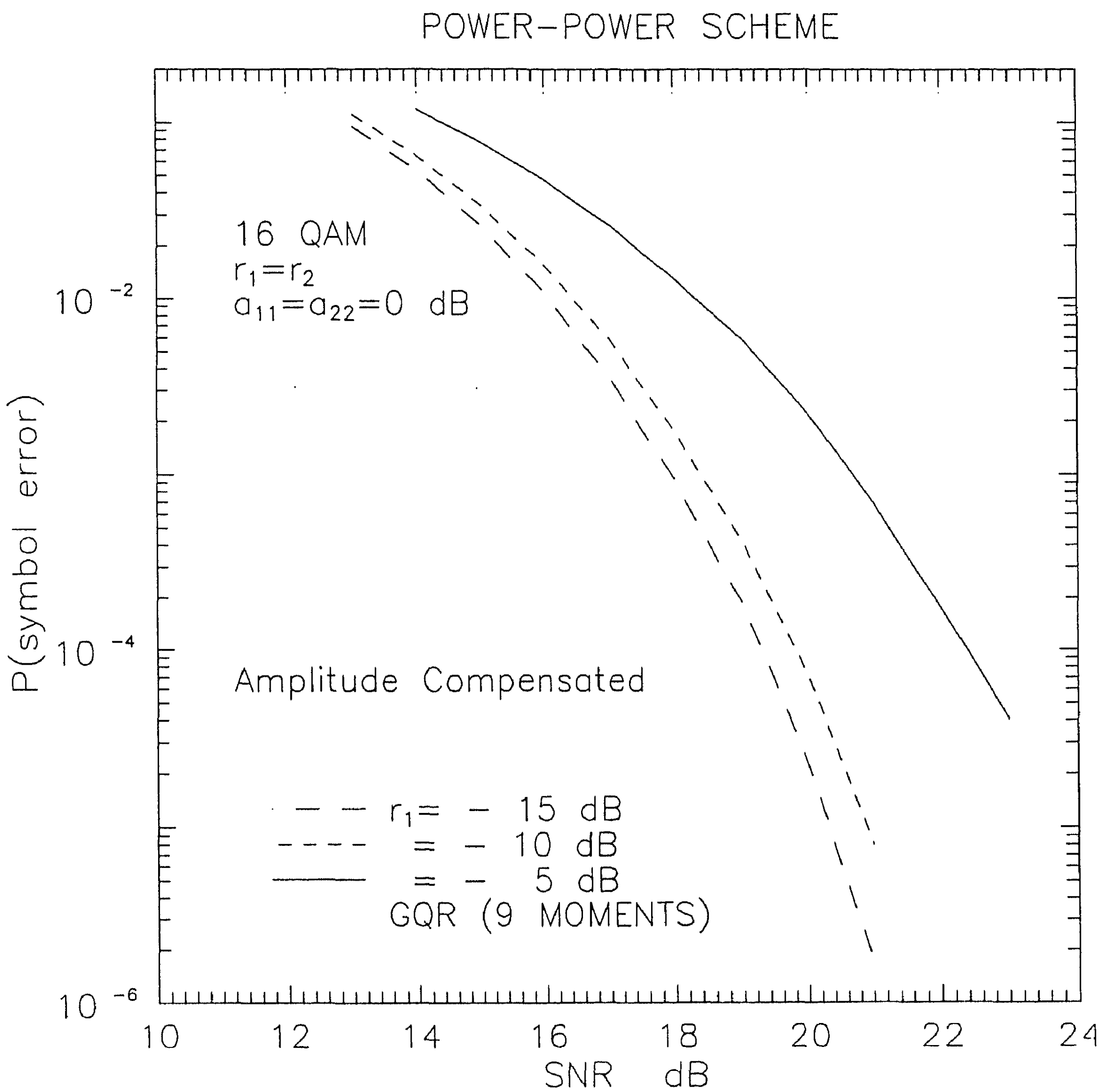

Figure 3.4: Power-Power Cross-Pol Canceler, GQR calculation, 16 QAM

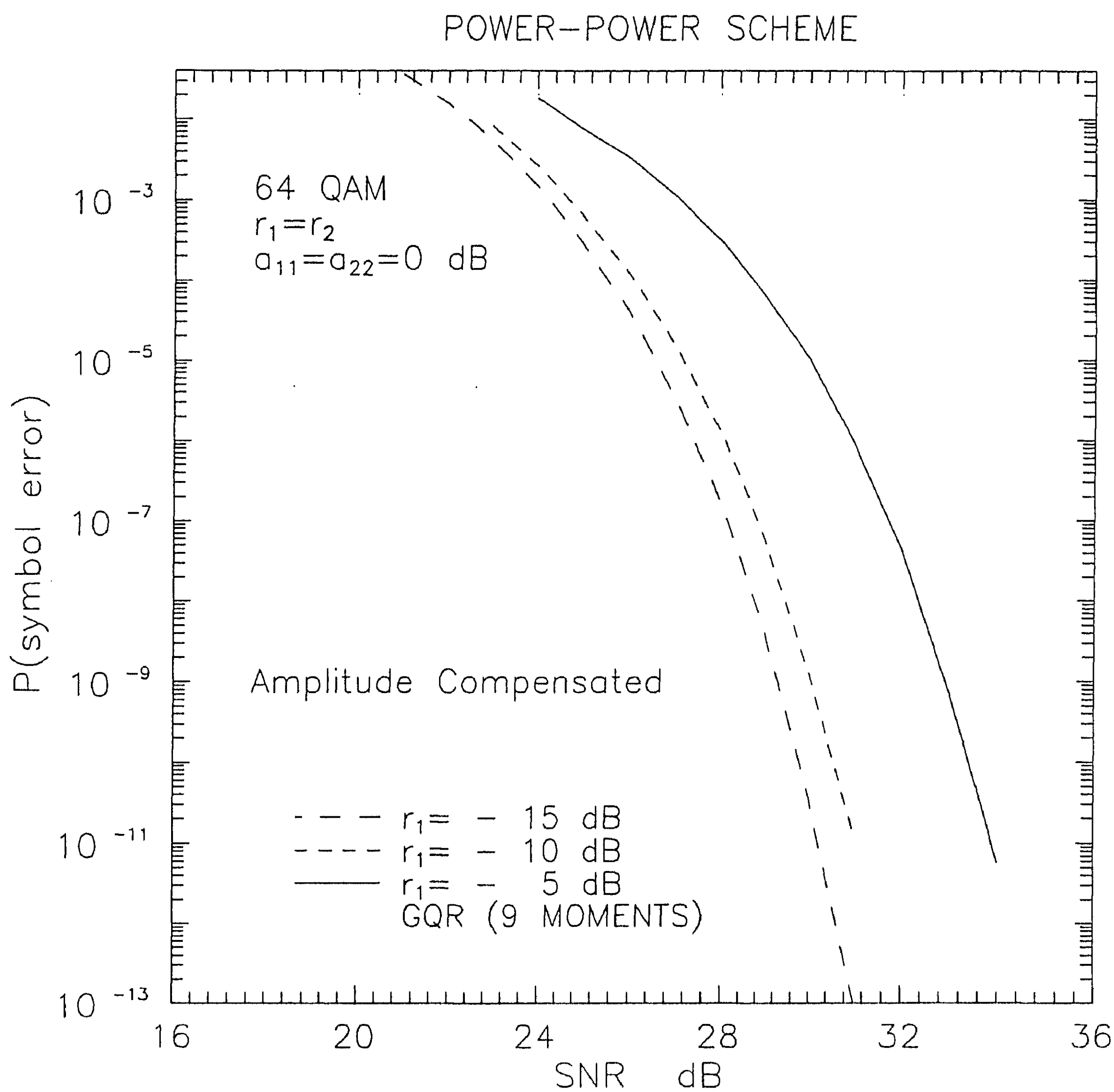

Figure 3.5: Power-Power Cross-Pol Canceler, GQR calculations, 64 QAM

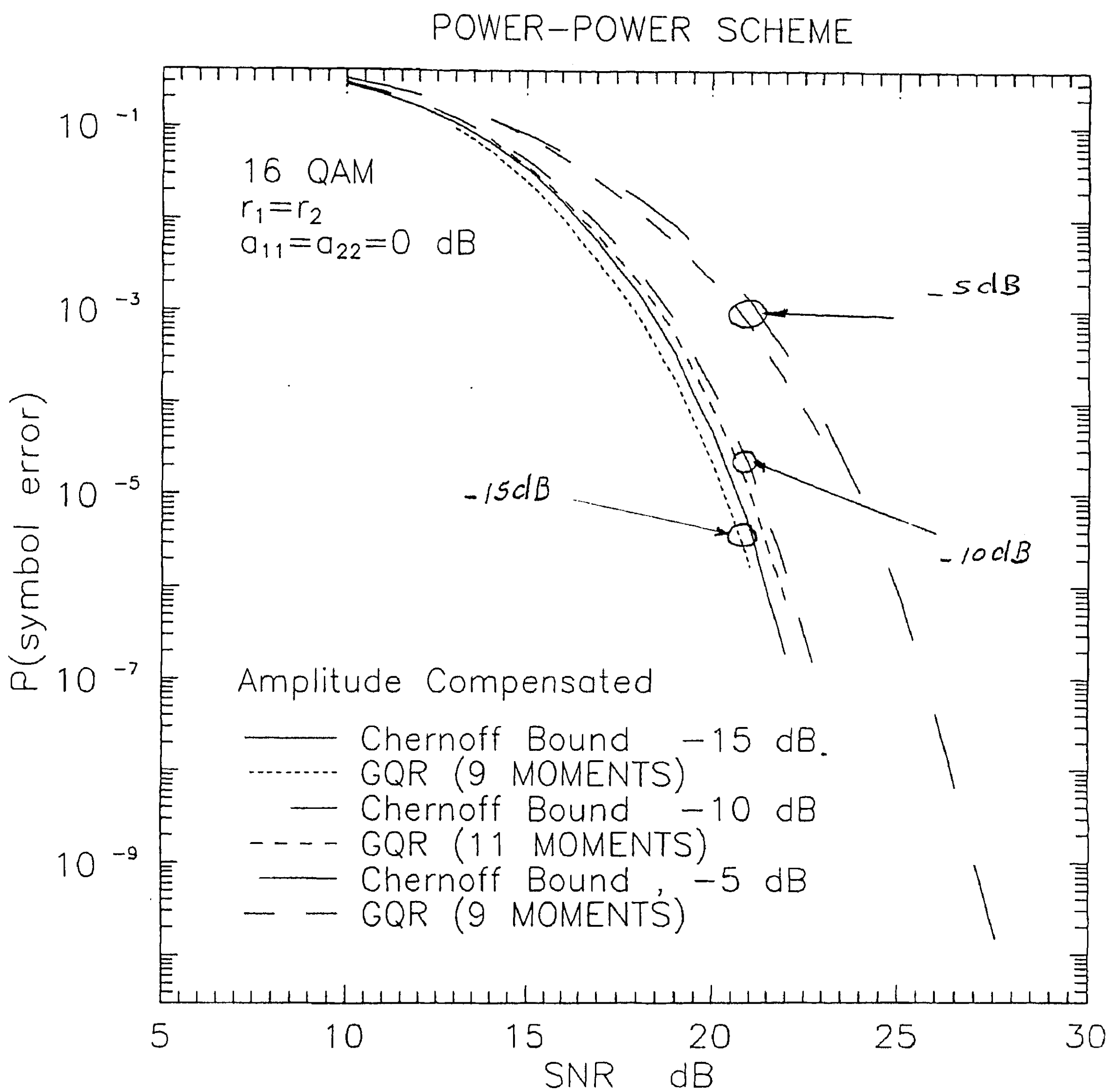

Figure 3.6: Power-Power Cross-Pol Canceler, Chernoff bound and GQR moment calculations comparison, 16 QAM

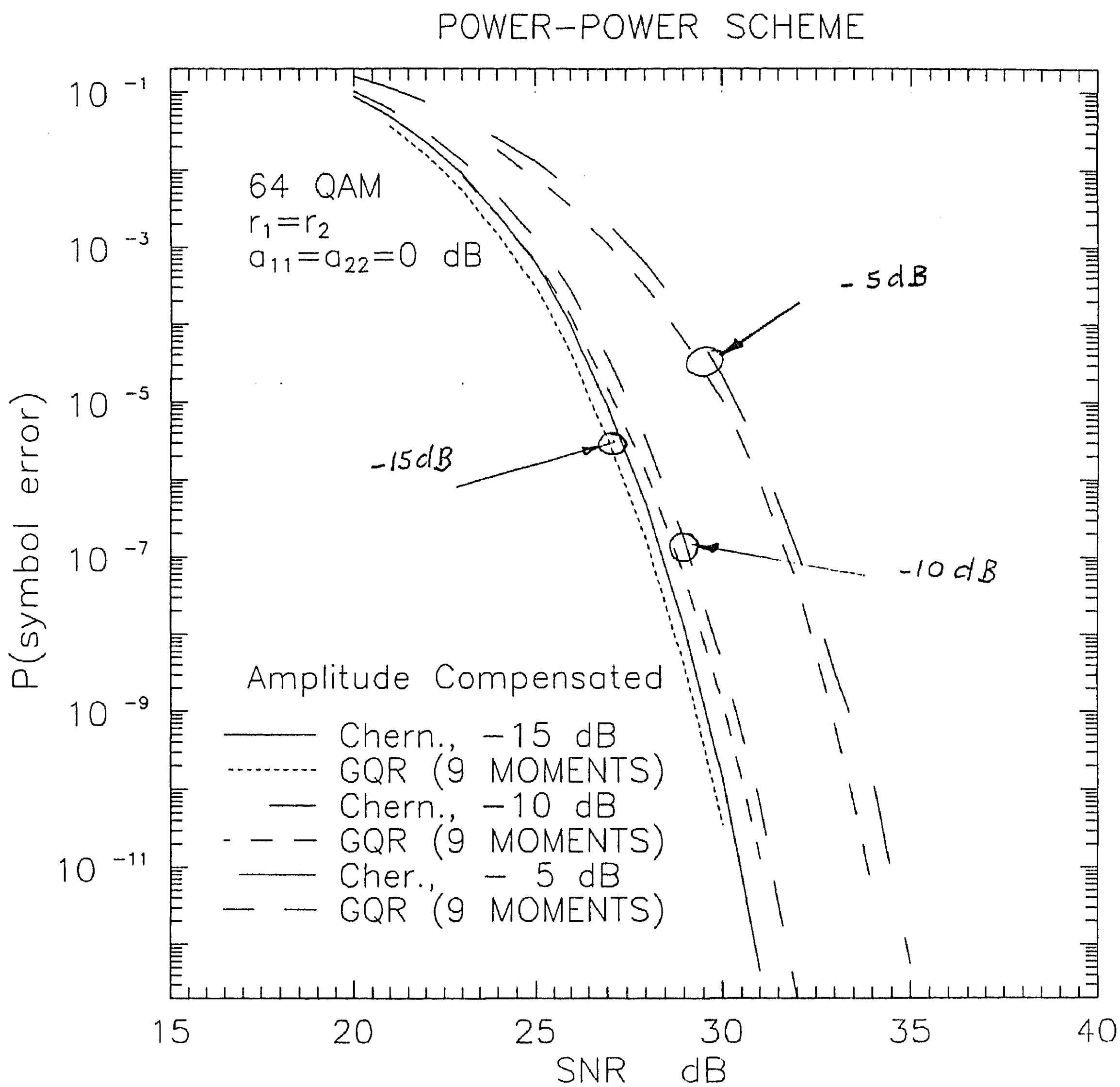

Figure 3.7: Power-Power Cross-Pol Canceler, Chernoff bound and GQR moment calculations comparison, 64 QAM

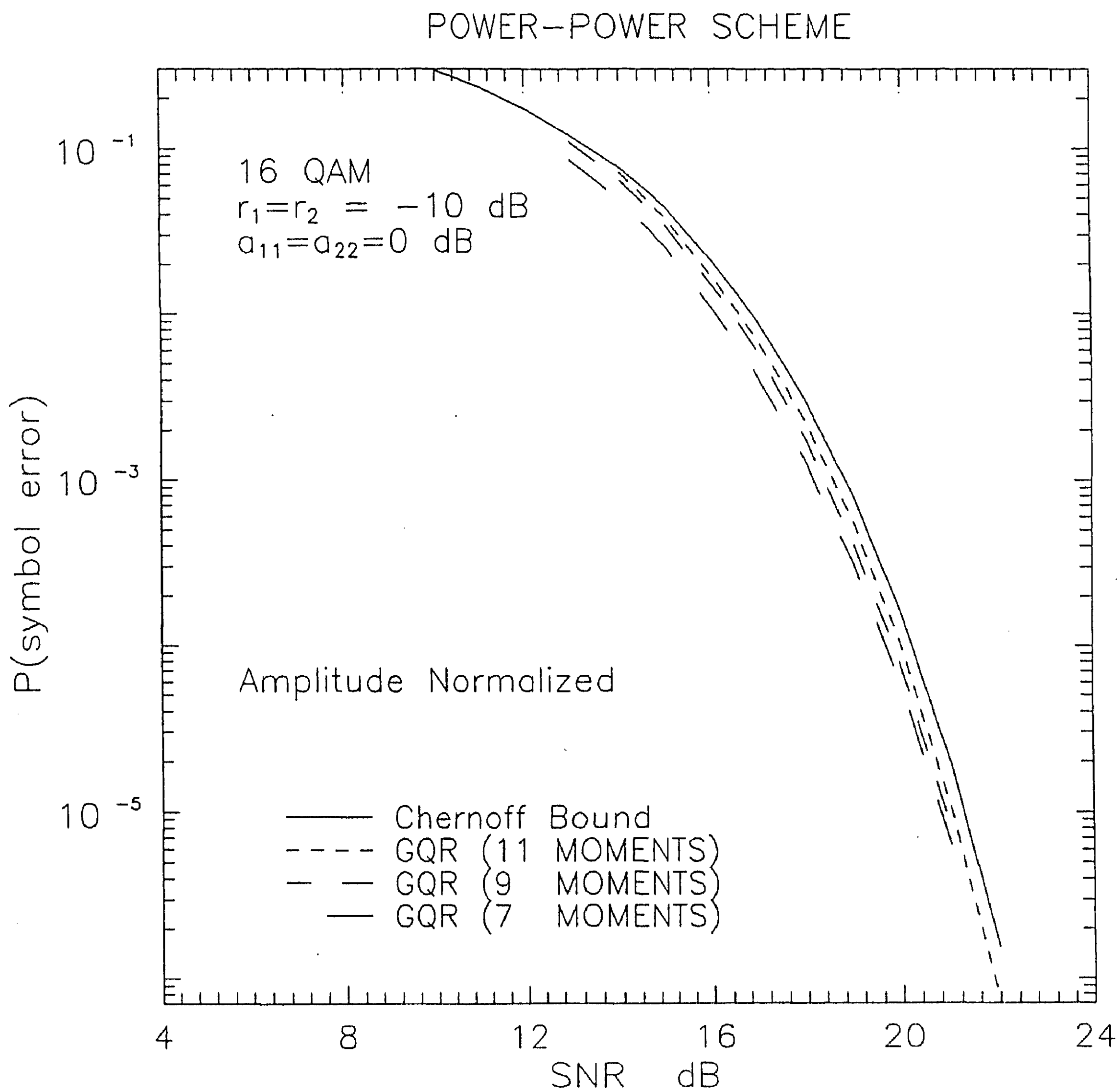

Figure 3.8: Power-Power Cross-Pol Canceler, effect of increasing number of moments on GQR calculation results, 16 QAM

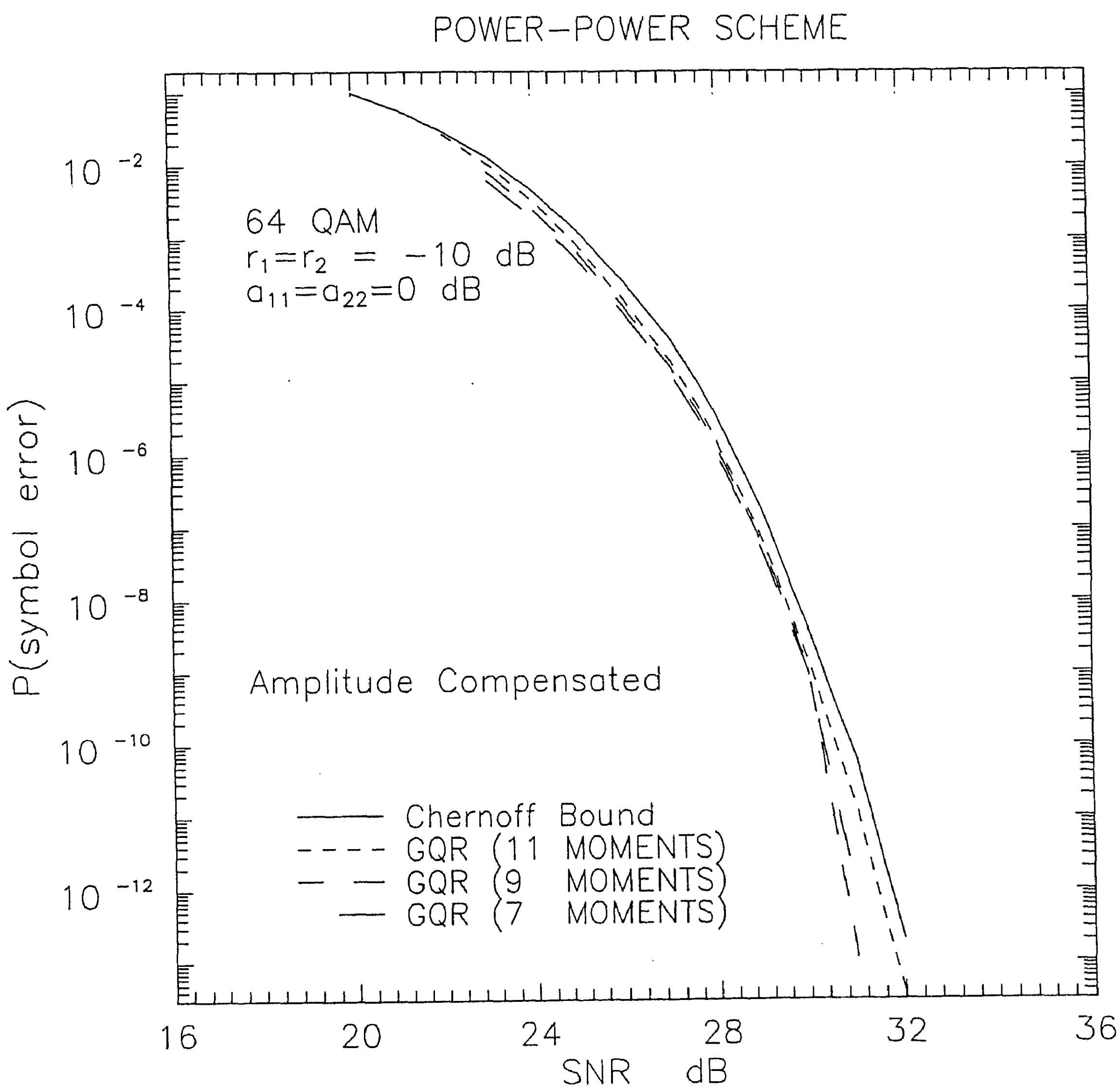

Figure 3.9: Power-Power Cross-Pol Canceler, effect of increasing number of moments on GQR calculation results, 64 QAM

Power-Power Scheme For 16 QAM		
$r_1 = r_2 =$ -15 dB		
SNR	Moment 9	Chernoff Bound
13	0.946E-1	1.078E-1
14	5.329E-2	6.541E-2
15	2.564E-2	3.491E-2
16	1.058E-2	1.586E-2
17	3.577E-3	5.881E-3
18	0.933E-3	1.692E-3
19	1.776E-4	3.540E-4
20	2.219E-5	4.973E-5
21	1.666E-6	4.244E-6
22	6.785E-8	1.943E-7

Table 3.1: Power-Power Cross-Pol Canceler, 16 QAM, performance calculation with Chernoff Bound and GQR methods, with amplitude compensation for cross coupling -15 dB

Power-Power Scheme For 16 QAM				
$r_1 = r_2 =$ -10 dB				
SNR	Moment 7	Moment 9	Moment 11	Chernoff Bound
13	8.615E-2	1.106E-1	1.261E-1	1.210E-1
14	4.888E-2	6.444E-2	7.172E-2	7.666E-2
15	2.480E-2	3.362E-2	3.724E-2	4.340E-2
16	1.102E-2	1.522E-2	1.719E-2	2.139E-2
17	4.158E-3	5.850E-3	6.830E-3	8.896E-3
18	1.281E-3	1.761E-3	2.235E-3	3.004E-3
19	3.067E-4	4.077E-4	5.496E-4	7.854E-4
20	5.367E-5	6.918E-5	0.936E-4	1.499E-4
21	6.343E-6	7.971E-6	1.041E-5	1.935E-5
22	0.4571E-6	0.564E-6	0.716E-6	1.532E-6

Table 3.2: Power-Power Cross-Pol Canceler, 16 QAM, performance calculation with Chernoff Bound and effect of increasing the number of moments on GQR calculation, with amplitude compensation for cross coupling -10 dB, 16 QAM

Power-Power Scheme For 16 QAM			
$r_1 = r_2 =$ -5 dB			
SNR	Moment 9	Moment 11	Chernoff Bound
14	1.185E-1	1.285E-1	1.194E-1
15	7.684E-2	8.555E-2	8.218E-2
16	4.746E-2	5.473E-2	5.317E-2
17	2.633E-2	3.292E-2	3.199E-2
18	1.299E-2	1.806E-2	1.760E-3
19	5.740E-3	8.119E-3	8.640E-3
20	2.199E-3	2.963E-3	3.667E-3
21	6.993E-4	0.905E-3	1.292E-3
22	1.749E-4	2.213E-4	3.605E-4

Table 3.3: Power-Power Cross-Pol Canceler , 16 QAM, performance calculation with Chernoff Bound and effect of increasing the number of moments on GQR calculation, with amplitude compensation for cross coupling -5 dB, 16 QAM

Power-Power Scheme For 64 QAM				
	$r_1 = r_2 =$ -15 dB		$r_1 = r_2 =$ -10 dB	
SNR	Moment 9	Chernoff Bound	Moment 11	Chernoff Bound
22	1.552E-2	2.274E-2	2.756E-2	3.048E-2
23	5.452E-3	8.878E-3	1.076E-2	1.324E-2
24	1.483E-3	2.724E-3	3.444E-3	4.718E-3
25	3.004E-4	6.180E-4	0.828E-3	1.318E-3
26	4.157E-5	9.606E-5	1.523E-4	2.726E-4
27	3.592E-6	9.302E-6	1.942E-5	3.890E-5
28	1.722E-7	4.987E-7	1.559E-6	3.489E-6
29	0.395E-8	1.277E-8	0.707E-7	1.745E-7
30	0.359E-10	1.376E-10	1.514E-9	4.177E-9
31	1.032E-13	4.576E-13	1.279E-11	6.350E-11
32	0.695E-16	3.629E-16	0.334E-13	2.063E-13

Table 3.4: Power-Power Cross-Pol Canceler, 64 QAM, performance calculation with Chernoff Bound and GQR methods, with amplitude compensation for cross coupling -15 dB and -10 dB

Power-Power Scheme For 64 QAM		
	$r_1 = r_2 =$ -5 dB	
SNR	Moment 9	Chernoff Bound
23	3.484E-2	4.518E-2
24	1.808E-2	2.549E-2
25	8.413E-3	1.291E-2
26	3.400E-3	5.693E-3
27	1.148E-3	2.106E-3
28	3.078E-4	6.227E-4
29	6.151E-5	1.388E-4
30	1.057E-5	2.165E-5
31	0.937E-6	2.154E-6
32	0.463E-7	1.216E-7
33	7.102E-10	3.366E-9
34	5.79E-12	1.386E-10

Table 3.5: Power-Power Cross-Pol Canceler, 64 QAM, performance calculation with Chernoff Bound and GQR methods, with amplitude compensation for cross coupling -5 dB

Chapter 4

PERFORMANCE ANALYSIS OF CORRELATION-CORRELATION SCHEME

4.1 Introduction

The purpose of this chapter is to study the performance of bootstrapped cross-pol canceller with correlation-correlation scheme. It differs from the power-power scheme in the way its weights are controlled. While in the former the output powers are minimized in searching for the optimal weights, here the correlation between the two outputs are used instead.

As in the case of power-power canceler, we first derive, in the next section the correlator-correlator canceler parameters; the optimal weight, with and without noise effects, the optimal output and the decision parameter with amplitude compensation only and with both amplitude and phase compensation. Using these decision parameter, we find in section 4.3, the least upper bound (Chernoff bound) on the probability of error. We also derive in this section an expression which is used to calculate an approximation to the probability of error by the method of moments.

Results on the performance of the correlator-correlator canceler using both Chernoff bound and the method of moments are presented in section 4.4 and depicted in figures and tables located at the end of this chapter. Comparisons of these results and conclusion are found in (4.5)

4.2 Canceler Scheme and Parameters

The correlation-correlation scheme is shown in Fig. 2.12. Its principle of operation was detailed in chapter 2.

4.2.1 Canceler Outputs

From Fig. 2.12, the outputs of the cross-polarization canceller are given by,

$$y_1(n) = x_1(n) + x_2(n)w_{12} \tag{4.1}$$

$$y_2(n) = x_2(n) + x_1(n)w_{21}, \tag{4.2}$$

where $x_1(n)$ and $x_2(n)$ are the received signals samples after match filtering given in (2.5). Substituting for $x_1(n)$ and $x_2(n)$, we get for the output of the canceler,

$$y_1(n) = I_1(n)(a_{11} + w_{12}a_{21}) + I_2(n)(a_{12} + w_{12}a_{22}) + n_1(n) + n_2(n)w_{12} \tag{4.3}$$

$$y_2(n) = I_1(n)(a_{21} + w_{21}a_{11}) + I_2(n)(a_{22} + w_{21}a_{12}) + n_1(n)w_{21} + n_2(n) \tag{4.4}$$

4.2.2 Optimal Weights

In contrast to the power-power canceler, here the control algorithm simultaneously minimizes square magnitude of the output correlations,

$$P_1(w_{12}^i, w_{21}^i) = |E\{y_{1d}^i(n)y_2^i(n)^*\}|^2 \tag{4.5}$$

$$Q_1(w_{12}^i, w_{21}^i) = |E\{y_{2d}^i(n) y_1^i(n)^*\}|^2 \tag{4.6}$$

where $y_1(n)$ and $y_2(n)$ are the samples of the corresponding outputs while $y_{1d}(n)$ and $y_{2d}(n)$ are the samples of these outputs after discriminations.

In fact, it simultaneously searches for $\partial|E\{y_{1d}(n)y_2(n)^*\}|^2/\partial w_{12} = 0$ and $\partial|E\{y_{2d}(n)y_1(n)^*\}|^2/\partial w_{21} = 0$, where $E\{\cdot\}$ and $|\cdot|$ denote the expected and magnitude respectively. The search for the optimum weights can be performed by successive use of the following recursive equations,

$$w_{12}^{i+1} = w_{12}^i - \mu_1 \frac{\partial}{\partial w_{12}^i} P_1(w_{12}^i, w_{21}^i) \tag{4.7}$$

$$w_{21}^{i+1} = w_{21}^i - \mu_2 \frac{\partial}{\partial w_{21}^i} Q_1(w_{12}^i, w_{21}^i), \tag{4.8}$$

where μ_1 and μ_2 are the constants which determine the stability of convergence.

The optimum weights that minimize the square magnitude of the output correlations are the steady state weights obtained from ,

$$\frac{\partial P_1(w_{12}^i, w_{21}^i)}{\partial w_{12}^i} = 0 \tag{4.9}$$

$$\frac{\partial Q_1(w_{12}^i, w_{21}^i)}{\partial w_{21}^i} = 0 \tag{4.10}$$

From (4.3) and (4.4), we first find the correlations between one output and the second output after discrimination:

$$\begin{aligned} A_1(w_{12}, w_{21}) \triangleq\ & E\{y_{1d}(n) y_2(n)^*\} = \delta_{11} E\{|I_1(n)|^2\}(a_{11} + w_{12}a_{21})(w_{21}a_{11} + a_{21})^* \\ & + \delta_{12} E\{|I_2(n)|^2\}(a_{12} + w_{12}a_{22})(a_{22} + w_{21}a_{12})^* + E\{|n_1(n)|^2\} w_{21}^* \\ & + E\{|n_2(n)|^2\} w_{12}, \end{aligned} \tag{4.11}$$

$$B_1(w_{12},w_{21}) \triangleq E\{y_{2d}(n)y_1(n)^*\} = \delta_{21}E\{|I_1(n)|^2\}(a_{11}+w_{12}a_{21})^*(w_{21}a_{11}+a_{21})$$

$$+\delta_{22}E\{|I_2(n)|^2\}(a_{12}+w_{12}a_{22})^*(a_{22}+w_{21}a_{12})+E\{|n_1(n)|^2\}w_{21}$$

$$+E\{|n_2(n)|^2\}w_{12}^*, \quad (4.12)$$

where $\delta_{i,j}$ i,j=1,2 denotes the effect of the i th discriminator on the different signals $I_1(n)$ or $I_2(n)$.

It has been shown in [13] that the optimum weights $w_{12\text{opt}}$ and $w_{21\text{opt}}$ that simultaneously minimize the square magnitude of the output correlation P_1 and Q_1 of (4.5) and (4.6), respectively can be obtained from equating simultaneously $A_1(w_{12},w_{21})$ and $B_1(w_{12},w_{21})$ of (4.11) and (4.12) to zero. This lead to;

$$w_{12\text{opt}} = \frac{-1}{Dw_{12\text{opt}}}\Big[a_{11}(a_{21}+w_{21\text{opt}}a_{11})^*E\{|I_1(n)|^2\}\delta_{11}$$

$$+a_{12}(a_{22}+w_{21\text{opt}}a_{12})^*E\{|I_2(n)|^2\}\delta_{12}+w_{21\text{opt}}^*E\{|n_1(n)|^2\}\Big], \quad (4.13)$$

with,

$$Dw_{12\text{opt}} = a_{21}(a_{21}+w_{21\text{opt}}a_{11})^*E\{|I_1(n)|^2\}\delta_{11}$$
$$+a_{22}(a_{22}+w_{21\text{opt}}a_{12})^*E\{|I_2(n)|^2\}\delta_{12}+E\{|n_2(n)|^2\}, \quad (4.14)$$

and

$$w_{21\text{opt}} = \frac{-1}{Dw_{21\text{opt}}}\Big[a_{21}(a_{11}+w_{12\text{opt}}a_{21})^*E\{|I_1(n)|^2\}\delta_{21}$$

$$+a_{22}(a_{12}+w_{12\text{opt}}a_{22})^*E\{|I_2(n)|^2\}\delta_{22}+w^*_{12\text{opt}}E\{|n_2(n)|^2\}\Big] \quad (4.15)$$

with

$$\begin{aligned} Dw_{21\text{opt}} &= a_{11}(a_{11}+w_{12\text{opt}}a_{21})^*E\{|I_1(n)|^2\}\delta_{21} \\ &\quad +a_{12}(a_{12}+w_{12\text{opt}}a_{22})^*E\{|I_2(n)|^2\}\delta_{22}+E\{|n_1(n)|^2\}. \quad (4.16) \end{aligned}$$

The effect of the discriminator are presented by $\delta_{i,j}\ \ i,j=1,2$ real valued and $\delta_{11}\delta_{22}<\delta_{12}\delta_{21}$. Note that, the first and second terms in (4.11) are complex conjugates of the terms in (4.12). Therefore, to find a unique solution for w_{12} and w_{21} using these equations, discriminators which enforce the constant $\delta_{i,j}\ \ i,j=1,2$ satisfying the above condition is essential. The simultaneous solution of these non-linear equations give two equilibrium points; $[w_{12\text{opt}1}, w_{21\text{opt}1}]$ and $[w_{12\text{opt}2}, w_{21\text{opt}2}]$. As it was discussed in power-power canceler, one of these points is a stable equilibrium which provide a solution to our problem.

4.2.3 Effect of Noise on Optimal Weight

In the absence of noise, that is when $E\{|n_1(n)|^2\}=E\{|n_2(n)|^2\}=0$ the stable equilibrium points can easily be found to be;

$$w_{12\text{opt}}=-\frac{a_{12}}{a_{22}}, \qquad w_{21\text{opt}}=-\frac{a_{21}}{a_{11}}. \quad (4.17)$$

By similar approach to that followed for power-power scheme, when noise is added, we will write

$$w_{12\text{opt}}=-\frac{a_{12}}{a_{22}}+\epsilon_1, \qquad w_{21\text{opt}}=-\frac{a_{21}}{a_{11}}+\epsilon_2, \quad (4.18)$$

where ϵ_1 and ϵ_2 are perturbations on the optimal weights due to the existence of input noise.

Notice that the optimal weights $w_{12\text{opt}}$ and $w_{21\text{opt}}$ for this scheme given by (4.13) and (4.15) are the same as these in (3.20) and (3.22) for the power-power scheme. Therefore, ϵ_1 and ϵ_2 are the same as in (3.67) and (3.76), respectively.

4.2.4 Canceler Optimal Outputs

Substituting (4.18) in (4.3), we get for $y_1(n)$,

$$\begin{aligned} y_1(n) &= I_1(n)[a_{11} + (\frac{-a_{12}}{a_{22}} + \epsilon_1)a_{21}] + I_2(n)[a_{12} + (\frac{-a_{12}}{a_{22}} + \epsilon_1)a_{22}] \\ & n_1(n) + n_2(n)[\frac{-a_{12}}{a_{22}} + \epsilon_1], \end{aligned} \tag{4.19}$$

and after combining term, we have,

$$\begin{aligned} y_1(n) &= I_1(n)a_{11}[1 - \frac{a_{12}a_{21}}{a_{22}a_{11}} + \frac{\epsilon_1 a_{21}}{a_{11}}] + I_2(n)a_{22}\epsilon_1 \\ & n_1(n) + n_2(n)[\frac{-a_{12}}{a_{22}} + \epsilon_1]. \end{aligned} \tag{4.20}$$

4.2.5 Decision Parameters with Amplitude Compensation at the Canceler Output

To compensate for the change in amplitude of the co-pol signal we use as a decision parameter $\hat{I}_1(n) = \frac{y_1(n)}{a_{11}}$. Therefore, from (4.20), we can write the amplitude compensated output,

$$\begin{aligned} \frac{y_1(n)}{a_{11}} &= I_1(n)[1 - \frac{a_{12}a_{21}}{a_{22}a_{11}} + \frac{\epsilon_1 a_{21}}{a_{11}}] + \frac{1}{a_{11}}\Big[I_2(n)a_{22}\epsilon_1 \\ & n_1(n) + n_2(n)[\frac{-a_{12}}{a_{22}} + \epsilon_1]\Big] \end{aligned} \tag{4.21}$$

Let the decision parameter be $Z_1(n) \triangleq \hat{I}_1(n) - I_1(n)$, then ,

$$Z_1(n) = I_1(n)[-\frac{a_{12}a_{21}}{a_{22}a_{11}} + \frac{\epsilon_1 a_{21}}{a_{11}}] + \frac{1}{a_{11}}\Big[I_2(n)a_{22}\epsilon_1 + n_1(n)$$

$$+n_2(n)[\frac{-a_{12}}{a_{22}} + \epsilon_1]\Big]. \quad (4.22)$$

Our aim is to find the probability of error $P_1\{|Z_1(n)| > c\}$, using $\frac{a_{12}}{a_{22}} = r_1 e^{j\phi_1}$, $\frac{a_{21}}{a_{11}} = r_2 e^{j\phi_2}$, and $V = \epsilon_1 \frac{a_{21}}{a_{11}}$ from (3.90) and taking $\epsilon_1 = \epsilon_{1R} + j\epsilon_{1I}$, we get

$$\begin{aligned} Z_1 &= (I_{1R} + jI_{1I})\Big[[V_R - r_1 r_2 cos(\phi_1 + \phi_2)] + j[V_I - r_1 r_2 sin(\phi_1 + \phi_2)] \\ &+\frac{1}{a_{11}}\Big[(I_{2R} + jI_{2I})a_{22}(\epsilon_{1R} + j\epsilon_{1I}) + n_{1R} + jn_{1I} \\ &+(n_{2R} + jn_{2I})[(\epsilon_{1R} - r_1 cos\phi_1) + j(\epsilon_{1I} - r_1 sin\phi_1)\Big] \end{aligned} \quad (4.23)$$

where V_R and V_I are specified in chapter 3. For the convenience of notation, we dropped the dependence on sampling time n. Finally from (4.23), we write the real and imaginary part of $Z_1(n)$ in terms of the real and imaginary part of signals and noises;

$$\begin{aligned} Z_{1R} &= I_{1R}[V_R - r_1 r_2 cos(\phi_1 + \phi_2)] - I_{1I}[V_I - r_1 r_2 sin(\phi_1 + \phi_2)] \\ &+\frac{a_{22}}{a_{11}}I_{2R}\epsilon_{1R} - \frac{a_{22}}{a_{11}}I_{2I}\epsilon_{1I} + \frac{1}{a_{11}}n_{1R} \\ &+\frac{1}{a_{11}}n_{2R}[\epsilon_{1R} - r_1 cos\phi_1] - \frac{1}{a_{11}}n_{2I}[\epsilon_{1I} - r_1 sin\phi_1], \end{aligned} \quad (4.24)$$

and

$$\begin{aligned} Z_{1I} &= I_{1R}[V_I - r_1 r_2 sin(\phi_1 + \phi_2)] + I_{1I}[V_R - r_1 r_2 cos(\phi_1 + \phi_2)] \\ &+\frac{a_{22}}{a_{11}}I_{2R}\epsilon_{1I} + \frac{a_{22}}{a_{11}}I_{2I}\epsilon_{1R} + \frac{1}{a_{11}}n_{1I} \\ &+\frac{1}{a_{11}}n_{2R}[\epsilon_{1I} - r_1 sin\phi_1] + \frac{1}{a_{11}}n_{2I}[\epsilon_{1R} - r_1 cos\phi_1]. \end{aligned} \quad (4.25)$$

The Decision Parameters Final Expression

From, (4.24) and (4.25), we can write the real and imaginary of $Z_1(n)$ in term of the random variables representing the real and imaginary part of the signals and noises;

$$Z_{1R} = I_{1R}Y_1 + I_{1I}Y_2 + I_{2R}Y_3 + I_{2I}Y_4 + n_{1R}Y_5 + n_{2R}Y_6 + n_{2I}Y_7, \tag{4.26}$$

$$Z_{1I} = -I_{1R}Y_2 + I_{1I}Y_1 - I_{2R}Y_4 + I_{2I}Y_3 + n_{1I}Y_5 - n_{2R}Y_7 + n_{2I}Y_6 \tag{4.27}$$

where

$$Y_1 = V_R - r_1 r_2 cos(\phi_1 + \phi_2) \tag{4.28}$$

$$Y_2 = -[V_I - r_1 r_2 sin(\phi_1 + \phi_2)] \tag{4.29}$$

$$Y_3 = \frac{a_{22}}{a_{11}}\epsilon_{1R} \tag{4.30}$$

$$Y_4 = -\frac{a_{22}}{a_{11}}\epsilon_{1I} \tag{4.31}$$

$$Y_5 = \frac{1}{a_{11}} \tag{4.32}$$

$$Y_6 = \frac{1}{a_{11}}[\epsilon_{1R} - r_1 cos\phi_1] \tag{4.33}$$

$$Y_7 = -\frac{1}{a_{11}}[\epsilon_{1I} - r_1 sin\phi_1]. \tag{4.34}$$

4.2.6 Decision Parameter with Amplitude and Phase Compensation at the Canceler Output

By compensating for the changes in both amplitude and phase of the co-pol signal, we can write (4.20) ,

$$\frac{y_1(n)}{\Delta_y} = I_1(n) + \frac{1}{\Delta_y}\Big[I_2(n)a_{22}\epsilon_1 + n_1(n) + n_2(n)[\frac{-a_{12}}{a_{22}} + \epsilon_1]\Big], \tag{4.35}$$

where,

$$\Delta_y = a_{11}[1 - \frac{a_{12}a_{21}}{a_{22}a_{11}} + \frac{\epsilon_1 a_{21}}{a_{11}}]. \tag{4.36}$$

We take $\frac{y_1(n)}{\Delta_y}$ as an estimate for the co-pol signal $I_1(n)$. Hence, the decision parameter $Z_{1AP}(n)$ becomes,

$$Z_{1AP}(n) \triangleq \hat{I}_1(n) - I_1(n) = \frac{1}{\Delta_y}\Big[I_2(n)a_{22}\epsilon_1 + n_1(n) + n_2(n)[\frac{-a_{12}}{a_{22}} + \epsilon_1]\Big] \tag{4.37}$$

Comparing (4.37) together with (4.36) to (3.127) (3.125), we conclude that when both amplitude and phase compensation is employed the decision parameters for the power-power and the correlator-correlator cancelers are the same. Therefore, the real and imaginary part of $Z_{1AP}(n)$ are given by (3.138) and (3.139) together with (3.140) to (3.146).

