ABSTRACT

Title of Thesis : Two and Three-Dimensional Isoparametric Finite Elements for Axisymmetric Micropolar Elasticity

FUANG YUAN HUANG : Doctor ofEngineering Science, 1986

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Finite element analysis program for isotropic and orthotropic axisymmetric micropolar (Cosserat) elastic solids are developed in this thesis. Isoparametric elements of 8- and 20-node are used to solve general three-dimensional problems, and both 4- and 8-node elements are used for two-dimensional cases. Three-dimensional finite element formulation for cylindrical coordinate system is derived. Corresponding Fortran programs are then developed. Patch tests are performed for two-dimensional cases to verify the applicability of the finite element method to non-rectangular geometries. Several two-dimensional and three-dimensional problems for micropolar elastic solids are solved to verify the formulations and computer program.

Good agreements were obtained in all cases, confirming the validity of the finite element method.

TWO AND THREE-DIMENSIONAL ISOPARAMETRIC FINITE ELEMENTS FOR AXISYMMETRIC MICROPOLAR ELASTICITY

BY

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CHAPTER I

INTRODUCTION

1.1 Introductory Comments

Micropolar elastic solid is an elastic solid whose deformation can be described by a "macro" displacement, together with a "micro" rotation. Micropolar elastic materials are the elastic materials with extra independent degree of freedom for the local rotations. Micropolar elasticity materials include certain classes of materials with fibrous and elongated grains.

Voigt [1] and F. Cosserat [2] defined the Cosserat continuum many years ago. Since then about 500 papers have been published on the micropolar elasticity. However, most of the works were restricted to isotropic case. Furthermore, they are also restricted to simple geometries only. Possible reason is that it is extremely difficult to solve problems of complex geometries using the analytical methods. This difficulty, however, can be overcome with the application of the finite element method. The finite element method is an efficient tool to numerically solve the engineering problems. In fact, finite element method has been applied to complex geometries and orthotropic problems in the classical elasticity. As in the classical elasticity, finite element method is expected to be one of the most powerful solution techniques in micropolar elasticity theory.

The present study develops the finite element method for axisymmetric micropolar elasticity based on the variational principle obtained by S. Nakamura et al.[3]. In this thesis, stress concentration problem will be solved and micro-rotation effect in the cylinder with semi-circular groove will be demonstrated. Since classical case has been solved for this problem in the literatures, the numerical results are compared with the one corresponding to the classical cases.

1.2 Literature Review

Voigt and F. Cosserat developed the theory for Cosserat continuum many years ago. However it was not until 1960's that fully developed microstructure theories evolved. In 1964, Eringen and Suhubi [4] introduced a nonlinear theory of microelastic solids. Similiar results were also obtained by Mindlin [5] in 1964 who derived a linear theory using variational principles. In 1962 Mindlin and Tiersten [6] advanced a couple stress theory in which the rotation of material point is equal to the local rotation of the surrounding medium. The couple stress theory presented by Mindlin and Mindlin and Tiersten is known to be a special case of the Cosserat continuum theory. Eringen renamed the Cosserat continuum theory as micropolar elasticity.

The symmetrical bending at laterally loaded circular isotropic micropolar plates was analytically solved by Arimann in 1964 [7]. In the paper by Kaloni and Ariman the micropolar theory is called Eringen's theory and the couple stress theory is called Mindlin's theory. Later Khan and Dhaliwal obtained the analytical solution for the isotropic micropolar elasticity of half-space subjected to an arbitrary normal pressure [8]. Kishida, Sazaki and Hanzawa solved stress concentration problem for a circular cylinder with a semicircular annular groove under uniaxial tension of linear isotropic couple stress elastic solids [9]. For this purpose, they used indirect fictious boundary integral method. In 1969 Gauthier analytically solved the isotropic axisymmetric micropolar elasticity of a cylinder subjected to axial tension and torsion and cylindrical bending of a rectangular plate [10]. Guathier and his co-worker also performed experiment to obtain micropolar elastic constants of isotropic composite materials [11].

S. Nakamura [12] was the first in solving the orthotropic micropolar elasticity using finite element method. Two finite element programs have been developed for plane Cosserat elasticity theory. The earlier program [13] used triangular constant strain element and the second used 4- and 8-node isoparametric elements [14]. In the following chapter, similar finite element formulation to Ref.[12] is applied to develop finite element method for three-dimensional and axisymmetric micropolar elasticity.

1.3 Scope of the thesis

First, finite element methods are reviewed from Ref. [15] in chapter II.

Equations of general micropolar elasticity, variational method and displacement type finite element formulation for micropolar elasticity are reviewed in Chapter III.

In chapter IV, micropolar elasticity and matrix finite element formulation for cylindrical coordinate system is developed. Numerical examples are also included.

Axisymmetric elements of 4-node and 8-node elements are used in Chapter V. Numerical results are compared graphically with the results of classical elasticity. Numerical examples and programming organization are also illustrated.

Finally, conclusions and recommendations for research are suggested in Chapter VI.

CHAPTER II

REVIEW OF THE FINITE ELEMENT METHOD

2.1 Introductory Comments

For the last two decades finite element methods have received much attention, due to the increasing use of high-speed computers and the growing emphasis on numerical methods for engineering analysis. This is completely understandable, since it is not possible to obtain analytical solutions for many practical engineering problems.

An analytical solution is a mathematical or functional expression that can give the values of the desired unknown variables at any location in a continuum, and as a consequence it is valid for an infinite number of locations in the body. However, analytical solutions can be obtained only for certain simple problems. For problems involving nonisotropic material properties and complex boundary conditions, one has to resort to numerical methods that provide approximate solutions with reasonable accurracies. In most of the numerical methods, the solutions yield approximate values of the unknown variables only at a discrete number of points in the continuum. The process of selecting finite number of discrete points in the continuum can be termed "discretization". One way of discretizing an entire body or structure is to divide it into a set of small bodies, or units. The assemblage of such units then represents the original body. Instead of solving the problem for the entire body in one operation, the solutions could be formulated for each constituent unit and then combined to obtain the solution for the original body or structure.

The finite element method is applicable to a wide range of boundary value problems in engineering. In boundary value problems, solutions are sought in the region of the body, while on the boundaries the values of the unknown variables (or their derivatives) are prescribed. Problems in the field of solid mechanics are usually tackled by one of the three approaches: the displacement method, the equilibrium method, or the mixed method. Displacement are assumed as primary unknown quantities in the displacement method; stress are assumed as primary unknown quantities in the equilibrium method; and some displacements and some stresses are assumed as unknown quantities in the mixed method.

In the following of this Chapter, isoparametric finite formulation and axisymmetric finite element method are reviewed first. Variational formulation of micropolar elasticity and finite element formulation for micropolar elasticity are then reviewed.

2.2 General Formulation of Finite Element Method

In this section finite element method is reviewed from Ref.[16]. The concept of finite element methods consist of a discretization of a continuous media to describe the state of those discretized continuum. There are two matrix method approaches associated with the finite element method:

a. The force type finite element method assumes the internal forces as the unknowns variables. To generate the governing equations, the equilibrium conditions are used first; then to develop the additional equations which might be necessary to obtain the solutions, and compatibility conditions are introduced.

b. The displacement type finite element methods assumes the displacements of the nodes as the unknown variables. In this approach the compatibility conditions in and among elements are initially satisfied. Then the governing equations in terms of nodal displacements are written for each nodal point using the equilibrium conditions.

The difference between force and displacement type finite element methods lies on the selection of the unknowns of the analysis and the variations in the matrix quantities associated with their formulations.

Since most engineers are familiar with the displacement analysis invloving such terms as stresses, strains, and equilibrium, the great majority of literature on the finite element methods has been written in terms of the displacement method. In this thesis displacement-type finite element method is used.

2.2.1 Displacement Finite Element Method

In this section, displacement finite element method is reviewed from Ref. [15].

The displacements of the finite elements are always described in the local coordinate system as shown in Fig.2.1

For one-dimensional truss elements one can use

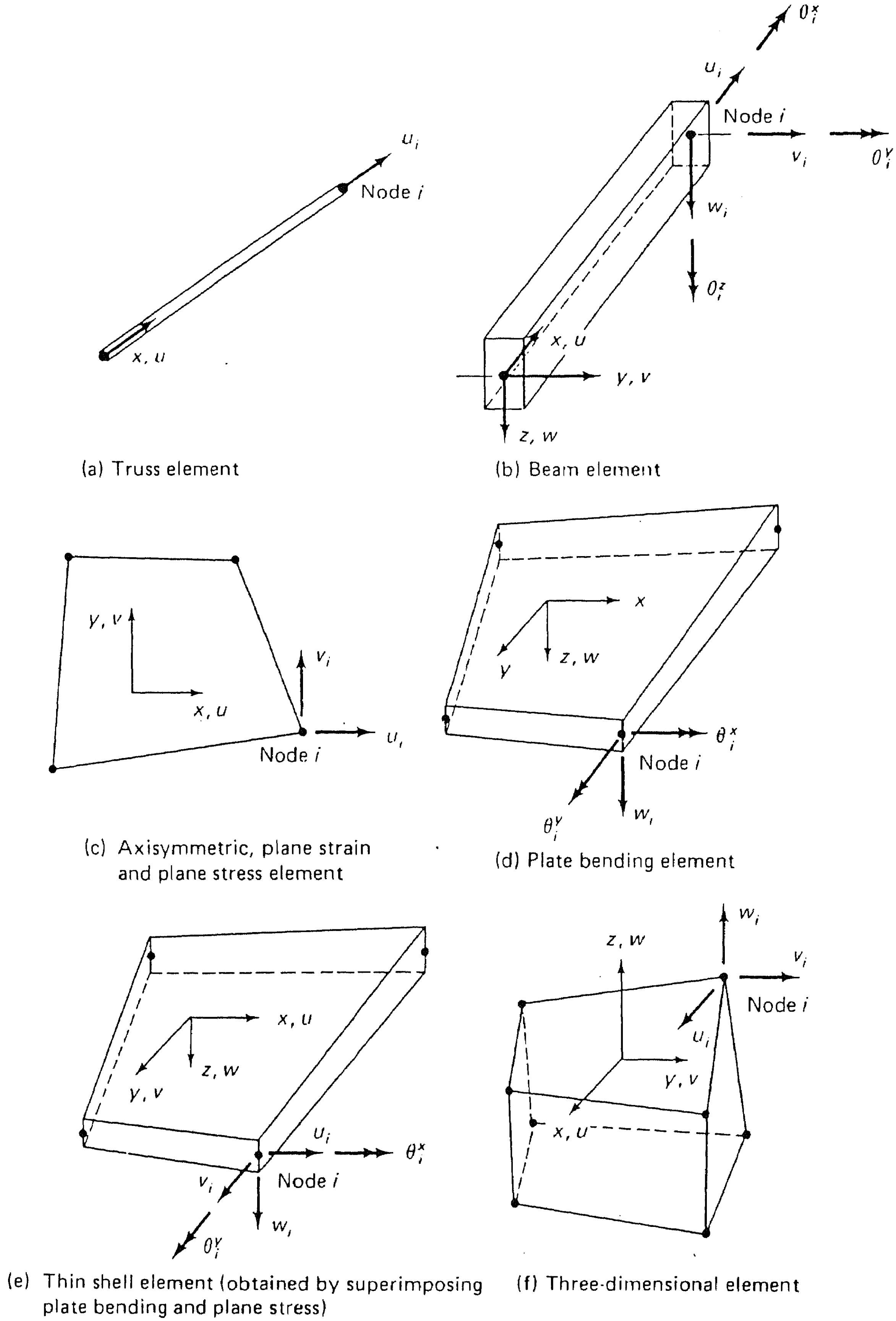

Fig. 2.1 Some Typical Finite Elements

$$u(x) = \alpha_1 + \alpha_2 x + \alpha_3 x^2 + \ldots\ldots$$

where x varies over the length of the element. u is the local element displacement and α_1, α_2, ... are generalized coordinates.

For two-dimensional elements like plane stress, plain strain and axisymmetric elements, one need two displacement variables u and v as a function of x and y coordinates,

$$u(x,y) = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 xy + \alpha_5 x^2 + \ldots$$
$$v(x,y) = \beta_1 + \beta_2 x + \beta_3 y + \beta_4 xy + \beta_5 x^2 + \ldots$$

where $\alpha_1, \alpha_2, \ldots$ and β_1, β_2 are generalized coordinates.

In the case of plate bending , the transverse deflection w is needed as a function of coordinates x and y;

$$w(x,y) = \gamma_1 + \gamma_2 x + \gamma_3 y + \gamma_4 xy + \gamma_5 x^2 + \ldots\ldots.$$

where $\gamma_1, \gamma_2 \ldots$ are generalized coordinates.

Finally, for general 3-dimensional elements in which u,v, w, are displacement variables at x,y,and z coordinates;

$$u(x,y,z) = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 z + \alpha_5 xy + \ldots\ldots$$
$$v(x,y,z) = \beta_1 + \beta_2 x + \beta_3 y + \beta_4 z + \beta_5 xy + \ldots\ldots$$
$$w(x,y,z) = \gamma_1 + \gamma_2 x + \gamma_3 y + \gamma_4 z + \gamma_5 xy + \ldots\ldots$$

where $\alpha_1, \alpha_2, \ldots, \beta_1, \beta_2, \ldots, \gamma_1, \gamma_2 \ldots$ are generalized coordinates.

2.2.2 Isoparametric Finite Element Method

The most widely used finite element method for general application is the isoparametric finite element method. The basic idea of isoparametric finite element formulation is to achieve the relationship between the element displacement at any point and the element nodal point displacement directly through the use of a shape function. The procedure using isoparametric finite element formulation is to express the element coordinates and element displacements in the form of interpolations using the natural coordinate system of each element. A natural coordinate system is a local system which permits the specification of a point within the element by a set of dimensionless numbers whose magnitudes never exceed unity. Such a coordinate system can generalize and simplify the formulation, and also facilitates the numerical integration required to obtain the element stiffness matrix. This coordinate system can be one-. two-, or three-dimensional, depending on the dimensionality of the element. The formulation of the element matrices is basically the same for a one-,two-,or three-dimensional element.

In this section, three-dimensional element is presented. However, the one- and two-dimensional elements are also included using only the relevant coordinate axis and the appropriate interpolation functions.

Considering a general three-dimensional element, the coordinate interpolations are

$$x = \sum_{i=1}^{q} N_i \, x_i$$
$$y = \sum_{i=1}^{q} N_i \, y_i \qquad (2.1)$$
$$z = \sum_{i=1}^{q} N_i \, z_i$$

where x, y, and z are the coordinates at any point of the element, and x_i, y_i, z_i, i=1,........, q, are the coordinates of the q element nodes. The interpolation functions h_i are defined in the natural coordinate system of the element, which has variables r, s, and t whose ranges are between -1 to +1. The interpolation functions for one- and two-dimensional elements are given in Appendix I. The interpolation functions for two-dimensional elements are applicable to axisymmetric analysis and used in Chapter 5.

In the classical elastic theory, for one-, or two-dimensional elements only the relevant equations in (2.1) would be employed, and the interpolation functions would depend only on the natural coordinate variables r , s and t.

In the isoparametric formulation, the same interpolation functions used in geometry, is used to express displacements:

$$u = \sum_{i=1}^{q} N_i \, u_i$$
$$v = \sum_{i=1}^{q} N_i \, v_i \qquad (2.2)$$
$$w = \sum_{i=1}^{q} N_i \, w_i$$

where u, v, and w are the local element displacement at any point of the element, and u_i, v_i, and w_i, i=1,.....,q, are the corresponding element displacements at each nodes. Therefore, it is assumed that to each nodal point coordinate necessary to describe the geometry of the element, there corresponds one nodal point displacement.

2.3 Finite-element matrices formulation

In general, the calculation of the element matrices should be carried out in the global coordinate system, using global displacement components if the number of natural coordinate variables is equal to the number of global variables.

To evaluate the stiffness matrix of an element, one needs to calculate the strain-displacement transformation matrix. The element strains are obtained in terms of derivatives of element displacements with respect to local coordinates. Because the element displacements are defined in the natural coordinate system using Eqn. (2.2), one has to relate the displacement with repect to x, y, and z to the ones with respect to r, s, and t . Let eqn (2.1) has the form

$$\begin{aligned} x &= f_1(r,s,t) \\ y &= f_2(r,s,t) \\ z &= f_3(r,s,t) \end{aligned} \qquad (2.3)$$

where f_i denotes "function of ". The inverse relationship is

$$\begin{aligned} r &= f_4(x,y,z) \\ s &= f_5(x,y,z) \\ t &= f_6(x,y,z) \end{aligned} \qquad (2.4)$$

To obtain the derivatives of $\partial/\partial x$, $\partial/\partial y$, and $\partial/\partial z$, one uses the chain rule:

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial}{\partial t}\frac{\partial t}{\partial x}$$

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial r}\frac{\partial r}{y} + \frac{\partial}{\partial s}\frac{\partial s}{\partial y} + \frac{\partial}{\partial t}\frac{\partial t}{\partial y} \qquad (2.5)$$

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial r}\frac{\partial r}{\partial z} + \frac{\partial}{\partial s}\frac{\partial s}{\partial z} + \frac{\partial}{\partial t}\frac{\partial t}{\partial z}$$

In matrix form,

$$\begin{Bmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial s} \\ \frac{\partial}{\partial t} \end{Bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} & \frac{\partial z}{\partial r} \\ \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} & \frac{\partial z}{\partial s} \\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} & \frac{\partial z}{\partial t} \end{bmatrix} \begin{Bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{Bmatrix}$$

where J is the Jacobian matrix relating the natural coordinate derivatives to the local coordinate derivatives. The Jacobian matrix can be found using (2.1)

$$\frac{\partial}{\partial x} = J^{-1}\frac{\partial}{\partial r}$$

which requires that the inverse of J exists. The inverse exists provided that there is a one-to-one correspondence between the natural and local coordinates of the elements (2.3.2.4).

Using Equations (2.5) and (2.6), one evaluates $\partial u/\partial x$, $\partial u/\partial y$, $\partial u/\partial z$, $\partial v/\partial x$, ..., $\partial w/\partial z$ and therefore constructs the strain-displacement transformation matrix B;

$$\epsilon = B\,u$$

where u is a vector listing the element nodal point displacement of Equation (2.2).

The element stiffness matrix corresponding to the local element degree of freedom is

$$K = \int_V B^T \, D \, B \, dv \qquad (2.9)$$

Here the elements of B are functions of the natural coordinates r, s, and t. Therefore, the volume integration extends over the natural coordinate volume, and the volume differential dv needs also be written in the terms of the natural coordinates. In general, we have

$$dv = \det J \, dr \, ds \, dt \qquad (2.10)$$

where detJ is the determinant of the Jacobin matrix in Equation (2.7).
Since an explicit evaluation of the volume integral in Equation (2.10) is not possible, numerical integration is used.

First, we rewrite Equation (2.9) in the form

$$K = F \, dr \, ds \, dt \qquad (2.11)$$

where $F = B^T D B \det J$ and integration is performed in the natural coordinate system of the element. F depends on r, s, and t, but the actual functional relationship is, in general, unknown. Using numerical integration, the stiffness matrix is now

$$k = \sum_{ijk} \alpha_{ijk} F_{ijk} \qquad (2.12)$$

where F_{ijk} is the matrix F evaluated at point r_i, s_j, and t_k.

The sampling point r_i, s_j, t_k of the function and the corresponding weighting factors $_{ijk}$ are chosen to obtain maximum accuracy for the interval -1 to +1, and are as given in Table [2.1].

The mass and load vectors are now given by

$$M = \int_V \rho \, H^T \, dv \qquad (2.13)$$

$$R_B = \int_V H^T \, f^B \, dv \qquad (2.14)$$

$$R_S = \int_S H^S \, f^S \, ds \qquad (2.15)$$

$$R_I = \int_V B^T \tau^I \, dv \qquad (2.16)$$

where H is a matrix of the interpolation functions. The above matrices are evaluated using numerical integration.

To calculate the body force vector, R_B, we use

$$F = H^T \, f^B \, \det J$$

For the surface force vector, we use $F = H^S \; f^S \det J$ and for the initial stress load vector we use $F = B^T \tau^I \det J$, and for the mass matrix one has $F = H^T \det J$. The weight coefficients α_{ijk} are the same as in the stiffness matrix evaluation and the same order of numerical integration is used for different order, values are obtained from the Table 2.1.

2.4 Convergence Considerations

The two requirements for monotonic convergence of a finite element analysis are that the elements must be compatible and complete. Completeness requires that the rigid body displacements and constant strain states be possible [15].

TABLE 2.1

SAMPLING POINTS AND WEIGHTS IN GAUSS-LEGENDRE NUMERICAL INTEGRATION

n	ri	si
1	0. (15 zeros)	2. (15 zeros)
2	+0.57735 02691 89626	1.00000 00000 00000
3	+0.77459 66692 41483	0.55555 55555 55556
	+0.00000 00000 00000	0.88888 88888 88889
4	+0.86113 63115 94053	0.34785 48451 37454
	+0.33998 10435 84856	0.65214 51548 62546

Note : Sampling Points and Weights till Order 4 is given since the thesis involves integration upto order 3.

The necessity for the constant strain states can physically be understood. When the limit as each element approaches a very small size, the strain in each element approaches a constant value, and any complex variation of strain within the structure can be approximated. The requirement of compatibility means that the displacements within the elements and across the element boundaries must be continuous [15].

In the isoparametric formulation, one has the displacement interpolation

$$
\begin{aligned}
u &= \sum_{i=1}^{q} N_i\, u_i \\
u_i &= a_1 + b_1 x_i + c_1 y_i + d_1 z_i \\
v &= \sum_{i=1}^{q} N_i\, v_i \\
v_i &= a_2 + b_2 x_i + c_2 y_i + d_2 z_i \\
w &= \sum_{i=1}^{q} N_i\, w_i \\
w_i &= a_3 + b_3 x_i + c_3 y_i + d_3 z_i
\end{aligned}
\tag{2.17}
$$

which can be reduced to

$$
\begin{aligned}
u &= a_1 \sum_{i=1}^{q} N_i + b_1 \sum_{i=1}^{q} N_i x_i + c_1 \sum_{i=1}^{q} N_i y_i + d_1 \sum_{i=1}^{q} N_i z_i \\
v &= a_2 \sum_{i=1}^{q} N_i + b_2 \sum_{i=1}^{q} N_i x_i + c_2 \sum_{i=1}^{q} N_i y_i + d_2 \sum_{i=1}^{q} N_i z_i \\
w &= a_3 \sum_{i=1}^{q} N_i + b_3 \sum_{i=1}^{q} N_i x_i + c_3 \sum_{i=1}^{q} N_i y_i + d_3 \sum_{i=1}^{q} N_i z_i
\end{aligned}
\tag{2.18}
$$

Since in the isoparametric formulation, the coordinates are interpolated in the same way as the displacements,

$$u = a_1 \sum_{i=1}^{q} N_i + b_1 x_i + c_1 y_i + d_1 z_i$$

$$v = a_2 \sum_{i=1}^{q} N_i + b_2 x_i + c_2 y_i + d_2 z_i \qquad (2.19)$$

$$w = a_3 \sum_{i=1}^{q} N_i + b_3 x_i + c_3 y_i + d_3 z_i$$

also

$$\sum_{i=1}^{q} N_i = 1$$

The above relation is the condition on the interpolation functions for the completeness requirement to be satisfied.

Chapter III

REVIEW OF MICROPOLAR ELASTICITY

3.1 Introductory Comments

A micropolar elastic material differs from classical elastic material solids in that each point has extra rotational degree of freedom independent of translation, and that a micropolar elastic material can transmit couple stress as well as the usual force stress [1].

The theory of micropolar elasticity is hoped to be applicable to many new industrial materials with microstructures. A dramatic increase in the applications of light-weight materials in industry is expected in the future, thus enhancing the demand of vast amounts of basic researches in the related area.

In this chapter, the micropolar elasticity theory is reviewed. Erigen has studied a comprehensive recapulation of micropolar elasticity theory based largely on earlier works by him and his co-workers [4]. These equations are written in rectangular cartesian tensor notation. His treatise provides an excellent starting point for further investigations into linear theory and is served as the main starting grounds for this study.

3.2 Basic Equations of Micropolar Elasticity

In 1964, Erigen and Suhubi first constructed the linear theory of micropolar elasticity. The equilibrium equations of micropolar elasticity is given [4]:

$$t_{ji,j} + \rho (f_k - \dot{v}_k) = 0 \qquad (3.1)$$

$$m_{ji,j} + e_{ikm} t_{km} + \rho (l_k - \dot{\sigma}_k) = 0$$

where e_{ikm} is the permutation tensor. Since only quasi-static problems are considered in this study, the inertia terms can be eliminated. The equilibrium equations thus become:

$$t_{ji,j} = 0 \qquad (3.2)$$

$$m_{ji,j} + e_{ikm} t_{km} = 0$$

3.2.1 Constitutive equations

The linear forms of the stress and couple stress constitutive equations for anisotropic micropolar elastic solids are [1]:

$$t_{kl} = A_{kl} + A_{klmn} \epsilon_{mn} \qquad (3.3)$$

$$m_{kl} = B_{lkmn} \phi_{m,n}$$

Where ϕ is the microrotation vector and ϵ_{mn} is the Kronecker delta. When the initial stress is zero, one has $A_{kl} = 0$. Thus, for the micropolar solid free of initial stress and couple stress, we have

$$t_{kl} = A_{klmn} \epsilon_{mn} \qquad (3.4)$$

$$m_{kl} = B_{lkmn} \phi_{m,n}$$

Various material symmetry conditions place further restrictions on the constitutive coefficients A_{klmn} and B_{lkmn}. These restrictions are found in the same manner as in classical elasticity [17]. If the body is isotropic with respect to both force and couple stress, the solid is called isotropic. In this case, the constitutive coefficients must be isotropic tensors. For second and forth order isotropic tensors, one has the most general forms:

$$A_{kl} = A\delta_{kl}$$
$$A_{klmn} = A_1\delta_{kl}\delta_{mn} + A_2\delta_{km}\delta_{ln} + A_3\delta_{kn}\delta_{lm} \qquad (3.5)$$
$$B_{klmn} = B_1\delta_{kl}\delta_{mn} + B_2\delta_{km}\delta_{ln} + B_3\delta_{kn}\delta_{lm}$$

where A, A_1,....B_2 and B_3 are functions of θ only. In this case, Equations (3.4) and (3.5) take the special forms:

$$t_{kl} = A\delta_{kl} + A_1\epsilon_{rr}\delta_{kl} + A_2\epsilon_{kl} + A_3\epsilon_{lk} \qquad (3.6)$$
$$m_{kl} = B_1\phi_{r,r}\delta_{kl} + B_2\phi_{l,k} + B_3\phi_{k,l}$$

For vanishing initial stress A=0.

