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Abstract<br>Name: Elisha Yegal Bar-Ness<br>Advisor: Prof. Erdal Panayirci<br>Thesis Title: Performance Analysis of Symbol Timing Recovery Circuits Employed in Digital Communications Systems

An analytical approach is presented for the jitter performance of a timing recovery circuit consisting of a prefilter, a zero-memory nonlinear device, and a narrow band postfilter tuned to the pulse repetition frequency. Assuming first a squarer type of nonlinearity, analytical expressions for the rms jitter in the timing wave are obtained as a function of the pre and post filtering characteristics. These expressions are suitable for judging the case where the baseband signal is bandlimited. Also for some specific examples, the jitter performance of this kind of STR circuit is evaluated.

Secondly, a general type of nonlinerity is assumed, and the rms jitter expressions are obtained in terms of the higher order moments of the input signal. The higher moments themselves are shown to be computed iteratively.

Finally some numerical results are obtained for the fourth order nonlinearity and the rms jitter curves are plotted as a function of the excess bandwidth factor $\gamma$, for several values of the quality factor $Q$ of the postfilter

# Performance Analysis of Symbol Timing Recovery Circuits Employed in Digital Communication Systems 

by

Elisha Yegal Bar-Ness

Thesis submitted to the Faculty of the Graduate School of the New Jersey Institute of Technology in partial fulfillment of the requirements for the degree of Master of Science in Electrical and

Computer Engineering

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To my Parents

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## Chapter 1

## Introduction

Synchronization in Latin means "time together". It is the process of aligning together the time scales between two or more periodic processes which occur at two spatially separated points. The synchronization process plays an important role in the field of communications, data transmission, radar, sonar, and navigation.

In the field of digital communications, we distinguish between a hierarchy of synchronization problems as follows:

1. Carrier synchronization
2. Clock synchronization
3. Codeword and node synchronization
4. Frame synchronization
5. Network synchronization

The third and fourth categories deal with the problem of identifying the beginning
and end of a code word (few characters), and a group of code words, respectively. Together with the carrier and clock synchronization, these two categories form different levels of synchronization in point-to-point transmission. The fifth type of synchronization, also called "packet synchronization," is needed when digital data are received from several sources, processed and transmitted to one or more users, (e.g. in switched digital communications networks). A variety of techniques can be used for this purpose.

A feature which distinguishes the latter types from those of carrier and bit (clock) synchronization is that they are usually solved by means of special design of the message format. This involves the repetitive insertion of bit or words into the data sequence, which serves the purpose of synchronization only. On the other hand, it is desirable, in carrier and bit synchronization, not to add such a special timing sequence which is a data overhead that causes a reduction in the available channel capacity. Cases of bit or carrier synchronization, wherein the transmitted signal contains an unmodulated component of sinusoidal carrier (pilot), can be handled by using a phase locked loop (PLL). Such a PLL will lock onto the carrier component and, due to its narrow bandwidth, will reject sideband components. There is a vast literature, including several textbooks, which deal with synchronization using PLL [1]-[7].

### 1.1 Carrier and Clock Synchronization [7]

Digital communication systems which are efficient in power requirement and bandwidth, employ "synchronous" (uniformly spaced) signaling pulses and "coherent
demodulation" of the received signal. For adequate demodulation of the passband signal, one needs a local carrier reference which has the same frequency, and phase, as the received signal. Similarly, for adequate detection of the data symbols, one needs a local clock which is accurately time aligned with the received pulses. The circuits which generate the local carrier and clock references are called Carrier and Clock Synchronizers.

As previously emphasized, in carrier and clock synchronization, we are mainly interested in cases where the reference must be extracted from the incoming signal which does not contain a pilot carrier or training sequences. This is because, with an efficient signal design, any components of carrier or clock are suppressed. Nonlinear devices are necessary to regenerate these components from signals in which carrier and clock were suppressed. Because of the need for nonlinearity, synchronizers are difficult to treat mathematically.

The history of synchronizers followed two possible approaches; the "ad hoc structure" approach and the "derived structure" approach. With the first approach, different structures were created by engineering inventions followed, by the analysis of some specific circuit configuration operating on specialized signal format. These analysis were directed towards predicting the behavior and performance of these circuits for the particular modulation/coding combinations.

With the derived structure approach, no a priori guesses are made on the structure of the synchronizer. In order to determine the performance achievable for a given signal format, channel response, and disturbances, we need a general theory of synchronization. We also need a theory which enables us to perform synthesis
of new configurations. The derived structure approach have several important advantages.

1. We can systematically derive algorithms for a given modulation/coding scheme and channel. These algorithms are, with respect to the chosen criterion, the best. The most powerful of such algorithms is the "maximum likelihood (ML) estimation."
2. Even if the algorithms are found to be unfeasible (due to complexity), one can use particular simplification which renders a feasible algorithm, though only sub-optimal.
3. Lower bounds (Cramer - Rao bounds) can be obtained, which are of great importance, particularly where it is very difficult to compute the performance of an algorithm. Lower bounds are also important in comparing the performance of sub-optimal algorithm to the theoretically best solution.

Maximum likelihood methods usually require heavy mathematical analysis. However, such approach is particularly useful when advanced modulation / coding schemes are employed, and/or when unknown channel parameters have to be extracted from the received signal.

With the "ad-hoc" structure approach;

1. The synchronizer consists of relatively simple basic building blocks such as nonlinearity, filters, voltage control oscillators, etc. The type and technology of these blocks mostly depends on the data rate. Particularly for high data rate, analog circuitry are usually used.
2. The technique for solving ad-hoc structure problems are straight forward. One allows some parameters within the structure to vary and chooses those whose values are optimum for a given performance criterion.

There are two main advantages to this approach

- It uses well understood building blocks in the synchronizer which greatly simplifies design, testing, and manufacturing.
- It uses only partial information about the channel and therefore it is not very sensitive in detecting the modulation errors.

The most serious disadvantage of using a fixed structure is that it is often impossible to tell whether the chosen structure is close to optimum.

Using the ad-hoc structure approach we introduce, in the next two sections, the principle of carrier and clock synchronization

### 1.1.1 Carrier Synchronization

Let the transmitter signal be a passband signal, wherein a carrier, $\cos \omega_{0} t$ is modulated by a baseband (See Fig. 1.1).

$$
\begin{equation*}
s(t)=\sqrt{2} s_{L}(t) A \cos \left(\omega_{0} t-\theta_{0}\right) \tag{1.1}
\end{equation*}
$$

where

$$
\begin{equation*}
s_{L}(t)=\sum_{k} a_{k} g_{T}\left(t-k T-\epsilon_{0} T\right) \tag{1.2}
\end{equation*}
$$



Figure 1.1: Carrier modulated transmission link
The symbols. $a_{k}= \pm 1$, amplitude modulate a sequence of pulses of $T$ seconds duration and each hare a pulse shape $g_{T}(t)$. The constant $\epsilon_{0}$ describes the time shift of the transmitted baseband time axis relative to a hypothetical reference time point (observer) and $\theta_{T}$ is the transmitter phase relative to the reference phase point.

We assume that the channel $(C(\omega))$ causes no distortion $(|C(\omega)|=1)$ but a delay of $D$ seconds. $D=M T+\epsilon_{c} T$. Since the receiver is only interested in the sequence of symbols, the delay $(M T)$ is of no concern and can be omitted. Therefore, the received signal is given by;

$$
\begin{equation*}
y(t)=\sqrt{2} s_{L}\left(t-\epsilon_{0} T\right) \cos \left(\omega_{0} t-\theta_{T}\right) \tag{1.3}
\end{equation*}
$$

where $\theta_{T}=\theta_{c}+\theta_{0}$, and $\theta_{c}$ is the $\omega_{0} D$ modulu $2 \pi$.

At the receiver, we assume having a perfect oscillator with frequency $\omega_{0}$ and phase $\theta_{R}$ relative to the reference phase point. Multiplying the received signal by this locally generated signal, $\sqrt{2} \cos \left(\omega_{0} t-\theta_{R}\right)$, we get

$$
\begin{equation*}
y(t) \cdot \sqrt{2} \cos \left(\omega_{R} t-\theta_{r}\right)=Y_{L}(t) \cos \left(\theta_{T}-\theta_{R}\right)+\text { high frequency terms } \tag{1.4}
\end{equation*}
$$

where

$$
\begin{equation*}
Y_{L}(t)=s_{L}\left(t-\epsilon_{T} T\right) \tag{1.5}
\end{equation*}
$$

Rejecting the high frequency terms, we obtain back the baseband signal, $s_{L}(t)$, time shifted relative to the hypothetical reference time point by $\epsilon_{T}=\epsilon_{0}+\epsilon_{c}$. Tracking this time is the duty of a clock synchronizer. It may seem that the multiplying factor, $\cos \left(\theta_{T}-\theta_{R}\right)$, is only a constant attenuation. In fact, due to the random nature of $\theta_{T}$ and $\theta_{R}$, this factor can cause serious distortion. The task of a "Carrier Synchronizer" is, therefore, to minimize the phase difference,

$$
\begin{equation*}
\Phi=\theta_{T}-\theta_{R} \tag{1.6}
\end{equation*}
$$

This is done by first estimating the difference between the received signal and the locally generated oscillation. The estimate is then used in controlling the phase of the reference oscillator to have $\theta_{R}=\theta_{T}$. Notice that only the difference phase error $\Phi$, not $\theta_{T}$ and $\theta_{R}$, is involved in our synchronization process.

Circuit implementation of such carrier synchronizer is depicted by the "error tracking" system of Fig. 1.2. An error signal, as a function of the alignment error (phase $\Phi$ ), is detected by the error detector. The error is then fed back to adjust the carrier frequency (voltage control oscillator). If properly designed, the error signal will be forced to zero as required.


Figure 1.2: Basic building blocks of an error tracking system
Let us assume that the receive signal is a pure sinusoid;

$$
\begin{equation*}
y(t)=\sqrt{2} A \sin \left(\omega_{0} t-\theta_{T}\right) \tag{1.7}
\end{equation*}
$$

Multiplying by the locally generated reference,

$$
\begin{equation*}
r(t)=\sqrt{2} \cos \left(\omega_{0} t-\theta_{R}\right) \tag{1.8}
\end{equation*}
$$

yields

$$
\begin{equation*}
y(t) r(t)=A \sin \left(\theta_{T}-\theta_{R}\right)+A \sin \left(2 \omega_{0} t-\theta_{T}-\theta_{R}\right) \tag{1.9}
\end{equation*}
$$

The double frequency term can be filtered out using a lowpass filter. The other term vanishes only when $\Phi=\theta_{T}-\theta_{R}$ is zero or $\pm \pi$. Additional circuits are needed to distinguish between the correct tracking point, zero phase, and the false error points $( \pm \pi)$.

The multiplier serves as a phase error detector, provided that the initial frequency difference between that of the local oscillator (adjustable clock) and of the received signal is small enough. Otherwise, auxiliary circuits are needed for frequency acquisition. The simple discussion presented so far deals with the principle operation of PLL's.

For modulation formats which exhibit high bandwidth and/or power efficiency, such as double sideband suppressed carrier (DSB), verstigal sideband (VSB), and quadrature amplitude modulation (QAM), we obtain, using the same error tracking as before, different values of phase accuracy measurements [8]. For DSB, we suffer reduction in signal-to-noise at the detector output, proportional to $\cos ^{2} \Phi$, when additive noise is present with the received signal. For VSB, we are also faced with an extra term of interference, called "quadrature distortion," when $\Phi \neq 0$. In the case of QAM, wherein two DSB signals are superimposed, we are faced with a "crosstalk" interference when $\Phi \neq 0$. Therefore the price of doubling the bandwidth efficiency in VSB or QAM, relative to DSB, causes an increase in phase error sensitivity. We must also add that error detection in carrier recovery process is more complex in VSB and QAM, relative to DSB.

### 1.1.2 Clock Synchronization (Principles)

To present the principles behind clock synchronization, we refer to Fig. 1.1. Assuming we have an accurate carrier recovery, we get, from (1.4) amd (1.5) together with (1.2);

$$
\begin{equation*}
y_{L}(t)=\sum_{k} a_{k} g_{T}\left(t-k T-\epsilon_{T} T\right)+n_{L}(t) \tag{1.10}
\end{equation*}
$$

where, again, $\epsilon_{T}=\epsilon_{0}+\epsilon_{c}$, is the time shift of the retrieved signal relative to the hypothetical reference time, and $n_{L}(t)$ is baseband additive noise. In order to obtain the sequence $\left\{a_{k}\right\}$, we must sample $y_{L}(t)$ at instances when the amplitude of the pulse is maximum. If the sampling times, using the local clock generator, are given by;

$$
\begin{equation*}
r(t)=\sum_{k} \delta\left(t-k T-\epsilon_{R} T\right) \tag{1.11}
\end{equation*}
$$



Figure 1.3: "Square and Filter" clock synchronizer
where $\delta(t)$ are delta impulses, we must then adjust $\epsilon_{R}$, using the difference $\epsilon=$ $\epsilon_{T}-\epsilon_{R}$, to make $\epsilon=0$ Therefore an ad-hoc structure, similar to Fig 1.2, with delay control of the adjustable local clock, instead of phase control, can do the job. This is termed in the literature, "delay locked loop (DLL)."

Instead of the DLL, a practical, and frequently used, ad-hoc method to recover the timing information from the data-carrying signal, is the squarer filter approach depicted in Fig. 1.3. From (1.10), we get (asssuming zero noise),

$$
\begin{align*}
y^{2}(t)= & {\left[\sum_{k} a_{k} g_{T}\left(t-k T-\epsilon_{T}\right)\right]^{2} } \\
= & \sum_{k} a_{k}^{2} g_{T}^{2}\left(t-k T-\epsilon_{T} T\right) \\
& +\sum_{k} \sum_{m \neq 0} a_{k} a_{k+m} g\left(t-k T-\epsilon_{T} T\right) g\left[t-(k+m) T-\epsilon_{T} T\right] \tag{1.12}
\end{align*}
$$

If the data symbols are statistically independent and equiprobable, then the second term has a zero mean. Since $a_{k}^{2}=1$, the first term is periodic with period $1 / T$.

After the lowpass filter, we get the fundamental of this periodic term

$$
\begin{equation*}
z(t)=\sqrt{2} A_{z} \sin \left[\frac{2 \pi}{T}\left(t-\epsilon_{T} T-\Delta\right)\right]+\text { disturbance } \tag{1.13}
\end{equation*}
$$

The time shift $\Delta$, depends on the pulse shape $g_{T}(t)$ and is independent of $\epsilon_{T}$. The zero crossing of $z(t)$ will give us the time reference needed for sampling $y(t)$. Clearly, due to the added disturbance, the nominal zero crossings of $z(t)$ will fluctuate (i.e. jitter). Also, $A_{z}$ may depend on the parameters of the system.

Notice, with the ad-hoc structure of Fig. 1.3,

1. We used a memoryless nonlinearity to regenerate the timing wave, and a linear time invariant filter to extract the clock rate component.
2. Within this class of systems, for example, one might look for system parameters which minimize the random fluctuation; order of nonlinearity, filter response, etc.
3. The random fluctuations of the zero crossings depend on the pulse shape $g_{T}(t)$ and the statistical properties of the symbols.

### 1.2 Ad-hoc Symbol Timing Recovery (STR) Circuits

Since the objective of this thesis is to evaluate the performance of $S T R$ circuits using a general type of nonlinearity, we scan, in this section, the different ad-hoc $S T R$ circuits which have the principle structure of the previous section. In fact, timing circuits might simply contain a narrowband filter tuned to a harmonic of the pulse repetition frequency. Such a scheme works in situations where the data
sequence has discrete spectral components. The performance of a single, high $Q$, resonant circuit tuned to the pulse repetition rate has received considerable attention, particularly in connection with its use in regenerative repeaters for PCM systems, [9]-[11]. Assuming that a passband signal is modulated by a random sequence, then the existence of a discrete spectral components would require that the data sequence have a non zero mean value, and that the Fourier transform of the data pulse not vanish at some multiple of the pulse repetition rate. [12]

As it was mentioned in the previous section, for power and bandwidth efficiency, none of the above cited conditions hold. Nevertheless, it has been recognized (see also previous section) that the previous scheme will work if a nonlinear element such as, square law device [13], absolute value[14], fourth-law rectifier [15], the threshold crossing device [16], and half bit delay detector [17], is placed before the narrowband filter. For the special case when the pulses are duration limited and nonoverlapping, the previous method of analysis used in [9]-[11] can be applied. This is so, because the output of the memoryless nonlinear element can be interpreted as a new data sequence with a modified pulse shape. In situations of practical interest in data communications, the signals are sharply bandlimited (duration unlimited) and hence, will experience considerable pulse overlap. This results in more difficulties in analyzing the performance of timing extraction circuits.

Franks and Bubrouski [13] considered a timing circuit involving only a square-law device followed by a narrowband filter. They showed that such a circuit gives satisfactory performance even when the data sequence is bandlimited to less than the pulse repetition frequency. They used a different, untraditional, approach in analyzing the degree of fluctuations in the position of the zero crossings of the
timing wave, timing jitter.

It was recognized in the literature that the amount of timing jitter depends jointly on the shape of the data pulses and the particular sequence of pulse amplitudes (data patterns). Takasaki [18], attempting to separate these factors, suggested to examine the phase of the timing wave resulting from a repetitive pulse pattern of a given length. The entire set of patterns, for a given pulse shape, needs to be examined in order to determine the maximum deviation in phase shift. Takasaki found an interesting frequency domain criteria for bounding the maximum deviation. But these criteria are easily applied only when the pattern length is short enough.

In chapter 2, we follow Frank and Bubrouski [13] in analyzing an STR circuit employing a squarer type of nonlinearity followed by a narrow band filter under certain band limiting conditions, for an arbitrary pulse shape. We assume certain statistical properties of the random data sequence and examine the statistical property of the resulting cyclostationary timing wave. It is the cyclostationarity property of the timing wave that allows us to extract timing information. In fact the function of $S T R$ circuits is to emphasize the degree of cyclostationarity of the data sequence.

First, in this chapter, we evaluate the mean value of the extracted timing wave, whose zero crossings determine the nominal timing instants. Next, we evaluate the mean-squared value of the timing wave, both in the time and frequency domains. Finally, we derive the final rms jitter expression. These results are obtained as a function of the Fourier transform of the pulse shape, prefilter response,
and the postfilter transfer function. We use our analytical expression to evaluate the jitter performance for three different prefilter responses;

1. Ideal Square type prefilter
2. Raised cosine prefilter
3. Trapezoidal prefilter

The first case was completed analytically with performance results being a direct function of the excess bandwidth, $\gamma$, and the quality factor, $Q$, of the postfilter. For the other two cases, we used conventional, numerical integration, computer routines.

The results of these calculations show that, other parameters kept equal, the behavior of the circuit depends on the excess bandwidth of the data pulses, in such a way that, the performance is satisfactory for medium and long values of excess bandwidth. However the performance becomes poorer as this factor decreases. In the extreme case of minimum bandwidth, Nyquist bandwidth, this method of clock recovery fails. Unfortunately, clock circuits implemented with non square-law devices are hardly tractable, mathematically, and their performance has only been evaluated by using computer simulation. [19]

In chapter 3, we concentrate on the performance of $S T R$ circuits, implemented by a memoryless device having an even, high order, nonlinearity. For analysis, we use a new approach that is based on the moments of the input to the nonlinear device. As in chapter 2, we first evaluate the mean value of the extracted timing wave, using the moment method. We show that for the special case of a
squarer, the results are the same as those obtained using the method of chapter 2. The mean squared value of the timing wave is related to the Fourier transforms, $R_{0}(f)$ and $R_{2}(f)$, of the Fourier coefficients, $r_{0}(\tau)$ and $r_{2}(\tau)$, of the periodic autocorrelation function, $R_{y}(t, s)$, of the output of the nonlinear device. Although not simple, this autocorrelation function depends on the joint cross moments of the input, $x(t)$ to the nonlinearity. The cross moments of $x(t)$, at two time points, are related to the derivatives of the joint characteristic function. A recursive formula is introduced, which can be used to derive these derivatives, for any type of even nonlinearity. As in chapter 2 , the rms jitter can be calculated from the mean and mean squared values of the extracted timing wave.

## Chapter 2

## Jitter Performance of STR Circuit Employing a Squarer Type of Nonlinearity

Symbol Timing Recovery ( $S T R$ ) circuits, was the subject of investigation for a long time, by many people. A summary of their work, relative to the subject of this thesis, was mentioned in the previous chapter. In this chapter, an analytical approach is presented for the jitter performance of an STR circuit consisting of a prefilter, a zero-memory nonlinear device, and a narrow band postfilter tuned to the pulse repetition frequency. Assuming first a squarer type of nonlinearity, analytical expressions for the mean timing wave and the rms jitter in the timing wave are obtained as a function of the pre and post filtering characteristics. Based on these results, exact analytical expressions are obtained for the rms jitter, for the case of a square type of prefilter pulse shape. Then numerical computer routines are used to evaluate the jitter performance for a raised cosine and trapezoidal types of pulse shapes. Finally, the rms jitter curves are obtained as a function of the postfilter quality factor, $Q$, for some values of rolloff factor $\gamma$.


Figure 2.1: STR circuit employing a squarer type of nonlinearity

### 2.1 Evaluation of rms Timing Jitter

The jitter of timing wave is defined as the deviation, $\Delta \tau$, of the zero crossing of $z(t)$ from the nominal zero crossing, $t_{0}$, of the mean timing wave, $E[z(t)]$.

The main objective of this chapter is to calculate the root mean square (rms) of the jitter of the timing wave, extracted from the output, $z(t)$, of the Symbol Timing Recovery (STR) circuit depicted in Fig. 2.1. Notice that $z(t)$, the timing wave at the output of the $S T R$ circuit, is a random process. It will be shown later, in the next section, that the mean of the timing wave, $E[z(t)]$, is a sinusoidal function. We begin our derivation of the timing jitter by referring to Fig. 2.2, in which $E[z(t)]$ is sketched, along with another sample function of $z(t)$ whose zero crossing is at $t_{0}+\Delta \tau$.

Clearly $\Delta \tau$ is a random variable depending on the particular sample function. If $\Delta \tau$ is small then almost all of the sample functions will cross the horizontal


Figure 2.2: Timing wave sample functions
line with slopes that are almost equal, and given by

$$
\begin{equation*}
\text { slope }\{E[z(t)]\}=E\left[\dot{z}\left(t_{0}\right)\right] \tag{2.1}
\end{equation*}
$$

From Fig. 2.2,

$$
\begin{equation*}
\tan \theta=E\left[\dot{z}\left(t_{0}\right)\right]=\frac{z\left(t_{0}\right)}{\Delta \tau} . \tag{2.2}
\end{equation*}
$$

therefore

$$
\begin{equation*}
\frac{\Delta \tau}{T}=\frac{1}{T} \frac{z\left(t_{0}\right)}{E\left[\dot{z}\left(t_{0}\right)\right]} \tag{2.3}
\end{equation*}
$$

where we normalized the random variable $\Delta \tau$ to the period $T$. Hence

$$
\begin{align*}
\left(\frac{\Delta \tau}{T}\right)_{r m s} & \triangleq\left[E\left(\frac{\Delta \tau}{T}\right)^{2}\right]^{1 / 2} \\
& =\frac{1}{T} \frac{\left(E\left[z^{2}\left(t_{0}\right)\right]\right)^{1 / 2}}{\mid E\left[\dot{z}\left(t_{0}\right) \mid\right.} \tag{2.4}
\end{align*}
$$

In the next sections we will first evaluate the mean value of the timing wave, $E[z(t)]$, and equate this to zero to determine the zero crossing points, $t_{0}$. Next
we will evaluate the mean squared value, $E\left[z^{2}(t)\right]$ and using its Fourier domain representation, we will obtain our results in terms of the prefilter and post filter transfer functions. In deriving our final equation, we use certain bandlimiting conditions which are acceptable in practical systems. Finally, we use these terms in (2.4) to obtain the rms timing jitter.