4.3 The Performance Analysis

4.3.1 Chernoff Bound

With Amplitude Compensation

When amplitude compensation is employed, we obtain by using (4.26) in (3.158) with the values of Y_i calculated in previous section,

$$P(|Z_{1R}| > c) \leq E_{\phi_1,\phi_2}\Big\{\exp[-\lambda c + \frac{\lambda^2 c^2}{2}\frac{M-1}{3}(Y_1^2 + Y_2^2 + Y_3^3 + Y_4^2) + \frac{\lambda^2 c^2}{2}\sigma_n^2(Y_5^2 + Y_6^2 + Y_7^2)]\Big\}, \tag{4.38}$$

with c is half the distance between two signal space, Y_i $i = 1, 2.., 6$ are given in (4.28) to (4.34) and M is the size of the QAM constellation. The existence of a λ

to satisfy the bound is inherent in the definition of Chernoff bound. As a function of λ, we write in compact form;

$$P(|Z_{1R}| > c) \leq E_{\phi_1,\phi_2}\{\exp(-\lambda c + \lambda^2[U_c(\phi_1,\phi_2) + W_c(\phi_1,\phi_2)])\} \tag{4.39}$$

where

$$\begin{aligned} U_c(\phi_1,\phi_2) &= \frac{c^2}{2}\frac{M-1}{3}(Y_1^2 + Y_2^2 + Y_3^2 + Y_4^2) \\ W_c(\phi_1,\phi_2) &= \frac{\sigma_n^2 c^2}{2}(Y_5^2 + Y_6^2 + Y_7^2) \end{aligned} \tag{4.40}$$

Minimizing the exponent of (4.39) with respect to λ, we obtain the least upper bound;

$$P(|Z_{1R}| > c) \leq E_{\phi_1,\phi_2}\Big\{\exp\Big[\frac{-c^2}{4[U_c(\phi_1,\phi_2) + W_c(\phi_1,\phi_2)]}\Big]\Big\} \tag{4.41}$$

then we can write

$$P(|Z_{1R}| > c) \leq E_{\phi_1,\phi_2}\Big\{\exp\Big(\frac{-c^2}{4[\frac{c^2}{2}\frac{M-1}{3}U_{1c}(\phi_1,\phi_2) + \frac{\sigma_n^2}{2}W_{1c}(\phi_1,\phi_2)]}\Big)\Big\} \tag{4.42}$$

where

$$\begin{aligned} U_{1c}(\phi_1,\phi_2) &= Y_1^2 + Y_2^2 + Y_3^2 + Y_4^2 \\ W_{1c}(\phi_1,\phi_2) &= Y_5^2 + Y_6^2 + Y_7^2 \end{aligned} \tag{4.43}$$

Rearranging terms, we get

$$\begin{aligned} P(|Z_{1R}| > c) &\leq E_{\phi_1,\phi_2}\Big\{\exp\Big[\frac{\frac{-c^2}{\sigma_n^2}}{2[c^2\frac{M-1}{3\sigma_n^2}U_{1c}(\phi_1,\phi_2) + W_{1c}(\phi_1,\phi_2)]}\Big]\Big\} \\ &= E_{\phi_1,\phi_2}\Big\{\exp\Big[\frac{-3(SNR)}{2(M-1)[(SNR)U_{1c}(\phi_1,\phi_2) + W_{1c}(\phi_1,\phi_2)]}\Big]\Big\} \end{aligned}$$

where in the last step, we used,

$$SNR = \frac{S}{N} = \frac{M-1}{3}\frac{c^2}{\sigma_n^2} \tag{4.44}$$

Continuing as in section 3.3.1, we obtain the following bound;

$$\begin{aligned} P_1(e) \;\; &\leq \;\; (1 - \frac{1}{\sqrt{M}})\frac{1}{4\pi^2} \cdot \\ &\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \exp\left[\frac{-3\,(SNR)}{2(M-1)}\frac{1}{(SNR)U_{1c}(\phi_1,\phi_2) + W_{1c}(\phi_1,\phi_2)}\right] d\phi_1 d\phi_2 \end{aligned} \tag{4.45}$$

which is the same form as 3.166 except for the function $U_{1c}(\phi_1, \phi_2)$ and $W_{1c}(\phi_1, \phi_2)$ from (4.43) instead of $U_1(\phi_1, \phi_2)$ and $W_1(\phi_1, \phi_2)$ in chapter 3.

With Amplitude and Phase Compensation

Clearly when both amplitude and phase compensation are employed then the Chernoff bounds are the same as for the power-power canceler, and can obtain from section 3.3.1. In particular (3.168) can be used to calculate the least upper bound on the probability of error.

4.3.2 Method of Moments for Probability of Error Calculation

With Amplitude Compensation

When only amplitude compensation is used, the average probability of error is different from those obtained with power-power. The steps of calculating these error probability using the method of moments are the same as these described in section 3.3.2 of the previous chapter. In the following, we will summarize the results as they relate to the correlator-correlator canceler.

From (3.178),

$$P_1(e|\phi_1, \phi_2, I_{1R}, I_{1I}, I_{2R}, I_{2I}) = 2(1 - \frac{1}{\sqrt{M}})Q\Big(\frac{c - X_{Ic}}{\sigma_c(\phi_1, \phi_2)}\Big). \tag{4.46}$$

where

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty \exp(\frac{-t^2}{2})\, dt \tag{4.47}$$

$$X_{Ic} = I_{1R}Y_1 + I_{1I}Y_2 + I_{2R}Y_3 + I_{2I}Y_4 \tag{4.48}$$

$$\sigma_c^2(\phi_1, \phi_2) = (Y_5^2 + Y_6^2 + Y_7^2)\sigma_n^2 \tag{4.49}$$

with Y_1 to Y_7 defined in (4.28) to (4.34).

We define a random variable

$$\mathrm{x_c} = \frac{c - X_{Ic}}{\sigma_{oc}^2(\phi_1, \phi_2)} \tag{4.50}$$

with

$$\sigma_{oc}^2(\phi_1, \phi_2) = (Y_5^2 + Y_6^2 + Y_7^2)c^2 \tag{4.51}$$

Using $\mathrm{x_c}$ in (4.46), we get

$$P_1(e|\mathrm{x_c}) = 2(1 - \frac{1}{\sqrt{M}})Q(\frac{c}{\sigma_n}\mathrm{x_c}) \tag{4.52}$$

Hence,

$$P_1(e) = \int_x P_1(e|x) f_{\mathrm{x_c}}(x)\, dx \tag{4.53}$$

with $f_{\mathrm{x_c}}(x)$ is the pdf of the random variable $\mathrm{x_c}$. Using GQR for evaluating (4.52), we get

$$P_1(e) = 2(1 - \frac{1}{\sqrt{M}}) \sum_{i=1}^{N} w_i Q\Big(\sqrt{\frac{3(SNR)}{M-1}} x_i\Big) \tag{4.54}$$

where x_i and w_i are the nodes and the weights of the GQR obtained from the moments of the random variable $\mathbf{x_c}$. These moments can be obtained in a similar way as in section (3.3.2), (moment of $\mathbf{x}$).

4.4 Results

As in the case of power-power canceler, in this section, we present results of calculations of average probability of error using Chernoff upper bound and the Gauss quadrature rule. In these calculation which are presented in the following figures and tables dual-polarized 16 and 64 QAM signal are used. The calculation are done for different signal-to-noise ratios and with different cross coupling constants.

In Fig. 4.1, we depict the Chernoff bound for 16 QAM signal as a function of SNR and with cross coupling r=-15 dB, -10 dB and -5 dB. These bounds are shown for the case when amplitude and both amplitude and phase compensations are employed. Equation (4.45) is used for the case with amplitude compensation. For the case when both amplitude and phase compensation is employed, the decision parameter turn out to be the same as the one for power-power canceler. Hence, equation (3.168) from chapter 3 is used to plot the corresponding curve. In fact, these results are simply repeated in this chapter to facilitate comparison to the results obtained with amplitude only compensation. Fig. 4.2 is the same as Fig. 4.1 except for the use of 64 QAM instead of 16 QAM. Comparing of the bound for 16 QAM and 64 QAM are done in Fig. 4.3 and Fig. 4.4, for the case with amplitude compensation and the case with amplitude and phase compensation, respectively. The separation of these results to two figures was done to emphasize the fact that in the correlator-correlator canceler unlike power-power different kind of compensation plays an important role in performance. Fig. 4.5, Fig. 4.6, Fig. 4.7 and Fig. 4.8 depict the probability of error as they are calculated using the Gauss quadrature rule. The first two use 16 QAM with amplitude compensation only and with both amplitude and phase compensation, respectively. The other two are same except for using 64 QAM. Fig. 4.9, 10, 11 and 12 compare the results obtained with GQR moment to their corresponding Chernoff bound for 16 QAM with amplitude compensation only, for 16 QAM with both amplitude and phase compensation, for 64 QAM with amplitude compensation only and for 64 QAM with both amplitude and phase compensation, respectively. Finally, we listed in Tables 4.1 to 4.8 some of the results shown in the aforementioned figures. Notice that, in Fig. 9 and Fig. 11 when comparing GQR results with the Chernoff bounds, a cross over occurs. That is the approximation with the moment method results in higher error at low SNR with relatively high cross coupling than the Chernoff bound. This could be due to inaccuracy in the GQR results occur particularly when amplitude compensation is used.

4.5 Conclusion

The correlator-correlator bootstrapped canceler was analysed and its performance was studied in this chapter. As in the previous chapter a Chernoff bound on the probability of error at the output of the canceler was found. These errors were also calculated using the moment method in the Gauss quadrature rule. As before two kinds of channel compensation were employed in the performance studies; amplitude compensation and both amplitude and phase compensation. Unlike the result obtained with power-power cancelers, the two method of compensation depict large difference in performance. Also, it was shown that decision parameters of this canceler when both amplitude and phase compensations are employed are the same as the corresponding parameter for power-power canceler. It is also shown that when only amplitude compensation is employed the Chernoff bound is not sufficiently tight particularly when cross coupling is high (-5 dB). This may have led to the cross over of the Chernoff bound and GQR result discussed in previous section.

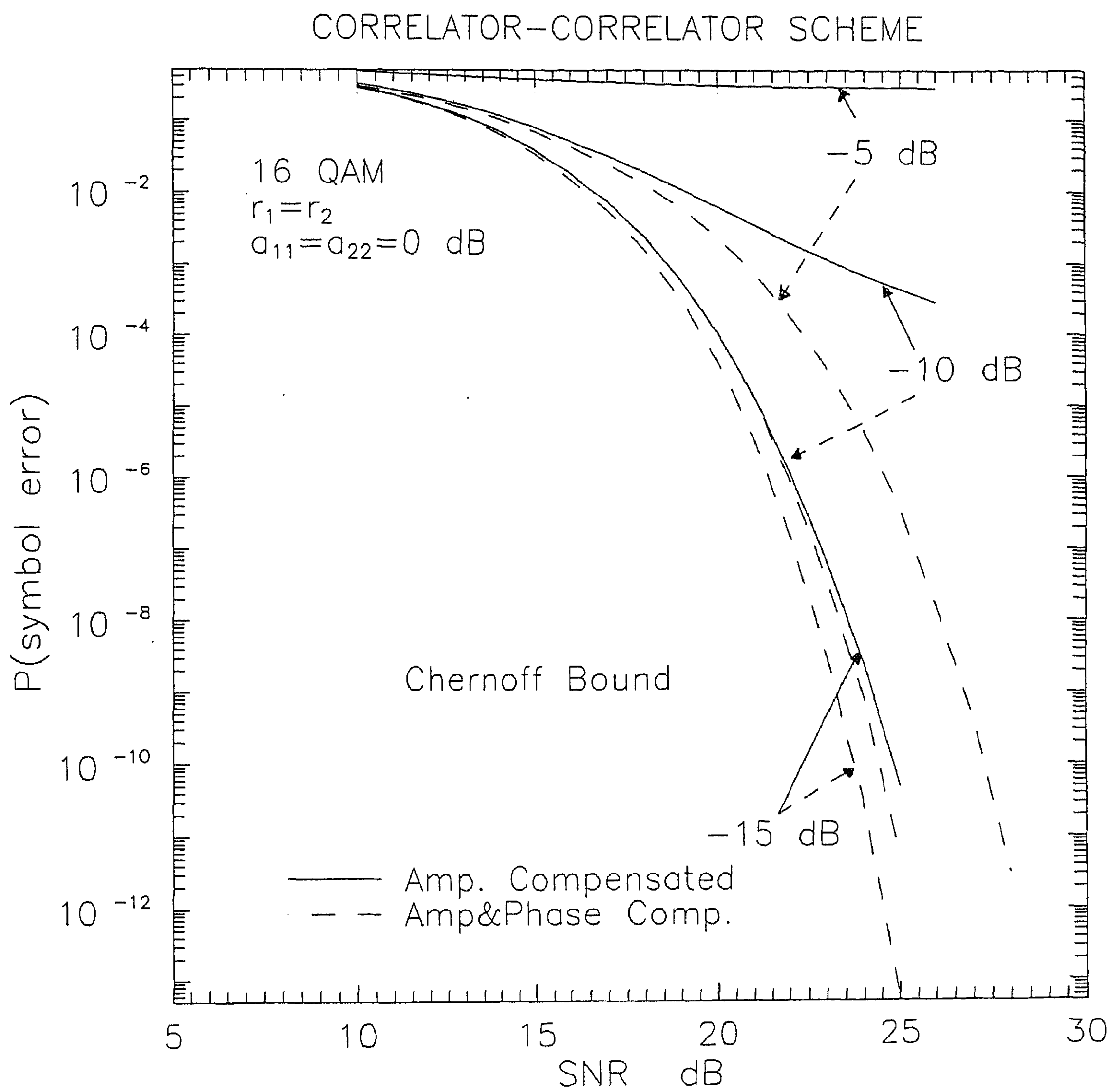

Figure 4.1: Correlator-Correlator Cross-Pol Canceler, Chernoff bound 16 QAM

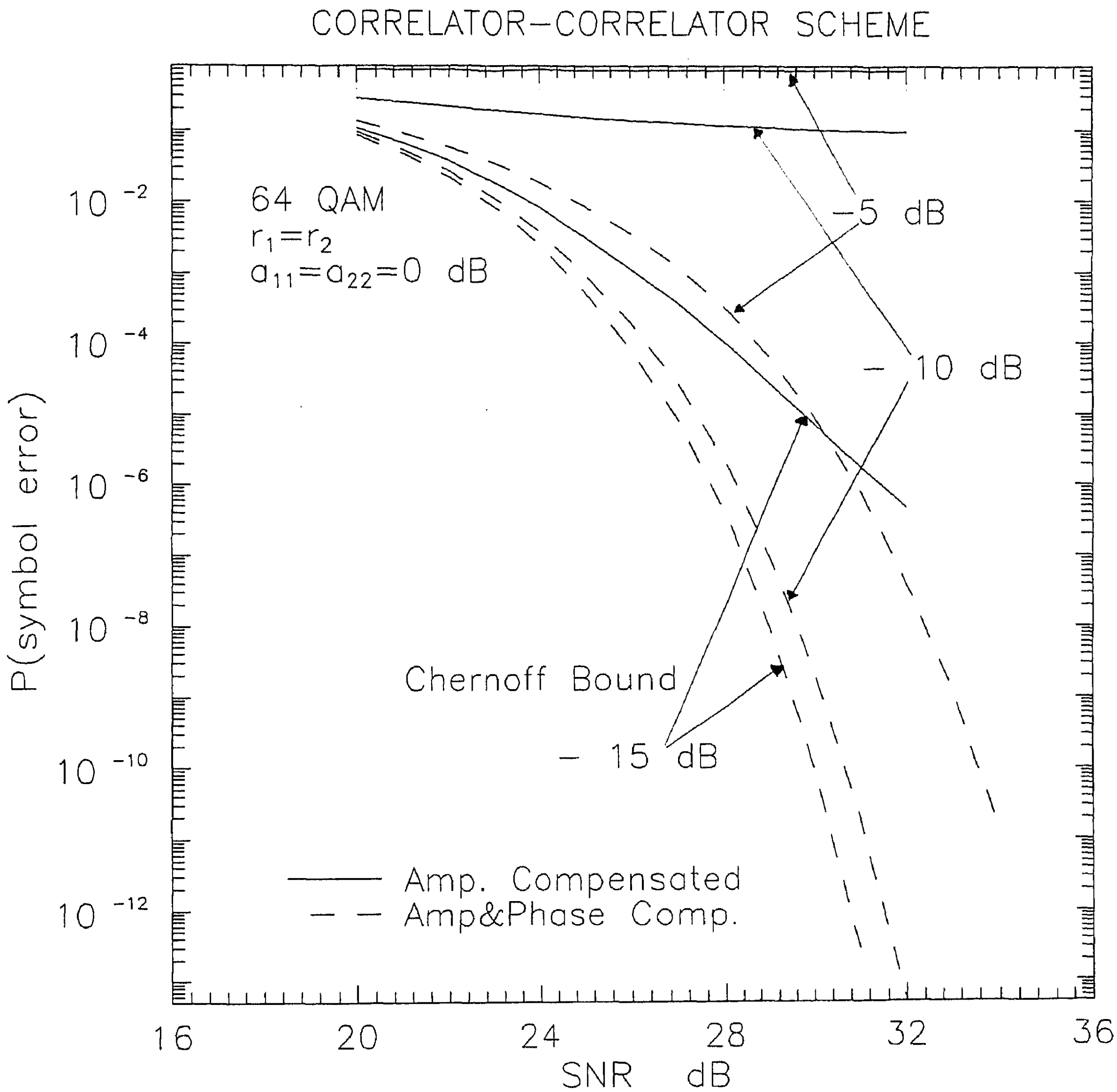

Figure 4.2: Correlator-Correlator Cross-Pol Canceler, Chernoff bound, 64 QAM

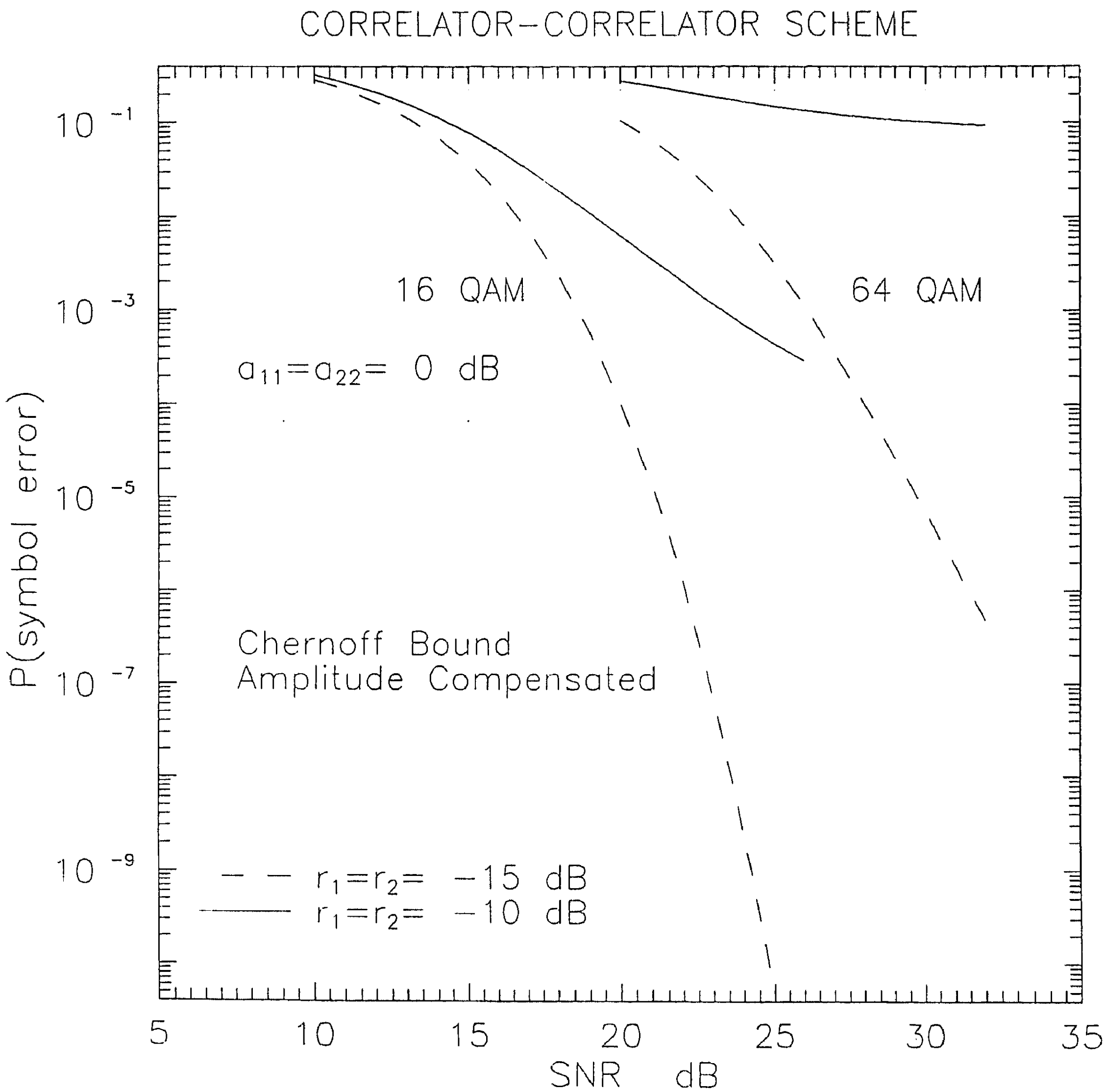

Figure 4.3: Correlator-Correlator Cross-Pol Canceler, Chernoff bound comparison 16 QAM v.s 64 QAM, with amplitude compensation

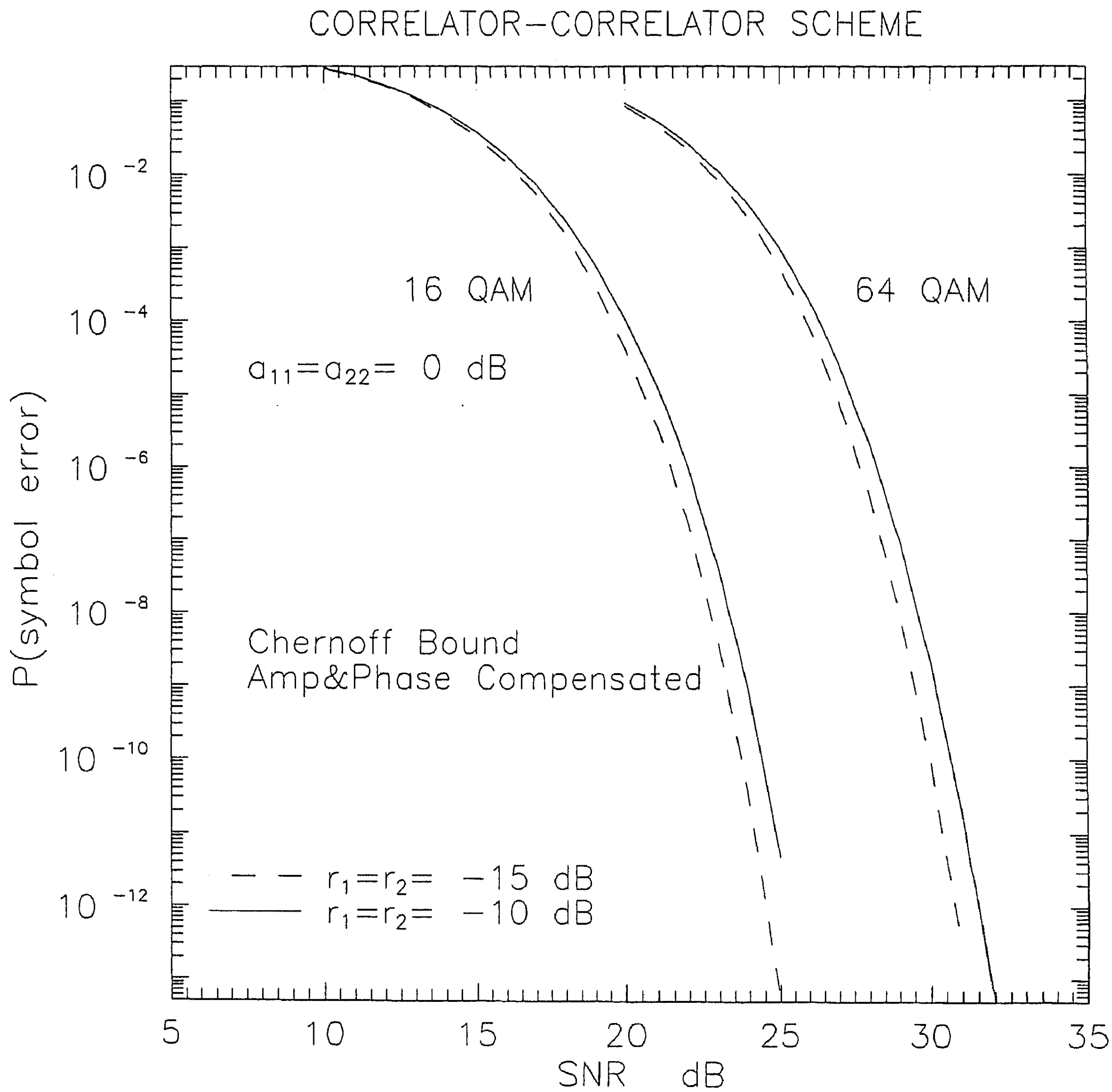

Figure 4.4: Correlator-Correlator Cross-Pol Canceler, Chernoff bound comparison 16 QAM v.s 64 QAM, with both amplitude and phase compensation

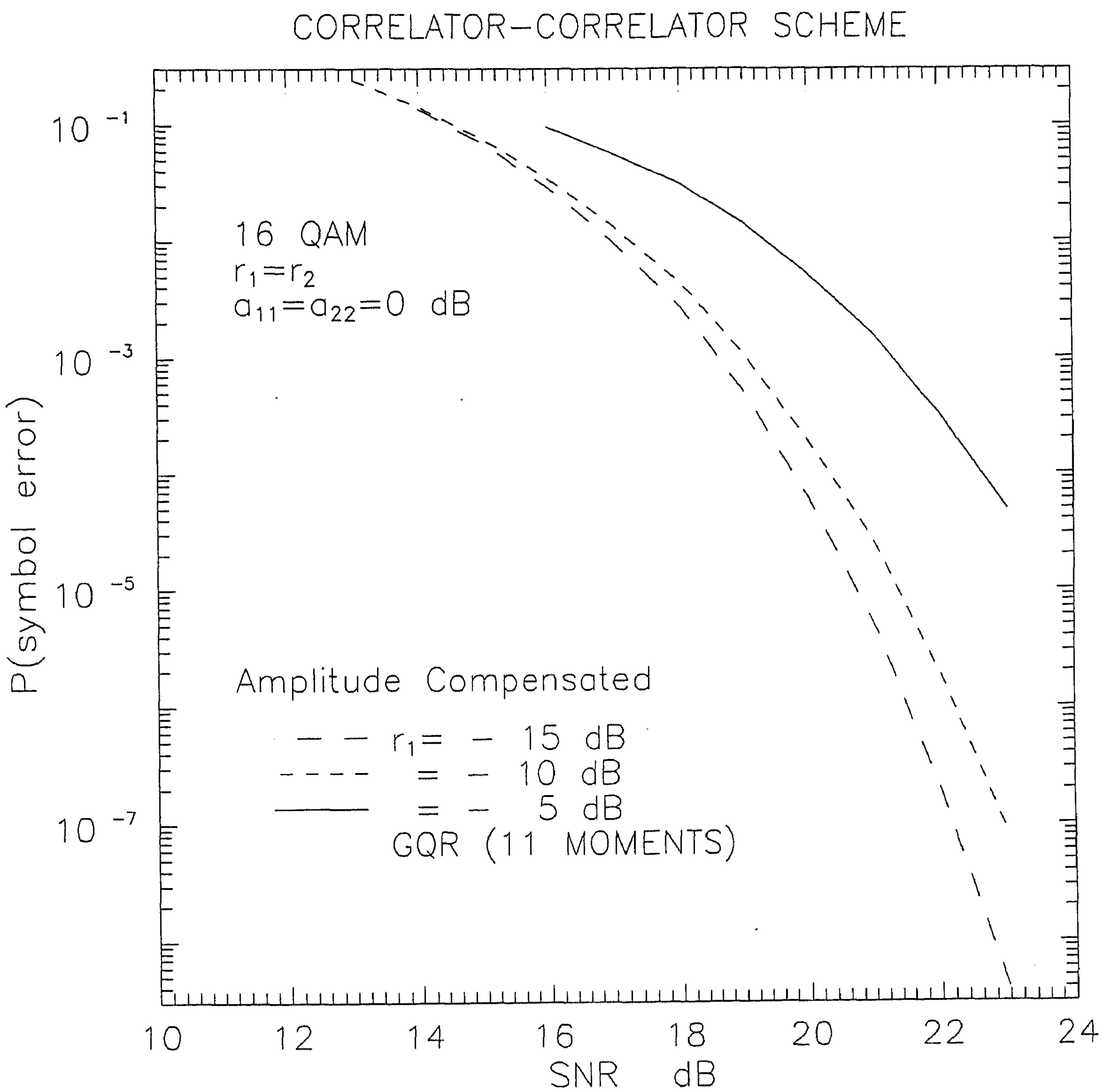

Figure 4.5: Correlator-Correlator Cross-Pol Canceler, GQR calculation, 16 QAM, with amplitude compensation

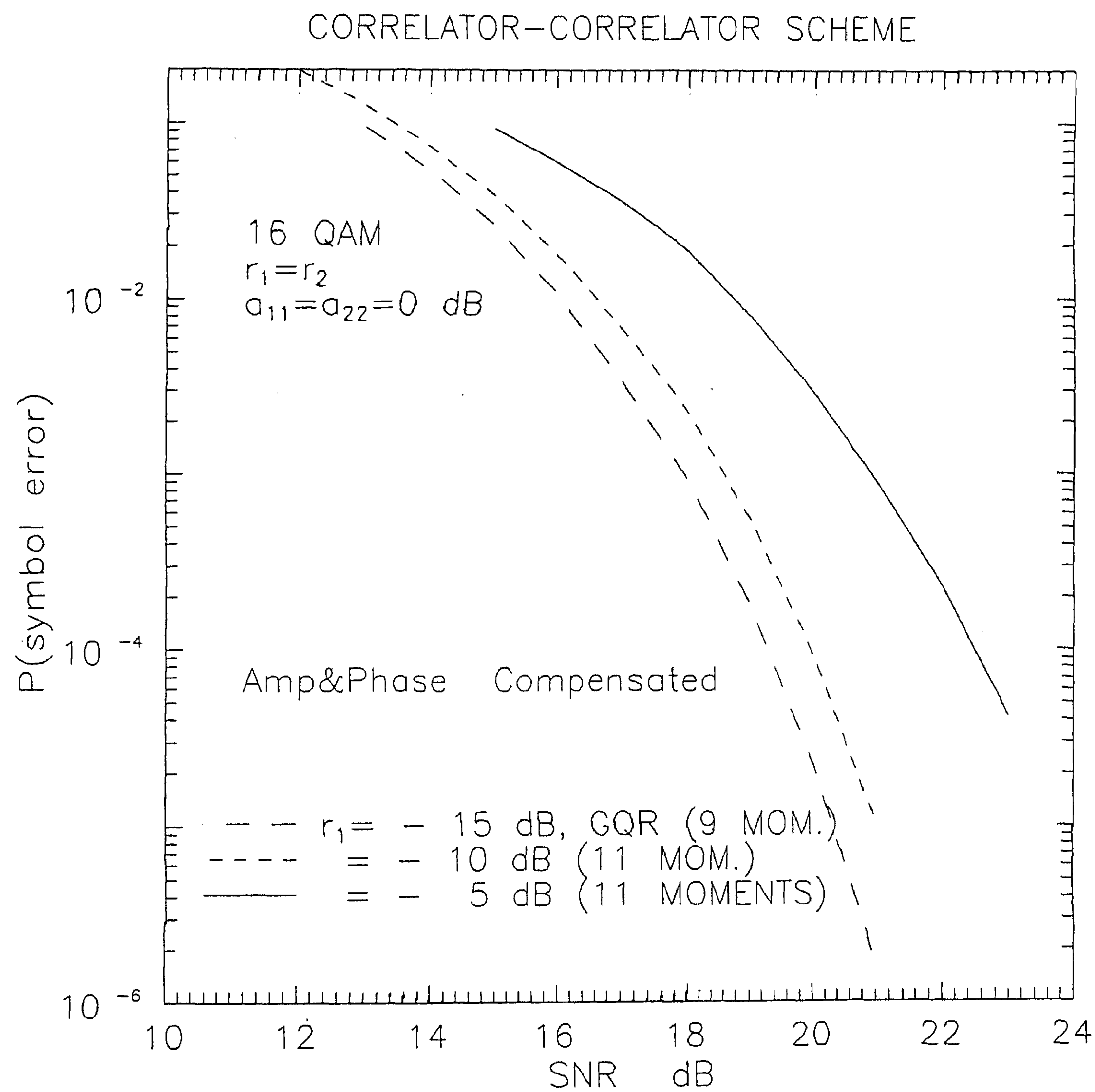

Figure 4.6: Correlator-Correlator Cross-Pol Canceler, GQR calculation, 16 QAM, both amplitude and phase compensation

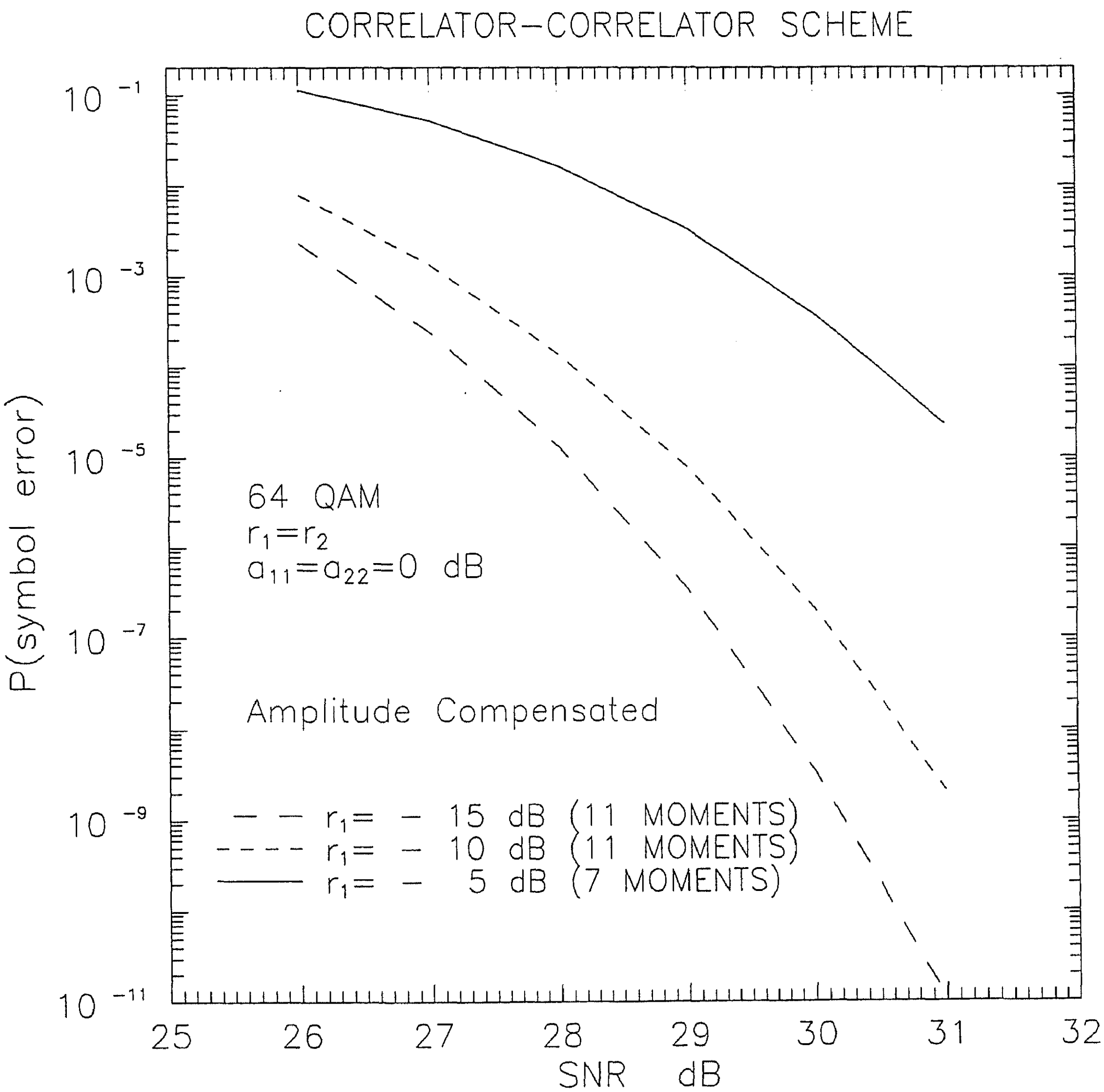

Figure 4.7: Correlator-Correlator Cross-Pol Canceler, GQR calculations, 64 QAM, with amplitude compensation

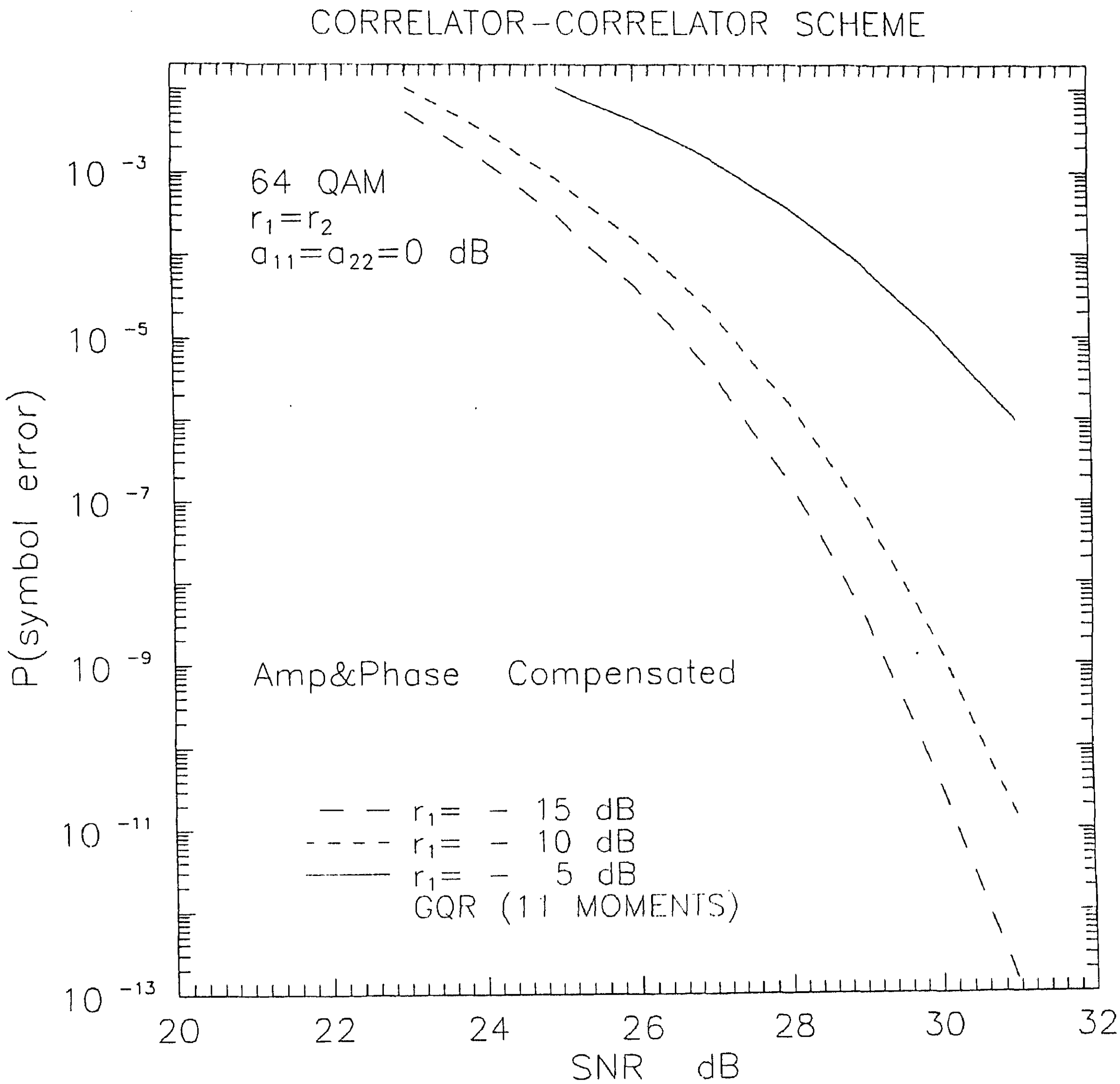

Figure 4.8: Correlator-Correlator Cross-Pol Canceler, GQR calculations, 64 QAM, with both amplitude and phase compensation

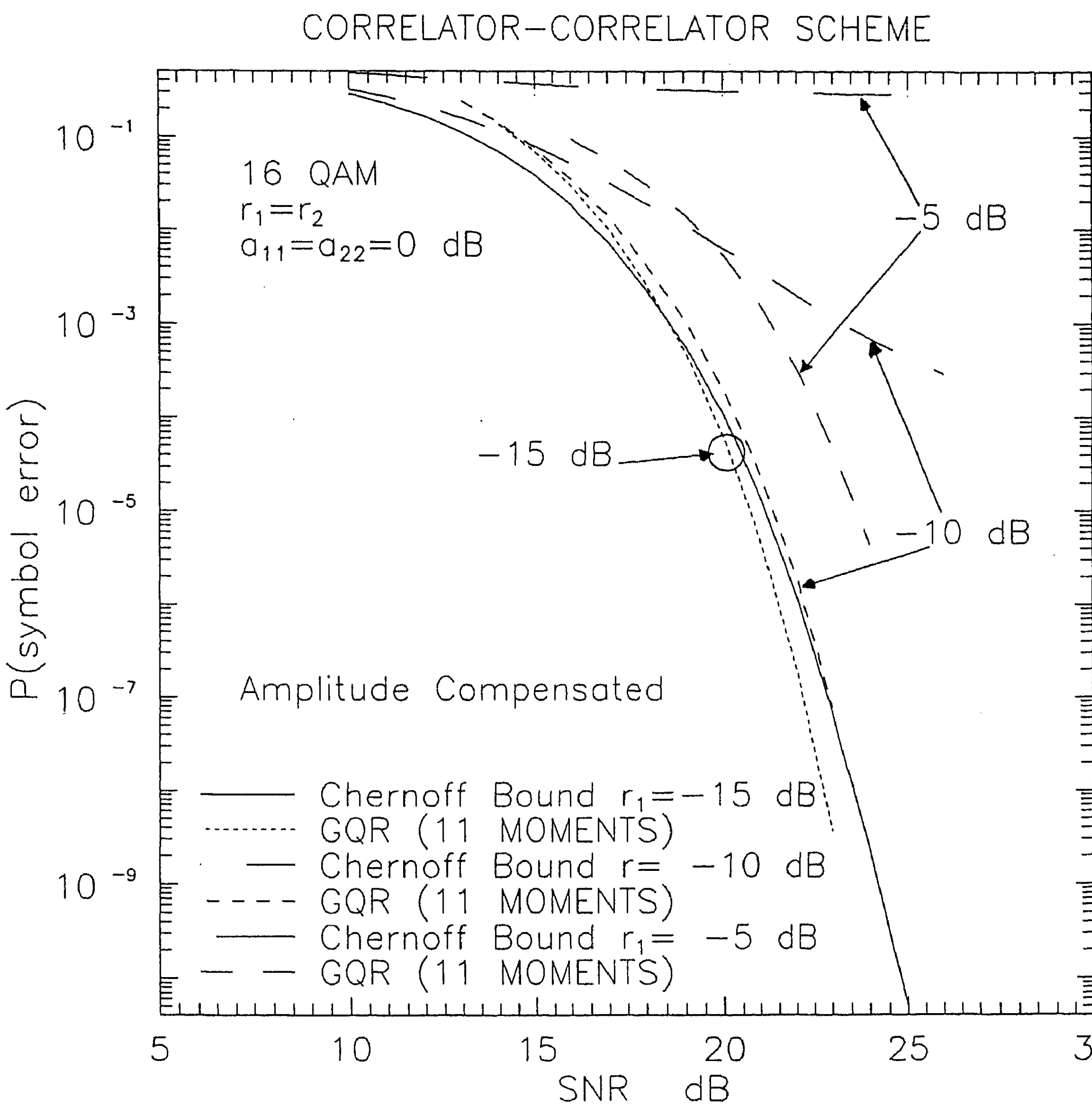

Figure 4.9: Correlator-Correlator Cross-Pol Canceler, Chernoff Bound and GQR calculation comparison, 16 QAM, with amplitude compensation

