

Copyright Warning & Restrictions

The copyright law of the United States (Title 17, United States Code) governs the making of photocopies or other reproductions of copyrighted material.

Under certain conditions specified in the law, libraries and archives are authorized to furnish a photocopy or other reproduction. One of these specified conditions is that the photocopy or reproduction is not to be “used for any purpose other than private study, scholarship, or research.” If a user makes a request for, or later uses, a photocopy or reproduction for purposes in excess of “fair use” that user may be liable for copyright infringement,

This institution reserves the right to refuse to accept a copying order if, in its judgment, fulfillment of the order would involve violation of copyright law.

Please Note: The author retains the copyright while the New Jersey Institute of Technology reserves the right to distribute this thesis or dissertation

Printing note: If you do not wish to print this page, then select “Pages from: first page # to: last page #” on the print dialog screen

The Van Houten library has removed some of the personal information and all signatures from the approval page and biographical sketches of theses and dissertations in order to protect the identity of NJIT graduates and faculty.

INFORMATION TO USERS

This reproduction was made from a copy of a document sent to us for microfilming. While the most advanced technology has been used to photograph and reproduce this document, the quality of the reproduction is heavily dependent upon the quality of the material submitted.

The following explanation of techniques is provided to help clarify markings or notations which may appear on this reproduction.

1. The sign or "target" for pages apparently lacking from the document photographed is "Missing Page(s)". If it was possible to obtain the missing page(s) or section, they are spliced into the film along with adjacent pages. This may have necessitated cutting through an image and duplicating adjacent pages to assure complete continuity.
2. When an image on the film is obliterated with a round black mark, it is an indication of either blurred copy because of movement during exposure, duplicate copy, or copyrighted materials that should not have been filmed. For blurred pages, a good image of the page can be found in the adjacent frame. If copyrighted materials were deleted, a target note will appear listing the pages in the adjacent frame.
3. When a map, drawing or chart, etc., is part of the material being photographed, a definite method of "sectioning" the material has been followed. It is customary to begin filming at the upper left hand corner of a large sheet and to continue from left to right in equal sections with small overlaps. If necessary, sectioning is continued again—beginning below the first row and continuing on until complete.
4. For illustrations that cannot be satisfactorily reproduced by xerographic means, photographic prints can be purchased at additional cost and inserted into your xerographic copy. These prints are available upon request from the Dissertations Customer Services Department.
5. Some pages in any document may have indistinct print. In all cases the best available copy has been filmed.

**University
Microfilms
International**

300 N. Zeeb Road
Ann Arbor, MI 48106

8517769

Mayk, Israel

INVESTIGATION OF PRE-DETECTION SIGNAL PROCESSING OF
PSEUDONOISE COMMUNICATION SIGNALS IN THE PRESENCE OF
ADDITIVE WHITE GAUSSIAN NOISE AND CW AND BURSTY INTERFERENCE

New Jersey Institute of Technology

D.ENG.SC.

1985

University
Microfilms
International

300 N. Zeeb Road, Ann Arbor, MI 48106

PLEASE NOTE:

In all cases this material has been filmed in the best possible way from the available copy. Problems encountered with this document have been identified here with a check mark .

1. Glossy photographs or pages
2. Colored illustrations, paper or print
3. Photographs with dark background
4. Illustrations are poor copy
5. Pages with black marks, not original copy
6. Print shows through as there is text on both sides of page
7. Indistinct, broken or small print on several pages
8. Print exceeds margin requirements
9. Tightly bound copy with print lost in spine
10. Computer printout pages with indistinct print
11. Page(s) _____ lacking when material received, and not available from school or author.
12. Page(s) _____ seem to be missing in numbering only as text follows.
13. Two pages numbered _____. Text follows.
14. Curling and wrinkled pages
15. Dissertation contains pages with print at a slant, filmed as received
16. Other _____

University
Microfilms
International

INVESTIGATION OF PRE-DETECTION SIGNAL PROCESSING
OF
PSEUDONOISE COMMUNICATION SIGNALS
IN THE PRESENCE OF
ADDITIVE WHITE GAUSSIAN NOISE
AND
CW AND BURSTY INTERFERENCE

by

Israel Mayk

Dissertation submitted to the Faculty of the Graduate School
of New Jersey Institute of Technology in partial fulfillment
of the requirements for the degree of
Doctor of Engineering Science
1985

APPROVAL OF DISSERTATION

Investigation of Pre-Detection Signal Processing
of Pseudonoise Communication Signals in the Presence of
Additive White Gaussian Noise and CW and Bursty Interference

by Israel Mayk

for Department of Electrical Engineering

New Jersey Institute of Technology

Approved: _____ Chairman

Newark, N.J.
May 1985

ABSTRACT

Title of Dissertation: INVESTIGATION OF PRE-DETECTION SIGNAL PROCESSING OF PSEUDONOISE COMMUNICATION SIGNALS IN THE PRESENCE OF ADDITIVE WHITE GAUSSIAN NOISE AND CW AND BURSTY INTERFERENCE

Israel Mayk, Doctor of Engineering Science, 1985

Dissertation directed by: Dr. Solomon Rosenstark

Associate Professor

Department of Electrical Engineering

New Jersey Institute of Technology

By comparison to conventional communication systems, spread-spectrum systems are known to be less affected by interference because of their large dimensionality in signal space. Nevertheless, significant performance degradation is experienced when large interference exists in a few or even one signal coordinates. In this case, interference reduction techniques are also known to provide additional processing gain. A novel class of pseudonoise (PN) invariant algorithms is derived to reduce the impact of interference and restore much of the structure of PN signals received in the presence of interference and noise. A PN signal received by a pre-detection signal process (PDSP) implementing a PN invariant algorithm remains unchanged at the output. When an interference waveform is added to the PN signal, most of the DC bias as well as other smooth components of the interference may be significantly reduced at the output of the same PDSP. If n is the longest run in the PN sequence of maximal length N , and R_c is the chip rate, it is shown that the algorithms work well when the interference is sinusoidal with a frequency deviation from the carrier up to R_c/N . At such a low frequency deviation, the processing gain is observed to be relatively high and independent of the phase deviation. As the frequency deviation increases to nR_c/N , the performance of the spread-spectrum system decreases to the level that would have been obtained in the absence of the PDSP.

VITA

Name: Israel Mayk

Degree and date to be conferred: D.Engr. Sc., 1985

Secondary education: Highland Park High School, N.J June 1966

Collegiate Institutions	Date	Degree	Major	Date of Degree
Rutgers College	66-70	BA	Physics	June, 1970
Weizmann Institute	70-73	MS	Physics	June, 1973
NJIT	79-85	D.Engr.Sc.	EE	June, 1985

Position held: Electronics Engineer, U.S. Army
Fort Monmouth, New Jersey 07703

DEDICATION

To my darling babes,
Shyella Joy and Liatl Natanya,
who all your lives have had a father who was
"doing his Doctorate",
I thank you.

For it was only through
the "presence" of your joyful "noises"
and "bursty interferences"
that I managed to find the strength
to complete this work.

ACKNOWLEDGEMENTS

For the drive and perseverance not only to dream of this accomplishment but also to fulfill it, I thank my mother and father. They instilled in me the thirst for knowledge that enabled me to reach this goal.

The challenge of pure proof reading fell to my father-in-law, Marty Graham. I wish to express my appreciation to him and Grandma Bea for "freeing" us in many ways so that I could devote my attention to the research leading to this dissertation.

My deep gratitude goes to the Government for its excellent educational benefit program and, in particular, to Messrs. Jerry Dressner and Loren Diedrichsen who were also sensitive to the demands of my research pursuits.

The very foundations of my research are based on the informative course on Statistical Communication Theory taught by Professor Joseph Frank, who also provided helpful comments during this project.