### 2.2 Evaluation of the Mean Value of the Timing Wave

From Fig. 2.1.

$$
\begin{align*}
y(t) & =x^{2}(t) \\
& =\left(\sum_{k=-\infty}^{\infty} a_{k} g(t-k T)\right)\left(\sum_{l=-\infty}^{\infty} a_{l} g(t-l T)\right) \tag{2.5}
\end{align*}
$$

Making the variable change, $l=k+m$ we get

$$
\begin{align*}
y(t) & =\sum_{k=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} a_{k} a_{k+m} g(t-k T) g(t-(k+m) T) \\
& =\sum_{k=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} a_{k} a_{k+m} p_{m}(t-k T) \tag{2.6}
\end{align*}
$$

where,

$$
\begin{equation*}
p_{m}(t)=g(t) g(t-m T) \tag{2.7}
\end{equation*}
$$

At the output of the postfilter we have the timing wave, $z(t)$,

$$
\begin{equation*}
z(t)=y(t) \otimes h(t) \tag{2.8}
\end{equation*}
$$

where the $\otimes$ means convolution, and $h(t)$ is the impulse response of the postfilter. If we substitute the expression for $y(t)$ in (2.8), we get

$$
\begin{equation*}
z(t)=\sum_{k=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} a_{k} a_{k+m} q_{m}(t-k T) \tag{2.9}
\end{equation*}
$$

where

$$
\begin{equation*}
q_{m}(t)=p_{m}(t) \otimes h(t) \tag{2.10}
\end{equation*}
$$

The sequence of random variables $\left\{a_{k}\right\}$ are assumed to be independent of each other with mean zero, hence

$$
E\left[a_{k} a_{k+m}\right]= \begin{cases}E\left[a_{k}\right] E\left[a_{k+m}\right]=0 & m \neq 0  \tag{2.11}\\ E\left[a_{k}^{2}\right] & m=0\end{cases}
$$

For our case where $a_{k}$ being a binary sequence $( \pm 1)$ with equal probabilities, $E\left[a_{k}^{2}\right]=$ 1 and $E\left[a_{k}\right]=0$, and the mean timing wave can be expressed as

$$
\begin{align*}
E[z(t)] & =\sum_{k=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} E\left[a_{k} a_{k+m}\right] q_{m}(t-k T) \\
& =\sum_{k=-\infty}^{\infty} q_{0}(t-k T) \tag{2.12}
\end{align*}
$$

The next step will be to modify (2.12), by expressing it in the frequency domain, using the Poisson sum formula. This formula states that for any real time function $x(t)$ whose Fourier transform is $X(f)$;

$$
\begin{equation*}
\sum_{k=-\infty}^{\infty} x(t-k T)=\frac{1}{T} \sum_{l=-\infty}^{\infty} X\left(\frac{l}{T}\right) \exp \left(j \frac{2 \pi l t}{T}\right) \tag{2.13}
\end{equation*}
$$

Therefore, using (2.12) in (2.13) yields,

$$
\begin{equation*}
E[z(t)]=\frac{1}{T} \sum_{l=-\infty}^{\infty} Q_{0}\left(\frac{l}{T}\right) \exp \left(j \frac{2 \pi l t}{T}\right) \tag{2.14}
\end{equation*}
$$

where $Q_{0}(f)$ is the Fourier transform of $q_{0}(t)$. By using the Fourier transform of (2.10), with $m=0$, we have

$$
\begin{equation*}
q_{0}(t)=p_{0}(t) \otimes h(t) \Longleftrightarrow Q_{0}(f)=P_{0}(f) H(f) \tag{2.15}
\end{equation*}
$$

where $H(f)$ is the transfer function of the postfilter. Substituting (2.15) into (2.14) we obtain,

$$
\begin{equation*}
E[z(t)]=\frac{1}{T} \sum_{l=-\infty}^{\infty} P_{0}\left(\frac{l}{T}\right) H\left(\frac{l}{T}\right) \exp \left(j \frac{2 \pi l t}{T}\right) \tag{2.16}
\end{equation*}
$$

The filter $H(f)$ is a narrowband bandpass filter centered at the symbol rate frequency $1 / T$ and satisfies the bandlimiting condition, (see Fig. 2.3)

$$
\begin{equation*}
H(f)=0 \text { for }\left||f|-\frac{1}{T}\right|>\frac{1}{2 T} \tag{2.17}
\end{equation*}
$$

That is the frequency response of the filter at the second and higher harmonics of the symbol rate frequency is zero. Taking the band limiting condition (2.17) into


Figure 2.3: Transfer function of postfilter $H(f)$
consideration, it is clear that all the terms in the summation of (2.16) equal zero. except for $l= \pm 1$. Therefore,

$$
\begin{equation*}
E[z(t)]=\frac{1}{T}\left[P_{0}\left(-\frac{1}{T}\right) H\left(-\frac{1}{T}\right) \exp \left(-j \frac{2 \pi t}{T}\right)+P_{0}\left(\frac{1}{T}\right) H\left(\frac{1}{T}\right) \exp \left(j \frac{2 \pi t}{T}\right)\right] \tag{2.18}
\end{equation*}
$$

Note that $p(t)$ and $h(t)$ are real functions which implies that $P(-f)$ and $H(-f)$ are the complex conjugates of $P(f)$ and $H(f)$ respectively. Hence.

$$
\begin{equation*}
E[z(t)]=\frac{1}{T}\left[P_{0}\left(\frac{1}{T}\right) H\left(\frac{1}{T}\right) \exp \left(j \frac{2 \pi t}{T}\right)+P_{0}^{*}\left(\frac{1}{T}\right) H^{*}\left(\frac{1}{T}\right) \exp \left(-j \frac{2 \pi t}{T}\right)\right] \tag{2.19}
\end{equation*}
$$

Define

$$
\begin{align*}
P_{0}\left(\frac{1}{T}\right) & =\left|P_{0}\left(\frac{1}{T}\right)\right| \exp \left(j \phi_{1}\right)  \tag{2.20}\\
H\left(\frac{1}{T}\right) & =\left|H\left(\frac{1}{T}\right)\right| \exp \left(j \phi_{2}\right) \tag{2.21}
\end{align*}
$$

Then (2.19) simplifies to

$$
\begin{equation*}
E[z(t)]=\frac{2}{T}\left|P_{0}\left(\frac{1}{T}\right)\right|\left|H\left(\frac{1}{T}\right)\right| \cos \left(\frac{2 \pi t}{T}+\phi\right) \tag{2.22}
\end{equation*}
$$

where $\phi=\phi_{1}+\phi_{2}$. Notice that from (2.7), $p_{0}(t)=g^{2}(t)$ so that,

$$
\begin{equation*}
P_{0}(f)=G(f) \otimes G(f) \tag{2.23}
\end{equation*}
$$

and hence

$$
\begin{equation*}
P_{0}\left(\frac{1}{T}\right)=\int_{-\infty}^{\infty} G(\alpha) G\left(\frac{1}{T}-\alpha\right) d \alpha \tag{2.24}
\end{equation*}
$$

From (2.22) we can conclude that the mean of the timing wave, $E[z(t)]$ is a sinusoid whose magnitude is

$$
\begin{equation*}
\frac{2}{T}\left|P_{0}\left(\frac{1}{T}\right)\right|\left|H\left(\frac{1}{T}\right)\right| \tag{2.25}
\end{equation*}
$$

and whose zero crossings, $t_{0}$, satisfies

$$
\begin{align*}
\frac{2 \pi t_{0}}{T}+\phi & =n \frac{\pi}{2} \\
t_{0} & =T\left(\frac{n}{4}-\frac{\phi}{2 \pi}\right) \tag{2.26}
\end{align*}
$$

where $n$ is an odd integer. It is at these zero crossing instants $\left(t_{0}\right)$ that we evaluate the mean of the timing waveform (see Eq. 2.4).

### 2.3 Evaluation of the Mean-Squared Value of the Timing Wave

Starting from (2.9);

$$
\begin{equation*}
z(t)=\sum_{k=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} a_{k} a_{k+m} q_{m}(t-k T) \tag{2.27}
\end{equation*}
$$

where $q_{m}(t-k T)$ was previously defined in (2.10), to obtain the general expression for the mean square;

$$
\begin{equation*}
E\left[z^{2}(t)\right]=\sum_{k} \sum_{m} \sum_{j} \sum_{l} E\left[a_{k} a_{k+m} a_{k+j} a_{k+j+l}\right] q_{m}(t-k T) q_{l}(t-k T-j T) \tag{2.28}
\end{equation*}
$$

Checking all the possibilities for $m, j$, and $l$ we can reduce the expectation terms according to the following conditions (see appendix A)

$$
E\left[a_{k} a_{k+m} a_{k+j} a_{k+j+l}\right]= \begin{cases}E\left[a^{4}\right] & \text { for } m=l=j=0  \tag{2.29}\\ \left\{E\left[a^{2}\right]\right\}^{2} \triangleq \alpha_{0}^{2} & \text { for } m=l=0 j \neq 0 \\ \alpha_{0}^{2} & \text { for } m=l \neq 0 j=0 \\ \alpha_{0}^{2} & \text { for } m=j \neq 0 l=-j \\ 0 & \\ \text { otherwise }\end{cases}
$$

Substituting in (2.28) we get:

$$
\begin{align*}
E\left[z^{2}(t)\right]= & E\left[a_{k}^{4}\right] \sum_{k} q_{0}^{2}(t-k T)+\alpha_{0}^{2} \sum_{k} \sum_{j \neq 0} q_{0}(t-k T) q_{0}(t-k T-j T) \\
& +\alpha_{0}^{2} \sum_{k} \sum_{m} q_{m \neq 0}(t-k T) q_{m}(t-k T) \\
& +\alpha_{0}^{2} \sum_{k} \sum_{m} \sum_{m \neq 0} q_{m}(t-k T) q_{-m}(t-k T-m T)  \tag{2.30}\\
E\left[z^{2}(t)\right]= & E\left[a_{k}^{4}\right] \sum_{k} q_{0}^{2}(t-k T)+\alpha_{0}^{2} \sum_{k} \sum_{j} q_{0}(t-k T) q_{0}(t-k T-j T) \\
& -\alpha_{0}^{2} \sum_{k} q_{0}^{2}(t-k T)+\alpha_{0}^{2} \sum_{k} \sum_{m} q_{m}^{2}(t-k T) \\
& -\alpha_{0}^{2} \sum_{k} q_{0}^{2}(t-k T)+\alpha_{0}^{2} \sum_{k} \sum_{m} q_{m}(t-k T) q_{-m}(t-k T-m T) \\
& -\alpha_{0}^{2} \sum_{k} q_{0}^{2}(t-k T) \tag{2.31}
\end{align*}
$$

By letting $l=k+j$, in the second term of the right hand expression of (2.31), we obtain;

$$
\begin{align*}
\sum_{k} \sum_{j} q_{0}(t-k T) q_{0}(t-k T-j T) & =\sum_{k} \sum_{l} q_{0}(t-k T) q_{0}(t-l T) \\
& =\left(\sum_{k} q_{0}(t-k T)\right)^{2} \tag{2.32}
\end{align*}
$$

Also, noting that (see appendix B)

$$
\begin{equation*}
q_{-m}(t-k T-m T)=q_{m}(t-k T) \tag{2.33}
\end{equation*}
$$

we get the following expression for the mean squared value of the timing wave,

$$
\begin{align*}
E\left[z^{2}(t)\right]= & {\left[\alpha_{0} \sum_{k} q_{0}(t-k T)\right]^{2}+\left(\bar{a}_{k}^{4}-3 \alpha_{0}^{2}\right) \sum_{k} q_{0}^{2}(t-k T) } \\
& +2 \alpha_{0}^{2} \sum_{k} \sum_{m} q_{m}^{2}(t-k T) \tag{2.34}
\end{align*}
$$

Using the definition of the variance,

$$
\begin{equation*}
\operatorname{var}[z(t)]=E\left[z^{2}(t)\right]-\{E[z(t)]\}^{2} \tag{2.35}
\end{equation*}
$$

and noting, from (2.27), that

$$
\begin{equation*}
E[z(t)]=\alpha_{0} \sum_{k} q_{0}(t-k T) \tag{2.36}
\end{equation*}
$$

is related to the first term on the right hand side of (2.34), we obtain the expression for the variance of the timing wave,

$$
\begin{equation*}
\operatorname{var}[z(t)]=2 \alpha_{0}^{2} \sum_{k} \sum_{m} q_{m}^{2}(t-k T)+\left(\overline{a_{k}^{4}}-3 \alpha_{0}^{2}\right) \sum_{k} q_{0}^{2}(t-k T) \tag{2.37}
\end{equation*}
$$

where (see 2.10 and 2.7),

$$
\begin{equation*}
q_{m}(t-k T)=[g(t-k T) g[t-k T-m T)] \otimes h(t) \tag{2.38}
\end{equation*}
$$

and

$$
\begin{align*}
\bar{a}_{k}^{4}=\bar{a}^{4} & \triangleq E\left[a^{4}\right] \\
\alpha_{0} & =E\left[a^{2}\right] \tag{2.39}
\end{align*}
$$

It should be noted that since $E\left[z\left(t_{0}\right)\right]=0$, then at the zero crossings, $t_{0}$, of the mean both the variance and the mean squared values are equal. Therefore when
we calculate the rms jitter at $t_{0}$ (Eq. 2.4) we can use the variance expression rather than the mean squared.

The variance expression we obtained thus far, is a time domain representation which is rather difficult to evaluate. As a matter of fact this can probably be done by using a computer program. However our main goal in this chapter is to determine the jitter analytically. We therefore use, in the next section, the frequency domain approach in attempt to simplify our analytical expression further.

### 2.3.1 Frequency Domain Representation of the Variance of the Timing Wave

The purpose of this section is to transform the time domain expression of the variance of the timing wave into frequency domain. This will be achieved by using the Poisson sum formula discussed earlier. The final expression will be analytically simple. For the convenience of the reader we will separate the process into two parts.

Part 1, Frequency domain representation of $\sum \sum q_{m}^{2}(t-k T)$

Let

$$
\begin{equation*}
a(t)=\sum_{m} q_{m}^{2}(t) \tag{2.40}
\end{equation*}
$$

where

$$
\begin{equation*}
q_{m}(t)=[g(t) g(t-m T)] \otimes h(t) \tag{2.41}
\end{equation*}
$$

If we look at $a(t)$ as a pulse in time domain, we could say that

$$
\begin{equation*}
\sum_{k=-\infty}^{\infty} a(t-k T)=\sum_{k} \sum_{m} q_{m}^{2}(t-k T) \tag{2.42}
\end{equation*}
$$

is periodic with period $T$. Using the Poisson formula we can write

$$
\begin{equation*}
\sum_{k=-\infty}^{\infty} a(t-k T)=\sum_{r} C_{r} \exp \left(j 2 \pi f_{0} r t\right) \tag{2.43}
\end{equation*}
$$

where $f_{0}=\frac{1}{T}$, and $C_{r}^{\prime} s$ are the Fourier coefficients given by

$$
\begin{equation*}
C_{r}=\frac{A\left(r f_{0}\right)}{T} \tag{2.44}
\end{equation*}
$$

with $A(f)$ being the Fourier transform of $a(t)$. By substituting (2.43) and (2.44) in (2.42) we obtain the time-frequency relation

$$
\begin{equation*}
\sum_{k} \sum_{m} q_{m}^{2}(t-k T)=\frac{1}{T} \sum_{r=-\infty}^{\infty} A\left(\frac{r}{T}\right) \exp \left(j \frac{2 \pi r t}{T}\right) \tag{2.45}
\end{equation*}
$$

From (2.40),

$$
\begin{align*}
A(f) & =\mathcal{F}[a(t)] \\
& =\sum_{m}\left[Q_{m}(f) \otimes Q_{m}(f)\right] \\
& =\sum_{m}^{m}\left\{\left[P_{m}(f) H(f)\right] \otimes\left[P_{m}(f) H(f)\right]\right\} \tag{2.46}
\end{align*}
$$

where $\mathcal{F}[\cdot]$ represents Fourier transform. Note that we used the fact that $q_{m}(t)=$ $p_{m}(t) \otimes h(t)$. Our next step will be to simplify the expression for $Q_{m}(f)$. We know that

$$
\begin{equation*}
p_{m}(t)=g(t) g(t-m T) \tag{2.47}
\end{equation*}
$$

implies that

$$
\begin{align*}
P_{m}(f) & =G(f) \otimes[G(f) \exp (-j 2 \pi f m T)] \\
& =\int_{\nu} G(\nu) \exp (-j 2 \pi \nu m T) G(f-\nu) d \nu \tag{2.48}
\end{align*}
$$

and

$$
\begin{equation*}
Q_{m}(f)=H(f) \int_{\nu} G(\nu) \exp (-j 2 \pi \nu m T) G(f-\nu) d \nu \tag{2.49}
\end{equation*}
$$

Therefore

$$
\begin{align*}
A(f)= & \sum_{m} \int_{\alpha} Q_{m}(\alpha) Q_{m}(f-\alpha) d \alpha \\
= & \sum_{m}\left\{\int_{\alpha}\left[H(\alpha) \int_{\nu} G(\nu) \exp (-j 2 \pi \nu m T) G(\alpha-\nu) d \nu\right]\right. \\
& \left.\quad\left[H(f-\alpha) \int_{\beta} G(\beta) \exp (-j 2 \pi \beta m T) G(f-\alpha-\beta) d \beta\right] d \alpha\right\} . \tag{2.50}
\end{align*}
$$

We now group and rearrange the order or summation and integration to produce a more understandable form for our equation.

$$
\begin{gather*}
A(f)=\int_{\alpha} \int_{\nu} \int_{\beta}[H(\alpha) H(f-\alpha) G(\nu) G(\alpha-\nu) G(\beta) G(f-\alpha-\beta) \\
\left.\sum_{m} \exp (-j 2 \pi(\nu+\beta) m T)\right] d \beta d \nu d \alpha \tag{2.51}
\end{gather*}
$$

If we use the Poisson sum formula again, we can get rid of the summation term. In fact,

$$
\begin{equation*}
\sum_{k} x\left(t-k T^{\prime}\right)=\frac{1}{T^{\prime}} \sum_{r} X\left(\frac{r}{T^{\prime}}\right) \exp \left(j \frac{2 \pi r t}{T^{\prime}}\right) \tag{2.52}
\end{equation*}
$$

leads for $x(t)=\delta(t)$,

$$
\begin{equation*}
\sum_{k} \delta\left(t-k T^{\prime}\right)=\frac{1}{T^{\prime}} \sum_{r} 1 \cdot \exp \left(j \frac{2 \pi r t}{T^{\prime}}\right) \tag{2.53}
\end{equation*}
$$

Substituting $T=\frac{1}{T^{\prime}}$, we get

$$
\begin{equation*}
\frac{1}{T} \sum_{k} \delta\left(t-\frac{k}{T}\right)=\sum_{r} \exp (j 2 \pi r t T) \tag{2.54}
\end{equation*}
$$

Now make the variable change $m=-r$,

$$
\begin{equation*}
\frac{1}{T} \sum_{k} \delta\left(t-\frac{k}{T}\right)=\sum_{m} \exp (-j 2 \pi m t T) \tag{2.55}
\end{equation*}
$$

A further variable change, $t=\nu+\beta$, results in,

$$
\begin{equation*}
\sum_{m} \exp (-j 2 \pi(\nu+\beta) m T)=\frac{1}{T} \sum_{k} \delta\left(\nu+\beta-\frac{k}{T}\right) \tag{2.56}
\end{equation*}
$$

Note that the left hand part of (2.56) is identical to the summation term inside the integral of (2.51). If we substitute(2.56) in (2.51) we will get a new expression for $A(f)$,

$$
\begin{gather*}
A(f)=\frac{1}{T} \sum_{k}\left\{\int_{\alpha} \int_{\nu} \int_{\beta} H(\alpha) H(f-\alpha) G(\nu) G(\alpha-\nu) G(\beta) G(f-\alpha-\beta)\right. \\
\left.\delta\left(\nu+\beta-\frac{k}{T}\right) d \beta d \nu d \alpha\right\} \tag{2.57}
\end{gather*}
$$

Integrating (2.57) with respect to $\nu$, we get

$$
\begin{gather*}
A(f)=\frac{1}{T} \sum_{k}\left[\int_{\alpha} \int_{\beta} H(\alpha) H(f-\alpha) G\left(\frac{k}{T}-\beta\right) G\left(\alpha+\beta-\frac{k}{T}\right)\right. \\
G(\beta) G(f-\alpha-\beta) d \beta d \alpha] \tag{2.58}
\end{gather*}
$$

Equation 2.58 is the frequency domain representation of $a(t)=\sum_{m} q_{m}^{2}(t)$. Substituting (2.58) with $f=\frac{r}{T}$, in (2.45), we finally obtain

$$
\begin{gather*}
\sum_{k} \sum_{m} q_{m}^{2}(t-k T)=\frac{1}{T^{2}} \sum_{r} \sum_{k} \int_{\alpha} \int_{\beta} H(\alpha) H\left(\frac{r}{T}-\alpha\right) G\left(\frac{k}{T}-\beta\right) \\
G\left(\alpha+\beta-\frac{k}{T}\right) G(\beta) G\left(\frac{r}{T}-\alpha-\beta\right) \\
\exp \left(j \frac{2 \pi r t}{T}\right) d \alpha d \beta \tag{2.59}
\end{gather*}
$$

This is the frequency domain representation of the first sum in the expression for the variance of $z(t)$, in (2.37). In the next section we find the frequency domain representation of the second sum of (2.37).