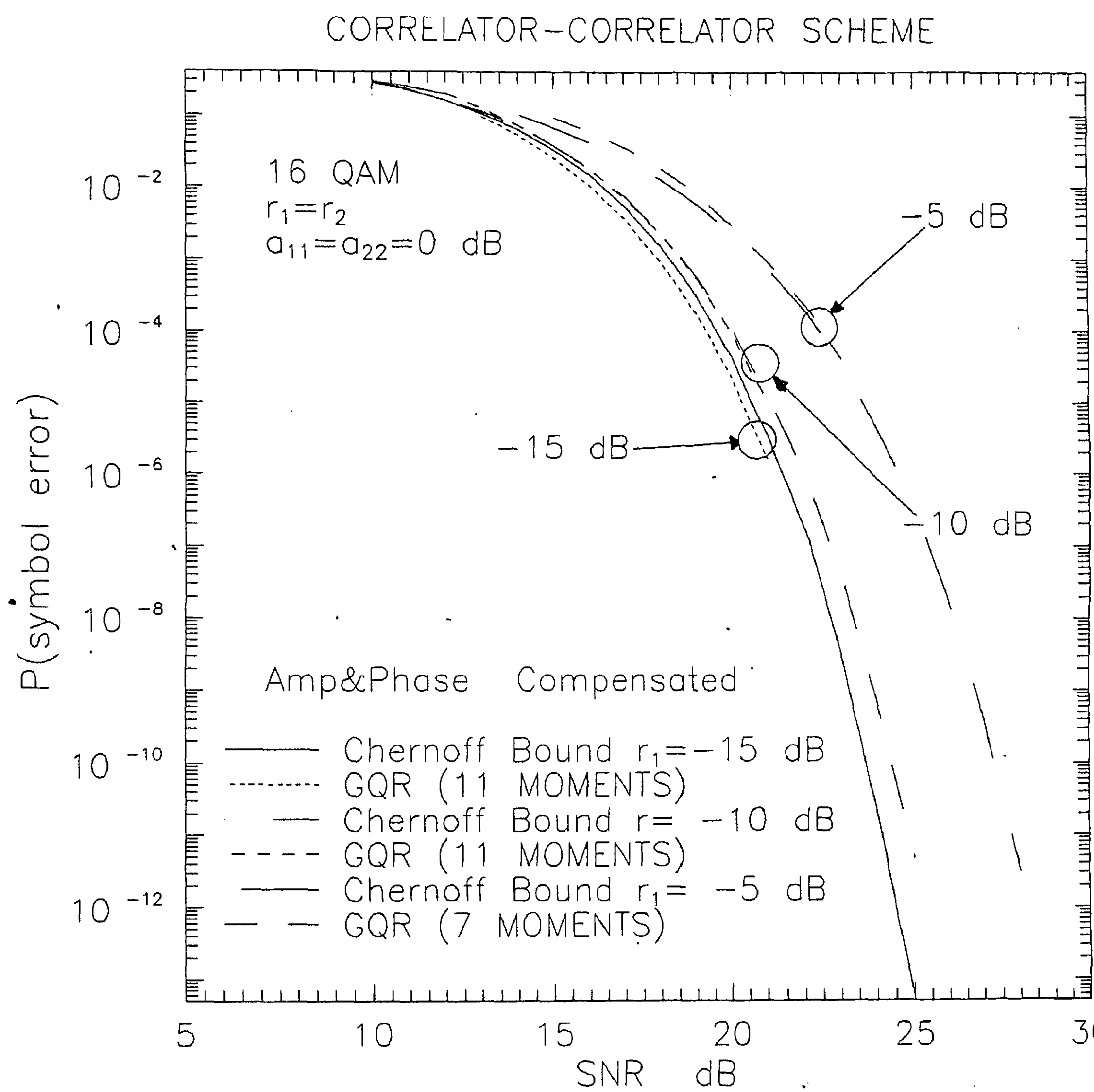

Figure 4.10: Correlator-Correlator Cross-Pol Canceler, Chernoff Bound and GQR calculation comparison, 16 QAM, with both amplitude and phase compensation

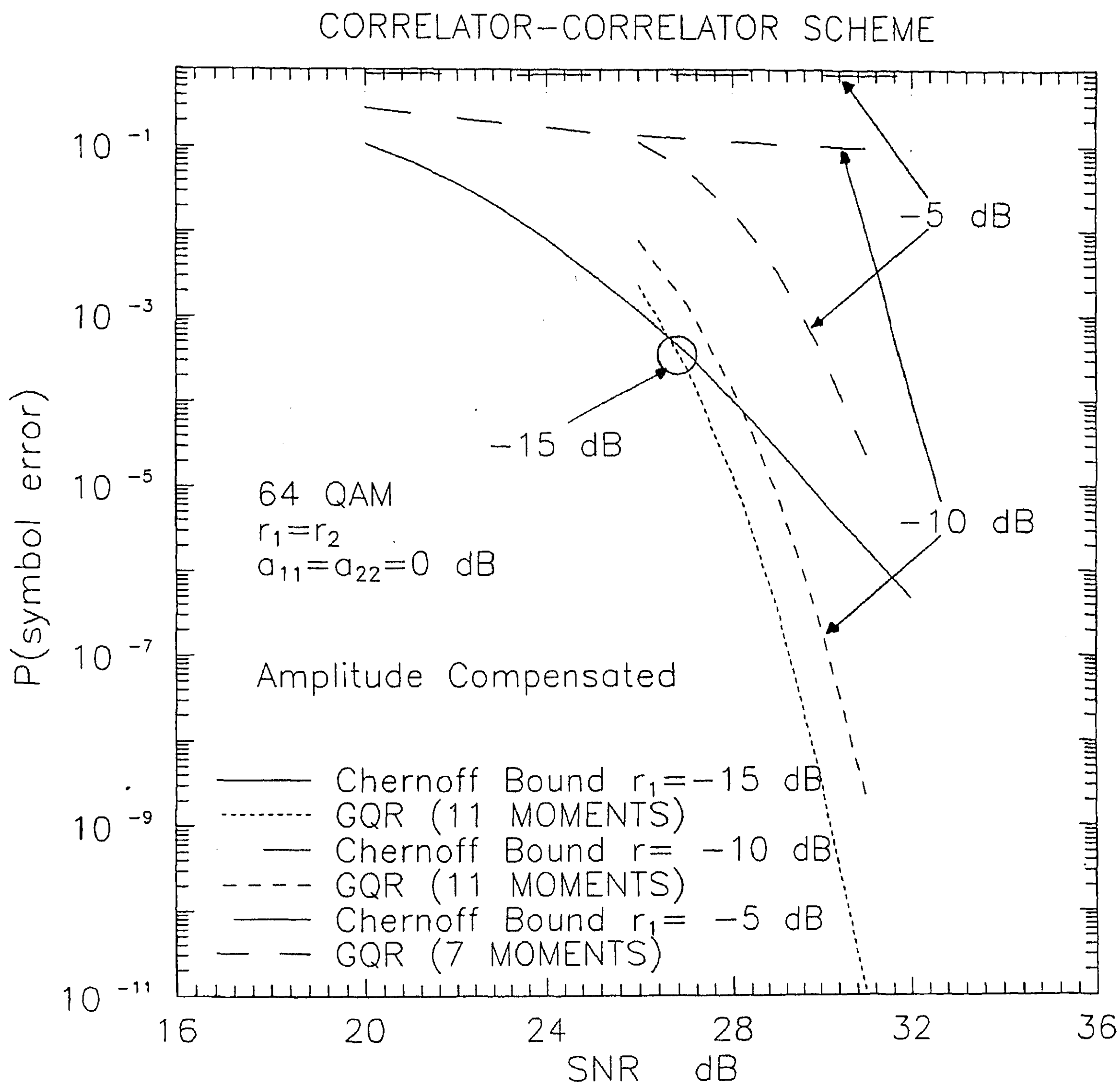

Figure 4.11: Correlator-Correlator Cross-Pol Canceler, Chernoff Bound and GQR calculation comparison, 64 QAM, with amplitude compensation

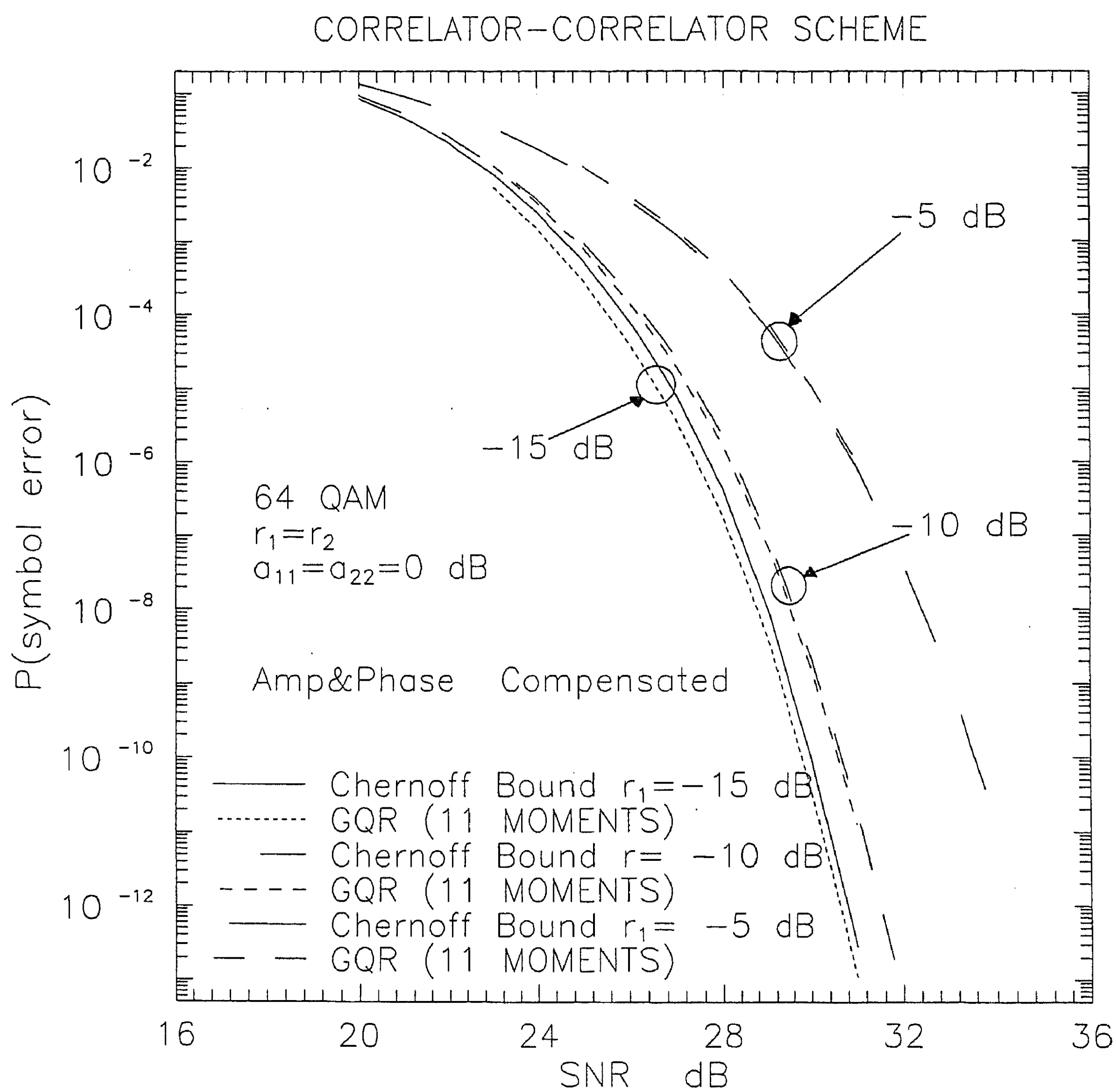

Figure 4.12: Correlator-Correlator Cross-Pol Canceler, Chernoff Bound and GQR calculation comparison, 64 QAM, with both amplitude and phase compensation

Correlator-Correlator Scheme For 16 QAM				
	$r_1 = r_2 =$ -15 dB		$r_1 = r_2 =$ -10 dB	
SNR	Moment 11	Chernoff Bound	Moment 11	Chernoff Bound
13	2.376E-1	1.121E-1	2.400E-1	1.593E-1
14	1.364E-1	6.937E-2	1.419E-1	1.150E-1
15	6.856E-2	3.816E-2	7.502E-2	7.898E-2
16	2.925E-2	1.817E-2	3.474E-2	5.156E-2
17	1.018E-2	7.252E-3	1.389E-2	3.208E-2
18	2.708E-3	2.339E-3	4.314E-3	1.914E-2
19	5.044E-4	5.844E-4	1.076E-3	1.105E-2
20	6.287E-5	1.080E-4	1.940E-4	6.257E-3
21	4.756E-6	1.403E-5	2.455E-5	3.522E-3
22	1.938E-7	1.219E-6	1.891E-6	2.001E-3
23	3.655E-9	6.765E-8	7.765E-8	1.164E-3

Table 4.1: Correlator-Correlator Cross-Pol Canceler, 16 QAM, performance calculation with Chernoff Bound and GQR method, with amplitude compensation for cross coupling -15 dB and -10 dB

Correlator-Correlator Scheme For 16 QAM		
$r_1 = r_2$= -5 dB		
SNR	Moment 11	Chernoff Bound
16	9.548E-2	3.485E-1
17	5.754E-2	3.353E-1
18	3.102E-2	3.242E-1
19	1.426E-2	3.150E-1
20	5.326E-3	3.074E-1
21	1.551E-3	3.012E-1
22	3.334E-4	2.962E-1
23	4.871E-5	2.922E-1
24	4.388E-6	2.889E-1

Table 4.2: Correlator-Correlator Cross-Pol Canceler, 16 QAM, performance calculation with Chernoff Bound and GQR method, with amplitude compensation for cross coupling -5 dB

Correlator-Correlator Scheme For 64 QAM				
$r_1 = r_2$ = -15 dB			$r_1 = r_2$ = -10 dB	
SNR	Moment 11	Chernoff Bound	Moment 11	Chernoff Bound
26	2.325E-3	1.150E-3	7.914E-3	1.349E-1
27	2.381E-4	3.619E-4	1.343E-3	1.238E-1
28	1.318E-5	1.036E-4	1.385E-4	1.149E-1
29	3.404E-7	2.761E-5	7.789E-6	1.078E-1
30	3.415E-9	7.074E-6	2.063E-7	1.022E-1
31	1.059E-11	1.804E-6	2.112E-9	9.780E-2

Table 4.3: Correlator-Correlator Cross-Pol Canceler, 64 QAM, performance calculation with Chernoff Bound and GQR method, with amplitude compensation for cross coupling -15 dB and -10 dB

Correlator-Correlator Scheme For 64 QAM		
$r_1 = r_2 =$ -5 dB		
SNR	Moment 7	Chernoff Bound
26	1.143E-1	6.945E-1
27	5.217E-2	6.935E-1
28	1.675E-2	6.927E-1
29	3.374E-3	6.921E-1
30	3.896E-4	6.916E-1
31	2.354E-5	6.912E-1

Table 4.4: Correlator-Correlator Cross-Pol Canceler, 64 QAM, performance calculation with Chernoff Bound and GQR method, with amplitude compensation for cross coupling -5 dB

Correlator-Correlator Scheme For 16 QAM				
$r_1 = r_2 =$ -15 dB			$r_1 = r_2 =$ -10 dB	
SNR	Moment 9	Chernoff Bound	Moment 11	Chernoff Bound
13	9.508E-2	1.051E-1	1.261E-1	1.128E-1
14	5.331E-2	6.329E-1	7.301E-2	7.008E-2
15	2.558E-2	3.346E-2	3.821E-2	3.874E-2
16	1.060E-2	1.502E-2	1.774E-2	1.855E-2
17	3.586E-3	5.489E-3	7.069E-3	7.449E-3
18	9.369E-4	1.550E-3	2.315E-3	2.412E-3
19	1.785E-4	3.171E-4	5.506E-4	5.998E-4
20	2.221E-5	4.332E-5	9.330E-5	1.078E-4
21	1.670E-6	3.572E-6	1.044E-5	1.296E-5

Table 4.5: Correlator-Correlator Cross-Pol Canceler, 16 QAM, performance calculation with Chernoff Bound and GQR method, with amplitude and phase compensation for cross coupling -15 dB and -10 dB

Correlator-Correlator Scheme For 16 QAM		
$r_1 = r_2$= -5 dB		
SNR	Moment 11	Chernoff Bound
15	9.337E-2	6.805E-2
16	6.016E-2	4.258E-2
17	3.615E-2	2.460E-2
18	1.857E-2	1.287E-2
19	8.062E-3	5.938E-3
20	2.979E-3	2.338E-3
21	9.231E-4	7.544E-4
22	2.278E-4	1.899E-4
23	4.191E-5	3.508E-5

Table 4.6: Correlator-Correlator Cross-Pol Canceler, 16 QAM, performance calculation with Chernoff Bound and GQR method, with amplitude and phase compensation for cross coupling -5 dB

Correlator-Correlator Scheme For 64 QAM				
$r_1 = r_2$ = -15 dB			$r_1 = r_2$ = -10 dB	
SNR	Moment 11	Chernoff Bound	Moment 11	Chernoff Bound
23	5.403E-3	8.285E-3	1.063E-2	1.104E-2
24	1.471E-3	2.498E-3	3.333E-3	3.776E-3
25	2.970E-4	5.545E-4	8.247E-4	1.004E-3
26	4.119E-5	8.391E-5	1.521E-4	1.961E-4
27	3.567E-6	7.862E-6	1.942E-5	2.611E-5
28	1.712E-7	4.048E-7	1.580E-6	2.159E-6
29	3.928E-9	9.761E-9	7.086E-8	9.816E-8
30	3.580E-11	9.432E-11	1.519E-9	2.107E-9
31	1.098E-13	2.851E-13	1.283E-11	1.988E-11

Table 4.7: Correlator-Correlator Cross-Pol Canceler, 64 QAM, performance calculation with Chernoff Bound and GQR method, with amplitude and phase compensation for cross coupling -15 dB and -10 dB

Correlator-Correlator Scheme For 64 QAM		
$r_1 = r_2 =$ -5 dB		
SNR	Moment 11	Chernoff Bound
25	1.055E-2	8.693E-3
26	4.212E-3	3.567E-3
27	1.411E-3	1.212E-3
28	3.775E-4	3.250E-4
29	7.603E-5	6.497E-5
30	1.067E-5	9.013E-6
31	9.458E-7	7.934E-7

Table 4.8: Correlator-Correlator Cross-Pol Canceler, 64 QAM, performance calculation with Chernoff Bound and GQR method, with amplitude and phase compensation for cross coupling -5 dB

Chapter 5

PERFORMANCE ANALYSIS OF POWER-CORRELATOR SCHEME

5.1 Introduction

In this chapter, we present the performance of the third scheme of the bootstrapped cross-polarization canceller for a dual M-QAM system over non-dispersive fading channels; The power-correlator scheme.

In this scheme, the two optimal weights are found; one by minimizing the output power of one channel while the other by minimizing the correlation between the outputs.

In the next section, after presenting the equations of the canceler two outputs, we derive the formula for the optimal weights, then after estimating the effects of noise on these weights and calculating the optimal outputs of each channel, we give the expression for the corresponding decision parameters. Performance of this canceler is estimated, as in the other cancelers, by the Chernoff bound on the probability of errors and via direct calculations of approximates to these probabilities using the quadrature rule. Because of the unsymmetry of this canceler, the decision parameter for two outputs are different. Nevertheless, the procedure of performing these calculations are the same as in previous chapters, and therefore will not be repeated. The reader is referred to the corresponding section of chapter 3 for a complete detail of these procedures.

Results will be depicted in section 5.4, followed by conclusion.

5.2 Canceler Scheme and Parameters

The power-correlation canceler scheme is shown in Fig. 2.13 and its principle of operation was explained in chapter 2.

5.2.1 Canceler Output

From Fig. 2.13 the outputs $y_1(n)$ and $y_2(n)$ are given by;

$$y_1(n) = x_1(n)(1 + w_{12}w_{21}) + x_2(n)w_{12} \tag{5.1}$$

$$y_2(n) = x_2(n) + x_1(n)w_{21} \tag{5.2}$$

where $x_1(n)$ and $x_2(n)$ are the received signals samples after the match filter and given by (2.5). Substituting for $x_1(n)$ and $x_2(n)$ in (5.1) and (5.2) , we get

$$\begin{aligned} y_1(n) &= I_1(n)[a_{11} + w_{12}(a_{21} + w_{21}a_{11})] + I_2(n)[a_{12} + w_{12}(a_{22} + w_{21}a_{12})] \\ &\quad + n_1(n)(1 + w_{12}w_{21}) + n_2(n)w_{12} \end{aligned} \tag{5.3}$$

$$y_2(n) = I_1(n)(a_{21} + w_{21}a_{11}) + I_2(n)(a_{22} + w_{21}a_{12}) + n_2(n) + n_1(n)w_{21} \tag{5.4}$$

5.2.2 Optimal Weights

The control algorithm simultaneously minimizes the output power

$$P_2(w_{12}^i, w_{21}^i) = E\{|y_{1d}^i(n)|^2\}$$

$$Q_2(w_{12}^i, w_{21}^i) \;\; = \;\; |B_2(w_{12}^i, w_{21}^i)|^2, \tag{5.5}$$

where $B_2(w_{12}^i, w_{21}^i) \triangleq E\{y_{2d}^i(n) y_1^i(n)^*\}$ and $y_{1d}(n)$ and $y_{2d}(n)$ are the samples of the corresponding output after the discrimination. $E\{\cdot\}$ and $|\cdot|$ denote the expected and magnitude of these arguments respectively. For optimum weights solution $\partial P_2(w_{12}, w_{21})/\partial w_{12} = 0$ and $\partial Q_2(w_{12}, w_{21})/\partial w_{21} = 0$ are searched simultaneously.

The search for the optimum weights can be performed by successive use of the following recursive equations,

$$w_{12}^{i+1} = w_{12}^i - \mu_1 \frac{\partial}{\partial w_{12}^i} P_2(w_{12}^i, w_{21}^i), \tag{5.6}$$

$$w_{21}^{i+1} = w_{21}^i - \mu_2 \frac{\partial}{\partial w_{21}^i} Q_2(w_{12}^i, w_{21}^i), \tag{5.7}$$

where μ_1 and μ_2 are the constants which determine the stability of convergence.

The optimum weights that minimize the power P_2 and the square magnitude of the output correlation Q_2 are obtained at the steady state from

$$\frac{\partial P_2(w_{12}^i, w_{21}^i)}{\partial w_{12}^i} = 0, \tag{5.8}$$

$$\frac{\partial Q_2(w_{12}^i, w_{21}^i)}{\partial w_{21}^i} = 0. \tag{5.9}$$

It has been shown in [13] (5.9) is true iff $B_2(w_{12}^i, w_{21}^i) = 0$. Therefore, the optimum solution is obtained from simultaneously solving

$$\frac{\partial P_2(w_{12}^i, w_{21}^i)}{\partial w_{12}^i} = 0 \tag{5.10}$$

$$B_2(w_{12}^i, w_{21}^i) = 0 \tag{5.11}$$

From (5.3), we find the power P_2 at the output of the first discriminator. We use the fact that $I_1(n)$, $n_1(n)$, $I_2(n)$ and $n_2(n)$ are all independent processes.

$$
\begin{aligned}
P_2(w_{12}, w_{21}) &= \delta_{11} E\{|I_1(n)|^2\}|a_{11} + w_{12}(a_{21} + w_{21}a_{11})|^2 \\
&+\delta_{12} E\{|I_2(n)|^2\}|a_{12} + w_{12}(a_{22} + w_{21}a_{12})|^2 \\
&+E\{|n_1(n)|^2\}|1 + w_{12}w_{21}|^2 + E\{|n_2(n)|^2\}|w_{12}|^2\Big] \qquad (5.12)
\end{aligned}
$$

Also, from (5.3) and (5.4), we get,

$$
B_2(w_{12}, w_{21}) = \delta_{21} E\{|I_1(n)|^2\}(a_{21} + w_{21}a_{11})a_{11} + \delta_{21} E\{|I_1(n)|^2\}|a_{21} + w_{21}a_{11}|^2 w_{12}^*
$$

$$
\begin{aligned}
&+\delta_{22} E\{|I_2(n)|^2\}(a_{22} + w_{21}a_{12})a_{12}^* + \delta_{22} E\{|I_2(n)|^2\}|a_{22} + w_{21}a_{12}|^2 w_{12}^* \\
&+E\{|n_1(n)|^2\}w_{21}(1 + w_{12}w_{21})^* + E\{|n_2(n)|^2\}w_{12}^* \qquad (5.13)
\end{aligned}
$$

where $\delta_{i,j}$ i,j=1,2 denotes the effect of the i th discriminator on the different signals $I_1(n)$ or $I_2(n)$ powers.

From (5.12) using the derivative of real function with respect to complex variable described in chapter 3, we get,

$$
\begin{aligned}
\frac{\partial P_2(w_{12}, w_{21})}{\partial w_{12}} &= 2\Big[\delta_{11} E\{|I_1(n)|^2\}[a_{11} + w_{12}(a_{21} + w_{21}a_{11})](a_{21} + w_{21}a_{11})^* \\
&+\delta_{12} E\{|I_2(n)|^2\}[a_{12} + w_{12}(a_{22} + w_{21}a_{12})](a_{22} + w_{21}a_{12})^* \\
&+E\{|n_1(n)|^2\}(1 + w_{12}w_{21})w_{21}^* + E\{|n_1(n)|^2\}w_{12}\Big] \qquad (5.14)
\end{aligned}
$$

Equating (5.14) to zero and separating terms, we get for $w_{12\text{opt}}$;

$$
\begin{aligned}
w_{12\text{opt}} &= \frac{-1}{Dw_{12\text{opt}}}\Big[a_{11}(a_{21} + w_{21\text{opt}}a_{11})^* E\{|I_1(n)|^2\}\delta_{11} \\
&+a_{12}(a_{22} + w_{21\text{opt}}a_{12})^* E\{|I_2(n)|^2\}\delta_{12} + w_{21\text{opt}}^* E\{|n_1(n)|^2\}\Big] (5.15)
\end{aligned}
$$

where,

$$Dw_{12\text{opt}} = |a_{21} + w_{21\text{opt}}a_{11}|^2 E\{|I_1(n)|^2\}\delta_{11} + |a_{22} + w_{21\text{opt}}a_{12}|^2 . \\ E\{|I_2(n)|^2\}\delta_{12} + E\{|n_2(n)|^2\} + E\{|n_1(n)|^2\}|w_{21}|^2, \qquad (5.16)$$

It is difficult to find from (5.13) an explicit function of $w_{21\text{opt}}$ in terms of $w_{12\text{opt}}$ as it was done in the other cancelers.

Nevertheless, from (5.13), we write,

$$B_2(w_{12}, w_{21})|_{w_{12\text{opt}}, w_{21\text{opt}}} = \delta_{21}E\{|I_1(n)|^2\}(a_{21} + w_{21\text{opt}}a_{11})a_{11} \\ +\delta_{21}E\{|I_1(n)|^2\}|a_{21} + w_{21\text{opt}}a_{11}|^2 w^*_{12\text{opt}} + \delta_{22}E\{|I_2(n)|^2\}(a_{22} + w_{21\text{opt}}a_{12})a^*_{12} \\ +\delta_{22}E\{|I_2(n)|^2\}|a_{22} + w_{21\text{opt}}a_{12}|^2 w^*_{12\text{opt}} + E\{|n_1(n)|^2\}w_{21\text{opt}}(1 + w_{12\text{opt}}w_{21\text{opt}})^* \\ +E\{|n_2(n)|^2\}w^*_{12\text{opt}} \qquad (5.17)$$

In appendix C, we show that the second term can be ignored in comparison to the other term and so is $w_{12\text{opt}}w_{21\text{opt}}$ in the fifth term of (5.17). Equating the sum of the remaining terms to zero, we have

$$\delta_{21}E\{|I_1(n)|^2\}a^2_{11}w_{21\text{opt}} \approx -\delta_{21}E\{|I_1(n)|^2\}a_{21}a_{11} \\ -[a_{12} + w_{12\text{opt}}(a_{22} + w_{21\text{opt}}a_{12})]^*(a_{22} + w_{21\text{opt}}a_{12})\delta_{22}E\{|I_2(n)|^2\} \\ -E\{|n_1(n)|^2\}w_{21\text{opt}} - E\{|n_2(n)|^2\}w^*_{12\text{opt}} \qquad (5.18)$$

The effect of the discriminator are presented by $\delta_{i,j} \quad i,j = 1,2$ real valued and $\delta_{22}\delta_{11} < \delta_{21}\delta_{12}$. Note that, the terms in (5.13) are complex conjugates of the terms in (5.14). Therefore, to find a unique solution for w_{12} and w_{21} using these equations, discriminators which enforce the constant $\delta_{i,j} \; i,j = 1,2$ satisfying the above condition is essential. The simultaneous solution of these non-linear equations give two equilibrium points; $[w_{12\text{opt}1}, w_{21\text{opt}1}]$ and $[w_{12\text{opt}2}, w_{21\text{opt}2}]$. One is a stable equilibrium which provide a solution to our problem.

5.2.3 Effect of Noise On Optimal Weights

In the absence of noise, that is when $E\{|n_1(n)|^2\} = E\{|n_2(n)|^2\} = 0$ the stable equilibrium points can easily found from equating (5.13) and (5.14) to zero;

$$w_{12\text{opt}} = -\frac{a_{12}}{a_{22}(1 - \frac{a_{12}a_{21}}{a_{11}a_{22}})} \qquad w_{21\text{opt}} = -\frac{a_{21}}{a_{11}} \tag{5.19}$$

In fact, it can easily be noticed that when the noise terms are zero, $w_{21\text{opt}} = -\frac{a_{21}}{a_{11}}$ will make the terms in (5.12) equal to zero while substituting this value of $w_{21\text{opt}}$ in (5.14) will result in $w_{12\text{opt}}$ as in (5.19). By similar approach followed for other cancelers when noise is added, we will write

$$w_{12\text{opt}} = -\frac{a_{12}}{a_{22}(1 - \frac{a_{12}a_{21}}{a_{11}a_{22}})} + \epsilon_1 \qquad w_{21\text{opt}} = -\frac{a_{21}}{a_{11}} + \epsilon_2 \tag{5.20}$$

where ϵ_1 and ϵ_2 are perturbations on the optimal weights due to added noise.

Perturbation On Optimal Weight, $w_{12\text{opt}}$

From (5.20),

$$\epsilon_1 = w_{12\text{opt}} + \frac{a_{12}}{a_{22}k} \tag{5.21}$$

where

$$k \triangleq 1 - \frac{a_{12}a_{21}}{a_{11}a_{22}} \tag{5.22}$$

Writing for $w_{21\text{opt}}$ from (5.20) in (5.15), we get,

$$\begin{aligned} w_{12\text{opt}} &= \frac{-1}{Dw_{12\text{opt}}}\Big[a_{11}[a_{21} + (-\frac{a_{21}}{a_{11}} + \epsilon_2)a_{11}]^*\delta_{11}E\{|I_1(n)|^2\} \\ &\quad + a_{12}[a_{22} + (-\frac{a_{21}}{a_{11}} + \epsilon_2)a_{12}]^*\delta_{12}E\{|I_2(n)|^2\} \\ &\quad + E\{|n_1(n)|^2\}[-\frac{a_{21}}{a_{11}} + \epsilon_2]^*\Big] \end{aligned} \tag{5.23}$$

and after simplification,

$$w_{12\text{opt}} = \frac{-1}{Dw_{12\text{opt}}}\Big[a_{11}^2\delta_{11}E\{|I_1(n)|^2\}\epsilon_2^* + \delta_{12}E\{|I_2(n)|^2\}|a_{12}|^2\epsilon_2^* + \delta_{12}E\{|I_2(n)|^2\}a_{12}a_{22}k^* + E\{|n_1(n)|^2\}[-\frac{a_{21}}{a_{11}}+\epsilon_2]^*\Big] \tag{5.24}$$

Substituting (5.21) in (5.16), we have,

$$Dw_{12\text{opt}} = |a_{21} + (-\frac{a_{21}}{a_{11}}+\epsilon_2)a_{11}|^2E\{|I_1(n)|^2\}\delta_{11} + |a_{22} + (-\frac{a_{21}}{a_{11}}+\epsilon_2)a_{12}|^2. E\{|I_2(n)|^2\}\delta_{12} + E\{|n_2(n)|^2\} + E\{|n_1(n)|^2\}|(-\frac{a_{21}}{a_{11}}+\epsilon_2)|^2, \tag{5.25}$$

or,

$$Dw_{12\text{opt}} = a_{11}^2E\{|I_1(n)|^2\}\delta_{11}|\epsilon_2|^2 + a_{22}^2|k + \frac{a_{12}}{a_{22}}\epsilon_2|^2E\{|I_2(n)|^2\}\delta_{12} + E\{|n_2(n)|^2\} + E\{|n_1(n)|^2\}|(-\frac{a_{21}}{a_{11}}+\epsilon_2)|^2, \tag{5.26}$$

Finally, we write for the denominator,

$$Dw_{12\text{opt}} = a_{11}^2|\epsilon_2|^2E\{|I_1(n)|^2\}\delta_{11} + \Big[|k|^2 + k^*\frac{a_{12}}{a_{22}}\epsilon_2 + k(\frac{a_{12}}{a_{22}})^*\epsilon_2^* + |\frac{a_{12}}{a_{22}}\epsilon_2|^2\Big]. a_{22}^2E\{|I_2(n)|^2\}\delta_{12} + E\{|n_2(n)|^2\} + E\{|n_1(n)|^2\}\Big[|\frac{a_{21}}{a_{11}}|^2 + |\epsilon_2|^2 - \frac{a_{21}}{a_{11}}\epsilon_2^* - (\frac{a_{21}}{a_{11}})^*\epsilon_2\Big] \tag{5.27}$$

Defining;

$$\Delta_{1A} \triangleq E\{|n_1(n)|^2\}\Big[a_{11}^2\frac{E\{|I_1(n)|^2\}}{E\{|n_1(n)|^2\}}\delta_{11} + a_{22}^2|\frac{a_{12}}{a_{22}}|^2\frac{E\{|I_2(n)|^2\}}{E\{|n_1(n)|^2\}}\delta_{12} + 1]\Big] \tag{5.28}$$

$$\Delta_{1B} \triangleq E\{|n_1(n)|^2\}\Big[|k|^2a_{22}^2\frac{E\{|I_2(n)|^2\}}{E\{|n_1(n)|^2\}}\delta_{12} + |\frac{a_{21}}{a_{11}}|^2 + \frac{E\{|n_2(n)|^2\}}{E\{|n_1(n)|^2\}}\Big] \tag{5.29}$$

$$\Delta_{1C} \triangleq E\{|n_1(n)|^2\}\Big[\frac{E\{|I_2(n)|^2\}}{E\{|n_1(n)|^2\}}\delta_{12}a_{22}^2k^*\frac{a_{12}}{a_{22}} - (\frac{a_{21}}{a_{11}})^*\Big] \tag{5.30}$$

and we can write (5.27),

$$Dw_{12\text{opt}} \triangleq \Delta_{1B} + \Delta_{1A}|\epsilon_2|^2 + \Delta_{1C}\epsilon_2 + \Delta_{1C}^*\epsilon_2^*. \tag{5.31}$$

For high signal-to-noise ratio, we can approximate (5.31) (see appendix C) by

$$Dw_{12\text{opt}} \approx \Delta_{1B} \tag{5.32}$$

Next, defining,

$$A \triangleq \delta_{11}E\{|I_1(n)|^2\}a_{11}^2 + \delta_{12}E\{|I_2(n)|^2\}|a_{12}|^2 + E\{|n_1(n)|^2\} \tag{5.33}$$

$$B \triangleq \delta_{12}E\{|I_2(n)|^2\}a_{12}a_{22}k^* - E\{|n_1(n)|^2\}(\frac{a_{21}}{a_{11}})^* \tag{5.34}$$

Using (5.32), (5.33) (5.34) in (5.24), we get,

$$w_{12\text{opt}} \approx -\frac{A\epsilon_2^* + B}{\Delta_{1B}}, \tag{5.35}$$

and substituting this in (5.21), we have

$$\epsilon_1 \approx -\frac{\epsilon_2^* A + B}{\Delta_{1B}} + \frac{a_{12}}{a_{22}k} \tag{5.36}$$

or

$$\epsilon_1 \approx \frac{\epsilon_2^* U + Y}{T} \tag{5.37}$$

where

$$T \triangleq \Delta_{1B}a_{22}k \tag{5.38}$$

$$U \triangleq -Aa_{22}k \tag{5.39}$$

$$Y \triangleq \Delta_{1B}a_{12} - Ba_{22}k \tag{5.40}$$

Using (2.6) in (5.34), we get

$$B = \delta_{12}E\{|I_2(n)|^2\}a_{22}^2 r_1 e^{j\phi_1}[1 - r_1 r_2 e^{-j(\phi_1+\phi_2)}] - E\{|n_1(n)|^2\}r_2 e^{-j\phi_2}, \quad (5.41)$$

and for the real and imaginary parts of B,

$$B = B_R + jB_I \quad (5.42)$$

we have,

$$B_R = \delta_{12}E\{|I_2(n)|^2\}a_{22}^2 r_1[cos\phi_1 - r_1 r_2 cos\phi_2] - E\{|n_1(n)|^2\}r_2 cos\phi_2 \quad (5.43)$$

$$B_I = \delta_{12}E\{|I_2(n)|^2\}a_{22}^2 r_1[sin\phi_1 + r_1 r_2 sin\phi_2] + E\{|n_1(n)|^2\}r_2 sin\phi_2 \quad (5.44)$$

From (5.22) , we can write k in terms of its real and imaginary parts,

$$k = k_R + jk_I \quad (5.45)$$

$$k_R = 1 - r_1 r_2 cos(\phi_1 + \phi_2) \quad (5.46)$$

$$k_I = -r_1 r_2 sin(\phi_1 + \phi_2) \quad (5.47)$$

Substituting (5.42) and (5.46) into (5.40), we have,

$$Y = a_{22}[\Delta_{1B}\frac{a_{12}}{a_{22}} - (B_R + jB_I)(k_R + jk_I)] \quad (5.48)$$

and by using (2.6) and combining real and imaginary terms together, we get

$$Y = a_{22}(\Delta_{1B} r_1 cos\phi_1 - B_R k_R + B_I k_I) + ja_{22}(\Delta_{1B} r_1 sin\phi_1 - B_R k_I - B_I k_R) \quad (5.49)$$

Therefore, the real and imaginary part of Y are;

$$Y_R = a_{22}(\Delta_{1B} r_1 cos\phi_1 - B_R k_R + B_I k_I)$$
$$Y_I = a_{22}(\Delta_{1B} r_1 sin\phi_1 - B_R k_I - B_I k_R) \quad (5.50)$$

with Δ_{1B}, B_R, B_I, k_R and k_I given by, (5.29), (5.43), (5.44), (5.46) and (5.47), respectively. Similarly, we write T from (5.38) in terms of its real and imaginary parts,

$$T = T_R + jT_I \quad (5.51)$$

where,

$$T_R = \Delta_{1B} a_{22} k_R \quad (5.52)$$

and

$$T_I = \Delta_{1B} a_{22} k_I \quad (5.53)$$

Finally, for U in (5.39)

$$U = U_R + jU_I, \quad (5.54)$$

with

$$U_R = -A a_{22} k_R \quad (5.55)$$
$$U_I = -A a_{22} k_I \quad (5.56)$$

Perturbation On Optimal Weight, $w_{21\text{opt}}$

To find the perturbation ϵ_2, of $w_{21\text{opt}}$, we proceed as follows:

From (5.20)

$$\epsilon_2 = w_{21\text{opt}} + \frac{a_{21}}{a_{11}} \quad (5.57)$$

Substituting from (5.57) and (5.21) for $w_{21\text{opt}}$ and $w_{12\text{opt}}$ respectively in (5.18), we obtain after arranging terms,

$$\epsilon_2 \approx \Big[-\Big(a_{12}+(-\frac{a_{12}}{a_{22}k}+\epsilon_1)[a_{22}+(-\frac{a_{21}}{a_{11}}+\epsilon_2)a_{12}]\Big)^*(a_{22}+(-\frac{a_{21}}{a_{11}}+\epsilon_2)a_{12}).$$

$$\delta_{22}E\{|I_2(n)|^2\} - E\{|n_1(n)|^2\}(-\frac{a_{21}}{a_{11}}+\epsilon_2)$$

$$-E\{|n_2(n)|^2\}(-\frac{a_{12}}{a_{22}k}+\epsilon_1)^*\Big]\frac{1}{\delta_{21}E\{|I_1(n)|^2\}a_{11}^2} \quad (5.58)$$

In section (2) of appendix C, we show that the first term in parenthesis of (5.58) can be approximated by $-a_{22}^2\delta_{22}E\{|I_2(n)|^2\}|k|^2\epsilon_1^*$. Therefore,

$$\epsilon_2 \approx \Big[-a_{22}^2\delta_{22}E\{|I_2(n)|^2\}|k|^2\epsilon_1^* + \frac{a_{21}}{a_{11}}E\{|n_1(n)|^2\} - E\{|n_1(n)|^2\}\epsilon_2$$
$$+E\{|n_2(n)|^2\}(\frac{a_{12}}{a_{22}k})^* - E\{|n_2(n)|^2\}\epsilon_1^*\Big]\frac{1}{\delta_{21}E\{|I_1(n)|^2\}a_{11}^2} \quad (5.59)$$

We write (5.59) in compact form;

$$\epsilon_2 \approx \frac{\epsilon_1^*U_1+Y_1}{T_1} \quad (5.60)$$

where

$$U_1 \triangleq -\delta_{22}E\{|I_2(n)|^2\}a_{22}^2|k|^2 - E\{|n_2(n)|^2\} \quad (5.61)$$
$$Y_1 \triangleq E\{|n_1(n)|^2\}\frac{a_{21}}{a_{11}} + E\{|n_2(n)|^2\}(\frac{a_{12}}{a_{22}k})^* \quad (5.62)$$
$$T_1 \triangleq \delta_{21}E\{|I_1(n)|^2\}a_{11}^2 + E\{|n_1(n)|^2\} \quad (5.63)$$

Using (2.6) in (5.62), we can write Y_1 in terms of its real and imaginary parts, Y_{1R} and Y_{1I}, respectively.

$$Y_{1R} = E\{|n_1(n)|^2\}r_2cos\phi_2 + \frac{E\{|n_2(n)|^2\}}{|k|^2}r_1[k_Rcos\phi_1+k_Isin\phi_1] \quad (5.64)$$
$$Y_{1I} = E\{|n_1(n)|^2\}r_2sin\phi_2 - \frac{E\{|n_2(n)|^2\}}{|k|^2}r_1[k_Rsin\phi_1-k_Icos\phi_1] \quad (5.65)$$

where using (2.6) and (5.22), we have for $|k|^2$,

$$\begin{aligned} |k|^2 &= k_R^2 + k_I^2 \\ &= 1 - 2r_1r_2cos(\phi_1 + \phi_2) + r_1^2r_2^2 \end{aligned} \quad (5.66)$$

Determination of Perturbations Final Expressions

From (5.37) and (5.60) , we write,

$$\epsilon_1 T - \epsilon_2^* U = Y \quad (5.67)$$

$$\epsilon_2 T_1 - \epsilon_1^* U_1 = Y_1, \quad (5.68)$$

From (5.61) and (5.63), we notice that U_1 and T_1 are real. Solving (5.67) and (5.68) give,

$$\epsilon_1 = \frac{YT_1 + UY_1^*}{TT_1 - UU_1} \quad (5.69)$$

$$\epsilon_2 = \frac{Y_1T^* + U_1Y^*}{T^*T_1 - U^*U_1} \quad (5.70)$$

Let

$$\begin{aligned} \Delta &\triangleq TT_1 - UU_1 \\ &= \Delta_R + j\Delta_I, \end{aligned} \quad (5.71)$$

then,

$$\Delta_R = T_RT_1 - U_RU_1 \quad (5.72)$$

$$\Delta_I = T_IT_1 - U_IU_1. \quad (5.73)$$

From (5.69),

$$\epsilon_1 = \frac{1}{\Delta}[(Y_R + jY_I)T_1 + (U_R + jU_I)(Y_{1R} - jY_{1I})] \tag{5.74}$$

Defining

$$\epsilon_1 = \frac{\epsilon_{1AR} + j\epsilon_{1AI}}{\Delta_R + j\Delta_I}, \tag{5.75}$$

then,

$$\epsilon_{1AR} = Y_R T_1 + U_R Y_{1R} + U_I Y_{1I} \tag{5.76}$$

$$\epsilon_{1AI} = Y_I T_1 - U_R Y_{1I} + U_I Y_{1R} \tag{5.77}$$

and can write (5.75) in terms of its; real and imaginary parts.

$$\epsilon_1 = \epsilon_{1R} + j\epsilon_{1I} \tag{5.78}$$

where,

$$\epsilon_{1R} = \frac{1}{|\Delta|^2}[\epsilon_{1AR}\Delta_R + \epsilon_{1AI}\Delta_I] \tag{5.79}$$

$$\epsilon_{1I} = \frac{1}{|\Delta|^2}[\epsilon_{1AI}\Delta_R - \epsilon_{1AR}\Delta_I] \tag{5.80}$$

Similarly, for ϵ_2, from (5.70), we write

$$\epsilon_2 = \frac{1}{\Delta^*}[(Y_{1R} + jY_{1I})(T_R - jT_I) + (Y_R - jY_I)U_1] \tag{5.81}$$

Defining,

$$\epsilon_2 = \frac{\epsilon_{2AR} + j\epsilon_{2AI}}{\Delta_R - j\Delta_I} \tag{5.82}$$

then,

$$\epsilon_{2AR} = Y_{1R}T_R + Y_{1I}T_I + U_1Y_R \quad (5.83)$$

$$\epsilon_{2AI} = Y_{1I}T_R - Y_{1R}T_I - Y_IU_1 \quad (5.84)$$

and we can write (5.82) in terms of its real and imaginary parts;

$$\epsilon_2 = \epsilon_{2R} + j\epsilon_{2I} \quad (5.85)$$

where,

$$\epsilon_{2R} = \frac{1}{|\Delta|^2}[\epsilon_{2AR}\Delta_R - \epsilon_{2AI}\Delta_I] \quad (5.86)$$

$$\epsilon_{2I} = \frac{1}{|\Delta|^2}[\epsilon_{2AI}\Delta_R + \epsilon_{2AR}\Delta_I] \quad (5.87)$$

5.2.4 Canceler Optimal Outputs

Unlike other two schemes of bootstrapped cancelers, power-correlator scheme is not symmetric, its outputs and hence, the decision parameters for each output are different.