I certainly wish to thank Professor Solomon Rosenstark, my advisor, for his invaluable guidance through the labyrinth of the NJIT Doctoral Program. His knowledge, suggestions, sense of humor and confidence in me provided the guideposts necessary to reach my destination.

My wife, Nissan, shared with me the anxieties and late nights inherent in juggling a job and writing a thesis of this scope and she graciously postponed most requests on my time except the absolute crises. Now I can mow the lawn.

TABLE OF CONTENTS

<u>Chapter</u>	<u>Page</u>
I : A PRE-DETECTION SIGNAL PROCESSING PROBLEM	
1.1 Introduction	1
1.2 Background	2
1.3 PN Communication Systems	8
1.3.1 Direct Sequencing(DS)	9
1.3.2 Frequency Hopping(FH)	13
1.3.3 FH/DS Hybrid Systems	16
1.4 Statement of the Problem	17
1.4.1 Suppression of Continuous Wave(CW) Interference	18
1.4.2 The Mathematical Framework and Scope	20
1.4.3 The Pre-Detection Signal Process(PDSP)	31
1.5 Dissertation Outline	35
II : RANDOMNESS PROPERTIES	
2.1 Introduction	38
2.2 Properties of Noise Vectors	40
2.2.1 The Polarity Property	44

TABLE OF CONTENTS (Cont.)

<u>Chapter</u>	<u>Page</u>
2.2.2 The Zero Crossing Property	47
2.2.3 The Autocorrelation Property	53
2.2.4 The Run Properties.....	58
2.3 Noise Invariance Transformations.....	66
2.4 Properties of PN Vectors.....	70
2.4.1 The Polarity Property.....	73
2.4.2 The Zero Crossing Property.....	75
2.4.3 The Autocorrelation Property.....	78
2.4.4 The Run Properties.....	80
2.5 PN Invariance Transformations.....	81
III : DETECTION IN THE PRESENCE OF AWGN	
3.1 Introduction.....	86
3.2 Signal Space Formulation.....	88
3.3 The Binary Likelihood Ratio Test(BLRT).....	91
3.3.1 The Logarithmic BLRT in a Low Noise Environment ..	101
3.3.2 The Logarithmic BLRT in a high Noise Environment ..	102
3.4 Errors in Additive White Gaussian Noise(AWGN).....	104

TABLE OF CONTENTS (Cont.)

<u>Chapter</u>	<u>Page</u>
3.5 Erasures in AWGN.....	110
3.5.1 Analog Signal Detection.....	114
3.5.2 Analog/Digital Hybrid Signal Detection.....	117
IV : DETECTION IN THE PRESENCE OF INTERFERENCE AND NOISE	
4.1 Introduction.....	127
4.2 The Vector Communication System with Interference and Noise.....	129
4.3 The Processing Gain of Pre-detection Signal Processes (PDSPs).....	135
4.4 Interference Suppression Using Randomness Properties	145
4.4.1 The Randomness Invariant Erasure Algorithm.....	147
4.4.2 The Randomness Invariant Average Algorithm.....	149
4.4.3 The Randomness Invariant Piece-wise Average Algorithm.....	153
4.4.4 The Randomness Invariant Piece-wise Linear-Average Correction Algorithm.....	156
4.5 Simulated PDSPs Performance Results	159
4.6 Performance Degradation in AWGN due to PDSP	167

TABLE OF CONTENTS (Cont.)

<u>Chapter</u>	<u>Page</u>
V : CONCLUSIONS AND RECOMMENDATIONS	
5.1 Introduction.....	170
5.2 Conclusions.....	170
5.3 Recommendations.....	171
APPENDIX A Functions Defined for the Normal Curve.....	172
APPENDIX B Invariant Algorithms Demonstration, Computer Program Listing.....	178
APPENDIX C Invariant Algorithms Demonstration, Statistical Results.....	207
APPENDIX D Invariant Algorithms Demonstration, Waveform Results.....	214
REFERENCES	222

LIST OF TABLES

<u>Table</u>	<u>Page</u>
4-1	160
Characteristic Parameters of Example PN Vector \underline{b} , Off-tone CW Interference Vector \underline{i} , and AWGN Vector \underline{n} and Their Linear Combinations	
4-2	162
The Sample Processing Gain (PG) of PDSPs F1 – F4 Corresponding to the Example PN Vector \underline{b} , and Off-tone CW Interference Vector \underline{i} as a Function of the Noise Standard Deviation σ	
4-3	164
The Sample PG of PDSP F4 and the Impact upon the Sufficient Statistic as a Function of the Amplitude Ratio α/b of Example Off-tone Interference and PN Signal	
4-4	165
The Sample PG of PDSP F4 and the Impact upon the Sufficient Statistic as a Function of the Relative Phase $\Delta\theta_k$ between Example PN Signal Carrier and Off-tone Interference.....	
4-5	166
The Sample PG of PDSP F4 and the Impact upon the Sufficient Statistic as a Function of the Relative Frequency Δf_k between Example PN Signal Carrier and Off-tone Interference.....	

LIST OF FIGURES

<u>Figure</u>	<u>Page</u>
1.3-1 A Basic DS PN Communication System	12
1.3-2 A Basic FH PN Communication System	15
1.3-3 A Basic DS/FH Hybrid PN Communication System	17
1.4-1 A Basic Coherent PSK DS PN Communication System	20
1.4-2 A Basic Coherent PSK DS PN Communication System with a PDSP Inserted	32
2.4-1 An n-Stage Feedback Shift Register (FBSR)	71
2.4-2 A 3-Stage Binary Maximal Length FBSR Corresponding to the Generating Polynomial $G(x) = x^3 + x + 1$	72
2.4-3 A 7-Stage Binary Maximal Length FBSR Corresponding to the Generating Polynomial $G(x) = x^7 + x^6 + x^4 + x + 1$	72
2.5-1 Transforming Higher Order PN Vectors into Lower Order Vectors	85
3.2-1 A Basic Communication System in the Presence of Noise: (a) Vector Representation; (b) N-Dimensional Signal Space Decomposition	91
3.4-1 The Binary Error Channel Model	105
3.4-2 The Sufficient Statistic Probability Density Functions in the Presence of Gaussian Noise	105
3.4-3 A 2-D Abstraction of Error Regions in DC and PN Systems	108

LIST OF FIGURES (Cont.)

<u>Figure</u>	<u>Page</u>
3.5-1 The Binary Error-Erasure Channel Model	110
3.5-2 A 2-D Abstraction of Errors and Erasure Regions in PN Systems	112
4.2-1 A Basic Vector Communication System in the Presence of Noise and Interference: (a) Vector Representation; (b) N-Dimensional Signal Space Decomposition	131
4.2-2 Example Mean Sufficient Statistic $I(T) = [\underline{i} + \underline{b}, \underline{b}]$ in the Presence of Off-tone ($\Delta f_k = 1/T$) CW Interference as a function of $\Delta \theta_k$	134
4.1-3 The Impact of Interference on the Sufficient Statistic for Binary Decisions in a Gaussian Noise Environment	135
A-1 The Q-Function.....	175
A-2 The Normal Probability Density Function.....	177
D-1 PN-Coded Waveform at Input to PDSPs.....	215
D-2 Relevant Interference at Input to PDSPs.....	215
D-3 Relevant Noise at Input to PDSPs.....	216
D-4 Received Signal at Input to PDSPs.....	216
D-5 Received and Distorted PN Waveforms at Output of PDSP 1.....	217
D-6 Received Waveform at Output of PDSP F 1 after Iteration *1.....	217

LIST OF FIGURES (Cont.)