Part 2. Frequency domain representation of $\sum q_{0}^{2}(t-k T)$

Just as in part 1, we start by letting

$$
\begin{equation*}
b(t)=q_{0}^{2}(t) \tag{2.60}
\end{equation*}
$$

where

$$
\begin{equation*}
q_{0}(t)=[g(t) g(t)] \otimes h(t) \tag{2.61}
\end{equation*}
$$

Therefore by using the Poisson formula we can write

$$
\begin{equation*}
\sum_{k} b(t-k T)=\sum_{r} C_{r} \exp \left(j 2 \pi f_{0} r t\right) \tag{2.62}
\end{equation*}
$$

where

$$
\begin{equation*}
C_{r}=\frac{B\left(r f_{0}\right)}{T} \tag{2.63}
\end{equation*}
$$

$B(f)$ is the Fourier transform of $b(t)$, and $f_{0}=\frac{1}{T}$. Using (2.63) in (2.62), together with (2.60), we get

$$
\begin{equation*}
\sum_{k} q_{0}^{2}(t-k T)=\frac{1}{T} \sum_{r} B\left(\frac{r}{T}\right) \exp \left(j \frac{2 \pi r t}{T}\right) \tag{2.64}
\end{equation*}
$$

From(2.60) and (2.61),

$$
\begin{align*}
B(f) & =Q_{0}(f) \otimes Q_{0}(f) \\
& =\left[P_{0}(f) H(f)\right] \otimes\left[P_{0}(f) H(f)\right] \tag{2.65}
\end{align*}
$$

Note that we used the fact that $q_{0}(t)=p_{0}(t) \otimes h(t)$. Also,

$$
\begin{equation*}
p_{0}(t)=g(t) g(t) \tag{2.66}
\end{equation*}
$$

implies,

$$
\begin{align*}
P_{0}(f) & =G(f) \otimes G(f) \\
& =\int_{\nu} G(\nu) G(f-\nu) d \nu \tag{2.67}
\end{align*}
$$

and

$$
\begin{equation*}
Q_{0}(f)=H(f) \int_{\nu} G(\nu) G(f-\nu) d \nu \tag{2.68}
\end{equation*}
$$

Therefore

$$
\begin{align*}
B(f)= & \int_{\alpha} Q_{0}(\alpha) Q_{0}(f-\alpha) d \alpha \\
= & \int_{\alpha}\left[H(\alpha) \int_{\nu} G(\nu) G(\alpha-\nu) d \nu\right] \\
& {\left[H(f-\alpha) \int_{\beta} G(\beta) G(f-\alpha-\beta) d \beta\right] d \alpha } \tag{2.69}
\end{align*}
$$

We now rearrange the order of integrations to produce

$$
\begin{array}{r}
B(f)=\int_{\alpha} \int_{\nu} \int_{\beta} H(\alpha) H(f-\alpha) G(\nu) G(\alpha-\nu) \\
G(\beta) G(f-\alpha-\beta) d \beta d \nu d \alpha . \tag{2.70}
\end{array}
$$

Substituting (2.70) with $f=\frac{r}{T}$ in (2.64) we obtain

$$
\begin{align*}
& \sum_{k} q_{0}^{2}(t-k T)=\frac{1}{T} \sum_{r} \int_{\alpha} \int_{\nu} \int_{\beta} H(\alpha) H\left(\frac{r}{T}-\alpha\right) G(\nu) G(\alpha-\nu) \\
& G(\beta) G\left(\frac{r}{T}-\alpha-\beta\right) \exp \left(j \frac{2 \pi r t}{T}\right) d \beta d \nu d \alpha \tag{2.71}
\end{align*}
$$

### 2.3.2 Effect of the Prefilter and Postfilter on the Frequency Representation of the Variance

Equations (2.45) and (2.64) determine the final frequency domain representation of the variance of the timing wave. Substituting these equations in (2.37) we get,

$$
\begin{align*}
\operatorname{var}[z(t)]= & \frac{2 \alpha_{0}^{2}}{T} \sum_{r} A\left(\frac{r}{T}\right) \exp \left(j \frac{2 \pi r t}{T}\right) \\
& +\frac{\overline{a^{4}}-3 \alpha_{0}^{2}}{T} \sum_{r} B\left(\frac{r}{T}\right) \exp \left(j \frac{2 \pi r t}{T}\right) \tag{2.72}
\end{align*}
$$

where, from (2.58) and (2.70) respectively,

$$
\begin{gather*}
A\left(\frac{r}{T}\right)=\frac{1}{T} \sum_{k} \int_{\alpha} \int_{\beta} H(\alpha) H\left(\frac{r}{T}-\alpha\right) G\left(\frac{k}{T}-\beta\right) G\left(\alpha+\beta-\frac{k}{T}\right) \\
G(\beta) G\left(\frac{r}{T}-\alpha-\beta\right) d \alpha d \beta \tag{2.73}
\end{gather*}
$$

and

$$
\begin{array}{r}
B\left(\frac{r}{T}\right)=\int_{\alpha} \int_{\nu} \int_{\beta} H(\alpha) H\left(\frac{r}{T}-\alpha\right) G(\nu) G(\alpha-\nu) \\
G(\beta) G\left(\frac{r}{T}-\alpha-\beta\right) d \nu d \beta d \alpha \tag{2.74}
\end{array}
$$

Equation (2.72) represents a frequency domain expression for the variance of the timing wave $z(t)$. Note that this equation depends on the responses of both the prefilter and the postfiler. We will now use the bandlimiting conditions of both responses, to determine a more simple version for the variance. We will start by applying the conditions to the first part of the expression and then apply the same procedure for the second part.

For the purpose of our derivations, we use a prefilter bandlimited to $1 / T$, that is

$$
\begin{equation*}
G(f)=0 \quad \text { for } \quad|f| \geq \frac{1}{T} \tag{2.75}
\end{equation*}
$$

and a postfilter with center frequency $1 / T$ and bandwidth $T$, whose low-pass equivalent is

$$
\begin{equation*}
H_{L P}(f)=0 \quad \text { for } \quad|f| \geq \frac{1}{2 T} \tag{2.76}
\end{equation*}
$$

Using the condition in (2.76) we note that the product

$$
\begin{equation*}
H(\alpha) H\left(\frac{r}{T}-\alpha\right) \tag{2.77}
\end{equation*}
$$

yields zero except for $r=0$ and $r= \pm 2$. Therefore

$$
\begin{equation*}
A\left(\frac{r}{T}\right) \neq 0 \quad \text { only for } \quad r=0, \pm 2 \tag{2.78}
\end{equation*}
$$

## Derivation of $A(0)$

For $r=0$ we have, from (2.73),

$$
\begin{equation*}
A(0)=\frac{1}{T} \sum_{k} \int_{\alpha} \int_{\beta}|H(\alpha)|^{2} G\left(\frac{k}{T}-\beta\right) G\left(\alpha-\left(\frac{k}{T}-\beta\right)\right) G(\beta) G(-\alpha-\beta) d \alpha d \beta \tag{2.79}
\end{equation*}
$$

Notice that $G(\beta)$ is a lowpass transfer function extending to a maximum of $f= \pm \frac{1}{T}$. This includes the case of a raised cosine pulse shape with rolloff factor $\gamma \leq 1$. Using such assumption, it is clear that $G\left(\frac{k}{T}-\beta\right) G(\beta)$ vanishes for $k>1$ and hence (2.79) reduces to

$$
\begin{align*}
A(0)= & \frac{1}{T} \int_{\alpha} \int_{\beta}|H(\alpha)|^{2} G(-\beta) G(\alpha+\beta) G(\beta) G(-\alpha-\beta) d \alpha d \beta \\
& +\frac{1}{T} \int_{\alpha} \int_{\beta}|H(\alpha)|^{2} G\left(\frac{1}{T}-\beta\right) G\left(\alpha+\beta-\frac{1}{T}\right) G(\beta) G(-\alpha-\beta) d \alpha d \beta \\
& +\frac{1}{T} \int_{\alpha} \int_{\beta}|H(\alpha)|^{2} G\left(-\frac{1}{T}-\beta\right) G\left(\alpha+\beta+\frac{1}{T}\right) G(\beta) G(-\alpha-\beta) d \alpha d \beta \tag{2.80}
\end{align*}
$$

Since $g(t)$ is real, $G^{*}(f)=G(-f)$ for every $f$. By using algebraic manipulation (See Eqn. C.10, appendix C) we rewrite (2.80),

$$
\begin{align*}
A(0)= & \frac{1}{T}\left\{\int _ { \alpha } \int _ { \beta } | H ( \alpha ) | ^ { 2 } \left(|G(\beta)|^{2}|G(\alpha+\beta)|^{2}\right.\right. \\
& \left.+2 \mathcal{R e}\left[C(\beta) C^{*}(\alpha+\beta)\right] d \alpha d \beta\right\} \tag{2.81}
\end{align*}
$$

where

$$
\begin{equation*}
C(f)=G(f) G\left(\frac{1}{T}-f\right) \tag{2.82}
\end{equation*}
$$

## Derivation of $A\left(\frac{2}{T}\right)$

For $r=2$ we have, from (2.73),
$A\left(\frac{2}{T}\right)=\frac{1}{T} \sum_{k} \int_{\alpha} \int_{\beta} H(\alpha) H\left(\frac{2}{T}-\alpha\right) G\left(\frac{k}{T}-\beta\right) G\left(\alpha-\left(\frac{k}{T}-\beta\right)\right) G(\beta) G\left(\frac{2}{T}-\alpha-\beta\right) d \alpha d \beta$.

It can easily be seen that

$$
\begin{equation*}
G(\beta) G\left(\frac{k}{T}-\beta\right)=0 \quad \text { for } \quad|k|>1 \tag{2.84}
\end{equation*}
$$

Therefore the right hand part of (2.84) can be expanded into

$$
\begin{align*}
& \frac{1}{T}\left[\int_{\alpha} \int_{\beta} H(\alpha) H\left(\frac{2}{T}-\alpha\right) G(-\beta) G(\alpha+\beta) G(\beta) G\left(\frac{2}{T}-\alpha-\beta\right) d \alpha d \beta\right. \\
+ & \int_{\alpha} \int_{\beta} H(\alpha) H\left(\frac{2}{T}-\alpha\right) G\left(\frac{1}{T}-\beta\right) G\left(-\frac{1}{T}+\alpha+\beta\right) G(\beta) G\left(\frac{2}{T}-\alpha-\beta\right) d \alpha d \beta \\
+ & \left.\int_{\alpha} \int_{\beta} H(\alpha) H\left(\frac{2}{T}-\alpha\right) G\left(-\frac{1}{T}-\beta\right) G\left(\frac{1}{T}+\alpha+\beta\right) G(\beta) G\left(\frac{2}{T}-\alpha-\beta\right) d \alpha d \beta\right] \tag{2.85}
\end{align*}
$$

Since $G(f)$ is limited to $|f|<1 / T$ and $|G(f)|=|G(-f)|$, the terms $G(\alpha+\beta)$ and $G\left(\frac{2}{T}-\alpha-\beta\right)$ do not overlap, resulting in a zero product in the first integral of (2.85). Similarly the product of $G\left(\frac{1}{T}+\alpha+\beta\right)$ and $G\left(\frac{2}{T}-\alpha-\beta\right)$ also result in a zero product for the third integral. We are thereby left with only the second integral. Using (2.82), we have

$$
\begin{equation*}
C\left(\alpha+\beta-\frac{1}{T}\right)=G\left(\alpha+\beta-\frac{1}{T}\right) G\left(\frac{1}{T}-\alpha-\beta+\frac{1}{T}\right) \tag{2.86}
\end{equation*}
$$

Therefore

$$
\begin{equation*}
A\left(\frac{2}{T}\right)=\frac{1}{T} \int_{\alpha} \int_{\beta} H(\alpha) H\left(\frac{2}{T}-\alpha\right) C(\beta) C\left(\alpha+\beta-\frac{1}{T}\right) d \alpha d \beta \tag{2.87}
\end{equation*}
$$

## Derivation of $A\left(-\frac{2}{T}\right)$

For $r=-2$ we have, from (2.73),
$\left.A\left(-\frac{2}{T}\right)=\frac{1}{T} \sum_{k} \int_{\alpha} \int_{\beta} H(\alpha) H\left(-\frac{2}{T}\right)-\alpha\right) G\left(\frac{k}{T}-\beta\right) G\left(\alpha+\beta-\frac{k}{T}\right) G(\beta) G\left(-\frac{2}{T}-\alpha-\beta\right) d \alpha d \beta$

Noting again that this equation is only valid for $k=0$ and $k= \pm 1$, the right hand side of (2.88) can be expanded into

$$
\begin{align*}
& \frac{1}{T}\left[\int_{\alpha} \int_{\beta} H(\alpha) H\left(-\frac{2}{T}-\alpha\right) G(-\beta) G(\alpha+\beta) G(\beta) G\left(-\frac{2}{T}-\alpha-\beta\right) d \alpha d \beta\right. \\
+ & \int_{\alpha} \int_{\beta} H(\alpha) H\left(-\frac{2}{T}-\alpha\right) G\left(\frac{1}{T}-\beta\right) G\left(-\frac{1}{T}+\alpha+\beta\right) G(\beta) G\left(-\frac{2}{T}-\alpha-\beta\right) d \alpha d \beta \\
+ & \left.\int_{\alpha} \int_{\beta} H(\alpha) H\left(-\frac{2}{T}-\alpha\right) G\left(-\frac{1}{T}-\beta\right) G\left(\frac{1}{T}+\alpha+\beta\right) G(\beta) G\left(-\frac{2}{T}-\alpha-\beta\right) d \alpha d \beta\right] \tag{2.89}
\end{align*}
$$

Similar to the previous part, note that the term $G(\alpha+\beta) G\left(-\frac{2}{T}-\alpha-\beta\right)$ causes the first integral expression to vanish. Also the product $G\left(-\frac{1}{T}+\alpha+\beta\right) G\left(-\frac{2}{T}-\alpha-\beta\right)$, in the second integral is zero. We are therefore left with only the third integral expression which becomes

$$
\begin{equation*}
\frac{1}{T} \int_{\alpha} \int_{\beta} H(-\alpha) H\left(-\frac{2}{T}+\alpha\right) G\left(-\frac{1}{T}+\beta\right) G\left(\frac{1}{T}-\alpha-\beta\right) G(-\beta) G\left(-\frac{2}{T}+\alpha+\beta\right) d \alpha d \beta \tag{2.90}
\end{equation*}
$$

Note that in (2.90) we made the variable changes $\alpha=-\alpha$ and $\beta=-\beta$ to aid us in the following steps. From (2.82) and (2.86) we get, respectively,

$$
\begin{equation*}
C^{*}(\beta)=G(-\beta) G\left(-\frac{1}{T}+\beta\right) \tag{2.91}
\end{equation*}
$$

and

$$
\begin{align*}
C^{*}\left(\alpha+\beta-\frac{1}{T}\right) & =G^{*}\left(\alpha+\beta-\frac{1}{T}\right) G^{*}\left(\frac{1}{T}-\alpha-\beta+\frac{1}{T}\right) \\
& =G\left(\frac{1}{T}-\alpha-\beta\right) G\left(\alpha+\beta-\frac{2}{T}\right) \tag{2.92}
\end{align*}
$$

Substituting (2.91) and (2.92) in (2.90), we get

$$
\begin{equation*}
A\left(-\frac{2}{T}\right)=\frac{1}{T} \int_{\alpha} \int_{\beta} H^{*}(\alpha) H^{*}\left(\frac{2}{T}-\alpha\right) C^{*}(\beta) C^{*}\left(\alpha+\beta-\frac{1}{T}\right) d \alpha d \beta \tag{2.93}
\end{equation*}
$$

Note that, as expected $A\left(-\frac{2}{T}\right)$ is the conjugate of $A\left(\frac{2}{T}\right)$ and the sum of these two terms is real.

## Derivation of $B(0)$

Similar to the previous cases, $B\left(\frac{r}{T}\right) \neq 0$ only for $r=0, \pm 2$. For $r=0$, we have from (2.74)

$$
\begin{equation*}
B(0)=\int_{\alpha} \int_{\beta} \int_{\nu} H(\alpha) H(-\alpha) G(\nu) G(\alpha-\nu) G(\beta) G(-\alpha-\beta) d \alpha d \beta d \nu \tag{2.94}
\end{equation*}
$$

Rearranging terms, we rewrite

$$
\begin{align*}
B(0)= & \int_{\alpha} H(\alpha) H(-\alpha)\left[\int_{\nu} G(\nu) G(\alpha-\nu) d \nu\right] \\
& {\left[\int_{\beta} G(\beta) G(-\alpha-\beta) d \beta\right] d \alpha } \tag{2.95}
\end{align*}
$$

We now change the variable $\beta$ to $-\beta$ in the second internal integral and use the fact that $G^{*}(\beta)=G(-\beta)$ and obtain,

$$
\begin{align*}
B(0)= & \int_{\alpha} H(\alpha) H(-\alpha)\left[\int_{\nu} G(\nu) G(\alpha-\nu) d \nu\right] \\
& {\left[\int_{\beta} G^{*}(\beta) G^{*}(\alpha-\beta) d \beta\right] d \alpha } \\
= & \int_{\alpha}|H(\alpha)|^{2}\left|P_{0}(\alpha)\right|^{2} d \alpha \tag{2.96}
\end{align*}
$$

In the last step we used,

$$
\begin{equation*}
P_{0}(\alpha)=G(\alpha) \otimes G(\alpha) \tag{2.97}
\end{equation*}
$$

## Derivation of $B\left(\frac{2}{T}\right)$

For $r=2$, we have, from (2.74)

$$
\begin{align*}
B\left(\frac{2}{T}\right) & =\int_{\alpha} \int_{\beta} \int_{\nu} H(\alpha) H\left(\frac{2}{T}-\alpha\right) G(\nu) G(\alpha-\nu) G(\beta) G\left(\frac{2}{T}-\alpha-\beta\right) d \alpha d \beta d \nu \\
& =\int_{\alpha} H(\alpha) H\left(\frac{2}{T}-\alpha\right) P_{0}(\alpha) P_{0}\left(\frac{2}{T}-\alpha\right) d \alpha \tag{2.98}
\end{align*}
$$

with $P_{0}(\alpha)$ as defined in (2.97).

## Derivation of $B\left(-\frac{2}{T}\right)$

For $r=-2$, from (2.74),

$$
\begin{align*}
B\left(-\frac{2}{T}\right) & =\int_{\alpha} \int_{\beta} \int_{\nu} H(\alpha) H\left(-\frac{2}{T}-\alpha\right) G(\nu) G(\alpha-\nu) G(\beta) G\left(-\frac{2}{T}-\alpha-\beta\right) d \alpha d \beta d \nu \\
& =\int_{\alpha} \int_{\beta} \int_{\nu} H^{*}(-\alpha) H^{*}\left(\frac{2}{T}+\alpha\right) G^{*}(-\nu) G^{*}(-\alpha+\nu) G^{*}(-\beta) G^{*}\left(\frac{2}{T}+\alpha+\beta\right) d \alpha d \beta d \nu \tag{2.99}
\end{align*}
$$

Changing the variables

$$
\begin{aligned}
& \alpha=-\alpha \\
& \beta=-\beta \\
& \nu=-\nu
\end{aligned}
$$

we get

$$
\begin{align*}
B\left(-\frac{2}{T}\right) & =\left[\int_{\alpha} \int_{\beta} \int_{\nu} H(\alpha) H\left(\frac{2}{T}-\alpha\right) G(\nu) G(\alpha-\nu) G(\beta) G\left(\frac{2}{T}-\alpha-\beta\right)\right]^{*} d \alpha d \beta d \nu \\
& =B^{*}\left(\frac{2}{T}\right) \tag{2.100}
\end{align*}
$$

Just as in the expression for $A\left(\frac{r}{T}\right), B\left(-\frac{2}{T}\right)$ and $B\left(\frac{2}{T}\right)$ are complex conjugates and therefore their sum is real.

### 2.3.3 Final Frequency Domain Expression of the Variance

In our process, trying to determine the frequency domain expression for the variance of the timing wave $z(t)$, we managed to reduce the expression from a complicated one (Eq. 2.72) involving summations and triple integrations to one involving merely a double integration. The resulting expression consists of only six terms. Next, we will further reduce this expression to one with only two terms. Before doing so, lets note that since we are using a binary sequence $\left\{a_{k}\right\}$ with equal probability, then

$$
\begin{align*}
\alpha_{0}^{2} & =E\left[a_{k}^{2}\right]=\frac{1}{2}(1)^{2}+\frac{1}{2}(-1)^{2}=1  \tag{2.101}\\
& =E\left[a_{k}^{4}\right]=\frac{1}{2}(1)^{4}+\frac{1}{2}(-1)^{4}=1 \tag{2.102}
\end{align*}
$$

Using these values in (2.72), we obtain

$$
\begin{align*}
\operatorname{var}[z(t)] & =\frac{2}{T} \sum_{r=0 \pm 2} A\left(\frac{r}{T}\right) \exp \left(j \frac{2 \pi r t}{T}\right)-\frac{2}{T} \sum_{r=0 \pm 2} B\left(\frac{r}{T}\right) \exp \left(j \frac{2 \pi r t}{T}\right. \\
& =\sum_{r=0 \pm 2} V_{r} \exp \left(j \frac{2 \pi r t}{T}\right) \tag{2.103}
\end{align*}
$$

where

$$
\begin{equation*}
V_{r}=\left[\frac{2}{T} A\left(\frac{r}{T}\right)-\frac{2}{T} B\left(\frac{r}{T}\right)\right] \tag{2.104}
\end{equation*}
$$

for $r=0$ and $\pm 2$. Since $V_{-2}=V_{2}^{*}$, we finally obtain

$$
\begin{equation*}
\operatorname{Var}[z(t)]=V_{0}+\left|V_{2}\right| \cos \left(\frac{4 \pi t}{T}+\theta\right) \tag{2.105}
\end{equation*}
$$

where

$$
\begin{align*}
V_{0} & =\frac{2}{T}[A(0)-B(0)]  \tag{2.106}\\
\left|V_{2}\right| & =\frac{2}{T}\left|A\left(\frac{2}{T}\right)-B\left(\frac{2}{T}\right)\right|  \tag{2.107}\\
\theta & =\arctan \left[A\left(\frac{2}{T}\right)-B\left(\frac{2}{T}\right)\right] \tag{2.108}
\end{align*}
$$

with $A(0)$ and $B(0)$ defined in (2.81) and (2.96), respectively, while $A\left(\frac{2}{T}\right)$ and $B\left(\frac{2}{T}\right)$ are defined in (2.87) and (2.98), respectively.