From (5.3) with the substitution of $w_{21} = w_{21\text{opt}}$ from (5.20), we obtain $y_1(n)$

$$\begin{aligned} y_1(n) &= I_1(n)a_{11}[1 - \frac{a_{12}}{a_{22}k}\epsilon_2 + \epsilon_1\epsilon_2] + I_2(n)a_{22}[-(\frac{a_{12}}{a_{22}})^2\frac{1}{k}\epsilon_2 + k\epsilon_1 + \epsilon_1\epsilon_2\frac{a_{12}}{a_{22}}] \\ &+n_1(n)[1 + \frac{a_{12}a_{21}}{a_{22}a_{11}}\frac{1}{k} - \frac{a_{21}}{a_{11}}\epsilon_1 - \frac{a_{12}}{a_{22}}\frac{\epsilon_2}{k} + \epsilon_1\epsilon_2] + n_2(n)[\frac{-a_{12}}{a_{22}k} + \epsilon_1], \end{aligned} \quad (5.88)$$

Similarly from (5.4);

$$\begin{aligned} y_2(n) &= I_1(n)\epsilon_2a_{11} + I_2(n)a_{22}[1 - \frac{a_{21}a_{12}}{a_{11}a_{22}} + \epsilon_2\frac{a_{12}}{a_{22}}] + n_2(n) \\ &+n_1(n)(\frac{-a_{21}}{a_{11}} + \epsilon_2). \end{aligned} \quad (5.89)$$

Decision Parameter for the Canceler Output-1 with Both Amplitude and Phase Compensation

We will only study the performance of the canceler when both amplitude and phase compensation is used on co-pol signal. Therefore, we will take $\hat{I}_1 = \frac{y_1(n)}{\Delta_{y1}}$ (Δ_{y1} defined later) as an estimate of the transmitted signal I_1 and we will neglect $\epsilon_1 \epsilon_2$ with respect to the other terms in the first, second and third terms of (5.88) equation.

Therefore, from (5.88),

$$\begin{aligned} \frac{y_1(n)}{\Delta_{y1}} &= I_1(n) + \frac{1}{\Delta_{y1}}\Big[I_2(n)a_{22}[-(\frac{a_{12}}{a_{22}})^2\frac{1}{k}\epsilon_2 + \epsilon_1 k] \\ &\quad + n_1(n)[1 + \frac{a_{12}a_{21}}{a_{22}a_{11}}\frac{1}{k} - \frac{a_{21}}{a_{11}}\epsilon_1 - \frac{a_{12}}{a_{22}}\frac{\epsilon_2}{k}] + n_2(n)[\frac{-a_{12}}{a_{22}k} + \epsilon_1]\Big], \end{aligned} \tag{5.90}$$

where

$$\Delta_{y1} \triangleq a_{11}[1 - \frac{a_{12}}{a_{22}k}\epsilon_2], \tag{5.91}$$

and,

$$Z_1(n) \triangleq \hat{I}_1(n) - I_1(n), \tag{5.92}$$

is taken as the decision parameter. Therefore, the probability of error is given by $P_1(e) = P\{|Z_1(n)| > c\}$ with c is half of the distance between two signals in the corresponding signal space.

Finally, from (5.92) together with (5.90), we have

$$\begin{aligned} Z_1(n) &= \frac{1}{\Delta_{y1}k}\Big[I_2(n)a_{22}[-(\frac{a_{12}}{a_{22}})^2\epsilon_2 + \epsilon_1 k^2] \\ &\quad n_1(n)[1 - \frac{a_{21}}{a_{11}}\epsilon_1 k - \frac{a_{12}}{a_{22}}\epsilon_2] + n_2(n)[-\frac{a_{12}}{a_{22}} + k\epsilon_1]\Big], \end{aligned} \tag{5.93}$$

where we also use the fact that $(1 + \frac{a_{12}a_{21}}{a_{11}a_{22}}) = k$.

In order to be able to calculate the probability of error, we must find the real and imaginary part of $Z_1(n)$. Before we do this, for the sake of simplicity, we will define the following terms,

$$\textbf{1.}\ K \triangleq \epsilon_2 \frac{a_{12}}{a_{22}}, \tag{5.94}$$

$$K = K_R + jK_I \tag{5.95}$$
$$= (\epsilon_{2R} + j\epsilon_{2I})r_1(cos\phi_1 + j sin\phi_1), \tag{5.96}$$

and hence,

$$K_R = r_1[\epsilon_{2R}cos\phi_1 - \epsilon_{2I}sin\phi_1] \tag{5.97}$$
$$K_I = r_1[\epsilon_{2R}sin\phi_1 + \epsilon_{2I}cos\phi_1]. \tag{5.98}$$

$$\textbf{2.}\ W \triangleq \epsilon_1 k^2, \tag{5.99}$$

$$W = W_R + jW_I \tag{5.100}$$
$$= (\epsilon_{1R} + j\epsilon_{1I})(k_R^2 - k_I^2 + j2k_Rk_I), \tag{5.101}$$

and hence,

$$\begin{aligned} W_R &= \epsilon_{1R}(k_R^2 - k_I^2) - 2\epsilon_{1I}k_Rk_I \\ W_I &= \epsilon_{1I}(k_R^2 - k_I^2) + 2\epsilon_{1R}k_Rk_I. \end{aligned} \tag{5.102}$$

$$\textbf{3.}\ L \triangleq -\epsilon_2(\frac{a_{12}}{a_{22}})^2, \tag{5.103}$$

$$L = L_R + jL_I \tag{5.104}$$

$$= -(\epsilon_{2R} + j\epsilon_{2I})r_1^2(cos2\phi_1 + jsin2\phi_1) \tag{5.105}$$

and hence,

$$L_R = -r_1^2[\epsilon_{2R}cos2\phi_1 - \epsilon_{2I}sin2\phi_1] \tag{5.106}$$

$$L_I = -r_1^2[\epsilon_{2R}sin2\phi_1 + \epsilon_{2I}cos2\phi_1]. \tag{5.107}$$

4. $S \triangleq \epsilon_1 k,$ (5.108)

$$S \triangleq S_R + jS_I \tag{5.109}$$

$$= (\epsilon_{1R} + j\epsilon_{1I})(k_R + jk_I), \tag{5.110}$$

and hence;

$$S_R = \epsilon_{1R}k_R - \epsilon_{1I}k_I \tag{5.111}$$

$$S_I = \epsilon_{1R}k_I + \epsilon_{1I}k_R. \tag{5.112}$$

5. $S_1 \triangleq S\frac{a_{21}}{a_{11}},$ (5.113)

$$S_1 \triangleq S_{1R} + jS_{1I} \tag{5.114}$$

$$= (S_R + jS_I)r_2(cos\phi_2 + jsin\phi_2), \tag{5.115}$$

and hence,

$$S_{1R} = r_2(S_R cos\phi_2 - S_I sin\phi_2) \tag{5.116}$$

$$S_{1I} = r_2(S_R sin\phi_2 + S_I cos\phi_2), \tag{5.117}$$

Notice that in the derivation of these terms, we used from (2.6) $\frac{a_{12}}{a_{22}} = r_1 e^{j\phi_1}$ and $\frac{a_{21}}{a_{11}} = r_2 e^{j\phi_2}$. Using (5.95), (5.100), (5.104), (5.109) and (5.114) in (5.93), we get

$$Z_1 = \frac{1}{Z_{D1}}\Big[(I_{2R} + jI_{2I})[(L_R + W_R) + j(L_I + W_I)]a_{22} + (n_{1R} + jn_{1I}).$$
$$[1 - (S_{1R} + jS_{1I}) - (K_R + jK_I)] + (n_{2R} + jn_{2I})[(S_R - r_1 cos\phi_1) + j(S_I - r_1 sin\phi_1)], \tag{5.118}$$

with,

$$Z_{D1} \triangleq \Delta_{y1} k \tag{5.119}$$

and where we used $I_i = I_{iR} + jI_{iI}$ and $n_i = n_{iR} + jn_{iI}$, $\quad i = 1, 2$ and in our notation. We also drop the dependence of terms on the sampling time n.

Also using (5.95) and (5.91) in (5.119), the real and imaginary part of the denominator Z_{D1},

$$Z_{D1R} = a_{11}[k_R - K_R] \tag{5.120}$$

$$Z_{D1I} = a_{11}[k_I - K_I]. \tag{5.121}$$

Clearly Z_{D1R} and Z_{D1I} are functions of the random variables ϕ_1 and ϕ_2.

Now, the real and imaginary part of numerator of (5.118), Z_{N1} are given by

$$Z_{N1R} = [I_{2R}(L_R + W_R) - I_{2I}(L_I + W_I)]a_{22} + n_{1R}(1 - S_{1R} - K_R) + n_{1I}(S_{1I} + K_I)$$
$$+ n_{2R}(S_R - r_1 cos\phi_1) - n_{2I}(S_I - r_1 sin\phi_1) \tag{5.122}$$

$$Z_{N1I} = [I_{2R}(L_I + W_I) + I_{2I}(L_R + W_R)]a_{22} - n_{1R}(S_{1I} + K_I) + n_{1I}(1 - S_{1R} - K_R)$$
$$+ n_{2R}(S_I - r_1 sin\phi_1) + n_{2I}(S_R - r_1 cos\phi_1) \tag{5.123}$$

Z_{N1R} and Z_{N1I} beside being function of the random variables ϕ_1 and ϕ_2, they are also function of the signals and noises random variables.

Finally, we can write (5.118) as

$$Z_1 = Z_{1R} + jZ_{1I} = \frac{Z_{N1R} + jZ_{N1I}}{Z_{D1R} + jZ_{D1I}} \tag{5.124}$$

with

$$Z_{1R} = \frac{Z_{N1R}Z_{D1R} + Z_{N1I}Z_{D1I}}{Z_{D1R}^2 + Z_{D1I}^2} \tag{5.125}$$

$$Z_{1I} = \frac{Z_{N1I}Z_{D1R} - Z_{N1R}Z_{DI}}{Z_{D1R}^2 + Z_{D1I}^2} \tag{5.126}$$

Using (5.120), (5.121), (5.122) and (5.123) in (5.124), we can get after some simplification which emphasizes the dependency of the different terms on the different random variables, the real part of the decision variable:

$$\begin{aligned} Z_{1R} = \frac{1}{|\Delta_{Z1}|^2}\Big[& I_{2R}a_{22}[(L_R + W_R)Z_{D1R} + (L_I + W_I)Z_{D1I}] \\ & + I_{2I}a_{22}[-(L_I + W_I)Z_{D1R} + (L_R + W_R)Z_{D1I}] \\ & + n_{1R}[(1 - S_{1R} - K_R)Z_{D1R} - (S_{1I} + K_I)Z_{D1I}] \\ & + n_{1I}[(S_{1I} + K_I)Z_{D1R} + (1 - S_{1R} - K_R)Z_{D1I}] \\ & + n_{2R}[(S_R - r_1 cos\phi_1)Z_{D1R} + (S_I - r_1 sin\phi_1)Z_{D1I}) \\ & + n_{2I}[-(S_I - r_1 sin\phi_1)Z_{D1R} + (S_R - r_1 cos\phi_1)Z_{D1I})]\Big] \end{aligned} \tag{5.127}$$

Similar expression for the imaginary part Z_{1I} ;

$$\begin{aligned} Z_{1I} = \frac{1}{|\Delta_{Z1}|^2}\Big[& I_{2R}a_{22}[(L_I + W_I)Z_{D1R} - (L_R + W_R)Z_{D1I})] \\ & + I_{2I}a_{22}[(L_R + W_R)Z_{D1R} + (L_I + W_I)Z_{D1I}) \\ & + n_{1R}[-(S_{1I} + K_I)Z_{D1R} - (1 - S_{1R} - K_R)Z_{D1I}] \\ & + n_{1I}[(1 - S_{1R} - K_R)Z_{D1R} - (S_{1I} + K_I)Z_{D1I}] \end{aligned}$$

$$+n_{2R}[(S_I - r_1 sin\phi_1)Z_{D1R} - (S_R - r_1 cos\phi_1)Z_{D1I})]$$
$$+n_{2I}[(S_R - r_1 cos\phi_1)Z_{D1R} + (S_I - r_1 sin\phi_1)Z_{D1I})\Big] \qquad (5.128)$$

The Decision Parameters Final Expressions

Finally we write the real and imaginary parts of $Z_1(n)$ in terms of the random variable representing the real and imaginary part of signal and noises of channel 1;

$$Z_{1R} = I_{2R}Y_{11} + I_{2I}Y_{12} + n_{1R}Y_{13} + n_{1I}Y_{14} + n_{2R}Y_{15} + n_{2I}Y_{16} \qquad (5.129)$$

$$Z_{1I} = -I_{2R}Y_{12} + I_{2I}Y_{11} - n_{1R}Y_{14} + n_{1I}Y_{13} - n_{2R}Y_{16} + n_{2I}Y_{15} \qquad (5.130)$$

where

$$Y_{11} = a_{22}\frac{(L_R + W_R)Z_{D1R} + (L_I + W_I)Z_{D1I}}{|\Delta_{Z1}|^2} \qquad (5.131)$$

$$Y_{12} = a_{22}\frac{-(L_I + W_I)Z_{D1R} + (L_R + W_R)Z_{D1I}}{|\Delta_{Z1}|^2} \qquad (5.132)$$

$$Y_{13} = \frac{[(1 - S_{1R} - K_R)Z_{D1R} - (S_{1I} + K_I)Z_{D1I}]}{|\Delta_Z|^2} \qquad (5.133)$$

$$Y_{14} = \frac{[(S_{1I} + K_I)Z_{D1R} + (1 - S_{1R} - K_R)Z_{D1I}]}{|\Delta_{Z1}|^2} \qquad (5.134)$$

$$Y_{15} = \frac{[(S_R - r_1 cos\phi_1)Z_{D1R} + (S_I - r_1 sin\phi_1)Z_{D1I}}{|\Delta_{Z1}|^2} \qquad (5.135)$$

$$Y_{16} = \frac{[-(S_I - r_1 sin\phi_1)Z_{D1R} + (S_R - r_1 cos\phi_1)Z_{D1I})}{|\Delta_{Z1}|^2} \qquad (5.136)$$

$$|\Delta_{Z1}|^2 = Z_{D1R}^2 + Z_{D1I}^2 \qquad (5.137)$$

Notice that $Y_i, \quad i = 1, 2..6$ depend only on the random variables ϕ_1 and ϕ_2.

5.2.5 Decision Parameter for Canceler Output-2 with Both Amplitude and Phase Compensation

We will take $\hat{I}_2(n) = \dfrac{y_2(n)}{\Delta_{y2}}$ where as an estimate of the transmitted signal $I_2(n)$, and

$$Z_2(n) \triangleq \hat{I}_2(n) - I_2(n) \tag{5.138}$$

as the decision parameter.

From (5.138) and (5.89), we have

$$Z_2(n) = \frac{1}{Z_{D2}}\left[I_1(n)a_{11}\epsilon_2 + n_2(n) + n_1(n)[-\frac{a_{21}}{a_{11}} + \epsilon_2]\right], \tag{5.139}$$

where $Z_{D2} \triangleq \Delta_{y2}$

$$\Delta_{y2} = a_{22}[1 - \frac{a_{12}a_{21}}{a_{11}a_{22}} + \epsilon_2\frac{a_{12}}{a_{22}}] \tag{5.140}$$

Define

$$Z_{D2} = Z_{D2R} + jZ_{D2I}, \tag{5.141}$$

and using K from (5.94), we get,

$$Z_{D2R} = a_{22}[k_R + K_R] \tag{5.142}$$

$$Z_{D2I} = a_{22}[k_I + K_I] \tag{5.143}$$

Substituting for the real and imaginary parts of the different terms in (5.139), we have,

$$Z_2 = \frac{1}{Z_{D2}}\Big[a_{11}(I_{1R}+jI_{2I})(\epsilon_{2R}+j\epsilon_{2I})+n_{2R}+jn_{2I}$$
$$+(n_{1R}+jn_{1I})[(\epsilon_{2R}-r_2cos\phi_2)+j(\epsilon_{2I}-r_2sin\phi_2)]\Big] \quad (5.144)$$

The real and imaginary part of numerator of (5.144), Z_{N2} are given by

$$Z_{N2R} = a_{11}(I_{1R}\epsilon_{2R}-I_{1I}\epsilon_{2I})+n_{2R}+n_{1R}(\epsilon_{2R}-r_2cos\phi_2)-n_{1I}(\epsilon_{2I}-r_2sin\phi_2)$$
$$Z_{N2I} = a_{11}(I_{1R}\epsilon_{2I}+I_{1I}\epsilon_{2R})+n_{2I}+n_{1R}(\epsilon_{2I}-r_2sin\phi_2)+n_{1I}(\epsilon_{2R}-r_2cos\phi_2)$$
$$(5.145)$$

Z_{N2R} and Z_{N2I} beside being function of the random variables ϕ_1 and ϕ_2, they are also function of the signals and noises random variables.

Finally, we can write (5.144) as

$$Z_2 = Z_{2R}+jZ_{2I} = \frac{Z_{N2R}+jZ_{N2I}}{Z_{D2R}+jZ_{D2I}} \quad (5.146)$$

with

$$Z_{2R} = \frac{Z_{N2R}Z_{D2R}+Z_{N2I}Z_{D2I}}{Z_{D2R}^2+Z_{D2I}^2} \quad (5.147)$$

$$Z_{2I} = \frac{Z_{N2I}Z_{D2R}-Z_{N2R}Z_{D2I}}{Z_{D2R}^2+Z_{D2I}^2} \quad (5.148)$$

Substituting for Z_{N2R} and Z_{N2I} from (5.145), we get

$$Z_{2R} = \frac{1}{|\Delta_{Z2}|^2}\Big[I_{1R}a_{11}(\epsilon_{2R}Z_{D2R}+\epsilon_{2I}Z_{D2I})+I_{1I}a_{11}(-\epsilon_{2I}Z_{D2R}+\epsilon_{2R}Z_{D2I})$$
$$+n_{1R}[(\epsilon_{2R}-r_2cos\phi_2)Z_{D2R}+(\epsilon_{2I}-r_2sin\phi_2)Z_{D2I}]$$
$$+n_{1I}[-(\epsilon_{2I}-r_2sin\phi_2)Z_{D2R}+(\epsilon_{2R}-r_2cos\phi_2)Z_{D2I}]$$
$$+n_{2R}Z_{D2R}+n_{2I}Z_{D2I}\Big] \quad (5.149)$$

and,

$$
\begin{aligned}
Z_{2I} &= \frac{1}{|\Delta_{Z2}|^2}\Big[I_{1R}a_{11}(\epsilon_{2I}Z_{D2R} - \epsilon_{2R}Z_{D2I}) + I_{1I}a_{11}(\epsilon_{2R}Z_{D2R} + \epsilon_{2I}Z_{D2I}) \\
&+n_{1R}[(\epsilon_{2I} - r_2 sin\phi_2)Z_{D2R} - (\epsilon_{2R} - r_2 cos\phi_2)Z_{D2I}] \\
&+n_{1I}[(\epsilon_{2R} - r_2 cos\phi_2)Z_{D2R} + (\epsilon_{2I} - r_2 sin\phi_2)Z_{D2I}] \\
&-n_{2I}Z_{D2R} + n_{2I}Z_{D2R}\Big] \qquad (5.150)
\end{aligned}
$$

The Decision Parameters Final Expressions

We write the real and imaginary parts of $Z_2(n)$ in terms of the random variable representing the real and imaginary part of signal and noises of channel 2;

$$Z_{2R} = I_{1R}Y_{21} + I_{1I}Y_{22} + n_{1R}Y_{23} + n_{1I}Y_{24} + n_{2R}Y_{25} + n_{2I}Y_{26} \qquad (5.151)$$

$$Z_{2I} = -I_{1R}Y_{22} + I_{1I}Y_{21} - n_{1R}Y_{24} + n_{1I}Y_{23} - n_{2R}Y_{26} + n_{2I}Y_{25} \qquad (5.152)$$

with

$$Y_{21} = \frac{a_{11}(\epsilon_{2R}Z_{D2R} + \epsilon_{2I}Z_{D2I})}{|\Delta_{Z2}|^2} \qquad (5.153)$$

$$Y_{22} = \frac{a_{11}(\epsilon_{2R}Z_{D2I} - \epsilon_{2I}Z_{D2R})}{|\Delta_{Z2}|^2} \qquad (5.154)$$

$$Y_{23} = \frac{[(\epsilon_{2R} - r_2 cos\phi_2)Z_{D2R} + (\epsilon_{2I} - r_2 sin\phi_2)Z_{D2I}]}{|\Delta_{Z2}|^2} \qquad (5.155)$$

$$Y_{24} = \frac{[-(\epsilon_{2I} - r_2 sin\phi_2)Z_{D2R} + (\epsilon_{2R} - r_2 cos\phi_2)Z_{D2I}}{|\Delta_{Z2}|^2} \qquad (5.156)$$

$$Y_{25} = \frac{Z_{D2R}}{|\Delta_{Z2}|^2} \qquad (5.157)$$

$$Y_{26} = \frac{Z_{D2I}}{|\Delta_{Z2}|^2} \tag{5.158}$$

$$|\Delta_{Z2}|^2 = Z_{D2R}^2 + Z_{D2I}^2 \tag{5.159}$$

Notice that $Y_{2i}, \quad i = 1, 2..6$ depend only on the random variables ϕ_1 and ϕ_2.

5.3 The Performance Analysis

The Chernoff error performance bound as well as its approximate to of these error using GQR moment method will be found in this section. Due to the fact that the power-correlator canceler is not symmetric, we will perform these calculation separately for each output.

5.3.1 Chernoff Bound

The procedure for finding the Chernoff bound is the same as in the previous chapters, except for the decision parameter to be used.

Canceler Output-1

From (3.168), we write the bound for output-1;

$$P_1(e) \leq (1 - \frac{1}{\sqrt{M}})\frac{1}{4\pi^2}. \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \exp\left[\frac{-3\,(SNR)}{2(M-1)} \frac{1}{(SNR)U_{cp}(\phi_1,\phi_2) + W_{cp}(\phi_1,\phi_2)}\right] d\phi_1 d\phi_2 \tag{5.160}$$

where,

$$U_{cp}(\phi_1,\phi_2) = Y_{11}^2 + Y_{12}^2$$

$$W_{cp}(\phi_1,\phi_2) = Y_{13}^2 + Y_{14}^2 + Y_{15}^2 + Y_{16}^2 \tag{5.161}$$

with Y_{1i} $i = 1,..6$ as they are defined in (5.131) to (5.136).

Canceler Output-2

Similarly, the Chernoff bound on the error at this output is given by (5.160) with

$$U_{cp}(\phi_1,\phi_2) = Y_{21}^2 + Y_{22}^2$$

$$W_{cp}(\phi_1,\phi_2) = Y_{23}^2 + Y_{24}^2 + Y_{25}^2 + Y_{26}^2 \tag{5.162}$$

and Y_{2i} $i = 1,..6$ as they are defined in (5.153) to (5.158).

5.3.2 Method of Moments for Probability of Error Calculation

For using the moments method, we follow the same steps as in section 3.3.2. From there, the probability of error is given by the Gauss quadrature rule equation (3.182);

$$P_1(e) = 2(1 - \frac{1}{\sqrt{M}})\sum_{i=1}^{N} w_i Q\left(\sqrt{\frac{3(SNR)}{M-1}}x_i\right) \tag{5.163}$$

where x_i and w_i are nodes and weights of the GQR obtained from the moment of the random variable,

$$\mathbf{x_{cp}} = \frac{c - X_{Icp}}{\sigma_o}, \tag{5.164}$$

$$\sigma_o^2(\phi_1,\phi_2) = (Y_{13}^2 + Y_{14}^2 + Y_{15}^2 + Y_{16}^2)c^2 \tag{5.165}$$

and

$$X_{Icp} = I_{2R}Y_{11} + I_{2I}Y_{12}, \tag{5.166}$$

for canceler output-1, while

$$\sigma_o^2(\phi_1, \phi_2) = (Y_{23}^2 + Y_{24}^2 + Y_{25}^2 + Y_{26}^2)c^2 \tag{5.167}$$

and

$$X_{Icp} = I_{2R}Y_{21} + I_{2I}Y_{22}, \tag{5.168}$$

for canceler output-2. Again Y_{1i} $i = 1, ..6$ are given in (5.131) to (5.136) and Y_{2i} $i = 1, ..6$ are given in (5.153) to (5.158).

5.4 Results

As in the case of other cancelers, we present in this section results of calculations performed using the power-correlator canceler. This done by either finding the Chernoff bound on the probability of error or by obtaining approximates to these error using the moments method. Again, we use 16 and 64 QAM signals, with variable SNR and different values of cross coupling.

Fig. 5.1, we depict the Chernoff bound for 16 QAM signal as a function of SNR and with cross coupling r= -10 dB and -15 dB. These bounds are shown for the case when both amplitude and phase compensation are employed. Equations (5.160) with (5.161) are used to calculate this bound for canceler output-1 and (5.160) with (5.162) are used to calculate the Chernoff bound for canceler output-2. Fig. 5.2 is the same for 64 QAM. The same equations are used to obtain the Chernoff bounds. In Fig. 5.3, we compare the bounds for 16 QAM to these for 64 QAM case. Since the GQR results are shown to be very similar to these of the power-power canceler, we will not plot these result again rather refer the reader to the results in section in chapter 3.

In Fig. 5.4 and 5.5, we will compare the results obtained with the moments method to their corresponding Chernoff bound for 16 QAM and 64 QAM , respectively. Finally, we list in tables 5.1 to 5.6 some of the results shown in the aforementioned figures. Particularly, we draw the attention to moments results in these tables and how they compare to the corresponding moments results shown in the tables of section 3.4.

5.5 Conclusion

The power-correlator bootstrapped canceler was analysed and its performance was studied in this chapter. As in the previous chapters, both the Chernoff bound on the probability of errors as well as approximate values of these errors are calculated. Amplitude and phase compensation was implemented. The case with amplitude can be easily considered. Because of asymmetry of this canceler, we consider separately the two different cancelers outputs. Nevertheless it turns out that the performances of these outputs are very close to one another. This might be due to the fact that with both amplitude and phase compensation, the errors in the steady state become of the same order. Here, as in the other canceler, particularly when both amplitude and phase compensation is used the Chernoff bound turns out sufficiently tight.

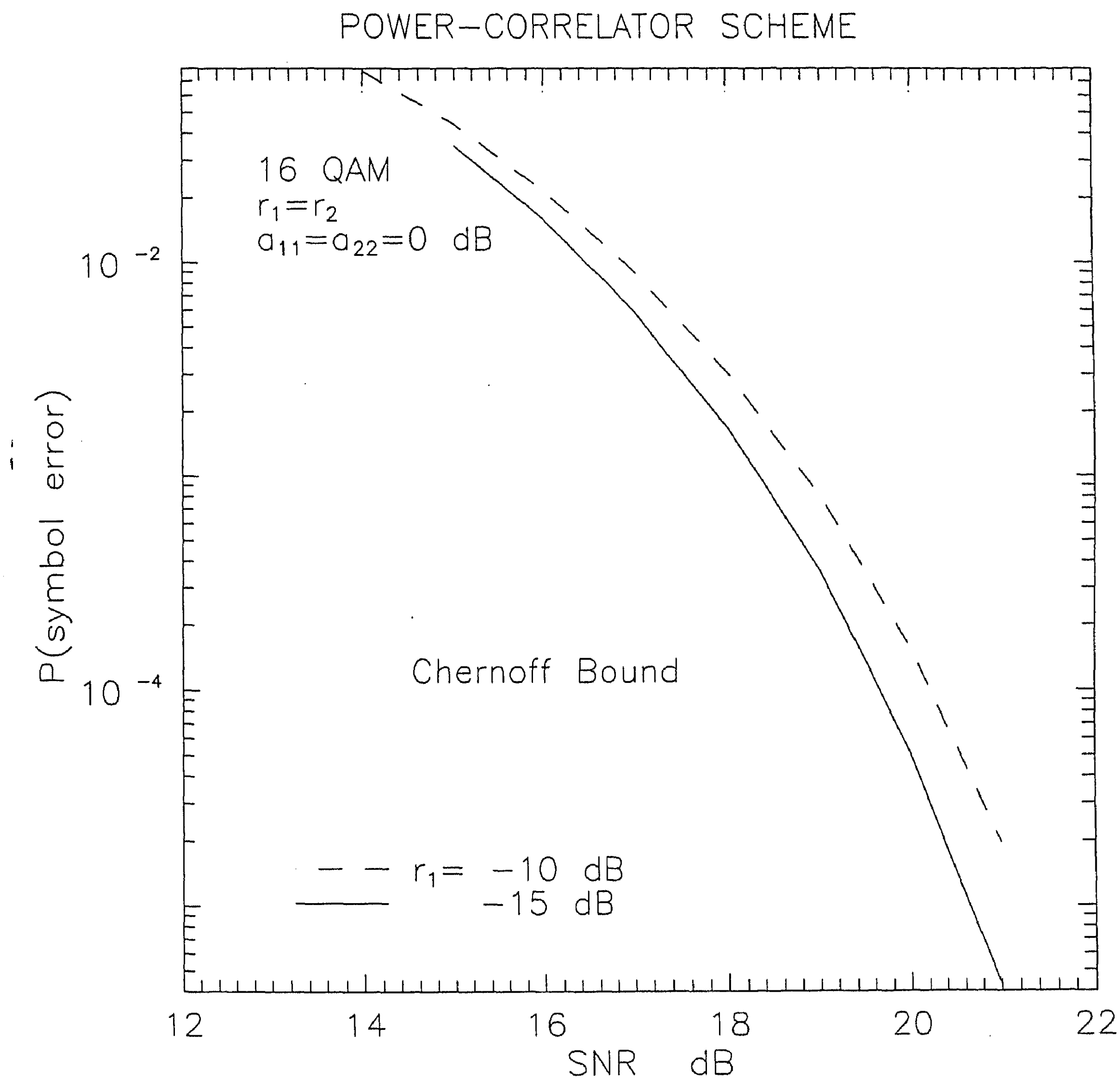

Figure 5.1: Power-Correlator Cross-Pol Canceler, Chernoff bound 16 QAM

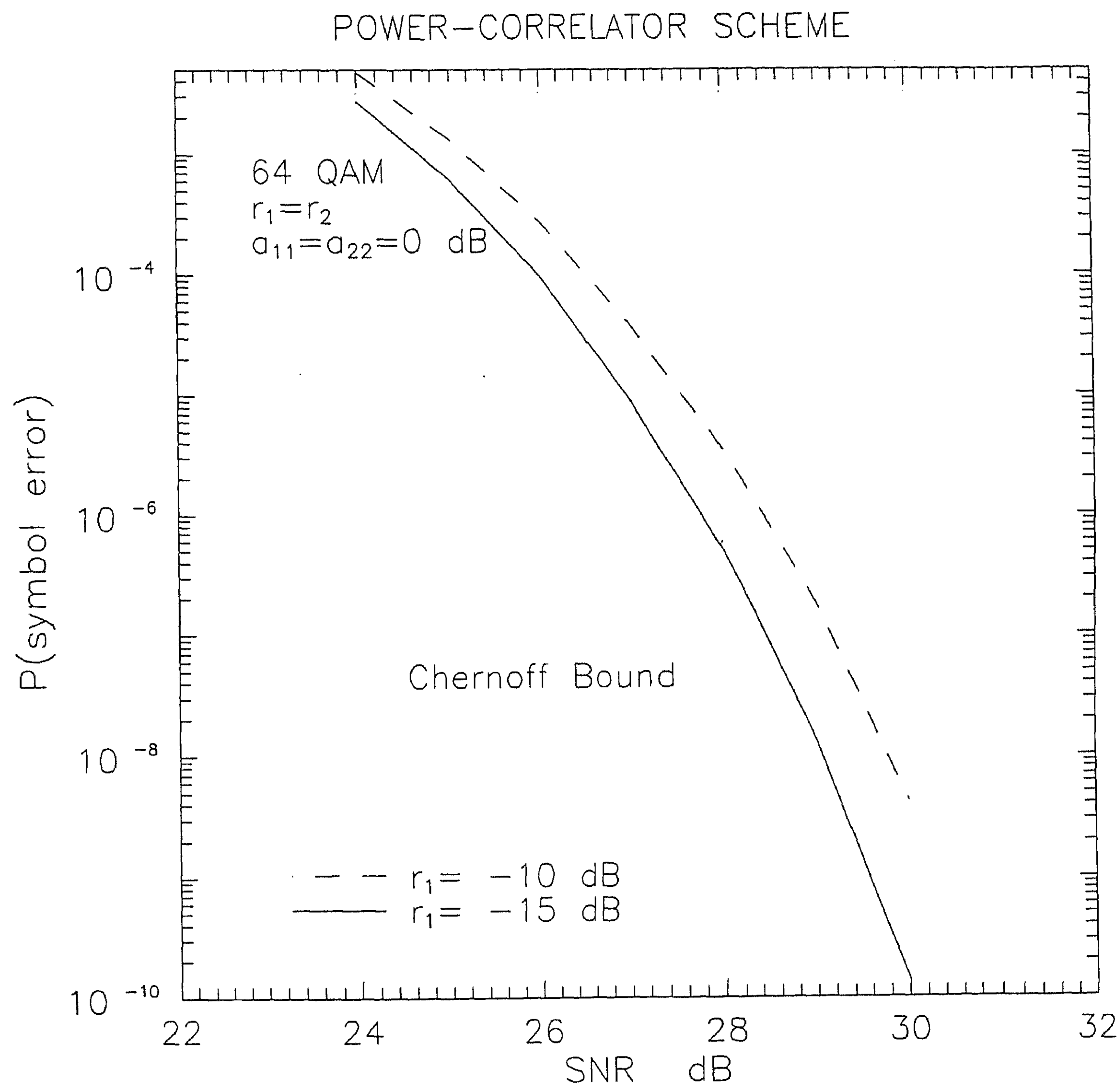

Figure 5.2: Power-Correlator Cross-Pol Canceler, Chernoff bound, 64 QAM

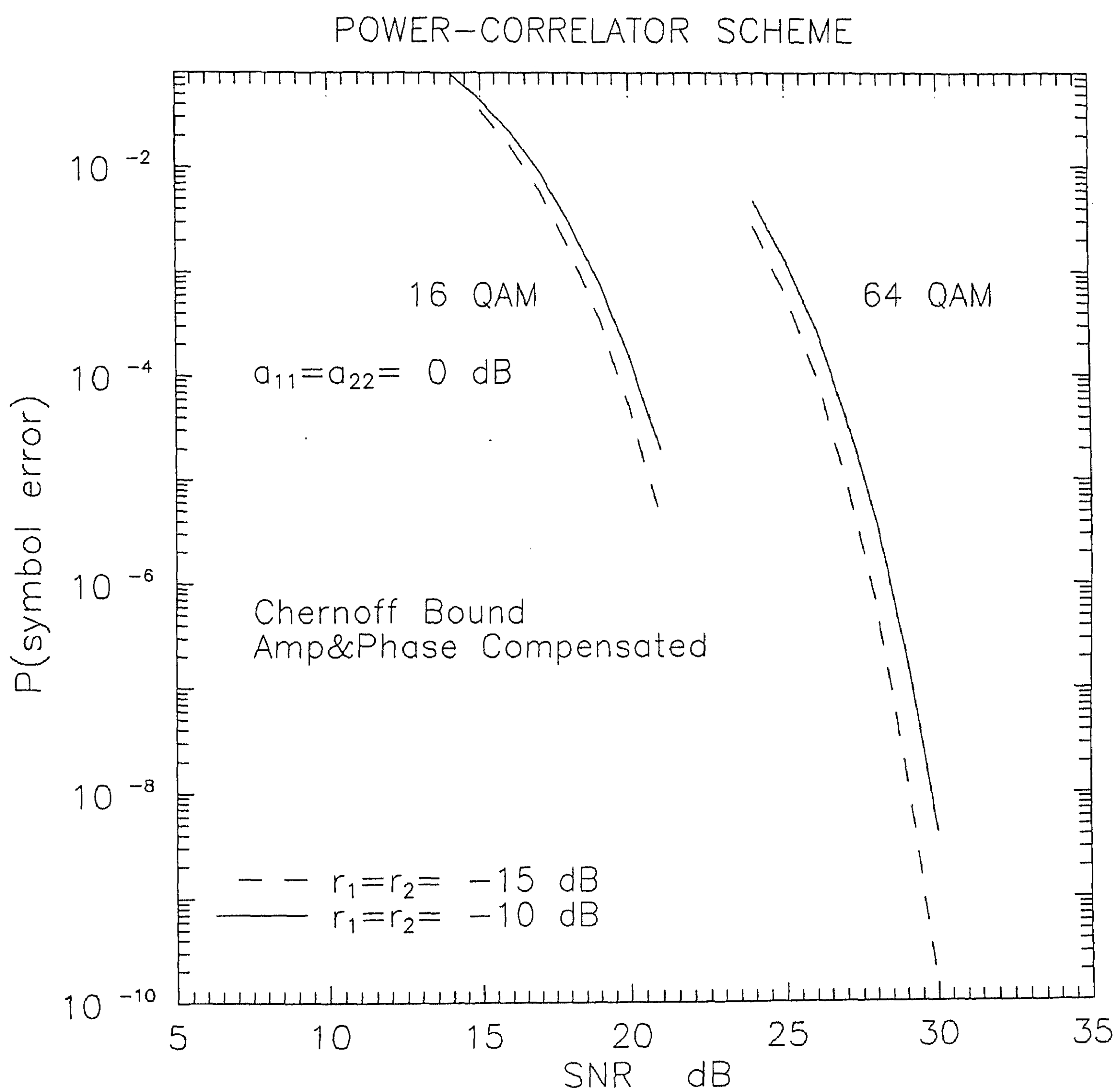

Figure 5.3: Power-Correlator Cross-Pol Canceler, Chernoff bound comparison 16 QAM v.s 64 QAM

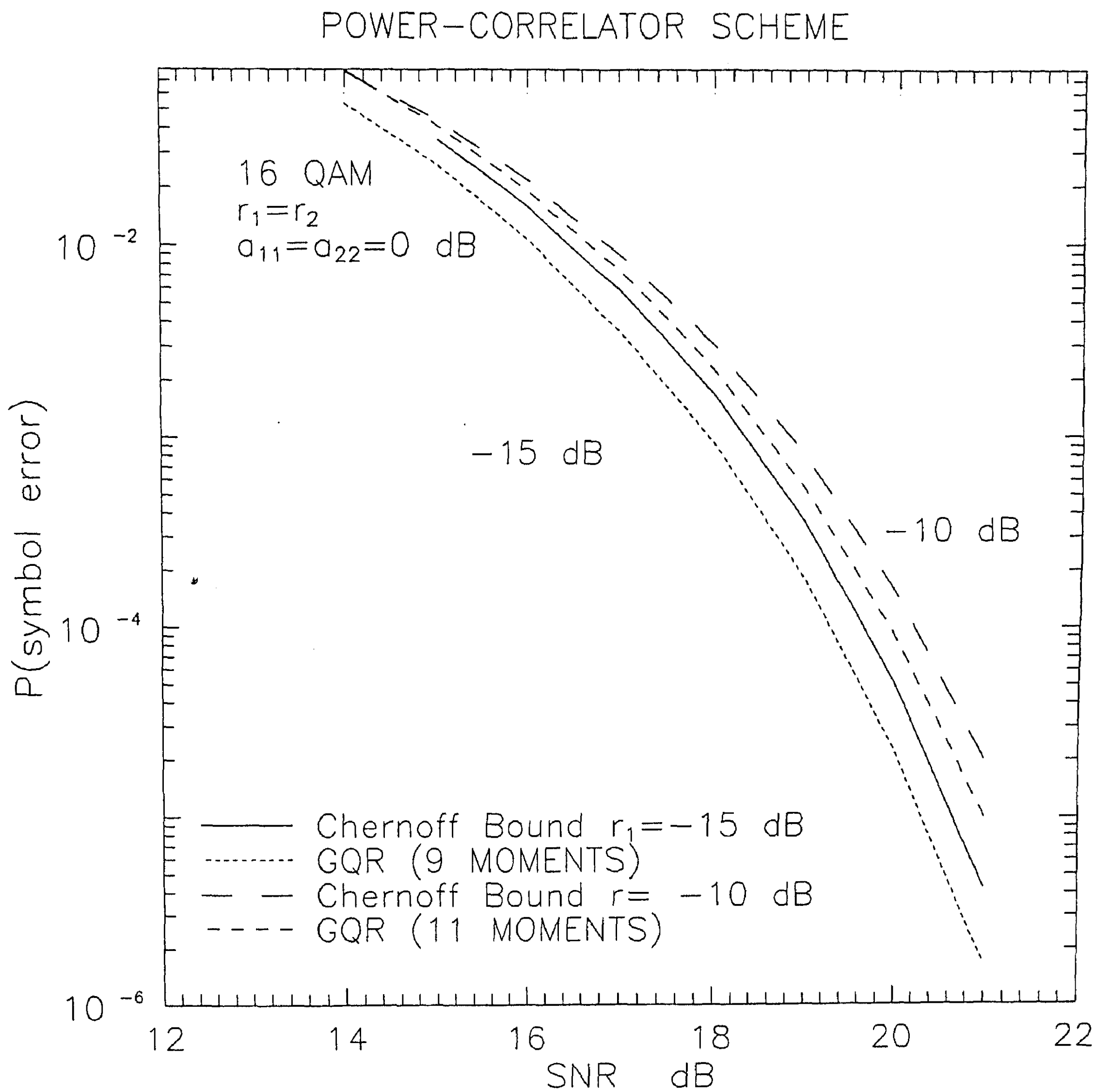

Figure 5.4: Power-Correlator Cross-Pol Canceler, Chernoff Bound and GQR calculation comparison, 16 QAM

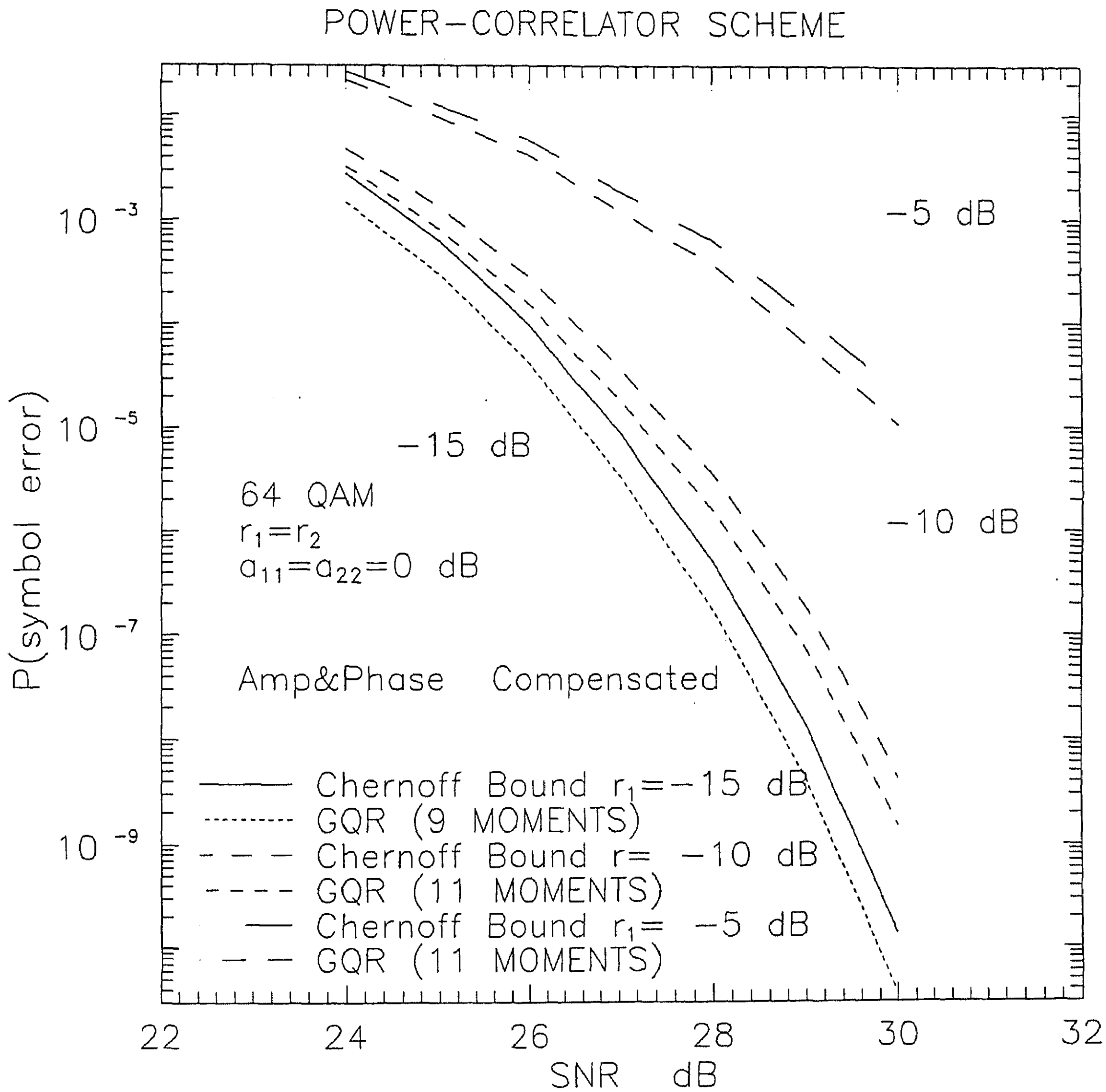

Figure 5.5: Power-Correlator Cross-Pol Canceler, Chernoff Bound and GQR calculation comparison, 64 QAM

Power-Correlator Scheme, Output-1 For 16 QAM				
$r_1 = r_2 =$ -15 dB			$r_1 = r_2 =$ -10 dB	
SNR	Moment 9	Chernoff Bound	Moment 11	Chernoff Bound
14	5.345E-2	6.563E-1	7.895E-2	7.766E-2
15	2.579E-2	3.503E-2	4.120E-2	4.399E-2
16	1.068E-2	1.591E-2	1.897E-2	2.170E-2
17	3.604E-3	5.903E-3	7.507E-3	9.031E-3
18	9.402E-4	1.698E-3	2.277E-3	3.052E-3
19	1.774E-4	3.554E-4	5.316E-4	7.987E-4
20	2.194E-5	4.990E-5	8.866E-5	1.526E-4
21	1.651E-6	4.260E-6	1.005E-5	1.971E-5

Table 5.1: Power-Correlator Cross-Pol Canceler, 16 QAM, performance calculation with Chernoff Bound and GQR method for output-1, with amplitude and phase compensation for cross coupling -15 dB and -10 dB,

Power-Correlator Scheme, Output-2 For 16 QAM				
	$r_1 = r_2 =$ -15 dB		$r_1 = r_2 =$ -10 dB	
SNR	Moment 9	Chernoff Bound	Moment 11	Chernoff Bound
14	5.356E-2	6.562E-1	7.619E-2	7.750E-2
15	2.573E-2	3.5036E-2	3.950E-2	4.390E-2
16	1.065E-2	1.591E-2	1.818E-2	2.165E-2
17	3.599E-3	5.902E-3	7.183E-3	9.010E-3
18	9.391E-4	1.698E-3	2.330E-3	3.045E-3
19	1.784E-4	3.554E-4	5.479E-4	7.967E-4
20	2.207E-5	4.993E-5	9.257E-5	1.522E-4
21	1.659E-6	4.262E-6	1.038E-5	1.966E-5

Table 5.2: Power-Correlator Cross-Pol Canceler, 16 QAM, performance calculation with Chernoff Bound and GQR method for output-2, with amplitude and phase compensation for cross coupling -15 dB and -10 dB,

Power-Correlator Scheme, Output-1 For 64 QAM				
	$r_1 = r_2 =$ -15 dB		$r_1 = r_2 =$ -10 dB	
SNR	Moment 9	Chernoff Bound	Moment 11	Chernoff Bound
24	1.466E-3	2.727E-3	3.226E-3	4.736E-3
25	2.961E-4	6.186E-4	8.054E-4	1.323E-3
26	4.109E-5	9.615E-5	1.496E-4	2.737E-4
27	3.560E-6	9.312E-6	1.921E-5	3.907E-5
28	1.709E-7	4.993E-7	1.551E-6	3.505E-6
29	3.932E-9	1.279E-8	6.975E-8	1.753E-7
30	3.584E-11	1.378E-10	1.499E-9	4.197E-9

Table 5.3: Power-Correlator Cross-Pol Canceler, 64 QAM, performance calculation with Chernoff Bound and GQR method for output-1, with amplitude and phase compensation for cross coupling -15 dB and -10 dB,

Power-Correlator Scheme, Output-2 For 64 QAM				
	$r_1 = r_2 =$ -15 dB		$r_1 = r_2 =$ -10 dB	
SNR	Moment 9	Chernoff Bound	Moment 11	Chernoff Bound
24	1.481E-3	2.727E-3	3.319E-3	4.734E-3
25	2.986E-4	6.186E-4	8.222E-4	1.322E-3
26	4.136E-5	9.615E-5	1.518E-4	2.736E-4
27	3.578E-6	9.312E-6	1.940E-5	3.904E-5
28	1.716E-7	4.992E-7	1.560E-6	3.503E-6
29	3.935E-9	1.279E-8	7.002E-8	1.752E-7
30	3.586E-11	1.376E-10	1.502E-9	4.194E-9

Table 5.4: Power-Correlator Cross-Pol Canceler, 64 QAM, performance calculation with Chernoff Bound and GQR method for output-2, with amplitude and phase compensation for cross coupling -15 dB and -10 dB,

Power-Correlator Scheme, Output-1 For 64 QAM		
$r_1 = r_2 =$ -5 dB		
SNR	Moment 11	Chernoff Bound
24	2.171E-2	2.588E-2
26	4.120E-3	5.785E-3
28	3.757E-4	6.330E-4
30	1.069E-5	2.201E-5

Table 5.5: Power-Correlator Cross-Pol Canceler, 64 QAM, performance calculation with Chernoff Bound and GQR method for output-1, with amplitude and phase compensation for cross coupling -5 dB

Power-Correlator Scheme, Output-2 For 64 QAM		
$r_1 = r_2 =$ -5 dB		
SNR	Moment 11	Chernoff Bound
24	2.292E-2	2.574E-2
26	4.202E-3	5.753E-3
28	3.773E-4	6.293E-4
30	1.067E-5	2.188E-5

Table 5.6: Power-Correlator Cross-Pol Canceler, 64 QAM, performance calculation with Chernoff Bound and GQR method for output-1, with amplitude and phase compensation for cross coupling -5 dB

Chapter 6

THE STABILITY OF BOOTSTRAPPED ALGORITHM

6.1 Introduction

In previous chapter, we found the equilibrium points for the weights of the bootstrapped algorithm. The question which arise is that whether these points are stable steady state points. We will answer this question for the three schemes of bootstrapped algorithm, the power-power, correlator-correlator and power-correlator schemes separately. We will restrict our discussion to the case of no noise. That is the dual channel noises $E\{n_1(n)^2\}$ and $E\{n_2(n)^2\}$ are zero. Also, for the sake of simplicity, we will consider the signal to be real. This is the case for example when the transmitted data is an M-ary signal.