By introducing

$$A_1 \equiv \lambda, \quad A_2 \equiv \mu + k \quad A_3 \equiv \mu \qquad (3.7)$$
$$B_1 \equiv \alpha \quad B_2 \equiv \gamma \quad B_3 \equiv \beta$$

the above equations can be rewritten as:

$$t_{kl} = \lambda\epsilon_{rr}\delta_{kl} + (\mu+k)\epsilon_{kl} + \mu\epsilon_{lk} \qquad (3.8)$$
$$m_{kl} = \alpha\phi_{r,r}\delta_{kl} + \beta\phi_{k,l} + \gamma\phi_{l,k}$$

Isotropic micropolar elasticity can therefore be distinguished from classical elasticity by the presence of four extra elastic moduli; namely, k,α ,β, and γ. When these four extra elastic moduli are set equal to zero, Equation (3.5) reduce to the well-known Hooke's law of the linear isotropic classical solid.

3.2.2 Restrictions on Micropolar Elastic Moduli

The necessary and sufficient conditions for the internal energy to be non-negative are [2]:

$$\begin{aligned} &0 \leq 3\lambda + 2\mu + k, \quad && 0 \leq 2\mu + k, \quad && 0 \leq k \\ &0 \leq 3\alpha + \beta + r, \quad && -r \leq \beta \leq r, \quad && 0 \leq r \end{aligned} \qquad (3.9)$$

The micropolar strain tensor indicates the relations among the strain , displacement and microrotation. Since there are 33 equations with 33 variables, the formulation is therefore complete.

3.2.3 Boundary conditions

Many different types of boundary conditions are suggested by the nature of various applications. For example, one may prescribe the displacement u_i and microrotation ϕ_j on a boundary surface s of a body [10]. Equally permissible is the alternate prescription of the tractions and couples on s, i.e.

$$\left.\begin{aligned} t_{lk}\, n_t &= t_{(n)}\, K \\ m_{lk}\, n_t &= m_{(n)}\, K \end{aligned}\right\} \text{ on } S \qquad (3.10)$$

where t_{lk} and m_{lk} are surface stress and surface couple vectors, respectively, and $t_{(n)}\, K$ and $m_{(n)}\, K$ are the prescribed tractions and couples on the bounding surface whose exterior is unit normal vector n. In some other problems, it is possible to prescribe the above two types of conditions as mixed boundary conditions.

3.2.4 Compatibility Conditions

The displacement u_i and microrotation ϕ_j are linked to the microstrain tensor ϵ_{ij} through the strain-displacement relations:

$$\epsilon_{ij} = u_{j,i} + e_{jik}\phi_k \qquad (3.11)$$

When the six quantities u_i and ϕ_j are prescribed, these strain fields are determined uniquely from Equation (3.11) by mere substitution. On the other hand, specification of ϵ_{ij} does not determine the displacements and microrotations uniquely as the system is overdetermined. In order to assure single-valuedness and continuity in the displacement and microrotation fields, it is necessary to apply constrains to ϵ_{ij} and $\phi_{i,j}$, limiting the arbitrariness with which we may prescribe these quantities. These conditions are known as the compatibility conditions, and they are given by

$$\epsilon_{ij,kl} + \epsilon_{kl,ij} + \epsilon_{ik,jl} + \epsilon_{jl,ik} = 0 \qquad (3.12)$$

3.3 Variational Formulation of Micropolar Elasticity

In the following, variational formulation of micropolar elasticity as originally derived by Nakamura et al. [3] is reviewed.

The potential energy Π of an elastic body is defined as:

$$\Pi = U - V \qquad (3.13)$$

where U is the strain energy and V is the work done by external load acting on the body.

The principle of minimum potential energy claims, at the equibrium states,

$$\delta \Pi = \delta U - \delta V = 0 \tag{3.14}$$

The work done by the body force B_i, body couple C_i, surface traction T_i and surface couple M_i can be expressed as

$$V = \iint_v (B_i U_i + C_i \phi_i) dv + \int_s (T_i^V U_i + M_i^V \phi_i) ds \tag{3.15}$$

where

$$T_i^V = t_{ji}$$

and

$$M_i^V = m_{ji} V_j$$

Under virtual displacement and virtual microrotation, the virtual work done by external load becomes

$$V = \iint_v (B_i\ U_i + C_i\ \phi_i) dv + \int_s (T_i^V\ U_i + M_i^V\ \phi_i) ds$$

$$\begin{aligned} \delta U = \delta V & \\ &= \iint_v (B_i\ U_i + C_i\ \phi_i) dv + \int_s (T_i^V\ U_i + M_i^V\ \phi_i) ds \\ &= \iint_v ((t_{ji,j} + B_i)\ \ U_i + (m_{ji,j} + E_{ijk} t_{jk} + C_i)\ \ \phi_i \\ &\quad + (t_{ji}\ U_{i,j} + m_{ji}\ \phi_{i,j} - E_{ijk} t_{jk}\ \phi_i)) dv \end{aligned} \tag{3.16}$$

The equilibrium equations for Cosserat elasticity are given in Equations (3.2):

$$t_{ji,j} + B_i = 0$$

$$E_{ijk}\ t_{jk} + m_{ji,j} + C_i = 0$$

Substituting the equilibrium equations into the above equations, the first and second terms vanish, and the virtual work becomes:

$$U=\iint_v (t_{ji}\, U_{i,j}+m_{ji}\, \phi_{i,j}-E_{ijk}\, \phi_i)\,dv = \iint_v (t_{ji}\, E_{ji}+m_{ji}\, \phi_{i,j})\,dv \quad (3.17)$$

The strain energy for linear constitutive relation is

$$U= \iint_v (t_{ji}E_{ji}+m_{ji}\phi_{i,j})\,dv \quad (3.18)$$

Then, we have a minimum potential energy functional of micropolar elasticity:

$$\Pi = U-V = \iint_v (t_{ji}E_{ji}+m_{ji}\phi_{i,j})\,dv - \iint_v (B_iU_i+C_i\phi_i)\,dv-\int_s (T_i^v U_i+M_i^v\phi_i)\,ds \quad (3.19)$$

3.4 Finite Element Formulation of Micropolar Elasticity

In this section, the finite element formulation for general micropolar elasticity based on the previous section derived by Nakamura et al. [12] is reviewed.

The total potential energy for a micropolar elastic solid are:

$$\Pi = 1/2 \iint_v (t_{ji}\quad_{ji} + m_{ji}\, \phi_{i,j})\, dv - \iint_v (B_iu_i + C_i\, \phi_i)\, dv - \int_s (T_i^{(v)}\, u_i+ M_i^{(v)}\phi_i)\, ds \quad (3.20)$$

Here t_{ji} and m_{ji} in the first term are the force stress and couple stress, respectively; and ϵ_{ij} and $\phi_{i,j}$ are the micropolar strain tensor and microrotation gradient, respectively. B_i and C_i in the second term are the applied body force and body couple, respectively; and U_i and ϕ_i are displacement and microrotation for the i-direction, respectively. T_i and M_i in the third term are the surface force and surface couple tractions, respectively. The first two integrals are volume integrals, while the last one is a surface integral for which the surface traction and surface couples are prescribed.

Extremizing the above total potential energy expression with respect to displacement field u_i and microrotation field ϕ_i gives the following four equations:

Force Equilibrium Equation:

$$t_{ji,j} + G_i = 0 \tag{3.21}$$

Moment Equilibrium Equation:

$$e_{ijk}\, t_{jk} + m_{ji,j} + C_i = 0 \tag{3.22}$$

Cauchy's Formular for Force Stress:

$$T_i^{(v)} = t_{ji}\, v_j \tag{3.23}$$

Cauchy's Formular for Couple stress:

$$M_i^{(v)} = m_{ji}\, v_j \tag{3.24}$$

Here e_{ijk} in Equation (3.21) is a permutation tensor and v_j in Equation (3.24) is a j-th component of unit normal vector.

One introduces appropriate shape functions N_u and N_0 such that

$$\forall_x \in V_e \ , \ u(x)=N_u(x)u^e \quad \text{and} \quad \phi(x)= N_0(x) \ \phi^e \tag{3.25}$$

Here u(x) and ϕ(x) are field variable vectors of displacement and microrotation in the finite domain. U^e and ϕ^e are nodal field variable vectors corresponding to them. Using the expression for micropolar strain tensor

$$\epsilon = L\,u(x) + M\,\phi\,(x) \tag{3.26}$$

where L is a differential operator and M is a permutation matrix, and general constitutive equations

$$t = D_0\,\epsilon$$

$$m = D_1 \nabla \phi \tag{3.27}$$

discretized equilibrium equation is obtained:

$$k^e\,U^e = F^e{}_v + F^e{}_s \tag{3.28}$$

Here the nodal value vector is $U^e = (\, u^{e^T},\ o^{e^T})$ and the element stiffness matrix k^e for micropolar elasticity is

$$k^e = \iint_v \left(B_0{}^T D_0\,B_0 + B_1{}^T D_1\,B_1\right)\,dv \tag{3.29}$$

where

$$B_0 = [\ L\,N_u\ ,\ M\,N_0\ ]$$

and

$$B_1 = [\ 0\ ,\ N_0\ ]$$

are strain-displacement matrix and microrotation gradient matrix, respectively.

CHAPTER IV

THREE-DIMENSIONAL ISOPARAMETRIC ELEMENTS FOR MICROPOLAR ELASTICITY

4.1 Introductory Comments

Micropolar theory can be applied to many structure materials, including fibrous, lattice and granular microstructures [1]. Although numerous boundary value problems have been solved analytically, the solutions are mainly limited to the isotropic cases. Moreover, it is difficult to apply the analytic methods to complex geometries. In addition, it is generally difficult to solve analytically a three-dimensional problem with the exception of a few special cases, such as the stress and displacement boundary value problems with Volterra's dislocation which can be treated as a plane problem and thus be solved as a two-dimensional problem [18]. Thus finite element methods are powerful tools for solving three-dimensional micropolar elasticity problems with arbitrary geometries.

In this chapter, a general three-dimensional micropolar elasticity theory in cylindrical system is reviewed first. A general purpose three-dimensional finite element method is then formulated, and a computer program is developed. The program is used to solve a micropolar elastic cylinder with a semi-circular groove under uniaxial tension.

4.2 General 3-D Micropolar Elasticity in Cylindrical coordinate System

In this section, micropolar elasticity in Cartesian

Coordinate system is transformed into Cylindrical Coordinate system.

Eringen's micropolar strain tensors

$$\epsilon_{ij} = u_{j,i} - e_{ijk}\,\phi_k \qquad (3.11)$$

can be expressed in vector form:

$$\epsilon = L\,u(x) + M\,\phi(x) \qquad (4.1)$$

where L is a gradient operator in Cartesian Coordinate (x,y,z):

$$L^T = \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right]$$

Equation (3.11) can be transformed into cylindrical coordinate system (r,θ,z), using the gradient operator:

$$L = \left[\frac{\partial}{\partial r}, \frac{1\partial}{r\partial \theta}, \frac{\partial}{\partial z} \right]^T$$

Thus

$$\epsilon = \begin{pmatrix} \frac{\partial u_r}{\partial r} & \frac{\partial u_0}{\partial r} - \phi_z , & \frac{\partial u_z}{\partial r} + \phi_\theta \\ \frac{\partial u_r}{r\partial \theta} - \frac{u_\theta}{r} + \phi_z & \frac{\partial u_\theta}{r\partial \theta} + \frac{u_r}{r} & \frac{\partial u_z}{r\partial \theta} - \phi_r \\ \frac{\partial u_r}{\partial z} - \phi_\theta & -\frac{\partial u_\theta}{\partial z} + \phi_r & -\frac{\partial u_z}{\partial z} \end{pmatrix} \qquad (4.2)$$

For Cylindrical Coordinate system (r,θ,z) the constitutive relationship has a form of:

$$
\begin{Bmatrix} t_{rr} \\ t_{\theta\theta} \\ t_{zz} \\ t_{r\theta} \\ t_{\theta r} \\ t_{rz} \\ t_{zr} \\ t_{\theta z} \\ t_{z\theta} \end{Bmatrix}
=
\begin{bmatrix}
A^{rr}_{rr} & A^{rr}_{\theta\theta} & A^{rr}_{zz} & & & & & & \\
 & A^{\theta\theta}_{\theta\theta} & A^{\theta\theta}_{zz} & & & & & & \\
 & & A^{zz}_{zz} & & & & 0 & & \\
 & & & A^{r\theta}_{r\theta} & A^{r\theta}_{\theta r} & & & & \\
 & & & & A^{\theta r}_{\theta r} & & & & \\
 & & & & & A^{rz}_{rz} & A^{rz}_{zr} & & \\
 & \text{symmetry} & & & & & A^{zr}_{zr} & & \\
 & & & & & & & A^{\theta z}_{\theta z} & A^{\theta z}_{z\theta} \\
 & & & & & & & & A^{z\theta}_{z\theta}
\end{bmatrix}
\begin{Bmatrix} \epsilon_{rr} \\ \epsilon_{\theta\theta} \\ \epsilon_{zz} \\ \epsilon_{r\theta} \\ \epsilon_{\theta r} \\ \epsilon_{rz} \\ \epsilon_{zr} \\ \epsilon_{\theta z} \\ \epsilon_{z\theta} \end{Bmatrix}
= D_0 \epsilon \qquad (4.3)
$$

and

$$
\begin{Bmatrix} m_{rr} \\ m_{\theta\theta} \\ m_{zz} \\ m_{\theta r} \\ m_{r\theta} \\ m_{zr} \\ m_{rz} \\ m_{z\theta} \\ m_{\theta z} \end{Bmatrix}
=
\begin{bmatrix}
B^{rr}_{rr} & B^{rr}_{\theta\theta} & B^{rr}_{zz} & & & & & & \\
 & B^{\theta\theta}_{\theta\theta} & B^{\theta\theta}_{zz} & & & & & & \\
 & & B^{zz}_{zz} & & & 0 & & & \\
 & & & B^{\theta r}_{\theta r} & B^{r\theta}_{\theta r} & & & & \\
 & & & & B^{\theta r}_{r\theta} & & & & \\
 & \text{symmetry} & & & & B^{rz}_{zr} & B^{zr}_{zr} & & \\
 & & & & & & B^{zr}_{zr} & & \\
 & & & & & & & B^{z\theta}_{z\theta} & B^{\theta z}_{z\theta} \\
 & & & & & & & & B^{\theta z}_{z\theta}
\end{bmatrix}
\begin{Bmatrix} \phi_{r,r} \\ \dfrac{\phi_r}{r} + \dfrac{\partial \phi_\theta}{r \partial \theta} \\ \phi_{z,z} \\ \phi_{\theta,r} \\ -\dfrac{\phi_\theta}{r} + \dfrac{\partial \phi_r}{r \partial \theta} \\ \phi_{z,r} \\ \phi_{r,z} \\ \phi_{\theta,z} \\ \phi_{z,\theta}/r \end{Bmatrix}
$$

$$
= D_1 \nabla \phi \qquad (4.4)
$$

4.3 Formulation of the Finite Element Method for General 3-D Micropolar Elasticity in Cylindrical Coordinate

In this section finite element method for 3-D micropolar elasticity in cylindrical coordinate is derived.

The coordinates transformation from (x,y,z) to (r,θ,z) can be calculated by using coordinate transformations:

$$x = r\cos\theta$$
$$y = r\sin\theta \tag{4.5}$$
$$z = z$$

The shape functions for the isoparametric element for eight to twenty variable-number-nodes are discussed in detail in Chapter 2. Using those shape functions N_i, displacement and microrotation field inside each element can be interpolated:

$$\begin{aligned} u_r &= \sum_{i=1}^{q} N_i \, u_{ri} \\ u_\theta &= \sum_{i=1}^{q} N_i \, u_{\theta i} \\ u_z &= \sum_{i=1}^{q} N_i \, u_{zi} \\ \phi_r &= \sum_{i=1}^{q} N_i \, \phi_{ri} \\ \phi_\theta &= \sum_{i=1}^{q} N_i \, \phi_{\theta i} \\ \phi_z &= \sum_{i=1}^{q} N_i \, \phi_{zi} \end{aligned} \tag{4.6}$$

Here q is a total number of the nodes of the element. For the development of computer programs for 3-D micropolar elasticity, only 8- and 20- node element are used in this study as described in Chapter 2. u_{ri}, $u_{\theta i}$ and u_{zi} are nodal displacements along x, y,and z direction, respectively, and ϕ_{ri}, $\phi_{\theta i}$ and ϕ_{zi}, are nodal microrotation about x, y, and z direction.

Defining the nodal value vector U^e, as

$$U^e = [\ u_{r1},\ u_{\theta 1},\ u_{z1},\ \phi_{r1},\ \phi_{\theta 1},\ \phi_{z1},$$
$$\ldots,\ u_{xq},\ u_{yq},\ u_{zq},\ \phi_{xq},\ \phi_{yq},\ \phi_{zq}]^T \quad (q=8 \text{ or } 20)$$

one can express Equation (14) in compact form:

$$\begin{matrix} u_x\ (R,S,T) \\ u_y\ (R,S,T) \\ u_z\ (R,S,T) \\ \phi_x\ (R,S,T) \\ \phi_y\ (R,S,T) \\ \phi_z\ (R,S,T) \end{matrix} = \begin{matrix} N_1\ (R,S,T) & \cdot & N_q\ (R,S,T) \\ N_1\ (R,S,T) & \cdot & N_q\ (R,S,T) \\ N_1\ (R,S,T) & \cdot & N_q\ (R,S,T) \\ N_1\ (R,S,T) & \cdot & N_q\ (R,S,T) \\ N_1\ (R,S,T) & \cdot & N_q\ (R,S,T) \\ N_1\ (R,S,T) & \cdot & N_q\ (R,S,T) \end{matrix} \quad U^e \qquad (4.7)$$

Here u_x, u_y, u_z, ϕ_x,ϕ_y, ϕ_z, and each shape function N_i are expressed using natural coordinate system in each element. The following shape functions are used for $q = 8$.

$$\begin{aligned} N_1 &= 1/8\ (1+R)\ (1+S)\ (1-T) \\ N_2 &= 1/8\ (1+R)\ (1-S)\ (1-T) \\ N_3 &= 1/8\ (1+R)\ (1+S)\ (1-T) \\ N_4 &= 1/8\ (1-R)\ (1+S)\ (1-T) \\ N_5 &= 1/8\ (1-R)\ (1-S)\ (1+T) \\ N_6 &= 1/8\ (1-S)\ (1+R)\ (1+T) \\ N_7 &= 1/8\ (1+R)\ (1+S)\ (1+T) \\ N_8 &= 1/8\ (1+S)\ (1-R)\ (1+T) \end{aligned} \qquad (4.8)$$

To calculate the derivatives with respect to the global coordinate system, one needs coordinate transformation from natural coordinate system to global coordinate system:

$$\begin{Bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{Bmatrix} = J^{-1} \begin{Bmatrix} \frac{\partial}{\partial R} \\ \frac{\partial}{\partial S} \\ \frac{\partial}{\partial T} \end{Bmatrix}$$

Here J is called Jacobian matrix defined by

$$J = \begin{bmatrix} \frac{\partial x}{\partial R} & \frac{\partial x}{\partial S} & \frac{\partial x}{\partial T} \\ \frac{\partial y}{\partial R} & \frac{\partial y}{\partial S} & \frac{\partial y}{\partial T} \\ \frac{\partial z}{\partial R} & \frac{\partial z}{\partial S} & \frac{\partial z}{\partial T} \end{bmatrix}$$

Thus

$$\begin{Bmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial \theta} \\ \frac{\partial}{\partial z} \end{Bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} & \frac{\partial z}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} & \frac{\partial z}{\partial \theta} \\ \frac{\partial x}{\partial z} & \frac{\partial y}{\partial z} & \frac{\partial z}{\partial z} \end{bmatrix} \begin{Bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{Bmatrix} \qquad (4.9)$$

$$= \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -r\sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{Bmatrix}$$

$$
= \begin{bmatrix} \nabla_{11} \\ \nabla_{22} \\ \nabla_{33} \end{bmatrix}
$$

where

$$
\nabla_{11} = \cos\theta * \nabla_1 + \sin\theta * \nabla_2
$$

$$
\nabla_{22} = -r\sin\theta * \nabla_1 + r\cos\theta * \nabla_2
$$

$$
\nabla_{33} = \nabla_3
$$

Here ∇_1, ∇_2 and ∇_3 are differential operators defined by:

$$
\nabla_1 = J^{-1}(1,1)\frac{\partial}{\partial R} + J^{-1}(1,2)\frac{\partial}{\partial S} + J^{-1}(1,3)\frac{\partial}{\partial T}
$$

$$
\nabla_2 = J^{-1}(2,1)\frac{\partial}{\partial R} + J^{-1}(2,2)\frac{\partial}{\partial S} + J^{-1}(2,3)\frac{\partial}{\partial T}
$$

$$
\nabla_3 = J^{-1}(3,1)\frac{\partial}{\partial R} + J^{-1}(3,2)\frac{\partial}{\partial S} + J^{-1}(3,3)\frac{\partial}{\partial T}
$$

Substituting equations (4.7) and (4.9) into strain displacement equation (4.2), one obtains the following strain-displacement matrix B_0 :

$$
\begin{Bmatrix} \epsilon_{rr} \\ \epsilon_{\theta\theta} \\ \epsilon_{zz} \\ \epsilon_{r\theta} \\ \epsilon_{\theta r} \\ \epsilon_{rz} \\ \epsilon_{zr} \\ \epsilon_{\theta z} \\ \epsilon_{z\theta} \end{Bmatrix} = B_0 \, U^e \qquad (4.10)
$$

where

$$B_0(1,1) = B_0(1,7) = \ldots\ldots = B_0(1,1+6*(q-1)) = \nabla_{11} N_q$$

$$B_0(2,1) = B_0(2,7) = \ldots\ldots = B_0(2,1+6*(q-1)) = N_q/r$$

$$B_0(2,2) = B_0(2,8) = \ldots\ldots = B_0(2,2+6*(q-1)) = \nabla_{22} N_q/r$$

$$B_0(3,3) = B_0(3,9) = \ldots\ldots = B_0(3,3+6*(q-1)) = \nabla_{33} N_q$$

$$B_0(4,2) = B_0(4,8) = \ldots\ldots = B_0(4,2+6*(q-1)) = \nabla_{11} N_q$$

$$B_0(4,6) = B_0(4,12) = \ldots\ldots = B_0(4,6+6*(q-1)) = -N_q$$

$$B_0(5,1) = B_0(5,7) = \ldots\ldots = B_0(5,1+6*(q-1)) = \nabla_{22} N_q/r$$

$$B_0(5,2) = B_0(5,8) = \ldots\ldots = B_0(5,2+6*(q-1)) = -N_q/r$$

$$B_0(5,6) = B_0(5,12) = \ldots\ldots = B_0(5,6+6*(q-1)) = N_q$$

$$B_0(6,5) = B_0(6,11) = \ldots\ldots = B_0(6,5+6*(q-1)) = N_q$$

$$B_0(6,3) = B_0(6,9) = \ldots\ldots = B_0(6,3+6*(q-1)) = \nabla_{11} N_q$$

$$B_0(7,1) = B_0(7,7) = \ldots\ldots = B_0(7,1+6*(q-1)) = \nabla_{33} N_q$$

$$B_0(7,5) = B_0(7,11) = \ldots\ldots = B_0(5,2+6*(q-1)) = -N_q$$

$$B_0(8,3) = B_0(8,9) = \ldots\ldots = B_0(8,3+6*(q-1)) = \nabla_{22} N_q/r$$

$$B_0(8,4) = B_0(8,10) = \ldots\ldots = B_0(8,4+6*(q-1)) = -N_q$$

$$B_0(9,2) = B_0(9,8) = \ldots\ldots = B_0(9,2+6*(q-1)) = \nabla_{33} N_q$$

$$B_0(9,4) = B_0(9,10) = \ldots\ldots = B_0(9,4+6*(q-1)) = N_q$$

for $q=1,2,\ldots\ldots 8$

For the axisymmetric case, $B_0(2,2)=\ldots =B_0(5,1)=\ldots =B_0(8,3)\ldots=0$

Similary for microrotation gradient, one obtains the following B_1 matrix:

$$\begin{bmatrix} \phi_{r,r} \\ \phi_{\theta,\theta}/r + \phi_r/r \\ \phi_{z,z} \\ \phi_{\theta,r} \\ \phi_{r,\theta}/r - \phi_\theta/r \\ \phi_{z,r} \\ \phi_{r,z} \\ \phi_{z,\theta}/r \\ \phi_{\theta,z} \end{bmatrix} = B^1 U^e \tag{4.11}$$

where B_1 is a 9 by 48 matrics in q=8 case.

and

$B_1(1,4) = B_1(1,10) = \ldots\ldots = B_1(1,4+6*(q-1)) = \nabla_{11} N_q$

$B_1(2,4) = B_1(2,10) = \ldots\ldots = B_1(2,4+6*(q-1)) = N_q/r$

$B_1(2,5) = B_1(2,11) = \ldots\ldots = B_1(2,5+6*(q-1)) = \nabla_{22} N_q/r$

$B_1(3,6) = B_1(3,12) = \ldots\ldots = B_1(3,6+6*(q-1)) = \nabla_{33} N_q$

$B_1(4,5) = B_1(4,11) = \ldots\ldots = B_1(4,5+6*(q-1)) = \nabla_{11} N_q$

$B_1(5,4) = B_1(4,10) = \ldots\ldots = B_1(4,4+4*(q-1)) = \nabla_{22} N_q/r$

$B_1(5,5) = B_1(5,11) = \ldots\ldots = B_1(5,5+6*(q-1)) = -N_q/r$

$B_1(6,6) = B_1(6,12) = \ldots\ldots = B_1(6,6+6*(q-1)) = \nabla_{11} N_q$

$B_1(7,4) = B_1(7,10) = \ldots\ldots = B_1(7,4+6*(q-1)) = \nabla_{33} N_q$

$B_1(8,6) = B_1(8,12) = \ldots\ldots = B_1(8,6+6*(q-1)) = \nabla_{22} N_q/r$

$B_1(9,5) = B_1(9,11) = \ldots\ldots = B_1(9,5+6*(q-1)) = \nabla_{33} N_q$

where q=1,2,.....8

The B_0- and B_1- matrices derived above can be substituted into equation (3.29) to obtain element stiffness matrix k^e. To carry out volume integral of equation (3.29), numerical integration of Gaussian quadrature shown in Table 2.1 is used in the program:

$$
\begin{aligned}
k^e = & \sum_{i,j}^{g} t_{ij}\,\alpha_{ij}\,(B_0{}^T{}_{ij}\; D_0\; B_0{}_{\,ij})\;\mathrm{DET} \\
& + \sum_{i,j}^{q} t_{ij}\,\alpha_{ij}\,(B_1{}^T{}_{ij}\; D_1\; B_1{}_{\,ij})\;\mathrm{DET}
\end{aligned}
\qquad (4.12)
$$

where α_{ij} is the weighting factors of Gaussian Quadrature.