### 2.4 Final Rms Jitter Expression

By definition

$$
\begin{equation*}
E\left[z^{2}\left(t_{0}\right)\right]=\operatorname{var}\left[z\left(t_{0}\right)\right]+\left\{E\left[z\left(t_{0}\right)\right]\right\}^{2} \tag{2.109}
\end{equation*}
$$

Since $\left\{E\left[z\left(t_{0}\right)\right]\right\}^{2}$ is zero, by using (2.105), we have

$$
\begin{align*}
E\left[z^{2}\left(t_{0}\right)\right] & =\operatorname{var}\left[z\left(t_{0}\right)\right] \\
& =V_{0}+2\left|V_{2}\right| \cos \left(\frac{4 \pi t_{0}}{T}+\theta\right) \tag{2.110}
\end{align*}
$$

Substituting $t_{0}$ from (2.26) we have

$$
\begin{equation*}
E\left[z^{2}\left(t_{0}\right)\right]=V_{0}+2\left|V_{2}\right| \cos (n \pi-2 \phi+\theta), \quad \mathrm{n} \text { odd } \tag{2.111}
\end{equation*}
$$

Also using (2.22), we get

$$
\begin{align*}
E\left[\dot{z}\left(t_{0}\right)\right] & =-\frac{4 \pi}{T^{2}}\left|P_{0}\left(\frac{1}{T}\right) \| H\left(\frac{1}{T}\right)\right| \sin \left(\frac{2 \pi t_{0}}{T}+\phi\right) \\
& =-\frac{4 \pi}{T^{2}}\left|P_{0}\left(\frac{1}{T}\right) \| H\left(\frac{1}{T}\right)\right| \sin \left(\frac{n \pi}{2}\right) \quad \mathrm{n} \text { odd } \tag{2.112}
\end{align*}
$$

Using (2.111) and (2.112) in (2.4), we finally obtain

$$
\begin{equation*}
\left(\frac{\triangle \tau}{T}\right)_{r m s}=\frac{T}{4 \pi} \frac{\left[V_{0}+2\left|V_{2}\right| \cos (n \pi-2 \phi+\theta)\right]^{1 / 2}}{\left|P_{0}\left(\frac{1}{T}\right)\right|\left|H\left(\frac{1}{T}\right)\right|\left|\sin \left(\frac{n \pi}{2}\right)\right|} \tag{2.113}
\end{equation*}
$$

The minimum rms jitter occurs when $2 \phi=\theta$. For that case

$$
\begin{equation*}
\left(\frac{\triangle \tau}{T}\right)_{r m s \min }=\frac{\left(V_{0}-2\left|V_{2}\right|\right)^{1 / 2}}{4 \pi u_{1}} \tag{2.114}
\end{equation*}
$$

where, from (2.106) together with (2.81) and (2.95),

$$
\begin{align*}
V_{0}= & \frac{2}{T^{2}} \int_{-\infty}^{\infty}|H(f)|^{2}\left\{\int_{-\infty}^{\infty}|G(f+\nu)|^{2}|G(\nu)|^{2} d \nu\right. \\
& -T\left|\int_{-\infty}^{\infty} G(f-\nu) G(\nu) d \nu\right|^{2} \\
& \left.+2 \operatorname{Re} \int_{-\infty}^{\infty} C^{*}(f+\nu) C(\nu) d \nu\right\} d f \tag{2.115}
\end{align*}
$$

from (2.107), together with (2.87) and (2.98),

$$
\begin{align*}
V_{2}= & \frac{2}{T^{2}} \int_{-\infty}^{\infty} H\left(\frac{2}{T}-f\right) H(f)\left\{\int_{-\infty}^{\infty} C\left(f-\frac{1}{T}+\nu\right) C(\nu) d \nu\right. \\
& \left.-T P_{0}\left(\frac{2}{T}-f\right) P_{0}(f)\right\} d f \tag{2.116}
\end{align*}
$$

and

$$
\begin{equation*}
u_{1}=\frac{1}{T}\left|H\left(\frac{1}{T}\right)\right|\left|P_{0}\left(\frac{1}{T}\right)\right| \tag{2.117}
\end{equation*}
$$

with $C(f)$, as was defined in (2.82), $P_{0}(f)$ in (2.99) and $\mathrm{H}(\mathrm{f})$ being the transfer function of the postfilter.

### 2.5 Example

In this section we apply the results obtained in the previous sections of this chapter to a specific, and practical, example. We use a prefilter whose bandwidth is varied between $1 / T$ and $2 / T$ hertz. The evaluation will be performed using analytical tools. Also note that we selected a bandpass postfilter, whose lowpass equivalent, $H_{L P}(f)$, has a single pole, given by

$$
\begin{equation*}
H_{L P}(f)=\frac{1}{1+j 2 f T Q} \tag{2.118}
\end{equation*}
$$

where $1 / T$ is the baud rate and $Q$ is the quality factor.

### 2.5.1 Ideal Bandlimiting Prefilter Response

Let $G(f)$ be given by,

$$
G(f)= \begin{cases}\frac{T}{1+\gamma} & \left.|f| \leq \frac{1+\gamma}{2 T} \right\rvert\,  \tag{2.119}\\ 0 & \text { elsewhere }\end{cases}
$$

and depicted in Fig. 2.4, $\gamma=0$ corresponds to the minimum transmission band-


Figure 2.4: $G(f)$, prefilter frequency response
width needed for zero Inter Symbol Interference ISI, Nyquist bandwidth, and $\gamma=1$ corresponds to the case of 100 percent excess bandwidth. Also notice that the area under $G(f)$ is taken to be unity so that $g(0)=1$.

## Evaluation of $V_{0}$

Using (2.115) we first rewrite

$$
\begin{equation*}
V_{0}=\frac{2}{T^{2}} \int_{-\infty}^{\infty}|H(f)|^{2}[X(f)-T Y(f)+2 \mathcal{R e} e Z(f)] d f \tag{2.120}
\end{equation*}
$$

where

$$
\begin{equation*}
X(f)=\int_{-\infty}^{\infty}|G(f+\nu)|^{2}|G(\nu)|^{2} d \nu \tag{2.121}
\end{equation*}
$$

can be shown to have the form

$$
X(f)=\left\{\begin{array}{lc}
\frac{T^{3}(1+\gamma+f T)}{(1+\gamma)^{4}} & -\frac{1+\gamma}{T}<f<0  \tag{2.122}\\
\frac{T^{3}(1+\gamma-f T)}{(1+\gamma)^{4}} & 0<f<\frac{1+\gamma}{T} \\
0 & \text { elsewhere }
\end{array}\right.
$$

which is depicted in Fig. 2.5, for $\gamma=0.5$ and $T=1$.


Figure 2.5: $\mathrm{X}(\mathrm{f})$, Evaluated for $\gamma=0.5$ and $T=1$

Also,

$$
\begin{equation*}
Y(f)=\left|\int_{-\infty}^{\infty} G(f-\nu) G(\nu) d \nu\right|^{2} \tag{2.123}
\end{equation*}
$$

can be shown to have the form

$$
Y(f)= \begin{cases}\frac{T^{2}}{(1+\gamma)^{4}}(1+\gamma+f T)^{2} & -\frac{1+\gamma}{T}<f<0  \tag{2.124}\\ \frac{T^{2}}{(1+\gamma)^{4}}(1+\gamma-f T)^{2} & 0<f<\frac{1+\gamma}{T} \\ 0 & \text { elsewhere }\end{cases}
$$

which is depicted in Fig. 2.6, for $\gamma=0.5$ and $T=1$.


Figure 2.6: $\mathrm{Y}(\mathrm{f})$, Evaluated for $\gamma=0.5$ and $T=1$

Finally,

$$
\begin{equation*}
Z(f)=\int_{-\infty}^{\infty} C^{*}(f+\nu) C(\nu) d \nu \tag{2.125}
\end{equation*}
$$

where, by definition

$$
\begin{align*}
C(f) & =G(f) G\left(\frac{1}{T}-f\right) \\
& =\frac{T^{2}}{(1+\gamma)^{2}} \quad \text { for } \quad \frac{1-\gamma}{2 T} \leq f \leq \frac{1+\gamma}{2 T} \tag{2.126}
\end{align*}
$$

Therefore $Z(f)$ can be shown to have the form

$$
Z(f)= \begin{cases}\frac{2 T^{3}}{(1+\gamma)^{2}}(\gamma+f T) & -\frac{\gamma}{T}<f<0  \tag{2.127}\\ \frac{2 T^{3}}{(1+\gamma)^{2}}(\gamma-f T) & 0<f<-\frac{\gamma}{T} \\ 0 & \text { elsewhere }\end{cases}
$$

which is depicted in Fig. 2.7, for $\gamma=0.5$ and $T=1$.


Figure 2.7: Z(f), Evaluated for $\gamma=0.5$ and $T=1$

In order to calculate $V_{0}$ we need to obtain $|H(f)|^{2}$, the bandpass response of the postfilter, whose center frequency is at $1 / T$. Notice that

$$
\begin{equation*}
H(f)=H_{L P}\left(f+\frac{1}{T}\right)+H_{L P}\left(f-\frac{1}{T}\right) \tag{2.128}
\end{equation*}
$$

where $H_{L P}(f)$ was defined in (2.118). One can easily observe that

$$
\begin{align*}
|H(f)|^{2} & =\left[H_{L P}\left(f+\frac{1}{T}\right)+H_{L P}\left(f-\frac{1}{T}\right)\right]\left[H_{L P}^{*}\left(f+\frac{1}{T}\right)+H_{L P}^{*}\left(f-\frac{1}{T}\right)\right] \\
& =\left|H_{L P}\left(f+\frac{1}{T}\right)\right|^{2}+\left|H_{L P}\left(f-\frac{1}{T}\right)\right|^{2} \tag{2.129}
\end{align*}
$$

For sake of simplicity we separate the integral expression for $V_{0},(2.120)$, into three parts as follows

$$
\begin{equation*}
V_{0}=I_{1}-I_{2}+I_{3} \tag{2.130}
\end{equation*}
$$

where it can be observed that

$$
\begin{align*}
& I_{1}=\frac{2}{T^{2}} \int_{-\infty}^{\infty}|H(f)|^{2} X(f) d f  \tag{2.131}\\
& I_{2}=\frac{2}{T} \int_{-\infty}^{\infty}|H(f)|^{2} Y(f) d f  \tag{2.132}\\
& I_{3}=\frac{4}{T^{2}} \int_{-\infty}^{\infty}|H(f)|^{2} \operatorname{Re} Z(f) d f \tag{2.133}
\end{align*}
$$

$X(f), Y(f)$, and $Z(f)$ were calculated previously and their results were given in (2.122), (2.124), and (2.127) respectively.

## 1. Evaluation of $I_{1}$

Substituting (2.122) and (2.129) into (2.131) we can write

$$
\begin{equation*}
\left.I_{1}=\frac{2}{T^{2}} \cdot 2 \int_{0}^{\frac{1+\gamma}{T}} \right\rvert\, H_{L P}\left(f-\left.\frac{1}{T}\right|^{2}(-a f+b) d f\right. \tag{2.134}
\end{equation*}
$$

where

$$
\begin{align*}
a & =\left(\frac{T}{1+\gamma}\right)^{4}  \tag{2.135}\\
b & =\left(\frac{t}{1+\gamma}\right)^{3} \tag{2.136}
\end{align*}
$$

Note that we also used the fact that the integrand in (2.131) is an even function of $f$. Using (2.119) to substitute for $H_{L P}(f)$ we get

$$
\begin{equation*}
I_{1}=\frac{4}{T^{2}} \int_{0}^{\frac{1+\gamma}{T}} \frac{-a f+b}{1+4 Q^{2}(f T-1)^{2}} d f \tag{2.137}
\end{equation*}
$$

This integral can be evaluated by first making a change of variable and then using table of integrals. Doing so, the final result obtained is,

$$
\begin{equation*}
I_{1}=\frac{4}{T^{4}}\left[\frac{b t-a}{2 Q}\left(\tan ^{-1} 2 Q \gamma+\tan ^{-1} 2 Q\right)+\frac{a}{8 Q^{2}} \ln \frac{4 Q^{2}+1}{4 Q^{2} \gamma^{2}+1}\right] \tag{2.138}
\end{equation*}
$$

where $a$ and $b$ defined in (2.135) and (2.136), respectively.

## 2. Evaluation of $I_{2}$

Substituting (2.124) and (2.129) into (2.132) we obtain, by using the fact that the integrand of (2.132) is an even function of $f$,

$$
\begin{equation*}
I_{2}=\frac{4 T}{1+\gamma} \int_{0}^{\frac{1+\gamma}{f}} \frac{\left(1-\frac{f T}{1+\gamma}\right)^{2}}{1+4 Q^{2}(f T-1)^{2}} d f \tag{2.139}
\end{equation*}
$$

Again, this integral can be evaluated by first changing variables and then applying a suitable table of integrals. The final expression for $I_{2}$ yields
$I_{2}=\frac{4}{(1+\gamma)^{2} Q}\left[\frac{1+\gamma}{4 Q}-\left(\frac{1}{8 Q^{2}}-\frac{\gamma^{2}}{2}\right)\left(\tan ^{-1} 2 Q \gamma+\tan ^{-1} 2 Q\right)+\frac{\gamma}{4 Q} \ln \frac{4 Q^{2}+1}{4 Q^{2} \gamma^{2}+1}\right]$

## 3. Evaluation of $I_{3}$

Substituting Eqns. (2.127) and (2.129) into (2.133) we obtain, by using even symmetry of the integrand,

$$
\begin{equation*}
I_{3}=\frac{4}{T^{2}} \int_{0}^{\frac{\gamma}{T}} \frac{-a_{1} f+b_{1}}{1+4 Q^{2}(f T-1)^{2}} d f \tag{2.141}
\end{equation*}
$$

where

$$
\begin{align*}
a_{1} & =2\left(\frac{T}{1+\gamma}\right)^{4}  \tag{2.142}\\
b_{1} & =2 \frac{T^{3} \gamma}{(1+\gamma)^{4}} \tag{2.143}
\end{align*}
$$

Just as in the evaluation of $I_{1}$ and $I_{2}$, we again use the table of integrals to produce

$$
\begin{equation*}
I_{3}=\frac{4}{T^{4}}\left[\frac{T b_{1}-a_{1}}{2 Q}\left(\tan ^{-1} 2 Q(\gamma-1)+\tan ^{-1} 2 Q\right)+\frac{a_{1}}{8 Q^{2}} \ln \frac{4 Q^{2}+1}{4 Q^{2}(\gamma-1)^{2}+1}\right] \tag{2.144}
\end{equation*}
$$

## Evaluation of $V_{2}$

Similar to the process used for evaluating $V_{0}$, we rewrite (2.116) as

$$
\begin{equation*}
V_{2}=\frac{2}{T^{2}} \int_{-\infty}^{\infty} H\left(\frac{2}{T}-f\right) H(f)[V(f)-T W(f)] d f \tag{2.145}
\end{equation*}
$$

where,

$$
\begin{equation*}
V(f)=\int_{-\infty}^{\infty} C\left(f-\frac{1}{T}+\nu\right) C(\nu) d \nu \tag{2.146}
\end{equation*}
$$

can be shown to have the form

$$
V(f)=\left\{\begin{array}{lc}
\frac{T^{3}}{(1+\gamma)^{4}}(\gamma-1+f T) & \frac{1-\gamma}{T}<f<\frac{1}{T}  \tag{2.147}\\
\frac{T^{3}}{(1+\gamma)^{4}}(\gamma+1-f T) & \frac{1}{T}<f<\frac{1+\gamma}{T} \\
0 & \text { elsewhere }
\end{array}\right.
$$

which is depicted in Fig. 2.8, for $\gamma=0.5$ and $T=1$. Also

$$
\begin{equation*}
W(f)=P_{0}\left(\frac{2}{T}-f\right) P_{0}(f) \tag{2.148}
\end{equation*}
$$

where, by definition,

$$
\begin{equation*}
P_{0}(f)=G(f) \otimes G(f) \tag{2.149}
\end{equation*}
$$



Figure 2.8: $V(\mathrm{f})$, Evaluated for $\gamma=0.5$ and $T=1$
can be shown to have the form

$$
P_{0}(f)=\left\{\begin{array}{lc}
\frac{T}{(1+\gamma)^{2}}(1+\gamma+f T) & -\frac{1+\gamma}{T}<f<0  \tag{2.150}\\
\frac{T}{(1+\gamma)^{2}}(1+\gamma-f T) & 0<f<\frac{1+\gamma}{T} \\
0 & \text { elsewhere }
\end{array}\right.
$$

Therefore $W(f)$ can be shown to have the form

$$
W(f)=\left\{\begin{array}{lc}
-\frac{T^{2}}{(1+\gamma)^{4}}\left[f^{2} T^{2}-2 f T+\left(1-\gamma^{2}\right)\right] & \frac{1-\gamma}{T}<f<\frac{1+\gamma}{T}  \tag{2.151}\\
0 & \text { elsewhere }
\end{array}\right.
$$

which is depicted in Fig. 2.9, for $\gamma=0.5$ and $T=1$.

In order to evaluate $V_{2}$, we first need to calculate the term $H(f) H\left(\frac{2}{T}-f\right)$. By using (2.128) we get

$$
\begin{aligned}
H(f) H\left(\frac{2}{T}-f\right) & =\left[H_{L P}\left(\frac{3}{T}-f\right)+H_{L P}\left(\frac{1}{T}-f\right)\right]\left[H_{L P}\left(f+\frac{1}{T}\right)+H_{L P}\left(f-\frac{1}{T}\right)\right] \\
& =H_{L P}\left(\frac{1}{T}-f\right)+H_{L P}\left(f-\frac{1}{T}\right)
\end{aligned}
$$



Figure 2.9: W(f), Evaluated for $\gamma=0.5$ and $T=1$

$$
\begin{equation*}
=\left|H_{L P}\left(f-\frac{1}{T}\right)\right|^{2} \tag{2.152}
\end{equation*}
$$

where we assumed that $H_{L P}(f)=0$ for $|f|>\frac{1}{T}$, and used the fact that $H_{L P}(-f)=$ $H_{L P}^{*}(f)$.

For the sake of simplicity, we rewrite $V_{2}$ of (2.145) as a sum of two integrals.

$$
\begin{equation*}
V_{2}=L_{1}+L_{2} \tag{2.153}
\end{equation*}
$$

where it can be observed that

$$
\begin{equation*}
L_{1}=\frac{2}{T^{2}} \int_{-\infty}^{\infty}\left|H_{L P}\left(f-\frac{1}{T}\right)\right|^{2} V(f) d f \tag{2.154}
\end{equation*}
$$

and

$$
\begin{equation*}
L_{2}=-\frac{2}{T^{2}} \int_{-\infty}^{\infty}\left|H_{L P}\left(f-\frac{1}{T}\right)\right|^{2} W(f) d f \tag{2.15.5}
\end{equation*}
$$

with $V(f)$ and $W(f)$ given in (2.147) and (2.151), respectively, and relation (2.1.52) also taken into consideration.

## 1. Evaluation of $L_{1}$

Substituting (2.147) into (2.154) we write

$$
\begin{equation*}
L_{1}=\frac{2}{T^{2}} \cdot \frac{2 T^{3}}{(1+\gamma)^{4}} \int_{\frac{1}{T}}^{\frac{1+\gamma}{T}}\left|H_{L P}\left(f-\frac{1}{T}\right)\right|^{2}(1+\gamma-f T) d f \tag{2.156}
\end{equation*}
$$

where we used the fact that the integrand in (2.154) has an even symmetry with respect to $1 / T$. Using the expression for $H_{L P}(f)$ we get

$$
\begin{equation*}
L_{1}=\frac{4 T}{(1+\gamma)^{4}} \int_{\frac{1}{T}}^{\frac{1+\gamma}{T}} \frac{1+\gamma-f T}{1+4 Q^{2}(f T-1)^{2}} d f \tag{2.157}
\end{equation*}
$$

Finally, by making a suitable change of variables and applying an appropriate table of integrals, we obtain

$$
\begin{equation*}
L_{1}=\frac{1}{2 Q^{2}(1+\gamma)^{4}}\left[4 Q \gamma \tan ^{-1} 2 Q \gamma-\ln \left(4 Q^{2} \gamma^{2}+1\right)\right] \tag{2.158}
\end{equation*}
$$

## 2. Evaluation of $L_{2}$

Substituting (2.151) into (2.155) we can write,

$$
\begin{equation*}
L_{2}=-\frac{2}{T} \cdot 2 \frac{T^{2}}{\left(1+\gamma^{4}\right)} \int_{\frac{1}{T}}^{\frac{1+\gamma}{T}}\left|H_{L P}\left(f-\frac{1}{T}\right)\right|^{2}\left[f^{2} T^{2}-2 f T+\left(1-\gamma^{2}\right)\right] d f \tag{2.159}
\end{equation*}
$$

where, again, we used the symmetric property of the integrand of (2.155) with respect to $1 / T$. Using the expression for $H_{L P}$ we get

$$
\begin{equation*}
L_{2}=-\frac{4 T}{(1+\gamma)^{4}} \int_{\frac{1}{T}}^{\frac{1+\gamma}{T}} \frac{f^{2} T^{2}-2 f T+\left(1-\gamma^{2}\right)}{1+4 Q^{2}(f T-1)^{2}} d f \tag{2.160}
\end{equation*}
$$

which, using table of integrals, it can be shown that,

$$
\begin{equation*}
L_{2}=\frac{2}{Q\left(1+\gamma^{2}\right)}\left[\left(\gamma^{2}+\frac{1}{4 Q^{2}} \tan ^{-1} 2 Q \gamma-\frac{\gamma}{2 Q}\right]\right. \tag{2.161}
\end{equation*}
$$

## Evaluating the Denominator of (2.114)

From (2.117), the denominator is given by

$$
\begin{equation*}
\mu_{1}=\frac{1}{T}\left|H\left(\frac{1}{T}\right)\right|\left|P_{0}\left(\frac{1}{T}\right)\right| \tag{2.162}
\end{equation*}
$$

Clearly $H\left(\frac{1}{T}\right)=1$, and from (2.150)

$$
\begin{equation*}
P_{0}\left(\frac{1}{T}\right)=\frac{\gamma T}{(1+\gamma)^{2}} \tag{2.163}
\end{equation*}
$$

therefore

$$
\begin{equation*}
4 \pi \mu_{1}=4 \pi \frac{\gamma}{(1+\gamma)^{2}} \tag{2.164}
\end{equation*}
$$

Magnitude of the Extracted Timing Wave

From (2.22)

$$
\begin{equation*}
E[z(t)]=\frac{2}{T}\left|P_{0}\left(\frac{1}{T}\right) \| H\left(\frac{1}{T}\right)\right| \cos \left(\frac{2 \pi t}{T}+\phi\right) . \tag{2.165}
\end{equation*}
$$

Using (2.163) for $P_{0}\left(\frac{1}{T}\right)$ and knowing that $\left|H\left(\frac{1}{T}\right)\right|=1$ we determine

$$
\begin{equation*}
|E[z(t)]|=\frac{2 \gamma}{(1+\gamma)^{2}} \tag{2.166}
\end{equation*}
$$

## Rms Jitter

Gathering all the parts together we determine the rms jitter expression as

$$
\begin{equation*}
\left(\frac{\triangle \tau}{T}\right)_{r m s, \min }=\frac{\left[\left(I_{1}-I_{2}+I_{3}\right)-2\left(L_{1}+L_{2}\right)\right]^{1 / 2}}{4 \pi \mu_{1}} \tag{2.167}
\end{equation*}
$$

where $I_{1}, I_{2}, I_{3}, L_{1}, L_{2}$, and $u_{1}$, are given by (2.138), (2.140), (2.144), (2.158), (2.161), and (2.164) respectively.

In conclusion, we notice that the minimum rms jitter in this example depends solely on the postfilter's quality factor $Q$ and the prefilter's excess bandwidth parameter $\gamma$. A simple computer program was written in order to obtain the values of the jitter, (2.167), as a function of $Q$ and $\gamma$.

### 2.6 Results

In this section we depict the results obtained from the analytical approach for a squarer type of nonlinearity employing an ideal type of prefilter. We then proceed by evaluating the performance using different type of prefilters such as raised cosine and trapezoidal.