From (2.5), the channel response in no noise case is,

$$x_1(n) = a_{11}I_1(n) + a_{12}I_2(n)$$

$$x_2(n) = a_{21}I_1(n) + a_{22}I_2(n) \tag{6.1}$$

where $x_1(n)$ and $x_2(n)$ are the sampled received signals at the first and second channels respectively. $I_i(n) \quad i = 1, 2$ are the inputs of the channel, which are taken to be real, equally likely distributed from the set $\{\pm 1c, \pm 3c, ... \pm(\sqrt{M}-1)c\}$, c is a constant which determines the distance to the decision boundary from each signal location. Also, from (2.6) when the channel co-pol and cross-pol responses are taken to be real, we have

$$\frac{a_{12}}{a_{22}} = r_1 \ , \qquad \frac{a_{21}}{a_{11}} = r_2 \tag{6.2}$$

where r_1, r_2 denote the magnitude of the normalized cross-pol interference (XPI) constants.

6.2 Equilibrium Points

6.2.1 Power-Power Scheme

With the arrangement of power-power cross-pol canceler (XPC) of Fig. 2.11, the canceler outputs, $y_1(n)$ and $y_2(n)$ are given by (see 3.2.1),

$$y_1(n) = \frac{x_1(n) + x_2(n)w_{12}}{1 - w_{12}w_{21}} \tag{6.3}$$

$$y_2(n) = \frac{x_2(n) + x_1(n)w_{21}}{1 - w_{12}w_{21}} \tag{6.4}$$

Substituting for $x_1(n)$ and $x_2(n)$ from (6.1), we get

$$y_1(n) = \frac{I_1(n)(a_{11} + w_{12}a_{21}) + I_2(n)(a_{12} + w_{12}a_{22})}{1 - w_{12}w_{21}} \tag{6.5}$$

$$y_2(n) = \frac{I_1(n)(a_{21} + w_{21}a_{11}) + I_2(n)(a_{22} + w_{21}a_{12})}{1 - w_{12}w_{21}} \tag{6.6}$$

The control algorithm simultaneously minimizes the output powers P and Q. In fact it simultaneously searches for $\partial E\{y_{1d}(n)^2\}/\partial w_{12} = 0$ and $\partial E\{y_{2d}(n)^2\}/\partial w_{21} = 0$, where $E\{\cdot\}$ the expected and y_{1d}, y_{2d} are the outputs of the discriminators where, $\delta_{ij} \quad i,j = 1,2$ are enforced. This can be performed by successive use of the following recursive equations, provided that $1 - w_{12}w_{21} \neq 0$.

$$w_{12}^{i+1} = w_{12}^{i} + \mu_1 \frac{\partial}{\partial w_{12}^{i}} E\{y_{1d}^{i}(n)^2\} \tag{6.7}$$

$$w_{21}^{i+1} = w_{21}^{i} + \mu_2 \frac{\partial}{\partial w_{21}^{i}} E\{y_{2d}^{i}(n)^2\} \tag{6.8}$$

where μ_1 and μ_2 are the constants which determine the stability of convergence. Due to the assumption that the channel responses and the signals are real w_{12} and w_{21} are also real. Clearly the equilibrium points must simultaneously satisfy the following equations

$$\mu_1 \frac{\partial P(w_{12}, w_{21})}{\partial w_{12}} = \mu_1 \frac{\partial}{\partial w_{12}} E\{y_{1d}(n)^2\} = 0 \tag{6.9}$$

$$\mu_2 \frac{\partial Q(w_{12}, w_{21})}{\partial w_{21}} = \mu_2 \frac{\partial}{\partial w_{21}} E\{y_{2d}(n)^2\} = 0 \tag{6.10}$$

Using (6.5) and (6.6), we get the power at the output of the discriminator;

$$P(w_{12}, w_{21}) = \frac{\delta_{11} E\{I_1(n)^2\}(a_{11} + w_{12}a_{21})^2 + \delta_{12} E\{I_2(n)^2\}(a_{12} + w_{12}a_{22})^2}{(1 - w_{12}w_{21})^2} \tag{6.11}$$

$$Q(w_{12}, w_{21}) = \frac{\delta_{21} E\{I_1(n)^2\}(a_{21} + w_{21}a_{11})^2 + \delta_{22} E\{I_2(n))^2\}(a_{22} + w_{21}a_{12})^2}{(1 - w_{12}w_{21})^2}. \tag{6.12}$$

When w_{12} and w_{21} are taken to be the steady state value of these weights from (6.7) and (6.8) respectively. Hence these points must satisfy (6.9) and (6.10). Taking the derivative of (6.11) and (6.12) with respect to w_{12} and w_{21} and multiplying with the convergence constants respectively we get;

$$\mu_1 \frac{\partial P(w_{12}, w_{21})}{\partial w_{12}} = \frac{2\mu_1}{(1 - w_{12}w_{21})^3} \Big[\delta_{11} E\{I_1(n)^2\}(a_{11} + w_{12}a_{21})(a_{21} + w_{21}a_{11})$$

$$+ \delta_{12} E\{I_2(n)^2\}(a_{12} + w_{12}a_{22})(a_{22} + w_{21}a_{12}) \Big] \tag{6.13}$$

$$\mu_2 \frac{\partial Q(w_{12}, w_{21})}{\partial w_{21}} = \frac{2\mu_2}{(1 - w_{12}w_{21})^3} \Big[\delta_{21} E\{I_1(n)^2\}(a_{11} + w_{12}a_{21})(a_{21} + w_{21}a_{11})$$

$$+ \delta_{22} E\{I_2(n)^2\}(a_{12} + w_{12}a_{22})(a_{22} + w_{21}a_{12}) \Big] \tag{6.14}$$

where $\delta_{ij} \quad i, j = 1, 2$ denotes the effect of the discrimination on the different signals $I_1(n)$ or $I_2(n)$. Note that without the inclusion of the discriminator the two equations in (6.13) and (6.14) are dependent. If the discriminator are chosen such that $\delta_{11}\delta_{22} \neq \delta_{21}\delta_{12}$ then (6.13) and (6.14) are independent. These solutions determine the equilibrium points for (6.7) and (6.8). Simple inspection of (6.13) and (6.14) shows that there are two such equilibrium points.

$$w_{12}\text{opt1} = -\frac{a_{12}}{a_{22}} \quad w_{21}\text{opt1} = -\frac{a_{21}}{a_{11}} \tag{6.15}$$

and

$$w_{12}\text{opt2} = -\frac{a_{11}}{a_{21}} \quad w_{21}\text{opt2} = -\frac{a_{22}}{a_{12}} \tag{6.16}$$

6.2.2 Correlator-Correlator Scheme

From Fig. 2.12, the outputs of the canceler can be written as,

$$y_1(n) = x_1(n) + w_{12}x_2(n)$$

$$y_2(n) = x_2(n) + w_{21}x_1(n) \tag{6.17}$$

Substituting (6.1) in (6.17), we get

$$y_1(n) = I_1(n)[a_{11} + w_{12}a_{21}] + I_2(n)[a_{12} + w_{12}a_{22}]$$

$$y_2(n) = I_1(n)[a_{21} + w_{21}a_{11}] + I_2(n)[a_{22} + w_{21}a_{12}] \tag{6.18}$$

Again the control algorithm simultaneously minimizes the output correlation powers $P_1 = E\{A_1(w_{12}, w_{21})\}^2$ and $Q_1 = E\{B_1(w_{12}, w_{21})\}^2$. It simultaneously searches for $\partial E\{A_1(w_{12}, w_{21})\}^2/\partial w_{12} = 0$ and $\partial E\{B_1(w_{12}, w_{21})\}^2/\partial w_{21} = 0$, where $E\{\cdot\}$ the expected value of $\{.\}$. This can be performed by successive use of the following recursive equations.

$$w_{12}^{i+1} = w_{12}^{i} + \mu_1 \frac{\partial}{\partial w_{12}^{i}} E\{A_1(w_{12}^{i}, w_{21}^{i})\}^2 \tag{6.19}$$

$$w_{21}^{i+1} = w_{21}^{i} + \mu_2 \frac{\partial}{\partial w_{21}^{i}} E\{B_1(w_{12}^{i}, w_{21}^{i})\}^2 \tag{6.20}$$

where $A_1(w_{12}, w_{21})$ and $B_1(w_{12}, w_{21})$ are the correlation between the output of channel 1 and the output of channel 2 after discrimination and vise versa, respectively. That is;

$$\begin{aligned} A_1(w_{12}, w_{21}) = E\{y_{1d}(n)y_2(n)\} &= \delta_{11}E\{I_1(n)^2\}(a_{11} + w_{12}a_{21})(w_{21}a_{11} + a_{21}) \\ &\quad + \delta_{12}E\{I_2(n)^2\}(a_{12} + w_{12}a_{22})(w_{21}a_{12} + a_{22}) \end{aligned} \tag{6.21}$$

$$\begin{aligned} B_1(w_{12}, w_{21}) = E\{y_1(n)y_{2d}(n)\} &= \delta_{21}E\{I_1(n)^2\}(a_{11} + w_{12}a_{21})(w_{21}a_{11} + a_{21}) \\ &\quad + \delta_{22}E\{I_2(n)^2\}(a_{12} + w_{12}a_{22})(w_{21}a_{12} + a_{22}) \end{aligned} \tag{6.22}$$

Notice again that for equations (6.21) and (6.22) to be independent, discrimination constants δ_{ij} are inserted.

The optimum weights can be obtained from

$$\mu_1 \frac{\partial P_1(w_{12}, w_{21})}{\partial w_{12}} = 2\mu_1 A_1(w_{12}, w_{21}) \frac{\partial}{\partial w_{12}} A_1(w_{12}, w_{21}) = 0 \quad (6.23)$$

$$\mu_2 \frac{\partial Q_1(w_{12}, w_{21})}{\partial w_{21}} = 2\mu_2 B_1(w_{12}, w_{21}) \frac{\partial}{\partial w_{21}} B_1(w_{12}, w_{21}) = 0 \quad (6.24)$$

But from (6.21) and (6.22),

$$\begin{aligned} \frac{\partial A_1(w_{12}, w_{21})}{\partial w_{12}} &= \delta_{11} E\{I_1(n)^2\} a_{21}(w_{21}a_{11} + a_{21}) \\ &+ \delta_{12} E\{I_2(n)^2\} a_{22}(w_{21}a_{12} + a_{22}) \end{aligned} \quad (6.25)$$

$$\begin{aligned} \frac{B_1(w_{12}, w_{21})}{\partial w_{21}} &= \delta_{21} E\{I_1(n)^2\} a_{11}(a_{11} + w_{12}a_{21}) \\ &+ \delta_{22} E\{I_2(n)^2\} a_{12}(a_{12} + w_{12}a_{22}) \end{aligned} \quad (6.26)$$

Therefore, equations (6.23) and (6.24) are simultaneously zero if and only if $A_1(w_{12}, w_{21})$ and $B_1(w_{12}, w_{21})$ are equal to zero [13]. Therefore, the optimum weights $w_{12\text{opt}}$ and $w_{21\text{opt}}$ are found by equating (6.25) and (6.26) to zero, respectively. It can be easily found that the optimum weights are the same with the ones found from power-power scheme (6.15) and (6.16).

6.2.3 Power-Correlator Scheme

From Fig. 2.13 , the canceler outputs, when the noise power is zero are given by

$$\begin{aligned} y_1(n) &= x_1(n)(1 + w_{12}w_{21}) + w_{21}x_2(n) \\ y_2(n) &= x_1(n)w_{21} + x_2(n) \end{aligned} \quad (6.27)$$

Substituting (6.1) in (6.27), we get

$$y_1(n) = I_1(n)[a_{11} + w_{12}(a_{21} + w_{21}a_{11})] + I_2(n)[a_{12} + w_{12}(a_{22} + w_{21}a_{12})] \tag{6.28}$$

$$y_2(n) = I_1(n)[a_{21} + w_{21}a_{11}] + I_2(n)[a_{22} + w_{21}a_{12}] \tag{6.29}$$

In this case, the control algorithm simultaneously minimizes the output power $P_2 = E\{y_{1d}(n)^2\}$ and the correlation power $Q_2 = B_2^2(w_{12}, w_{21}) = E\{y_1(n)y_{2d}(n)\}^2$. It simultaneously searches for $\partial E\{y_{1d}(n)^2\}/\partial w_{12} = 0$ and $\partial E\{y_{2d}(n)y_1(n)\}^2/\partial w_{21} = 0$, where $E\{\cdot\}$ is the expected value of (.), and y_{1d}, y_{2d} are the discriminator outputs. This can be performed by successive use of the following recursive equations.

$$w_{12}^{i+1} = w_{12}^i + \mu_1 \frac{\partial}{\partial w_{12}^i} E\{y_{1d}^i(n)^2\} \tag{6.30}$$

$$w_{21}^{i+1} = w_{21}^i + \mu_2 \frac{\partial}{\partial w_{21}^i} E\{y_{2d}^i(n)y_1^i(n)\}^2 \tag{6.31}$$

where μ_1 and μ_2 are the constants which determine the stability of convergence.

$$\mu_1 \frac{\partial P_2(w_{12}, w_{21})}{\partial w_{12}} = \mu_1 \frac{\partial}{\partial w_{12}} E\{y_{1d}(n)^2\} = 0 \tag{6.32}$$

$$\mu_2 \frac{\partial Q_2(w_{12}, w_{21})}{\partial w_{21}} = \mu_1 \frac{\partial}{\partial w_{12}} E\{y_1(n)y_{2d}(n)\}^2 = 0 \tag{6.33}$$

From (6.28) and (6.29), we write the power at the output of the channel 1 discriminator

$$\begin{aligned} P_2(w_{12}, w_{21}) &= \delta_{11} E\{I_1(n)^2\}[a_{11} + w_{12}(a_{21} + w_{21}a_{11})]^2 \\ &\quad + \delta_{12} E\{I_2(n)^2\}[a_{12} + w_{12}(a_{22} + w_{21}a_{12})]^2 \end{aligned} \tag{6.34}$$

and the correlation between the output of the channel 1 and that of the channel 2 after discrimination;

$$B_2(w_{12}, w_{21}) = \delta_{21} E\{I_1(n)^2\}[a_{11} + w_{12}(a_{21} + w_{21}a_{11})][a_{21} + w_{21}a_{11}]$$
$$+\delta_{22} E\{I_2(n)^2\}[a_{12} + w_{12}(a_{22} + w_{21}a_{12})][a_{22} + w_{21}a_{12}] \quad (6.35)$$

Taking the derivative of (6.34) with respect to w_{12} and and multiplying by the convergence constant μ_1, we get;

$$\mu_1 \frac{\partial P_2(w_{12}, w_{21})}{\partial w_{12}} = 2\mu_1 \Big[\delta_{11} E\{I_1(n)^2\}[a_{11} + w_{12}(a_{21} + w_{21}a_{11})](a_{21} + w_{21}a_{11})$$
$$+\delta_{12} E\{I_2(n)^2\}[a_{12} + w_{12}(a_{22} + w_{21}a_{12})][a_{22} + w_{21}a_{12}]\Big]$$
$$(6.36)$$

From (6.33) and the definition of $B_2(w_{12}, w_{21})$, we have

$$\mu_2 \frac{\partial Q_2(w_{12}, w_{21})}{\partial w_{21}} = 2\mu_2 B_2(w_{12}, w_{21}) \frac{\partial}{\partial w_{21}} B_2(w_{12}, w_{21}) = 0 \quad (6.37)$$

Also taking the derivative of (6.35) with respect to w_{21}, we get;

$$\frac{\partial B_2(w_{12}, w_{21})}{\partial w_{21}} = \delta_{21} E\{I_1(n)^2\}[a_{11}^2 + 2w_{12}(a_{21} + w_{21}a_{11})a_{11}]$$
$$+\delta_{22} E\{I_2(n)^2\}[a_{12}^2 + 2w_{12}(a_{22} + w_{21}a_{12})a_{12}] \quad (6.38)$$

$w_{12\text{opt}}$ is obtained by equating (6.36) to zero. Since equation (6.37) becomes zero if and only if (6.35) becomes zero, therefore $w_{21\text{opt}}$ is obtained by equating $B_2(w_{12}, w_{21})$ to zero [13].

Notice that as in the other cases if all discrimination constants δ_{ij} $i, j = 1, 2$ are taken to equal to each other, then (6.35) and (6.36) become dependent equations.

For a unique solution to the optimum weights, these two equations must be made independent from each other. This is the reason for the discriminators and the resulting constants. Two equilibrium points can be found from simultaneously equating (6.35) and (6.36) to zero;

$$w_{12\text{opt}1} = -\frac{a_{12}}{a_{22}(1-\frac{a_{21}a_{12}}{a_{11}a_{22}})} \quad w_{21\text{opt}1} = -\frac{a_{21}}{a_{11}} \tag{6.39}$$

and;

$$w_{12\text{opt}2} = -\frac{a_{11}}{a_{21}(1-\frac{a_{22}a_{11}}{a_{21}a_{12}})} \quad w_{21\text{opt}2} = -\frac{a_{22}}{a_{12}} \tag{6.40}$$

6.3 Stability Parameters

Equations (6.7) (6.8), (6.19) ,(6.20) and (6.30), (6.31) are all nonlinear in w_{12} and w_{21}. Therefore to classify the equilibrium points of these equations , we will consider a small deviation from the equilibrium points, i.e, by varying w_{12} and w_{21} to $w_{12\text{opt}}+ \Delta w_{12}$ and $w_{21\text{opt}} + \Delta w_{21}$, respectively with Δw_{12} and Δw_{21} very small. For Δw_{12} and Δw_{21} and any twice differentiable function $X(w_{12}, w_{21})$ can be approximated, by

$$\frac{\partial X_1(w_{12}+\Delta w_{12}, w_{21}+\Delta w_{21})}{\partial w_{12}} = \frac{\partial^2 X_1}{\partial w_{12}^2}.|_{\mathbf{w}_{\text{opt}}}\Delta w_{12} + \frac{\partial^2 X_1}{\partial w_{12}\partial w_{21}}.|_{\mathbf{w}_{\text{opt}}}\Delta w_{21} \tag{6.41}$$

(6.42)

$$\frac{\partial X_2(w_{12}+\Delta w_{12}, w_{21}+\Delta w_{21})}{\partial w_{21}} = \frac{\partial^2 X_2}{\partial w_{12}\partial w_{21}}\cdot|_{\mathbf{w}_{\text{opt}}}\Delta w_{12} + \frac{\partial^2 X_2}{\partial w_{21}^2}\cdot|_{\mathbf{w}_{\text{opt}}}\Delta w_{21} \tag{6.43}$$

where

$$\mathbf{w}_{\text{opt}} = [w_{12\text{opt}}, w_{21\text{opt}}]^T \tag{6.44}$$

In matrix notation

$$\underline{\dot{\mathbf{X}}} = \mathbf{A}\underline{\mathbf{\Delta w}} \tag{6.45}$$

$$\begin{aligned}\underline{\mathbf{\Delta w}} &= [\Delta w_{12}, \Delta w_{21}]^T \\ \underline{\dot{\mathbf{X}}} &= \left[\frac{\partial X_1(w_{12}+\Delta w_{12}, w_{21}+\Delta w_{21})}{\partial w_{12}}, \frac{\partial X_2(w_{12}+\Delta w_{12}, w_{21}+\Delta w_{21})}{\partial w_{21}}\right]^T\end{aligned} \tag{6.46}$$

where

$$\mathbf{A} = \begin{bmatrix} \frac{\partial^2 X_1}{\partial w_{12}^2}|_{\mathbf{w}_{\text{opt}}} & \frac{\partial^2 X_1}{\partial w_{12}\partial w_{21}}|_{\mathbf{w}_{\text{opt}}} \\ \frac{\partial^2 X_2}{\partial w_{12}\partial w_{21}}|_{\mathbf{w}_{\text{opt}}} & \frac{\partial^2 X_2}{\partial w_{21}^2}|_{\mathbf{w}_{\text{opt}}} \end{bmatrix} \tag{6.47}$$

The stability of equilibrium points depend on the eigenvalues of matrix $\mathbf{A}$. Considering the characteristics equation of the matrix $\mathbf{A}$ from $|\lambda\mathbf{I} - \mathbf{A}| = 0$, we can find the eigenvalues of $\mathbf{A}$ by solving

$$\lambda^2 - \mathbf{b}\lambda + \mathbf{c} = 0 \tag{6.48}$$

where

$$\mathbf{b} = \frac{\partial^2 X_1}{\partial w_{12}^2}|_{\mathbf{w}_{\text{opt}}} + \frac{\partial^2 X_2}{\partial w_{21}^2}|_{\mathbf{w}_{\text{opt}}} \tag{6.49}$$

$$\mathbf{c} = \frac{\partial^2 X_1}{\partial w_{12}^2}|_{\mathbf{w}_{\text{opt}}} \frac{\partial^2 X_2}{\partial w_{21}^2}|_{\mathbf{w}_{\text{opt}}} - \frac{\partial^2 X_1}{\partial w_{12} \partial w_{21}}|_{\mathbf{w}_{\text{opt}}} \frac{\partial^2 X_2}{\partial w_{12} \partial w_{21}}|_{\mathbf{w}_{\text{opt}}} \quad (6.50)$$

The nature of the eigenvalues of $\mathbf{A}$ in the complex plane, or equivalently the relation between $\mathbf{b}$ and $\mathbf{c}$ defined in (6.48) and (6.49) determines the classification of the equilibrium point. Next, we intend to find the different entries of $\mathbf{A}$ for the different bootstrapped schemes.

6.3.1 Stability Parameters of Power-Power Scheme

Here, $X_1(w_{12}, w_{21})$ and $X_2(w_{12}, w_{21})$ are given by $\mu_1 P(w_{12}, w_{21})$ and $\mu_2 Q(w_{12}, w_{21})$, respectively from (6.13) and (6.14). First notice that for any rational function in x, $f(x) = \frac{N(x)}{D(x)}$

$$\begin{aligned}
\frac{df(x)}{dx}|_{N(x)=0} &= \frac{D(x)\frac{dN(x)}{dx} - N(x)\frac{dD(x)}{dx}}{D^2(x)}|_{N(x)=0} \\
&= \frac{dN(x)}{dx}\frac{1}{D(x)} \quad (6.51)
\end{aligned}$$

Using this relation in (6.13) and (6.14), we get

$$\mu_1 \frac{\partial^2 P(w_{12}, w_{21})}{\partial w_{12}^2}|_{\mathbf{w}_{\text{opt}}} = \frac{2\mu_1}{(1 - w_{12\text{opt}} w_{21\text{opt}})^3} \Big[\delta_{11} E\{I_1(n)^2\}$$

$$(a_{21} + w_{21\text{opt}} a_{11}) a_{21} + \delta_{12} E\{I_2(n)^2\} (a_{22} + w_{21\text{opt}} a_{12}) a_{22}) \Big] \quad (6.52)$$

$$\mu_2 \frac{\partial^2 Q(w_{12}, w_{21})}{\partial w_{21}^2}|_{\mathbf{w}_{\text{opt}}} = \frac{2\mu_2}{(1 - w_{12\text{opt}} w_{21\text{opt}})^3} \Big[\delta_{21} E\{I_1(n)^2\}$$

$$(a_{11} + w_{12\text{opt}} a_{21}) a_{11} + \delta_{22} E\{I_2(n)^2\} (a_{12} + w_{12\text{opt}} a_{22}) a_{12} \Big] \quad (6.53)$$

Substituting for $\underline{\mathbf{w}}_{\text{opt}}$ from (6.15) and (6.16) in (6.51), we obtain respectively

$$\mu_1 \frac{\partial^2 P(w_{12}, w_{21})}{\partial w_{12}^2}|_{\mathbf{w}_{\text{opt1}}} = 2\mu_1 \frac{\delta_{12} E\{I_2(n)^2\} a_{22}^2}{(1 - r_1 r_2)^2} \quad (6.54)$$

$$\mu_1 \frac{\partial^2 P(w_{12}, w_{21})}{\partial w_{12}^2}|_{\mathbf{w}_{\text{opt2}}} = 2\mu_1 \frac{\delta_{11} E\{I_1(n)^2\} a_{21}^2}{(1 - \frac{1}{r_1 r_2})^2} \quad (6.55)$$

and substituting $\underline{\mathbf{w}}_{\mathbf{opt}}$ from (6.15) and (6.16) in (6.52), we obtain respectively

$$\mu_2 \frac{\partial^2 Q(w_{12}, w_{21})}{\partial w_{21}^2}|_{\mathbf{w}_{\text{opt1}}} = 2\mu_2 \frac{\delta_{21} E\{I_1(n)^2\} a_{11}^2}{(1 - r_1 r_2)^2} \quad (6.56)$$

$$\mu_2 \frac{\partial^2 Q(w_{12}, w_{21})}{\partial w_{21}^2}|_{\mathbf{w}_{\text{opt2}}} = 2\mu_2 \frac{\delta_{22} E\{I_2(n)^2\} a_{12}^2}{(1 - \frac{1}{r_1 r_2})^2} \quad (6.57)$$

Similarly by applying (6.50) to (6.13) and (6.14) with respect to w_{21} and w_{12} respectively;

$$\mu_1 \frac{\partial^2 P(w_{12}, w_{21})}{\partial w_{21} \partial w_{12}}|_{\mathbf{w}_{\text{opt}}} = \frac{2\mu_1}{(1 - w_{12\text{opt}} w_{21\text{opt}})^3} \Big[\delta_{11} E\{I_1(n)^2\}$$

$$(a_{11} + w_{12\text{opt}} a_{21}) a_{11} + \delta_{12} E\{I_2(n)^2\}(a_{12} + w_{12\text{opt}} a_{22}) a_{12}\Big] \quad (6.58)$$

$$\mu_2 \frac{\partial^2 Q(w_{12}, w_{21})}{\partial w_{21} \partial w_{12}}|_{\mathbf{w}_{\text{opt}}} = \frac{2\mu_2}{(1 - w_{12\text{opt}} w_{21\text{opt}})^3} \Big[\delta_{21} E\{I_1(n)^2\}$$

$$(a_{21} + w_{21\text{opt}} a_{11}) a_{21} + \delta_{22} E\{I_2(n)^2\}(a_{22} + w_{21\text{opt}} a_{12}) a_{22}\Big] \quad (6.59)$$

Also, substituting (6.15) and (6.16) in (6.57) , we obtain respectively

$$\mu_1 \frac{\partial^2 P(w_{12}, w_{21})}{\partial w_{21} \partial w_{12}}|_{\mathbf{w}_{\text{opt1}}} = 2\mu_1 \frac{\delta_{11} E\{I_1(n)^2\} a_{11}^2}{(1 - r_1 r_2)^2} \tag{6.60}$$

$$\mu_1 \frac{\partial^2 P(w_{12}, w_{21})}{\partial w_{21} \partial w_{12}}|_{\mathbf{w}_{\text{opt2}}} = 2\mu_1 \frac{\delta_{12} E\{I_2(n)^2\} a_{12}^2}{(1 - \frac{1}{r_1 r_2})^2} \tag{6.61}$$

and using (6.15) and (6.16) in (6.58), we obtain respectively

$$\mu_2 \frac{\partial^2 Q(w_{12}, w_{21})}{\partial w_{12} \partial w_{21}}|_{\mathbf{w}_{\text{opt1}}} = 2\mu_2 \frac{\delta_{22} E\{I_2(n)^2\} a_{22}^2}{(1 - r_1 r_2)^2} \tag{6.62}$$

$$\mu_2 \frac{\partial^2 Q(w_{12}, w_{21})}{\partial w_{12} \partial w_{21}}|_{\mathbf{w}_{\text{opt2}}} = 2\mu_2 \frac{\delta_{21} E\{I_1(n)^2\} a_{21}^2}{(1 - \frac{1}{r_1 r_2})^2} \tag{6.63}$$

Using (6.53) and (6.55) in (6.48), we can calculate $\mathbf{b}_{\text{opt1}}$ and by using these equations together with (6.59) and (6.61) in (6.49) we can calculate $\mathbf{c}_{\text{opt1}}$.

Let $\mu_1 = \mu_2 = \mu$, then for the first equilibrium point $(w_{12\text{opt1}}, w_{21\text{opt1}})$, we have,

$$\mathbf{b}_{\text{opt1}} = \frac{2\mu}{(1 - r_1 r_2)^2} \left[\delta_{21} a_{11}^2 E\{I_1(n)^2\} + \delta_{12} a_{22}^2 E\{I_2(n)^2\} \right] \tag{6.64}$$

$$\mathbf{c}_{\text{opt1}} = \frac{4\mu^2}{(1 - r_1 r_2)^4} a_{22}^2 a_{11}^2 E\{I_2(n)^2\} E\{I_1(n)^2\} (\delta_{21}\delta_{12} - \delta_{22}\delta_{11}) \tag{6.65}$$

For the second equilibrium point, by using (6.54) and (6.56) in (6.48), we calculate $\mathbf{b}_{\text{opt2}}$ and by using these equations together with (6.60) and (6.62) in (6.49), we calculate $\mathbf{c}_{\text{opt2}}$;

$$\mathbf{b}_{\text{opt2}} = \frac{2\mu}{(1-\frac{1}{r_1 r_2})^2}\Big[\delta_{22}a_{12}^2 E\{I_2(n)^2\} + \delta_{11}a_{21}^2 E\{I_1(n)^2\}\Big] \quad (6.66)$$

$$\mathbf{c}_{\text{opt2}} = \frac{-4\mu^2}{(1-\frac{1}{r_1 r_2})^4} a_{12}^2 a_{21}^2 E\{I_2(n)^2\}E\{I_1(n)^2\}(\delta_{21}\delta_{12} - \delta_{11}\delta_{22}) \quad (6.67)$$

6.3.2 Stability Parameters of Correlator-Correlator Scheme

For this scheme, $X_1(w_{12}, w_{21})$ and $X_2(w_{12}, w_{21})$ are given by $\mu_1 P_1(w_{12}, w_{21}) = \mu_1 A_1^2(w_{12}, w_{21})$ and $\mu_2 Q_1(w_{12}, w_{21}) = \mu_1 B_1^2(w_{12}, w_{21})$ from (6.21) and (6.22), respectively.

From (6.23),

$$\mu_1 \frac{\partial^2 P_1(w_{12}, w_{21})}{\partial w_{12}^2} = 2\mu_1 \Big[\Big(\frac{\partial A_1(w_{12}, w_{21})}{\partial w_{12}}\Big)^2 + A_1(w_{12}, w_{21})\frac{\partial^2 A_1(w_{12}, w_{21})}{\partial w_{12}^2}\Big] \quad (6.68)$$

In section 6.2.2, we concluded that for the optimum weight, we must have $A_1(w_{12}, w_{21})|_{\mathbf{w}_{\text{opt}}} = 0$. Therefore, (6.67) results in

$$\mu_1 \frac{\partial^2 P_1(w_{12}, w_{21})}{\partial w_{12}^2}|_{\mathbf{w}_{\text{opt}}} = 2\mu_1 \Big(\frac{\partial A_1(w_{12}, w_{21})}{\partial w_{12}}|_{\mathbf{w}_{\text{opt}}}\Big)^2 \quad (6.69)$$

Similar argument lead from (6.23) to

$$\mu_1 \frac{\partial^2 P_1(w_{12}, w_{21})}{\partial w_{21}\partial w_{12}}|_{\mathbf{w}_{\text{opt}}} = 2\mu_1 \frac{\partial A_1(w_{12}, w_{21})}{\partial w_{21}}|_{\mathbf{w}_{\text{opt}}} \frac{\partial A_1(w_{12}, w_{21})}{\partial w_{12}}|_{\mathbf{w}_{\text{opt}}} \quad (6.70)$$

Similarly using (6.24), we get

$$\mu_2 \frac{\partial^2 Q_1(w_{12}, w_{21})}{\partial w_{21}^2}|_{\mathbf{w}_{\text{opt}}} = 2\mu_2 \Big(\frac{\partial B_1(w_{12}, w_{21})}{\partial w_{21}}\Big)^2|_{\mathbf{w}_{\text{opt}}} \quad (6.71)$$

Notice that,

$$\mu_2 \frac{\partial^2 Q_1(w_{12}, w_{21})}{\partial w_{21} \partial w_{12}}|_{\mathbf{w}_{\text{opt}}} = 2\mu_2 \frac{\partial B_1(w_{12}, w_{21})}{\partial w_{21}}|_{\mathbf{w}_{\text{opt}}} \frac{\partial B_1(w_{12}, w_{21})}{\partial w_{12}}|_{\mathbf{w}_{\text{opt}}} \tag{6.72}$$

Also using (6.21) and (6.22), we can write,

$$\begin{aligned} \frac{\partial A_1(w_{12}, w_{21})}{\partial w_{21}} &= \delta_{11} E\{I_1(n)^2\} a_{11}(w_{12}a_{21} + a_{11}) \\ &\quad + \delta_{12} E\{I_2(n)^2\} a_{12}(w_{12}a_{22} + a_{12}) \end{aligned} \tag{6.73}$$

$$\begin{aligned} \frac{B_1(w_{12}, w_{21})}{\partial w_{12}} &= \delta_{21} E\{I_1(n)^2\} a_{21}(a_{21} + w_{21}a_{11}) \\ &\quad + \delta_{22} E\{I_2(n)^2\} a_{22}(a_{22} + w_{21}a_{12}) \end{aligned} \tag{6.74}$$

Substituting (6.15), (6.16) in (6.68), we get respectively, for the two equilibrium points,

$$\mu_1 \frac{\partial^2 P_1(w_{12}, w_{21})}{\partial w_{12}^2}|_{\mathbf{w}_{\text{opt1}}} = 2\mu_1 \left[\delta_{12} E\{I_2(n)^2\} a_{22}^2 (1 - \frac{a_{12}a_{21}}{a_{22}a_{11}})\right]^2 \tag{6.75}$$

$$\mu_1 \frac{\partial^2 P_1(w_{12}, w_{21})}{\partial w_{12}^2}|_{\mathbf{w}_{\text{opt2}}} = 2\mu_1 \left[\delta_{11} E\{I_1(n)^2\} a_{21}^2 (1 - \frac{a_{22}a_{11}}{a_{12}a_{21}})\right]^2 \tag{6.76}$$

Similarly from (6.70), we get

$$\mu_2 \frac{\partial^2 Q_1(w_{12}, w_{21})}{\partial w_{21}^2}|_{\mathbf{w}_{\text{opt1}}} = 2\mu_2 \left[\delta_{21} E\{I_1(n)^2\} a_{11}^2 (1 - \frac{a_{12}a_{21}}{a_{22}a_{11}})\right]^2 \tag{6.77}$$

$$\mu_2 \frac{\partial^2 Q_1(w_{12}, w_{21})}{\partial w_{21}^2}|_{\mathbf{w}_{\text{opt2}}} = 2\mu_2 \left[\delta_{22} E\{I_2(n)^2\} a_{12}^2 (1 - \frac{a_{22}a_{11}}{a_{12}a_{21}})\right]^2 \tag{6.78}$$

Finally, from (6.69), we obtain respectively, for the two equilibrium points,

$$\mu_1 \frac{\partial^2 P_1(w_{12}, w_{21})}{\partial w_{12} \partial w_{21}}|_{\mathbf{w}_{\text{opt1}}} = 2\mu_1 [E\{I_1(n)^2\} E\{I_2(n)^2\} a_{11}^2 a_{22}^2.$$

$$(1-\frac{a_{12}a_{21}}{a_{22}a_{11}})^2\delta_{12}\delta_{11} \quad (6.79)$$

$$\mu_1\frac{\partial^2 P_1(w_{12},w_{21})}{\partial w_{12}\partial w_{21}}|_{\mathbf{w}_{\text{opt2}}} = 2\mu_1[E\{I_1(n)^2\}E\{I_2(n)^2\}a_{12}^2a_{21}^2 \cdot (1-\frac{a_{11}a_{22}}{a_{12}a_{21}})^2\delta_{12}\delta_{11} \quad (6.80)$$

$$\mu_2\frac{\partial^2 Q_1(w_{12},w_{21})}{\partial w_{12}\partial w_{21}}|_{\mathbf{w}_{\text{opt1}}} = 2\mu_2[E\{I_1(n)^2\}E\{I_2(n)^2\}a_{11}^2a_{22}^2 \cdot (1-\frac{a_{12}a_{21}}{a_{22}a_{11}})^2\delta_{21}\delta_{22} \quad (6.81)$$

$$\mu_2\frac{\partial^2 Q_1(w_{12},w_{21})}{\partial w_{12}\partial w_{21}}|_{\mathbf{w}_{\text{opt2}}} = 2\mu_2[E\{I_1(n)^2\}E\{I_2(n)^2\}a_{12}^2a_{21}^2 \cdot (1-\frac{a_{11}a_{22}}{a_{12}a_{21}})^2\delta_{21}\delta_{22} \quad (6.82)$$

Finally using (6.74) and (6.76) in (6.48), we calculate $\mathbf{b}_{1\text{opt1}}$ and using these equations together with (6.78) and (6.80) in (6.49), we can calculate $\mathbf{c}_{1\text{opt1}}$.

Let $\mu_1 = \mu_2 = \mu$, then for the first equilibrium point,

$$\mathbf{b}_{1\text{opt1}} = 2\mu(1-\frac{a_{21}a_{12}}{a_{22}a_{11}})^2\Big[(\delta_{12}E\{I_2(n)^2\}a_{22}^2)^2 + (\delta_{21}E\{I_1(n)^2\}a_{11}^2)^2\Big] \quad (6.83)$$

$$\mathbf{c}_{1\text{opt1}} = 4\mu^2\Big[E\{I_2(n)^2\}E\{I_1(n)^2\}a_{11}^2a_{22}^2(1-\frac{a_{12}a_{21}}{a_{22}a_{11}}\Big]^2\delta_{21}\delta_{12}(\delta_{12}\delta_{21}-\delta_{11}\delta_{22}) \quad (6.84)$$

Similarly, for the second equilibrium (6.16) , using (6.75), (6.77) in (6.48) and together with (6.79), (6.81) in (6.49), we can calculate $\mathbf{b}_{1\text{opt2}}$ and $\mathbf{c}_{1\text{opt2}}$, respectively.

$$\mathbf{b}_{1\text{opt2}} = 2\mu(1-\frac{a_{11}a_{22}}{a_{12}a_{21}})^2\Big[(\delta_{11}E\{I_1(n)^2\}a_{21}^2)^2 + (\delta_{22}E\{I_2(n)^2\}a_{12}^2)^2\Big] \quad (6.85)$$

$$\mathbf{c}_{1\text{opt1}} = -4\mu^2\Big[E\{I_2(n)^2\}E\{I_1(n)^2\}a_{12}^2a_{21}^2(1-\frac{a_{11}a_{22}}{a_{12}a_{21}}\Big]^2\delta_{22}\delta_{11}(\delta_{12}\delta_{21}-\delta_{11}\delta_{22}) \tag{6.86}$$

6.3.3 Stability Parameters of Power-Correlator Scheme

For this scheme, $X_1(w_{12}, w_{21})$ and $X_2(w_{12}, w_{21})$ are given by $\mu_1 P_2(w_{12}, w_{21})$ from (6.36) and $\mu_2 Q_2(w_{12}, w_{21}) = \mu_1 B_2^2(w_{12}, w_{21})$ from (6.35), respectively.