<u>Figure</u>	<u>Page</u>
D-7 Received Waveform at Output of PDSP F2 after Iteration #1.....	218
D-8 Received Waveform at Output of PDSP F2 after Iteration #2.....	218
D-9 Received Waveform at Output of PDSP F2 after Iteration #3.....	219
D-10 Distorted PN Signal at Output of PDSP F2.....	219
D-11 Received Waveform at Output of PDSP F3 after Iteration #1.....	220
D-12 Distorted PN Signal at Output of PDSP F3.....	220
D-13 Received Waveform at Output of PDSP F4 after Iteration #1.....	221
D-14 Distorted PN Signal at Output of PDSP F4.....	221

CHAPTER I

A PRE-DETECTION SIGNAL PROCESSING PROBLEM

1.1 Introduction

The communications of digital data through spread-spectrum systems has been studied extensively over the last forty years. The subject, however, continues to influence technological discoveries as well as theoretical developments. In practical applications digital data is communicated over inherently noisy channels which are corrupted by intentional or unintentional interference from other users or polluters. The noise and /or the interference may introduce digital errors and erasures. The type of noise has been known to vary with the type of channel under consideration. In space applications, the channel noise is predominantly additive white Gaussian noise (AWGN). In contrast, the noise which contributes to errors and erasures on telephone lines is known to be predominantly impulsive [1]. Due to the impulse response of telephone lines, high intensity bursts of short duration typically occur at random statistically more frequently than bursts due to Gaussian noise. Fortunately, the inter-arrival time between bursts is long compared to typical burst durations. In terrestrial radio applications, the noise is likely to include both AWGN and impulsive noise. Analysis of received signals is further complicated by jamming, multipath fading, and doppler degradations. To combat impulsive noise, and bursty and/or sinusoidal CW interference, the use of basic pseudonoise (PN) systems is investigated in conjunction with the possible application of a novel

class of time domain pre-detection signal processes (PDSPs) designed to improve detection in a mix of noise and interference. It is shown that in the presence of Gaussian noise only, the amount of theoretical degradation in performance of a matched filter or correlative receiver, due to the insertion of a PDSP as an integral part of a PN system receiver, is insignificant and decreases rapidly with higher dimensionality of the PN signal space. In the presence of impulsive noise, bursty and/or sinusoidal CW interference mixed with AWGN, the amount of theoretical improvement due to the insertion of a PDSP preceding a matched filter or correlative receiver can be significant and increases as the dimensionality of the PN signal space increases as well as the average, slowly fluctuating, content of the interference increases.

1.2. Background

In a tutorial on the theory of spread spectrum communications, Pickholtz, *et al.* [2] define spread-spectrum as follows:

Spread-spectrum is a means of transmission in which the signal occupies a bandwidth in excess of the minimum necessary to send the information; the band spread is accomplished by means of a pseudorandom code which is independent of the data, and a synchronized reception with the code at the receiver is used for despreading and subsequent data recovery.

The underlined "pseudorandom" characterization of the code missing in the

reference quoted is an important criterion which is implicitly understood by all who study spread-spectrum systems, but should have been included in a fundamental definition. In a monograph on the origin of spread-spectrum communication, Scholtz [3] identifies the basic signal characteristics of modern spread-spectrum systems as follows:

- 1) The carrier is an unpredictable, or pseudorandom, wide-band signal.
- 2) The bandwidth of the carrier is much wider than the bandwidth of the data modulation.
- 3) Reception is accomplished by cross correlation of the received wide band carrier.

In this definition, strictly speaking, the carrier could not possibly be totally unpredictable since it must be known at the receiver. "Unpredictable" is not synonymous with "pseudorandom" which may be easy or hard to predict depending upon the pseudorandom code generator. While both definitions are consistent with each other, the fact that different wording is used to describe the same class of communication techniques implies that the theory of spread-spectrum system has not matured to provide a common set of terms and definitions as might be expected of a more established area of research.

More serious issues exist when comparing performance among spread-spectrum systems. Typically, the processing gain is used as a figure of merit. The processing gain in general communication theory is the ratio of the signal-to-noise ratio at the output of the receiver (SNR_o) to the signal-to-noise ratio at the input of the receiver (SNR_i). In the presence of interference,

however, it is understood that the noise term in the SNR must include interference in addition to random noise. Some authors, therefore, use the term signal-to-interference ratio (SIR). To make matters worse, the definition of the spread-spectrum system processing gain has been limited by many researchers [4], [5], [6], to the ratio of the dimensionality of the pseudorandom coded symbol to the dimensionality of the data code. This ratio has also been dubbed by some as the "spreading factor". It is the lack of commonly acceptable standard terminology which is believed to be, by this author, at the root of many misinterpretations of results among researchers and which may also contribute to the evolution of some of the myths attributed to spread-spectrum systems [7], [8].

The lack of standard terminology in spread-spectrum is most likely due to the fact that many applications utilizing a wide variety of techniques have been identified for spread-spectrum systems and developed independently over a relatively short period of time. These include communications in interference environments, covert communications, multiple-access communications, identification, ranging and relative navigation.

Historically, it seems that the most important impetus to the development of spread-spectrum systems is the requirement to communicate in the presence of interference. Interference may be intentional in which case it is also referred to as jamming. Unintentional interference is sometimes referred to as self-jamming. The spreading of the minimum essential communication bandwidth by wideband pseudorandom code, however, is limited by the ability of

technology to provide wider band components as well as frequency allocation for operational use of the spectrum resource. In operating within a limited spread-spectrum, interference effects are mitigated but continue to plague spread-spectrum systems. Other means, therefore, must be investigated to achieve greater processing gains beyond that afforded by basic spread-spectrum systems. Here again, many techniques have been proposed and investigated on an *ad hoc* basis [9]-[37].

Since spread-spectrum systems are extremely complex systems, there are numerous implementation alternatives. Each alternative will have its own processing gain corresponding to each interference phenomenon. Even the insertion of a simple device such as a limiter may significantly affect the processing gain [38].

The aim of this dissertation is to investigate the basic properties common to random noise and pseudorandom noise, hereafter referred to as pseudonoise (PN), and to exploit these properties in rejecting or suppressing interference phenomena in spread-spectrum communications. The approach is to use digital signal processing in the time-domain to perform randomness-invariant or almost invariant operations which randomize and, thereby, reduce non-random interference. It is shown that such signal processing would have minimal or no impact upon the PN coded signal in the absence of interference. This type of filtering is robust since it is independent of the detailed interference structure. As a figure of merit, the processing gain for individually received symbols and subsymbols are used as the lowest common denominator for evaluation of performance.

Detection of signals corrupted by random noise such as AWGN has been studied extensively and exhaustively by many researchers [39, p. 707]. This is true particularly for AWGN because AWGN lends itself to analytical analysis. This is also true because AWG-like noise is prevalent in many communication channels, if not as the primary then as a secondary source of noise. Therefore, in evaluating new techniques which process signals corrupted by any noise or interference phenomena, one cannot ignore AWGN as an important case. AWG-like noise may result in several ways. AWG-like noise may arise in benign environments due to thermal characteristics of the channel including some of the receiver components [40, p. 196]. In spread-spectrum systems, when subjected to tone interference or jamming, AWG-like noise may develop as a consequence of the PN despreading algorithm [2]. Broad-band jammers also transmit AWG-like noise of high power spectral density. It is important to remember that if interference were pure white random noise or broadband jamming with equivalent spectral density, the processing gain of spread-spectrum systems in general would be unity.

Detection of signals polluted by interference has also been studied extensively. The interest in such problems is derived from the reality of electromagnetic interference due to other communicators using the same frequency resource, electromagnetic impulses due to a variety of electrical phenomena, and intentional acts of jamming. Typically the approach to investigating performance characteristics assumes specific forms of interference, such as single-tone continuous wave (CW), multi-tone, self-interference, broadband interference, partial-band interference,

repeat-back interference, and electromechanical impulses. Similarly to the AWGN case, if a parametric model for the interference is known, one may design an optimum receiver. We note, however, that for arbitrarily fluctuating interference which is independent of the source, one cannot constrain the interference to the model. Invariably, when the actual interference deviates from the assumed interference, a significant performance deviation may result.