Due to the complexity of the terms, for the raised cosine and trapezoidal responses, we can not derive $V_{0}$ and $V_{2}$ as a function of $\gamma$ and $Q$ directly. Instead we use equations 2.115 and 2.116 to calculate $V_{0}$ and $V_{2}$, respectively, and then, together with the results of the denominator (2.117), we determine the rms jitter.

### 2.6.1 Ideal Square Type of Prefilter

In Fig. 2.10 we depict the values of the minimum rms jitter as a function of $Q$, between $25-150$ in steps of 25 . The excess bandwidth, $0.2-0.8$ in steps of 0.2 ,
was used as a parameter for these curves. Also, for the convenience of the reader, tabulated results are given in Table (2.1). Last we plot, in Fig. 2.11, the mag-

| $Q$ | $\gamma=0.2$ | $\gamma=0.4$ | $\gamma=0.6$ | $\gamma=0.8$ |
| :---: | :---: | :---: | :---: | :---: |
| 25 | 0.034799 | 0.017216 | 0.011643 | 0.009436 |
| 50 | 0.019663 | 0.00975 | 0.006574 | 0.005245 |
| 75 | 0.013932 | 0.006915 | 0.004656 | 0.00369 |
| 100 | 0.010868 | 0.005397 | 0.003631 | 0.002867 |
| 125 | 0.008947 | 0.004445 | 0.002989 | 0.002353 |
| 150 | 0.007624 | 0.003789 | 0.002547 | 0.002001 |

Table 2.1: rms jitter as a function of $Q$, for an ideal bandlimiting filter response


Figure 2.10: rms jitter for an ideal square type of prefilter
nitude of the extracted timing wave as a function of the excess bandwidth $\gamma$, (2.164).


Figure 2.11: Magnitude of the mean timing wave for ideal bandlimiting filter response

From these figures, we notice that the magnitude of the extracted timing wave decreases as $\gamma$ decreases and finally vanishes as $\gamma$ approaches zero. The rms jitter, on the other hand, increases as $\gamma$ decreases, for a fixed quality factor $Q$ of the postfiler. Also for a fixed $\gamma$ a better quality factor for the postfilter produces a better jitter performance. Note that, with the reduction of the excess bandwidth $\gamma$, not only are we worsening the jitter but we are also left with a smaller amplitude for the extracted timing wave and hence are more susceptible to noise. In order to improve both of these performance factors, amplitude and jitter, we need to increase our excess bandwidth and use a postfilter with a high qualify factor.

### 2.6.2 Raised Cosine Filter

For the second example we use a prefilter whose frequency response is

$$
G(f)=\left\{\begin{array}{lc}
T & |f|<\frac{1-\gamma}{2 T}  \tag{2.168}\\
\frac{T}{2}\left\{1-\sin \frac{\pi T}{\gamma}\left(|f|-\frac{1}{2 T}\right)\right\} & \frac{1-\gamma}{2 T} \leq|f| \leq \frac{1+\gamma}{2 T} \\
0 & \text { elsewhere }
\end{array}\right.
$$

which is depicted in Fig. 2.12, for $\gamma=0.5$ and $T=1$. As in the first example.


Figure 2.12: Raised cosine prefilter response for $\gamma=0.5$
$g(0)=1$. It is known that the raised cosine response, like the ideal bandlimited prefilter response (with $\gamma=0$ ), satisfies the Nyquist condition $g(k T)=0$ for $k \neq 0$ and hence has zero ISI. However, because of the smaller sidelobes of the pulse (in comparison to the ideal bandlimited response), a system containing a raised cosine type of filter is less sensitive to sampling errors.

As in the previous example, we depict, in Fig. 2.13, the rms jitter as the
function of $Q$ with $\gamma$ used as a parameter. Again, for the convenience of the reader we tabulated the results in Table 2.2. Finally, the magnitude of the extracted timing wave is plotted in Fig. 2.14.

| $Q$ | $\gamma=0.2$ | $\gamma=0.4$ | $\gamma=0.5$ | $\gamma=0.6$ | $\gamma=0.8$ | $\gamma=1.0$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 25 | 0.121379 | 0.044075 | 0.030396 | 0.021595 | 0.011361 | 0.005725 |
| 50 | 0.062815 | 0.022594 | 0.015548 | 0.011094 | 0.005819 | 0.002956 |
| 75 | 0.039757 | 0.015028 | 0.010439 | 0.007309 | 0.003915 | 0.001950 |
| 100 | 0.028647 | 0.011533 | 0.007868 | 0.005641 | 0.002884 | 0.001497 |
| 125 | 0.022475 | 0.009549 | 0.006324 | 0.004506 | 0.001865 | 0.001298 |
| 150 | 0.019066 | 0.007303 | 0.005298 | 0.003935 | 0.002126 | 0.000999 |

Table 2.2: rms jitter values, raised cosine response


Figure 2.13: rms jitter for a raised cosine prefilter response


Figure 2.14: Magnitude of mean of timing wave, raised cosine prefilter response

### 2.6.3 Trapezoidal prefilter

As a last example we use a filter whose response is as follows;

$$
G(f)=\left\{\begin{array}{lc}
T & |f|<\frac{1-\gamma}{2 T}  \tag{2.169}\\
-\frac{T^{2}}{\gamma}\left(\left|\frac{1+\gamma}{2 T}-f\right|\right) & \frac{1-\gamma}{2 T} \leq|f| \leq \frac{1+\gamma}{2 T} \\
0 & \text { elsewhere }
\end{array}\right.
$$

which is depicted in Fig. 2.15, for $\gamma=0.5$ and $T=1$. Again, notice that, $g(0)=1$. The results for this kind of response are depicted in Table 2.3 and Figs. 2.16 and 2.17 , respectively.


Figure 2.15: Trapezoidal prefilter response for $\gamma=0.5$

| $Q$ | $\gamma=0.2$ | $\gamma=0.4$ | $\gamma=0.5$ | $\gamma=0.6$ | $\gamma=0.8$ | $\gamma=1.0$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 25 | 0.081620 | 0.027248 | 0.017698 | 0.011686 | 0.004464 | 0.000813 |
| 50 | 0.042157 | 0.013994 | 0.009087 | 0.006003 | 0.002315 | 0.000496 |
| 75 | 0.028432 | 0.009421 | 0.006118 | 0.004044 | 0.001571 | 0.000364 |
| 100 | 0.021447 | 0.007104 | 0.004614 | 0.003050 | 0.001193 | 0.000295 |
| 125 | 0.017169 | 0.005704 | 0.003706 | 0.002453 | 0.000964 | 0.000252 |
| 150 | 0.014422 | 0.004767 | 0.003098 | 0.002052 | 0.000810 | 0.000223 |

Table 2.3: rms jitter values, trapezoidal response


Figure 2.16: rms jitter for a raised trapezoidal prefilter response


Figure 2.17: Magnitude of mean of timing wave, trapezoidal prefilter response

### 2.6.4 Performance Comparison

Based on the results of the previous section, the performance of $S T R$ circuits in extracting the timing wave, when using a square, raised cosine, and trapezoidal pulse shapes, are compared in Figs. 2.18 and 2.19 (for $\gamma=0.5$ ). In the Fig. 2.18, we show the amplitude of the mean value of the extracted timing wave as a function of the excess bandwidth parameter, $\gamma$. The ideal type of prefilter response resulted in larger extracted timing amplitude than the trapezoidal response, which in turn was larger than the raised cosine response. Also note that, except for the ideal type of prefilter response, all responses yielded a linear amplitude gain as a function of the excess bandwidth.

An important category of our research involves timing extraction at lower transmission bandwidth. Paying particular attention to the lower part of Fig. 2.18, we notice how, around $\gamma<0.2$, the amplitude of the extracted timing wave slowly vanishes and becomes insignificant for extraction purposes. This is where timing recovery fails, and we have to look for other methods of extractions. Such method is the topic of the next chapter.

In Fig. 2.19, comparison of the three responses is made for the Rms jitter, as a function of the quality factor $Q$. Again, the ideal square type of response seems to outperform all the others. We also note that the using the trapezoidal response, due to it's side characteristics, is slightly better than the raised cosine type of response. We should emphasize, however, that the square pulse shape, with non zero excess bandwidth, does not satisfy the Nyquist pulse shape condition and hence it is not a zero ISI pulse. Therefore, we shall conclude that for the purpose
of timing extraction using a squarer type of nonlinearity, a trapezoidal prefilter response is the best selection.


Figure 2.18: Comparison of amplitudes of the extract timing wave for the different types of prefilter responses


Figure 2.19: Comparison of rms jitter of the extracted timing wave for the different types of prefilter responses

## Chapter 3

## The Moment Method for Evaluating the Jitter Performance of STR Circuits Employing High Order Nonlinearity

The investigations on clock recovery employing a square-law device, performed in the previous chapter, show that other parameters being kept the same, it's behavior depends on the excess bandwidth of the input pulses. The results obtained showed us that a satisfactory performance is achieved for medium and large values of excess bandwidth $\gamma$. However, the performance of the circuit deteriorates as $\gamma$ decreases. In the extreme case of $\gamma=0$, Nyquist bandwidth, the magnitude of the recovered timing wave becomes zero and hence this method of timing recovery fails. Therefore, when dealing with such strongly bandlimited pulses, we must consider other types of nonlinearity. Unfortunately, clock circuits implemented with non square law devices are difficult to evaluate analytically, using the method of chapter 2. In fact,
such circuits are hardly tractable mathematically and their performance has only been evaluated by computer simulations [19].

In this chapter, we describe a new technique for evaluating the performance of a zero memory $S T R$ circuit which employs an even, high order, nonlinearity. The technique implements the moments of the input signal to the timing circuit. We begin the following section, using the same steps used in chapter 2 , by deriving the expression for the jitter performance.

In section 3.2 the moments of the input to the nonlinear device are used to find the mean of the extracted timing wave at the output of the $S T R$ circuit. The particular case of second order nonlinearity results are compared to those obtained in chapter 2 . The mean squared value of the timing wave is related to the autocorrelation function, $R_{Y^{\prime}}(t, s)$, of the output of the nonlinear device. Using the fact that this function is periodic, its Fourier series coefficients are obtained and the Fourier transform of those coefficient is used to obtain our results. Taking the transfer function of the post filter into effect, only the Fourier transform of the zero, $r_{0}(\tau)$, and second, $r_{2}(\tau)$, order coefficients enter into the final expression for the mean squared value of the timing wave.

To pursue determining $r_{0}(\tau)$ and $r_{2}(\tau)$, we relate $R_{Y}(t, s)$ to the joint moments of the inputs to the nonlinear device. Then, in section 3.4 , we relate these moments to the joint characteristic function and its derivatives, which could be obtained by some recursive formulation. These recursions can be used for any type of even order nonlinearity.


Figure 3.1: Basic STR circuit employing a general type of nonlinearity

### 3.1 Evaluation of the rms timing jitter

As in Fig 2.1, the basic model is redrawn in Fig 3.1 where, instead of $y(t)=x^{2}(t)$, we used $y(t)=f[x(t)]$. The transformation $f[\cdot]$ is a high order, zero memory, nonlinear device represented by the finite power series of the form

$$
\begin{align*}
y(t) & =f[x(t)] \\
& =\sum_{n=0}^{N} C_{n} x^{2 n}(t) \tag{3.1}
\end{align*}
$$

where the $C_{n}$ 's. $n=1,2 \cdots N$ are given real constants and $2 N$ is the order of the nonlinearity. The postfilter to be used, same as in chapter 2 , is a narrowband bandpass filter whose transfer function, $H(f)$, is centered at the symbol rate frequency $1 / T$ and satisfies the band limiting condition

$$
\begin{equation*}
H(f)=0 \quad \text { for } \quad\left||f|-\frac{1}{T}\right|>\frac{1}{2 T} \tag{3.2}
\end{equation*}
$$

Equation (3.2) reflects the condition that the second and higher harmonics of the symbol rate are eliminated.

For the evaluation of the rms timing jitter, we refer to equation (2.2) of the
previous chapter

$$
\begin{equation*}
\left(\frac{\triangle \tau}{T}\right)_{r m s} \triangleq \frac{1}{T} \frac{\left\{E\left[z^{2}\left(t_{0}\right)\right]\right\}^{2}}{E\left[\dot{z}\left(t_{0}\right)\right]} \tag{3.3}
\end{equation*}
$$

In the following sections, we will follow the same steps as in chapter 2, using the moments of the input to the nonlinear device.

### 3.2 Evaluation of the Mean Value of the Timing Wave

From Fig. 3.1, $z(t)$ is the convolution of $y(t)$ with $h(t)$, the impulse response of the postfilter.

$$
\begin{equation*}
E[z(t)]=\int_{-\infty}^{\infty} E[y(\alpha)] h(t-\alpha) d \alpha \tag{3.4}
\end{equation*}
$$

However, from (3.1)

$$
\begin{align*}
E[y(t)] & =\sum_{n=0}^{N} C_{n} E\left[x^{2 n}(t)\right] \\
& =\sum_{n=0}^{N} C_{n} M_{2 n}(t) \tag{3.5}
\end{align*}
$$

where $M_{2 n}(t)$ is the $2 n$ - th moment of the random variable $x$. The moment, $M_{2 n}(t)$ can be obtained by using the recursive equation [20], (Also see Appendix D, in particular, D. 13 and D.19).

$$
\begin{equation*}
M_{2 n}(t)=-\left\{\sum_{i=1}^{n}\binom{2 n-1}{2 i-1}(-1)^{i} M_{2(n-i)}(t) \cdot \lambda(0)^{2 i-1}\right\} \tag{3.6}
\end{equation*}
$$

where

$$
\begin{equation*}
\lambda^{2 i-1}(0)=\frac{2^{i}\left(2^{i}-1\right)}{2}\left|B_{2 i}\right| \sum_{-\infty}^{\infty}[g(t-l T)]^{2 i}, \tag{3.7}
\end{equation*}
$$

with $B_{i}$ being the Bernouli numbers [21], $g(t)$ the impulse response of the prefilter, and $M_{0}$ is constant and defined to equal one. We should also note that in obtaining
the moments in (3.6), we used the fact that

$$
\begin{equation*}
x(t)=\sum_{-\infty}^{\infty} a_{k} g(t-k T) \tag{3.8}
\end{equation*}
$$

with the data sequence, $\left\{a_{k}\right\}$, being an independent random variable whose values are $\pm 1$, with equal probability. Since $x(t)$ is a cyclostationary process (CT), the high order components, $x^{2 n}(t)$, at the output of the non-linear device are also CT processes. Therefore their mean functions, $M_{2 n}(t)=E\left[x^{2 n}(t)\right]$, are periodic in time, with period $T$, and they can be expanded into Fourier series as:

$$
\begin{equation*}
M_{2 n}(t)=\sum_{k} m_{k}^{(2 n)} \exp \left(j \frac{2 \pi t k}{T}\right), \quad n=0,1, \cdots, N \tag{3.9}
\end{equation*}
$$

where the complex Fourier coefficients $m_{k}^{(2 n)}$ are computed from

$$
\begin{equation*}
m_{k}^{(2 n)}=\frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} M_{2 n}(\alpha) \exp \left(-j \frac{2 \pi \alpha k}{T}\right) d \alpha . \tag{3.10}
\end{equation*}
$$

Using (3.5) and (3.4) we get

$$
\begin{equation*}
E[z(t)]=\int_{-\infty}^{\infty} \sum_{n=0}^{N} C_{n} M_{2 n}(\alpha) h(t-\alpha) d \alpha \tag{3.11}
\end{equation*}
$$

and by using (3.9) we obtain

$$
\begin{align*}
E[z(t)] & =\int_{-\infty}^{\infty} \sum_{n=0}^{N} C_{n}\left(\sum_{-\infty}^{\infty} m_{k}^{(2 n)} \exp \left(j \frac{2 \pi \alpha k}{T}\right) h(t-\alpha)\right) d \alpha \\
& =\sum_{n=0}^{N} C_{n}\left(\sum_{k=-\infty}^{\infty} m_{k}^{(2 n)} \int_{-\infty}^{\infty} h(t-\alpha) \exp \left(j \frac{2 \pi \alpha k}{T}\right) d \alpha\right) \\
& =\sum_{n=0}^{N} C_{n}\left(\sum_{k=-\infty}^{\infty} m_{k}^{(2 n)} H\left(\frac{k}{T}\right) \exp \left(j \frac{2 \pi k t}{T}\right)\right) \tag{3.12}
\end{align*}
$$

However, $H(f)$ is centered around $1 / T$, and as a result of (3.2) we have $H(k / T) \neq 0$ only for $k= \pm 1$ and therefore,

$$
\begin{align*}
E[z(t)] & =\sum_{n=0}^{N} C_{n}\left[m_{-1}^{(2 n)} H\left(-\frac{1}{T}\right) \exp \left(-j \frac{2 \pi t}{T}\right)+m_{1}^{(2 n)} H\left(\frac{1}{T}\right) \exp \left(j \frac{2 \pi t}{T}\right)\right] \\
& =\left(H\left(\frac{1}{T}\right) \sum_{n=0}^{N} C_{n} m_{1}^{(2 n)}\right) \exp \left(j \frac{2 \pi t}{T}\right)+\left(H\left(-\frac{1}{T}\right) \sum_{n=0}^{N} C_{n} m_{-1}^{(2 n)}\right) \exp \left(-j \frac{2 \pi t}{T}\right) . \tag{3.13}
\end{align*}
$$

Defining

$$
\begin{equation*}
\mu_{1}=\sum_{n=0}^{N} C_{n} m_{1}^{(2 n)}, \quad \mu_{-1}=\sum_{n=0}^{N} C_{n} m_{-1}^{(2 n)} \tag{3.14}
\end{equation*}
$$

and noting that $\mu_{-1}=\mu_{1}^{*}$ and $H\left(-\frac{1}{T}\right)=H^{*}\left(\frac{1}{T}\right)$, we obtain the following expression for $E[z(t)]$,

$$
\begin{equation*}
E[z(t)]=2 H\left(\frac{1}{T}\right)\left|\mu_{1}\right| \cos \left(\frac{2 \pi t}{T}+\phi\right) \tag{3.15}
\end{equation*}
$$

where $\phi$ is the phase of $\mu_{1}$. When $C_{n}$ 's are zero, except for $C_{N}$ (only $2 N-t h$ order component), then

$$
\begin{equation*}
E[z(t)]=2 C_{2 N} H\left(\frac{1}{T}\right)\left|m_{1}^{(2 N)}\right| \cos \left(\frac{2 \pi t}{T}+\phi\right) \tag{3.16}
\end{equation*}
$$

Equation (3.15) represents the mean value of the timing waveform from which it is clear that the larger the $\left|\mu_{1}\right|$, the better the timing recovery circuit will perform. As a specific case, if $N=1$ and $C_{2}=1,(3.16)$ becomes

$$
\begin{equation*}
E[z(t)]=2 H\left(\frac{1}{T}\right)\left|m_{1}^{(2)}\right| \cos \left(\frac{2 \pi t}{T}+\phi\right) \tag{3.17}
\end{equation*}
$$

It can be shown (see appendix E ) that $m_{1}^{(2)}=\frac{1}{T} P_{0}\left(\frac{1}{T}\right)$ and therefore (3.17) is the same equation as (2.20) of chapter 2.

### 3.3 Evaluation of the Mean squared Value of the Timing Wave

From Fig. 3.1 we can write

$$
\begin{align*}
E\left[z^{2}(t)\right] & =E\left[\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\alpha) h(\beta) y(t-\alpha) y(t-\beta) d \alpha d \beta\right]  \tag{3.18}\\
& =\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\alpha) h(\beta) E[y(t-\alpha) y(t-\beta)] d \alpha d \beta \tag{3.19}
\end{align*}
$$

Note that the term inside the expectation is actually the auto-correlation function of the random process $y(t)$. Since we can not assume stationarity of this process
we must use

$$
\begin{equation*}
R(t-\alpha, t-\beta) \triangleq E[y(t-\alpha) y(t-\beta)] \tag{3.20}
\end{equation*}
$$

We will now rewrite (3.19) as

$$
\begin{equation*}
E\left[z^{2}(t)\right]=\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\alpha) h(\beta) R(t-\alpha, t-\beta) d \alpha d \beta \tag{3.21}
\end{equation*}
$$

Since $x(t)$ is a $C T$ process, so is $y(t)$ and therefore it's correlation function $R(t-$ $\alpha, t-\beta)$ is periodic with period $T$. Hence we can represent the auto-correlation function by the Fourier series expansion as follows,

$$
\begin{equation*}
R_{y}(t-\alpha, t-\beta)=\sum_{k=-\infty}^{\infty} r_{k}(\alpha-\beta) \exp \left[j \frac{\pi k}{T}(2 t-\alpha-\beta)\right] \tag{3.22}
\end{equation*}
$$

where the Fourier coefficients are given by (See appendix F)

$$
\begin{equation*}
r_{k}(\tau)=\frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} R_{y}\left(s-\frac{\tau}{2}, s+\frac{\tau}{2}\right) \exp \left(-j \frac{2 \pi k s}{T}\right) d s \tag{3.23}
\end{equation*}
$$

Using (3.22) and (3.23) we rewrite (3.21) as,

$$
\begin{align*}
E\left[z^{2}(t)\right] & =\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\alpha) h(\beta) \sum_{k=-\infty}^{\infty} r_{k}(\alpha-\beta) \exp \left[j \frac{\pi k}{T}(2 t-\alpha-\beta)\right] d \alpha d \beta  \tag{3.24}\\
& =\sum_{k=-\infty}^{\infty} \exp \left(j \frac{2 \pi k t}{T} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\alpha) h(\beta) r_{k}(\alpha-\beta) \exp \left[-j \frac{\pi k}{T}(\alpha+\beta)\right] d \alpha d \beta\right. \tag{3.25}
\end{align*}
$$

Equation (3.25) represents a time domain expression of the mean squared function of the extracted timing wave $z(t)$. By using the Fourier transform relation between $h(t)$ and its transfer function $H(f)$, we write (3.25) as follows

$$
\begin{align*}
E\left[z^{2}(t)\right]= & \left.\sum_{k} \exp \left(j \frac{2 \pi k t}{T}\right) \int_{\alpha} \int_{\beta}\left[\int_{f} H(f) \exp (j 2 \pi f \alpha)\right) d f\right] \\
& {\left[\int_{\nu} H(\nu) \exp (j 2 \pi \nu \beta) d \nu\right] \cdot r_{k}(\alpha-\beta) \exp \left[-j \frac{\pi k}{T}(\alpha+\beta)\right] d \alpha d \beta } \tag{3.26}
\end{align*}
$$

Rearranging the order of integrations, we obtain

$$
\begin{gather*}
E\left[z^{2}(t)\right]=\sum_{k} \exp \left(j \frac{2 \pi k t}{T}\right) \int_{f} \int_{\nu} H(f) H(\nu)\left[\int_{\alpha} \int_{\beta} \exp (j 2 \pi f \alpha) \exp (j 2 \pi \nu \beta)\right. \\
\left.r_{k}(\alpha-\beta) \exp \left[-j \frac{\pi k}{T}(\alpha+\beta)\right] d \alpha d \beta\right] d f d \nu \tag{3.27}
\end{gather*}
$$

and changing the variable $\tau=\alpha-\beta$ in (3.27) yields,

$$
\begin{gather*}
E\left[z^{2}(t)\right]=\sum_{k} \exp \left(j \frac{2 \pi k t}{T}\right)\left[\int_{f} \int_{\nu} H(f) H(\nu)\left[\int_{\alpha} \exp \left[j 2 \pi \alpha\left(f+\nu-\frac{k}{T}\right)\right] d \alpha\right]\right. \\
\left.\left[\int_{\tau} r_{k}(\tau) \exp \left[-j 2 \pi \tau\left(\nu-\frac{k}{2 T}\right)\right] d \tau\right]\right] d f d \nu \tag{3.28}
\end{gather*}
$$

Note that the integral on $\tau$ is a Fourier transform, $R_{k}(f)$, of $r_{k}(\tau)$. Therefore

$$
\begin{gather*}
E\left[z^{2}(t)\right]=\sum_{k} \exp \left(j \frac{2 \pi k t}{T}\right)\left\{\int_{f} \int_{\nu}\left[\int_{\alpha} H(f) H(\nu) \exp \left[j 2 \pi \alpha\left(f+\nu-\frac{k}{T}\right)\right] d \alpha\right]\right. \\
\left.R_{k}\left(\nu-\frac{k}{2 T}\right) d f d \nu\right\} . \tag{3.29}
\end{gather*}
$$

The integration on $\alpha$ yields $\delta\left(f+\nu-\frac{k}{T}\right)$, so that, after finally integrating on $f$ we obtain

$$
\begin{equation*}
E\left[z^{2}(t)\right]=\sum_{k} \exp \left(j \frac{2 \pi k t}{T}\right) \int_{\nu} H\left(\frac{k}{T}-\nu\right) H(\nu) R_{k}\left(\nu-\frac{k}{2 T}\right) d \nu \tag{3.30}
\end{equation*}
$$

where $H(f)$ is the transfer function of the postfilter and $R_{k}(f)$ is the Fourier transform of $r_{k}(\tau)$, the Fourier coefficients of $R_{y}(\alpha, \beta)$.