From (6.36);

$$\mu_1\frac{\partial^2 P_2(w_{12}, w_{21})}{\partial w_{12}^2} = 2\mu_1\Big[\delta_{11}E\{I_1(n)^2\}(a_{21}+w_{21\text{opt}}a_{11})^2 + \delta_{12}E\{I_2(n)^2\}(a_{22}+w_{21\text{opt}}a_{12})^2\Big] \tag{6.87}$$

and from (6.35)

$$\mu_2\frac{\partial^2 Q_2(w_{12}, w_{21})}{\partial w_{21}^2} = 2\mu_2\Big[\Big(\frac{\partial B_2(w_{12}, w_{21})}{\partial w_{21}}\Big)^2 + B_2(w_{12}, w_{21})\frac{\partial^2 B_2(w_{12}, w_{21})}{\partial w_{21}^2}\Big] \tag{6.88}$$

But in section 6.2.3, we concluded that at the optimum weights, we must have

$$B_2(w_{12}, w_{21})|_{\mathbf{w}_{\text{opt}}} = 0 \tag{6.89}$$

Therefore, (6.89) becomes

$$\mu_2\frac{\partial^2 Q_2(w_{12}, w_{21})}{\partial w_{21}^2} = 2\mu_2\Big(\frac{\partial B_2(w_{12}, w_{21})}{\partial w_{21}}\Big)^2 \tag{6.90}$$

Substituting (6.39) and (6.40) in (6.88), we get, for the two equilibrium points,

$$\mu_1\frac{\partial^2 P_2(w_{12}, w_{21})}{\partial w_{12}^2}|_{\mathbf{w}_{\text{opt1}}} = 2\mu_1\delta_{12}E\{I_2(n)^2\}a_{22}^2(1-\frac{a_{12}a_{21}}{a_{11}a_{22}})^2 \tag{6.91}$$

$$\mu_1\frac{\partial^2 P_2(w_{12}, w_{21})}{\partial w_{12}^2}|_{\mathbf{w}_{\text{opt2}}} = 2\mu_1\delta_{11}E\{I_1(n)^2\}a_{21}^2(1-\frac{a_{22}a_{11}}{a_{12}a_{21}})^2 \tag{6.92}$$

From (6.35) together with (6.39) with (6.40) in (6.91), we have for the two equilibrium points,

$$\mu_2 \frac{\partial^2 Q_2(w_{12}, w_{21})}{\partial w_{21}^2}|_{\mathbf{w}_{\text{opt1}}} = 2\mu_2[\delta_{21} E\{I_1(n)^2\} a_{11}^2 - \delta_{22} E\{I_2(n)^2\} a_{12}^2]^2 \tag{6.93}$$

$$\mu_2 \frac{\partial^2 Q_2(w_{12}, w_{21})}{\partial w_{21}^2}|_{\mathbf{w}_{\text{opt2}}} = 2\mu_2[-\delta_{21} E\{I_1(n)^2\} a_{11}^2 + \delta_{22} E\{I_2(n)^2\} a_{12}^2]^2 \tag{6.94}$$

Also taking the derivative of (6.36) with respect to w_{21}, we get

$$\mu_1 \frac{\partial^2 P_2(w_{12}, w_{21})}{\partial w_{12} \partial w_{21}}|_{\mathbf{w}_{\text{opt}}} = 2\mu_1 \Big[\delta_{11} E\{I_1(n)^2\}[a_{11}^2 + 2w_{12\text{opt}} \cdot (a_{21} + w_{21\text{opt}} a_{11}) a_{11}] + \delta_{12} E\{I_2(n)^2\}[a_{12}^2 + 2w_{12\text{opt}}(a_{22} + w_{21\text{opt}} a_{12}) a_{12}]\Big] \tag{6.95}$$

From the definition of $Q_2(w_{12}, w_{21})$, we have,

$$\mu_2 \frac{\partial^2 Q_2(w_{12}, w_{21})}{\partial w_{21} \partial w_{12}} = 2\mu_2 \left[\frac{\partial B_2(w_{12}, w_{21})}{\partial w_{21}} \frac{\partial B_2(w_{12}, w_{21})}{\partial w_{12}} + B_2(w_{12}, w_{21}) \frac{\partial B_2(w_{12}, w_{21})}{\partial w_{12} \partial w_{21}}\right] \tag{6.96}$$

But at equilibrium point $B_2(w_{12}, w_{21})$ is equal to zero and we get,

$$\mu_2 \frac{\partial^2 Q_2(w_{12}, w_{21})}{\partial w_{21} \partial w_{12}}|_{\mathbf{w}_{\text{opt1}}} = 2\mu_2 \frac{\partial B_2(w_{12}, w_{21})}{\partial w_{21}}|_{\mathbf{w}_{\text{opt1}}} \frac{\partial B_2(w_{12}, w_{21})}{\partial w_{12}}|_{\mathbf{w}_{\text{opt1}}} \tag{6.97}$$

where, from (6.35)

$$\frac{B_2(w_{12}, w_{21})}{\partial w_{12}} = \delta_{21} E\{I_1(n)^2\}(a_{21} + w_{21} a_{11})^2 + \delta_{22} E\{I_2(n)^2\}(a_{22} + w_{21} a_{12})^2 \tag{6.98}$$

Substituting (6.39), (6.40) in (6.96), we get respectively,

$$\mu_1 \frac{\partial^2 P_2(w_{12}, w_{21})}{\partial w_{12} \partial w_{21}}|_{\mathbf{w}_{\text{opt1}}} = 2\mu_1[\delta_{11} E\{I_1(n)^2\} a_{11}^2 - \delta_{12} E\{I_2(n)^2\} a_{12}^2] \tag{6.99}$$

$$\mu_1 \frac{\partial^2 P_2(w_{12}, w_{21})}{\partial w_{12} \partial w_{21}}|_{\mathbf{w}_{\text{opt2}}} = -2\mu_1[\delta_{11} E\{I_1(n)^2\} a_{11}^2 - \delta_{12} E\{I_2(n)^2\} a_{12}^2] \tag{6.100}$$

Also, using (6.39) and (6.40) with (6.38) and (6.97) in (6.98) , we get respectively,

$$\mu_2 \frac{\partial^2 Q_2(w_{12}, w_{21})}{\partial w_{12} \partial w_{21}}|_{\mathbf{w}_{\text{opt1}}} = 2\mu_2[\delta_{21} E\{I_1(n)^2\} a_{11}^2 - \delta_{22} E\{I_2(n)^2\} a_{12}^2]. \delta_{22} E\{I_2(n)^2\} a_{22}^2 (1 - \frac{a_{12}a_{21}}{a_{11}a_{22}})^2 \tag{6.101}$$

$$\mu_2 \frac{\partial^2 Q_2(w_{12}, w_{21})}{\partial w_{12} \partial w_{21}}|_{\mathbf{w}_{\text{opt2}}} = -2\mu_2[\delta_{21} E\{I_1(n)^2\} a_{11}^2 - \delta_{22} E\{I_2(n)^2\} a_{12}^2] \delta_{21} E\{I_1(n)^2\} a_{21}^2 (1 - \frac{a_{22}a_{11}}{a_{12}a_{21}})^2 \tag{6.102}$$

Finally using (6.92), (6.94) in (6.46) , together with the definition of X_1 and X_2 for this scheme, we calculate $\mathbf{b_2}_{\text{opt1}}$ and then by using these equations together with (6.100) and (6.102) in (6.47), we calculate $\mathbf{c_2}_{\text{opt1}}$.

Let $\mu_1 = \mu_2 = \mu$, then

$$\mathbf{b_2}_{\text{opt1}} = 2\mu\Big[\delta_{12} a_{22}^2 E\{I_2(n)^2\}(1 - \frac{a_{12}a_{21}}{a_{22}a_{11}})^2 + \Big[\delta_{21} E\{I_1(n)^2\} a_{11}^2 - \delta_{22} E\{I_2(n)^2\} a_{12}^2\Big]^2\Big]$$

$$\mathbf{c_2}_{\text{opt1}} = 4\mu^2 E\{I_2(n)^2\} E\{I_1(n)^2\} a_{11}^2 a_{22}^2 (1 - \frac{a_{12}a_{21}}{a_{22}a_{11}})^2. \Big[\delta_{21} E\{I_1(n)^2\} a_{11}^2 - \delta_{22} E\{I_2(n)^2\} a_{12}^2\Big](\delta_{21}\delta_{12} - \delta_{11}\delta_{22}) \tag{6.103}$$

For the second equilibrium point in (6.40) we calculate $\mathbf{b_2}_{\text{opt2}}$ by using (6.93) and (6.95) in (6.46) and by using these equations together with (6.101) and (6.103) in (6.47), we calculate $\mathbf{c_2}_{\text{opt1}}$.

$$\mathbf{b}_{2\mathrm{opt}2} = 2\mu\Big[\delta_{11}a_{21}^2 E\{I_1(n)^2\}(1-\frac{a_{22}a_{11}}{a_{21}a_{12}})^2 + \Big[-\delta_{21}E\{I_1(n)^2\}a_{11}^2 + \delta_{22}E\{I_2(n)^2\}a_{12}^2\Big]^2\Big]$$

$$\mathbf{c}_{2\mathrm{opt}2} = 4\mu^2 E\{I_2(n)^2\}E\{I_1(n)^2\}a_{21}^2 a_{12}^2(1-\frac{a_{22}a_{11}}{a_{21}a_{12}})^2. \Big[\delta_{21}E\{I_1(n)^2\}a_{11}^2 - \delta_{22}E\{I_2(n)^2\}a_{12}^2\Big](\delta_{21}\delta_{12}-\delta_{11}\delta_{22}) \quad (6.104)$$

6.4 Stability Conditions

From the characteristics equation in (6.45), the two eigenvalues λ_1 and λ_2 are related to $\mathbf{b}$ and $\mathbf{c}$ as they are defined at the two equilibrium points;

The stability condition can be summarized as follows,

• If $\mathbf{c} < 0$, the eigenvalues of (6.45), λ_1 and λ_2 are both real and $\lambda_1\lambda_2 < 0$. Since one of the eigenvalue is positive , then the equilibrium is unstable.

• If $\mathbf{c} > 0$,the two eigenvalues are either both real or complex-conjugate pair, and $\lambda_1\lambda_2 > 0$. Both eigenvalues (if they are real) or their real part (if they are complex) are negative if $\mathbf{b} < 0$, positive if $\mathbf{b} > 0$. Therefore, the system is stable if $\mathbf{c} > \mathbf{0}$ and $\mathbf{b} < \mathbf{0}$ and unstable if $\mathbf{c} > \mathbf{0}$ and $\mathbf{b} > \mathbf{0}$.

• If $\mathbf{c} = 0$, then one of the eigenvalue is zero an the other one is equal to $\mathbf{b}$ and the system is unstable [20].

6.4.1 Stability Conditions for Power-Power Scheme

The equilibrium points for this scheme are given by (6.15) and (6.16). First, for $\mathbf{c}$ at the first equilibrium point to be positive , we must have from (6.64); $\delta_{21}\delta_{12} > \delta_{11}\delta_{22}$ and for $\mathbf{b}$ to be negative, we must have from (6.63) $\mu < 0$. These two conditions will result in convergence of the algorithm at point (6.15). However, from (6.65) and (6.66) these conditions will result in divergence at the second equilibrium point in (6.16) (saddle point).

6.4.2 Stability Conditions for Correlator-Correlator Scheme

The equilibrium for this scheme is the same as the previous scheme and given by (6.15) and (6.16). Again at the first equilibrium point it is convergent if $\delta_{21}\delta_{12} > \delta_{11}\delta_{22}$ and μ_1, μ_2 or μ are negative. This results from the signs of $\mathbf{c_1}$ and $\mathbf{b_1}$ given in (6.82) and (6.83), respectively. The same condition leads to divergence as a result of the signs of $\mathbf{c_1}$ and $\mathbf{b_1}$ given in (6.84) and (6.85).

6.4.3 Stability Conditions for Power-Correlator Scheme

The equilibrium point for this scheme are given by (6.39) and (6.40). For $\mathbf{c_2}$ at the first equilibrium point in (6.39) to be positive, we must have from (6.104);

- $\delta_{21}\delta_{12} > \delta_{11}\delta_{22}$, and
- $\frac{\delta_{21}}{\delta_{22}} > \frac{E\{I_2(n)^2\}}{E\{I_1(n)^2\}}(\frac{a_{12}}{a_{11}})^2$,

and for $\mathbf{b_2} < 0$ to be negative, we must have μ_1, μ_2 or μ must be less than zero.

Furthermore, for the second equilibrium point from (6.105), the same conditions lead to convergence.

Chapter 7

DYNAMIC ANALYSIS OF BXPC FOR M-ary SIGNALS USING PERTURBATION SEQUENCES

7.1 Introduction

In previous chapters, we studied the steady state behaviour of the different configurations of the bootstrapped schemes. In each scheme, the computation of the optimal weights require the knowledge of the channel model of (2.4) as well as the signal and the noise powers.

Alternative procedures to find these optimal weights are to use the recursive relations given in (3.9) for power-power, in (4.8) for correlator-correlator and in (5.7) for power-correlator schemes of bootstrapped algorithm. All these recursive procedures require knowledge of the gradient of the output powers or the gradient of correlation between the outputs.

In this chapter, we will study the dynamic analysis of the power-power scheme. We will present a technique for finding the optimal weights with a recursive weight updating procedure using estimates of the gradients. With this technique, the estimate of the gradients are obtained by applying orthogonal perturbation sequences to the weights simultaneously, and measuring the corresponding changes at the output powers P and Q of the power-power scheme.

7.2 Dynamic Power-Power Scheme

For the convenient of the reader, we will repeat in this section some equations from chapter 3 rather than referring to them. The received signals sampled after matched filters, were denoted in chapter 3 by;

$$\begin{aligned} x_1(n) &= a_{11}I_1(n) + a_{12}I_2(n) + n_1(n) \\ x_2(n) &= a_{21}I_1(n) + a_{22}I_2(n) + n_2(n) \end{aligned} \tag{7.1}$$

where $x_1(n)$ and $x_2(n)$ are the sampled received signals at the first and second channels respectively. $I_i(n)$ and $n_i(n)$ are the corresponding signals and noises at these outputs such that $I_i\ i = 1,2$ are M-ary signals from set $\{. \pm 1, \pm 3, -1, ... \pm (\sqrt{M} - 1)\}$ and $a_{ij} \quad i,j = 1,2$ are real channel couplings.

Also $n_1(n)$ and $n_2(n)$ are independent samples of zero mean Gaussian with

$$E\{n_i^2(n)\} = \sigma_i^2, \quad i = 1,2 \tag{7.2}$$

7.2.1 Output of the Canceler

From Fig. 7.1, the outputs $y_1(n)$ and $y_2(n)$ are as follows

$$y_1(n) = x_1(n) + y_2(n)w_{12}(n) \tag{7.3}$$

$$y_2(n) = x_2(n) + y_1(n)w_{21}(n) \tag{7.4}$$

$$\begin{aligned} y_1(n) &= \frac{x_1(n) + x_2(n)w_{12}(n)}{1 - w_{12}(n)w_{21}(n)} \\ y_2(n) &= \frac{x_2(n) + x_1(n)w_{21}(n)}{1 - w_{12}(n)w_{21}(n)} \end{aligned} \tag{7.5}$$

Substituting for $x_1(n)$ and $x_2(n)$ from (7.1) we get the outputs after the discrimination at the n th instant of time, respectively,

$$y_{1d}(n) = \frac{\delta_{11}I_1(n)(a_{11}+w_{12}(n)a_{21}) + \delta_{12}I_2(n)(a_{12}+w_{12}(n)a_{22}) + n_1(n) + n_2(n)w_{12}(n)}{1 - w_{12}(n)w_{21}(n)} \tag{7.6}$$

$$y_{2d}(n) = \frac{\delta_{21}I_1(n)(a_{21}+w_{21}(n)a_{11}) + \delta_{22}I_2(n)(a_{22}+w_{21}(n)a_{12}) + n_1(n)w_{21}(n) + n_2(n)}{1 - w_{12}(n)w_{21}(n)} \tag{7.7}$$

Also, from Fig. 7.1 ,we can write the weights $w_{12}(n)$ and $w_{21}(n)$ as a sum of a nominal value $w_{12}(i)$ and $w_{21}(i)$ plus perturbation sequences $p_1(n)$ and $p_2(n)$ whose magnitudes are Λ. That is

$$w_{12}(n) = w_{12}(i) + \Lambda p_1(n) \tag{7.8}$$

$$w_{21}(n) = w_{21}(i) + \Lambda p_2(n) \tag{7.9}$$

7.2.2 Approximate Canceler Outputs

In chapter 3, we found that, in the no noise environment the optimal weights are; $w_{21}\text{opt} = -\frac{a_{21}}{a_{11}}$ and $w_{12}\text{opt} = -\frac{a_{12}}{a_{22}}$

Given that the cross-couplings $\frac{a_{21}}{a_{11}}$ $\frac{a_{12}}{a_{22}} << 1$, then,

$$w_{12}(n)w_{21}(n) << 1 \tag{7.10}$$

The assumption in (7.10) will be used to simplify the analysis of dynamic study. With this approximation, we can write (7.6) and (7.7) as

$$y_{1d}(n) \approx \Big[\delta_{11}I_1(n)[a_{11}+w_{12}(n)a_{21}] + \delta_{12}I_2(n)[a_{12}+w_{12}(n)a_{22}]$$

$$+n_1(n) + n_2(n)w_{12}(n)\Big][1 + w_{12}(n)w_{21}(n)] \tag{7.11}$$

$$y_{2d}(n) \approx \Big[\delta_{21}I_1(n)[a_{21} + w_{21}(n)a_{11}] + \delta_{22}I_2(n)[a_{22} + w_{21}(n)a_{12}]$$

$$+n_2(n) + n_1(n)w_{21}(n)\Big][1 + w_{12}(n)w_{21}(n)] \tag{7.12}$$

7.2.3 Mean Output Powers

In the steady state, using (7.11) and (7.12) respectively, we can write the mean output powers with weights fixed at w_{12} and w_{21}; $P(w_{12}, w_{21}) = E\{y_{1d}^2(n)\}$ and $Q(w_{12}, w_{21}) = E\{y_{2d}^2(n)\}$, respectively,

$$P(w_{12}, w_{21}) \approx \Big[\delta_{11}E\{I_1^2(n)\}(a_{11} + w_{12}a_{21})^2 + \delta_{12}E\{I_2^2(n)\}(a_{12} + w_{12}a_{22})^2$$

$$+E\{n_1^2(n)\} + E\{n_2^2(n)\}w_{12}^2\Big](1 + w_{12}w_{21})^2, \tag{7.13}$$

and output power Q at the second channel,

$$Q(w_{12}, w_{21}) \approx \Big[\delta_{21}E\{I_1^2(n)\}(a_{21} + w_{21}a_{11})^2 + \delta_{22}E\{I_2^2(n)\}(a_{22} + w_{21}a_{12})^2$$

$$+E\{n_2^2(n)\} + E\{n_1^2(n)\}w_{21}^2\Big](1 + w_{12}w_{21})^2 \tag{7.14}$$

where $E\{(.)\}$ denotes the expected value of (.).

7.2.4 Optimum Weights

The optimal weight vector; $\mathbf{w}_{\text{opt}} = [w_{12\text{opt}}, w_{21\text{opt}}]^T$ which minimizes the mean output powers P and Q are found by taking the derivative of P and Q with respect to w_{12} and w_{21} and equating the result to zero, respectively.

From (7.13) and (7.14), we get respectively,

$$\frac{\partial P(w_{12},w_{21})}{\partial w_{12}} = 2\Big[\delta_{11}E\{I_1^2(n)\}\Big((1+w_{12}w_{21})^2(a_{11}+w_{12}a_{21})a_{21}$$

$$+(1+w_{12}w_{21})(a_{11}+w_{12}a_{21})^2w_{21}\Big)$$

$$+\delta_{12}E\{I_2^2(n)\}\Big((1+w_{12}w_{21})^2(a_{12}+w_{12}a_{22})a_{22}$$

$$+(1+w_{12}w_{21})(a_{12}+w_{12}a_{22})^2w_{21}\Big) + E\{n_1^2(n)\}(1+w_{12}w_{21})w_{21}$$

$$+E\{n_2^2(n)\}\Big((1+w_{12}w_{21})^2w_{12}+(1+w_{12}w_{21})w_{12}^2w_{21}\Big)\Big], \quad (7.15)$$

and,

$$\frac{\partial Q(w_{12},w_{21})}{\partial w_{21}} = 2\Big[\delta_{21}E\{I_1^2(n)\}\Big((1+w_{12}w_{21})^2(a_{21}+w_{21}a_{11})a_{11}$$

$$+(1+w_{12}w_{21})(a_{21}+w_{21}a_{11})^2w_{12}\Big)$$

$$+\delta_{22}E\{I_2^2(n)\}\Big((1+w_{12}w_{21})^2(a_{22}+w_{21}a_{12})a_{12}$$

$$+(1+w_{12}w_{21})(a_{22}+w_{21}a_{12})^2w_{12}\Big) + E\{n_1^2(n)\}(1+w_{12}w_{21})w_{12}$$

$$+E\{n_2^2(n)\}\Big((1+w_{12}w_{21})^2w_{21}+(1+w_{12}w_{21})w_{21}^2w_{12}\Big)\Big] \quad (7.16)$$

7.3 Gradient Descent

An alternative way to get to the optimum weights is to use gradient descent method. With this technique, the weight vector at time i+1 is computed by using the gradients (7.15) and (7.16) according to the following recursive relation,

$$w_{12}^{i+1} = w_{12}^{i} - \mu_1 \frac{\partial}{\partial w_{12}^{i}} E\{y_{1d}^{i}(n)^2\} \tag{7.17}$$

$$w_{21}^{i+1} = w_{21}^{i} - \mu_2 \frac{\partial}{\partial w_{21}^{i}} E\{y_{2d}^{i}(n)^2\} \tag{7.18}$$

7.3.1 Gradient Estimation Using Perturbation Sequences

The use of random search and weight perturbation techniques for gradient estimation in adaptive systems have been reported by many authors [21], [22]. In this section , we will follow Cantoni's gradient estimation definition [22], in which the estimate gradient is obtained by perturbing the weights (different sequence for each weight) simultaneously with different perturbation sequences and correlating the outputs with the same sequences.

That is with this procedure, one can find the gradients of P and Q with respect to w_{12} and w_{21} , respectively by perturbing the weights with mutually orthogonal zero mean sequences and then correlating the corresponding instantaneous output powers with the corresponding sequences.

Therefore, we can obtain estimates of the gradients of the output from;

$$g_l(i) = \frac{1}{N\Lambda} \sum_{m=iN+1}^{(i+1)N} y_{ld}^2(m)\, p_l(m) \quad l = 1, 2 \tag{7.19}$$

where $y_{1d}^2(m)$ and $y_{2d}^2(m)$ are the instantaneous output powers P and Q, respectively. $g_1(i)$ and $g_2(i)$ are taken to be the estimates of the true gradients given in (7.15) and (7.16), respectively.

7.3.2 Properties of the Orthogonal Perturbation Sequences

The desired sequence to be used in the estimation of the gradient will be mutually orthogonal and zero mean over N cycle. For normalization, we divide by the cycle period N to obtain unit average power. Such sequence will satisfy the following,

$$\frac{1}{N}\sum_{n=1}^{N} p_k(n)p_l(n) = \begin{cases} 1 & k = l \\ 0 & k \neq l \end{cases} \tag{7.20}$$

$$\frac{1}{N}\sum_{n=1}^{N} p_l(n) = 0 \quad l = 1, 2, ... \tag{7.21}$$

and for higher moments

$$\frac{1}{N}\sum_{n=1}^{N} p_k^x(n)p_l^\nu(n) = \begin{cases} 1 & k = l \text{ if both } \nu \text{ and } x \text{ are even} \\ 0 & k \neq l \text{ if at least one of them is odd} \end{cases} \tag{7.22}$$

where p_i is a periodic perturbation sequence with period N and perturbation size Λ. Λ is a positive real constant and $\Lambda << 1$. The sequence can be selected to yield an unbiased gradient estimate (for example, the rows of Hadamard matrix [22]) .

7.3.3 Weight Updating Using the Gradient Estimates

The weight vector is updated at time i by using the estimated gradients $g_1(i)$ and $g_2(i)$ in the following recursive relation,

$$w_{12}(i+1) = w_{12}(i) - \mu g_1(i) \tag{7.23}$$

$$w_{21}(i+1) = w_{21}(i) - \mu g_2(i) \quad i = 1, 2, ... \tag{7.24}$$

$$w_{12}(i+1) = \text{clip}(w_{12}(i+1)) \tag{7.25}$$

$$w_{21}(i+1) = \text{clip}(w_{21}(i+1)) \tag{7.26}$$

where, clip$(\cdot)$ is a clipping function such that

$$\text{clip}(w) = \begin{cases} w & |w| \leq \alpha \\ \dfrac{w\alpha}{|w|} & |w| > \alpha \end{cases} \tag{7.27}$$

and μ is the constant which determines the stability of convergence. The clipping operation ensures that each weight is bounded by a constant α to be less than one for the desired equilibrium point.

7.3.4 Gradient Estimates, $g_1(i)$ and $g_2(i)$

From (7.19), we write the expected value of gradient *estimates* conditioned on the weights,

$$E\{g_l(i)|\mathbf{w}(i)\} = \frac{1}{N\Lambda} \sum_{m=iN+1}^{(i+1)N} E\{y_{ld}^2(m)|\mathbf{w}(i)\}\ p_l(m) \quad l = 1,2 \tag{7.28}$$

Substituting (7.11) in (7.28), and taking the expectation of both sides conditioned on the weight vector $\mathbf{w}(i)$, we get

$$\begin{aligned} E\{g_1(i)|\mathbf{w}(i)\} &= \frac{1}{N\Lambda} \sum_{m=iN+1}^{(i+1)N} \Big[\delta_{11} E\{I_1^2(m)\}(a_{11} + w_{12}(m)a_{21})^2 \\ &\quad + \delta_{12} E\{I_2^2(m)\}(a_{12} + w_{12}(m)a_{22})^2 + E\{n_1(m)^2\} \\ &\quad + E\{n_2^2(m)\} w_{12}^2(m)\Big] [1 + w_{12}(m)w_{21}(m)]^2 p_1(m) \end{aligned} \tag{7.29}$$

and

$$\begin{aligned} E\{g_2(i)|\mathbf{w}(i)\} &= \frac{1}{N\Lambda} \sum_{m=iN+1}^{(i+1)N} \Big[\delta_{21} E\{I_1^2(m)\}(a_{21} + w_{21}(m)a_{11})^2 \\ &\quad + \delta_{22} E\{I_2^2(m)\}(a_{22} + w_{21}(m)a_{12})^2 + E\{n_2(m)^2\} \end{aligned}$$

$$+E\{n_1^2(m)\}w_{21}^2(m)\Big][1+w_{12}(m)w_{21}(m)]^2 p_2(m) \quad (7.30)$$

where $w_{12}(m)$ and $w_{21}(m)$ are the perturbed weights and related to the nominal values $w_{12}(i)$ and $w_{21}(i)$ by equation (7.8) and (7.9). Using these relation and the orthogonal properties of the perturbation sequence stated in (7.20) to (7.22), we find in appendix D,

$$\begin{aligned} E\{g_1(i)|\mathbf{w}(i)\} \;=\; & 2\Big[\delta_{11}E\{I_1(i)^2\}\Big([1+w_{12}(i)w_{21}(i)]^2[a_{11}+w_{12}(i)a_{21}]a_{21} \\ & +[1+w_{12}(i)w_{21}(i)][a_{11}+w_{12}(i)a_{21}(i)]^2 w_{21}(i)\Big) \\ & +\delta_{12}E\{I_2(i)^2\}\Big([1+w_{12}(i)w_{21}(i)]^2[a_{12}+w_{12}(i)a_{22}]a_{22} \\ & +[1+w_{12}(i)w_{21}(i)][a_{12}+w_{12}(i)a_{22}]^2 w_{21}(i)\Big) \\ & +E\{n_1(i)^2\}[1+w_{12}(i)w_{21}(i)]w_{21}(i) \\ & +E\{n_2(i)^2\}\Big([1+w_{12}(i)w_{21}(i)]^2 w_{12}(i) \\ & +[1+w_{12}(i)w_{21}(i)]w_{12}^2(i)w_{21}(i)\Big)\Big], \quad (7.31) \end{aligned}$$

Similar evaluation as well as exploiting symmetry properties between $y_{1d}(m)$ and $y_{2d}(m)$, we get for the expected value of gradient estimator $g_2(i)$ conditioned on $\mathbf{w}(i)$;

$$E\{g_2(i)|\mathbf{w}(i)\} \;=\; 2\Big[\delta_{21}E\{I_1(i)^2\}\Big([1+w_{12}(i)w_{21}(i)]^2[a_{21}+w_{21}(i)a_{11}]a_{11}$$

$$+[1 + w_{12}(i)w_{21}(i)][a_{21} + w_{21}(i)a_{11}]^2 w_{12}(i)\Big)$$

$$+\delta_{22}E\{I_2(i)^2\}\Big([1 + w_{12}(i)w_{21}(i)]^2[a_{22} + w_{21}(i)a_{12}]a_{12}$$

$$+[1 + w_{12}(i)w_{21}(i)][a_{22} + w_{21}(i)a_{12}]^2 w_{12}(i)\Big)$$

$$+E\{n_1(i)^2\}[1 + w_{12}(i)w_{21}(i)]w_{12}(i)$$

$$+E\{n_2(i)^2\}\Big([1 + w_{12}(i)w_{21}(i)]^2 w_{21}(i)$$

$$+[1 + w_{12}(i)w_{21}(i)]w_{21}^2(i)w_{12}(i)\Big)\Big] \qquad (7.32)$$

7.4 Convergence in the Mean

In this section, we will investigate the convergence of the recursive relation in (7.23) (7.24) to the optimum weights in the mean. We will also find an upper bound for the step size μ to satisfy the mean weight convergence for the stability of the recursive weight updating algorithm.

7.4.1 The Error's Mean

Convergence in the mean of weight vector; $\mathbf{w}(i) = [w_{12}(i), w_{21}(i)]^T$ to optimum $\mathbf{w}_{\text{opt}} = [w_{12\text{opt}}, w_{21\text{opt}}]^T$, means

$$\lim_{i \to \infty} E\{\mathbf{w}(i)\} = \mathbf{w}_{\text{opt}} \qquad (7.33)$$

We begin our analysis by defining a weight error vector at time i as $\mathbf{e}(i) = [e_1(i), e_2(i)]^T$, with

$$e_1(i) = w_{12}(i) - w_{12\text{opt}} \quad (7.34)$$

$$e_2(i) = w_{21}(i) - w_{21\text{opt}} \quad (7.35)$$

We know that at the optimum weights, the gradients of the mean output powers P and Q are equal to zero, i.e

$$\frac{\partial P(w_{12}, w_{21})}{\partial w_{12}}|_{\mathbf{w}_{\text{opt}}} = 0 \quad (7.36)$$

$$\frac{\partial Q(w_{12}, w_{21})}{\partial w_{12}}|_{\mathbf{w}_{\text{opt}}} = 0 \quad (7.37)$$

Subtracting the optimum weights $w_{12\text{opt}}$ and $w_{21\text{opt}}$ from both sides of (7.23) and (7.24), respectively. and the gradients in (7.15) and (7.16) from the estimate of the gradients $g_1(i)$ and $g_2(i)$ in these equation respectively, we can write,

$$w_{12}(i+1) - w_{12\text{opt}} = w_{12}(i) - w_{12\text{opt}} - \mu\Big[g_1(i) - \frac{\partial P(w_{12}, w_{21})}{\partial w_{12}}|_{\mathbf{w}_{\text{opt}}}\Big] \quad (7.38)$$

$$w_{21}(i+1) - w_{21\text{opt}} = w_{21}(i) - w_{21\text{opt}} - \mu\Big[g_2(i) - \frac{\partial Q(w_{12}, w_{21})}{\partial w_{21}}|_{\mathbf{w}_{\text{opt}}}\Big] \quad (7.39)$$

By using the definition in (7.34) and (7.35) in (7.38) and (7.39), respectively, we write,

$$e_1(i+1) = e_1(i) - \mu\Big[g_1(i) - \frac{\partial P(w_{12}, w_{21})}{\partial w_{12}}|_{\mathbf{w}_{\text{opt}}}\Big] \quad (7.40)$$

$$e_2(i+1) = e_2(i) - \mu\Big[g_2(i) - \frac{\partial Q(w_{12}, w_{21})}{\partial w_{21}}|_{\mathbf{w}_{\text{opt}}}\Big] \quad (7.41)$$

In order to investigate the convergence in the mean, we take the expectation of (7.40) and (7.41) conditioned on the weight vector, $\mathbf{w}(i)$. We get,

$$E\{e_1(i+1)|\mathbf{w}(i)\} = E\{e_1(i)|\mathbf{w}(i)\} - \mu\Big[E\{g_1(i)|\mathbf{w}(i)\} - E\{\frac{\partial P(w_{12}, w_{21})}{\partial w_{12}}|_{\mathbf{w}_{\text{opt}}}|\mathbf{w}(i)\}\Big] \quad (7.42)$$

$$E\{e_2(i+1)|\mathbf{w}(i)\} = E\{e_2(i)|\mathbf{w}(i)\} - \mu\Big[E\{g_2(i)|\mathbf{w}(i)\} - E\{\frac{\partial Q(w_{12}, w_{21})}{\partial w_{21}}|_{\mathbf{w}_{\text{opt}}}|\mathbf{w}(i)\}\Big] \quad (7.43)$$

7.4.2 Approximate Terms for the True and Estimate Gradients

In the case when the cross coupling constants $|\frac{a_{12}}{a_{22}}|$ and $|\frac{a_{21}}{a_{11}}|$ are -10 to -15 dB then $w_{12}w_{21} << 1$, and we can approximate the true gradients from (7.15) and (7.16), by;

$$\frac{\partial E\{y_{1d}(n)^2\}}{\partial w_{21}}|_{\mathbf{w}_{\text{opt}}} \approx 2\Big[\delta_{11}E\{I_1(n)^2\}[(a_{11} + w_{12\text{opt}}a_{21})a_{21} + a_{11}^2 w_{21\text{opt}}] + \delta_{12}E\{I_2(n)^2\}[(a_{12} + a_{22}w_{12\text{opt}})a_{22} + a_{12}^2 w_{21\text{opt}}] + E\{n_1(n)^2\}w_{21\text{opt}} + E\{n_1(n)^2\}w_{12\text{opt}}\Big] \quad (7.44)$$

and

$$\frac{\partial E\{y_{2d}(n)^2\}}{\partial w_{21}}|_{\mathbf{w}_{\text{opt}}} \approx 2\Big[\delta_{21}E\{I_1(n)^2\}[(a_{21} + a_{11}w_{21\text{opt}})a_{11} + a_{21}^2 w_{12\text{opt}}] + \delta_{22}E\{I_2(n)^2\}[(a_{22} + a_{12}w_{21\text{opt}})a_{12} + a_{22}^2 w_{12\text{opt}}] + E\{n_1(n)^2\}w_{21\text{opt}} + E\{n_1(n)^2\}w_{12\text{opt}}\Big] \quad (7.45)$$

Similarly, under the assumption that $E\{I_k^2(n)\} = E\{I_k^2(i)\}$ and $E\{n_k^2(n)\} = E\{n_k^2(i)\}$ $k = 1,2$, then, the estimated gradients can be approximated from (7.29), (7.30) to get

$$
\begin{aligned}
E\{g_1(i)|\mathbf{w}(i)\} \approx\ & 2\Big[\delta_{11}E\{I_1(n)^2\}[(a_{11}+a_{21}w_{12}(i))a_{21}+a_{11}^2w_{21}(i)] \\
& +\delta_{12}E\{I_2(n)^2\}[(a_{12}+a_{22}w_{12}(i))a_{22}+a_{12}^2w_{21}(i)] \\
& +E\{n_2(n)^2\}w_{21}(i)+E\{n_1(n)^2\}w_{12}(i)\Big] \qquad (7.46)
\end{aligned}
$$

and

$$
\begin{aligned}
E\{g_2(i)|\mathbf{w}(i)\} \approx\ & 2\Big[\delta_{21}E\{I_1(n)^2\}[(a_{21}+a_{11}w_{21}(i))a_{11}+a_{21}^2w_{12}]+\delta_{22}E\{I_2(n)^2\} \\
& [(a_{22}+a_{12}w_{21}(i))a_{12}+a_{22}^2w_{12}(i)] \\
& +E\{n_2(n)^2\}w_{21}(i)+E\{n_1(n)^2\}w_{12}(i)\Big] \qquad (7.47)
\end{aligned}
$$

Subtracting the true gradients in (7.44) and (7.45) from the estimate gradients in (7.46) and (7.47) , we get respectively,

$$
\begin{aligned}
E\{g_1(i)|\mathbf{w}(i)\} - \frac{\partial E\{y_{1d}(n)^2\}}{\partial w_{12}}|_{\mathbf{w}_{\text{opt}}} = 2\Big[\delta_{11}E\{I_1(n)^2\}a_{11}^2[w_{21}(i)-w_{21\text{opt}}] \\
\delta_{11}E\{I_1(n)^2\}a_{21}^2[w_{12}(i)-w_{12\text{opt}}]+\delta_{12}E\{I_2(n)^2\}a_{22}^2[w_{12}(i)-w_{12\text{opt}}] \\
+\delta_{12}E\{I_2(n)^2\}a_{12}^2[w_{21}(i)-w_{21\text{opt}}]+E\{n_1(n)^2\}[w_{21}(i)-w_{21\text{opt}}] \\
+E\{n_2(n)^2\}[w_{12}(i)-w_{12\text{opt}}]\Big] \quad (7.48)
\end{aligned}
$$

and also

$$
E\{g_2(i)|\mathbf{w}(i)\} - \frac{\partial E\{y_{2d}(n)^2\}}{\partial w_{21}}|_{\mathbf{w}_{\text{opt}}} = 2\Big[\delta_{21}E\{I_1(n)^2\}a_{11}^2[w_{21}(i)-w_{21\text{opt}}]
$$

$$\delta_{21}E\{I_1(n)^2\}a_{21}^2[w_{12}(i)-w_{12\text{opt}}]+\delta_{22}E\{I_2(n)^2\}a_{22}^2[w_{12}(i)-w_{12\text{opt}}]$$
$$+\delta_{22}E\{I_2(n)^2\}a_{12}^2[w_{21}(i)-w_{21\text{opt}}]+E\{n_1(n)^2\}[w_{21}(i)-w_{21\text{opt}}]$$
$$\left.+E\{n_2(n)^2\}[w_{12}(i)-w_{12\text{opt}}]\right] \quad (7.49)$$

Using (7.48) , (7.49) in (7.42) and (7.43) respectively, , we have

$$\begin{aligned} E\{e_1(i+1)|\mathbf{w}(i)\} &= (1-\mu a)E\{e_1(i)|\mathbf{w}(i)\} \\ &\quad -\mu bE\{e_2(i)|\mathbf{w}(i)\} \end{aligned} \quad (7.50)$$

$$\begin{aligned} E\{e_2(i+1)|\mathbf{w}(i)\} &= -\mu cE\{e_1(i)|\mathbf{w}(i)\} \\ &\quad +(1-\mu d)E\{e_2(i)|\mathbf{w}(i)\} \end{aligned} \quad (7.51)$$

where;

$$a = 2[\delta_{12}E\{I_2(n)^2\}a_{22}^2+\delta_{11}E\{I_1(n)^2\}a_{21}^2+E\{n_2(n)^2\}] \quad (7.52)$$

$$b = 2[\delta_{11}E\{I_1(n)^2\}a_{11}^2+\delta_{12}E\{I_2(n)^2\}a_{12}^2+E\{n_1(n)^2\}] \quad (7.53)$$

$$c = 2[\delta_{22}E\{I_2(n)^2\}a_{22}^2+\delta_{21}E\{I_1(n)^2\}a_{21}^2+E\{n_2(n)^2\}] \quad (7.54)$$

$$d = 2[\delta_{21}E\{I_1(n)^2\}a_{11}^2+\delta_{22}E\{I_2(n)^2\}a_{12}^2+E\{n_1(n)^2\}] \quad (7.55)$$

In matrix notation, we can write

$$E\{\mathbf{e}(i+1)|\mathbf{w}(i)\} = (\mathbf{I}-\mu\mathbf{A})E\{\mathbf{e}(i)|\mathbf{w}(i)\} \quad (7.56)$$

where **I** and **A** are the identity and the weight error matrices, respectively.

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad (7.57)$$

Taking the expected value of (7.56) over the weights, $\mathbf{w}(i)$ we can write ,

$$E\{\mathbf{e}(i+1)\} = (\mathbf{I}-\mu\mathbf{A})E\{\mathbf{e}(i)\} \quad (7.58)$$

If $||\mathbf{I} - \mu\mathbf{A}|| < 1$ then $\lim_{i\to\infty} E\{\mathbf{e}(i)\} \to 0$. Therefore, we can establish an upper bound for the convergence constant μ

$$0 < \mu < \frac{1}{\lambda_{\text{max}}} \tag{7.59}$$

where λ_{max} is the maximum eigenvalue of the weight error matrix $\mathbf{A}$.

7.5 Results

Using computer, we simulated nondispersive fading channel and employed power-power canceler to eliminate the effect of cross-pol interference. Perturbation sequences used in the computer simulation are chosen from the rows of Hadamard matrix (i.e $p_1 = [1, -1, 1, -1, ..]$ and $p_2 = [1, 1, -1, -1, 1, 1, ..]$).

The block diagram of power-power canceler using perturbation sequences in the control algorithm is given in Fig. 7.1. We applied two independent uniformly distributed bipolar data to the nondispersive channel. Then, corrupted data is applied to the canceler. In Fig. 7.2, the interference power residue versus data sample is given for -14 dB cross-pol interference with perturbation length N=8 and different perturbation magnitudes Λ. Same experiment is done for different perturbation sequence sizes and depicted in Fig. 7.3. The results depicted in these figures aforementioned are the average of four random experiments.

7.6 Conclusion

In this chapter, we studied the dynamic analysis of power-power canceler by using orthogonal perturbation sequences in the control algorithm. The results of the computer analysis shows that as the perturbation magnitude is reduced, the interference power residue decreases. Also, as the perturbation sequence length is increased a smooth estimate of the gradient is obtained, but the convergence time takes longer, as expected.

We conclude perturbation sequences can be used effectively in cross-pol interference cancelation.