If body force is neglected, the force vector can becomes

$$F^e = F_s{}^e.$$

The force vector can be calculated by

$$
F_s{}^e = \int_s \begin{bmatrix} N_1 & & \\ & N_1 & \\ & & N_1 \\ \hline & \cdot & \\ & \cdot & \\ & \cdot & \\ \hline N_q & & \\ & N_q & \\ & & N_q \end{bmatrix} \begin{Bmatrix} P_r \\ P_\theta \\ P_z \end{Bmatrix} ds \qquad (4.13)
$$

The element stiffness matrices calculated in equation (4.12) are assembled into a global stiffiness matrix in a symmetric banded form with only the upper triangular part stored. Boundary conditions of the prescribed displacements and microrotations are imposed by modification of the corresponding rows and columns of this global stiffness matrix. Similarly, the force vector is generated by superposing element force vector in equation (4.13). The linear algebraic equation is then solved for nodal displacement using the skyline technique [12].

4.4 Numerical examples

The finite element formulations developed in the previous section are implemented in Fortran programs(Appendix 2) with either 8- or 20- node element, as shown Fig.4.1 and Fig. 4.2 The eight nodes in the 8-node element represent the eight corners of a cubic finite element, and the additional 12 nodes are included in the 20- node element to bisect every two corners. In order to verify the validity of the proposed finite element formulations, two examples of axisymmetric micropolar elastic solids are solved. The axisymmetric examples are deliberately chosen so that they can be compared with existing two-dimensional analytical solutions [10].

Example 1

This example is to solve a simple tension of cylindrical Cosserat solids. The 8-node element is used. The finite element meshes used are depicted in Fig.4.3 and Fig.4.4. The material parameters are:

Coupling factor N=0.0, 0.25, 0.50, 0.7, 0.9, characteristic length $l=8.333\times10^{-3}$ inch and Poissons ratio=0.3. $\mu=1\times10^{3}$ (psi), k=0.0 (psi) β =666.6666 (psi), γ =0.185185185 (pound)

Dimensions used are: radius R=2 (inches) and length L=2 (inches).

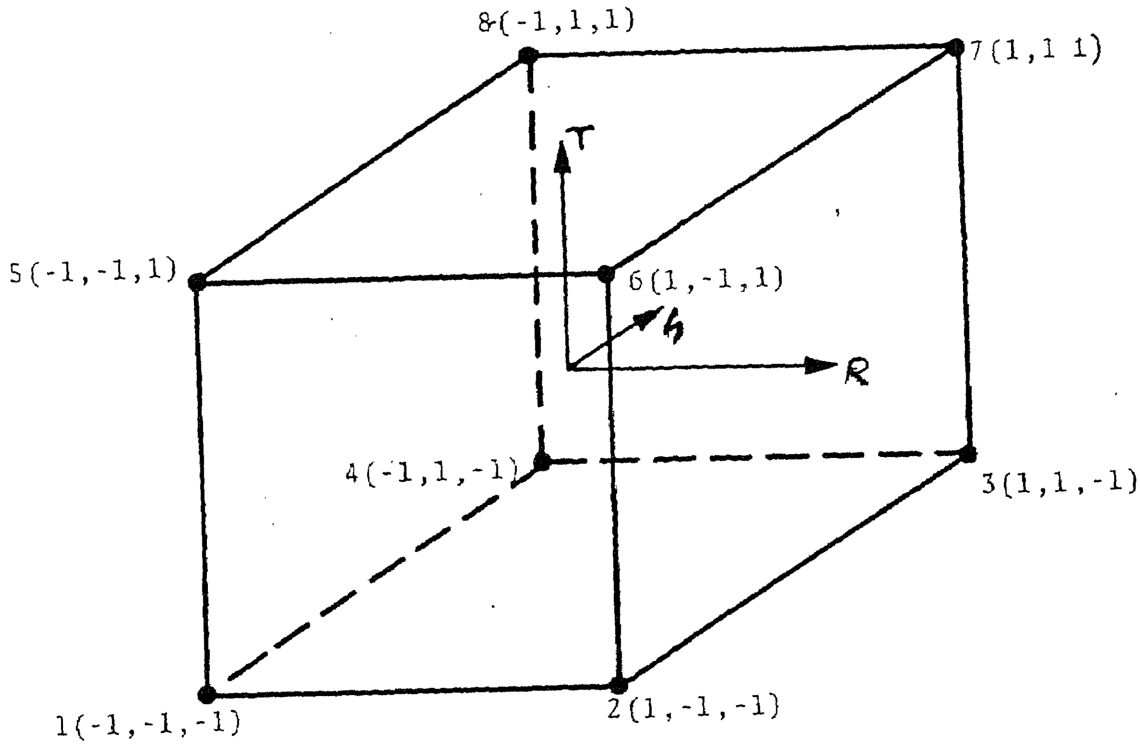

Fig. 4.1 Eight Node Three-dimensional Element

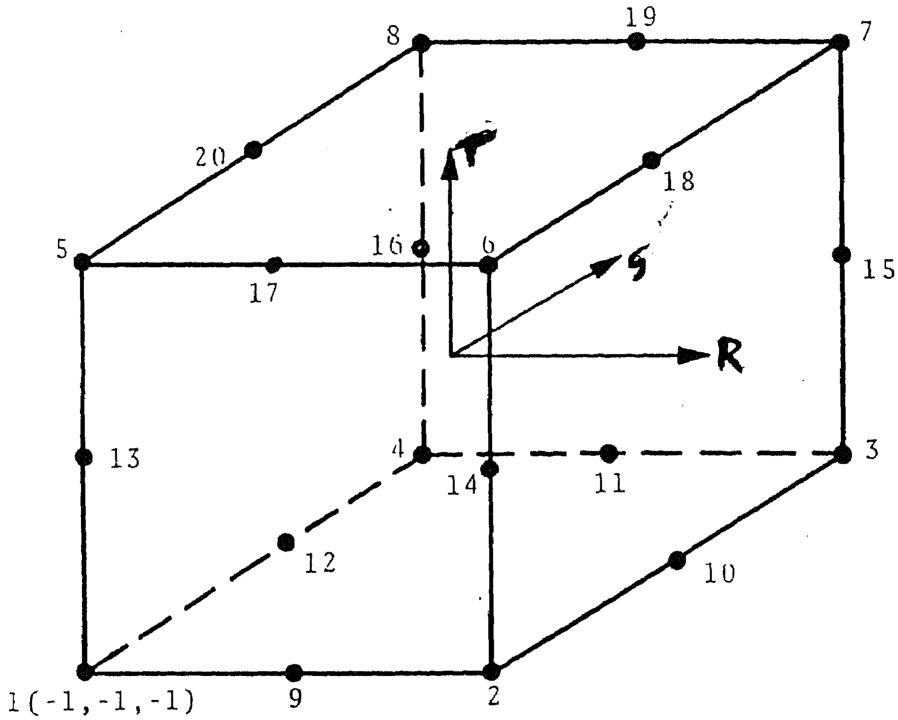

Fig. 4.2 Twenty Node Three-dimensional Element

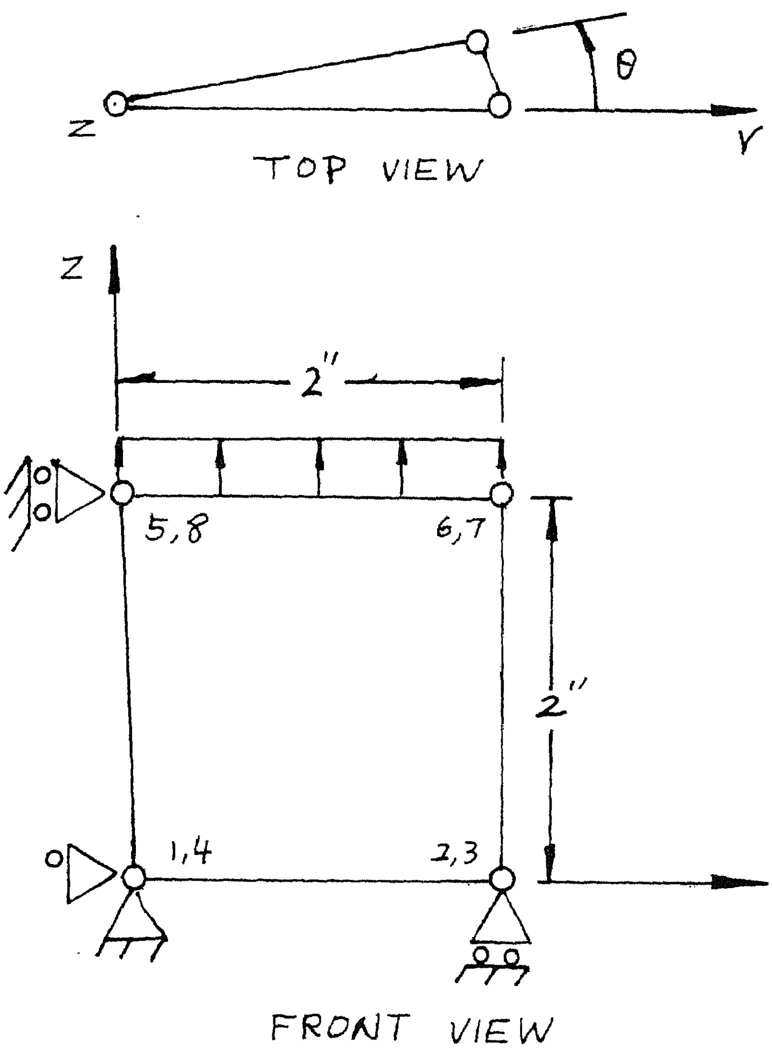

Fig.4.3. 8-Node Three-dimensional Element for Simple Tension.

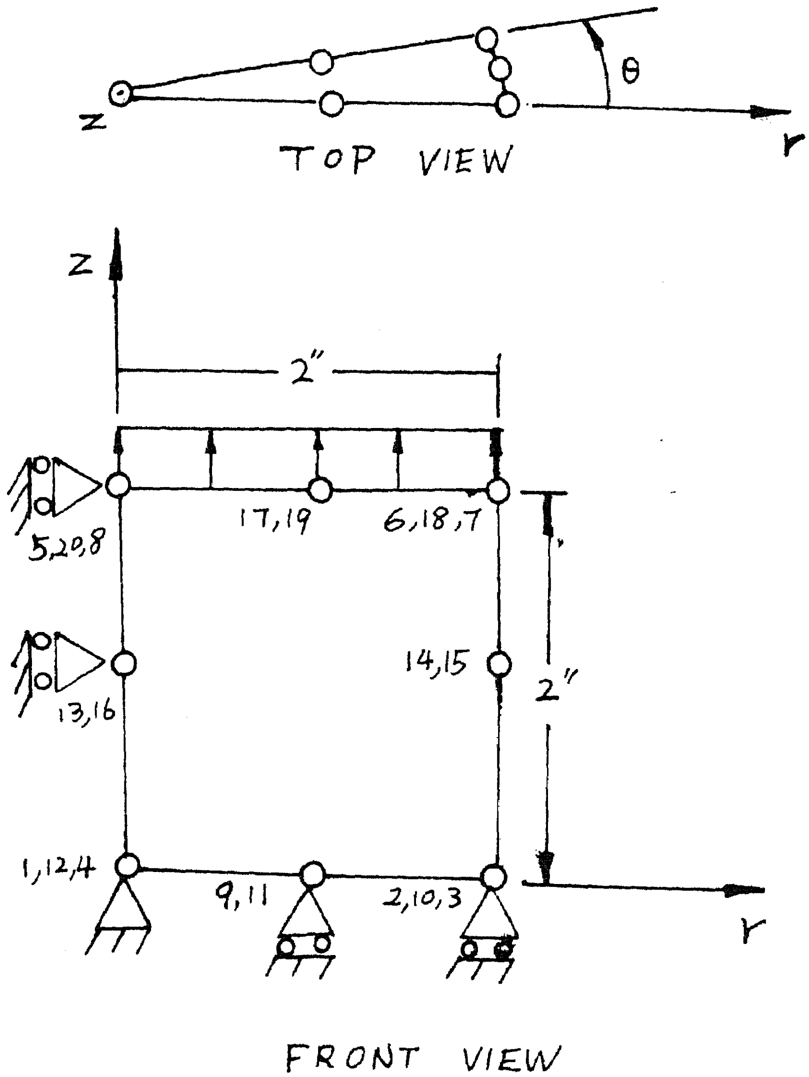

Fig.4.4. 20-Node Three-dimensional Element for Simple Tension

TABLE 4.1 Numerical results from Fig. 4.3 c.p. Analytical solution

		numerical results	analytical solution
(1) displacement (inch)			
nodal no.	coordinate r (inch)	u_{rr} (inch)	$u_{rr} = - \frac{v_m r}{E_m}$ (inch)
2,3,6,7	2.0	$-0.34615386 \times 10^{-3}$	$-0.34615386 \times 10^{-3}$
nodal no.	z	u_{zz}	$v_{zz} = \frac{z}{E_m}$
5,6,7,8	2.0	$0.11538462 \times 10^{-2}$	$0.11538462 \times 10^{-2}$
(2) stress			
element no.		t_{zz}	$t_{zz} = \frac{P_z}{A}$
1		1.0	1.0

TABLE 4.2 Numerical results from Fig. 4.4
c.p. Analytical solution

		numerical results	analytical solution
(1) displacement (inch)			
nodal no.	coordinate r (inch)	u_{rr} (inch)	$u_{rr} = - \frac{v_m r}{E_m}$ (inch)
9,11,17,19	1.0	$-0.17307693 \times 10^{-3}$	$-0.17307693 \times 10^{-3}$
2,6,3,7 10,14,18,15	2.0	$-0.34615386 \times 10^{-3}$	$-0.34615386 \times 10^{-3}$
nodal no.	z (inch)	u_{zz}	$v_{zz} = \frac{z}{E_m}$
13,14,15,16	1.0	$0.57692309 \times 10^{-3}$	$0.57692309 \times 10^{-3}$
5,6,7,8 17,18,19,20		$0.11538462 \times 10^{-2}$	$0.11538462 \times 10^{-2}$
(2) stress			
element no.		t_{zz}	$t_{zz} = \frac{P_z}{A}$
1		1.0	1.0

The numerical results obtained in this study are summarized in Table 4.1 and Table 4.2. Also shown in these tables are the analytical solutions of the same isotropic micropolar elastics solids derived by Gauthier:

Force stress: $t_{zz} = \frac{P_z}{A}$

r-displacement: $u = -\frac{\nu_m P_z r}{E_m A}$

z-displacement: $v = \frac{P_z z}{E_m A}$

where A is the area of crossection and Young's modulus and Poission's ratio are defined by

$$E_m = \frac{(2\mu+k)\ (3\lambda+2\mu+\kappa)}{(2\mu+2\lambda+k\ \ )}\ ,\qquad \nu_m = \frac{\lambda}{(2\mu+2\lambda+k\ \ )}$$

The significant findings are the following:

1. The numerical results for displacements are identical with those from the analytical solution. This confirms the validity of the proposed isoparametric finite element formulation.
2. The displacements are not effected by the coupling factor N. This indicates that the micropolar effects do not affect the simple test, meaning that during the simple test, micropolar effects vanish.
3. The tension stress in the z-direction are found to be equal to unity everywhere in every element.

Example 2

(1) The geometry is simple.

(2)The shape function and numerical integration are exact for lower displacement field.

Example 2

To demonstrate the capability and validity of the proposed finite element formulation, an isotropic micropolar elastic cylinder with a semi-circular groove is used as the second example. To the best of the author's knowledge, this particular problem has never been solved before, probably due to the difficulty in obtaining analytical solutions with such a complex geometry . In the following it is shown that the numerical results obtained in this study indeed coincides with the available experimental results when the coupling factor is equal to zero, i.e., when the micropolar solid reduces to a classical material.

The isotropic micropolar elastic cylinder with a semi-circular groove used in this example is shown in Fig.4.5. The diameter of the cylinder D is fixed to 0.1 inch. The radius ratio k is defined by k = r/d, where r is the radius of groove, and d is D - 2r. In this study, k is varied in the range of 0.05 to 0.5. Finite element mesh used for the analysis is shown in Fig.4.6 for radius ratio k=0.5. The material properties used for the isotropic case are as listed in Table.4.3. The characteristic length l is fixed to $8.333\text{x}10^{-3}$, and the Poisson ratio of 0.3 is used. These properties are summarized in Table 4.3.

The numerical results of K_c, or sigma max/sigma nom,

Table 4.3 Material properties used for the analysis of isotropic micropolar elasticity

N	λ		μ		κ	
0.0	1.000×10^3	(psi)	6.666×10^2	(psi)	0.0	(psi)
	6.895×10^6	(N/m^2)	4.596×10^6	(N/m^2)	0.0	(N/m^2)
0.25	1.125×10^4	(psi)	7.000×10^3	(psi)	1.000×10^3	(psi)
	7.757×10^7	(N/m^2)	4.826×10^7	(N/m^2)	6.895×10^6	(N/m^2)
0.50	2.250×10^3	(psi)	1.000×10^3	(psi)	1.000×10^3	(psi)
	5.063×10^6	(N/m^2)	6.895×10^6	(N/m^2)	6.895×10^6	(N/m^2)
0.75	5.833×10^2	(psi)	1.111×10^2	(psi)	1.000×10^3	(psi)
	4.022×10^6	(N/m^2)	7.660×10^6	(N/m^2)	6.895×10^6	(N/m^2)
0.90	1.795×10^2	(psi)	3.827×10^2	(psi)	1.000×10^3	(psi)
	1.213×10^6	(N/m^2)	2.639×10^5	(N/m^2)	6.895×10^6	(N/m^2)

N	γ	α	β
0.00	0.185185185 (pound)	0.0 (pound)	0.0 (pound)
0.25	2.083333333 (pound)	0.0 (pound)	0.0 (pound)
0.50	0.416666666 (pound)	0.0 (pound)	0.0 (pound)
0.75	0.108024691 (pound)	0.0 (pound)	0.0 (pound)
0.90	0.032578875 (pound)	0.0 (pound)	0.0 (pound)

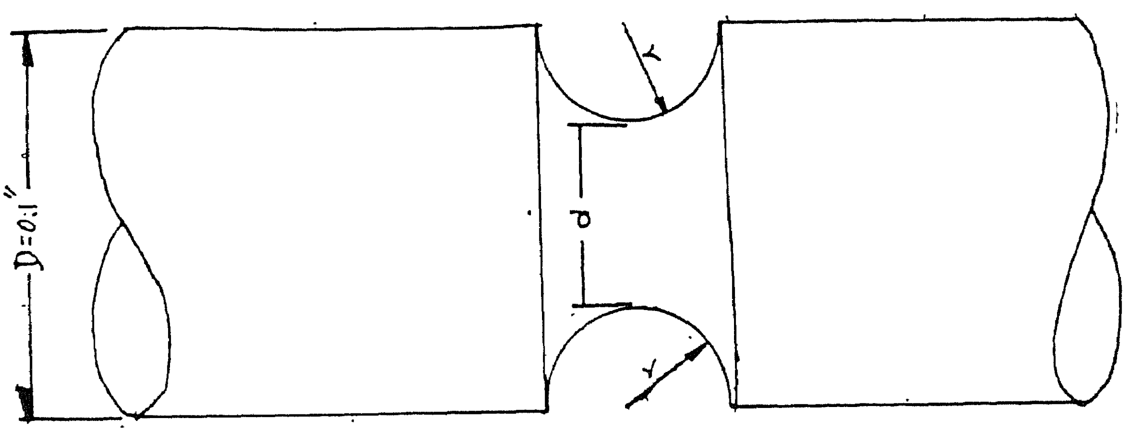

Fig. 4.5 Isotropic Micropolar Elastic Cylinder with Semi-circular Groove

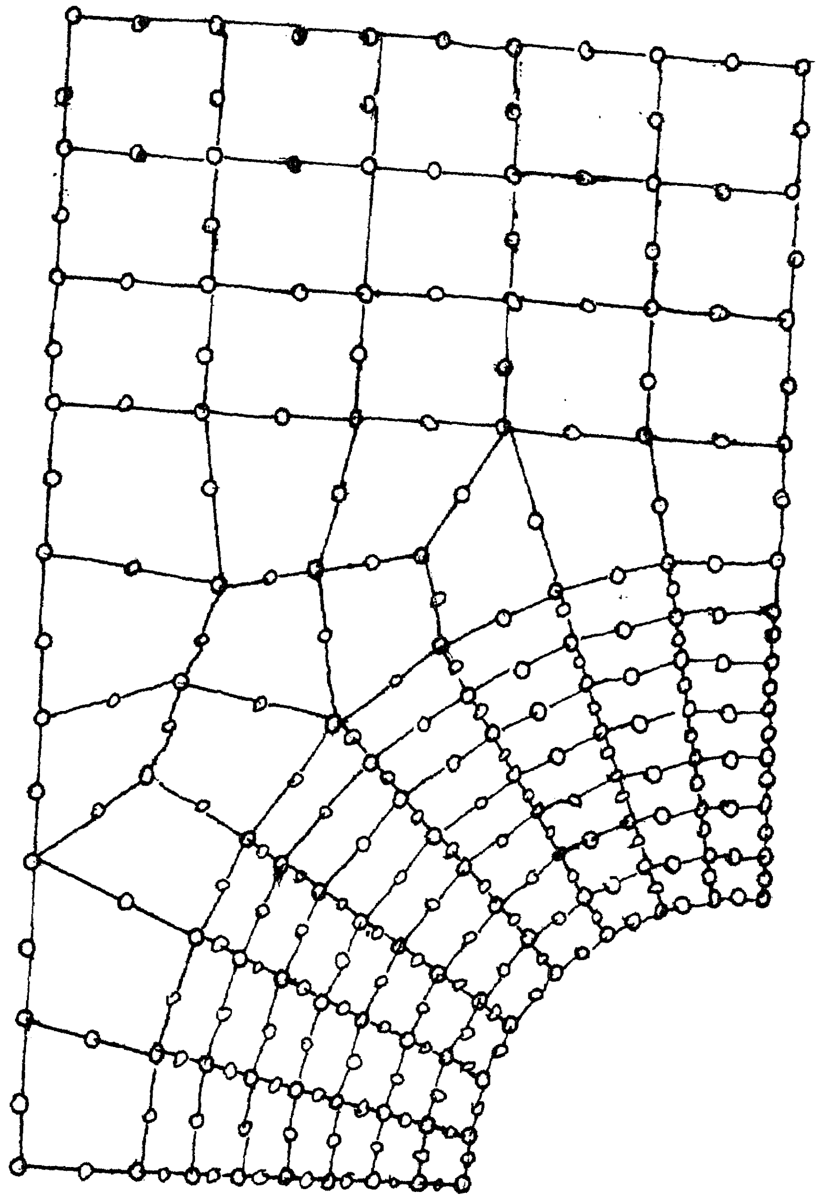

Fig.4.6 Finite element meshes

Table 4.4 Numerical results of stress concentration factor vs r/d. (isotropic for force stress and couple stress)

ratio r/d	N=0.90	N=0.75	N=0.50	N=0.25	N=0.00
0.05	1.75	1.92	2.26	2.58	2.73
0.08	1.65	1.81	2.11	2.40	2.53
0.10	1.60	1.75	2.03	2.28	2.39
0.13	1.53	1.67	1.91	2.13	2.22
0.17	1.47	1.59	1.80	1.98	2.04
0.20	1.43	1.54	1.73	1.88	1.93
0.23	1.40	1.50	1.66	1.79	1.84
0.27	1.36	1.45	1.60	1.70	1.74
0.30	1.33	1.42	1.55	1.65	1.68
0.35	1.28	1.35	1.47	1.56	1.59
0.40	1.25	1.31	1.42	1.49	1.52
0.45	1.23	1.29	1.38	1.44	1.46
0.50	1.20	1.26	1.34	1.40	1.42