Reviewing the many types of filters designed to combat interference, parametric as well as non-parametric signal processing algorithms/ filters have been identified [9]. Consider, for example, the parametric class of adaptive filters. A model of the interference is predicted at each chip time interval using estimates from previous samples of the received waveform. Adaptation results from monitoring increasingly more samples to refine the interference model. Thus, adaptive filters self tune to subtract the best available interference estimate. Even for this class of sophisticated receivers, severe degradation may result when the statistics of the interference changes faster than the monitoring period. In contrast, non-parametric filters which are non-adaptive also perform signal processing which ascertains the parameters of any interference independently from one time period to the next. Filters in this class typically include transversal filters which may employ fast Fourier techniques to remove spectral components which are inconsistent with the power spectral density of the spread-spectrum signal [9]. In this dissertation, we propose a third class of noise suppression/rejection techniques which is independent of the interference parameters, *i.e.*, no estimation of the interference parameters are required, parametric or non-parametric. Namely, pre-detection signal processing is performed upon the received wideband

waveform to make it consistent with the randomness properties of the transmission. Inconsistencies with the randomness properties may be easily detected and may be used to declare erasures. Interference which is narrow band relative to the spread-spectrum may also be "corrected" by removing most of the non-conforming bias introduced by the interference.

1.3 PN Communication Systems

The purpose of a PN communication system is two-fold: a) to reduce the reception probability of error P_e , in the presence of intentional and/or unintentional interference (*e.g.*, jamming) and b) to reduce the transmission probability of interception in the presence of noise. These objectives may be achieved simultaneously by introducing PN to modulate the RF signals in a variety of ways. By spreading the spectrum from narrow band to very wide band, the signal power spectral density is hidden in the omnipresent thermal noise. In addition, the signal space is expanded into many more dimensions which are more likely to be orthogonal to a narrower band interference.

Of the many different techniques utilizing PN in communication systems [3], two basic techniques, a) direct sequence (DS) and b) frequency hop (FH), have emerged as most commonly considered. Theoretically, both techniques achieve the same two-fold purpose mentioned above. Practically, however, technological limitations require trade-offs to be made which sometimes may even result in the merging of both techniques into a hybrid FH/DS architecture.

An example of one important advantage of DS is the feasibility of coherent modulation and detection which is well known to be more efficient than non-coherent modulation and detection [39]. An example of one important advantage of FH is the feasibility of implementing wider time-bandwidth products than currently possible with DS devices. This is accomplished by using many independent and/or programmable frequency synthesizers. In the following subsections, we discuss the basic design of DS, FH and FH/DS hybrid spread-spectrum systems which is necessary to provide the context for the results presented in this dissertation.

1.3.1 Direct sequencing (DS). Consider a basic direct sequence PN communication system as shown in Figure 1.3-1. A sample N-dimensional PN vector \underline{b} , consisting of binary variable elements b_i , $i = 1 \dots N$, is used to generate a periodic bipolar PN rectangular waveform $b(t)$. The transmission time duration of one period is given by $T = N T_c$. T_c is assumed to be the transmission time duration of a single PN variable element which is commonly referred to as a chip. The source data binary symbol, such as a '0' or a '1', is of transmission time duration T_b . Typically, $T_b \leq T$ for a variety of reasons [41]. Long PN codes are required to minimize the predictability of the PN sequence being used. The processing gain in the presence of narrowband interference, however, is maximized when $T_b = T$. For simplicity, we assume that once the length of the PN code has been established to meet the predictability criterion, the data rate may be adjusted such that $T_b = T$, to minimize the interference. Expressed in terms of N time-orthogonal unit rectangular pulse functions $q_i(t)$, $i = 1 \dots N$, of

transmission time duration T_c , for one period T , $b(t)$ is given by

$$b(t) = \sum_{i=1}^N b_i q_i(t) = \sum_{i=1}^N b_i q[t-(i-1)T_c] \quad (1-1)$$

where

$$b_i = \pm b \quad \text{where} \quad b > 0, \quad (1-2)$$

$$q_i(t) = q[t-(i-1)T_c], \quad i=1 \dots N, \quad (1-3)$$

$$q(t) = 1, \quad \text{when} \quad 0 \leq t \leq T_c, \quad (1-4a)$$

and

$$q(t) = 0, \quad \text{otherwise.} \quad (1-4b)$$

The set of functions $\{q_i(t), i=1 \dots N\}$ form a complete set of time-orthogonal basis functions spanning the entire time interval T . Therefore, the scalar products

$$\langle q_i(t), q_j(t) \rangle = \int_{-\infty}^{+\infty} q_i(t) q_j(t) dt = T_c \delta_{ij} \text{ for } i, j=1 \dots N. \quad (1-5)$$

Similarly, consider a sample data message vector \underline{u} of dimension J . It may be used to generate a bipolar source data waveform

$$u(t) = \sum_{j=1}^J u_j p_j(t) = \sum_{j=1}^J u_j p[t-(j-1)T] \quad (1-6)$$

where

$$u_j = \pm 1, \quad (1-7)$$

$$p_j(t) = p(t - (j-1)T), \quad j=1 \dots J, \quad (1-8)$$

$$p(t) = 1, \quad \text{when } 0 \leq t \leq T, \quad (1-9a)$$

and

$$p(t) = 0, \quad \text{otherwise.} \quad (1-9b)$$

The set of functions $\{ p_j(t), j=1 \dots J \}$ also form a complete set of time-orthogonal basis functions, spanning the entire time interval T .

The waveforms $u(t)$ and $b(t)$ may be mixed directly, as shown in Figure 1.3-1, to produce a direct sequence signal waveform

$$s(t) = u(t)b(t) = \sum_{j=1}^J \sum_{i=1}^N u_j b_i q[t - (i-1)T_c]. \quad (1-10)$$

The signal waveform $s(t)$ then modulates a single carrier

$$z(t) = \cos(\omega_0 t + \theta_0), \quad (1-11)$$

where ω_0 is the angular frequency and θ_0 is the phase of the carrier. The resulting coherently transmitted waveform

$$x(t) = s(t) z(t). \quad (1-12)$$

The transmitted waveform $x(t)$ is attenuated by propagation phenomena and corrupted by random noise $N(t)$ and interference $I(t)$. The received waveform $y(t)$ is mixed with a local phase locked oscillator $z(t+\Delta t_z)$ in order to remove the carrier and produce a wideband baseband signal $r(t)$. The wideband baseband signal $r(t)$ is subsequently correlated with a synchronized replica of the PN waveform $b(t+\Delta t_o)$, which spreads the interference and despreads the data symbol signal, resulting in a "sufficient statistic" for a symbol decision. With perfect synchronization of the carrier and the PN sequence, the time offsets Δt_z and Δt_o are expected to vanish. In this dissertation we shall assume this to be the case.