It is easy to show that, by using the band limiting conditions on $H(f)$, that the product $H(k / T-\nu) H(\nu)$ is identically zero except for the case when $k=0$ and $k= \pm 2$. Taking this into consideration, we simplify (3.30) to the summation of three terms as,

$$
\begin{align*}
E\left[z^{2}(t)\right]= & \int_{-\infty}^{\infty}|H(\nu)|^{2} R_{0}(\nu) d \nu \\
& +\exp \left(j \frac{4 \pi k t}{T}\right) \int_{-\infty}^{\infty} H\left(\frac{2}{T}-\nu\right) H(\nu) R_{2}\left(\nu-\frac{1}{T}\right) d \nu \\
& +\exp \left(-j \frac{4 \pi k t}{T}\right) \int_{-\infty}^{\infty} H\left(-\frac{2}{T}-\nu\right) H(\nu) R_{2}\left(\nu+\frac{1}{T}\right) d \nu \tag{3.31}
\end{align*}
$$

Finally, we can show (see appendix G) that the third term in (3.31) is the complex conjugate of the second term. Therefore we write our expression, similar to chapter 2 , as

$$
\begin{equation*}
V_{0}+V_{2} \exp \left(j \frac{4 \pi k t}{T}\right)+V_{2}^{*} \exp \left(-j \frac{4 \pi k t}{T}\right) \tag{3.32}
\end{equation*}
$$

where

$$
\begin{equation*}
V_{0}=\int_{-\infty}^{\infty}|H(\nu)|^{2} R_{0}(\nu) d \nu \tag{3.33}
\end{equation*}
$$

and

$$
\begin{equation*}
V_{2}=\int_{-\infty}^{\infty} H\left(\frac{2}{T}-\nu\right) H(\nu) R_{2}\left(\nu-\frac{1}{T}\right) d \nu \tag{3.34}
\end{equation*}
$$

Upon simplification of (3.32) our mean squared expression yield a final form of

$$
\begin{equation*}
E\left[z^{2}(t)\right]=V_{0}+2\left|V_{2}\right| \cos \left(\frac{2 \pi t}{T}+\theta\right) \tag{3.35}
\end{equation*}
$$

where $\left|V_{2}\right|$ and $\theta$ are the magnitude and phase of $V_{2}$.

From (3.34), together with (3.32) and (3.33), it can be observed that the mean squared value of the timing wave depends on the post filter's transfer function, $H(f)$, as well as the Fourier transforms, $R_{0}(f)$ and $R_{2}(f)$, of the Fourier coefficients, $r_{0}(\tau)$ and $r_{2}(\tau)$, of the auto-correlation function $R_{y}(t, s)$. Although not simple, this function depends on the joint cross moments of the input $x(t)$ to the non linear device. This is the case since, from (3.1),

$$
\begin{align*}
R_{y}(\alpha, \beta) & =E\left[\sum_{m} \sum_{n} C_{m} C_{n} x^{2 m}(\alpha) x^{2 n}(\beta)\right] \\
& =\sum_{m} \sum_{n} C_{m} C_{n} E\left[x^{2 m}(\alpha) x^{2 n}(\beta)\right] \tag{3.36}
\end{align*}
$$

and therefore, in order to determine $R_{0}(f)$ and $R_{2}(f)$, one must first evaluate the surface $R_{y}(\alpha, \beta)$, then using (3.23) determine it's Fourier coefficients $r_{0}(\tau)$ and $r_{2}(\tau)$, and finally take their Fourier transforms. Keeping in mind that the surface $R_{y}(\alpha, \beta)$, depicted in (3.36), depends on the joint moment $M_{2 m, 2 n}$ of $x(t)$, we must
determine this joint moment for any $m$ and $n$. This means we must derive a recursive formula similar to the one we had in the previous section for evaluating the joint moments.

### 3.4 Derivation of the Cross-Moments $M^{m, n}$

Let

$$
\begin{align*}
& x(t)=\sum_{k} a_{k} g(t-k T) \triangleq x_{1} \triangleq \sum_{k} a_{k} \alpha_{k} \\
& x(s)=\sum_{k} a_{k} g(s-l T) \triangleq x_{2} \triangleq \sum_{k} a_{k} \beta_{k} \tag{3.37}
\end{align*}
$$

The joint characteristic function of $x_{1}$ and $x_{2}$ is defined by;

$$
\begin{equation*}
\Phi\left(\omega_{1}, \omega_{2}\right)=E\left[\exp \left[j\left(\omega_{1} x_{1}+\omega_{2} x_{2}\right)\right]\right] \tag{3.38}
\end{equation*}
$$

and from the definition of $x_{1}$ and $x_{2}$ we get

$$
\begin{equation*}
\Phi\left(\omega_{1}, \omega_{2}\right)=E\left[\exp \left[j \sum_{k} a_{k}\left(\omega_{1} \alpha_{k}+\omega_{2} \beta_{2}\right)\right]\right] \tag{3.39}
\end{equation*}
$$

Now, if $a_{k}$ 's are independent, then

$$
\begin{equation*}
\Phi\left(\omega_{1}, \omega_{2}\right)=\prod_{k} E\left[\exp \left[j a_{k}\left(\omega_{1} \alpha_{k}+\omega_{2} \beta_{2}\right)\right]\right] \tag{3.40}
\end{equation*}
$$

and for $a_{k}= \pm 1$ with equal probabilities we have

$$
\begin{equation*}
\Phi\left(\omega_{1}, \omega_{2}\right)=\prod_{k} \cos \left(\omega_{1} \alpha_{k}+\omega_{2} \beta_{2}\right) \tag{3.41}
\end{equation*}
$$

One can easily show that, if $a_{k}$ 's are identically and independently distributed with zero mean, then

$$
E\left[x_{1}^{m} x_{2}^{n}\right]=M^{m, n}= \begin{cases}0 & \text { if } m+n \text { is odd }  \tag{3.42}\\ \neq 0 & \text { if } m+n \text { is even }\end{cases}
$$

Expanding (3.39) we get

$$
\begin{equation*}
\Phi\left(\omega_{1}, \omega_{2}\right)=E\left[1+j\left(\omega_{1} x_{1}+\omega_{2} x_{2}\right)+\frac{j^{2}}{2!}\left(\omega_{1} x_{1}+\omega_{2} x_{2}\right)^{2}+\cdots+\frac{j^{k}}{k!}\left(\omega_{1} x_{1}+\omega_{2} x_{2}\right)^{k} \cdots\right] \tag{3.43}
\end{equation*}
$$

But

$$
\begin{equation*}
M^{m, n}=E\left[x_{1}^{m} x_{2}^{n}\right]=E\left[x_{1}^{m}(t) x_{2}^{n}(s)\right] \tag{3.44}
\end{equation*}
$$

hence

$$
\begin{align*}
\Phi\left(\omega_{1}, \omega_{2}\right)= & 1+j\left(\omega_{1} M^{1,0}+\omega_{2} M^{0,1}\right)+\frac{j^{2}}{2!}\left(\omega_{1}^{2} M^{2,0}+2 \omega_{1} \omega_{2} M^{1,1}+\omega_{2}^{2} M^{0,2}\right)+\cdots \\
& +\frac{j^{k}}{k!} \sum_{i=0}^{k}\binom{k}{i} \omega_{1}^{k-i} \omega_{2}^{i} M^{k-i, i}+\cdots \tag{3.45}
\end{align*}
$$

From (3.42) we determine that the non zero cross moments are $M^{2 m, 2 n}$ and $M^{2 m+1,2 n+1}$ for any $m$ and $n$. By comparing (3.45) with (3.43) we conclude the following identities;

$$
\begin{equation*}
\text { 1. } \Phi(0,0)=1 \tag{3.46}
\end{equation*}
$$

2. $\left.\quad \frac{\partial^{2 m+2 n} \Phi\left(\omega_{1}, \omega_{2}\right)}{\partial \omega_{1}^{2 m} \partial \omega_{2}^{2 n}}\right|_{\omega_{1}=0, \omega_{2}=0} \triangleq \Phi^{2 m, 2 n}(0,0)=(-1)^{m+n} M^{2 m, 2 n}$
3. $\left.\frac{\partial^{2 m+2 n+2} \Phi\left(\omega_{1}, \omega_{2}\right)}{\partial \omega_{1}^{2 m+1} \partial \omega_{2}^{2 n+1}}\right|_{\omega_{1}=0, \omega_{2}=0} \triangleq \Phi^{2 m+1,2 n+1}(0,0)=(-1)^{m+n+1} M^{2 m+1,2 n+1}$

### 3.4.1 Derivation of $\Phi^{2 m, 2 n}(0,0)$

From (3.40), by taking the derivative with respect to $\omega_{1}$, we obtain

$$
\begin{align*}
\Phi^{1,0}\left(\omega_{1}, \omega_{2}\right) & =\frac{\partial}{\partial \omega_{1}} \Phi\left(\omega_{1}, \omega_{2}\right) \\
& =-\sum_{k=-L}^{L} \alpha_{k} \sin \left(\omega_{1} \alpha_{k}+\omega_{2} \beta_{k}\right) \prod_{l=-L}^{L} \cos \left(\omega_{1} \alpha_{L}+\omega_{2} \beta_{L}\right) \tag{3.49}
\end{align*}
$$

Multiplying and dividing by $\cos \left(\omega_{1} \alpha_{k}+\omega_{2} \beta_{k}\right)$ we get

$$
\begin{equation*}
\Phi^{1,0}\left(\omega_{1}, \omega_{2}\right)=-\lambda_{\alpha}^{0,0}\left(\omega_{1}, \omega_{2}\right) \Phi^{0,0}\left(\omega_{1}, \omega_{2}\right) \tag{3.50}
\end{equation*}
$$

where

$$
\begin{equation*}
\lambda_{\alpha}^{0,0}=\sum_{k=-L}^{L} \alpha_{k} \tan \left(\omega_{1} \alpha_{k}+\omega_{2} \beta_{k}\right) \tag{3.51}
\end{equation*}
$$

and $\Phi^{0,0}\left(\omega_{1}, \omega_{2}\right)$, as given in (3.40) is truncated from $-L$ to $L$. Similarly, by taking the derivative with respect to $\omega_{2}$ we get

$$
\begin{equation*}
\Phi^{0,1}\left(\omega_{1}, \omega_{2}\right)=-\lambda_{\beta}^{0,0}\left(\omega_{1}, \omega_{2}\right) \Phi^{0,0}\left(\omega_{1}, \omega_{2}\right) \tag{3.52}
\end{equation*}
$$

where

$$
\begin{equation*}
\lambda_{\beta}^{0,0}\left(\omega_{1}, \omega_{2}\right)=\sum_{k=-L}^{L} \beta_{k} \tan \left(\omega_{1} \alpha_{k}+\omega_{2} \beta_{k}\right) \tag{3.53}
\end{equation*}
$$

Recall that $\alpha_{k}=g(t-k T)$ and $\beta_{k}=g(s-l T)$.

Similar to the steps used in Appendix A for the one dimensional moment case, if we take the derivatives of $\lambda^{0,0}\left(\omega_{1}, \omega_{2}\right)$ iteratively, we get

$$
\begin{align*}
\lambda_{\alpha}^{p, q}(0,0) & =\left.\frac{\partial^{p+q}}{\partial \omega_{1}^{p} \partial \omega_{2}^{q}} \lambda_{\alpha}^{0,0}\left(\omega_{1}, \omega_{2}\right)\right|_{\omega_{1}=0, \omega_{2}=0} \\
& = \begin{cases}\frac{2^{p+q+1}\left(2^{p+q+1}-1\right)}{p+q+1}\left|B_{p+q+1}\right| \sum_{k=-L}^{L} \alpha_{k}^{p+1} \beta_{k}^{q} & \text { if } p+q \text { is odd } \\
0 & \text { if } p+q \text { is even }\end{cases} \tag{3.54}
\end{align*}
$$

Similarly

$$
\begin{align*}
\lambda_{\beta}^{p, q}(0,0) & =\left.\frac{\partial^{p+q}}{\partial \omega_{1}^{p} \partial \omega_{2}^{q}} \lambda_{\beta}^{0,0}\left(\omega_{1}, \omega_{2}\right)\right|_{\omega_{1}=0, \omega_{2}=0} \\
& = \begin{cases}\frac{\left.2^{p+q+1} 2^{p+q+1}-1\right)}{p+q+1}\left|B_{p+q+1}\right| \sum_{k=-L}^{L} \alpha_{k}^{p} \beta_{k}^{q+1} & \text { if } p+q \text { is odd } \\
0 & \text { if } p+q \text { is even }\end{cases} \tag{3.55}
\end{align*}
$$

By successive differentiation of (3.50) with respect to $\omega_{1}$ we get

$$
\begin{equation*}
\Phi^{p, 0}\left(\omega_{1}, \omega_{2}\right)=-\sum_{j=0}^{p-1}\binom{p-1}{j} \lambda_{\alpha}^{j, 0}\left(\omega_{1}, \omega_{2}\right) \Phi^{p-1-j, 0}\left(\omega_{1}, \omega_{2}\right) \tag{3.56}
\end{equation*}
$$

and following the same steps used in appendix D we end up, for the even order of derivatives, with

$$
\begin{equation*}
\Phi^{2 m, 0}(0,0)=-\sum_{i=1}^{m}\binom{2 m-1}{2 i-1} \lambda_{\alpha}^{2 i-1,0}(0,0) \Phi^{2(m-i), 0}(0,0) . \tag{3.57}
\end{equation*}
$$

Similarly, for the derivatives with respect to $\omega_{2}$

$$
\begin{equation*}
\Phi^{0,2 n}(0,0)=-\sum_{i=1}^{n}\binom{2 n-1}{2 i-1} \lambda_{\beta}^{0,2 i-1}(0,0) \Phi^{0,2(n-1)}(0,0) \tag{3.58}
\end{equation*}
$$

Furthermore, from (3.52)

$$
\begin{equation*}
\Phi^{p, 1}\left(\omega_{1}, \omega_{2}\right)=-\sum_{k=0}^{p}\binom{p}{k} \lambda_{\beta}^{p-k, 0}\left(\omega_{1}, \omega_{2}\right) \Phi^{k, 0}\left(\omega_{1}, \omega_{2}\right) \tag{3.59}
\end{equation*}
$$

and by successive differentiation of (3.59) with respect to $\omega_{2}, q-1$ times, we obtain

$$
\begin{equation*}
\Phi^{p, q}\left(\omega_{1}, \omega_{2}\right)=-\sum_{k=0}^{p}\binom{p}{k}\left[\sum_{l=0}^{q-1}\binom{q-1}{l} \lambda_{\beta}^{p-k, q-1-l}\left(\omega_{1}, \omega_{2}\right) \Phi^{k, l}\left(\omega_{1}, \omega_{2}\right)\right] \tag{3.60}
\end{equation*}
$$

Changing the order of summation, we get for the even order of derivatives, $p=2 m$ and $q=2 n$,

$$
\begin{equation*}
\Phi^{2 m, 2 n}\left(\omega_{1}, \omega_{2}\right)=-\sum_{l=0}^{2 n-1}\binom{2 n-1}{l}\left[\sum_{k=0}^{2 m}\binom{2 m}{k} \lambda_{\beta}^{2 m-k, 2 n-1-l}\left(\omega_{1}, \omega_{2}\right) \Phi^{k, l}\left(\omega_{1}, \omega_{2}\right)\right] \tag{3.61}
\end{equation*}
$$

We now separate the internal summation by grouping those with even $l(l=2 i)$ and those with odd $l(l=2 i+1)$, as follows;

$$
\begin{gather*}
R_{2 i}\left(\omega_{1}, \omega_{2}\right)=\sum_{k=0}^{2 m}\binom{2 m}{k} \lambda_{\beta}^{2 m-k, 2 n-1-2 i}\left(\omega_{1}, \omega_{2}\right) \Phi^{k, 2 i}\left(\omega_{1}, \omega_{2}\right)  \tag{3.62}\\
R_{2 i+1}\left(\omega_{1}, \omega_{2}\right)=\sum_{k=0}^{2 m}\binom{2 m}{k} \lambda_{\beta}^{2 m-k, 2 n-1-(2 i+1)}\left(\omega_{1}, \omega_{2}\right) \Phi^{k, 2 i+1}\left(\omega_{1}, \omega_{2}\right) . \tag{3.63}
\end{gather*}
$$

Therefore,

$$
\begin{equation*}
\Phi^{2 m, 2 n}\left(\omega_{1}, \omega_{2}\right)=-\sum_{i=0}^{n-1}\binom{2 n-1}{2 i} R_{2 i}\left(\omega_{1}, \omega_{2}\right)-\sum_{i=1}^{n}\binom{2 n-1}{2 i+1} R_{2 i+1}\left(\omega_{1}, \omega_{2}\right) \tag{3.64}
\end{equation*}
$$

At $\omega_{1}=0 \quad \omega_{2}=0$, some of the terms in $R_{2 i}(0,0)$ and $R_{2 i+1}(0,0)$ are zero. This depends on the two indices of $\Phi(0,0)$ and $\lambda_{\beta}(0,0)$ as follows

$$
\begin{align*}
& \lambda_{\beta}^{p, q}(0,0) \neq 0 \text { only when } p+q \text { is odd }  \tag{3.65}\\
& \Phi^{p, q}(0,0) \neq 0 \text { only when } p+q \text { is even } \tag{3.66}
\end{align*}
$$

From (3.62), in order to satisfy (3.66), $k$ must be even. Let $k=2 j$, then

$$
\begin{equation*}
R_{2 i}(0,0)=\sum_{j=0}^{m}\binom{2 m}{2 j} \lambda_{\beta}^{2 m-2 j, 2 n-1-2 i}(0,0) \Phi^{2 j, 2 i}(0,0) \tag{3.67}
\end{equation*}
$$

Notice that the sum of the indices of $\lambda_{\beta}$ are odd, satisfying (3.65). Also from (3.63), to satisfy (3.66) $k$ must be odd. Let $k=2 j-1$, then

$$
\begin{equation*}
R_{2 i+1}(0,0)=\sum_{j=1}^{m}\binom{2 m}{2 j-1} \lambda_{\beta}^{2 m-(2 j-1), 2 n-1-(2 i+1)}(0,0) \Phi^{2 j-1,2 i+1}(0,0) \tag{3.68}
\end{equation*}
$$

Again we notice that the indices of $\lambda_{\beta}$ are odd, satisfying (3.65).