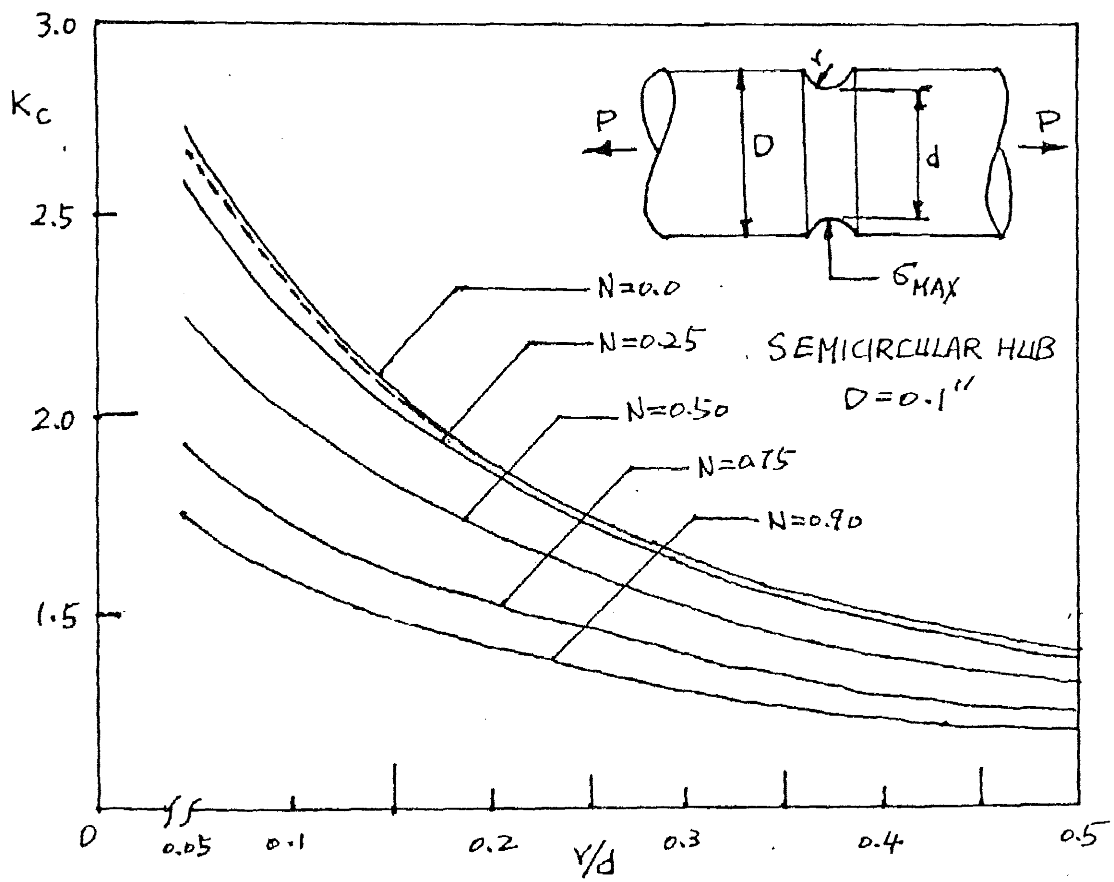

Fig. 4.7 Stress Concentraction Factor, K_c for a Round Tension Bar with a Semi-circular Groove in Three-dimensional case (Orthotropic for Force Stress and Couple Stress)

where sigma max is the maximum stress at the groove and sigma norm is the stress without the groove for different coupling factor N are listed in Table 4.4. The simulated concentration factor versus radius ratio with coupling factor as a parameter are also poltted in Fig.4.7. It can be seen that K_c decreeses monotonically as k increases. The experimental results for the classical elasticity material done by Peterson [19] are also plotted in Fig.4.7 with dashed line. The experimental data concide very well with the numerical results for the case of N = 0.0 in which the micropolar elasticity reduces to classical elasticity. This fact again confirms the validity of the finite element formulation.

The numerical results also show that for non-zero N, the stress concentration factor is reduced, indicating that the microrotation does affect the stress concentration factor. For a fixed radius ratio, a larger N results in a smaller stress concentration factor, suggesting that the microrotation effect increases when the coupling factor increases. It is also interesting to note that the effect of coupling factor on K_c decreases at high radius ratio. When the radius ratio becomes very large, the problem can be treated as a simple test which has no micropolar effect.

CHAPTER V

AXISYMMETRIC ELEMENTS

5.1 Introductory Comments

Many engineering problems involve solids of revolution subjected to axially asymmetric loading. It is therefore important to consider the asymmetric problems in the micropolar elasticity. The axisymmetric problems can be transformed to a two-dimensional plane case and solved by two-dimensional analytical methods. However, no universal technique is available for general boundary value problems, making the solutions procedure tedious. In this chapter it is demonstrated that the finite element method can overcome these short-comings by offering a general-purpose numerical solution, thus providing a powerful technique in solving axisymmetrical problems.

In the following sections, the simplified two-dimensional strain-displacement relationship for micropolar elasticity of the axisymmetric case due to axisymmetric load is derived, and then the isoparametric finite element method is applied to formulate the simplified strain-displacement relationship. Isoparametric elements of 4-node and 8-node are used. The programming organization is then described. The program is also verified by comparing the numerical results with analytical solutions for simple tension of cylindrical Cosserat solid and also for stress concentration of a round bar with semi-circular groove.

5.2 Axisymmetric Micropolar Elasticity

To derive the simplifed two-dimensional strain-displacement relationship for micropolar elasticity of the axisymmetric material properties under axisymmetric loading, one starts with micropolar strain tensors defined in Equation. 3.11.

$$\epsilon_{ij} = u_{j,i} - e_{ijk}\phi_k$$

This equation was transformed intothe cylindrical coordinate system (r,o,z) in chapter 4 and had the form:

$$\epsilon_{ij} = \begin{matrix} \dfrac{\partial u_r}{\partial r} & \dfrac{\partial u_\theta}{\partial r} - \phi_z & \dfrac{\partial u_z}{\partial r} + \phi_\theta \\ \dfrac{\partial u_r}{r\partial \theta} - \dfrac{u_\theta}{r} + \phi_z & \dfrac{\partial u_\theta}{r\partial \theta} + \dfrac{u_r}{r} & \dfrac{\partial u_z}{r\partial \theta} - \phi_r \\ \dfrac{\partial u_r}{\partial z} - \phi_\theta & \dfrac{\partial u_\theta}{\partial z} + \phi_r & \dfrac{\partial u_z}{\partial z} \end{matrix} \qquad (4.2)$$

For the axisymmetric case, dependency on θ vanishes and the above strain tensors reduce to the following form:

$$\epsilon_{ij} = \begin{bmatrix} \dfrac{\partial u_r}{\partial r} & \dfrac{\partial u_\theta}{\partial r} - \phi_z & \dfrac{\partial u_z}{\partial r} + \phi_\theta \\ -\dfrac{u_\theta}{r} + \phi_z & \dfrac{u_r}{r} & -\phi_r \\ \dfrac{\partial u_r}{\partial z} - \phi_\theta & \dfrac{\partial u_\theta}{\partial z} + \phi_r & \dfrac{\partial u_z}{\partial z} \end{bmatrix} \qquad (5.1)$$

In the case of axisymmetric load, microrotations about r and z axis vanish, thus Equation (5.1) can be further reduced :

$$(\epsilon_{rr}, \epsilon_{\theta\theta}, \epsilon_{zz}, \epsilon_{rz}, \epsilon_{zr})$$

$$= \left(\frac{\partial u_r}{\partial r}, \frac{u_r}{r}, \frac{\partial u_z}{\partial z}, \frac{\partial u_z}{\partial r} + \phi_\theta, \frac{\partial u_r}{\partial z} - \phi_\theta \right) \qquad (5.2)$$

From now on, the following simplified notations are used.

$$u = u_r \quad , \; v = u_z \quad \text{and} \; \phi = \phi_\theta$$

Thus expression (5.2) becomes

$$\epsilon = \left(\frac{\partial u}{\partial r}, \frac{u}{r}, \frac{\partial v}{\partial z}, \frac{\partial v}{\partial r} + \phi, \frac{\partial u}{\partial z} - \phi \right)^T \qquad (5.3)$$

The above kinematical considerations tell one that for elastic solids under axisymmetric loading, each material point has only three degrees of freedom, i.e., (u,v,ϕ), which represent displacements in the r, and z directions; and microrotation about O direction, respectively.

5.3 Axisymmetric Element:

The shape functions for the isoparametric element for four to nine variable-number-nodes were discussed in chapter 3, and also shown in Appendix 1. Using those shape functions N_i, displacement and microrotation field inside each element can be interpolated:

$$u = \sum_{i=1}^{q} N_i \, u^i$$

$$v = \sum_{i=1}^{q} N_i \, v^i \qquad (5.4)$$

$$\phi = \sum_{i=1}^{q} N_i \, \phi^i$$

Here q is the total number of nodes of the element and q= 4 or 8 For the development of the computer program for micropolar elasticity, both 4 node- and 8 node-element are used . u^i and v^i are nodal displacements along r and z direction, respectively; and ϕ is a nodal microrotation about θ direction.

By defining the nodal value vector U^e :

$$U^e = (u_1, v_1, \phi_1, \ . \ . \ ., u_q, v_q, \phi_q)^T, \quad (q=4 \text{ or } 8)$$

one can express Equation (5.4) in compact form:

$$\begin{Bmatrix} u\ (R,S) \\ v\ (R,S) \\ \phi\ (R,S) \end{Bmatrix} = \begin{bmatrix} N_1\ (R,S) & . & & N_q\ (R,S) & & \\ & N_1\ (R,S) & . & & N_q\ (R,S) & \\ & & N_1\ (R,S) & . & & N_q\ (R,S) \end{bmatrix} U^e \quad (5.5)$$

Here u,v,ϕ and each shape function N_i are expressed using natural coordinate system in each element. The following shape functions are used for q=8.

$$\begin{aligned} N_1 &= 1/4\ (1+R)\ (1+S)\ (R+S-1) \\ N_2 &= 1/4\ (1-R)\ (1+S)\ (-R+S-1) \\ N_3 &= 1/4\ (1-R)\ (1-S)\ (-R-S-1) \\ N_4 &= 1/4\ (1+R)\ (1-S)\ (R-S-1) \\ N_5 &= 1/2\ (1-R)\ (1+S) \\ N_6 &= 1/2\ (1-S)\ (1-R) \\ N_7 &= 1/2\ (1-R)\ (1-S) \\ N_8 &= 1/2\ (1-S)\ (1+R) \end{aligned} \quad (5.6)$$

To calculate the derivatives with respect to global coordinate system, one needs coordinate transformation:

$$\begin{Bmatrix} \dfrac{\partial}{\partial r} \\[2ex] \dfrac{\partial}{\partial z} \end{Bmatrix} = J^{-1} \begin{Bmatrix} \dfrac{\partial}{\partial R} \\[2ex] \dfrac{\partial}{\partial S} \end{Bmatrix}$$

Here J is the Jacobian matrix and is defined by

$$J = \begin{bmatrix} \dfrac{\partial r}{\partial R} & \dfrac{\partial r}{\partial S} \\[2ex] \dfrac{\partial z}{\partial R} & \dfrac{\partial z}{\partial S} \end{bmatrix}$$

thus

$$J^{-1} = \frac{1}{DET} \begin{bmatrix} \dfrac{\partial z}{\partial S} & -\dfrac{\partial z}{\partial R} \\[2ex] -\dfrac{\partial r}{\partial S} & \dfrac{\partial r}{\partial R} \end{bmatrix}$$

and

$$DET = \begin{vmatrix} \dfrac{\partial r}{\partial R} & \dfrac{\partial r}{\partial S} \\[2ex] \dfrac{\partial z}{\partial R} & \dfrac{\partial z}{\partial S} \end{vmatrix}$$

By substituting Equations (5.5) and (5.6) into strain displacement equation (5.3), one obtains the following strain-displacement matrix B_0 :

$$\begin{Bmatrix} \epsilon_{rr} \\ \epsilon_{\theta\theta} \\ \epsilon_{zz} \\ \epsilon_{rz} \\ \epsilon_{zr} \end{Bmatrix} = \begin{bmatrix} \frac{1}{DET}\Delta_1 N_1 & 0 & 0 & \cdots & \frac{1}{DET}\Delta_1 N_q & 0 & 0 \\ \frac{N_1}{r} & 0 & 0 & \cdots & \frac{N_q}{r} & 0 & 0 \\ 0 & \frac{1}{DET}\Delta_2 N_1 & 0 & \cdots & 0 & \frac{1}{DET}\Delta_2 N_q & 0 \\ 0 & \frac{1}{DET}\Delta_1 N_1 & N_1 & \cdots & 0 & \frac{1}{DET}\Delta_1 N_q & N_q \\ \frac{1}{DET}\Delta_2 N_1 & 0 & -N_1 & \cdots & \frac{1}{DET}\Delta_2 N_q & 0 & -N_q \end{bmatrix} U^e$$

$$= B_0 \, U^e \tag{5.7}$$

Here Δ_1 and Δ_2 are differential operators defined by:

$$\Delta_1 = \frac{\partial z}{\partial s}\frac{\partial}{\partial R} - \frac{\partial z}{\partial R}\frac{\partial}{\partial s}$$

$$\Delta_2 = -\frac{\partial r}{\partial s}\frac{\partial}{\partial R} + \frac{\partial r}{\partial R}\frac{\partial}{\partial s}$$

Similarly for microrotation gradient, one obtains the following B_1 matrix:

$$\begin{Bmatrix} \frac{\partial \phi}{\partial r} \\ -\frac{\phi}{r} \\ \frac{\partial \phi}{\partial z} \end{Bmatrix} = \begin{bmatrix} \frac{1}{DET}\Delta_1 \, (0 & 0 & N_1 & \cdots & 0 & 0 & N_q) \\ (0 & 0 & -\frac{N_1}{r} & \cdots & 0 & 0 & -\frac{N_q}{r}) \\ \frac{1}{DET}\Delta_2 \, (0 & 0 & N_1 & \cdots & 0 & 0 & N_q) \end{bmatrix} U^e$$

$$= B_1 U^e \tag{5.8}$$

The B_0- and B_1- matrices derived above can be substituted into equation (3.29) to obtain the element stiffness matrix k^e.

To carry out volume integral of Equation (3.29), numerical integration of Gaussian quadrature is used in the program. The sampling points and weighting factors for the interval -1 to +1 are given in Table 2.1.

If body force is neglected, the force vector of equation (3.28)) becomes

$$F^e = F_s^{\,e} \tag{5.9}$$

The force vector can be calculated by

$$F_s^{\,e} = \int_S N^T \begin{bmatrix} P_r \\ P_\theta \\ P_z \end{bmatrix} ds \tag{5.10}$$

$$= \int_s \begin{bmatrix} N_1 & & \\ & N_1 & \\ & & N_1 \\ \hline \cdot & & \\ & \cdot & \\ & & \cdot \\ \hline N_q & & \\ & N_q & \\ & & N_q \end{bmatrix} \begin{bmatrix} P_r \\ P_\theta \\ P_z \end{bmatrix} ds$$

If the load is applied along z-direction only, $P_r = P_\theta = 0$.

Thus

$$F_s^e = \int s \begin{Bmatrix} 0 \\ 0 \\ 1/4(1+R)(1+S)(R) \\ 0 \\ 0 \\ 1/4(1-R)(1+S)(-R) \\ 0 \\ 0 \\ 0 \\ 0 \\ (1-R) \\ 0 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{Bmatrix}_{24x1} \begin{Bmatrix} P_z \end{Bmatrix} rdr \qquad (5.11)$$

After integration, the nodal forces can be calculated by the following equations:

$$F_1 = \frac{1}{(r_2-r_1)^2}\left(\frac{r^4}{2} - \frac{4}{3} r^3 r_1 + r^2 r_1^2\right) - \frac{1}{(r_2-r_1)}\left(\frac{r^3}{3} - \frac{r^2}{2} r_1\right)$$

$$F_2 = \frac{1}{(r_2-r_1)^2}\left(\frac{r^4}{2} - \frac{4}{3} r^3 r_1 + r^2 r_1^2\right) - \frac{1}{(r_2-r_1)}\left(r^3 - \frac{3}{2} r^2 r_1\right) + \frac{r^2}{2}$$

$$F_5 = \frac{1}{(r_2-r_1)^2}\left(- r^4 + \frac{8}{3} r^3 r_1 - 2r^2 r_1^2\right) + \frac{1}{(r_2-r_1)}\left(\frac{4}{3} r^3 - 2r^2 r_1\right) \qquad (5.12)$$

If $r_1=0$ and $r_2=1$, then the above equation is reduced to

$$F_1 = 2/3$$

$$F_2 = 0$$

$$F_5 = 4/3$$

To evalate the element stiffness matrix k^e of equation (3.28), numerical integration can be used:

$$k^e = \sum_{i,j}^{8} t_{ij}\alpha_{ij} \; (B_0{}^T{}_{ij} \; D_0 \; B_0{}_{\,ij}) \; DET + \sum_{i,j}^{8} t_{ij}\alpha_{ij} \; (B_1{}^T{}_{ij} \; D_1 \; B_1{}_{\,ij}) \; DET \qquad (5.13)$$

where t_{ij} is the thickness of a portion of axisymmetric element corresponding to unit radian at each node.

$$t_{ij} = \alpha_{ij} \, N_k(R_i, S_j) \, r_k$$

and α_{ij} is the weighing factors of Gaussian Quadrature.

5.4 Numerical Examples

To verify the simplified two-dimensional axisymmetrical finite element formulation and computer program developed in this chapter, two numerical examples of axisymmetric micropolar elastic solids used in the previous chapter 4 are solved. The same examples are chosen in order to facilitate the comparison between the simplified two-dimensional finite element solutions obtained in this section and three-dimensional finite element solutions obtained in Section 4.4 of this study. The results are also used to compare with the analytical solutions available in Ref [10]. In addition, a third example with an orthotropic for force stress but isotropic for couple-stress is also included to demonstrate the micropolar effects in orthotropic materials.

Example 1

This first example solved is a simple tension of cylindrical

Cosserat solids as used in Chapter 4. Both 4- and 8-node elements are used in this study for comparison. As shown in Fig.5.1 and Fig.5.2, the four corner nodes in a 8-node element correspond with the four nodes in a 4-node element. The finite element meshes used is shown in Fig.5.1 and Fig.5.2. The material parameter used are: Coupling factor N=0.0, characteristic length $l=8.333 \times 10^{-3}$ inch and Poissons ratio=0.3. $\mu=1 \times 10^3$ (psi), k=0.0 (psi), $\beta=666.6666$ (psi), $\gamma=0.185185185$ (pound) Dimensions are: radius R=2 (inches) and length L=2 (inches). The r- and z-displacements obtained by two-dimensional numerical simulations in this study are summarized in Table 5.1 and Table 5.2, for the 4- and 8-node rectangular element, respectively. Again it is found that the displacements are independent of the coupling factor for axisymmetrical loading. It can also be seen that the displacements at the four corner nodes of the 8-node element (Table.5.2) also coincide with the three-dimensional simulations obtained with a 20-node three-dimensional element in the previous chapter(Table 4.2), and the 4-node two-dimensional results (Table 5.1) coincides with the 8-node three-dimensional results in Table 4.1. These results also coincide with Gauthier's analytical solution shown in those tables. All these confirm the adequacy and validity of the proposed finite element approach.

To confirm the applicability of the proposed finite element method to an arbitrary geometry, a patch test problem as defined in Fig.5.3 for 4-node and 8-node elements, respectively are performed. Five elements are assembled as shown in Fig.5.3 and subjected to an axial tension. The boundary conditions are

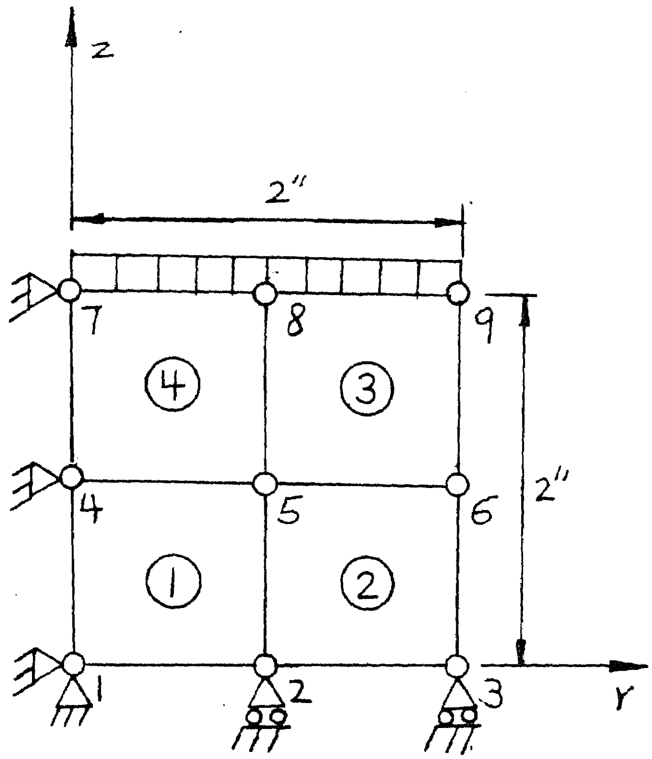

Fig. 5.1 4 Node Two-dimensional Element for Simple Tension

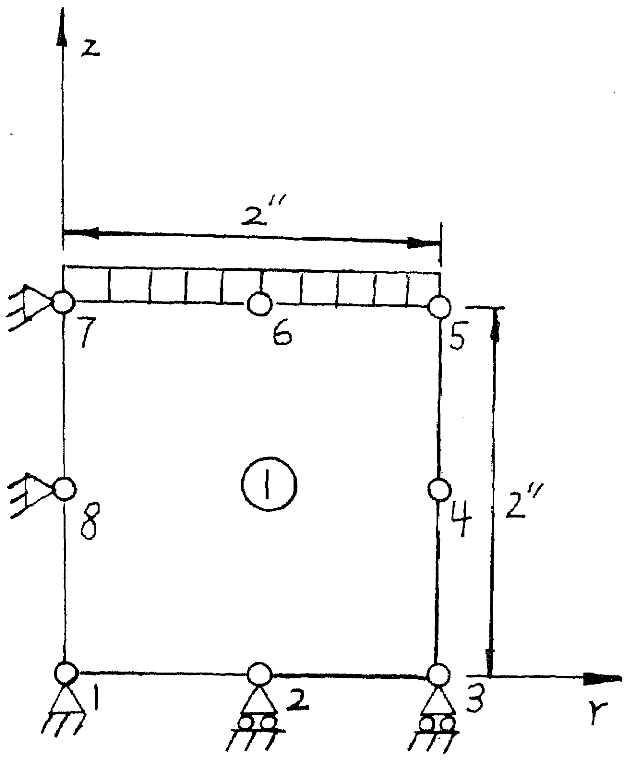

Fig. 5.2 8 Node Two-dimensional Element for Simple Tension

TABLE 5.1 Numerical results from case (a)
c.p. Analytical solution

(1) displacement (inch)

nodal no.	coordinate r (inch)	numerical results u (inch)	analytical solution $u = -\frac{\nu_m r}{E_m}$ (inch)
2,6,8	1.0	$-0.17307693 \times 10^{-3}$	$-0.17307693 \times 10^{-3}$
3,6,9	2.0	$-0.34615386 \times 10^{-3}$	$-0.34615386 \times 10^{-3}$

nodal no.	z	v	$v = \frac{z}{E_m}$
4,5,6	1.0	$0.57692309 \times 10^{-3}$	$0.57692309 \times 10^{-3}$
7,8,9	2.0	$0.11538462 \times 10^{-2}$	$0.11538462 \times 10^{-2}$

(2) stress

element no.	numerical results	analytical solution
	t_{zz}	$t_{zz} = \frac{P_z}{A}$
1,2,3,4	1.0	1.0
	t_{rr}	t_{rr}
1	-6.8313×10^{-15}	0.0
2	1.2629×10^{-15}	0.0
3	3.5631×10^{-15}	0.0
4	2.8189×10^{-14}	0.0

TABLE 5.2 Numerical results from case (b) c.p. Analytical solution

		numerical results	analytical solution
(1) displacement (inch)			
nodal no.	coordinate r (inch)	u (inch)	$u = -\frac{v_m r}{E_m}$ (inch)
2,6	1.0	-0.17307693×10^{-3}	-0.17307693×10^{-3}
3,4,5	2.0	-0.34615386×10^{-3}	-0.34615386×10^{-3}
nodal no.	z (inch)	v	$v = \frac{z}{E_m}$
4,8	1.0	0.57692309×10^{-3}	0.57692309×10^{-3}
5,6,7	2.0	0.11538462×10^{-2}	0.11538462×10^{-2}
(2) stress			
element no.			
		t_{zz}	$t_{zz} = \frac{P_z}{A}$
1		1.0	1.0
		t_{rr}	t_{rr}
1		-6.3630×10^{-15}	0.0