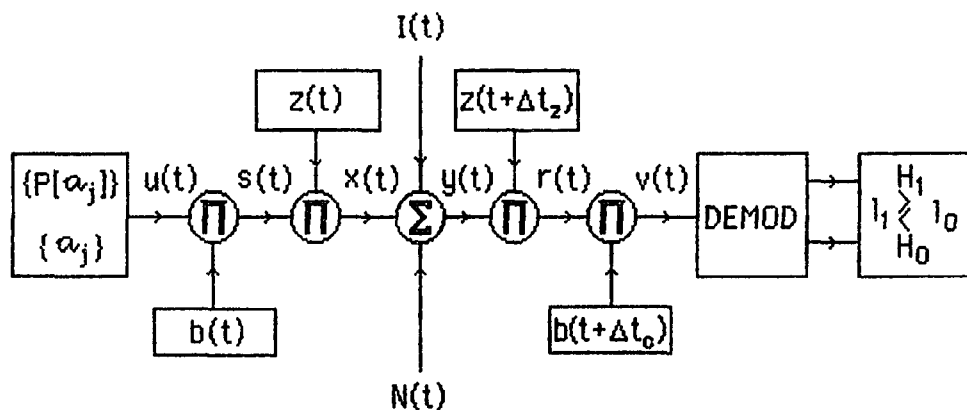


Figure 1.3-1. A Basic DS PN Communication System

1.3.2 Frequency hopping (FH). Consider a basic frequency hopping PN communication system as shown in Figure 1.3-2. A sample N-dimensional PN vector \underline{a} , consisting of unipolar binary variable elements a_i , $i = 1 \dots N$, is used to generate a periodic set of pseudorandom numbers $a(i)$, consisting of $n = \log_2(N+1)$ digits. Since $a(i) = a(i+N)$, the transmission time duration associated with one period is given by $T = N T_c$. T_c is assumed to be the transmission time duration associated with a single PN number $a(i)$, which is also commonly referred to as a chip. The number $a(i)$ is used to select and/or program one of $M < N$ frequency synthesizers available at the transmitter and the receiver. The frequency selected is then activated as the carrier of the source data waveform. We assume that the source data symbol a_j is binary, such as a '0' or a '1', and is of transmission time duration, T_b . Typically, $T_b \ll T$ since long PN codes are required to minimize the predictability of the PN sequence being used. The processing gain in the presence of partial band interference, however, is dependent predominantly upon M and should be independent of N . We assume that once the length of the PN code has been established to meet the predictability criterion, the frequency hops are uniformly distributed to minimize the interference. The number $a(i)$, therefore, corresponds to the m^{th} frequency synthesizer or carrier $z_m(t)$, which is modulated directly by a bipolar source data waveform $u(t)$, for T_c units of time. Typically,

$$z_m(t) = \cos(\omega_m t + \theta_m), \quad (1-13)$$

where ω_m is the angular frequency and θ_m is the phase of the carrier.

If the time duration of a source data symbol $T_b \leq T_c$, then the frequency hops occur only once per one or more symbols. This is known as slow hopping. For a message with J symbols and one hop per message, the transmitted waveform

$$x(t) = A \sum_{j=1}^J u_j z_m(t) q[t-(j-1)T_c]. \quad (1-14)$$

If the time duration of a source data symbol $T_b \gg T_c$, then many frequency hops occur per symbol. This is known as fast hopping. For a message with J symbols and N hops per symbol the transmitted waveform

$$x(t) = A \sum_{j=1}^J \sum_{i=1}^N u_j z_m(t) q[t-(i-1)T_c]. \quad (1-15)$$

The sample N -dimensional PN vector \underline{a} , consisting of unipolar binary variable elements a_i , $i = 1 \dots N$, may also be used to generate a periodic bipolar PN rectangular waveform $b_a(t)$, as in (1-1), and pseudo-randomize the source data prior to mixing with the selected carrier. For a message with J symbols and N hops per symbol the transmitted waveform is modified to

$$x(t) = A \sum_{j=1}^J \sum_{i=1}^N u_j b_i z_m(t) q[t-(i-1)T_c]. \quad (1-16)$$

At the receiver, the received waveform $y(t)$ consists of an attenuated $x(t)$ corrupted by random noise $N(t)$ and interference $I(t)$. $y(t)$ is mixed with the local oscillator $z_m(t+\Delta t_o)$ selected by the current hop number $a(i+\Delta i)$ to produce a narrowband or a wideband baseband signal corresponding to slow hopping or fast hopping, respectively. Both the hop number offset Δi and the carrier activation time offset Δt_o are expected to vanish if the current hop number at the receiver is synchronized with the transmitted hop. If required, the baseband signal $r(t)$ may be mixed with a synchronized replica of the PN waveform $b_s(t+\Delta t_o)$. The resulting waveform $v(t)$ is then processed by conventional detection techniques appropriate to the modulation format [40, p. 562].

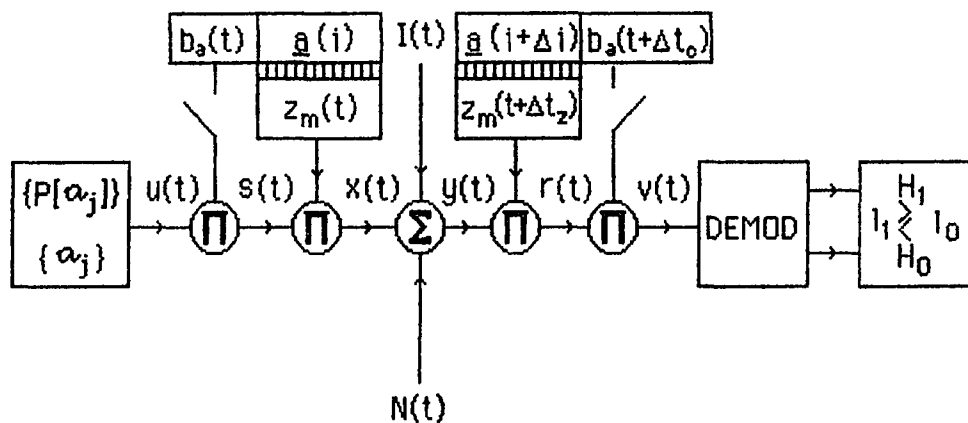


Figure 1.3-2. A Basic FH PN Communication System

1.3.3 FH/DS hybrid systems. A hybrid system results when features from both DS and FH systems are integrated into one system. The salient characteristic is the use of two independent PN variable vectors, \underline{a} and \underline{b} . \underline{b} is used to spread the source data waveform $u(t)$ with a periodic direct sequence $\{b_k, k = 1 \dots K\}$. \underline{a} is used to select from or "frequency hop" to different carriers $z_m(t)$. Within this class of systems various techniques exist, ranging from situations in which the period of the PN for direct sequence is totally contained within each hop as given by

$$x(t) = A \sum_{j=1}^J \sum_{i=1}^N \sum_{k=1}^K u_j b_k q[t-(k-1)T_b] z_m(t) q[t-(i-1)T_a], \quad (1-17)$$

to situations in which their clocks are identical. A basic FH/DS PN hybrid system is shown in Figure 1.3-3. As for all communication systems, the transmitted waveform $x(t)$ is attenuated by propagation losses and corrupted by random noise $N(t)$ and interference $I(t)$. The received waveform $y(t)$ is mixed with the local oscillators $z_m(t)$, switched by the PN FH generator replica to remove the carrier and produce a baseband signal $r(t)$. The DS signal $s(t)$, embedded in the baseband signal $r(t)$, is subsequently correlated with a replica of the PN waveform $b(t)$, which spreads the interference and despreads the source data signal $u(t)$, to produce the waveform $v(t)$. A conventional demodulation process will then result in a "sufficient statistic" for a symbol decision.