The result from (3.64) together with (3.67) and (3.68) are summarized in Table 3.1.

$$
\begin{equation*}
\Phi^{2 m, 2 n}(0,0)=-\sum_{i=0}^{n-1}\binom{2 n-1}{2 i} R_{2 i}(0,0)-\sum_{i=1}^{n}\binom{2 n-1}{2 i+1} R_{2 i+1}(0,0) \tag{3.69}
\end{equation*}
$$

where

$$
\begin{align*}
R_{2 i}(0,0) & =\sum_{j=0}^{m}\binom{2 m}{2 j} \lambda_{\beta}^{2 m-2 j, 2 n-1-2 i}(0,0) \Phi^{2 j, 2 i}(0,0)  \tag{3.70}\\
R_{2 i+1}(0,0) & =\sum_{j=1}^{m}\binom{2 m}{2 j-1} \lambda_{\beta}^{2 m-(2 j-1), 2 n-1-(2 i+1)}(0,0) \Phi^{2 j-1,2 i+1}(0,0) \tag{3.71}
\end{align*}
$$

Table 3.1

Equivalently, we can start the derivation of $M^{2 m, 2 n}$ from (3.50) instead of (3.52)

$$
\begin{equation*}
\Phi^{1, q}\left(\omega_{1}, \omega_{2}\right)=-\sum_{l=0}^{q}\binom{q}{l} \lambda_{\alpha}^{0, q-l}\left(\omega_{1}, \omega_{2}\right) \Phi^{0, l}\left(\omega_{1}, \omega_{2}\right) \tag{3.72}
\end{equation*}
$$

and by successive differentiation with respect to $\omega_{1}$, we obtain

$$
\begin{equation*}
\Phi^{p, q}\left(\omega_{1}, \omega_{2}\right)=-\sum_{l=0}^{q}\binom{q}{l}\left[\sum_{k=0}^{p-1}\binom{p-1}{k} \lambda_{\alpha}^{p-1-k, q-l}\left(\omega_{1}, \omega_{2}\right) \Phi^{k, l}\left(\omega_{1}, \omega_{2}\right)\right] . \tag{3.73}
\end{equation*}
$$

Changing the order of summation we have, for the even derivatives,

$$
\begin{equation*}
\Phi^{2 m, 2 n}\left(\omega_{1}, \omega_{2}\right)=-\sum_{k=0}^{2 m-1}\binom{2 m-1}{k}\left[\sum_{l=0}^{2 n}\binom{2 n}{l} \lambda_{\alpha}^{2 m-1-k, 2 n-l}(0,0) \Phi^{k, l}\left(\omega_{1}, \omega_{2}\right)\right] \tag{3.74}
\end{equation*}
$$

We separate the internal summation to those with even $k$ and odd $k$

$$
\begin{equation*}
\Phi^{2 m, 2 n}\left(\omega_{1}, \omega_{2}\right)=-\sum_{j=0}^{m-1}\binom{2 m-1}{2 j} R_{2 j}\left(\omega_{1}, \omega_{2}\right)-\sum_{j=1}^{m}\binom{2 m-1}{2 j+1} R_{2 j+1}\left(\omega_{1}, \omega_{2}\right) \tag{3.75}
\end{equation*}
$$

where

$$
\begin{equation*}
R_{2 j}\left(\omega_{1}, \omega_{2}\right)=\sum_{l=0}^{2 n}\binom{2 n}{l} \lambda_{\alpha}^{2 m-1-2 j, 2 n-l}\left(\omega_{1}, \omega_{2}\right) \Phi^{2 j, l}\left(\omega_{1}, \omega_{2}\right) \tag{3.76}
\end{equation*}
$$

and

$$
\begin{equation*}
R_{2 j+1}\left(\omega_{1}, \omega_{2}\right)=\sum_{l=0}^{2 n}\binom{2 n}{l} \lambda_{\alpha}^{2 m-1-(2 j+1), 2 n-l}\left(\omega_{1}, \omega_{2}\right) \Phi^{2 j+1, l}\left(\omega_{1}, \omega_{2}\right) \tag{3.77}
\end{equation*}
$$

Dropping the zero terms of $\lambda_{\alpha}(0,0)$ and $\Phi(0,0)$, according to (65) and (66), we get

$$
\begin{equation*}
R_{2 j}(0,0)=\sum_{l=0}^{n}\binom{2 n}{2 i} \lambda_{\alpha}^{2 m-1-2 j, 2 n-2 i}(0,0) \Phi^{2 j, 2 i}(0,0) \tag{3.78}
\end{equation*}
$$

and

$$
\begin{equation*}
R_{2 j+1}(0,0)=\sum_{l=0}^{2 n}\binom{2 n}{2 i-1} \lambda_{0,0)}^{2 m-1-(2 j+1), 2 n-(2 i-1)} \Phi^{2 j+1,2 i-1}(0,0) \tag{3.79}
\end{equation*}
$$

The Equivalent expression for $\Phi^{2 m, 2 n}$, of Table 3.1, are summarized in Table 3.2.

$$
\begin{equation*}
\Phi^{2 m, 2 n}(0,0)=-\sum_{j=0}^{m-1}\binom{2 m-1}{2 j} R_{2 j}(0,0)-\sum_{i=1}^{m}\binom{2 m-1}{2 j+1} R_{2 j+1}(0,0) \tag{3.80}
\end{equation*}
$$

where

$$
\begin{align*}
R_{2 j}(0,0) & =\sum_{i=0}^{n}\binom{2 n}{2 i} \lambda_{\alpha}^{2 m-1-2 j, 2 n-2 i}(0,0) \Phi^{2 j, 2 i}(0,0)  \tag{3.81}\\
R_{2 j+1}(0,0) & =\sum_{i=1}^{n}\binom{2 n}{2 i-1} \lambda_{\alpha}^{2 m-1-(2 j+1), 2 n-(2 i-1)}(0,0) \Phi^{2 j+1,2 i-1}(0,0)(3 \tag{3.82}
\end{align*}
$$

Table 3.2

The results depicted in Tables 3.1 and 3.2 are equivalent recursive equations, one uses $\lambda_{\beta}(0,0)$ and the other uses $\lambda_{\alpha}(0,0)$.

### 3.4.2 Derivation of $\Phi^{2 m+1,2 n+1}(0,0)$

From (3.60), with $p=2 m+1$ and $q=2 n+1$, we get
$\Phi^{2 m+1,2 n+1}\left(\omega_{1}, \omega_{2}\right)=\sum_{l=0}^{2 n}\binom{2 n}{l}\left[\sum_{k=0}^{2 m+1}\binom{2 m+1}{k} \lambda_{\beta}^{2 m+1-k, 2 n-l}\left(\omega_{1}, \omega_{2}\right) \Phi^{k, l}\left(\omega_{1}, \omega_{2}\right)\right]$

Again, separating the internal summation into terms with even $l(l=2 i)$ and odd $l(l=2 i+1)$, we define

$$
\begin{equation*}
S_{2 i}\left(\omega_{1}, \omega_{2}\right)=\sum_{k=0}^{2 m+1}\binom{2 m+1}{k} \lambda_{\beta}^{2 m+1-k, 2 n-2 i}\left(\omega_{1}, \omega_{2}\right) \Phi^{k, 2 i}\left(\omega_{1}, \omega_{2}\right) \tag{3.84}
\end{equation*}
$$

and

$$
\begin{equation*}
S_{2 i+1}\left(\omega_{1}, \omega_{2}\right)=\sum_{k=0}^{2 m+1}\binom{2 m+1}{k} \lambda_{\beta}^{2 m+1-k, 2 n-(2 i+1)}\left(\omega_{1}, \omega_{2}\right) \Phi^{k, 2 i+1}\left(\omega_{1}, \omega_{2}\right) \tag{3.85}
\end{equation*}
$$

We therefore write,

$$
\begin{equation*}
\Phi^{2 m+1,2 n+1}\left(\omega_{1}, \omega_{2}\right)=-\sum_{i=0}^{n}\binom{2 n}{2 i} S_{2 i}\left(\omega_{1}, \omega_{2}\right)-\sum_{i=1}^{n-1}\binom{2 n}{2 i+1} S_{2 i+1}\left(\omega_{1}, \omega_{2}\right) \tag{3.86}
\end{equation*}
$$

Applying the condition for $\lambda_{\beta}^{p, q}(0,0)$ and $\Phi^{p, q}(0,0)$, in (3.65) and (3.66), we are left, in (3.84), with terms having $k=2 j$, even, so that,

$$
\begin{equation*}
S_{2 i}(0,0)=\sum_{j=0}^{m}\binom{2 m+1}{2 j} \lambda_{\beta}^{2 m+1-2 j, 2 n-2 i}(0,0) \Phi^{2 j, 2 i}(0,0) \tag{3.87}
\end{equation*}
$$

In (3.85) we are left with terms having $k=2 j+1$, odd, so that

$$
\begin{equation*}
S_{2 i+1}(0,0)=\sum_{j=0}^{m}\binom{2 m+1}{2 j+1} \lambda_{\beta}^{2 m-2 j, 2 n-(2 i+1)}(0,0) \Phi^{2 j+1,2 i+1}(0,0) \tag{3.88}
\end{equation*}
$$

In summary we have

$$
\begin{equation*}
\Phi^{2 m+1,2 n+1}(0,0)=-\sum_{i=0}^{n}\binom{2 n}{2 i} S_{2 i}(0,0)-\sum_{i=0}^{n-1}\binom{2 n}{2 i+1} S_{2 i+1}(0,0) \tag{3.89}
\end{equation*}
$$

where

$$
\begin{align*}
S_{2 i}(0,0) & =\sum_{j=0}^{m}\binom{2 m+1}{2 j} \lambda_{\beta}^{2 m+1-2 j, 2 n-2 i}(0,0) \Phi^{2 j, 2 j}(0,0)  \tag{3.90}\\
S_{2 i+1}(0,0) & =\sum_{j=0}^{m}\binom{2 m+1}{2 j+1} \lambda_{\beta}^{2 m-2 j, 2 n-(2 i+1)}(0,0) \Phi^{2 j+1,2(j+1)}(0,0) \tag{3.91}
\end{align*}
$$

Table 3.3

Equivalently from (3.73), with $p=2 m+1$ and $q=2 n+1$, we get
$\Phi^{2 m+1,2 n+1}\left(\omega_{1}, \omega_{2}\right)=\sum_{k=0}^{2 m}\binom{2 m}{k}\left[\sum_{l=0}^{2 n+1}\binom{2 n+1}{l} \lambda_{\alpha}^{2 m-k, 2 n+1-l}\left(\omega_{1}, \omega_{2}\right) \Phi^{k, l}\left(\omega_{1}, \omega_{2}\right)\right]$

Separating the internal summation into terms with even and odd $k$, we have, respectively

$$
\begin{equation*}
S_{2 j}\left(\omega_{1}, \omega_{2}\right)=\sum_{l=0}^{2 n+1}\binom{2 n+1}{l} \lambda_{\alpha}^{2 m-2 j, 2 n+1-l}\left(\omega_{1}, \omega_{2}\right) \Phi^{2 j, l}\left(\omega_{1}, \omega_{2}\right) \tag{3.93}
\end{equation*}
$$

and

$$
\begin{equation*}
S_{2 j+1}\left(\omega_{1}, \omega_{2}\right)=\sum_{l=0}^{2 n+1}\binom{2 n+1}{l} \lambda_{\alpha}^{2 m-(2 j+1), 2 n+1-l}\left(\omega_{1}, \omega_{2}\right) \Phi^{2 j+1, l}\left(\omega_{1}, \omega_{2}\right) \tag{3.94}
\end{equation*}
$$

We therefore write

$$
\begin{equation*}
\Phi^{2 m+1,2 n+1}\left(\omega_{1}, \omega_{2}\right)=-\sum_{k=0}^{m}\binom{2 m}{2 j} S_{2 j}\left(\omega_{1}, \omega_{2}\right)-\sum_{k=0}^{m-1}\binom{2 m}{2 j+1} S_{2 j+1}\left(\omega_{1}, \omega_{2}\right) \tag{3.95}
\end{equation*}
$$

For the condition on $\lambda_{\beta}^{p, q}(0,0)$ and $\Phi^{p, q}(0,0)$ to be non zero, we are left, in (3.93), with terms having even $l(l=2 i)$, so that

$$
\begin{equation*}
S_{2 j}(0,0)=\sum_{i=0}^{n}\binom{2 n+1}{2 i} \lambda_{\alpha}^{2 m-2 j, 2 n+1-2 i}(0,0) \Phi^{2 j, 2 i}(0,0) \tag{3.96}
\end{equation*}
$$

In (3.94) we are left with terms having odd $l(l=2 i+1)$, so that

$$
\begin{equation*}
S_{2 j+1}(0,0)=\sum_{i=0}^{n}\binom{2 n+1}{2 i+1} \lambda_{\beta}^{2 m-(2 j+1), 2 n-2 i}(0,0) \Phi^{2 j+1,2 i+1}(0,0) \tag{3.97}
\end{equation*}
$$

In summary we have

$$
\begin{equation*}
\Phi^{2 m+1,2 n+1}(0,0)=-\sum_{k=0}^{m}\binom{2 m}{2 j} S_{2 j}(0,0)-\sum_{k=0}^{m-1}\binom{2 m}{2 j+1} S_{2 j+1}(0,0) \tag{3.98}
\end{equation*}
$$

where

$$
\begin{align*}
S_{2 j}(0,0) & =\sum_{i=0}^{n}\binom{2 n+1}{2 i} \lambda_{\alpha}^{2 m-2 j, 2 n+1-2 i}(0,0) \Phi^{2 j, 2 j}(0,0)  \tag{3.99}\\
S_{2 j+1}(0,0) & =\sum_{i=0}^{n}\binom{2 n+1}{2 i+1} \lambda_{\alpha}^{2 m-(2 j+1), 2 n-2 i}(0,0) \Phi^{2 j+1,2 i+1)}(0,0) \tag{3.100}
\end{align*}
$$

Table 3.4

In conclusion, in order to determine the moment, $M^{2 m, 2 n}$ or $M^{2 m+1,2 n+1}$, we first refer to (3.47) and (3.48) which relate these moments to the derivatives of the joint characteristic function;

$$
\begin{align*}
M^{2 m, 2 n} & =(-1)^{m+n} \Phi^{2 m, 2 n}(0,0)  \tag{3.101}\\
M^{2 m+1,2 n+1} & =(-1)^{m+n+1} \Phi^{2 m+1,2 n+1}(0,0) \tag{3.102}
\end{align*}
$$

The joint characteristic functions $\Phi^{2 m, 2 n}(0,0)$ and $\Phi^{2 m+1,2 n+1}(0,0)$ can be evaluated by using Tables 3.1 and 3.3 respectively. Equivalently, Tables 3.2 and 3.4 could also be used where $\lambda_{\alpha}(0,0)$ and its derivatives are used instead of $\lambda_{\beta}(0,0)$. The equations presented in Tables 3.1 and 3.1 are recursive formulas involving the terms $R_{2 i}$ and $R_{2 i+1}$, or $S_{2 i}$ and $S_{2 i+1}$ which are, in turn, recursive formulas involving $\Phi^{2 m, 2 n}(0,0)$ and $\Phi^{2 m+1,2 n+1}(0,0)$.

## Chapter 4

## Numerical Results of Jitter Performance for STR Circuits Using the Moment Method

In this chapter, we will present and compare some of the numerical results obtained by using the moment method for evaluating the performance of $S T R$ circuits as it is described in chapter 3.

For the purpose of our discussion, as a prefilter, we selected a raised cosine pulse shape (2.168), which is of most importance in practice;

$$
G(f)=\left\{\begin{array}{lc}
T & |f|<\frac{1-\gamma}{2 T}  \tag{4.1}\\
\frac{T}{2}\left\{1-\sin \frac{\pi T}{\gamma}\left(|f|-\frac{1}{2 T}\right)\right\} & \frac{1-\gamma}{2 T} \leq|f| \leq \frac{1+\gamma}{2 T} \\
0 & \text { elsewhere }
\end{array}\right.
$$

with $\gamma$ being the excess bandwidth, and $1 / T$ is the baud rate. For the post filter, we used the following bandpass filter, centered at $1 / T$,

$$
\begin{equation*}
H(f)=H_{L P}\left(f-\frac{1}{T}\right)+H_{L P}\left(f+\frac{1}{T}\right) \tag{4.2}
\end{equation*}
$$

where the lowpass equivalent transfer function, $H_{L P}(f)$, is given by

$$
\begin{equation*}
H_{L P}(f)=\frac{1}{1+j 2 f T Q} \tag{4.3}
\end{equation*}
$$

with $Q$ being the quality factor.

### 4.1 Calculating the Mean Value of the Timing Wave

Using the recursive formulas in (3.6) and (3.7), we first calculated the moments $M_{2 n}(t)$, for $n=1,2,3$, and 4. These results are depicted in Figs 4.1 and 4.2 for $\gamma=0.1$ and $\gamma=0.5$, respectively. Note that we did not perform the evaluation for $\gamma$ greater than 0.5 since bandwidth efficiency is one of our main concerns. Since the input to the nonlinear device is cyclostationary, the moments $M_{2 n}(t)=E\left[x^{2 n}(t)\right]$ are periodic with period $1 / T$. This is shown in Figs. 4.3 and 4.4 , for $n=1,2,3$, and 4 , for $\gamma=0.1$ and 0.5 , respectively.

As a function of the excess bandwidth $\gamma$, in Fig. 4.5, we depict the amplitude of the mean value of the extracted timing wave, where the order, $n=1,2,3,4$, is kept as a parameter. We notice from this figure that, the higher the order of the nonlinearity the larger the amplitude of the extracted timing wave. Particularly, for low values of $\gamma$ we see how the second order failed. This is one of the advantages of STR circuits employing a high order nonlinearity.


Figure 4.1: $M_{2 n}(t)$ for $n=1,2,3$, and $4, \gamma=0.1$


Figure 4.2: $M_{2 n}(t)$ for $n=1,2,3$, and $4, \gamma=0.5$


Figure 4.3: Few Cycles of $M_{2 n}(t), \gamma=0.1$


Figure 4.4: Few Cycles of $M_{2 n}(t), \gamma=0.5$


Figure 4.5: Amplitude of the mean value of extracted timing wave as a function of $\gamma$, for $n=1,2,3$, and 4

### 4.2 Calculating the Jitter Performance

Since calculating the jitter performance is a rather complex process, we must first calculate all the parts which make up of, and lead to, the final expression. This involves first using the equations in Tables 3.1 and 3.3 , or 3.2 and 3.4 , to calculate a 3-dimensional surface plots, which represents the joint moment, $M_{2 m, 2 n}(t, s)$, of the input to the nonlinear device.

For the case of second order nonlinearity ( $\mathrm{N}=2$ ), we show, in Fig. 4.6, the joint moment $M_{2,2}(t, s)$ for $\gamma=0.1$. We also include the same plot, rotated by 90 degrees, to emphasize its shape in other directions. Fig. 4.8 depicts the same results as 4.6 , except that the rolloff factor $\gamma$ is now 0.5. Last, in Fig. 4.10 we depict the results for $\gamma=0.9$. Notice that for the added convenience of the reader, we inserted the contour plots, corresponding to these 3-dimensional views, in Figs 4.7, 4.9, and 4.11 for the values of $\gamma=0.1,0.5,0.9$, respectively.

For the case of fourth order nonlinearity $(\mathrm{N}=4)$, we show, in Fig 4.12, the joint moment $M_{4,4}(t, s)$ for $\gamma=0.1$. We also included the same plot rotated by 90 degrees. Similarly, in Fig. 4.14, we show the plots for $\gamma=0.5$, and last, we show the plots for $\gamma=0.9$, in Fig. 4.16. Again, just as before, we added the contour plots, corresponding to the 3 -dimensional figures, in Figs. 4.13, 4.15, and 4.17, for $\gamma=0.1,0.5$, and 0.9 respectively.


Figure 4.6: (left) Joint Moment $M_{2,2}(t, s), \gamma=0.1$ (right) Joint Moment $M_{2,2}(t, s), \gamma=0.1$, rotated 90 degrees


Figure 4.7: $\gamma=0.1$, topographical map



Figure 4.8: (left) Joint Moment $M_{2,2}(t, s), \gamma=0.5$ (right) Joint Moment $M_{2,2}(t, s), \gamma=0.5$, rotated 90 degrees


Figure 4.9: $M_{2,2}(t, s), \gamma=0.5$, topographical map



Figure 4.10: (left) Joint Moment $M_{2,2}(t, s), \gamma=0.9$ (right) Joint Moment $M_{2,2}(t, s), \gamma=0.9$, rotated 90 degrees


Figure 4.11: $M_{2,2}(t, s), \gamma=0.9$, topographical map


Figure 4.12: (left) Joint Moment $M_{4,4}(t, s), \gamma=0.1$ (right) Joint Moment $M_{4,4}(t, s), \gamma=0.1$, rotated 90 degrees


Figure 4.13: $M_{4,4}(t, s), \gamma=0.1$, topographical map


Figure 4.14: (left) Joint Moment $M_{4,4}(t, s), \gamma=0.5$ (right) Joint Moment $M_{4,4}(t, s), \gamma=0.5$, rotated 90 degrees


Figure 4.15: $M_{4,4}(t, s), \gamma=0.5$, topographical map



Figure 4.16: (left) Joint Moment $M_{4,4}(t, s), \gamma=0.9$ (right) Joint Moment $M_{4,4}(t, s), \gamma=0.9$, rotated 90 degrees


Figure 4.17: $M_{4,4}(t, s), \gamma=0.9$, topographical map

From these figures, one can easily notice that the joint moments are periodic along lines $s=t+c$ ( $c$ is constant), with period $1 / T$. The peaks of these plots occur at the line $s=t(c=0)$, and are almost equal in amplitude. Along the $t$ or $s$ axis, the peaks are not equal. The value of the peaks become lower as $\gamma$ increases, and the difference between the peaks and valleys becomes smaller. Last we note that, perpendicular to the line $s=t$, the plot is no longer periodic but rather symmetric. Similar conclusion can be stated for the case of $N=4$, except that the peaks are more emphasized and the entire plot gains amplitude compared to thecase of $N=2$.

From (3.36) we have,

$$
R_{Y^{\cdot}}(t, s)= \begin{cases}M_{2,2}(t, s) & \text { for second order nonlinearity }  \tag{4.4}\\ M_{4,4}(t, s) & \text { for fourth order nonlinearity }\end{cases}
$$

We next use the previous results shown in Figs. 4.6-4.11, for $N=2$ and Figs. 4.12-4.17, for $N=4$, to calculate the Fourier coefficients, $r_{k}(\tau)$, of these moments, for $k=0$ and $k=2$. For the second order nonlinearity, we show in Figs. 4.18 and 4.19, the Fourier coefficients $r_{0}(\tau)$ and $r_{2}(\tau)$, respectively. Similar behavior is expected of $r_{0}(\tau)$ and $r_{2}(\tau)$ for $N=4$.


Figure 4.18: Fourier Coefficients $r_{0}(\tau) N=2, \gamma=0.5$


Figure 4.19: Fourier Coefficients $r_{2}(\tau), N=2, \gamma=0.5$

### 4.3 Final Jitter Results

The last step in determining the rms timing jitter is to gather all the results obtained from the previous figures, of the previous sections, and combine them to yield one final result. All the parts, which make up of the jitter, we evaluated using a fortran program, ran on the VAX computer.

In Fig. 4.20 we depict the rms jitter for $N=2$. This figure is drawn as a function of the quality factor of the postfiler, where the excess bandwidth, $\gamma$, was used as a parameter. In Fig. 4.21 we depict the jitter performance for $N=4$. Last, for sake of comparison we show, in Fig. 4.22, a comparison of the rms jitter function for $N=2$ vs $N=4$, for the same excess bandwidth, $\gamma$. In all the figures, we used an ideal square type of prefilter.

It can be observed from figs $4.20-4.22$ that the jitter performance gets better as we increase the quality factor of the postfilter. Furthermore, as we shift from one curve to the other, we note that the jitter performacnce also improves. The same characteristics apply to the case of fourth order, $N=4$, nonlinearity. Comparing the two type of nonlinearity in Fig. 4.22, we see that the second order nonlinearity superceeds the performance of the fourth order. However, we should keep in mind that we are not only concerned with the jitter performance, but also with bandwidth efficiency and timing extraction with significant amplitude. Therefore for our purposes we conclude that a fourth order nonlinearity, at low transmission bandwidth, superceed the second order.


Figure 4.20: Rms Jitter, $N=2$


Figure 4.21: Rms Jitter, $N=4$


Figure 4.22: Rms Jitter, $N=2$ vs $N=4$

## Chapter 5

## Conclusion

This thesis presents a general method for evaluating the jitter performance of a popular type of Symbol Timing Recovery ( $S T R$ ) circuit for baseband digital transmission systems. The STR circuit consists of any even symmetric, zero memory, non-linear device followed by a narrow band postfilter tuned to the signaling rate ( $1 / T$ ), along with a prefilter for reshaping the pulses entering the timing path. The output of the STR circuit is nearly a sinusoidal timing wave whose zero crossings indicate the appropriate sampling instants for demodulating the incoming signal. For a random data sequence, the timing wave exhibits phase fluctuations which strongly depend on the pulse shapes entering the timing path and the quality factor $(Q)$ of the postfilter.

In Chapter 2, after defining the rms jitter for a timing wave, we first considered a squaring type of nonlinearity and obtained exact analytical expressions for the rms phase jitter in the timing wave as a function of the rolloff factor ( $\gamma$ ) of the pulse shape. These expressions have a form which is especially suitable for
studying the case where the baseband signal is band limited to frequencies less than $1 / T$. The rms jitter, based on derived expressions, was computed as a function of $Q$, for various values of rolloff factors. In addition, numerically obtained rms jitter values for the raised cosine and trapezoidal pulse shapes were plotted on the same scale and compared to each other.