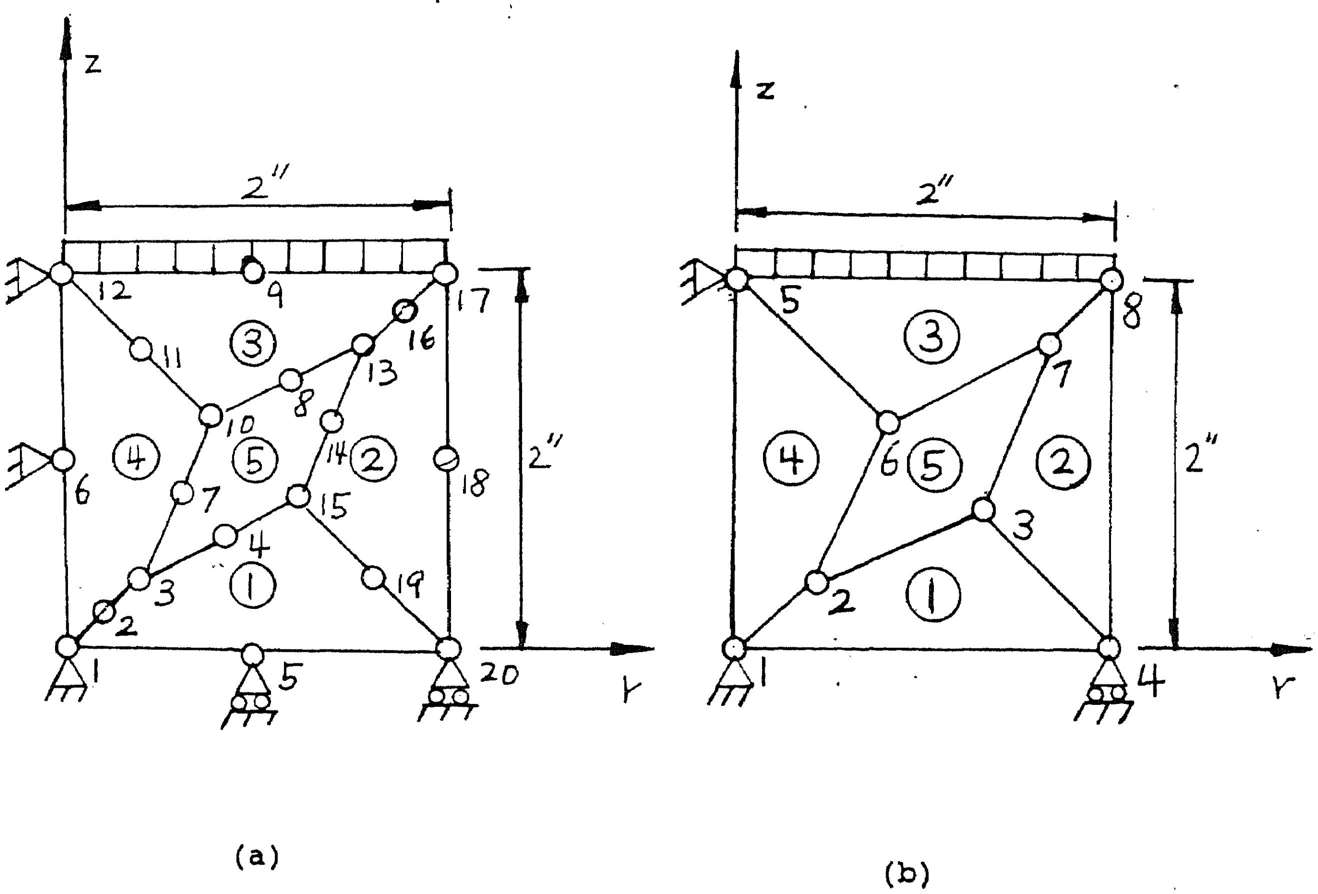

Fig. 5.3 Patch Test for 4- and 8- Node Two-dimensional Elements

TABLE 5.3 Numerical results from case (a)
c.p. Analytical solution

(1) displacement (inch)

nodal no.	coordinate r (inch)	u (inch) numerical results	$u = -\frac{v_m r}{E_m}$ (inch) analytical solution
2	0.2	$-0.34615386\text{x}10^{-4}$	$-0.34615386\text{x}10^{-4}$
3,11	0.4	$-0.69230771\text{x}10^{-4}$	$-0.69230771\text{x}10^{-4}$
4,10	0.8	$-0.13846154\text{x}10^{-3}$	$-0.13846154\text{x}10^{-3}$
5,9	1.0	$-0.17307693\text{x}10^{-3}$	$-0.17307693\text{x}10^{-3}$
7	0.6	$-0.10384616\text{x}10^{-3}$	$-0.10384616\text{x}10^{-3}$
8,15	1.2	$-0.20769231\text{x}10^{-3}$	$-0.20769231\text{x}10^{-3}$
13,19	1.6	$-0.27692309\text{x}10^{-3}$	$-0.27692309\text{x}10^{-3}$
14	1.4	$-0.24230770\text{x}10^{-3}$	$-0.24230770\text{x}10^{-3}$
16	1.8	$-0.31153847\text{x}10^{-3}$	$-0.31153847\text{x}10^{-3}$
17,18,20	2.0	$-0.34615386\text{x}10^{-3}$	$-0.34615386\text{x}10^{-3}$

TABLE 5.3 Numerical results from case (a) c.p. Analytical solution

(1) displacement (inch)

nodal no.	coordinate z (inch)	numerical results v (inch)	analytical solution $v = \frac{z}{E_m}$ (inch)
2	0.2	$0.11538462\text{x}10^{-3}$	0.11153846210^{-3}
3,19	0.4	$0.23076924\text{x}10^{-3}$	$0.23076924\text{x}10^{-3}$
4	0.6	$0.34615385\text{x}10^{-3}$	$0.34615385\text{x}10^{-3}$
6,18	1.0	$0.57692309\text{x}10^{-3}$	$0.57692309\text{x}10^{-3}$
7,15	0.8	$0.46153847\text{x}10^{-3}$	$0.46153847\text{x}10^{-3}$
8	1.4	$0.80769233\text{x}10^{-3}$	$0.80769233\text{x}10^{-3}$
9,12,17	2.0	$0.11538462\text{x}10^{-2}$	$0.11538462\text{x}10^{-2}$
10,14	1.2	$0.69230771\text{x}10^{-3}$	$0.69230771\text{x}10^{-3}$
11,13	1.6	$0.92307694\text{x}10^{-3}$	$0.92307694\text{x}10^{-3}$
16	1.8	$0.10384616\text{x}10^{-3}$	$0.10384616\text{x}10^{-3}$

(2) stress

element no.	numerical results	analytical solution
	t_{zz}	$t_{zz} = \frac{P_z}{A}$
1,2,3,4,5	1.0	1.0
	t_{rr}	t_{rr}
1	$2.4473\text{x}10^{-14}$	0.0
2	$-2.8082\text{x}10^{-14}$	0.0
3	$-8.9376\text{x}10^{-14}$	0.0
4	$-4.2917\text{x}10^{-14}$	0.0
5	$-3.6863\text{x}10^{-14}$	0.0

TABLE 5.3 Numerical results from case (b) c.p. Analytical solution

(1) displacement (inch)

nodal no.	coordinate r (inch)	numerical results u (inch)	analytical solution $u = - \frac{v_m r}{E_m}$ (inch)
2	0.4	$-0.69230771 \times 10^{-4}$	$-0.69230771 \times 10^{-4}$
3	1.2	$-0.20792310 \times 10^{-3}$	$-0.20792310 \times 10^{-3}$
4,8	2.0	$-0.34615386 \times 10^{-3}$	$-0.34615386 \times 10^{-3}$
6	0.8	$-0.13846186 \times 10^{-3}$	$-0.13846186 \times 10^{-3}$
7	1.6	$-0.27692309 \times 10^{-3}$	$-0.27692309 \times 10^{-3}$

nodal no.	z (inch)	v	$v = \frac{z}{E_m}$
2	0.4	$0.23076924 \times 10^{-3}$	$0.23076924 \times 10^{-3}$
3	0.8	$0.46153847 \times 10^{-3}$	$0.46153847 \times 10^{-3}$
5,8	2.0	$0.11538462 \times 10^{-2}$	$0.11538462 \times 10^{-2}$
6	1.2	$0.69230771 \times 10^{-3}$	$0.69230771 \times 10^{-3}$
7	1.6	$0.92307694 \times 10^{-3}$	$0.92307649 \times 10^{-3}$

specified in Fig. 5.3 the numerical results are summarized in Table 5.3 and Table 5.4 , for 4-node and 8-node patch elements, respectively. These results are found to be exactly identical with those obtained with rectangular element, confirming the applicability to an arbitrary geometry with the isoparametric finite element.

Example 2

The second example solved by the simplified two-dimensional method is an isotropic micropolar elastic cylinder with a semi-circular groove shown in the inset of Fig.5.4, subjected to an axisymmetric loading. The same example is solved in Chapter 4 by three-dimensional method. The material properties and geometrical dimensions are as given in Table 4.3 . Numerical results for different coupling factor N are listed in Table 5.4 , and ploted in Fig. 5.4. Again these results coincide exactly with the three-dimensional finite element results shown in Table 4.4. The experiement results from the literature for the classical material (N=0.0) are also plotted in the same figue which dased line. One can see excellent agreement.

Table 5.4 Numerical results of stress concentration factor vs r/d. (isotropic for force stress and couple stress)

ratio r/d	N=0.90	N=0.75	N=0.50	N=0.25	N=0.00
0.05	1.75	1.92	2.26	2.58	2.73
0.08	1.65	1.81	2.11	2.40	2.53
0.10	1.60	1.75	2.03	2.28	2.39
0.13	1.53	1.67	1.91	2.13	2.22
0.17	1.47	1.59	1.80	1.98	2.04
0.20	1.43	1.54	1.73	1.88	1.93
0.23	1.40	1.50	1.66	1.79	1.84
0.27	1.36	1.45	1.60	1.70	1.74
0.30	1.33	1.42	1.55	1.65	1.68
0.35	1.28	1.35	1.47	1.56	1.59
0.40	1.25	1.31	1.42	1.49	1.52
0.45	1.23	1.29	1.38	1.44	1.46
0.50	1.20	1.26	1.34	1.40	1.42

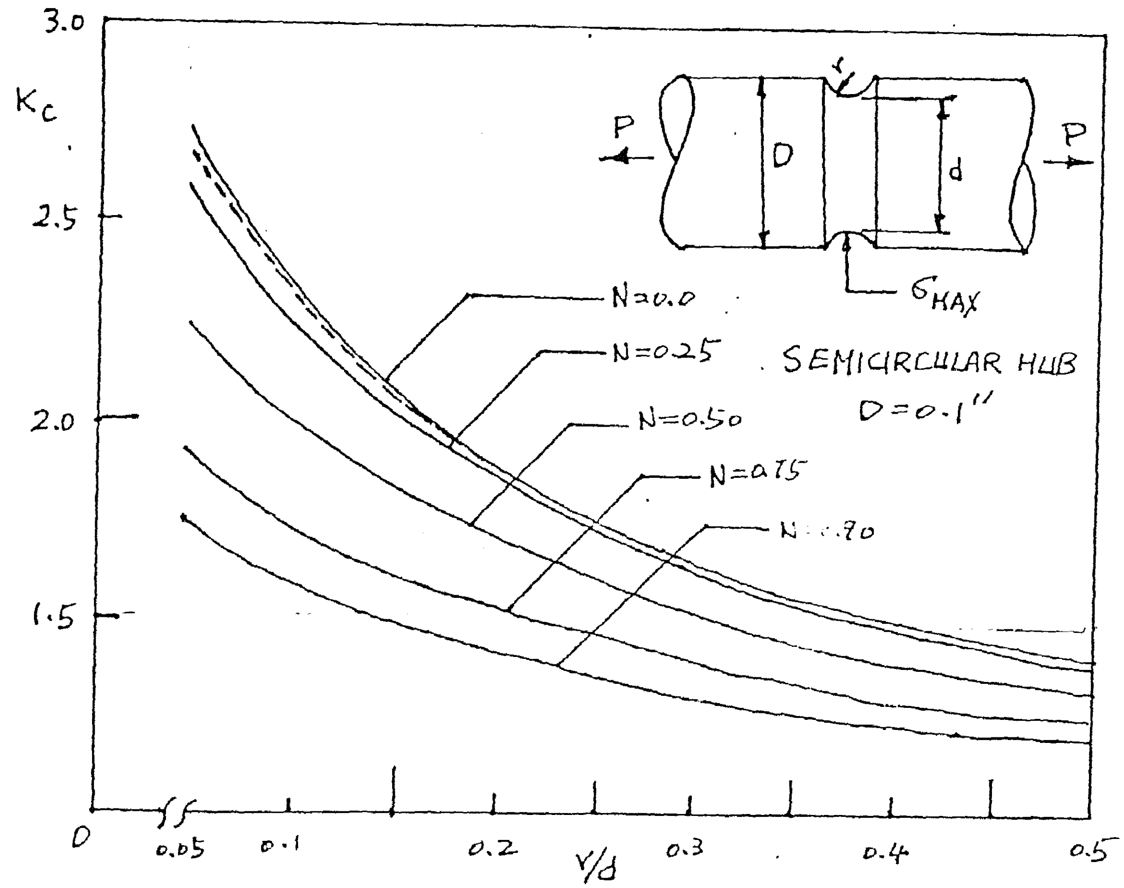

Fig. 5.4 Stress Concentraction Factor, K_c for a Round Tension Bar with a Semi-circular Groove in Three-dimensional case (Orthotropic for Force Stress and Couple Stress)

Example 3

The finite element method is applied to solve a micropolar elastic solid with an orthotropic for force stress but isotropic for couple-stress. The geometry and loading conditions are the same as in example 2. To generate the new material properties, classical technical constant of graphite-epoxy composite is chosen.

$$E_r = 20\text{x}10^6 \text{ psi} \quad E_z = 1\text{x}10^6$$
$$G_{rz} = 5.14035\text{x}10^6 \text{ psi}$$

Based on these classical material properties, some of the material parameters for force stress are chosen as:

$$A^{rr}_{\theta\theta} = A^{rr}_{zz} = 2.006\text{x}10^7 \text{ psi}$$
$$A^{rr}_{rr} = A^{\theta\theta}_{\theta\theta} = A^{zz}_{zz} = 2.508\text{x}10^5 \text{ psi}$$
$$A^{r\theta}_{r\theta} = 1.003\text{x}10^6 \text{ psi}$$

and the rest are list in Table 5.5 for different coupling factor N. Parameters for couple stress are fixed to

$$B^{rr}_{rr} = B^{\theta\theta}_{\theta\theta} = B^{zz}_{zz} = 1.4278750\text{x}10^3 \text{ lb}$$
$$B^{rr}_{\theta\theta} = B^{rr}_{zz} = 0.0 \text{ lb}$$

That is, the characteristic length is fixed to $8.333\text{x}10^{-3}$ in. for all N.

The numerical results are summarized in Table 5.6 and also plotted in Fig. 5.5. It is shown that as the radius ratio increase the stress concentration factor decrease, a similar trend as found in isotropic cases. However, the stress

Table 5.5 SOME OF MATERIALS PARAMETERS TO FORCE STRESS OF ANISOTROPIC MATERIALS

N \ A	$A^{r\theta}_{r\theta}$	$A^{r\theta}_{\theta r}$	$A^{\theta\theta}_{\theta\theta}$
N=0.90	3.2258064×10^{7} (psi)	-2.000×10^{7} (psi)	3.2258064×10^{7} (psi)
N=0.75	1.440×10^{7} (psi)	-1.800×10^{6} (psi)	1.440×10^{7} (psi)
N=0.50	9.800×10^{6} (psi)	4.900×10^{6} (psi)	9.800×10^{6} (psi)
N=0.25	5.760×10^{6} (psi)	5.04×10^{6} (psi)	5.760×10^{6} (psi)
N=0.00	5.14035×10^{6} (psi)	5.14035×10^{6} (psi)	5.14035×10^{6} (psi)

Table 5.6 Numerical results of stress concentration factor vs r/d. (othotropic for force stress isotropic isotropic for couple stress)

ratio r/d	N=0.90	N=0.75	N=0.50	N=0.25	N=0.00
0.05	1.77	1.90	2.24	3.10	3.73
0.08	1.67	1.81	2.14	3.08	3.81
0.10	1.62	1.76	2.08	3.02	3.79
0.17	1.48	1.61	1.89	2.77	3.59
0.23	1.40	1.51	1.77	2.58	3.35
0.30	1.34	1.43	1.66	2.38	3.06
0.40	1.27	1.35	1.55	2.15	2.68
0.50	1.23	1.30	1.46	1.97	2.36

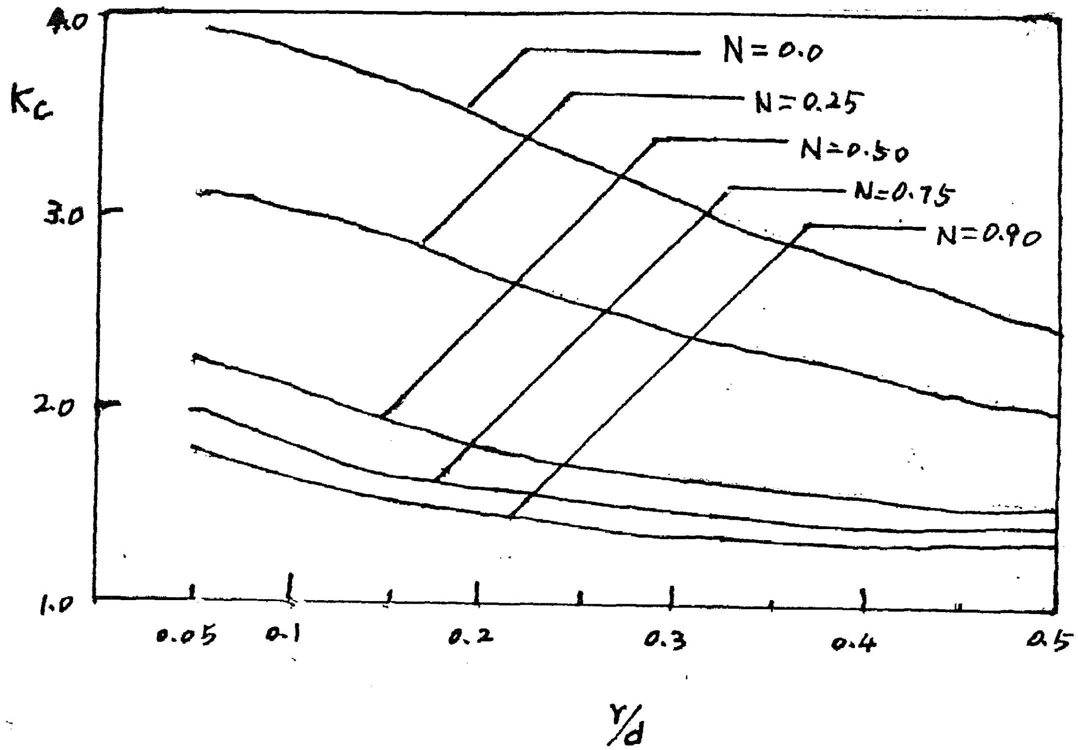

Fig. 5.5 Stress Concentraction Factor, K_C for a Round Tension Bar with a Semi-circular Groove in Two-dimensional case. (Orthotropic for Force Stress but Isotropic for Couple Stress)

concentraction factor for a fixed N are higher than that in the isotropic cases.

CHAPTER VI

CONCLUSION

6.1 Concluding Remarks

In this study isoparametric finite element method for isotropic and orthotropic axisymmetric micropolar (Cosserat) elastic solids was developed. Both 8- and 20-node elements were employed for solving general three-dimensional problems, and both 4- and 8-node elements were used for two-dimensional cases. Both three-dimensional and two-dimensional finite element formulation for cylindrical coordinate system were obtained. Corresponding Fortran programs were developed to solve several two-dimensional and three-dimensional problems for micropolar elastics solids.

The validity and compatibility of the developed three-dimensional finite element programs were established by comparing the numerical results with available analytical solutions for a cylindrical Cosserat solid subjected to a simple tension. The numerical results for displacements were found to be identical with the analytical solutions, and the displacements was independent of the coupling factor. The flexibility and capability of the three-dimensional finite element method are then demonstrated by solving the stress concentration for a classical round tension bar with a semi-circular groove subjected to an axial symmetrical load. For the first time, to the best of the author's knowledge, the effects of various coupling factor on stress concentration factor were obtained for the micropolar round tension bar with a semi-circular groove subjected to an axial symmetrical load. The stress concentration factor on the micropolar elastic solids was smaller than that in classical elastic solids for the same radius ratio. The stress concentration factor was also found to decrease monotonically as radius ratio increases. Again the validity and compatibility are confirmed by the excellent fit between the available experimental data for classical materials and the numerical results at zero coupling factor, as microelasticity reduces to a classical case when the coupling factor N is equal to zero. In the classical case, the effects of microrotation is vanished.

The three-dimensional finite element formulations can be simplified to a two-dimensional formulation in the case of an asymmetric object when subjected to an axisymmetric load, and can result in significant savings in CPU time. The simplified two-dimensional programs with both 4- and 8-node elements are first applied to solve the simple tension of a cylindrical Cosserat solid, which has also been solved by the three-dimensional programs in this study. It is found that the displacements at the four corner-nodes of the 8-node element coincides exactly with those obtained by the 4-node element, indicating the adequacy of the 4-node element approach, which is more economic in CPU time. In addition, the 8-node two-dimensional simulation results are also identical with the three-dimensional simulation results obtained with a 20-node three-dimensional element in this study; and the 4-node two-dimensional numerical results are identical with those with 8-node three-dimensional results in this study.

The simplified programs are also applied to solve a micropolar elastic solid with a semi-circular-groove subjecting to an axial symmetrical loading. Again the results coincide exactly with the three-dimensional finite element results, as well as the experimental data for the special case when the coupling factor is reduced to zero. The applicability of the proposed finite element method to an arbitrary geometry is also verified by the patch tests performed on a two-dimensional example. The results are found to be exactly identical with those obtained with the rectangular elements. Finally, the developed two-dimensional finite element formulation is applied to investigate the dependency of material properties for a material with an orthotropic for stress but isotropic for couple stress. It was verified that the new material depicts a similar trend as the isotropic material with the stress concentration factor decreases with increasing radius ratio. However, the stress concentration factor is larger for any particular coupling factor.

6.2 Future Study

Another interesting topic is the boundary value problem for the stress concentration at spherical cavity in a field of isotropic tension. This is solved by Bleustein [20] using the theory developed by Mindlin. It is found that the stress concentration factor is larger than the 3/2 of classical elasticity for a wide range of material properties and ratio of radius of cavity to a length parameter of the material with a critical ratio, nearly independent of the remaining material properties, for which the stress concentration factor is a maximum. Bleustein found that for the classical theory of elasticity the stress concentration factor is a constant, independent of material properties and the radius of cavity. However, the solution obtained by the theory of micropolar elasticity shows a stress concentration factor which depends on both material properties and radius. The stress concentration factor is higher than the classical value of 3/2. It is generally true that when the micropolar behavior is considered stress concentration factor becomes smaller than the classical elasticity. Therefore identifying the condition under which the stress concentration factor is larger than the classical elasticity is an interesting future study.