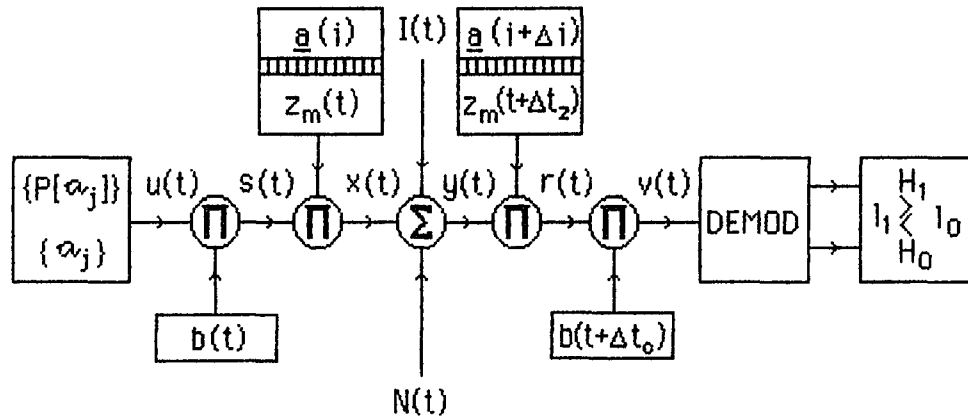


Figure 1.3-3. A Basic DS/FH Hybrid PN Communication System

1.4 Statement of the Problem

As discussed in the background, when compared to conventional systems, PN systems attempt to satisfy the much more demanding requirements of achieving a) low probability of interception $P_i \ll 1$, in the presence of sophisticated radiometers and b) low probability of error $P_e \ll 1$, in the presence of interference. In addition, the PN system must perform comparably as well as conventional systems in the presence of random noise. Given the basic design of PN systems, can we enhance the design to improve both performance measures? If not, can we improve in one area and not degrade the performance in the other area? Tradeoffs between the optimal solutions to these two problems depend upon the assigned objective-risk function which must include the relative importance between P_i and P_e . The design parameters are constrained by the source data symbol signal transmission time, available bandwidth and power

parameters which may also be adjusted within limits. By increasing the source data symbol signal transmission time or power, we may decrease P_e at the expense of P_i . By increasing the bandwidth, both requirements for P_i and P_e may be achieved.

Given a basic PN system with a time-bandwidth-power constraint, a class of robust pre-detection digital signal processing (PDSP) algorithms is proposed as one way to enhance receiver performance in typical interference environments with minimum degradation of performance in random noise environments and no degradation in P_i . Conversely, if in the absence of a PDSP algorithm, the maximum tolerated P_e is achieved with the basic available interference suppression processing gain margin, the introduction of a PDSP algorithm may significantly increase the processing gain which may then be readjusted to enhance P_i by the lowering of the data symbol signal transmission time and/or power requirements.

1.4.1 Suppression of Continuous Wave (CW) Interference. Consider the basic DS PN coherent PSK system shown in Figure 1.4-1. One figure of merit for the reduction in the impact of an interfering signal upon reception is the processing gain (PG). The PG is usually defined as the ratio given by

$$PG = SNR_{out} / SNR_{in} \quad (1-18)$$

where SNR_{in} denotes the ratio of the signal power to the sum of the interference

and noise power at the receiver input, and SNR_{out} denotes the ratio of the signal power to the sum of the interference and noise power at the output of the desreader which may be measured at the input to the conventional demodulator. As noted by other authors, [4, p. 348], [42, p. 140], the PG of a PN system is highly sensitive to the parameters of both modulation and interference waveforms, and therefore should only be used in the context of specific system design parameters subjected to interference of a specific structure and statistical properties. It should not be used to compare performances between different types of spread spectrum systems. For example, the PG of a basic coherent PSK DS PN system in the presence of a single frequency CW interference is critically dependent upon the phase relationship between the carrier and the interference which is assumed to be tuned to the signal carrier. For a phase relationship between the carrier and interference which is constant during the entire PN sequence, it was shown by Levitt [6] that

$$PG_{\theta} = (R_c / R_b) \cos^2(\Delta\theta_k) \quad (1-19)$$

where $R_b = 1/T_b$, $R_c = 1/T_c$, and $\Delta\theta_k$ is the phase difference between the PN carrier and the interference tone. The fact that the PG has such a wide range given by

$$(R_c / R_b) < PG < \infty \quad (1-20)$$

provides the challenge to search for robust signal processing techniques which

may reduce this range by increasing the worst case PG.

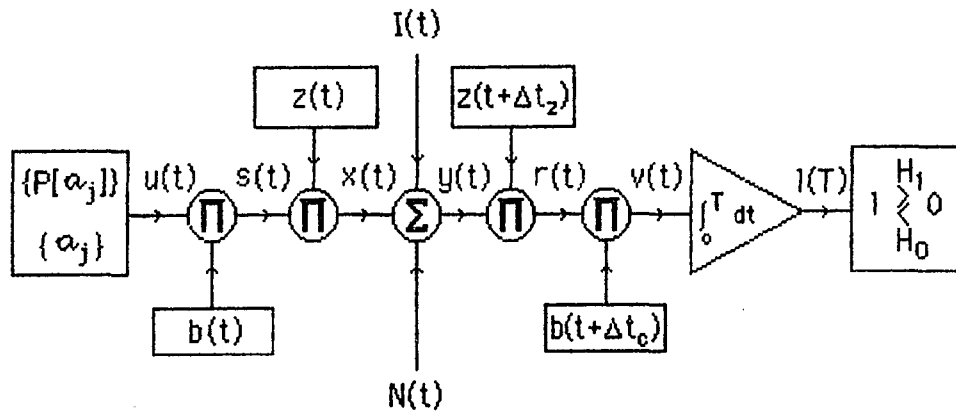


Figure 1.4-1. A Basic Coherent PSK DS PN Communication System

1.4.2 The mathematical framework and scope. Referring to Figure 1.4-1, the selection of hypothesis H_0 or H_1 depends upon the sufficient statistic

$$I(T) = \int_0^T v(t) dt. \quad (1-21)$$

Since

$$v(t) = r(t)b(t+\Delta t_0), \quad (1-22)$$

and

$$r(t) = y(t)z(t+\Delta t_2), \quad (1-23)$$

therefore,

$$v(t) = y(t)z(t+\Delta t_2)b(t+\Delta t_0). \quad (1-24)$$

The input to the receiver is given by

$$y(t) = Ax(t) + I(t) + N(t), \quad (1-25)$$

where A is the received amplitude of the transmitted waveform

$$x(t) = s(t) \quad z(t) = u(t)b(t)z(t). \quad (1-26)$$

Using (1-23) through (1-26), the output of the carrier mixer is given by

$$r'(t) = [Au(t)b(t)z(t) + I(t) + N(t)] z(t+\Delta t_z). \quad (1-27)$$

We assume coherent detection with perfect synchronization between the carrier and local oscillator, *i.e.* $\Delta t_z = 0$. Furthermore, we assume that the RF is filtered out of $r'(t)$ by passing it through a low pass filter (LPF) of bandwidth $R_o = 1/T_o$. The wideband baseband signal $r(t) = \text{LPF}[r'(t)]$ may be written, therefore, as the sum of three terms, *i.e.*

$$r(t) = r_s(t) + r_I(t) + r_N(t). \quad (1-28)$$

The first contribution to $r(t)$ is due to the signal and is given by

$$r_s(t) = \text{LPF}[Au(t)b(t)z^2(t)] = \frac{1}{2}As(t). \quad (1-29)$$

The second contribution to $r(t)$ is due to the interference and is given by

$$r_I(t) = \text{LPF}[I(t)z(t)] = i(t). \quad (1-30)$$

The third contribution to $r(t)$ is due to the random noise and is given by

$$r_N(t) = \text{LPF}[N(t)z(t)] = n(t). \quad (1-31)$$

We also assume coherent detection with perfect synchronization between the modulating sample PN vector and receiver replica, *i.e.* $\Delta t_0 = 0$. In addition, let the magnitude of the PN waveform $b = 1$. Since $[b(t)]^2 = 1$, mixing $r(t)$ with $b(t)$, we obtain

$$v(t) = \frac{1}{2}Au(t) + i(t)b(t) + n(t)b(t). \quad (1-32)$$

Note that $v(t)$ is also the sum of three terms, *i.e.*

$$v(t) = v_S(t) + v_I(t) + v_N(t), \quad (1-33)$$

where the first contribution to $v(t)$ is due to the signal and is given by

$$v_S(t) = \frac{1}{2}Au(t), \quad (1-34)$$

the second contribution is due to the interference and is given by

$$v_I(t) = i(t)b(t), \quad (1-35)$$

and the third contribution is due to the random noise and is given by

$$v_N(t) = n(t)b(t). \quad (1-36)$$

The sufficient statistic is, therefore, also the sum of three terms given by

$$I(T) = I_S(T) + I_I(T) + I_N(T) \quad (1-37)$$

where

$$I_S(T) = \int_0^T v_S(t) dt, \quad I_I(T) = \int_0^T v_I(t) dt, \quad \text{and} \quad I_N(T) = \int_0^T v_N(t) dt. \quad (1-38)$$