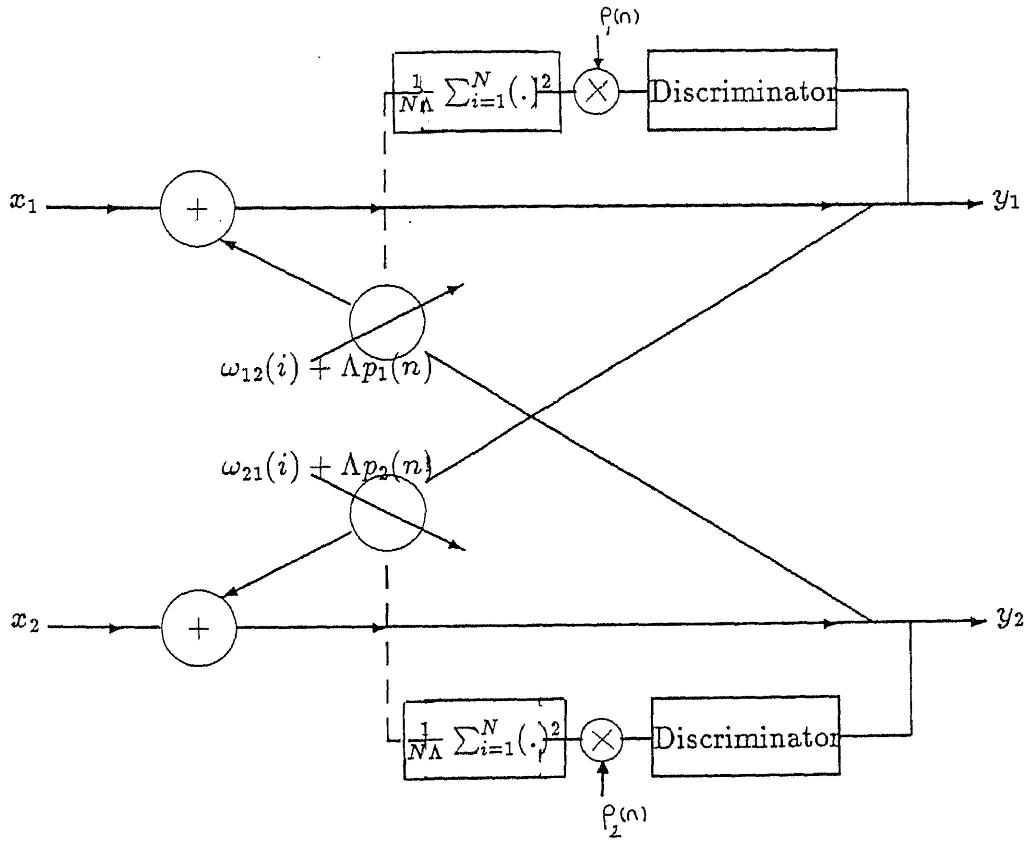

Figure 7.1: Power-Power Cross-Pol Interference Canceler controlled by orthogonal perturbation sequences

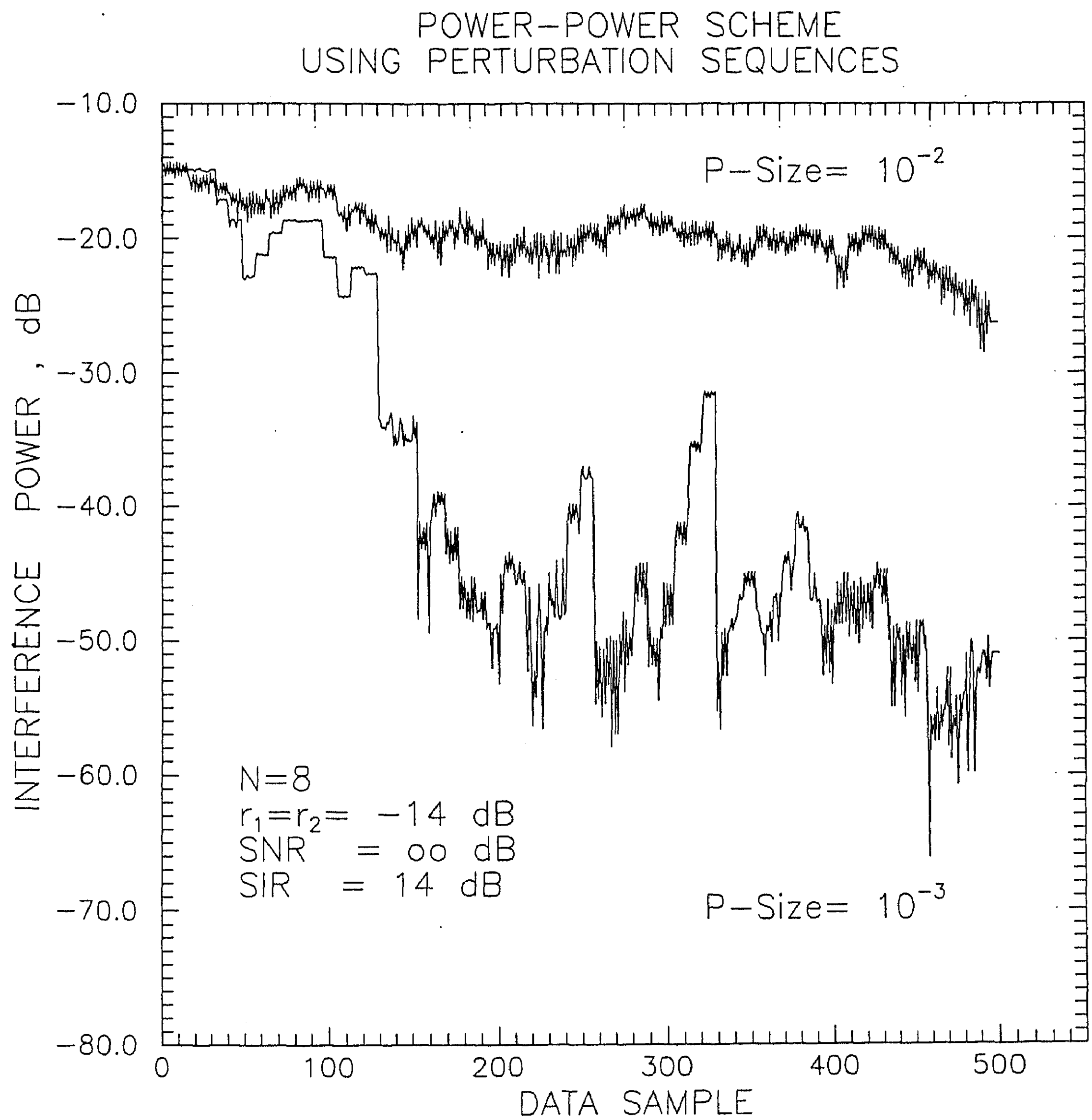

Figure 7.2: Effect of Different Perturbation magnitudes on convergence

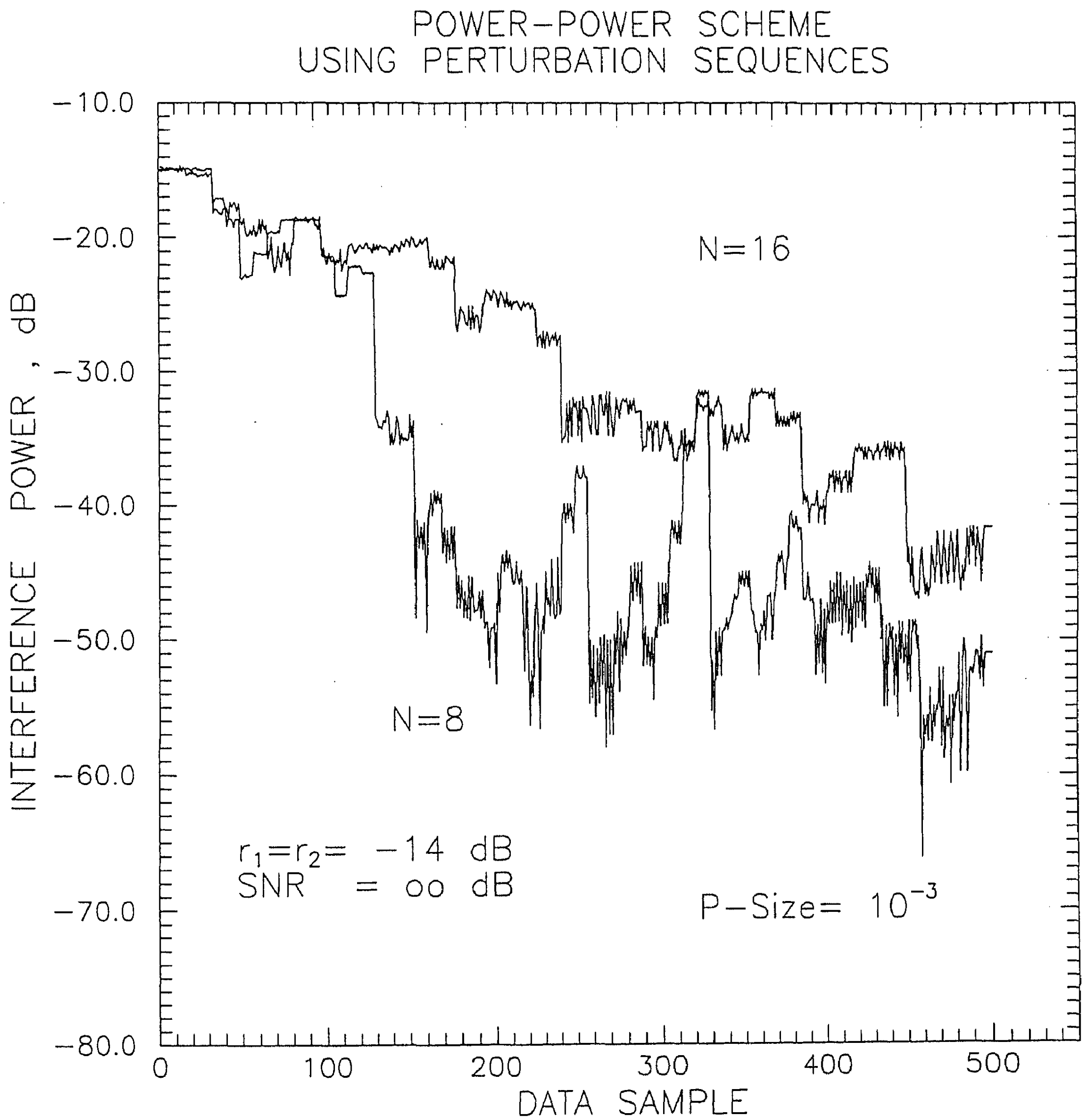

Figure 7.3: Comparison of Different Perturbation Lengths on convergence

Chapter 8

PERFORMANCE COMPARISON and CONCLUSIONS

We conclude our study by comparing the performance of the different bootstrapped cancelers, and then comparing the performances of these cancelers with those of the diagonalizer and LMS cancelers. The performance measure will be the symbol error probability.

It is important to emphasize that the error probability, although an important factor, is certainly not the only advantage of the bootstrapped canceler. We mention the following other points in favor of these cancelers:

1. Under the same system condition, the bootstrapped canceler steady state interference residue is smaller.

2. It is found under many practical conditions to be a faster algorithm.

3. To implement the bootstrapped algorithm one needs less complex hardware than for the diagonalizer, which needs a zero forcing algorithm, and for the LMS canceler, which needs decision feedback information. In fact, it is clear that adding a decision feedback to the bootstrap schemes will result in faster

convergence and still better performance than that which we obtained in the current analysis.

4. The fact that the bootstrapped cancelers do not need decision feedback makes them ideal for acquisition and, hence, suitable for channels with fast and deep fading which causes occasional system outage.

Before comparing the performances of the different bootstrapped schemes analysed in previous chapters, we observe the following: It was found that the three cancelers behave quite similarly when we implement both amplitude and phase compensation. This is not true when only amplitude compensation is applied to the outputs of the cancelers. Using amplitude compensation only is practical since it can be performed via simple AGC on the stronger, co-pol signal. Therefore, our comparisons are made for only this compensation.

In Fig. 8.1, for 16 QAM, we depict the Chernoff bound for the power-power canceler in comparison to that of the correlator-correlator canceler. Comparisons are made for $r = -15$ dB and $r = -10$ dB (we take $r_1 = r_2 = r$). From this figure, it is clear that the power-power canceler outperforms the correlator-correlator canceler, particularly, for higher cross coupling ($r = -10$ dB).

Fig 8.2 shows the same results when using the moment method. This figure does not depict the big difference we noticed with the Chernoff bound approach. Clearly, this indicates that the Chernoff bound approach is not tight for the case of the correlator-correlator canceler. Similar behavior is noticed in comparing Fig. 8.3, which depicts the Chernoff bound for 64 QAM, with Fig. 8.4, where the moment (or GQR) method is used for 64 QAM.

In Fig 8.5, we compare the performance of the power-power canceler to that of the LMS canceler, for 16 QAM and $r = -10$ dB. Fig 8.6 depicts the same comparison with $r = -15$ dB. To emphasize the need for cancelers in dual polarized systems, we add, to these two curves, the error performance without cancelers.

Figs. 8.7 and 8.8 are the same as Figs. 8.5 and 8.6, except for the use of 64 QAM instead of 16 QAM. In the last four figures, the moment method was used. For each curve, the number of the moments are marked in parentheses.

The three cancelers, power-power canceler, LMS canceler and the diagonalizer, are compared in Figs. 8.9 and 8.10. A 16 QAM signal is assumed in these figures, with $r = -10$ dB and $r = -15$ dB, respectively. Although GQR calculation has been done for amplitude and phase compensated diagonalizer (see chapter 2), the GQR calculation has not been done for amplitude compensated diagonalizer. Therefore, the comparisons are based on the Chernoff bound.

Finally, we would like to remark on some suggested future work:

1. The bootstrapped canceler can be extended to multi-input, multi-output systems. As such, it has a potential for implementation as an algorithm for neural network control.

2. The issue of adding decision feedback and assessing its effect upon performance still needs to be examined.

3. More results need to be obtained for different compensation approaches so as to facilitate complexity versus performance comparisons.

4. The issue of dynamic performance with different perturbation sequences needs to be studied further.

5. It is well known that LMS cancelers depend heavily upon the relative power of the received signals and their SNR's, as reflected in the eigenvalues of the cor-

relation matrix of the inputs. An interesting question is whether this behavior is better or worse in the case of bootstrapped algorithms.

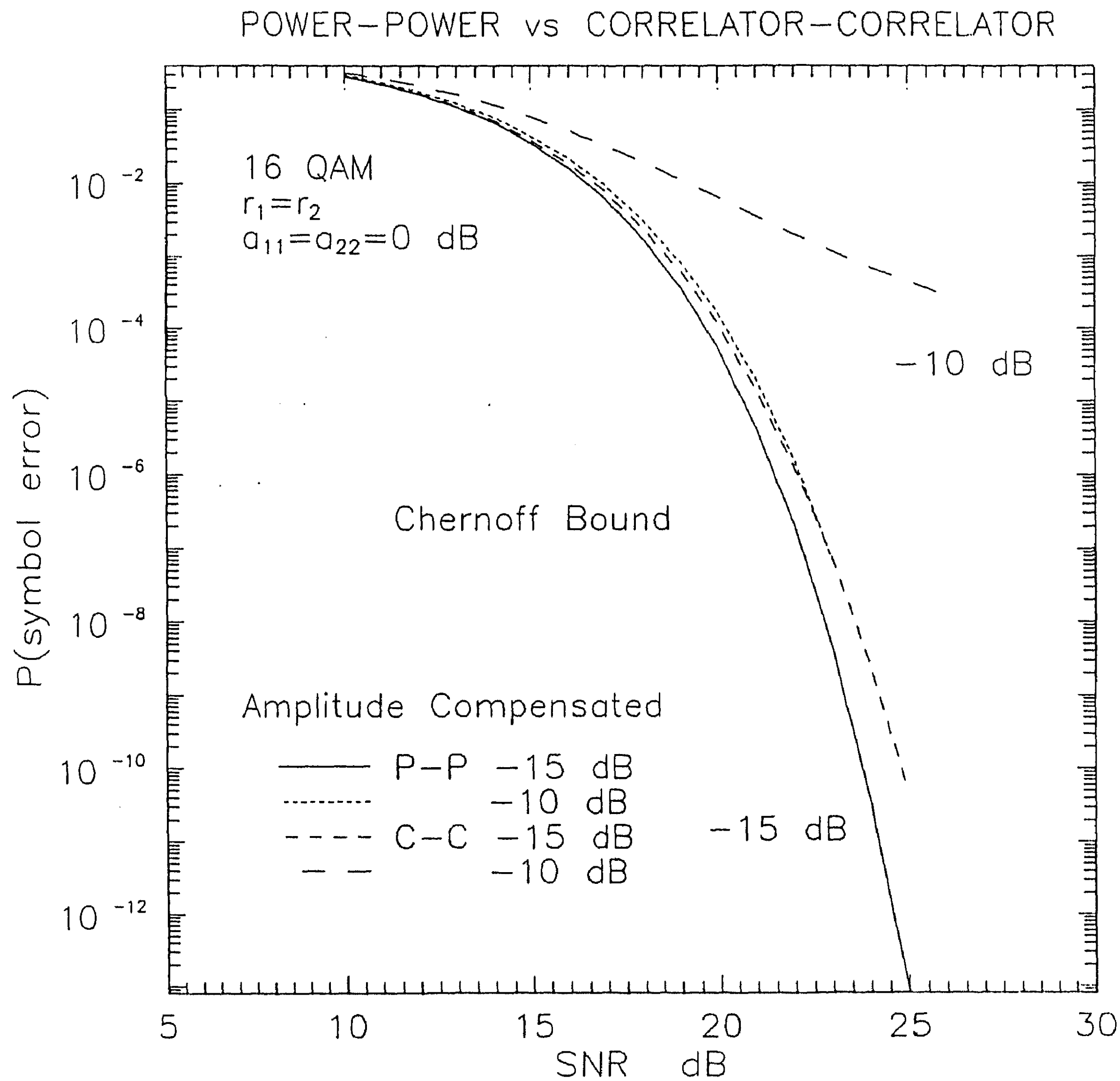

Figure 8.1: Performance Comparison of Power-Power with Correlator-Correlator cancelers, Chernoff bound, 16 QAM, with amplitude compensation, -15 dB, -10 dB

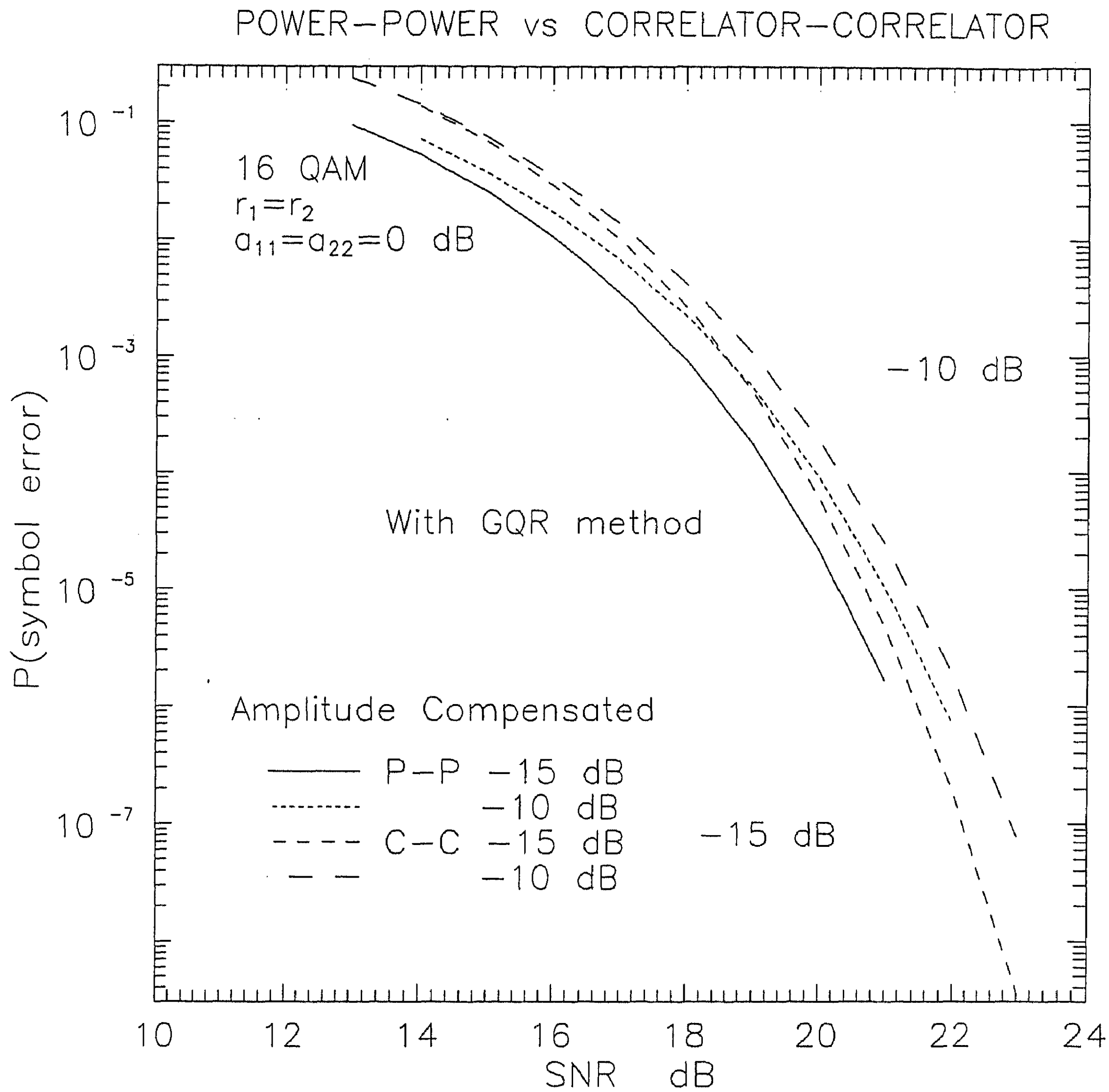

Figure 8.2: Performance Comparison of Power-Power with Correlator-Correlator cancelers, GQR calculation, 16 QAM, with amplitude compensation, -15 dB, -10 dB

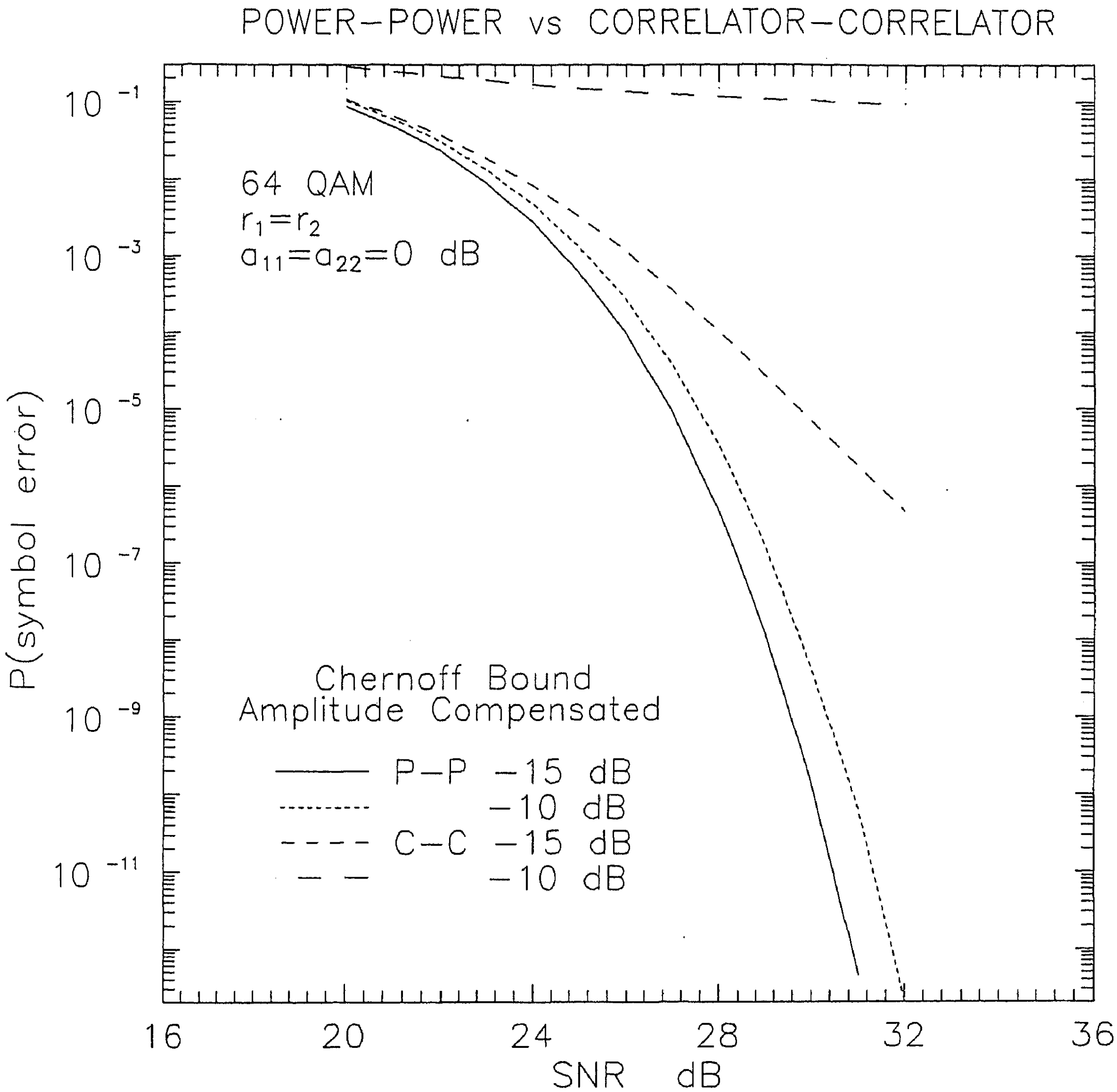

Figure 8.3: Performance Comparison of Power-Power with Correlator-Correlator cancelers, Chernoff bound, 64 QAM, with amplitude compensation, -15 dB, -10 dB

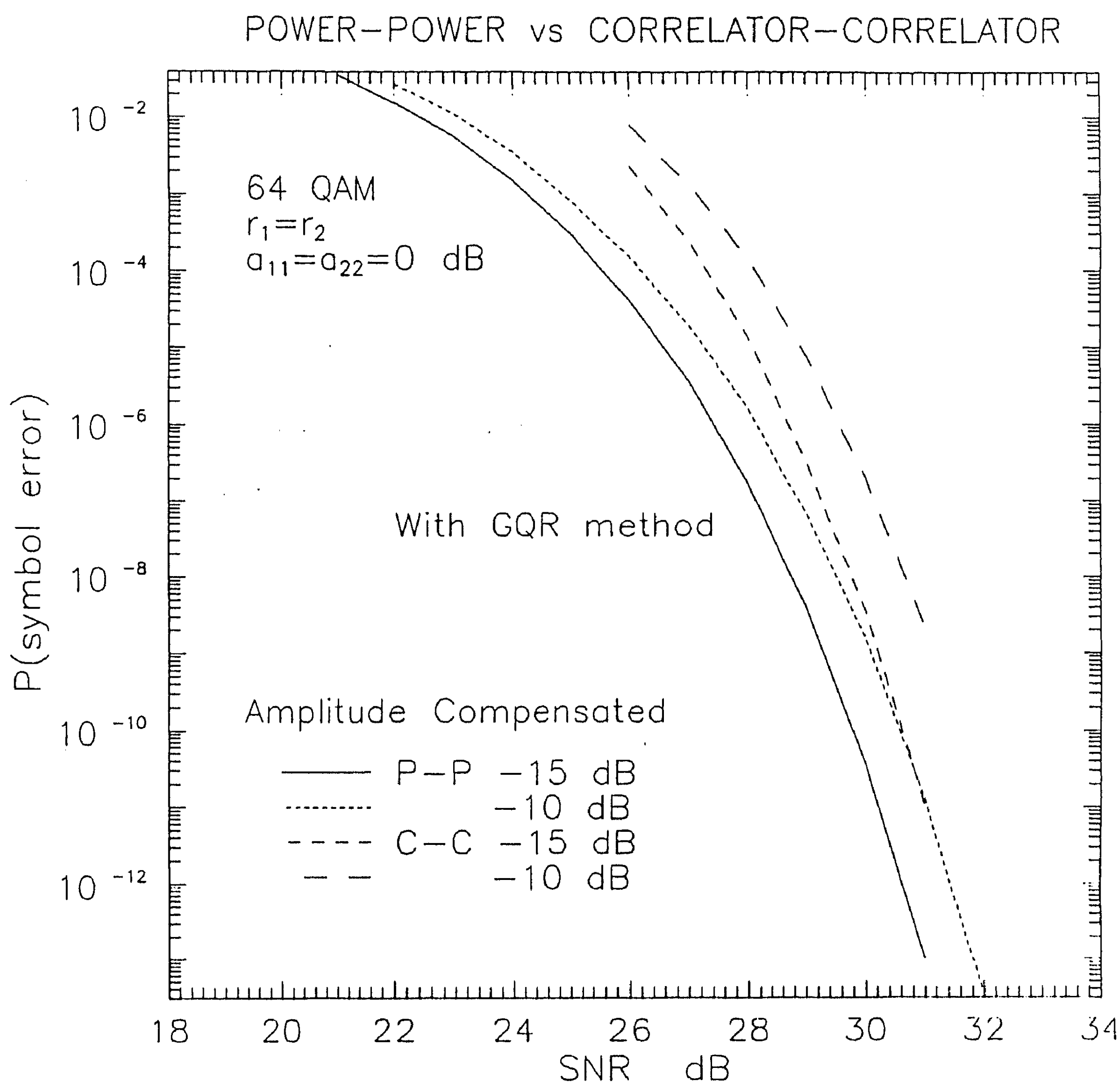

Figure 8.4: Performance Comparison of Power-Power with Correlator-Correlator cancelers, GQR calculation, 64 QAM, with amplitude compensation, -15 dB, -10 dB

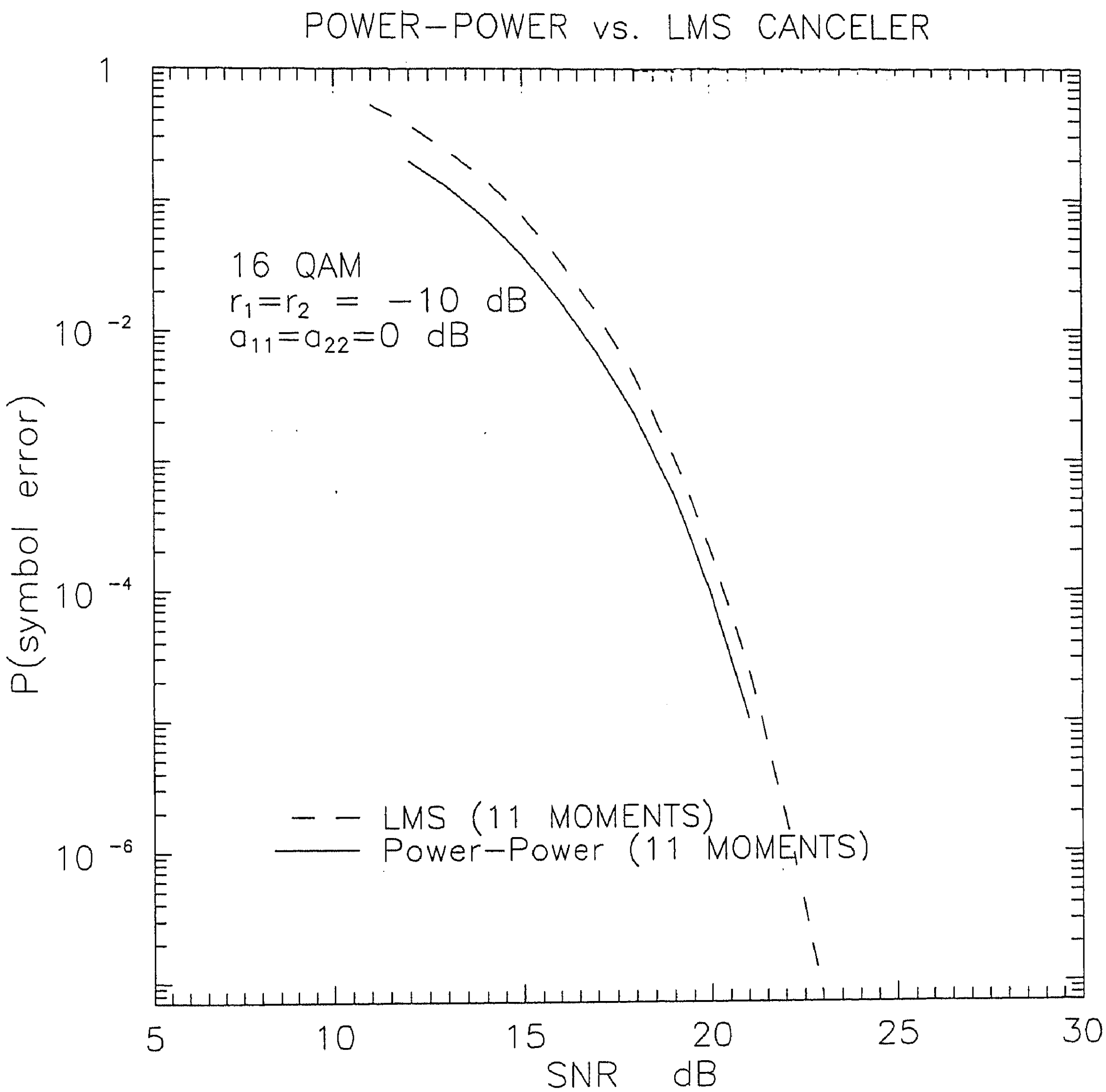

Figure 8.5: Performance Comparison of Power-Power with LMS cancelers, GQR calculation, 16 QAM, with amplitude compensation, -10 dB

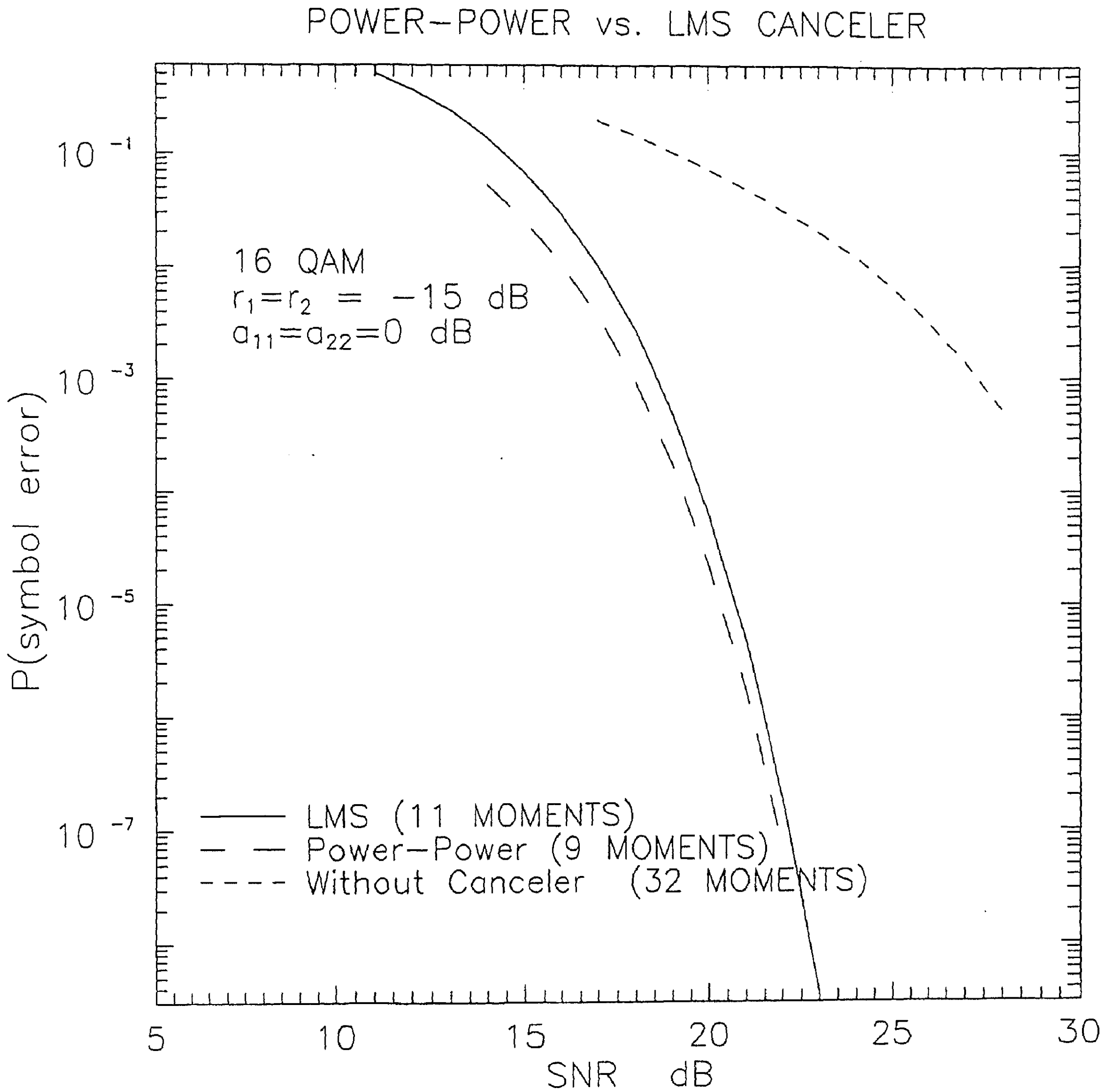

Figure 8.6: Performance Comparison of Power-Power with LMS cancelers, GQR calculation, 16 QAM, with amplitude compensation, -15 dB

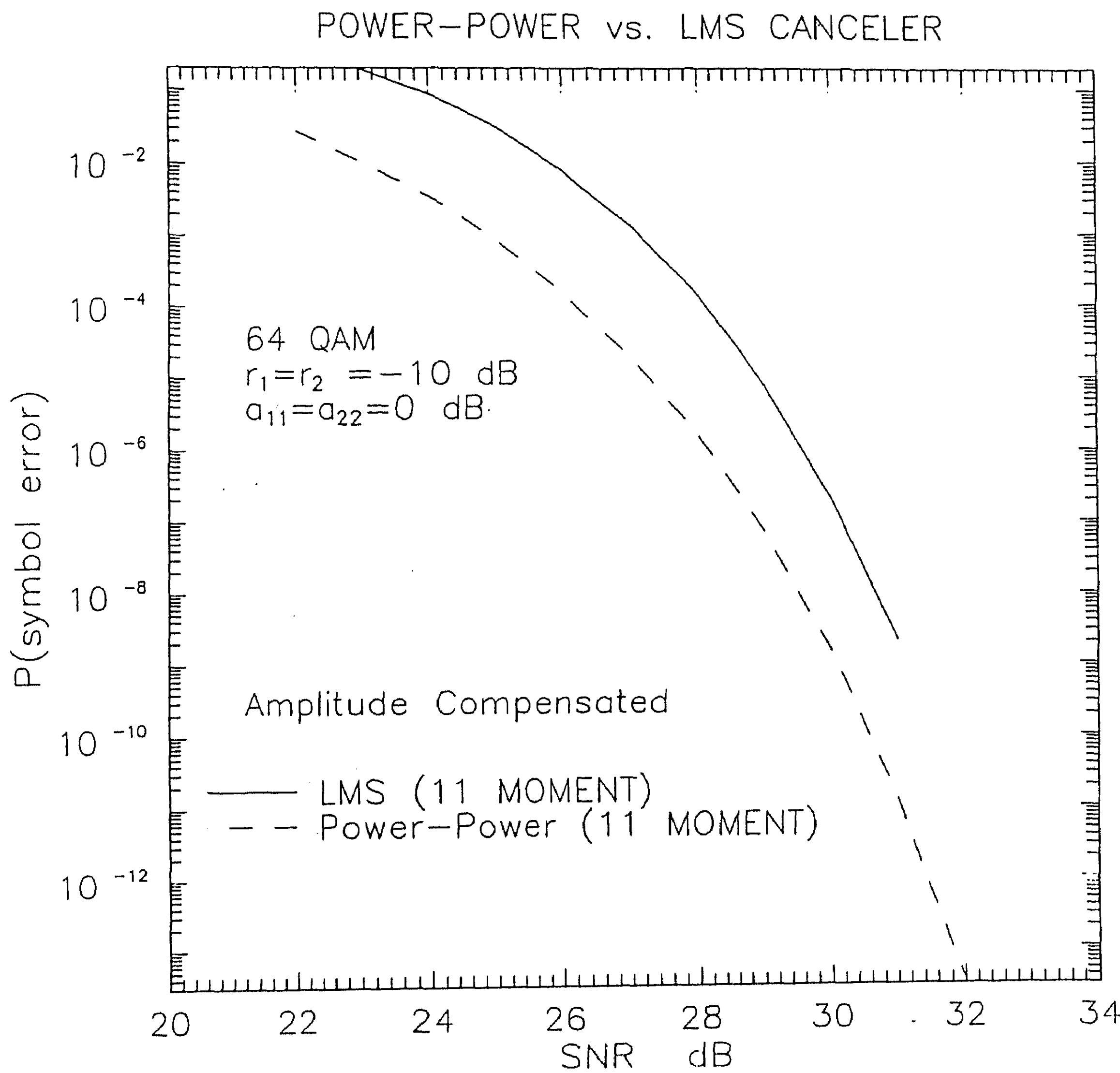

Figure 8.7: Performance Comparison of Power-Power with LMS cancelers, GQR calculation, 64 QAM, with amplitude compensation, -10 dB

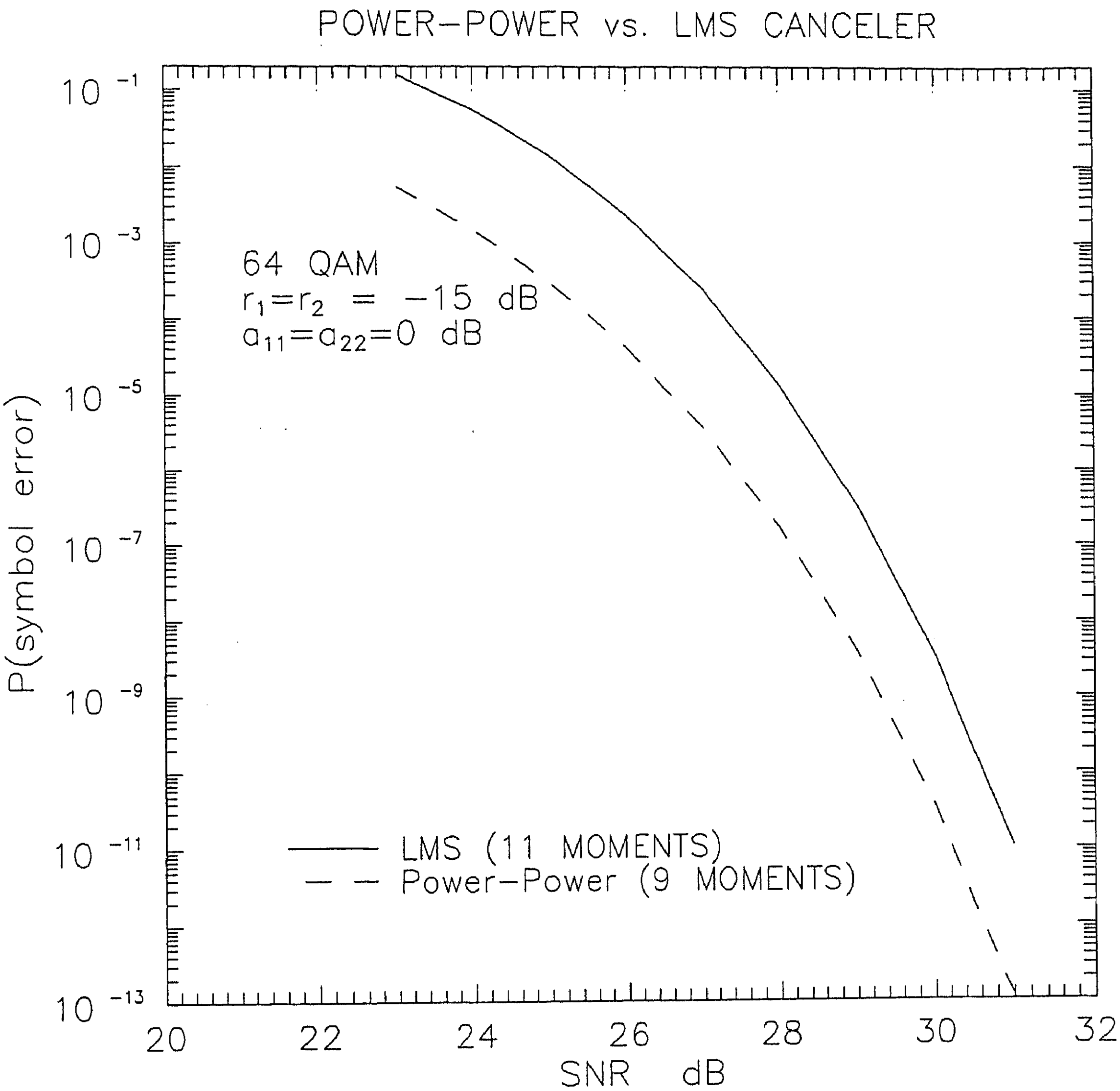

Figure 8.8: Performance Comparison of Power-Power with LMS cancelers, GQR calculation, 64 QAM, with amplitude compensation, -15 dB

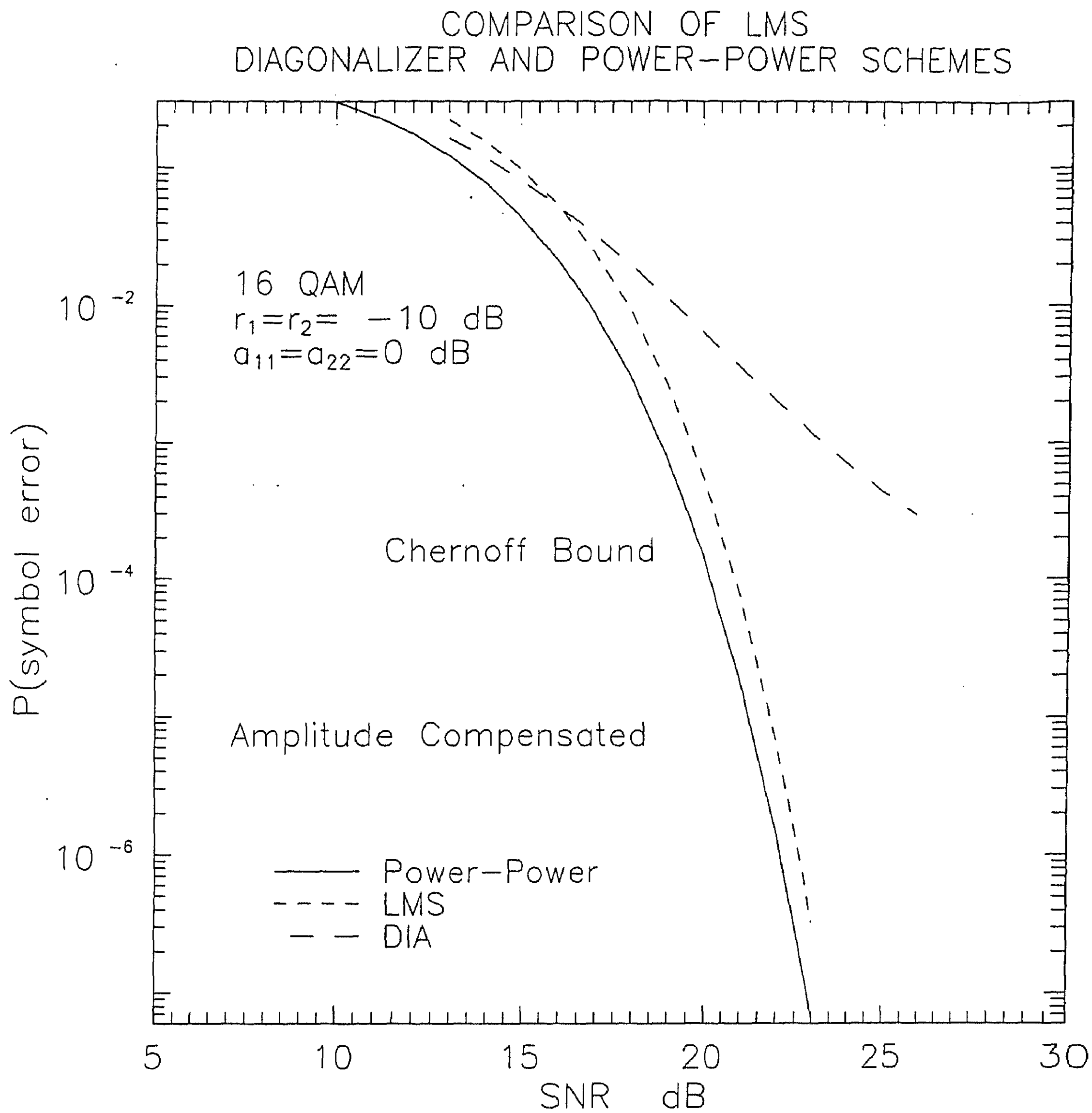

Figure 8.9: Performance Comparison of LMS, Diagonalizer and Power-Power cancelers, Chernoff bound, 16 QAM, with amplitude compensation, -10 dB

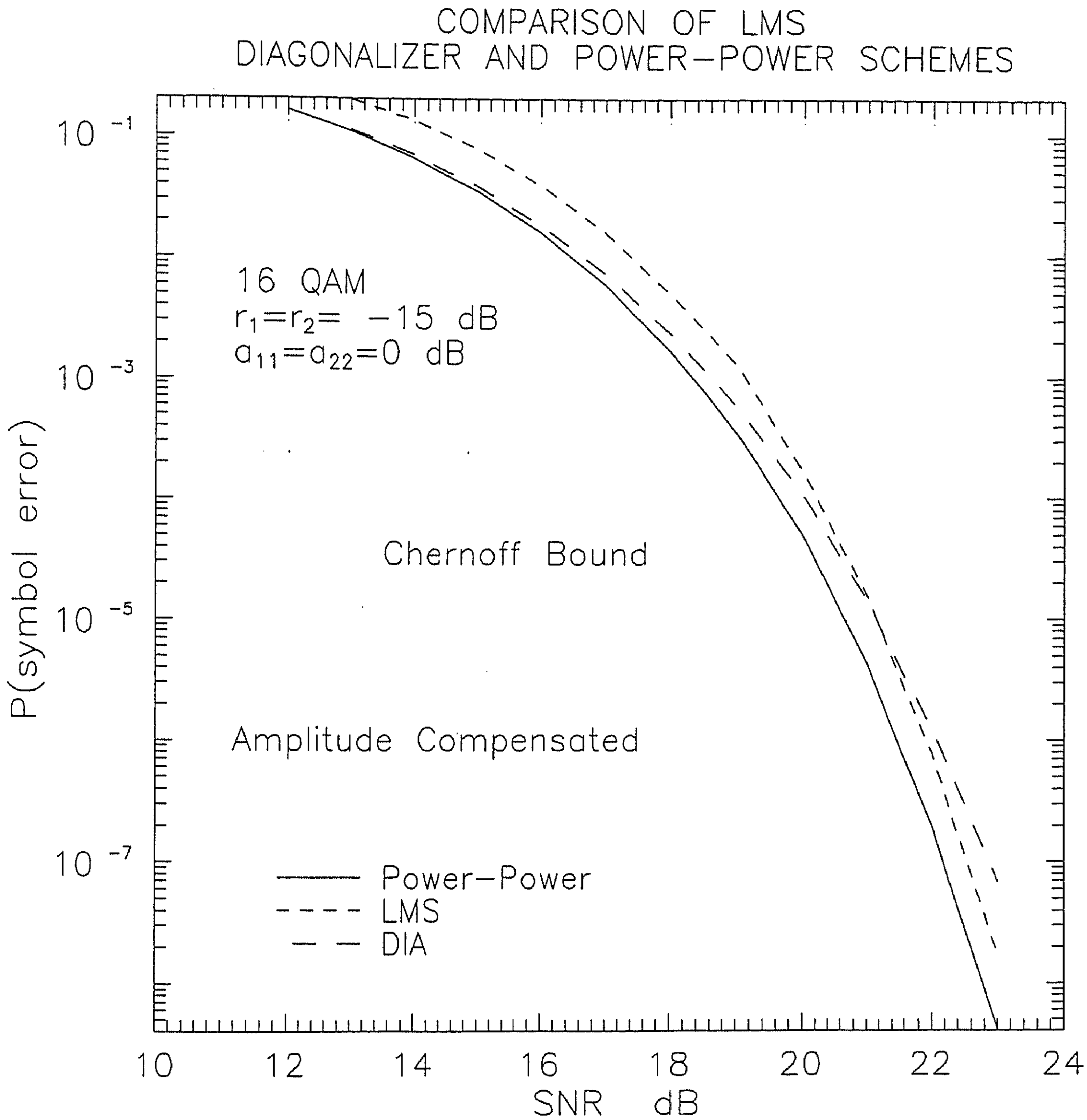

Figure 8.10: Performance Comparison of LMS, Diagonalizer and Power-Power cancelers, Chernoff bound, 16 QAM, with amplitude compensation, -15 dB

Appendix A

Derivation of Some Equations

•Derivation of (3.150)

For a discrete random variable **I** which takes the values $\{\pm 1c, \pm 3c, \ldots \pm(\sqrt{M}-1)c\}$ with equal probability $p(I) = \frac{1}{\sqrt{M}}$; we can write,

$$E\{\exp(a\mathbf{I})\} = \frac{1}{\sqrt{M}} \sum_{m=0}^{\sqrt{M}-1} e^{a(2m+1-\sqrt{M})c} \tag{A.1}$$

$$= \frac{1}{\sqrt{M}} e^{ac(1-\sqrt{M})} \sum_{m=0}^{\sqrt{M}-1} e^{2amc}$$

By using summation of geometric series, we get

$$E\{\exp(a\mathbf{I})\} = \frac{1}{\sqrt{M}} e^{ac(1-\sqrt{M})} \frac{e^{2ac\sqrt{M}}-1}{e^{2ac}-1}$$

$$= \frac{1}{\sqrt{M}} \frac{\sinh(ac\sqrt{M})}{\sinh(ac)} \tag{A.2}$$

Changing the variable of summation in (A.1), it is possible to show that,

$$E\{\exp(a\mathbf{I})\} = \frac{1}{\sqrt{M}} \sum_{m=1}^{\sqrt{M}/2} [e^{a(2m-1)c} + e^{-a(2m-1)c}]$$

$$= \frac{2}{\sqrt{M}} \sum_{m=1}^{\sqrt{M}/2} \cosh[(2m-1)ac] \tag{A.3}$$

Saltzberg [23] proved that the sum in (A.3) can be upper bounded as follows

$$E\{\exp(a\mathbf{I})\} \;\leq\; \exp(\frac{a^2}{2}c^2\frac{M-1}{3}) \tag{A.4}$$

•Derivation of (3.154)

For any normal random variable $\mathbf{n} = N(0, \sigma_n^2)$ and for any given constant a

$$E_n\{\exp(a\mathbf{n})\} = \int_{-\infty}^{\infty} \exp(an)\frac{1}{\sqrt{2\pi\sigma_n^2}} \exp(\frac{-n^2}{2\sigma_n^2})dn \tag{A.5}$$

where $E_n\{.\}$ denotes the expected value taken over the random variable, $\mathbf{n}$.