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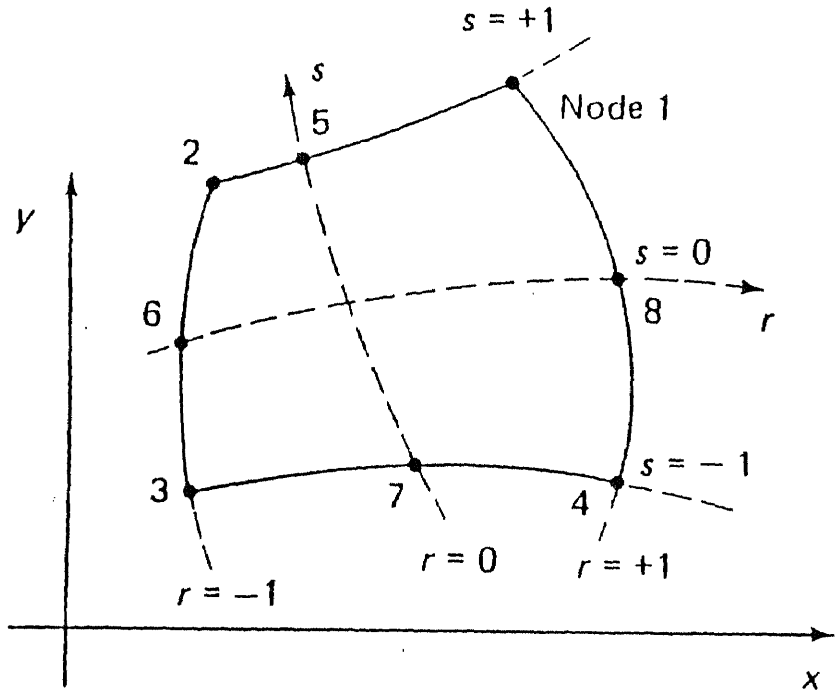
APPENDIX I

Interpolation Functions of Four to Eight Variable-Number-Nodes Two-Dimensional Element



(a) Four to 8 variable-number-nodes two-dimensional element

		Include only if node i is defined			
		$i = 5$	$i = 6$	$i = 7$	$i = 8$
$h_1 =$	$\frac{1}{4}(1+r)(1+s)$	$-\frac{1}{2}h_5$			$-\frac{1}{2}h_8$
$h_2 =$	$\frac{1}{4}(1-r)(1+s)$	$-\frac{1}{2}h_5$	$-\frac{1}{2}h_6$		
$h_3 =$	$\frac{1}{4}(1-r)(1-s)$		$-\frac{1}{2}h_6$	$-\frac{1}{2}h_7$	
$h_4 =$	$\frac{1}{4}(1+r)(1-s)$			$-\frac{1}{2}h_7$	$-\frac{1}{2}h_8$
$h_5 =$	$\frac{1}{2}(1-r^2)(1+s)$				
$h_6 =$	$\frac{1}{2}(1-s^2)(1-r)$				
$h_7 =$	$\frac{1}{2}(1-r^2)(1-s)$				
$h_8 =$	$\frac{1}{2}(1-s^2)(1+r)$				