The BPSK signal contribution. Of the many coherent data modulation techniques which exist, binary phase shift keying (BPSK) [40, p. 552] is very commonly used. Compared to other coherent modulation techniques such as minimum shift keying (MSK) [40, p. 556] , or quadriphase shift keying (QPSK) [40, p. 570], BPSK is the simplest to analyze as well as to implement. Other modulation formats, however, may be used when the signal-to-noise is high or when spectral efficiency is of concern. For BPSK signals, the source data waveform is given by

$$u(t) = +1 \quad \text{for a '1'} \quad (1-39a)$$

$$u(t) = -1 \quad \text{for a '0'}. \quad (1-39b)$$

Using (1-38) and (1-39),

$$I_S(T) = \pm \frac{1}{2} A \int_0^T u(t) dt = \pm \frac{1}{2} AT. \quad (1-40)$$

Note that the contribution of the signal to $I(T)$ is unaffected by the PN vector. For the remainder of this dissertation we shall consider only the BPSK modulation format.

The single-tone CW interference contribution. For a single-tone CW interference, the waveform at the input to the receiver is given by

$$I(t) = \alpha \cos(\omega_k t + \theta_k) \quad (1-41)$$

where α is the interference amplitude, ω_k is the constant angular frequency and θ_k is an arbitrary but constant phase. Using (1-30) and (1-41), (1-35) becomes

$$v_1(t) = \alpha \text{LPF}[\cos(\omega_k t + \theta_k) \cos(\omega_o t + \theta_o)] b(t). \quad (1-42)$$

Let $\psi_k = \omega_k t + \theta_k$, and $\psi_o = \omega_o t + \theta_o$. Using the trigonometric identity

$$\cos(\psi_k) \cos(\psi_o) = \frac{1}{2} \cos(\psi_k - \psi_o) + \frac{1}{2} \cos(\psi_k + \psi_o), \quad (1-43)$$

(1-42) becomes

$$v_I(t) = \frac{1}{2}\alpha b(t) \text{LPF}[\cos[(\omega_k - \omega_o)t + (\theta_k - \theta_o)] + \cos[(\omega_k + \omega_o)t + (\theta_k + \theta_o)]] \quad (1-44)$$

Assuming that $\omega_k \approx \omega_o$, the sum frequency component of $v_I(t)$ will be greatly attenuated by the receiver wideband baseband LPF. Ignoring the sum frequency term,

$$v_I(t) = \frac{1}{2}\alpha b(t) \cos(\Delta\omega_k t + \Delta\theta_k) \quad (1-45)$$

where

$$\Delta\omega_k = \omega_k - \omega_o \text{ and } \Delta\theta_k = \theta_k - \theta_o. \quad (1-46)$$

Using the trigonometric identity for the cosine of the sum of two angles, (1-45) may be rewritten as

$$v_I(t) = \frac{1}{2}\alpha b(t) [\cos(\Delta\omega_k t) \cos(\Delta\theta_k) - \sin(\Delta\omega_k t) \sin(\Delta\theta_k)]. \quad (1-47)$$

Note that $v_I(t)$ may contribute constructively or destructively depending upon the sign of $v_S(t)$ relative to the dominant inphase or quadrature interference term in (1-47). The contribution of the interference to the sufficient statistic $I(T)$, therefore, also consists of an inphase component given by

$$L_o = \frac{1}{2} \alpha \cos(\Delta\theta_k) I_o \quad (1-48)$$

and a quadrature component given by

$$L_s = \frac{1}{2} \alpha \sin(\Delta\theta_k) I_s, \quad (1-49)$$

where

$$I_c = \int_0^T b(t) \cos(\Delta\omega_k t) dt, \quad (1-50)$$

and

$$I_s = \int_0^T b(t) \sin(\Delta\omega_k t) dt. \quad (1-51)$$

Consider the following special cases:

$$a) \quad \Delta\omega_k = 0, \text{ and } \Delta\theta_k = \frac{1}{2}\pi, \Rightarrow I_I(T) = 0.$$

$$b) \quad \Delta\omega_k = 0, \text{ and } I_c = 0, \Rightarrow I_I(T) = 0.$$

If we define $SNR_{out} = [I_s(T) / (I_I(T) + I_N(T))]^2$, then using (1-18), note that $PG = \infty$ when a) and b) hold and when the random noise power is negligible compared to the interference power. When

$$c) \quad \Delta\omega_k \neq 0, \text{ and } \Delta\omega_k T \approx 0,$$

$$I_I(T) \approx \frac{1}{2} \alpha \cos(\Delta\theta_k) \int_0^T b(t) dt. \quad (1-52)$$

For a PN vector with $N^+ = N^-$, the integral vanishes and $I_I(T) \approx 0$. For a PN vector

with $N^+ = N \pm 1$, however,

$$I_I(T) \approx \pm \frac{1}{2}(\alpha T/N)\cos(\Delta\theta_k), \quad (1-53)$$

and, hence,

$$|I_I(T)| \leq \frac{1}{2}(\alpha T/N). \quad (1-54)$$

Since $SNR_{in} = (A/\alpha)^2$ when the random noise power is negligible compared to the interference power, $PG \geq N^2$ when c) holds. Note that this is a factor of N better than reported by other authors for the case where the PN period, T , is much longer than the symbol decision period, T_b .

To obtain the average of PG which is $E[PG]$, consider an interference only environment, in which $I(T) \approx I_{SI}(T) = I_S(T) + I_I(T)$. Therefore,

$$I(T) \approx \pm \frac{1}{2}AT + \frac{1}{2}\alpha\cos(\Delta\theta_k)\int_0^T b(t)dt. \quad (1-55)$$

The normalized sufficient statistic is given by

$$I_I(T) \approx \pm 1 + (\alpha/A)\cos(\Delta\theta_k)T^{-1}\int_0^T b(t)dt. \quad (1-56)$$

Note that if the period of $b(t)$ deviates from T , the integral of $b(t)$ may cause the

interference to significantly degrade the PG and affect the binary decision. This phenomenon is discussed by Levitt [6] and Singh [41]. We shall assume, therefore, that the period of $b(t)$ is T which results in

$$I_1(T) \approx \pm 1 + (\alpha/NA)\cos(\Delta\theta_k). \quad (1-57)$$

Since the interference is independent of the source, we assume that $\Delta\theta_k(t)$ is a slowly varying random process defined by samples $\Delta\theta_k(t_j)$ which are essentially constant within $(j-1)T < t < jT$, $j = 1 \dots J$, but are uniformly distributed between π and $-\pi$ when sampled every j^{th} symbol. Averaging over $\Delta\theta_k$, the following statistical results are obtained:

$$E[I_1(T)] = \pm 1 \text{ depending upon whether a '0' or a '1' is transmitted,}$$

and

$$\text{Var}[I_1(T)] = E[(I_1(T) - \pm 1)^2] = E[(\alpha/NA)\cos(\Delta\theta_k)]^2 = \frac{1}{2}(\alpha/NA)^2.$$

It follows that the average SNR_{out} associated with $I(T)$ is given by

$$\langle \text{SNR}_{\text{out}} \rangle = \{ E[I_1(T)] \}^2 / \{ \text{Var}[I_1(T)] \} = 2N^2(A/\alpha)^2 \quad (1-58)$$

Since $\text{SNR}_{\text{in}} = (A/\alpha)^2$, the average processing gain is given by $\langle \text{PG} \rangle = 2N^2$.