In Chapter 3, the results obtained for the squaring type of nonlinearity were extended for the case of any even symmetric, zero memory, nonlinear device. The rms jitter of the STR circuit was evaluated by means of higher order moments of the input signal. An iterative method was given for the evaluation of the higher order cross moments of the input signal, and then, these moments were expressed in the frequency domain to compute the rms jitter.

Chapter 4, discussed numerical jitter performance investigations for the STR circuit employing squaring and fourth order nonlinearities. The cross moments corresponding to these cases were obtained and their three dimensional plots were drawn. The rms jitter curves were obtained and plotted as a function of the quality factor, for several values of rolloff factors, and compared to each other.

The main conclusions drawn are as follows

1. rms jitter decreases as $Q$ increases.
2. rms jitter increases as $\gamma$ decreases. In fact, for the squarer type of STR circuit, the jitter takes very large values as $\gamma \rightarrow 0$.
3. Fourth order STR circuits work as well for small $\gamma$ values. This is one of the major advantages over the squaring STR circuit.

## For Further Research

1. Noise can be added into the analysis
2. Analysis can be extended to include multi level and correlated data
3. Other type of nonlinearities, such as absolute value and $\tan$ hyperbolic functions, can be investigated

## Appendix A

- Evaluation of $E\left[a_{k} a_{k+m} a_{k+j} a_{k+j+l}\right]$ of (2.29)

The value of $E\left[a_{k} a_{k+m} a_{k+j} a_{k+j+l}\right]$ can be evaluated by combining the different indices. We consider separately the different cases of $m, l$, and $j$, from the following table.

| case | m | l | j |
| :---: | :---: | :---: | :---: |
| 1 | 0 | 0 | 0 |
| 2 | 0 | 0 | $\neq 0$ |
| 3 | $m=l \neq 0$ |  | 0 |
| 4 | $m=l \neq 0$ | $\neq 0$ |  |
| 5 | $m=j \neq 0$ | $j=-l$ |  |
| 6 | $m=j \neq 0$ | $j \neq-l$ |  |
| 7 | $m \neq j$ |  |  |
|  | $m \neq l$ |  |  |

(1) when $m=l=j=0$ we get $E\left[a_{k}^{4}\right]$.
(2) when $m=l=0$ and $j \neq 0$ we get,

$$
\begin{align*}
E\left[a_{k} a_{k} a_{k+j} a_{k+j}\right] & =E\left[a_{k}^{2}\right] E\left[a_{k+j}^{2}\right]  \tag{A.1}\\
& =\left\{E\left[a_{k}^{2}\right]\right\}^{2}  \tag{A.2}\\
& \triangleq \alpha_{0}^{2} \tag{A.3}
\end{align*}
$$

(3) when $m=l \neq 0$ and $j=0$ we get

$$
\begin{align*}
E\left[a_{k} a_{k+m} a_{k} a_{k+m}\right. & =E\left[a_{k}^{2}\right] E\left[a_{k+m}^{2}\right]  \tag{A.4}\\
& =\left\{E\left[a_{k}^{2}\right]\right\}^{2}  \tag{A.5}\\
& \triangleq \alpha_{0}^{2} \tag{A.6}
\end{align*}
$$

(4) when $m=l \neq 0$ and $j \neq 0$ we get

$$
E\left[a_{k} a_{k+m} a_{k+j} a_{k+j+m}\right]= \begin{cases}E\left[a_{k}\right] E\left[a_{k+m} a_{k+j} a_{k+j+m}\right]=0 & \text { if } j=-m  \tag{A.7}\\ E\left[a_{k}^{2}\right] E\left[a_{k+m}\right] E\left[a_{k-m}\right]=0 & \text { if } j \neq-m\end{cases}
$$

(5) when $m=j \neq 0$ and $j=-l$ we get

$$
\begin{align*}
E\left[a_{k} a_{k+m} a_{k+j} a_{k+j+l}\right] & =E\left[a_{k} a_{k+m} a_{k+m} a_{k}\right]  \tag{A.8}\\
& =E\left[a_{k}^{2}\right] E\left[a_{k+m}^{2}\right]  \tag{A.9}\\
& =\alpha_{0}^{2} \tag{A.10}
\end{align*}
$$

(6) when $m=j \neq 0$ and $j \neq-l$

$$
\begin{align*}
E\left[a_{k} a_{k+m} a_{k+j} a_{k+j+l}\right] & =E\left[a_{k} a_{k+m}^{2} a_{k+j+l}\right]  \tag{A.11}\\
& =E\left[a_{k}\right] E\left[a_{k+m}^{2}\right] E\left[a_{k+j+l}\right]  \tag{A.12}\\
& =0 \tag{A.13}
\end{align*}
$$

(7) when $m \neq j$ and $m \neq l$

$$
E\left[a_{k} a_{k+m} a_{k+j} a_{k+j+l}\right]= \begin{cases}E\left[a_{k}^{2}\right] E\left[a_{k+m}\right] E\left[a_{k+j}\right]=0 & \text { if } j=-l  \tag{A.14}\\ E\left[a_{k}\right] E\left[a_{k+m}\right] E\left[a_{k+j}\right] E\left[a_{k+j+l}\right]=0 & \text { if } j \neq-l\end{cases}
$$

## Appendix B

## - Proof of (2.33)

From the definition of $q_{m}(t)$ and (2.7)

$$
\begin{equation*}
q_{m}(t-k T)=[g(t-k T) g(t-k T-m T)] \otimes h(t) \tag{B.1}
\end{equation*}
$$

Therefore

$$
\begin{align*}
q_{-m}(t-k T-m T) & =[g(t-k T-m T) g(t-k T-m T+m T)] \otimes h(t) \\
& =q_{m}(t-k T) \otimes h(t) \tag{B.2}
\end{align*}
$$

## Appendix C

## - Derivation of (2.82)

Let

$$
\begin{equation*}
C(f)=G(f) G\left(\frac{1}{T}-f\right) \tag{C.1}
\end{equation*}
$$

Therefore the conjugate

$$
\begin{align*}
C^{*}(f) & =G^{*}(f) G^{*}\left(\frac{1}{T}-f\right)  \tag{C.2}\\
& =G(-f) G\left(f-\frac{1}{T}\right) \tag{C.3}
\end{align*}
$$

and hence we can write

$$
\begin{equation*}
C^{*}(\alpha+\beta)=g(-\alpha-\beta) G\left(\alpha+\beta-\frac{1}{T}\right) \tag{C.4}
\end{equation*}
$$

Combining (C.1) and (C.4), we can rewrite the second expression in (2.80) using the fact that

$$
\begin{equation*}
G(\beta) G\left(\frac{1}{T}-\beta\right) G(-\alpha-\beta) G\left(\alpha+\beta-\frac{1}{T}\right)=C(\beta) C^{*}(\alpha+\beta) \tag{C.5}
\end{equation*}
$$

As for the third expression in (2.80) we first make the variable change $\beta=-\beta$ and $\alpha=-\alpha$ to get

$$
\begin{equation*}
\frac{1}{T} \int_{\alpha} \int_{\beta}|H(-\alpha)|^{2} G\left(-\frac{1}{T}+\beta\right) G\left(-\alpha-\beta+\frac{1}{T}\right) G(-\beta) G(\alpha+\beta) d \alpha d \beta \tag{C.6}
\end{equation*}
$$

Using (C.3)

$$
\begin{equation*}
C^{*}(\beta)=G(-\beta) G\left(\beta-\frac{1}{T}\right) \tag{C.7}
\end{equation*}
$$

and from (C.1)

$$
\begin{equation*}
C(\alpha+\beta)=G(\alpha+\beta) G\left(\frac{1}{T}-\alpha-\beta\right) \tag{C.8}
\end{equation*}
$$

Combining (C.7) and (C.8) we get

$$
\begin{equation*}
G(-\beta) G\left(\beta-\frac{1}{T}\right) G(\alpha+\beta) G\left(\frac{1}{T}-\alpha-\beta\right)=C^{*}(\beta) C(\alpha+\beta) \tag{C.9}
\end{equation*}
$$

Finally, using the fact that $|H(-\alpha)|^{2}=|H(\alpha)|^{2}$, we have, for the second and third terms of (2.80)

$$
\begin{equation*}
\frac{1}{T} \int_{\alpha} \int_{\beta}|H(\alpha)|^{2}\left[C(\beta) C^{*}(\alpha+\beta)+C^{*}(\beta) C(\alpha+\beta)\right] d \alpha d \beta \tag{C.10}
\end{equation*}
$$

## Appendix D

- Derivation of moment function $M_{2 n}(t)$, Eq. (3.6)

The characteristic function of a random variable $x$ is defined by

$$
\begin{equation*}
\Phi(\omega)=\int_{-\infty}^{\infty} \exp (j \omega x) p(x) d x \tag{D.1}
\end{equation*}
$$

where $p(x)$ is the probability density function of $x$.

$$
\begin{equation*}
\Phi(\omega)=1+j \omega M_{1}+\frac{(j \omega)^{2}}{2!} M_{2}+\cdots+\frac{(j \omega)^{2 n}}{n!} M_{n} \tag{D.2}
\end{equation*}
$$

with $M_{n}=E\left[x_{n}\right]$. It is easy to prove that $\Phi(0)=1$ and

$$
\begin{equation*}
\left.\frac{d^{n} \Phi(\omega)}{d \omega^{n}}\right|_{\omega=0}=(j \omega)^{n} M_{n} \tag{D.3}
\end{equation*}
$$

For our case, the random variable $x(t)$ is related to the data sequence, $\left\{a_{k}\right\}$ by

$$
\begin{equation*}
x(t)=\sum_{k=-\infty}^{\infty} a_{k} g(t-k T) \tag{D.4}
\end{equation*}
$$

therefore

$$
\begin{equation*}
\Phi_{X}(\omega)=E\left[\exp \left(j \omega \sum_{k=-\infty}^{\infty} a_{k} g(t-k T)\right)\right] \tag{D.5}
\end{equation*}
$$

If $\left\{a_{k}\right\}$ are independently distributed, then

$$
\begin{equation*}
\Phi_{X}(\omega)=\prod_{k=-\infty}^{\infty} E\left[\exp \left(j a_{k} \omega g(t-k T)\right]\right. \tag{D.6}
\end{equation*}
$$

For $\left\{a_{k}\right\}= \pm 1$ with equal probabilities, then

$$
\begin{align*}
\Phi_{X}(\omega) & =\prod_{k=-\infty}^{\infty}\left(\frac{1}{2} \exp (j \omega g(t-k T))+\frac{1}{2} \exp (j \omega g(t-k T))\right) \\
& =\prod_{k=-\infty}^{\infty} \cos (\omega g(t-k T)) \tag{D.7}
\end{align*}
$$

Taking derivatives

$$
\begin{align*}
\Phi^{\prime}(\omega) & =\frac{d \Phi(\omega)}{d \omega} \\
& =-\sum_{k=-\infty}^{\infty} g(t-k T) \sin (\omega g(t-k T)) \prod_{l=-\infty}^{\infty} \cos (\omega g(t-l T)) \tag{D.8}
\end{align*}
$$

Multiplying and dividing the left hand side of (D.8) by $\cos (\omega g(t-k T))$, we get

$$
\begin{align*}
\Phi^{\prime}(\omega) & =-\sum_{k=-\infty}^{\infty} g(t-k T) \tan (\omega g(t-k T)) \prod_{l=-\infty}^{\infty} \cos (\omega g(t-l T)) \\
& =-\left[\sum_{k=-\infty}^{\infty} g(t-k T) \tan (\omega g(t-k T))\right] \Phi(\omega) \\
& =-\lambda(\omega) \Phi(\omega) \tag{D.9}
\end{align*}
$$

where

$$
\begin{equation*}
\lambda(\omega)=\sum_{k=-\infty}^{\infty} g(t-k T) \tan (\omega g(t-k T)) \tag{D.10}
\end{equation*}
$$

The derivatives of $\lambda(\omega)$ can be obtained from the power series expansion of $\tan (\omega g(t-$ $k T)$ ) around the origin, (See 4.3.67 of [20]).

$$
\begin{equation*}
\tan (\omega g(t-k T))=\omega g(t-k T)+\frac{(\omega g(t-k T))^{3}}{3!}+\cdots+\frac{2^{2 l}\left(2^{2 l}-1\right)}{(2 l!)}\left|B_{2 l}\right|(\omega g(t-k T))^{2 l-1} \cdots \tag{D.11}
\end{equation*}
$$

where $B_{n}$ are the Bernouli numbers. It is possible to show that

$$
\left.\frac{d^{l}}{d \omega^{l}} \tan \omega g(t-k T)\right|_{\omega=0}= \begin{cases}{[g(t-k T)]^{\frac{2^{2+1}\left(2^{l+1)-1)}\right.}{(l+1)!}}\left|B_{l+1}\right|} & l \text { odd }  \tag{D.12}\\ 0 & l \text { even }\end{cases}
$$

Therefore

$$
\begin{align*}
\left.\frac{d^{l} \lambda(\omega)}{d \omega^{l}}\right|_{\omega=0} & =\left.\sum_{k=-\infty}^{\infty} g(t-k T) \frac{d^{l}}{d \omega^{l}} \tan \omega g(t-k T)\right|_{\omega=0} \\
& =\sum_{k=-\infty}^{\infty} g(t-k T)^{l+1} \frac{2^{l+1}\left(2^{l+1}-1\right)}{(l+1)}\left|B_{l+1}\right| \tag{D.13}
\end{align*}
$$

By successive differentiation of (D.9) we get

$$
\begin{equation*}
\Phi^{l}(\omega)=-\sum_{j=0}^{l-1}\binom{l-1}{j} \lambda^{j}(\omega) \Phi^{l-1-j}(\omega) \tag{D.14}
\end{equation*}
$$

One can show directly from the definition and the fact that $a_{k}$ has a zero mean, that the odd moments are zero. Hence from (D.3) we are interested only in the even order of the derivatives of $\Phi(\omega)$, such that,

$$
\begin{equation*}
\Phi^{2 l}(\omega)=-\left[\sum_{j=0}^{2 l-1}\binom{2 l-1}{j} \lambda^{j}(\omega) \Phi^{2 l-1-j}(\omega)\right] \tag{D.15}
\end{equation*}
$$

From the above summation, we take only the even derivatives of $\Phi(\omega)$. Consider $2 l-1-j=2 l-2 i$, or $1+j=2 i$. Using this index change,

$$
\begin{equation*}
\Phi^{2 l}(\omega)=-\sum_{2 i=l}^{2 l}\binom{2 l-1}{2 i-1} \lambda^{2 i-1}(\omega) \Phi^{2(l-i)}(\omega) \tag{D.16}
\end{equation*}
$$

The first terms contains $\lambda^{0}(\omega)$. Note, from (D.10) that $\lambda^{0}(0)=0$. Hence we get

$$
\begin{equation*}
\Phi^{2 l}(0)=-\sum_{i=1}^{l}\binom{2 l-1}{2 i-1} \lambda^{2 i-1}(0) \Phi^{2(l-i)}(0) \tag{D.17}
\end{equation*}
$$

Last, we note that

$$
\begin{equation*}
M_{2 l}=(-1)^{l} \Phi^{2 l}(0) \tag{D.18}
\end{equation*}
$$

therefore

$$
\begin{equation*}
M_{2 l}=-\sum_{i=1}^{l}\binom{2 l-1}{2 i-1}(-1)^{l} M_{2(l-i)} \lambda^{2 i-1}(0) \tag{D.19}
\end{equation*}
$$

## Appendix E

- Derivation of Mean Eq. (3.17) for 2-nd order nonlinearity

In order to compare (3.17) with (2.25), we notice from (3.10), with $n=1$ and by using (3.8), that

$$
\begin{equation*}
m_{1}^{(2)}=\frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} E\left[\sum a_{k} g(t-k T)\right]^{2} \exp \left(-j \frac{2 \pi t}{T}\right) d t \tag{E.1}
\end{equation*}
$$

But $a_{k}$ are, identically distributed, random variable. $a_{k}= \pm 1$ with probability $1 / 2$. Therefore

$$
\begin{align*}
m_{1}^{(2)}= & \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} \sum g^{2}(t-k T) \exp \left(-j \frac{2 \pi t}{T}\right) d t \\
= & \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} g^{2}(t) \exp \left(-j \frac{2 \pi t}{T}\right) d t \\
& +\frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} \sum_{k \neq 0} g^{2}(t-k T) \exp \left(-j \frac{2 \pi t}{T}\right) d t \tag{E.2}
\end{align*}
$$

Since $g(t)=0$ for $|t|>T / 2$, the first integral is the Fourier Transform of $g^{2}(t)$ at $f=1 / T$ and the second integral is zero. Hence

$$
\begin{equation*}
m_{1}^{(2)}=\frac{1}{T} P_{0}\left(\frac{1}{T}\right) \tag{E.3}
\end{equation*}
$$

where

$$
\begin{equation*}
P_{0}(f)=G(f) \otimes G(f) \tag{E.4}
\end{equation*}
$$

$\otimes$ stands for convolution.

## Appendix F

- Evaluation of (3.23)

Let

$$
\begin{equation*}
t-\alpha=s-\frac{\alpha-\beta}{2} \tag{F.1}
\end{equation*}
$$

then

$$
\begin{equation*}
t-\beta=s+\frac{\alpha-\beta}{2} \tag{F.2}
\end{equation*}
$$

and hence

$$
\begin{equation*}
2 t-\alpha-\beta=2 s \tag{F.3}
\end{equation*}
$$

Substituting in (3.23), we get

$$
\begin{equation*}
R_{Y}\left(s-\frac{\alpha-\beta}{2}, s+\frac{\alpha-\beta}{2}\right)=\sum_{k} r_{k}(\alpha-\beta) \exp \left(j \frac{2 \pi k}{T} s\right) . \tag{F.4}
\end{equation*}
$$

Therefore

$$
\begin{equation*}
r_{k}(\alpha-\beta)=\frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} R_{Y}\left(s-\frac{\alpha-\beta}{2}, s+\frac{\alpha-\beta}{2}\right) \exp \left(-j \frac{2 \pi k}{T} s\right) d s \tag{F.5}
\end{equation*}
$$

with $\alpha-\beta=\tau$, we finally get (3.23)

## Appendix G

- Derivation of (3.32)
we wish to show that

$$
\begin{align*}
{\left[\exp \left(j \frac{4 \pi k t}{T}\right)\right.} & \left.\int_{-\infty}^{\infty} H\left(\frac{2}{T}-\nu\right) H(\nu) R_{2}\left(\nu-\frac{1}{T}\right) d \nu\right]^{*}= \\
& \exp \left(-j \frac{4 \pi k t}{T}\right) \int_{-\infty}^{\infty} H\left(-\frac{2}{T}-\nu\right) H(\nu) R_{2}\left(\nu+\frac{1}{T}\right) d \nu \tag{G.1}
\end{align*}
$$

therefore

$$
\begin{align*}
& {\left[\exp \left(-j \frac{4 \pi k t}{T}\right) \int_{-\infty}^{\infty} H\left(-\frac{2}{T}-\nu\right) H(\nu) R_{2}\left(\nu+\frac{1}{T}\right) d \nu\right]^{*}=} \\
& \quad \int_{\nu} H\left(\frac{2}{T}+\nu\right) H(-\nu) R_{2}\left(-\nu-\frac{1}{T}\right) d \nu \tag{G.2}
\end{align*}
$$

changing the variable $\eta=-\nu, d \eta=d \nu$, yields

$$
\begin{equation*}
\int_{\eta} H\left(\frac{2}{T}-\eta\right) H(\eta) R_{2}\left(\eta-\frac{1}{T}\right) d \eta \tag{G.3}
\end{equation*}
$$

which is equal to the second term of (F.1). This therefore proves the validity of 3.32.

## Bibliography

[1] A.J. Viterbi, "Principle of coherent communication," New York: McGraw Hill 1966
[2] W.C. Lindsey, "Synchronization systems in communication and control," Englewood cliffs, NJ: Prentice Hall 1972
[3] W.C. Lindsey and M.K. Simon, "Telecommunication systems engineering," Englewood Cliffs, NJ: Prentice Hall 1973
[4] F.M. Gardner, "Phaselock techniques," 2nd ed., New York: Wiley 1979
[5] W.C. Lindsey and M.K. Simon (eds) "Phase-Locked Loops and Their Application," IEEE Press, New York 1978
[6] W.C. Lindsey and C.M. Chic (eds) "Phase-locked loops," IEEE Press, New York 1985
[7] H. Meyr and Gerd Ascheid, "Synchronization in digital communications," Vol 1, Phase-Frequency Locked Loops and Amplitude Control. Wiley Interscience, 1990
[8] L.E. Franks, "Carrier and bit synchronization in data communication - A tutorial review," IEEE Trans on Comm. Vol COM-28, No. 8, Aug 1980
[9] W.R. Bennet, "Statistics of regenerative digital transmission," Bell Syst. Tech J., Vol 37, Nov 1958
[10] M.R. Aron and J.R. Gray, "Probability distribution of the phase jitter in self timed reconstructive repeaters for PCM," Bell Syst. Tech. J., Vol 41, Jan 1962
[11] J.M. Manley, "The regeneration and accumulationof time noise in PCM systems - an experimental and theoretical study," Bell Syst. Tech. J., Vol 48, March 1969
[12] L.E. Franks, "Signal theory," Englewood Cliffs, NJ: Prentice Hall 1969
[13] L.E. Franks and J.P. Bubrouski, "Statistical properties of timing jitter in a PAM timing recovery scheme," IEE Trans. Commun., vol COM-22, pp. 913920, July 1974.
[14] W.R. Bennety, "Statistics of regenerative digital transmission," Bell Syst. Tech. J., pp. 1501-1542, Nov. 1958.
[15] J.E. Mazo, "Jitter comparison of tones generating by squarring and by fourth power circuits," Bell Syst. Tech. J., pp. 1489-1498, May-June 1978.
[16] B. Salzberg, "Timing recovery for synchronous binary data transmission," Bell Syst. Tech. J., pp. 593-615, May. 1967.
[17] E. Panayirci, and N. Ekmekcioglu, "Analysis of a serial symbol timing recovery technizue employing Exclusive-OR circuit," IEEE Trans. Commun., vol. COM38, pp. 915-924, June 1990.
[18] Y. Takasaki, "Timing extraction in baseband pulse transmission," IEEE Trans. on Commun., vol. COM-20, pp. 872-884, Oct. 1972.
[19] N.A. D'Andrea and U. Mengali, "A simulation study of clock recovery in QPSK and 9QPSK systems," IEEE Trans. on Commun., vol COM-33, pp. 1139-1141, Oct. 1985.
[20] E.Y. Ho, and Y.S. Yeh, "A new approach for evaluating error probability in the presence of intersymbol interference and additive Gaussian noise," Bell Syst. Tech. J., vol. 48, Nov. 1970.
[21] M. Abramowitz and I.A. Stegner, "Handbook of mathematical functions", New York.