APPENDIX II Program Listing for Two and Three-dimensional Isoparametric Finite Elements

```
C                   3-D MICROPOLAR FINITE ELEMENT METHOD
C
      IMPLICIT REAL*8(A-H,O-Z)
      INTEGER*4 ITIM
      INTEGER*2 TYPE,CODE
      COMMON/B01/B0(200,9,48),B1(200,9,48),INXY(200),ID(300)
      COMMON/XYZ/NNP(200,8),X(300),Y(300),Z(300)
      COMMON/CSTRN/KSTRN(1000),KSTRT(1000)
      COMMON/MAT/D0(9,9),D1(9,9),A11,A12,A22,A23,B11
      COMMON/AIN/NB,NC,ND,NE,NN,TH,NINT,R,S,T,DET
      COMMON/FSK/A(350000),V(1800),EK(48,48),MAXA(1801),NEIRE
      READ(5,100) NE
  100 FORMAT(I4)
      DO 10 N=1,NE
      READ(5,200)  (NNP(N,I),I=1,8),INXY(N)
  200 FORMAT(4X,8I4,3X,I5)
   10 CONTINUE
      READ(5,100) NN
      DO 20 N=1,NN
      READ(5,300) X(N),Y(N),Z(N)
   20 CONTINUE
  300 FORMAT(4X,3F20.10)
      READ(5,100) NC
      DO 30 N=1,NC
      READ(5,400) KSTRN(N),KSTRT(N)
  400 FORMAT(4X,I4,i7)
   30 CONTINUE
      READ(5,550)NINT
  550 FORMAT(I5)
      READ(5,600)A11,A12,A22,A23,B11
600   FORMAT(5E20.7)
      WRITE(6,610)A11,A12,A22,A23,B11
610   FORMAT(2X,'A11=',E20.7,'A12=',E20.7,'A22=',
     *E20.7,'A23=',E20.7,'B11=',E20.7)
      ND=NN*6
      NB=0
      DO 40 IB=1,NE
      IMAX=MAX0(NNP(IB,1),NNP(IB,2),NNP(IB,3),NNP(IB,4),NNP(IB,5),
     *NNP(IB,6),NNP(IB,7),NNP(IB,8))
      IMIN=MIN0(NNP(IB,1),NNP(IB,2),NNP(IB,3),NNP(IB,4),NNP(IB,5),
     *NNP(IB,6),NNP(IB,7),NNP(IB,8))
      NBCHEK=(IMAX-IMIN+1)*6
      IF(NBCHEK.GT.NB) NB=NBCHEK
      IF(NB .GT. 300) GO TO 99
   40 CONTINUE
      SCALE=1.0D0
      DO 41 I=1,NN
      X(I)=X(I)*SCALE
      Y(I)=Y(I)*SCALE
   41 Z(I)=Z(I)*SCALE
```

```
         WRITE(6,700)
     700 FORMAT(1H1)
         WRITE(6,800)
     800 FORMAT(3X,'*****************************************************
    *****************')
      WRITE(6,900)
  900 FORMAT(3X,'*',68X,'*')
      WRITE(6,1000)
 1000 FORMAT(3X,'*',13X,'3-D ORTHOTROPIC MICROPOLAR STRESS ANALYSIS',
     *        13X,'*')
      WRITE(6,1050)
 1050 FORMAT(3X,'*',28X,' SKYLINEMICRO',27X,'*')
      WRITE(6,900)
      WRITE(6,800)
      WRITE(6,1100)
 1100 FORMAT(/20X,'**** DISCRETIZATION NUMBER ****')
      WRITE(6,1200)
 1200 FORMAT(/13X,'ELEMT.#',3X,'NODES.#',3X,'CONSTR.#',3X,'THIKNES',
     *        3X,'BAND-WIDTH',3X,'GAUSS NUMERICAL INTEGRATION ORDER')
      WRITE(6,1300) NE,NN,NC,TH,NB,NINT
 1300 FORMAT(13X,I4,7X,I3,7X,I3,5X,E10.3,5X,I5,19X,I2)
      WRITE(6,1700)
 1700 FORMAT(//21X,'**** ELEMENT-NODE CONNECTION ****')
      WRITE(6,1800)
 1800 FORMAT(/3X,'ELM NP1 NP2 NP3 NP4 NP5 NP6 NP7 NP8    IXY   ELM
     * NP1 NP2 NP3 NP4 NP5 NP6 NP7 NP8    IXY')
      DO 45 I=1,NE
   45 ID(I)=I
      LINE=NE/2
      IRESID=NE-2*LINE
      DO 50 N=1,LINE
   50 WRITE(6,1900) (ID(2*(N-1)+I),NNP(2*(N-1)+I,1),NNP(2*(N-1)+I,2),
     *NNP(2*(N-1)+I,3),NNP(2*(N-1)+I,4),NNP(2*(N-1)+I,5),
     *NNP(2*(N-1)+I,6),NNP(2*(N-1)+I,7),NNP(2*(N-1)+I,8),
     *INXY(2*(N-1)+I),I=1,2)
 1900 FORMAT(2(2X,I4,1X,I3,1X,I3,1X,I3,1X,I3,1X,I3,1X,I3,1X,I3,1X,I3,
     *1X,I6))
      IF(IRESID .EQ. 0) GO TO 56
      WRITE(6,1900) (ID(2*LINE+I),NNP(2*LINE+I,1),NNP(2*LINE+I,2),
     *NNP(2*LINE+I,3),NNP(2*LINE+I,4),NNP(2*LINE+I,5),NNP(2*LINE+I,6),
     *NNP(2*LINE+I,7),NNP(2*LINE+I,8),INXY(2*LINE+I),I=1,IRESID)
   56 WRITE(6,2000)
 2000 FORMAT(//24X,'**** NODAL COORDINATE ****')
      WRITE(6,2100)
 2100 FORMAT(/1X,'NODE',5X,'X',8X,'Y',8X,'Z',4X,'NODE',5X,'X',8X,'Y',
     * 8X,'Z',4X,'NODE',5X,'X',8X,'Y',8X,'Z',4X,'NODE',5X,'X',
     * 8X,'Y',8X,'Z')
      DO 55 I=1,NN
   55 ID(I)=I
      LINE=NN/4
      IRESID=NN-4*LINE
      DO 60 N=1,LINE
   60 WRITE(6,2200) (ID(4*(N-1)+I),X(4*(N-1)+I),Y(4*(N-1)+I),
     * Z(4*(N-1)+I),I=1,4)
```

```
 2200 FORMAT(4(2X,I3,1X,F8.3,1X,F8.3,1X,F8.3))
      IF(IRESID .EQ. 0) GO TO 57
      WRITE(6,2200) (ID(4*LINE+I),X(4*LINE+I),Y(4*LINE+I),
     * Z(4*LINE+I),I=1,IRESID)
   57 WRITE(6,2300)
 2300 FORMAT(//27X,'**** CONSTRAINT ****')
      WRITE(6,2400)
 2400 FORMAT(/24X,'CNSTRND-NODE',2X,'CNSTRND-CODE')
      WRITE(6,2500) (KSTRN(N),KSTRT(N),N=1,NC)
 2500 FORMAT(28X,I3,10X,I6)
                           CALL STSTIF
                           CALL LOADER
                           CALL COLSOL
                           CALL STRESS
   99 WRITE(6,999) NB,IB
  999 FORMAT('*****STOP NB=',I5,'  AT ELEMENT=',I5)
      STOP
                                          END
C *********************************************************************
C *                                                                   *
                        SUBROUTINE STSTIF
C *                                                                   *
C *********************************************************************
      IMPLICIT REAL*8(A-H,O-Z)
      COMMON/CSTRN/KSTRN(1000),KSTRT(1000)
      COMMON/XYZ/NNP(200,8),X(300),Y(300),Z(300)
      COMMON/AIN/NB,NC,ND,NE,NN,TH,NINT,R,S,T,DET
      COMMON/FSK/A(350000),V(1800),EK(48,48),MAXA(1801),NEIRE
      DIMENSION MHT(1800)
      NBND=NB*ND
      DO 10 I=1,NBND
10    A(I)=0.0D0
      DO 20 NEIRE=1,NE
                                  CALL ELSTIF
      DO 20 INC=1,8
      INOC=NNP(NEIRE,INC)
      IBC=(NNP(NEIRE,INC)-1)*6
      DO 20 IDC=1,6
      ICEL=(INC-1)*6+IDC
      ICST=IBC+IDC
      IDI=0
      IF(ICST.GT.NB)IDI=ICST-NB
      IVC=(ICST-1)*NB
      DO 18 INR=1,8
      INOR=NNP(NEIRE,INR)
      IF(INOC.LT.INOR)GO TO 18
      IBR=(NNP(NEIRE,INR)-1)*6
      IDVC=IDC
      IF(INOC.GT.INOR)IDVC=6
      DO 15 IDR=1,IDVC
      IREL=(INR-1)*6+IDR
      IVV=IVC+IBR+IDR-IDI
      SS=A(IVV)+EK(IREL,ICEL)
15    A(IVV)=SS
```

```
18    CONTINUE
20    CONTINUE
      DO 140 N=1,NC
      IRCX=KSTRN(N)*6-5
      IRCY=KSTRN(N)*6-4
      IRCZ=KSTRN(N)*6-3
      IRCRX=KSTRN(N)*6-2
      IRCRY=KSTRN(N)*6-1
      IRCRZ=KSTRN(N)*6
      KCHK=KSTRT(N)
      IF(KCHK.LT.100000) GO TO 40
      ICB=ND-IRCX+1
      IF(ICB.GT.NB)ICB=NB
      DO 35 I=1,ICB
      IDI=0
      IRCXX=IRCX+I-1
      IF(IRCXX.GT.NB)IDI=IRCXX-NB
      IXV=(IRCX-2+I)*NB+IRCX-IDI
      IF(I.EQ.1)GO TO 25
      A(IXV)=0.0D0
      GO TO 35
25    A(IXV)=1.0D0
      IF(IRCX.EQ.1)GO TO 35
      DO 30 J=1,IRCX-IDI-1
      IXXV=IXV-J
30    A(IXXV)=0.0D0
35    CONTINUE
      KCHK=KCHK-100000
40    IF(KCHK.LT.010000) GO TO 60
      ICB=ND-IRCY+1
      IF(ICB.GT.NB)ICB=NB
      DO 55 I=1,ICB
      IDI=0
      IRCYY=IRCY+I-1
      IF(IRCYY.GT.NB)IDI=IRCYY-NB
      IYV=(IRCY-2+I)*NB+IRCY-IDI
      IF(I.EQ.1)GO TO 45
      A(IYV)=0.0D0
      GO TO 55
45    A(IYV)=1.0D0
      IF(IRCY.EQ.1)GO TO 55
      DO 50 J=1,IRCY-IDI-1
      IYYV=IYV-J
50    A(IYYV)=0.0D0
55    CONTINUE
      KCHK=KCHK-10000
60    IF(KCHK.LT.001000) GO TO 80
      ICB=ND-IRCZ+1
      IF(ICB.GT.NB)ICB=NB
      DO 75 I=1,ICB
      IDI=0
      IRCZZ=IRCZ+I-1
      IF(IRCZZ.GT.NB)IDI=IRCZZ-NB
      IZV=(IRCZ-2+I)*NB+IRCZ-IDI
```

```
      IF(I.EQ.1)GO TO 65
      A(IZV)=0.0D0
      GO TO 75
65    A(IZV)=1.0D0
      IF(IRCZ.EQ.1)GO TO 75
      DO 70 J=1,IRCZ-IDI-1
      IZZV=IZV-J
70    A(IZZV)=0.0D0
75    CONTINUE
      KCHK=KCHK-1000
80    IF(KCHK.LT.000100) GO TO 100
      ICB=ND-IRCRX+1
      IF(ICB.GT.NB)ICB=NB
      DO 95 I=1,ICB
      IDI=0
      IRCRXX=IRCRX+I-1
      IF(IRCRXX.GT.NB)IDI=IRCRXX-NB
      IRXV=(IRCRX-2+I)*NB+IRCRX-IDI
      IF(I.EQ.1)GO TO 85
      A(IRXV)=0.0D0
      GO TO 95
85    A(IRXV)=1.0D0
      IF(IRCRX.EQ.1)GO TO 95
      DO 90 J=1,IRCRX-IDI-1
      IRXXV=IRXV-J
90    A(IRXXV)=0.0D0
95    CONTINUE
      KCHK=KCHK-100
100   IF(KCHK.LT.000010) GO TO 120
      ICB=ND-IRCRY+1
      IF(ICB.GT.NB)ICB=NB
      DO 115 I=1,ICB
      IDI=0
      IRCRYY=IRCRY+I-1
      IF(IRCRYY.GT.NB)IDI=IRCRYY-NB
      IRYV=(IRCRY-2+I)*NB+IRCRY-IDI
      IF(I.EQ.1)GO TO 105
      A(IRYV)=0.0D0
      GO TO 115
105   A(IRYV)=1.0D0
      IF(IRCRY.EQ.1)GO TO 115
      DO 110 J=1,IRCRY-IDI-1
      IRYYV=IRYV-J
110   A(IRYYV)=0.0D0
115    CONTINUE
      KCHK=KCHK-10
120   IF(KCHK.LT.000001) GO TO 140
      ICB=ND-IRCRZ+1
      IF(ICB.GT.NB)ICB=NB
      DO 135 I=1,ICB
      IDI=0
      IRCRZZ=IRCRZ+I-1
      IF(IRCRZZ.GT.NB)IDI=IRCRZZ-NB
      IRZV=(IRCRZ-2+I)*NB+IRCRZ-IDI
```

```
      IF(I.EQ.1)GO TO 125
      A(IRZV)=0.0D0
      GO TO 135
125   A(IRZV)=1.0D0
      IF(IRCRZ.EQ.1)GO TO 135
      DO 130 J=1,IRCRZ-IDI-1
      IRZZV=IRZV-J
130   A(IRZZV)=0.0D0
135   CONTINUE
140   CONTINUE
      DO 160 I=1,ND
      IDI=0
      IF(I.GT.NB)IDI=I-NB
      IIV=(I-1)*NB
      DO 150 J=1,I
      IF(A(IIV+J).EQ.0.0D0)GO TO 150
      MHT(I)=I-J-IDI
      GO TO 160
150   CONTINUE
160   CONTINUE
      NM=ND+1
      DO 170 I=1,NM
170   MAXA(I)=0
      MAXA(1)=1
      MAXA(2)=2
      IF(ND.EQ.1)GO TO 190
      DO 180 I=2,ND
180   MAXA(I+1)=MAXA(I)+MHT(I)+1
190   NWK=MAXA(ND+1)-MAXA(1)
      IAN=0
      DO 200 I=1,ND
      ICK=MAXA(I+1)-MAXA(I)
      IDI=0
      IF(I.GT.NB)IDI=I-NB
      INBB=(I-1)*NB+I-IDI
      DO 200 II=1,ICK
      IAN=IAN+1
      IAV=INBB-II+1
      A(IAN)=A(IAV)
200   CONTINUE
      DO 210 I=NWK+1,NBND
210   A(I)=0.0D0
      RETURN
                                          END
C ***********************************************************************
C *                                                                     *
                         SUBROUTINE ELSTIF
C *                                                                     *
C ***********************************************************************
      IMPLICIT REAL*8(A-H,O-Z)
      COMMON/FSK/A(350000),V(1800),EK(48,48),MAXA(1801),NEIRE
      COMMON/AIN/NB,NC,ND,NE,NN,TH,NINT,R,S,T,DET
      COMMON/B01/B0(200,9,48),B1(200,9,48),INXY(200),ID(300)
      COMMON/XYZ/NNP(200,8),X(300),Y(300),Z(300)
```

```
      COMMON/MAT/D0(9,9),D1(9,9),A11,A12,A22,A23,B11
      DIMENSION XG(4,4),WGT(4,4),D0B0(9),D1B1(9)
      COMMON V1(100)
       COMMON H(8)
      DATA XG/0.0D0,0.0D0,0.0D0,0.0D0,-.5773502691896D0,
     *.5773502691896D0,0.0D0,0.0D0,-.7745966692415D0,
     *0.0D0,.7745966692415D0,0.0D0,-.8611363115941D0,
     *-.3399810435849D0,.3399810435849D0,.8611363115941D0/
      DATA WGT/2.0D0,0.0D0,0.0D0,0.0D0,1.0D0,1.0D0,0.0D0,
     *0.0D0,.5555555555556D0,.8888888888889D0,
     *.5555555555556D0,0.0D0,.3478548451375D0,
     *.6521451548625D0,.6521451548625D0,
     *.3478548451375D0/
      DO 43 I=1,9
      DO 43 J=1,9
      D0(I,J)=0.0D0
43    D1(I,J)=0.0D0
      D0(1,1)=A11
      D0(1,2)=A12
      D0(1,3)=A12
      D0(2,1)=A12
      D0(2,2)=A11
      D0(2,3)=A12
      D0(3,1)=A12
      D0(3,2)=A12
      D0(3,3)=A11
      D0(4,4)=A22
      D0(4,5)=A23
      D0(5,4)=A23
      D0(5,5)=A22
      D0(6,6)=A22
      D0(6,7)=A23
      D0(7,6)=A23
      D0(7,7)=A22
      D0(8,8)=A22
      D0(8,9)=A23
      D0(9,8)=A23
      D0(9,9)=A22
      DO 46 I=1,9
      D1(I,I)=B11
46    CONTINUE
      DO 39 I=1,8
      V(I)=0.0D0
39    CONTINUE
      DO 83 LX=1,NINT
      R=XG(LX,NINT)
      DO 83 LY=1,NINT
      S=XG(LY,NINT)
      T=1.0D0
      CALL STDM
      WS=WGT(LX,NINT)*WGT(LY,NINT)*DET
      DO 89 I=1,8
      V1(I)=V1(I)+H(I)*WS
       dO 199 J=1,8
```

```
      WRITE(6,99)V1(J)
  99    FORMAT(2X,F15.7)
  199    CONTINUE
 89     CONTINUE
 83     CONTINUE
        DO 30 I=1,48
        DO 30 J=1,48
 30     EK(I,J)=0.0D0
        DO 80 LX=1,NINT
        R=XG(LX,NINT)
        DO 80 LY=1,NINT
        S=XG(LY,NINT)
        DO 80 LZ=1,NINT
        T=XG(LZ,NINT)
        CALL STDM
        WT=WGT(LX,NINT)*WGT(LY,NINT)*WGT(LZ,NINT)*DET
        DO 70 J=1,48
        DO 40 K=1,9
        D0B0(K)=0.0D0
        D1B1(K)=0.0D0
        DO 40 L=1,9
        D0B0(K)=D0B0(K)+D0(K,L)*B0(NEIRE,L,J)
 40     D1B1(K)=D1B1(K)+D1(K,L)*B1(NEIRE,L,J)
        DO 60 I=J,48
        STIFF=0.0D0
        DO 50 L=1,9
 50     STIFF=STIFF+B0(NEIRE,L,I)*D0B0(L)+B1(NEIRE,L,I)*D1B1(L)
 60     EK(I,J)=EK(I,J)+STIFF*WT
 70     CONTINUE
 80     CONTINUE
        DO 90 J=1,48
        DO 90 I=J,48
 90     EK(J,I)=EK(I,J)
        IF(INXY(NEIRE).EQ.00001)GO TO 100
        IF(INXY(NEIRE).EQ.00010)GO TO 110
        IF(INXY(NEIRE).EQ.00011)GO TO 120
        IF(INXY(NEIRE).EQ.00100)GO TO 130
        IF(INXY(NEIRE).EQ.00101)GO TO 140
        IF(INXY(NEIRE).EQ.00110)GO TO 150
        IF(INXY(NEIRE).EQ.00111)GO TO 160
        IF(INXY(NEIRE).EQ.01000)GO TO 170
        IF(INXY(NEIRE).EQ.01001)GO TO 180
        IF(INXY(NEIRE).EQ.01010)GO TO 190
        IF(INXY(NEIRE).EQ.01011)GO TO 200
        IF(INXY(NEIRE).EQ.01100)GO TO 210
        IF(INXY(NEIRE).EQ.01101)GO TO 220
        IF(INXY(NEIRE).EQ.01110)GO TO 230
        IF(INXY(NEIRE).EQ.01111)GO TO 240
        IF(INXY(NEIRE).EQ.10000)GO TO 250
        IF(INXY(NEIRE).EQ.10001)GO TO 260
        IF(INXY(NEIRE).EQ.10010)GO TO 270
        IF(INXY(NEIRE).EQ.10011)GO TO 280
        IF(INXY(NEIRE).EQ.10100)GO TO 290
        R= 0.0D0
```

```
      S= 0.0D0
      T= 0.0D0
      GO TO 300
100   R= 1.0D0
      S= 1.0D0
      T= 1.0D0
      GO TO 300
110   R=-1.0D0
      S= 1.0D0
      T= 1.0D0
      GO TO 300
120   R=-1.0D0
      S=-1.0D0
      T= 1.0D0
      GO TO 300
130   R= 1.0D0
      S=-1.0D0
      T= 1.0D0
      GO TO 300
140   R= 1.0D0
      S= 1.0D0
      T=-1.0D0
      GO TO 300
150   R=-1.0D0
      S= 1.0D0
      T=-1.0D0
      GO TO 300
160   R=-1.0D0
      S=-1.0D0
      T=-1.0D0
      GO TO 300
170   R= 1.0D0
      S=-1.0D0
      T=-1.0D0
      GO TO 300
180   R= 0.0D0
      S= 1.0D0
      T= 1.0D0
      GO TO 300
190   R=-1.0D0
      S= 0.0D0
      T= 1.0D0
      GO TO 300
200   R= 0.0D0
      S=-1.0D0
      T= 1.0D0
      GO TO 300
210   R= 1.0D0
      S= 0.0D0
      T= 1.0D0
      GO TO 300
220   R= 0.0D0
      S= 1.0D0
      T=-1.0D0
```

```
      GO TO 300
  230 R=-1.0D0
      S= 0.0D0
      T=-1.0D0
      GO TO 300
  240 R= 0.0D0
      S=-1.0D0
      T=-1.0D0
      GO TO 300
  250 R= 1.0D0
      S= 0.0D0
      T=-1.0D0
      GO TO 300
  260 R= 1.0D0
      S= 1.0D0
      T= 0.0D0
      GO TO 300
  270 R=-1.0D0
      S= 1.0D0
      T= 0.0D0
      GO TO 300
  280 R=-1.0D0
      S=-1.0D0
      T= 0.0D0
      GO TO 300
  290 R= 1.0D0
      S=-1.0D0
      T= 0.0D0
  300 CALL STDM
      RETURN
      END
C ********************************************************************
C *                                                                  *
                        SUBROUTINE STDM
C *                                                                  *
C ********************************************************************
      IMPLICIT REAL*8(A-H,O-Z)
      COMMON/FSK/A(350000),V(1800),EK(48,48),MAXA(1801),NEIRE
      COMMON/AIN/NB,NC,ND,NE,NN,TH,NINT,R,S,T,DET
      COMMON/B01/B0(200,9,48),B1(200,9,48),INXY(200),ID(300)
      COMMON/XYZ/NNP(200,8),X(300),Y(300),Z(300)
      DIMENSION P(8,3),XYZ(8,3),XJ(3,3),XJI(3,3)
      COMMON H(8)
       I=NNP(NEIRE,1)
       J=NNP(NEIRE,2)
       K=NNP(NEIRE,3)
       L=NNP(NEIRE,4)
      IJ=NNP(NEIRE,5)
      JK=NNP(NEIRE,6)
      KL=NNP(NEIRE,7)
      LI=NNP(NEIRE,8)
      XYZ(1,1)=X( I)
      XYZ(2,1)=X( J)
      XYZ(3,1)=X( K)
```

```
XYZ(4,1)=X( L)
XYZ(5,1)=X(IJ)
XYZ(6,1)=X(JK)
XYZ(7,1)=X(KL)
XYZ(8,1)=X(LI)
XYZ(1,2)=Y( I)
XYZ(2,2)=Y( J)
XYZ(3,2)=Y( K)
XYZ(4,2)=Y( L)
XYZ(5,2)=Y(IJ)
XYZ(6,2)=Y(JK)
XYZ(7,2)=Y(KL)
XYZ(8,2)=Y(LI)
XYZ(1,3)=Z( I)
XYZ(2,3)=Z( J)
XYZ(3,3)=Z( K)
XYZ(4,3)=Z( L)
XYZ(5,3)=Z(IJ)
XYZ(6,3)=Z(JK)
XYZ(7,3)=Z(KL)
XYZ(8,3)=Z(LI)
RP=1.0D0+R
RM=1.0D0-R
SP=1.0D0+S
SM=1.0D0-S
TP=1.0D0+T
TM=1.0D0-T
H(7)=  0.125D0*RP*SP*TP
H(8)=  0.125D0*RM*SP*TP
H(5)=  0.125D0*RM*SM*TP
H(6)=  0.125D0*RP*SM*TP
H(3)=  0.125D0*RP*SP*TM
H(4)=  0.125D0*RM*SP*TM
H(1)=  0.125D0*RM*SM*TM
H(2)=  0.125D0*RP*SM*TM
P( 7,1)=  0.125D0*SP*TP
P( 8,1)= -0.125D0*SP*TP
P( 5,1)= -0.125D0*SM*TP
P( 6,1)=  0.125D0*SM*TP
P( 3,1)=  0.125D0*SP*TM
P( 4,1)= -0.125D0*SP*TM
P( 1,1)= -0.125D0*SM*TM
P( 2,1)=  0.125D0*SM*TM
P( 7,2) =  0.125D0*RP*TP
P( 8,2) =  0.125D0*RM*TP
P( 5,2) = -0.125D0*RM*TP
P( 6,2) = -0.125D0*RP*TP
P( 3,2) =  0.125D0*RP*TM
P( 4,2) =  0.125D0*RM*TM
P( 1,2) = -0.125D0*RM*TM
P( 2,2) = -0.125D0*RP*TM
P( 7,3) =  0.125D0*RP*SP
P( 8,3) =  0.125D0*RM*SP
P( 5,3) =  0.125D0*RM*SM
```

```
      P( 6,3) =  0.125D0*RP*SM
      P( 3,3) = -0.125D0*RP*SP
      P( 4,3) = -0.125D0*RM*SP
      P( 1,3) = -0.125D0*RM*SM
      P( 2,3) = -0.125D0*RP*SM
      DO 30 I=1,3
      DO 30 J=1,3
      DUM= 0.0D0
      DO 20 K=1,8
20    DUM=DUM+P(K,I)*XYZ(K,J)
30    XJ (I,J)=DUM
      DET=XJ(1,1)*XJ(2,2)*XJ(3,3)+XJ(1,3)*XJ(2,1)*XJ(3,2)
     *+XJ(1,2)*XJ(2,3)*XJ(3,1)
     *-XJ(1,3)*XJ(2,2)*XJ(3,1)-XJ(1,2)*XJ(2,1)*XJ(3,3)
     *-XJ(1,1)*XJ(2,3)*XJ(3,2)
      IF(DET.GT.1.0D-14) GO TO 40
      WRITE(6,1000)DET
 1000 FORMAT(5X,'DET= ',E14.7//)
      WRITE (6,2000) NEIRE
2000  FORMAT(3X,'*** ERROR, ZERO OR NEGATIVE JACOBIAN
     * DETERMINANT AT ELEMENT=',I4)
      STOP
40    DUM=1.0D0/DET
      XJI(1,1)= (XJ(2,2)*XJ(3,3)-XJ(2,3)*XJ(3,2))*DUM
      XJI(2,1)=-(XJ(2,1)*XJ(3,3)-XJ(2,3)*XJ(3,1))*DUM
      XJI(3,1)= (XJ(2,1)*XJ(3,2)-XJ(2,2)*XJ(3,1))*DUM
      XJI(1,2)=-(XJ(1,2)*XJ(3,3)-XJ(1,3)*XJ(3,2))*DUM
      XJI(2,2)= (XJ(1,1)*XJ(3,3)-XJ(1,3)*XJ(3,1))*DUM
      XJI(3,2)=-(XJ(1,1)*XJ(3,2)-XJ(1,2)*XJ(3,1))*DUM
      XJI(1,3)= (XJ(1,2)*XJ(2,3)-XJ(1,3)*XJ(2,2))*DUM
      XJI(2,3)=-(XJ(1,1)*XJ(2,3)-XJ(1,3)*XJ(2,1))*DUM
      XJI(3,3)= (XJ(1,1)*XJ(2,2)-XJ(1,2)*XJ(2,1))*DUM
      TH=0.0D0
      DO 49 K=1,8
      TH=TH+H(K)*XYZ(K,1)
 49   CONTINUE
      DO 50 I=1,9
      DO 50 J=1,48
      B0(NEIRE,I,J)=0.0D0
 50   B1(NEIRE,I,J)=0.0D0
      K6=0
      DO 65 K=1,8
      K6=K6+6
      B0(NEIRE,4,K6)=-H(K)
      B0(NEIRE,5,K6)=H(K)
      B0(NEIRE,6,K6-1)=H(K)
      B0(NEIRE,7,K6-1)=-H(K)
      B0(NEIRE,8,K6-2)=-H(K)
      B0(NEIRE,9,K6-2)=H(K)
      B0(NEIRE,2,K6-5)=H(K)/TH
      B0(NEIRE,5,K6-4)=-H(K)/TH
      B1(NEIRE,2,K6-2)=H(K)/TH
      B1(NEIRE,5,K6-1)=-H(K)/TH
      DO 64 I=1,3
```

```
      B0(NEIRE,1,K6-5)=B0(NEIRE,1,K6-5)+XJI(1,I)*P(K,I)
      B0(NEIRE,3,K6-3)=B0(NEIRE,3,K6-3)+XJI(3,I)*P(K,I)
      B0(NEIRE,4,K6-4)=B0(NEIRE,4,K6-4)+XJI(1,I)*P(K,I)
      B0(NEIRE,6,K6-3)=B0(NEIRE,6,K6-3)+XJI(1,I)*P(K,I)
      B0(NEIRE,7,K6-5)=B0(NEIRE,7,K6-5)+XJI(3,I)*P(K,I)
      B0(NEIRE,9,K6-4)=B0(NEIRE,9,K6-4)+XJI(3,I)*P(K,I)
      B0(NEIRE,2,K6-4)=B0(NEIRE,2,K6-4)+XJI(2,I)*P(K,I)
      B0(NEIRE,5,K6-5)=B0(NEIRE,5,K6-5)+XJI(2,I)*P(K,I)
      B0(NEIRE,8,K6-3)=B0(NEIRE,8,K6-3)+XJI(2,I)*P(K,I)
      B1(NEIRE,1,K6-2)=B1(NEIRE,1,K6-2)+XJI(1,I)*P(K,I)
      B1(NEIRE,2,K6-1)=B1(NEIRE,2,K6-1)+XJI(2,I)*P(K,I)
      B1(NEIRE,3,K6)=B1(NEIRE,2,K6)+XJI(3,I)*P(K,I)
      B1(NEIRE,4,K6-1)=B1(NEIRE,4,K6-1)+XJI(1,I)*P(K,I)
      B1(NEIRE,5,K6-2)=B1(NEIRE,5,K6-2)+XJI(2,I)*P(K,I)
      B1(NEIRE,6,K6)=B1(NEIRE,6,K6)+XJI(1,I)*P(K,I)
      B1(NEIRE,7,K6-2)=B1(NEIRE,7,K6-2)+XJI(3,I)*P(K,I)
      B1(NEIRE,8,K6)=B1(NEIRE,8,K6)+XJI(2,I)*P(K,I)
      B1(NEIRE,9,K6-1)=B1(NEIRE,9,K6-1)+XJI(3,I)*P(K,I)
 64   CONTINUE
 65   CONTINUE
      RETURN
      END
C ***********************************************************************
C *                                                                     *
                      SUBROUTINE LOADER
C *                                                                     *
C ***********************************************************************
      IMPLICIT REAL*8(A-H,O-Z)
      COMMON/AIN/NB,NC,ND,NE,NN,TH,NINT,R,S,T,DET
      COMMON/FSK/A(350000),V(1800),EK(48,48),MAXA(1801),NEIRE
      COMMON V1(100)
      DO 10 LO=1,ND
   10 V(LO)=0.0D0
      SCALE=1.0D0
      WRITE(6,1000)
 1000 FORMAT(//34X,'**** EXTERNAL LOAD ****')
      WRITE(6,2000)
 2000 FORMAT(/1X,'NODE',8X,'X-FORCE',8X,'Y-FORCE',8X,'Z-FORCE',
     *        8X,'X-MOMNT',8X,'Y-MOMNT',8X,'Z-MOMNT')
      IND=0
      DO 20 N=1,ND,6
      IND=IND+1
      CHECK= DABS(V(N))+DABS(V(N+1))+DABS(V(N+2))
     1     +DABS(V(N+3))+DABS(V(N+4))+DABS(V(N+5))
      IF(CHECK .EQ. 0.0D0) GO TO 20
      WRITE(6,3000) (IND,V(N),V(N+1),V(N+2),
     *               V(N+3),V(N+4),V(N+5))
   20 CONTINUE
 3000 FORMAT(2X,I3,6(5X,E10.3))
      RETURN
                              END
C ***********************************************************************
C *                                                                     *
                      SUBROUTINE COLSOL
```

```
   C *
C *************************************************************************
      IMPLICIT REAL*8(A-H,O-Z)
      COMMON/AIN/NB,NC,ND,NE,NN,TH,NINT,R,S,T,DET
      COMMON/FSK/A(350000),V(1800),EK(48,48),MAXA(1801),NEIRE
      DO 140 N=1,ND
      KN=MAXA(N)
      KL=KN+1
      KU=MAXA(N+1)-1
      KH=KU-KL
      IF(KH)110,90,50
50    K=N-KH
      IC=0
      KLT=KU
      DO 80 J=1,KH
      IC=IC+1
      KLT=KLT-1
      KI=MAXA(K)
      NND=MAXA(K+1)-KI-1
      IF(NND)80,80,60
60    KK=MIN0(IC,NND)
      C=0.0D0
      DO 70 L=1,KK
70    C=C+A(KI+L)*A(KLT+L)
      A(KLT)=A(KLT)-C
80    K=K+1
90    K=N
      B=0.0D0
      DO 100 KK=KL,KU
      K=K-1
      KI=MAXA(K)
      C=A(KK)/A(KI)
      B=B+C*A(KK)
100   A(KK)=C
      A(KN)=A(KN)-B
110   IF(A(KN))120,120,140
120   WRITE(6,2000)N,A(KN)
      STOP
140   CONTINUE
      DO 180 N=1,ND
      KL=MAXA(N)+1
      KU=MAXA(N+1)-1
      IF(KU-KL)180,160,160
160   K=N
      C=0.0D0
      DO 170 KK=KL,KU
      K=K-1
170   C=C+A(KK)*V(K)
      V(N)=V(N)-C
180   CONTINUE
      DO 200 N=1,ND
      K=MAXA(N)
200   V(N)=V(N)/A(K)
      IF(ND.EQ.1)RETURN
```

```
      N=ND
      DO 230 L=2,ND
      KL=MAXA(N)+1
      KU=MAXA(N+1)-1
      IF(KU-KL)230,210,210
  210 K=N
      DO 220 KK=KL,KU
      K=K-1
  220 V(K)=V(K)-A(KK)*V(N)
  230 N=N-1
      WRITE(6,1000)
 1000 FORMAT(//33X,'*** NODAL DISPLACEMENT ***')
      WRITE(6,1500)
 1500 FORMAT(/1X,'NODE',8X,'X-DISP',9X,'Y-DISP',9X,'Z-DISP',
     *        9X,'X-ROTN',9X,'Y-ROTN',9X,'Z-ROTN')
      IND=0
      DO 250 K=1,ND,6
      IND=IND+1
      WRITE(6,3000) IND,V(K),V(K+1),V(K+2),
     *              V(K+3),V(K+4),V(K+5)
  250 CONTINUE
      RETURN
 2000 FORMAT(//48H STOP - STIFFNESS MATRIX NOT POSITIVE DEFINITE   ,//
     *          32H NONPOSITIVE PIVOT FOR EQUATION ,I4,//
     *          10H PIVOT =   ,E20.12)
 3000 FORMAT(2X,I3,6(5X,E10.3))
      END
  C ******************************************************************
C *                                                                  *
                          SUBROUTINE STRESS
C *                                                                  *
C ********************************************************************
     IMPLICIT REAL*8(A-H,O-Z)
     COMMON/B01/B0(200,9,48),B1(200,9,48),INXY(200),ID(300)
     COMMON/XYZ/NNP(200,8),X(300),Y(300),Z(300)
     COMMON/MAT/D0(9,9),D1(9,9),A11,A12,A22,A23,B11
     COMMON/AIN/NB,NC,ND,NE,NN,TH,NINT,R,S,T,DET
     COMMON/FSK/A(350000),V(1800),EK(48,48),MAXA(1801),NEIRE
     DIMENSION EDISP(48),E(100,9),PHIJ(100,9),T(100,9),CM(100,9),
    *U(100)
     DO 300 IE=1,NE
      DO 20 IJM=1,8
      IEB=(IJM-1)*6
      ISB=(NNP(IE,IJM)-1)*6
       DO 20 IDOF=1,6
  20   EDISP(IEB+IDOF)=V(ISB+IDOF)