Completing the quadratic term, we get

$$E_n\{\exp(a\mathbf{n})\} = \exp(\frac{a^2\sigma_n^2}{2})\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma_n^2}} \exp[\frac{(n-a\sigma_n^2)^2}{2\sigma_n^2}]dn \tag{A.6}$$

The integral equal to unity and

$$E_n\{\exp(a\mathbf{n})\} = \exp(\frac{a^2\sigma_n^2}{2}) \tag{A.7}$$

Appendix B

Computing the Distribution of Random Variables/Gauss Quadrature Rules

In this appendix, we discuss numerical method to calculate integral of the form

$$E\{g(\mathrm{x})\} = \int_a^b g(x) f_{\mathrm{x}}(x)\, dx \tag{B.1}$$

and apply it for (3.181) in which $f_{\mathrm{x}}(x)$ is the probability density function of the random variable x and g(x) is continuous function of x, as in (3.179) for example.

B.1 Taylor Series Expansion

By knowing the moments of the random variable x, it is possible to find a numerical approximations to ($E\{g(\mathrm{x})\}$), where $g(.)$ is a known deterministic function and x defined on the interval [a,b].

In fact, for a function $g(.)$ analytic at x_o, one can use Taylor series expansion to present the random variable $Y = g(\mathrm{x})$;

$$g(\mathrm{x}) = g(x_o) + (\mathrm{x} - x_o)g'(x_o)....(\mathrm{x} - x_o)^n \frac{g^n(x_o)}{n!} + ... \tag{B.2}$$

where $g^n(x_o) = \dfrac{d^n g(\mathrm{x})}{d\mathrm{x}^n}|_{\mathrm{x}=x_o}$

By taking termwise expectation of (B.1), we get

$$E\{g(\mathrm{x})\} = g(x_o) + E\{(\mathrm{x} - x_x)\}g'(x_o)....E\{(\mathrm{x} - x_o)^n\}\frac{g^n(x_o)}{n!} + ... \tag{B.3}$$

where $E\{(\mathrm{x}-x_o)^n\}$ are the n th moment centered on x_o. Approximation of $E\{g(x)\}$ in (B.2) based on its N finite moments can be written as

$$E\{g(\mathrm{x})\} \approx \sum_{k=0}^{N} E\{(\mathrm{x} - x_o)^k\}\frac{g^k(x_o)}{k!} \tag{B.4}$$

That is $E\{g(\mathrm{x})\}$ can be evaluated on the basis of knowledge of its central moments provided that the series in (B.3) converges . It may seem that the error in approximating $E\{g(\mathrm{x})\}$ can be made small by taking N sufficiently large. Nevertheless, the higher moments are difficult to compute with sufficient accuracy. Hence, using higher N in (B.3), not necessarily improve approximation accuracy [19].

This motivates the use of another technique, known as Gauss quadrature rules, to compute approximation to the expectation of $g(\mathrm{x})$.

B.2 Gauss Quadrature Rules

With this method we write, provided $g(x)$ has continuous derivatives up to 2N [24];

$$\begin{aligned} E\{g(\mathrm{x})\} &= \int_a^b g(x)f_{\mathrm{x}}(x)\, dx \\ &= \sum_{i=1}^{N} w_i g(x_i) + R_N(r) \quad a < r < b \;\; a < x_i < b \;\; i = 1, 2, ..N \end{aligned} \tag{B.5}$$

where

$$R_N(r) = \frac{g^{2N}(r)}{(2N)!(C_N)^2} \tag{B.6}$$

$R_N(r)$ is a remainder which is equal to zero if $g(x)$ is a polynomial of degree $\leq 2N - 1$, in which case the Gauss quadrature rule (GQR) is exact. Otherwise

$$E\{g(\mathrm{x})\} \approx \sum_{i=1}^{N} w_i g(x_i) \;\; i = 1, 2, ..N \tag{B.7}$$

is only an approximation. The real values $\{x_i\}$ called the nodes of the GQR are the distinct real roots of the unique N th degree polynomial

$$P_N(x) = c_N \prod_{i=1}^{N} (x - x_i) \;\; c_N > 0 \tag{B.8}$$

The polynomials $P_n(x)$ are orthonormal with respect to $f_{\mathbf{x}}(x)$. That is;

$$E\{P_n(x)P_m(x)\} = \int_a^b P_n(x)P_m(x)f_{\mathbf{x}}(x) = \delta_{mn} \quad m, n = 1, 2... \tag{B.9}$$

The values $\{w_i\}$, called the weight of the GQR, are strictly positive and are given by,

$$w_i = -c_{N+1}\left[c_N P_{N+1}(x_i)\frac{dP_N(x)}{dx}|_{x=x_i}\right]^{-1} \;\; i = 1, 2..., N \tag{B.10}$$

The 2N-tuple $\{w_i, x_i\}_{i=1}^{N}$ are known as the N-point rule corresponding to $f_{\mathbf{x}}(x)$.

Generating the point rule

Several algorithms have been proposed to generate the N-point rule for a given p.d.f. The most useful ones are the modified moment algorithm of Gautchi [25] and the unmodified moment algorithm of Golub and Welsch [26]. Both algorithm rely on Cholesky decomposition of positive definite matrix of moments. Limitation in machine accuracy causes this matrix to become non-positive definite due to roundoff errors. Meyer [24] suggested an alternative method for performing the Cholesky decomposition that avoids taking square root at each step in the algorithm and hence, reduce the effect of roundoff.

For any set of orthonormal polynomial $\{P_n(x)\}_{i=1}^{N}\}$ we have the three terms recurrence relationship.

$$P_n(x) = (a_n x + b_n)P_{n-1}(x) - c_n P_{n-2}(x), \quad n = 1, 2..N, \quad P_{-1}(x) = 0, \; P_o(x) = 1 \tag{B.11}$$

It is simple to show that in matrix notation, (B.11) can be written as,

$$x\mathbf{P}(x) = \mathbf{T}\mathbf{P}(x) + \frac{1}{a_N} P_N(x)\mathbf{E} \tag{B.12}$$

where

$$\mathbf{P}(x) = [P_o(x), P_1(x), .., P_{N-1}(x)]^T \quad \mathbf{E}(x) = [0, 0, ..., 1]]^T \tag{B.13}$$

and

$$\mathbf{T} = \begin{bmatrix} -\frac{b_1}{a_1} & \frac{1}{a_1} & 0 & . & . & 0 \\ \frac{c_2}{a_2} & -\frac{b_2}{a_2} & \frac{1}{a_2} & . & . & 0 \\ 0 & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & & \frac{1}{a_{N-1}} \\ . & . & . & . & \frac{c_N}{a_N} & -\frac{b_N}{a_N} \end{bmatrix}$$

From (B.12), we note that $P_N(x_i) = 0$ if and only if

$$x_i\mathbf{P}(x_i) = \mathbf{T}\mathbf{P}(x_i) \tag{B.14}$$

That is the nodes of GQR are the eigenvalues of the tridiagonal matrix T, corresponding to the vector $\mathbf{P}(x_i)$. Wilf [27] also show that instead of (B.10), it is possible to obtain the GQR weights $\{w_i\}_{i=1}^N$ from the identity

$$w_i[\mathbf{P}^T(x_i)][\mathbf{P}(x_i)] = 1 \tag{B.15}$$

where $\mathbf{P}(x_i)$ are the eigenvector associate with the eigenvalue x_i. If we normalize this eigenvector,

$$\mathbf{e}_i = \frac{\mathbf{P}(x_i)}{K_i} \tag{B.16}$$

so that $\mathbf{e}_i^T\mathbf{e}_i = 1$ then,

$$\mathbf{e}_i^T\mathbf{e}_i = \sum_{n=0}^{N-1} \frac{P_n^2(x_i)}{K_i^2} = 1 \tag{B.17}$$

Comparing with (B.15), we conclude that

$$w_i = \frac{1}{K_i^2} \tag{B.18}$$

or by using (B.16)

$$w_i = \frac{e_{1i}^2}{P_0^2(x_i)} \tag{B.19}$$

Therefore, if we choose $P_o(x) = 1$ then the GQR weights $\{w_i\}_{i=1}^N$ are obtained from the square of firts elements of the normalized eigenvectors that correspond to x_i.

B.3 Determining the Three Term Relationship from the Moments

We show in the previous section that the GQR's 2N-tuple $\{w_i, x_i\}_{i=1}^N$ can be found from the eigenvalues and the corresponding eigenvectors of the matrix $\mathbf{T}$. $\mathbf{T}$ is a tridiagonal matrix whose elements depend on the coefficients $\{a_i, b_i, c_i\}$ of the three terms relationship. $\mathbf{T}$ which have been tabulated for a number of functions $f_x(x)$.

In this section, we describe a method of generating these coefficients, and hence the matrix $\mathbf{T}$, using Gautchi's modified moments technique [25], or using the special case of this technique suggested by Golub and Welsch [26] that uses the unmodified moments $E\{x^n\}$. This special case technique of Golub and Welsch which uses the nonorthogonal polynomial $P_n(x) = x^n$ is termed the GQR based on the moments method.

Let the modified moments matrix M be defined by

$$M = \begin{bmatrix} m_{11} & m_{12} & m_{13} & \cdot & \cdot & \cdot & \cdot & m_{1,N+1} \\ & m_{22} & \cdot & \cdot & \cdot & \cdot & \cdot & m_{2,N+1} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & m_{N+1,N+1} \end{bmatrix}$$

where

$$m_{ij} = \int_a^b T_{i-1}(x)T_{j-1}(x)f_{\mathbf{x}}(x)\,dx \quad i,j = 1,2,..N+1 \tag{B.20}$$

and $\{T_i(x)\}_{i=0}^{N}$ are the first (N+1) moments

Method to Evaluate the nodes x_i and the weights w_i

The procedure can be summarized as follows:

- compute $2N+1$ moments $\{\mu_n\}|_{n=0}^{2N}$ of random variable $\mathbf{x}$, that is,

$$\mu_n = \int_a^b x^n f_{\mathbf{x}}(x)dx \tag{B.21}$$

where $f_{\mathbf{x}}(x)$ is the probability density function of $\mathbf{x}$.

- generate $(N+1)$x$(N+1)$ positive definite matrix $\mathbf{M}$ where the entries of this matrix are the moments of random variable $\mathbf{x}$. where,

$$m_{ij} = \mu_{i+j-2} \quad i,j = 1,2..,N+1 \tag{B.22}$$

therefore,

$$\mathbf{M} = \begin{bmatrix} 1 & \mu_1 & \mu_2 & \mu_3 & . & . & . & \mu_N \\ \mu_1 & \mu_2 & \mu_3 & \mu_4 & . & . & . & \mu_{N+1} \\ \mu_2 & \mu_3 & \mu_4 & \mu_5 & . & . & . & \mu_{N+2} \\ \mu_3 & \mu_4 & \mu_5 & \mu_6 & . & . & . & . \\ . & \mu_5 & \mu_6 & \mu_7 & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & \mu_{2N} \end{bmatrix}$$

- Compute, $\mathbf{M} = \mathbf{R}^T\mathbf{R}$ the Cholesky decomposition of $\mathbf{M}$ where $\mathbf{R}$ is a upper triangular matrix and which is positive definite if $\mathbf{M}$ is positive definite. Particularly the diagonal terms of R is positive and are the square root of a diagonal matrix $\mathbf{D}$. $\mathbf{M} = \hat{\mathbf{R}}^T \text{diagonal}(r_{ii}^2)\hat{\mathbf{R}}$.

where $\hat{\mathbf{R}}$ is an upper triangular with units along the diagonal. In case M is ill conditioned, finite arithmetic may cause the matrix to appear singular.

A modified version of Cholesky decomposition can overcome this ill-conditioned by only requiring square roots to be computed at the end of the decomposition and not at each step [24].

With this approach, we find the elements r_{ij} of $\mathbf{R}$ by the following recursive equations:

$$\hat{m_{ij}} = \hat{r_{ij}} d_j \tag{B.23}$$

then

$$\begin{aligned} \hat{m_{ij}} &= m_{ij} - \sum_{k=1}^{j-1} \hat{m_{i,k}} \hat{r_{jk}} \quad j = 1, 2, ... i-1, \quad i > j \\ d_i &= m_{ii} - \sum_{k=1}^{i-1} \hat{m_{ik}} \hat{r_{ik}} \end{aligned} \tag{B.24}$$

where d_i and m_{ij} are the elements of the diagonal matrix D and the moment matrix M, respectively.

Notice that as $\mathbf{R} = \hat{\mathbf{R}}$ diag (r_{ii}) the square roots are not required until the final step which is the advantage of this modified decomposition.

- $\mathbf{J}$, symmetric NxN tridiagonal matrix with the entries α_n, β_n $n = 1, .., N$ is then formed,

$$J = \begin{bmatrix} \alpha_1 & \beta_1 & 0 & . & . & 0 \\ \beta_1 & \alpha_2 & \beta_2 & . & . & 0 \\ 0 & \beta_2 & 0 & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & \alpha_{N-1} & \beta_{N-1} \\ . & . & . & . & \beta_{N-1} & \alpha_N \end{bmatrix}$$

whose entries are given by the following relation

$$\begin{aligned} \alpha_n &= \frac{r_{n,n+1}}{r_{nn}} - \frac{r_{n-1,n}}{r_{n-1,n-1}} \qquad n = 1, 2, ..., N \\ \beta_n &= \frac{r_{n+1,n+1}}{r_{nn}} \qquad n = 1, 2, ..., N-1 \end{aligned} \tag{B.25}$$

- The final step is to obtain the nodes and the weights from the eigenvalues and eigenvectors of $\mathbf{J}$ matrix. The nodes x_i are the eigenvalues and the weights w_i are the square of the first components of the corresponding eigenvector of this tridiagonal matrix.

The eigenvector $\mathbf{e}_i$ corresponding to the eigenvalue x_i is found from the equation

$$\mathbf{J}\mathbf{e}_i = x_i \mathbf{e}_i \tag{B.26}$$

The eigenvalues of the $\mathbf{J}$ matrix are the nodes $\{x_i\}|_{i=1}^{N}$ and the positive weights are the square of the first elements of the corresponding eigenvectors $\mathbf{e}_i$. That is,

$$w_i = \mathbf{e}_{1i}^2 \mu_o \tag{B.27}$$

where $\mu_o = 1$

$$\mathbf{e}_i^T = (e_{1i}, e_{2i}, ...e_{Ni})$$

Appendix C

Approximation of some Equations

•Approximations in (5.18)

1. We intend to show that $w^*_{12\text{opt}}|a_{21}+w_{21\text{opt}}a_{11}|^2\delta_{21}E\{|I_1(n)|^2\}$ in (5.17) is small in comparison to other terms. Particularly, comparing this term with the fourth term of (5.17), we notice that by using (5.20), we have,

$$\begin{aligned}|a_{21}+w_{21\text{opt}}a_{11}|^2 &= |a_{21}+(-\frac{a_{21}}{a_{11}}+\epsilon_2)a_{11}|^2 \\ &= |a_{11}\epsilon_2|^2 \end{aligned} \tag{C.1}$$

also by using (5.20),

$$\begin{aligned}|a_{22}+w_{21\text{opt}}a_{12}|^2 &= a_{22}|1+(-\frac{a_{21}a_{12}}{a_{11}a_{22}}+\epsilon_2\frac{a_{12}}{a_{11}})|^2 \\ &= a_{22}^2|k+\epsilon_2 r_1 e^{j\phi_1}|^2\end{aligned}$$

But, for any complex number x; $|x|^2 \geq (\text{Real}(x))^2$

then,

$$\begin{aligned}|a_{22}+w_{21\text{opt}}a_{12}|^2 &\geq a_{22}^2\Big[\text{Real}(k+\epsilon_2 r_1 e^{j\phi_1})\Big]^2 \\ &= a_{22}^2[1-r_1r_2cos(\phi_1+\phi_2)+\epsilon_{2R}r_1cos\phi_1-\epsilon_{2I}r_1sin\phi_1]^2 \\ &\geq a_{22}^2[1-r_1r_2-r_1(\epsilon_{2R}+\epsilon_{2I})]^2\end{aligned} \tag{C.2}$$

Finally,

$$|a_{22} + w_{21\text{opt}}a_{12}|^2 \;\; \geq \;\; a_{22}^2[1 - 2r_1r_2 - 2r_1^2r_2(\epsilon_{2R} + \epsilon_{2I})]^2 \qquad \text{(C.3)}$$

For sufficiently small r_1 and r_2 and small ϵ_2 the lower bound of (C.2) is much larger than $|a_{11}\epsilon_2|^2$ of (C.1). Hence for $|I_1(n)|^2$ and $|I_2(n)|^2$ of the same order, we have,

$$\begin{aligned} \delta_{21}E\{|I_1(n)|^2\}|a_{21} + w_{21\text{opt}}a_{11}|^2 w_{12\text{opt}}^* &= \delta_{21}E\{|I_1(n)|^2\}|a_{11}\epsilon_2|^2 w_{12\text{opt}}^* \\ &\ll \delta_{22}E\{|I_2(n)|^2\}[1 - 2r_1r_2 - r_1^2r_2(\epsilon_{2R} + \epsilon_{2I})]w_{12\text{opt}}^* \\ &\leq \delta_{22}E\{|I_2(n)|^2\}|a_{22} + w_{21\text{opt}}a_{12}|^2 w_{12\text{opt}}^* \end{aligned} \qquad \text{(C.4)}$$

which complete our claim.

2. We also claim that the term $w_{21\text{opt}}w_{12\text{opt}}$ in the fifth term of (5.17) can be ignored. In $(1 + w_{21\text{opt}}w_{12\text{opt}})$, we can ignore the $\text{Real}(1 + w_{21\text{opt}}w_{12\text{opt}})$ in comparison to one, if we assume that the cross coupling r_1 and/or r_2 are of the order of at least -5 dB. For the imaginary part, we use in our comparison the imaginary part of the sixth term of this equation;

$$\begin{aligned} E\{|n_1(n)|^2\}Im(w_{21\text{opt}}w_{12\text{opt}}^*w_{21\text{opt}}^*) &= E\{|n_1(n)|^2\}Im(|w_{21\text{opt}}|^2 w_{12\text{opt}}^*) \\ &<< E\{|n_2(n)|^2\}Im(w_{12\text{opt}}^*) \end{aligned} \qquad \text{(C.5)}$$

provided, we use the same assumption on r_1 and r_2, so that $|w_{21\text{opt}}|^2$ will be much smaller than unity, and that the noise powers are of the same order.

- **Approximation in (5.32)**

From (5.31) and (5.30), we write,

$$\Delta_{1C}\epsilon_2 + \Delta_{1C}^*\epsilon_2^* = 2E\{|n_1(n)|^2\}Real\Big[\frac{E\{|I_2(n)|^2\}}{E\{n_1(n)|^2\}}\delta_{12}a_{22}^2 k^* \frac{a_{12}}{a_{22}} - (\frac{a_{21}}{a_{11}})^*\Big]\epsilon_2 \tag{C.6}$$

$$\textbf{1.}\ Real(k^*\frac{a_{12}}{a_{22}}\epsilon_2) = Real(k^*\frac{a_{12}}{a_{22}})\epsilon_{2R} - Im(k^*\frac{a_{12}}{a_{22}})\epsilon_{2I} \tag{C.7}$$

and using (5.46) and (5.45), we get,

$$\begin{aligned} Real(k^*\frac{a_{12}}{a_{22}}) &= [1 - r_1 r_2 cos(\phi_1+\phi_2)]r_1 cos\phi_1 - r_1 r_2 sin(\phi_1+\phi_2) r_1 sin\phi_1 \\ &= r_1 cos\phi_1 - r_1^2 r_2 cos\phi_2 \end{aligned} \tag{C.8}$$

$$\begin{aligned} Im(k^*\frac{a_{12}}{a_{22}}) &= [1 - r_1 r_2 cos(\phi_1+\phi_2)]r_1 sin\phi_1 + r_1 r_2 sin(\phi_1+\phi_2) r_1 cos\phi_1 \\ &= r_1 sin\phi_1 + r_1^2 r_2 sin\phi_2 \end{aligned} \tag{C.9}$$

and substituting (C.8) and (C.9) in (C.7), we have,

$$Real(k^*\frac{a_{12}}{a_{22}}\epsilon_2) = [r_1 cos\phi_1 - r_1^2 r_2 cos\phi_2]\epsilon_{2R} - [r_1 sin\phi_1 + r_1^2 r_2 sin\phi_2]\epsilon_{2I} \tag{C.10}$$

$$\textbf{2.}\ (\frac{a_{21}}{a_{11}})^*\epsilon_2 = r_2\epsilon_{2R} cos\phi_2 + r_2\epsilon_{2I} sin\phi_2 \tag{C.11}$$

Therefore,

$$\begin{aligned} \Delta_{1C}\epsilon_2 + \Delta_{1C}^*\epsilon_2^* &= 2E\{|n_1(n)|^2\}Real\Big[\frac{E\{|I_2(n)|^2\}}{E\{n_1(n)|^2\}}\delta_{12}a_{22}^2(r_1 cos\phi_1 - r_1^2 r_2 cos\phi_2)\epsilon_{2R} \\ &-(r_1 sin\phi_1 + r_1^2 r_2 cos\phi_2)\epsilon_{2I} + r_2\epsilon_{2R} cos\phi_2 + r_2\epsilon_{2I} sin\phi_2] \end{aligned} \tag{C.12}$$

Next, we write (5.28) by substituting for $|\frac{a_{12}}{a_{22}}|^2 = r_1^2$,

$$\Delta_{1A} = E\{|n_1(n)|^2\}\Big[a_{11}^2\frac{E\{|I_1(n)|^2\}}{E\{|n_1(n)|^2\}}\delta_{11} + a_{22}^2 r_1^2\frac{E\{|I_2(n)|^2\}}{E\{|n_1(n)|^2\}}\delta_{22} + 1], \quad (C.13)$$

and by substituting $|k|^2$ from (5.47) and for $|\frac{a_{21}}{a_{11}}|^2$, we write (5.29),

$$\begin{aligned}\Delta_{1B} &= E\{|n_1(n)|^2\}\Big[a_{22}^2\frac{E\{|I_2(n)|^2\}}{E\{n_1(n)|^2\}}\delta_{12}(1 - 2r_1r_2cos(\phi_1 + \phi_2) + r_1^2r_2^2) \\ &+ r_2^2 + \frac{E\{|n_2(n)|^2\}}{E\{n_1(n)|^2\}}\Big], \end{aligned} \quad (C.14)$$

It is easy to upper bound (C.12) by,

$$\begin{aligned}\Delta_{1C}\epsilon_2 + \Delta_{1C}^*\epsilon_2^* &\leq 2E\{|n_1(n)|^2\}\Big[2a_{22}^2\frac{E\{|I_2(n)|^2\}}{E\{n_1(n)|^2\}}\delta_{12}(2r_1 + 2r_1^2r_2)|\epsilon_2| + 2r_2|\epsilon_2|\Big] \\ &= 2E\{|n_1(n)|^2\}\Big[4a_{22}^2\frac{E\{|I_2(n)|^2\}}{E\{n_1(n)|^2\}}\delta_{12}(r_1 + r_1^2r_2)|\epsilon_2| + 2r_2|\epsilon_2|\Big]\end{aligned} \quad (C.15)$$

and lower bound (C.16) by,

$$\Delta_{1B} \geq E\{|n_1(n)|^2\}\Big[a_{22}^2\frac{E\{|I_2(n)|^2\}}{E\{n_1(n)|^2\}}\delta_{12}(1 - r_1r_2)^2 + r_2^2 + \frac{E\{|n_2(n)|^2\}}{E\{n_1(n)|^2\}}\Big], \quad (C.16)$$

Combining (C.13) with (C.15), we have,

$$\begin{aligned}\Delta_{1C}\epsilon_2 + \Delta_{1C}^*\epsilon_2^* + \Delta_{1A} &\leq E\{|n_1(n)|^2\}\Big[4a_{22}^2\frac{E\{|I_2(n)|^2\}}{E\{n_1(n)|^2\}}\delta_{12}(r_1 + r_1r_2)^2|\epsilon_2| + 2r_2|\epsilon_2| \\ &+ \Big(a_{11}^2\frac{E\{|I_1(n)|^2\}}{E\{n_1(n)|^2\}}\delta_{11}a_{22}^2\frac{E\{|I_2(n)|^2\}}{E\{n_1(n)|^2\}}\delta_{22}r_1^2 + 1\Big)|\epsilon_2|^2\Big]\end{aligned} \quad (C.17)$$

It is easy to see that if $|\frac{E\{|I_1(n)|^2\}}{E\{n_1(n)|^2\}}|^2$ and $|\frac{E\{|I_2(n)|^2\}}{E\{n_1(n)|^2\}}|^2$ are of the same order and for r_1r_2 small so that $(1 - r_1r_2) \approx 1$, then due to the fact that $|\epsilon|$ is small, the upper bound in (C.17) is much smaller than the lower bound of (C.16). Therefore,

$$\Delta_{1C}\epsilon_2 + \Delta_{1C}^*\epsilon_2^* + \Delta_{1A} \approx \Delta_{1B} \tag{C.18}$$

Appendix D

- Evaluating (7.29)

  Let

$$E\{g_1(i)|\mathbf{w}(i)\} \triangleq E\{g_{11}(i)|\mathbf{w}(i)\} + E\{g_{12}(i)|\mathbf{w}(i)\}$$
$$E\{g_{13}(i)|\mathbf{w}(i)\} + E\{g_{14}(i)|\mathbf{w}(i)\} \tag{D.1}$$

where $g_{1k} \quad k = 1, ..4$ are the different four terms in (7.29). For the sake of simplicity, we evaluate (7.29) term by term,

That is;

$$E\{g_{11}(i)|\mathbf{w}(i)\} \triangleq \frac{\delta_{11} E\{I_1^2(i)\}}{N\Lambda} \sum_{m=iN+1}^{(i+1)N} (a_{11} + w_{12}(m)a_{21})^2 .$$
$$[1 + w_{12}(m)w_{21}(m)]^2 p_1(m) \tag{D.2}$$

$$E\{g_{12}(i)|\mathbf{w}(i)\} \triangleq \frac{\delta_{12} E\{I_2^2(i)\}}{N\Lambda} \sum_{m=iN+1}^{(i+1)N} (a_{12} + w_{12}(m)a_{22})^2 .$$
$$[1 + w_{12}(m)w_{21}(m)]^2 p_1(m) \tag{D.3}$$

$$E\{g_{13}(i)|\mathbf{w}(i)\} \triangleq \frac{E\{n_1^2(i)\}}{N\Lambda} \sum_{m=iN+1}^{(i+1)N} [1 + w_{12}(m)w_{21}(m)]^2 p_1(m) \tag{D.4}$$

$$E\{g_{14}(i)|\mathbf{w}(i)\} \triangleq \frac{E\{n_2^2(i)\}}{N\Lambda} \sum_{m=iN+1}^{(i+1)N} w_{12}^2(m)[1+w_{12}(m)w_{21}(m)]^2 p_1(m) \tag{D.5}$$

where we also used the fact that the random sequences $I_l(i)$ and $n_l(i)$ $l = 1, 2$ are stationary.

Using (7.8), (7.9) in (D.2), we can get;

$$\begin{aligned} E\{g_{11}(i)|\mathbf{w}(i)\} &= \frac{\delta_{11} E\{I_1^2(i)\}}{N\Lambda} \sum_{m=iN+1}^{(i+1)N} (a_{11} + [w_{12}(i) + \Lambda p_1(m)]a_{21})^2. \\ & \left(1 + [w_{12}(i) + \Lambda p_1(m)][w_{21}(i) + \Lambda p_2(m)]\right)^2 p_1(m) \\ &= \frac{\delta_{11} E\{I_1^2(i)\}}{N\Lambda} \sum_{m=iN+1}^{(i+1)N} X_{11}(m) X_{12}(m) p_1(m) \end{aligned} \tag{D.6}$$

where,

$$X_{11}(m) \triangleq (a_{11} + [w_{12}(i) + \Lambda p_1(m)]a_{21})^2 \tag{D.7}$$

$$X_{12}(m) \triangleq \left(1 + [w_{12}(i) + \Lambda p_1(m)][w_{21}(i) + \Lambda p_2(m)]\right)^2 \tag{D.8}$$

As a function of the perturbation $p_1(m)$ and $p_2(m)$, we write (D.7) and (D.8) as

$$X_{11}(m) = A + B\Lambda p_1(m) + C\Lambda^2 p_1(m) p_1(m) \tag{D.9}$$

$$\begin{aligned} X_{12}(m) &= D + E\Lambda p_1(m) + F\Lambda p_2(m) + G\Lambda^2 p_1(m) p_2(m) + H\Lambda^2 p_1(m) p_1(m) \\ &+ I\Lambda^2 p_2(m) p_2(m) + J\Lambda^3 p_1(m) p_2(m) p_2(m) + K\Lambda^3 p_2(m) p_1(m) p_1(m) \\ &+ \Lambda^4 p_1(m) p_1(m) p_2(m) p_2(m) \end{aligned} \tag{D.10}$$

where,

$$
\begin{aligned}
A &= [a_{11} + a_{21}w_{12}(i)]^2 && \text{(D.11)}\\
B &= 2a_{21}[a_{11} + a_{21}w_{12}(i)] && \text{(D.12)}\\
C &= a_{21}^2 && \text{(D.13)}\\
D &= [1 + w_{12}(i)w_{21}(i)]^2 && \text{(D.14)}\\
E &= 2[1 + w_{12}(i)w_{21}(i)]w_{21}(i) && \text{(D.15)}\\
F &= 2[1 + w_{12}(i)w_{21}(i)]w_{12}(i) && \text{(D.16)}\\
G &= 2[1 + 2w_{12}(i)w_{21}(i)] && \text{(D.17)}\\
H &= w_{21}^2(i) && \text{(D.18)}\\
I &= w_{12}^2(i) && \text{(D.19)}\\
J &= 2w_{12}(i) && \text{(D.20)}\\
K &= 2w_{21}(i) && \text{(D.21)}
\end{aligned}
$$

Therefore, from (D.6), and by using the orthogonal properties of the perturbation sequences $p_1(m)$ and $p_2(m)$, we get

$$
\begin{aligned}
E\{g_{11}(i)|\mathbf{w}(i)\} &= \frac{\delta_{11}E\{I_1(i)^2\}}{N\Lambda}[(AE + BD)\Lambda N \\
&+(AJ + BH + BI + CE)\Lambda^3 N + (B + CJ)\Lambda^5 N] \qquad \text{(D.22)}
\end{aligned}
$$

For $\Lambda << 1$ the second and the third terms in the paranthesis is small in comparison to the first and we have,

$$E\{g_{11}(i)|\mathbf{w}(i)\} \approx \delta_{11}E\{I_1^2(i)\}(AE + BD) \qquad \text{(D.23)}$$

with A, B, D and E defined in (D.11) , (D.12) , (D.14) and (D.15) respectively. Therefore;

$$E\{g_{11}(i)|\mathbf{w}(i)\} \approx 2\delta_{11}E\{I_1^2(i)\}\Big[[a_{11}+a_{21}w_{12}(i)]^2[1+w_{12}(i)w_{21}(i)]w_{21}(i)$$

$$+[a_{11}+a_{21}w_{12}(i)]a_{21}[1+w_{12}(i)w_{21}(i)]^2\Big] \quad \text{(D.24)}$$

Similarly, using (7.8) and (7.9) in (D.3), we get,

$$\begin{aligned} E\{g_{12}(i)|\mathbf{w}(i)\} &= \frac{\delta_{12}E\{I_2^2(i)\}}{N\Lambda}\sum_{m=iN+1}^{(i+1)N}(a_{12}+w_{12}(m)a_{22})^2. \\ &\qquad [1+w_{12}(m)w_{21}(m)]^2 p_1(m) \\ &= \frac{\delta_{12}E\{I_2^2(i)\}}{N\Lambda}\sum_{m=iN+1}^{(i+1)N} X_{13}(m)X_{12}(m)p_1(m) \end{aligned} \quad \text{(D.25)}$$

where,

$$X_{13}(m) \triangleq (a_{12}+[w_{12}(i)+\Lambda p_1(m)]a_{22})^2 \quad \text{(D.26)}$$

and $X_{12}(m)$ as in (D.10) ,

As a function of the perturbation $p_1(m)$, we write (D.26) as

$$X_{13}(m) = A_1 + B_1\Lambda p_1(m) + C_1\Lambda^2 p_1(m)p_1(m) \quad \text{(D.27)}$$

where,

$$A_1 = [a_{12}+a_{22}w_{12}(i)]^2 \quad \text{(D.28)}$$

$$B_1 = 2[a_{12}+a_{22}w_{12}(i)]a_{22} \quad \text{(D.29)}$$

$$C_1 = a_{22}^2 \quad \text{(D.30)}$$

Therefore, from (D.25) and by using the orthogonal properties of $p_1(m)$ and $p_2(m)$, we get,

$$E\{g_{12}(i)|\mathbf{w}(i)\} = \frac{\delta_{12}E\{I_2^2(i)\}}{N\Lambda}[(A_1E+B_1D)\Lambda N$$
$$+(A_1J+B_1H+B_1I+C_1E)\Lambda^3N+(B_1+C_1J)\Lambda^5N] \quad \text{(D.31)}$$

For $\Lambda << 1$, the second and the third tems in the paranthesis is small in comparison to the first and we have,

$$E\{g_{12}(i)|\mathbf{w}(i)\} \approx \delta_{12}E\{I_2^2(i)\}(A_1E+B_1D) \quad \text{(D.32)}$$

with A_1, B_1, D and E defined in (D.28), (D.29), (D.14), (D.15), respectively. Therefore,

$$E\{g_{12}(i)|\mathbf{w}(i)\} \approx 2\delta_{12}E\{I_2^2(i)\}\Big[[a_{12}+a_{22}w_{12}(i)]^2[1+w_{12}(i)w_{21}(i)]w_{21}(i)$$
$$+[a_{12}+a_{22}w_{12}(i)]a_{22}[1+w_{12}(i)w_{21}(i)]^2\Big] \quad \text{(D.33)}$$

Again, using (7.8) and (7.9) in (D.4) , we get,

$$E\{g_{13}(i)|\mathbf{w}(i)\} = \frac{E\{n_1^2(i)\}}{N\Lambda}\sum_{m=iN+1}^{(i+1)N}\Big(1+[w_{12}(i)+\Lambda p_1(m)].$$
$$[w_{21}(i)+\Lambda p_2(m)]\Big)^2 p_1(m)$$
$$= \frac{E\{n_1^2(i)\}}{N\Lambda}\sum_{m=iN+1}^{(i+1)N}X_{12}(m)p_1(m) \quad \text{(D.34)}$$

with $X_{12}(m)$ as in (D.10) ,

Applying the orthogonal properties of the perturbation, we get

$$E\{g_{13}(i)|\mathbf{w}(i)\} \approx \frac{E\{n_1^2(i)\}}{N\Lambda}(E\Lambda+J\Lambda^3)N \quad \text{(D.35)}$$

For Λ small the second term can be neglected and using (D.15) and (D.20) in (D.35), we get;

$$E\{g_{13}(i)|\mathbf{w}(i)\} \approx 2E\{n_1^2(i)\}[1 + w_{12}(i)w_{21}(i)]w_{21}(i) \tag{D.36}$$

Finally, using (7.8) and (7.9) in (D.5), we get,

$$\begin{aligned} E\{g_{14}(i)|\mathbf{w}(i)\} &= \frac{E\{n_2^2(i)\}}{N\Lambda} \sum_{m=iN+1}^{(i+1)N} [w_{12}(i) + \Lambda p_1(m)]^2. \\ &\quad \Big(1 + [w_{12}(i) + \Lambda p_1(m)][w_{21}(i) + \Lambda p_2(m)]\Big)^2 p_1(m) \\ &= \frac{E\{n_2^2(i)\}}{N\Lambda} \sum_{m=iN+1}^{(i+1)N} X_{14}(m)X_{12}(m)p_1(m) \end{aligned} \tag{D.37}$$

where;

$$\begin{aligned} X_{14}(m) &\triangleq [w_{12}(i) + \Lambda p_1(m)]^2 \\ &= w_{12}^2(i) + 2w_{12}(i)\Lambda p_1(m) + \Lambda^2 p_1(m)p_1(m) \end{aligned} \tag{D.38}$$

and $X_{12}(m)$ as in (D.10)

Again applying the orthogonal properties of the perturbation, we get

$$\begin{aligned} E\{g_{14}(i)|\mathbf{w}(i)\} &= \frac{E\{n_2(i)^2\}}{N\Lambda}\Big[[Ew_{12}^2(i) + 2Dw_{12}(i)]\Lambda N \\ &\quad +[E + 2w_{12}(i)H + 2Iw_{12}(i) + Jw_{12}^2(i)]\Lambda^3 N \\ &\quad +(J + 2w_{12}(i))\Lambda^5 N \end{aligned} \tag{D.39}$$

With the same approximation in (D.33), we can write

$$E\{g_{14}(i)|\mathbf{w}(i)\} \approx E\{n_2^2(i)\}[2D + Ew_{12}(i)]w_{12}(i) \tag{D.40}$$

With D and E defined by (D.14) and (D.15), respectively. Therefore,

$$E\{g_{14}(i)|\mathbf{w}(i)\} \approx 2E\{n_2^2(i)\}\Big[[1+w_{12}(i)w_{21}(i)]^2 + [1+w_{12}(i)w_{21}(i)]w_{21}(i)w_{12}(i)\Big]w_{12}(i) \qquad \text{(D.41)}$$

Combining the four terms from (D.24), (D.32), (D.36) and (D.41) in (D.1), we write,

$$\begin{aligned} E\{g_1(i)|\mathbf{w}(i)\} = 2\Big[&\delta_{11}E\{I_1(i)^2\}\Big([1+w_{12}(i)w_{21}(i)]^2[a_{11}+w_{12}(i)a_{21}]a_{21} \\ &+[1+w_{12}(i)w_{21}(i)][a_{11}+w_{12}(i)a_{21}(i)]^2w_{21}(i)\Big) \\ &+\delta_{12}E\{I_2(i)^2\}\Big([1+w_{12}(i)w_{21}(i)]^2[a_{12}+w_{12}(i)a_{22}]a_{22} \\ &+[1+w_{12}(i)w_{21}(i)][a_{12}+w_{12}(i)a_{22}]^2w_{21}(i)\Big) \\ &+E\{n_1(i)^2\}[1+w_{12}(i)w_{21}(i)]w_{21}(i) \\ &+E\{n_2(i)^2\}\Big([1+w_{12}(i)w_{21}(i)]^2w_{12}(i) \\ &+[1+w_{12}(i)w_{21}(i)]w_{12}^2(i)w_{21}(i)\Big)\Big], \end{aligned} \qquad \text{(D.42)}$$

Similar lengthy manipulation can be applied to obtain the *estimate* of the gradient of Q with respect to w_{21}. However, due to the symmetry, we can obtain $Q(w_{12}, w_{21})$ from $P(w_{12}, w_{21})$ by taking $I_2(i)$, $n_2(i)$, a_{22}, a_{21}, w_{21} respectively instead of $I_1(i)$, $n_1(i)$, a_{11}, a_{12}, w_{12} and vise versa. Also, replacing δ_{11} and δ_{12} by δ_{22}, δ_{21}.

Applying these changes and replacements to (D.42) we get,

$$\begin{aligned} E\{g_2(i)|\mathbf{w}(i)\} &= 2\Big[\delta_{21}E\{I_1(i)^2\}\Big([1+w_{12}(i)w_{21}(i)]^2[a_{21}+w_{21}(i)a_{11}]a_{11} \\ &\quad +[1+w_{12}(i)w_{21}(i)][a_{21}+w_{21}(i)a_{11}(i)]^2 w_{12}(i)\Big) \\ &\quad +\delta_{22}E\{I_2(i)^2\}\Big([1+w_{12}(i)w_{21}(i)]^2[a_{22}+w_{21}(i)a_{12}]a_{12} \\ &\quad +[1+w_{12}(i)w_{21}(i)][a_{22}+w_{21}(i)a_{12}]^2 w_{12}(i)\Big) \\ &\quad +E\{n_1(i)^2\}[1+w_{12}(i)w_{21}(i)]w_{12}(i) \\ &\quad +E\{n_2(i)^2\}\Big([1+w_{12}(i)w_{21}(i)]^2 w_{21}(i) \\ &\quad +[1+w_{12}(i)w_{21}(i)]w_{21}^2(i)w_{12}(i)\Big)\Big], \end{aligned} \tag{D.43}$$

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