      DO 40 IC=1,9
      SUM=0.0D0
       DO 30 K=1,48
  30   SUM=SUM+B0(IE,IC,K)*EDISP(K)
  40  E(IE,IC)=SUM
      DO 60 IC=1,9
      SUM=0.0D0
       DO 50 K=1,48
```

```
 50     SUM=SUM+B1(IE,IC,K)*EDISP(K)
 60    PHIJ(IE,IC)=SUM
       DO 80 IC=1,9
       SUM=0.0D0
        DO 70 K=1,9
 70     SUM=SUM+D0(IC,K)*E(IE,K)
 80    T(IE,IC)=SUM
       DO 100 IC=1,9
       SUM=0.0D0
        DO 90 K=1,9
 90     SUM=SUM+D1(IC,K)*PHIJ(IE,K)
100    CM(IE,IC)=SUM
        SUM=0.0D00
        DO 200 IC=1,9
200     SUM=SUM+E(IE,IC)*T(IE,IC)+PHIJ(IE,IC)*CM(IE,IC)
       U(IE)=SUM*0.50D00
300 CONTINUE
    WRITE(6,1000)
1000 FORMAT(//19X,'**** STRESSES & STRAINS CALCULATED ****')
    WRITE(6,2000)
2000 FORMAT(/1X,'ELMT',1X,'COMP',1X,'DISP-STRAN',3X,'FORCE-STRS'
    *      ,2X,'COMP',1X,'ROTATN-GRAD',2X,'COUPLE-STRS'
    *      ,2X,'STRN-ENEGY')
    WRITE(6,3000)
3000 FORMAT(15X,'E',12X,'T',15X,'PHI,J',9X,'M',12X,'U')
    DO 400 IE=1,NE
    WRITE(6,4000) IE
4000 FORMAT(I5)
    WRITE(6,5000) E(IE,1),T(IE,1),PHIJ(IE,1),CM(IE,1),U(IE)
5000 FORMAT(7X,'rr',1X,1PE11.4,2X,1PE11.4,2X,'r,r',2X,1PE11.4,
    *        2X,1PE11.4,1X,1PE11.4)
    WRITE(6,6000) E(IE,2),T(IE,2),PHIJ(IE,2),CM(IE,2)
6000 FORMAT(4x,'theta',1X,1PE11.4,2X,1PE11.4,2X,'O,O',2X,1PE11.4,
    *        2X,1PE11.4)
    WRITE(6,7000) E(IE,3),T(IE,3),PHIJ(IE,3),CM(IE,3)
7000 FORMAT(7X,'ZZ',1X,1PE11.4,2X,1PE11.4,2X,'Z,Z',2X,1PE11.4,
    *        2X,1PE11.4)
    WRITE(6,8000) E(IE,4),T(IE,4),PHIJ(IE,4),CM(IE,4)
8000 FORMAT(7X,'ro',1X,1PE11.4,2X,1PE11.4,2X,'r,o',2X,1PE11.4,
    *        2X,1PE11.4)
    WRITE(6,9000) E(IE,5),T(IE,5),PHIJ(IE,5),CM(IE,5)
9000 FORMAT(7X,'or',1X,1PE11.4,2X,1PE11.4,2X,'o,r',2X,1PE11.4,
    *        2X,1PE11.4)
    WRITE(6,10000) E(IE,6),T(IE,6),PHIJ(IE,6),CM(IE,6)
10000 FORMAT(7X,'rz',1X,1PE11.4,2X,1PE11.4,2X,'r,z',2X,1PE11.4,
    *        2X,1PE11.4)
    WRITE(6,11000) E(IE,7),T(IE,7),PHIJ(IE,7),CM(IE,7)
11000 FORMAT(7X,'zr',1X,1PE11.4,2X,1PE11.4,2X,'z,r',2X,1PE11.4,
    *        2X,1PE11.4)
    WRITE(6,12000) E(IE,8),T(IE,8),PHIJ(IE,8),CM(IE,8)
12000 FORMAT(7X,'oz',1X,1PE11.4,2X,1PE11.4,2X,'o,z',2X,1PE11.4,
    *        2X,1PE11.4)
    WRITE(6,13000) E(IE,9),T(IE,9),PHIJ(IE,9),CM(IE,9)
13000 FORMAT(7X,'zo',1X,1PE11.4,2X,1PE11.4,2X,'z,o',2X,1PE11.4,
```

```
     *        2X,1PE11.4)
 400 CONTINUE
     RETURN
                                 END
```

```
C                 2-D MICROPOLAR FINITE ELEMENT METHOD
      IMPLICIT REAL*8(A-H,O-Z)
      INTEGER*4 ITIM
      INTEGER*2 TYPE,CODE
      COMMON/STR/EDISP(24),E(100,5),PHIJ(100,3),T(100,5)
     *,CM(100,3),U(100)
      COMMON/B01/B0(100,5,24),B1(100,4,24),INXY(200),ID(500)
      COMMON/XYZ/NNP(200,8),X(750),Y(750),XG(4,4),WGT(4,4)
     *,XY(8,2)
      COMMON/CSTRN/KSTRN(1000),KSTRT(1000)
      COMMON/MAT/D0(5,5),D1(4,4),A10,A11,A12,A22,A23,B11,B12
      COMMON/AIN/NB,NC,ND,NE,NN,TH,NINT,R,S,DET
      COMMON/FSK/A(185220),V(1890),MAXA(1891),
     *EK(24,24),NEIRE
        READ(5,100)NE
100     FORMAT(I4)
        DO 10 N=1,NE
      READ(5,200)NNP(N,1),NNP(N,2),NNP(N,3),NNP(N,4)
     *,NNP(N,5),NNP(N,6),NNP(N,7),NNP(N,8),INXY(N)
200     FORMAT(4X,8I4,I6)
10      CONTINUE
       WRITE(6,63)NNP(1,1),NNP(1,2),NNP(1,3),NNP(1,5)
     * ,NNP(1,6),NNP(1,7),NNP(1,8)
63     FORMAT(2X,'NNP=',8I4)
        READ(5,100)NN
        DO 20 N=1,NN
        READ(5,300) X(N), Y(N)
300     FORMAT(4X,F20.10,F20.11)
20      CONTINUE
        READ(5,100) NC
        DO 30 N=1,NC
        READ(5,400) KSTRN(N),KSTRT(N)
400     FORMAT(4X,2I4)
30      CONTINUE
       WRITE(6,73)KSTRN(1),KSTRT(1),KSTRN(2)
73     FORMAT(2X,'KSTR=',3I4)
      READ(5,550)NINT
550   FORMAT(I5)
      READ(5,600) A10,A11,A12,A22,A23,B11,B12
600   FORMAT(7E14.7)
      ND=NN*3
      NB=0
      DO 40 IB=1,NE
      IMAX=MAX0(NNP(IB,1),NNP(IB,2),NNP(IB,3),NNP(IB,4)
     *,NNP(IB,5),NNP(IB,6),NNP(IB,7),NNP(IB,8))
      IMIN=MIN0(NNP(IB,1),NNP(IB,2),NNP(IB,3),NNP(IB,4)
     *,NNP(IB,5),NNP(IB,6),NNP(IB,7),NNP(IB,8))
      NBCHEK=(IMAX-IMIN+1)*3
      IF(NBCHEK.GT.NB) NB=NBCHEK
      IF(NB.GT.2000) GO TO 99
40    CONTINUE
      SCALE=1.0D0
      DO 41 I=1,NN
      X(I)=X(I)*SCALE
```

```
41     Y(I)=Y(I)*SCALE
       WRITE(6,700)
700    FORMAT(1H1)
       WRITE(6,800)
800    FORMAT(3X,'*************************************************
      ***********')
       WRITE(6,900)
900    FORMAT(3X,'*',68X,'*')
       WRITE(6,1000)
1000   FORMAT(3X,'*',13X,'AXISYMMETRIC CASE                 ')
       WRITE(6,1050)
1050   FORMAT(3X,'*',28X,'SKYLINEMICRO',27X,'*')
       WRITE(6,900)
       WRITE(6,800)
       WRITE(6,1100)
1100   FORMAT(/20X,'**** DISCRETIZATION NUMBER ****')
       WRITE(6,1200)
1200   FORMAT(/13X,'ELEMT.#',3X,'NODES.#',3X,'CONSTR.#',3X,
      *6X,'BAND-WIDTH',3X,'GAUSS NUMBERICAL INTEGERATI
      *N ORDER')
       WRITE(6,1300)NE,NN,NC,NB,NINT
1300   FORMAT(13X,I4,7X,I3,7X,I3,20X,I5,19X,I2)
       WRITE(6,1400)
1400   FORMAT(/6X,'A11',9X,'A12',9X,'A22',8X,'A23')
       WRITE(6,1500) A10,A11,A12,A22,A23
1500   FORMAT(4E12.3)
       WRITE(6,1600)
1600   FORMAT(/6X,'B11',9X,'B12')
       WRITE(6,1550) B11,B12
1550   FORMAT(2E12.2)
         WRITE(6,234)KSTRN(1),KSTRT(1),KSTRN(2),KSTRT(2)
234      FORMAT(3X,'KSTR==',4I4)
       WRITE(6,1700)
1700   FORMAT(//21X, '**** ELEMENT-NODE CONNECTION ****')
       WRITE(6,1800)
1800   FORMAT(/3X, 'ELM NP1 NP2 NP3 NP4  IXY  elm NP1 NP2 NP3 NP4
      * IXY   ELM NP1 NP2 NP3 NP4 IXY   ELM NP1 NP2 NP3 NP4
      *ixy')
       DO 45 I=1,NE
45     ID(I)=I
       LINE=NE/2
       IRESID=NE-2*LINE
       DO 50 N=1,LINE,2
50     WRITE(6,1900)  (ID(2*(N-1)+I),NNP(2*(N-1)+I,1),NNP(2*(N-1)
      *+I,2),NNP(2*(N-1)+I,3),NNP(2*(N-1)+I,4),INXY(2*(N-1)+I),
      *i=1,4)
1900   FORMAT(4(2X,I4,1X,I3,1X,I3,1X,I3,1X,I3,1X,I3))
       IF(IRESID.EQ.0) GO TO 56
       WRITE(6,1900)(ID(4*LINE+I),NNP(4*LINE+I,1),
      *NNP(4*LINE+I,2)
      *,NNP(4*LINE+I,3),NNP(4*LINE+I,4),INXY(4*LINE+I),
      *i=1,iresid
      *)
56     WRITE(6,2000)
```

```
2000  FORMAT(//24X, '**** NODAL COORDINATE ****')
      WRITE(6,2100)
2100  FORMAT(/1X,'NODE',5X,'X',8X,'Y',4X,'NODE',5X,'X',8X,'Y',
     *4X,'NODE',5X,'X',8X,'Y',4X,'NODE',5X,'X',8X,'Y')
      DO 55 I=1,NN
55    ID(I)=I
      LINE=NN/4
      IRESID=NN-4*LINE
      DO 60 N=1,LINE
60    WRITE(6,2200) (ID(4*(N-1)+I),X(4*(N-1)+I),Y(4*(N-1)+I),
     *I=1,4)
2200  FORMAT(4(2X,I3,1X,F9.3,1X,F9.3))
      IF(IRESID.EQ.0)GO TO 57
      WRITE(6,2200) (ID(4*LINE+I),X(4*LINE+I),Y(4*LINE+I),
     *I=1,IRESID)
       WRITE(6,123)KSTRN(1),KSTRT(1),KSTRN(2),KSTRT(2)
123    FORMAT(4X,'KST=',4I4)
57    WRITE(6,2300)
2300  FORMAT(//27X,'**** CONSTRAINT ****')
      WRITE(6,2400)
2400  FORMAT(/24X,'CNSTRND-NODE',2X,'CNSTRND-CODE')
      DO 222 I=1,NC
      WRITE(6,2500)KSTRN(I),KSTRT(I)
2500  FORMAT(28X,I3,12X,I3)
222   CONTINUE
      CALL STSTIF
      CALL LOADER
      CALL COLSOL
      CALL STRESS
99    WRITE(6,999)NB,IB
999   FORMAT('*****STOP NB=',I5,' AT ELEMENT=', I5)
      STOP
      END
                SUBROUTINE STSTIF
      IMPLICIT REAL*8(A-H,O-Z)
      COMMON/CSTRN/KSTRN(1000),KSTRT(1000)
      COMMON/XYZ/NNP(200,8),X(750),Y(750),XG(4,4),
     *WGT(4,4),XY(8,2)
      COMMON/AIN/NB,NC,ND,NE,NN,TH,NINT,R,S,DET
      COMMON/FSK/A(185220),V(1890),MAXA(1891),
     *EK(24,24),NEIRE
      DIMENSION MHT(1890)
      NBND=NB*ND
      DO 10 I=1,NBND
10    A(I)=0.0D0
      DO 20 NEIRE=1,NE
      CALL ELSTIF
      DO 20 INC=1,8
      INOC=NNP(NEIRE,INC)
      IBC=(NNP(NEIRE,INC)-1)*3
      DO 20 IDC=1,3
      ICEL=(INC-1)*3+IDC
      ICST=IBC+IDC
      IDI=0
```

```
      IF(ICST.GT.NB)IDI=ICST-NB
      IVC=(ICST-1)*NB
      DO 18 INR=1,8
      INOR=NNP(NEIRE,INR)
      IF(INOC.LT.INOR) GO TO 18
      IBR=(NNP(NEIRE,INR)-1)*3
      IDVC=IDC
      IF(INOC.GT.INOR)IDVC=3
      DO 15 IDR=1,IDVC
      IREL=(INR-1)*3+IDR
      IVV=IVC+IBR+IDR-IDI
      SS=A(IVV)+EK(IREL,ICEL)
      IF(DABS(SS).LT.1.0D-14)SS=0.0D0
15    A(IVV)=SS
18    CONTINUE
20    CONTINUE
      DO 140 N=1,NC
      IRCX=KSTRN(N)*3-2
      IRCY=KSTRN(N)*3-1
      IRCZ=KSTRN(N)*3
      KCHK=KSTRT(N)
      IF(KCHK.LT.100) GO TO 60
      ICB=ND-IRCX+1
      IF(ICB.GT.NB)ICB=NB
      DO 50 I=1,ICB
      IDI=0
      IRCXX=IRCX+I-1
      IF(IRCXX.GT.NB) IDI=IRCXX-NB
      IXV=(IRCX-2+I)*NB+IRCX-IDI
      IF(I.EQ.1) GO TO 30
      A(IXV)=0.0D0
      GO TO 50
30    A(IXV)=1.0D0
      IF(IRCX.EQ.1)GO TO 50
      DO 40 J=1,IRCX-IDI-1
      IXXV=IXV-J
40    A(IXXV)=0.0D0
50    CONTINUE
      KCHK=KCHK-100
60    IF(KCHK.LT.010) GO TO 100
      ICB=ND-IRCY+1
      IF(ICB.GT.NB) ICB=NB
      DO 90 I=1,ICB
      IDI=0
      IRCYY=IRCY+I-1
      IF(IRCYY.GT.NB)IDI=IRCYY-NB
      IYV=(IRCY-2+I)*NB+IRCY-IDI
      IF(I.EQ.1)GO TO 70
      A(IYV)=0.0D0
      GO TO 90
70    A(IYV)=1.0D0
      IF(IRCY.EQ.1)GO TO 90
      DO 80 J=1,IRCY-IDI-1
      IYYV=IYV-J
```

```
80     A(IYYV)=0.0D0
90     CONTINUE
       KCHK=KCHK-10
100    IF(KCHK.LT.001)GO TO 140
       ICB=ND-IRCZ+1
       IF(ICB.GT.NB)ICB=NB
       DO 130 I=1,ICB
       IDI=0
       IRCZZ=IRCZ+I-1
       IF(IRCZZ.GT.NB)IDI=IRCZZ-NB
       IZV=(IRCZ-2+I)*NB+IRCZ-IDI
       IF(I.EQ.1)GO TO 110
       A(IZV)=0.0D0
       GO TO 130
110    A(IZV)=1.0D0
       IF(IRCZ.EQ.1)GO TO 130
       DO 120 J=1,IRCZ-IDI-1
       IZZV=IZV-J
120    A(IZZV)=0.0D0
130    CONTINUE
140    CONTINUE
       DO 160 I=1,ND
       IDI=0
       IF(I.GT.NB)IDI=I-NB
       IIV=(I-1)*NB
       DO 150 J=1,I
       IF(A(IIV+J).EQ.0.0D0) GO TO 150
       MHT(I)=I-J-IDI
       GO TO 160
150    CONTINUE
160    CONTINUE
       NM=ND+1
       DO 170 I=1,NM
170    MAXA(I)=0
       MAXA(1)=1
       MAXA(2)=2
       IF(ND.EQ.1)GO TO 190
       DO 180 I=2,ND
180    MAXA(I+1)=MAXA(I)+MHT(I)+1
190    NWK=MAXA(ND+1)-MAXA(1)
       IAN=0
       DO 200 I=1,ND
       ICK=MAXA(I+1)-MAXA(I)
       IDI=0
       IF(I.GT.NB)IDI=I-NB
       INBB=(I-1)*NB+I-IDI
       DO 200 II=1,ICK
       IAN=IAN+1
       IAV=INBB-II+1
       A(IAN)=A(IAV)
200    CONTINUE
       DO 210 I=NWK+1,NBND
210    A(I)=0.0D0
       RETURN
```

```
      END
                        SUBROUTINE ELSTIF
      IMPLICIT REAL*8(A-H,O-Z)
      COMMON/FSK/A(185220),V(1890),MAXA(1891),EK(24,
     *24),NEIRE
      COMMON/AIN/NB,NC,ND,NE,NN,TH,NINT,R,S,DET
      COMMON/B01/B0(100,5,24),B1(100,4,24),INXY(200),ID(500)
      COMMON/XYZ/NNP(200,8),X(750),Y(750),XG(4,4),WGT(4,4)
     *,XY(8,2)
      COMMON/DB01/D0B0(5),D1B1(4)
      COMMON/MAT/D0(5,5),D1(4,4),A10,A11,A12,A22,A23,B11,B12
      DATA XG/0.0D0,0.0D0,0.0D0,0.0D0,-.5773502691896D0
     *,.5773502691896D0,0.0D0,0.0D0,-.7745966692415D0
     *,.0D0,.7745966692415D0,0.0D0,-.8611363115941D0,
     *-.3399810435849D0,.3399810435849D0,.8611363115941
     *D0/
      DATA WGT/2.0D0,0.0D0,0.0D0,0.0D0,1.0D0,1.0D0,0.0D0,
     *0.0D0,.5555555555556D0,.888888888889D0,
     *.5555555555556D0,0.0D0,.3478548451375D0,
     *.6521451548625D0,.6521451548625D0,
     *.3478548451375D0/
      DO 10 I=1,5
      DO 10 J=1,5
10    D0(I,J)=0.0D0
      D0(1,1)=A11
      D0(1,2)=A12
      D0(2,1)=A12
      D0(2,2)=A10
      D0(1,3)=A12
      D0(3,1)=A12
      D0(2,3)=A12
      D0(3,2)=A12
      D0(3,3)=A11
      D0(4,4)=A22
      D0(4,5)=A23
      D0(5,4)=A23
      D0(5,5)=A22
      DO 2 I=1,4
      DO 2 J=1,4
      D1(I,J)=0.0D0
2     CONTINUE
      D1(1,1)=B11
      D1(1,2)=B12
      D1(2,1)=0.0D0
      D1(2,2)=B11
      D1(1,3)=0.0D0
      D1(2,3)=B12
      D1(3,1)=0.0D0
      D1(3,2)=B12
      D1(3,3)=B11
      I=NNP(NEIRE,1)
      J=NNP(NEIRE,2)
      K=NNP(NEIRE,3)
      L=NNP(NEIRE,4)
```

```
      MIJ=NNP(NEIRE,5)
      MJK=NNP(NEIRE,6)
      MKL=NNP(NEIRE,7)
      MLI=NNP(NEIRE,8)
      XY(1,1)=X(I)
      XY(2,1)=X(J)
      XY(3,1)=X(K)
      XY(4,1)=X(L)
      XY(5,1)=X(MIJ)
      XY(6,1)=X(MJK)
      XY(7,1)=X(MKL)
      XY(8,1)=X(MLI)
      XY(1,2)=Y(I)
      XY(2,2)=Y(J)
      XY(3,2)=Y(K)
      XY(4,2)=Y(L)
      XY(5,2)=Y(MIJ)
      XY(6,2)=Y(MJK)
      XY(7,2)=Y(MKL)
      XY(8,2)=Y(MLI)
      DO 30 I=1,24
      DO 30 J=1,24
30    EK(I,J)=0.0D0
      DO 80 LX=1,NINT
      R=XG(LX,NINT)
      DO 80 LY=1,NINT
      S=XG(LY,NINT)
      CALL STDM
      WT=WGT(LX,NINT)*WGT(LY,NINT)*TH*DET
      DO 70 J=1,24
      DO 40 K=1,5
      D0B0(K)=0.0D0
      DO 40 L=1,5
40    D0B0(K)=D0B0(K)+D0(K,L)*B0(NEIRE,L,J)
      DO 45 K=1,4
      D1B1(K)=0.0D0
      DO 45 L=1,4
45    D1B1(K)=D1B1(K)+D1(K,L)*B1(NEIRE,L,J)
      DO 60 I=J,24
      STIFF=0.0D0
      DO 50 L=1,5
50    STIFF=STIFF+B0(NEIRE,L,I)*D0B0(L)
      DO 55 L=1,4
55    STIFF=STIFF+B1(NEIRE,L,I)*D1B1(L)
60    EK(I,J)=EK(I,J)+STIFF*WT
70    CONTINUE
80    CONTINUE
      DO 90 J=1,24
      DO 90 I=J,24
90    EK(J,I)=EK(I,J)
      IF(INXY(NEIRE).EQ.100)GO TO 100
      IF(INXY(NEIRE).EQ.010)GO TO 110
      IF(INXY(NEIRE).EQ.001)GO TO 120
      IF(INXY(NEIRE).EQ.011)GO TO 130
```

```fortran
      IF(INXY(NEIRE).EQ.110)GO TO 135
      R=0.0D0
      S=0.0D0
      GO TO 140
100   R=0.0D0
      S=1.0D0
      GO TO 140
110   R=-1.0D0
      S=0.0D0
      GO TO 140
120   R=0.0D0
      S=-1.0D0
      GO TO 140
130   R=1.0D0
      S=0.0D0
      GO TO 140
135   R=-1.0D0
      S=-1.0D0
140   CALL STDM
      RETURN
      END
                        SUBROUTINE STDM
      IMPLICIT REAL*8(A-H,O-Z)
      COMMON/FSK/A(185220),V(1890),MAXA(1891),
     *EK(24,24),NEIRE
      COMMON/AIN/NB,NC,ND,NE,NN,TH,NINT,R,S,DET
      COMMON/B01/B0(100,5,24),B1(100,4,24),INXY(200),ID(500)
      COMMON/INF/H(8),P(8,2),XJ(2,2),XJI(2,2)
      COMMON/XYZ/NNP(200,8),X(750),Y(750),XG(4,4),WGT(4,4),
     *XY(8,2)
      RP=1.0D0+R
      RM=1.0D0-R
      SP=1.0D0+S
      SM=1.0D0-S
      RSM=1.0D0-(R*R)
      SSM=1.0D0-(S*S)
      H(1)=0.25D0*RP*SP*(R+S-1.0D0)
      H(2)=0.25D0*RM*SP*(-R+S-1.0D0)
      H(3)=0.25D0*RM*SM*(-R-S-1.0D0)
      H(4)=0.25D0*RP*SM*(R-S-1.0D0)
      H(5)=0.5D0*RSM*SP
      H(6)=0.5D0*SSM*RM
      H(7)=0.5D0*RSM*SM
      H(8)=0.5D0*SSM*RP
      P(1,1)=0.25D0*SP*(R+R+S)
      P(2,1)=0.25D0*SP*(R+R-S)
      P(3,1)=0.25D0*SM*(R+R+S)
      P(4,1)=0.25D0*SM*(R+R-S)
      P(5,1)=-R*SP
      P(6,1)=-0.5D0*SSM
      P(7,1)=-R*SM
      P(8,1)=0.5D0*SSM
      P(1,2)=0.25D0*RP*(R+S+S)
      P(2,2)=0.25D0*RM*(S+S-R)
```

```
      P(3,2)=0.25D0*RM*(R+S+S)
      P(4,2)=0.25D0*RP*(S+S-R)
      P(5,2)=0.5D0*RSM
      P(6,2)=-S*RM
      P(7,2)=-0.5D0*RSM
      P(8,2)=-S*RP
      DO 30 I=1,2
      DO 30 J=1,2
      DUM=0.0D0
      DO 20 K=1,8
20    DUM=DUM+P(K,I)*XY(K,J)
30    XJ(I,J)=DUM
      DET=XJ(1,1)*XJ(2,2)-XJ(2,1)*XJ(1,2)
      IF(DET.GT.1.0D-07)GO TO 40
      WRITE(6,2000) NEIRE
2000  FORMAT(3X,'*** ERROR, ZERO OR NEGATIVE JACOBIAN
     1DETERMINANT AT ELEMENT=',I4)
      STOP
40    DUM=1.0D0/DET
      XJI(1,1)=XJ(2,2)*DUM
      XJI(1,2)=-XJ(1,2)*DUM
      XJI(2,1)=-XJ(2,1)*DUM
      XJI(2,2)=XJ(1,1)*DUM
      TH=0.0D0
      DO 45 K=1,8
45    TH=TH+H(K)*XY(K,1)
      DO 50 I=1,5
      DO 50 J=1,24
50    B0(NEIRE,I,J)=0.0D0
      DO 55 I=1,4
      DO 55 J=1,24
55    B1(NEIRE,I,J)=0.0D0
      B0(NEIRE,1,1)=(XJI(1,1)*P(1,1)+XJI(1,2)*P(1,2))
      B0(NEIRE,1,4)=(XJI(1,1)*P(2,1)+XJI(1,2)*P(2,2))
      B0(NEIRE,1,7)=(XJI(1,1)*P(3,1)+XJI(1,2)*P(3,2))
      B0(NEIRE,1,10)=(XJI(1,1)*P(4,1)+XJI(1,2)*P(4,2))
      B0(NEIRE,1,13)=(XJI(1,1)*P(5,1)+XJI(1,2)*P(5,2))
      B0(NEIRE,1,16)=(XJI(1,1)*P(6,1)+XJI(1,2)*P(6,2))
      B0(NEIRE,1,19)=(XJI(1,1)*P(7,1)+XJI(1,2)*P(7,2))
      B0(NEIRE,1,22)=(XJI(1,1)*P(8,1)+XJI(1,2)*P(8,2))
      B0(NEIRE,4,2)=B0(NEIRE,1,1)
      B0(NEIRE,4,3)=H(1)
      B0(NEIRE,4,5)=B0(NEIRE,1,4)
      B0(NEIRE,4,6)=H(2)
      B0(NEIRE,4,8)=B0(NEIRE,1,7)
      B0(NEIRE,4,9)=H(3)
      B0(NEIRE,4,11)=B0(NEIRE,1,10)
      B0(NEIRE,4,12)=H(4)
      B0(NEIRE,4,14)=B0(NEIRE,1,13)
      B0(NEIRE,4,15)=H(5)
      B0(NEIRE,4,17)=B0(NEIRE,1,16)
      B0(NEIRE,4,18)=H(6)
      B0(NEIRE,4,20)=B0(NEIRE,1,19)
      B0(NEIRE,4,21)=H(7)
```

```
BO(NEIRE,4,23)=BO(NEIRE,1,22)
BO(NEIRE,4,24)=H(8)
BO(NEIRE,3,1)=H(1)/TH
BO(NEIRE,3,4)=H(2)/TH
BO(NEIRE,3,7)=H(3)/TH
BO(NEIRE,3,10)=H(4)/TH
BO(NEIRE,3,13)=H(5)/TH
BO(NEIRE,3,16)=H(6)/TH
BO(NEIRE,3,19)=H(7)/TH
BO(NEIRE,3,22)=H(8)/TH
BO(NEIRE,5,1)=(XJI(2,1)*P(1,1)+XJI(2,2)*P(1,2))
BO(NEIRE,5,3)=-H(1)
BO(NEIRE,5,4)=(XJI(2,1)*P(2,1)+XJI(2,2)*P(2,2))
BO(NEIRE,5,6)=-H(2)
BO(NEIRE,5,7)=(XJI(2,1)*P(3,1)+XJI(2,2)*P(3,2))
BO(NEIRE,5,9)=-H(3)
BO(NEIRE,5,10)=(XJI(2,1)*P(4,1)+XJI(2,2)*P(4,2))
BO(NEIRE,5,12)=-H(4)
BO(NEIRE,5,13)=(XJI(2,1)*P(5,1)+XJI(2,2)*P(5,2))
BO(NEIRE,5,16)=(XJI(2,1)*P(6,1)+XJI(2,2)*P(6,2))
BO(NEIRE,5,19)=(XJI(2,1)*P(7,1)+XJI(2,2)*P(7,2))
BO(NEIRE,5,22)=(XJI(2,1)*P(8,1)+XJI(2,2)*P(8,2))
BO(NEIRE,5,15)=-H(5)
BO(NEIRE,5,18)=-H(6)
BO(NEIRE,5,21)=-H(7)
BO(NEIRE,5,24)=-H(8)
BO(NEIRE,2,2)=(XJI(2,1)*P(1,1)+XJI(2,2)*P(1,2))
BO(NEIRE,2,5)=(XJI(2,1)*P(2,1)+XJI(2,2)*P(2,2))
BO(NEIRE,2,8)=(XJI(2,1)*P(3,1)+XJI(2,2)*P(3,2))
BO(NEIRE,2,11)=(XJI(2,1)*P(4,1)+XJI(2,2)*P(4,2))
BO(NEIRE,2,14)=(XJI(2,1)*P(5,1)+XJI(2,2)*P(5,2))
BO(NEIRE,2,17)=(XJI(2,1)*P(6,1)+XJI(2,2)*P(6,2))
BO(NEIRE,2,20)=(XJI(2,1)*P(7,1)+XJI(2,2)*P(7,2))
BO(NEIRE,2,23)=(XJI(2,1)*P(8,1)+XJI(2,2)*P(8,2))
B1(NEIRE,2,3)=-H(1)/TH
B1(NEIRE,2,6)=-H(2)/TH
B1(NEIRE,2,9)=-H(3)/TH
B1(NEIRE,2,12)=-H(4)/TH
B1(NEIRE,2,15)=-H(5)/TH
B1(NEIRE,2,18)=-H(6)/TH
B1(NEIRE,2,21)=-H(7)/TH
B1(NEIRE,2,24)=-H(8)/TH
B1(NEIRE,1,3)=BO(NEIRE,1,1)
B1(NEIRE,1,6)=BO(NEIRE,1,4)
B1(NEIRE,1,9)=BO(NEIRE,1,7)
B1(NEIRE,1,12)=BO(NEIRE,1,10)
B1(NEIRE,1,15)=BO(NEIRE,1,13)
B1(NEIRE,1,18)=BO(NEIRE,1,16)
B1(NEIRE,1,21)=BO(NEIRE,1,19)
B1(NEIRE,1,24)=BO(NEIRE,1,22)
B1(NEIRE,3,3)=BO(NEIRE,2,2)
B1(NEIRE,3,6)=BO(NEIRE,2,5)
B1(NEIRE,3,9)=BO(NEIRE,2,8)
B1(NEIRE,3,12)=BO(NEIRE,2,11)
```

```
      B1(NEIRE,3,15)=B0(NEIRE,2,14)
      B1(NEIRE,3,18)=B0(NEIRE,2,17)
      B1(NEIRE,3,21)=B0(NEIRE,2,20)
      B1(NEIRE,3,24)=B0(NEIRE,2,23)
      RETURN
      END
                        SUBROUTINE LOADER
      IMPLICIT REAL*8(A-H,O-Z)
      COMMON/CSTRN/KSTRN(1000),KSTRT(1000)
      COMMON/AIN/NB,NC,ND,NE,NN,TH,NINT,R,S,DET
      COMMON/FSK/A(185220),V(1890),MAXA(1891),
     *EK(24,24),NEIRE
      DO 10 LO=1,ND
10    V(LO)=0.0D0
      SCALE=1.0D0
      V(2)=20.D0/6.D0
      V(5)=12.0D0
      V(8)=32.D0/6.D0
      V(11)=56.D0/6.D0
      V(14)=4.D0
      V(17)=40.D0/6.D0
      V(20)=16.D0/6.D0
      V(23)=4.D0
      V(26)=8.D0/6.D0
      V(29)=8.D0/6.D0
      WRITE(6,1000)
1000  FORMAT(//26X,'**** EXTERNAL LOAD ****')
      WRITE(6,2000)
2000  FORMAT(/1X,'NODE',4X,'X-FORCE',4X,'Y-FORCE',4X,
     *'Z-MOMENT',1X,'NODE',4X,'X-FORCE',4X,'Y-FORCE',
     *4X,'Z-MOMNT')
      IND=-1
      INE=0
      DO 20 N=1,ND,6
      IND=IND+2
      INE=INE+2
      CHECK=DABS(V(N))+DABS(V(N+1))+DABS(V(N+2))+DABS(V(N+3))
     *+DABS(V(N+4))+DABS(V(N+5))
      IF(CHECK.EQ.0.0D0) GO TO 20
      WRITE(6,3000) (IND,V(N),V(N+1),V(N+2),INE,V(N+3),V(N+4),
     *V(N+5))
20    CONTINUE
3000  FORMAT(2(2X,I3,1X,E10.3,1X,E10.3,1X,E10.3))
      RETURN
      END
      SUBROUTINE COLSOL
      IMPLICIT REAL*8(A-H,O-Z)
      COMMON/AIN/NB,NC,ND,NE,NN,TH,NINT,R,S,DET
      COMMON/FSK/A(185220),V(1890),MAXA(1891),EK(24,24),NEIRE
      DO 140 N=1,ND
      KN=MAXA(N)
      KL=KN+1
      KU=MAXA(N+1)-1
      KH=KU-KL
```

```
      IF(KH)110,90,50
50    K=N-KH
      IC=0
      KLT=KU
      DO 80 J=1,KH
      IC=IC+1
      KLT=KLT-1
      KI=MAXA(K)
      NND=MAXA(K+1)-KI-1
      IF(NND)80,80,60
60    KK=MIN0(IC,NND)
      C=0.0D0
      DO 70 L=1,KK
70    C=C+A(KI+L)*A(KLT+L)
      A(KLT)=A(KLT)-C
80    K=K+1
90    K=N
      B=0.
      DO 100 KK=KL,KU
      K=K-1
      KI=MAXA(K)
      C=A(KK)/A(KI)
      B=B+C*A(KK)
100   A(KK)=C
      A(KN)=A(KN)-B
110   IF(A(KN))120,120,140
120   WRITE(6,2000)N,A(KN)
      STOP
140   CONTINUE
      DO 180 N=1,ND
      KL=MAXA(N)+1
      KU=MAXA(N+1)-1
      IF(KU-KL)180,160,160
160   K=N
      C=0.
      DO 170 KK=KL,KU
      K=K-1
170   C=C+A(KK)*V(K)
      V(N)=V(N)-C
180   CONTINUE
      DO 200 N=1,ND
      K=MAXA(N)
200   V(N)=V(N)/A(K)
      IF(ND.EQ.1)RETURN
      N=ND
      DO 230 L=2,ND
      KL=MAXA(N)+1
      KU=MAXA(N+1)-1
      IF(KU-KL)230,210,210
210   K=N
      DO 220 KK=KL,KU
      K=K-1
220   V(K)=V(K)-A(KK)*V(N)
230   N=N-1
```

```
      WRITE(6,1000)
1000  FORMAT(//25X,'*** NODAL DISPLACEMENT ***')
      WRITE(6,2001)
2001  FORMAT(/1X,'NODE',4X,'X-DISP',5X,'Y-DISP',5X,'Z-ROTN',
     *3X,'NODE',4X,'X-DISP',5X,'Y-DISP',5X,'Z-ROTN')
      IND=-1
      INE=0
      DO 251 K=1,ND,6
      IND=IND+2
      INE=INE+2
251    WRITE(6,3000)IND,V(K),V(K+1),V(K+2),INE,V(K+3),V(K+4),
     *V(K+5)
3000  FORMAT(2(2X,I3,1X,E10.3,1X,E10.3,1X,E10.3))
      RETURN
2000  FORMAT(//'STOP MATRIX NOT POSITIVE',I4,E20.12)
      END

                        SUBROUTINE STRESS
      IMPLICIT REAL*8(A-H,O-Z)
      COMMON/STR/EDISP(24),E(100,5),PHIJ(100,3),T(100,5),CM(100,
     *3),U(100)
      COMMON/B01/B0(100,5,24),B1(100,4,24),INXY(200),ID(500)
      COMMON/XYZ/NNP(200,8),X(750),Y(750),XG(4,4),WGT(4,4),
     *XY(8,2)
      COMMON/MAT/D0(5,5),D1(4,4),A10,A11,A12,A22,A23,B11,B12
      COMMON/AIN/NB,NC,ND,NE,NN,TH,NINT,R,S,DET
      COMMON/FSK/A(185220),V(1890),MAXA(1891),EK(24,
     *24),NEIRE
      DO 300 IE=1,NE
      DO 20 IJM=1,8
      IEB=(IJM-1)*3
      ISB=(NNP(IE,IJM)-1)*3
      DO 20 IDOF=1,3
20    EDISP(IEB+IDOF)=V(ISB+IDOF)
      DO 40 IC=1,5
      SUM=0.0D0
      DO 30 K=1,24
30    SUM=SUM+B0(IE,IC,K)*EDISP(K)
40    E(IE,IC)=SUM
      DO 60 IC=1,3
      SUM=0.0D0
      DO 50 K=1,24
50    SUM=SUM+B1(IE,IC,K)*EDISP(K)
60    PHIJ(IE,IC)=SUM
      DO 80 IC=1,5
      SUM=0.0D0
      DO 70 K=1,5
70    SUM=SUM+D0(IC,K)*E(IE,K)
80    T(IE,IC)=SUM
      DO 100 IC=1,3
      SUM=0.0D0
      DO 90 K=1,3
90    SUM=SUM+D1(IC,K)*PHIJ(IE,K)
100   CM(IE,IC)=SUM
```

```
      SUM=0.0D0
      DO 200 IC=1,5
200   SUM=SUM+E(IE,IC)*T(IE,IC)
      DO 250 IC=1,3
250   SUM=SUM+PHIJ(IE,IC)*CM(IE,IC)
      U(IE)=SUM*0.50D0
300   CONTINUE
      WRITE(6,1000)
1000  FORMAT(//19X,' **** STRESS &STRAIN CALCULATED ****')
      WRITE(6,2000)
2000  FORMAT(/,'ELMT',1X,'COMP',1X,'DIS-STRAN',3X,'FORCE-STRS'
     *,2X,'COMP',1X,'ROTAN-GRAD',2X,'COUPLE-STRS',2X,'STRN-
     *ENEGY')
      WRITE(6,3000)
3000  FORMAT(14X,'E',12X,'T',15X,'PHI.J',9X,'M',12X,'U')
      DO 400 IE=1,NE
      WRITE(6,4000) IE
4000  FORMAT(I4)
      WRITE(6,5000) E(IE,1),T(IE,1),PHIJ(IE,1),CM(IE,1),U(IE)
5000  FORMAT(6X,'XX',1X,1PE11.4,2X,1PE11.4,2X,'Z,X',1X,1PE11.4,
     12X,1PE11.4,2X,1PE11.4)
      WRITE(6,6000) E(IE,2),T(IE,2),PHIJ(IE,2),CM(IE,2)
6000  FORMAT(6X,'YY',1X,1PE11.4,2X,1PE11.4,2X,'Z/Y',1X,
     11PE11.4,2X,1PE11.4)
      WRITE(6,7000) E(IE,3),T(IE,3),PHIJ(IE,3),CM(IE,3)
7000  FORMAT(6X,'ZZ',1X,1PE11.4,2X,1PE11.4,2X,'Z,Y',1X,
     *1PE11.4,2X,1PE11.4)
      WRITE(6,8000) E(IE,4),T(IE,4)
8000  FORMAT(6X,'XY',1X,1PE11.4,2X,1PE11.4)
      WRITE(6,9000) E(IE,5),T(IE,5)
9000  FORMAT(6X,'YX',1X,1PE11.4,2X,1PE11.4)
400   CONTINUE
      RETURN
      END
```