The Random Noise Contribution. Consider, finally, the random noise term $I_N(T)$, given by (1-38). Recall that $n(t) = \text{LPF}[N(t)z(t)]$ where $N(t)$ is a stationary

white Gaussian noise process of zero-mean, and power spectral density $S_N(f) = \frac{1}{2}N_0$. The autocorrelation of $N(t)$, therefore is given by

$$R_N(t_1, t_2) = \mathcal{F}^{-1} \{ S_N(f) \} = \frac{1}{2}N_0 \delta(t_1 - t_2), \quad (1-59)$$

where \mathcal{F}^{-1} denotes the inverse Fourier transform. Since $z(t)$ is a zero-mean unit amplitude sinusoid, $N(t)z(t)$ is also a stationary white Gaussian noise process of zero-mean, and power spectral density $S_{Nz}(f) = \frac{1}{2}N_0$. Thus, the autocorrelation of $N(t)z(t)$ is similarly given by

$$R_{Nz}(t_1, t_2) = \mathcal{F}^{-1} \{ S_{Nz}(f) \} = \frac{1}{2}N_0 \delta(t_1 - t_2). \quad (1-60)$$

Since the process $n(t)$ is the output of low-pass filtering of a white zero-mean Gaussian process, $n(t)$ is no longer white but it is still zero-mean Gaussian with power spectral density

$$S_n(f) = \frac{1}{2}N_0 \quad \text{for } |f| \leq f_b, \quad (1-61a)$$

$$S_n(f) = 0 \quad \text{for } |f| > f_b, \quad (1-61b)$$

where $f_b = 1/T_b$ is the lowpass filter cutoff frequency. The autocorrelation of $n(t)$, therefore, is given by

$$R_n(t_1, t_2) = \mathcal{F}^{-1} \{ S_n(f) \} = \frac{1}{2}N_0 \sin[2\pi f_b(t_1 - t_2)] / [\pi(t_1 - t_2)], \quad (1-62)$$

and the variance of $n(t)$ is given by $(\sigma_n)^2 = R_n(0) = N_0 f_b = N_0 / T_b$. Since the $b(t)$ is an independent almost zero-mean binary psuedo-random process, the process $v_N(t) = n(t)b(t)$ is also expected to be an almost zero-mean Gaussian process equivalent to $n(t)$. It follows then from the definition of a Gaussian process that the integration output $I_N(T)$ is a sample value of the zero mean Gaussian random variable $I_N(T)$. The variance of $I_N(T)$ is given by

$$(\sigma_N)^2 = \text{Var}[I_N(T)] = E[(I_N(T) - E[I_N(T)])^2] = E[I_N(T)^2] \quad (1-63)$$

$$(\sigma_N)^2 = E\left[\int_0^T \int_0^T n(t_1)n(t_2)b(t_1)b(t_2)dt_1 dt_2\right] \quad (1-64)$$

$$(\sigma_N)^2 = \int_0^T \int_0^T R_n(t_1, t_2)dt_1 dt_2 . \quad (1-65)$$

Since $n(t)$ is is band limited, if it is sampled at twice the highest frequency f_b , the autocorrelation function vanishes at these sample points, *i.e.*

$$\begin{aligned} R_n(i/2f_b, j/2f_b) &= \delta_{ij}(\sigma_n)^2 = \delta_{ij}R_n(0) \\ &= \delta_{ij}N_0f_b = \delta_{ij}N_0 / T_b \quad \text{for any integer } i, j, \end{aligned} \quad (1-66)$$

where

$$\delta_{ij} = 1 \text{ for } i = j \text{ and } \delta_{ij} = 0 \text{ for } i \neq j.$$

Assuming that sampling occurs at the Nyquist rate, integration of (1-65) yields

$$(\sigma_N)^2 = N_0 T. \quad (1-67)$$

Note that, as expected, no processing gain is possible for PN systems in which the interference may be neglected and the random noise is dominant.

1.4.3 The pre-detection signal process (PDSP). Consider the insertion of a signal process, \mathcal{D} , as shown in Figure 1.4-2. The input to \mathcal{D} is given by $r(t)$, which is the received wide-band baseband waveform output of the RF LPF. The output of \mathcal{D} will then continue into the PN correlator which consists of the PN mixer and integrator. The selection of hypothesis H_0 or H_1 will now depend upon a new sufficient statistic given by

$$\ell(t) = \int_0^T v(t) dt \quad (1-68)$$

where

$$v(t) = \rho(t)b(t+\Delta t_c) \quad (1-69)$$

and

$$\rho(t) = \mathcal{D}[r(t)]. \quad (1-70)$$

The PDSP, therefore, is completely characterized by the operator $\mathcal{D}[\]$.

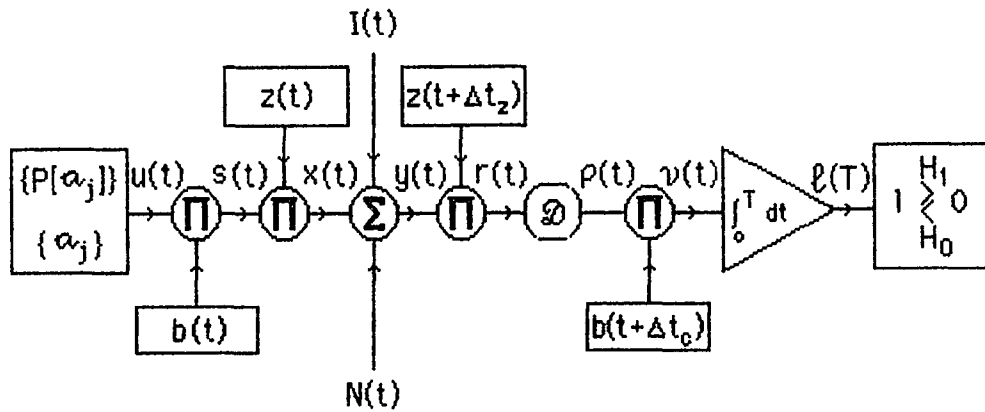


Figure 1.4-2. A Basic Coherent PSK DS PN Communication System with a PDSP \mathcal{D} Inserted

Assuming perfect synchronization as before, *i.e.* $\Delta t_z = \Delta t_c = 0$,

$$v(t) = \mathcal{D}[r_S(t) + r_I(t) + r_N(t)] b(t). \quad (1-71)$$

Thus, \mathcal{D} operates on the sum of three terms. If \mathcal{D} operated individually upon each of the three terms in the sum, we would have

$$\rho_S(t) = \mathcal{D}[r_S(t)], \quad \rho_I(t) = \mathcal{D}[r_I(t)], \quad \text{and} \quad \rho_N(t) = \mathcal{D}[r_N(t)]. \quad (1-72)$$

The ideal output of \mathcal{D} , therefore, would be given by

$$\rho_S(t) = r_S(t), \quad \rho_I(t) = 0, \quad \text{and} \quad \rho_N(t) = r_N(t). \quad (1-73)$$

The first term implies that the source data signal is passed undistorted. The second term implies that the interference is totally rejected. Finally, the third term implies that the noise is undistorted as required for optimum detection.

Unfortunately, the contribution of each waveform to the total waveform cannot be isolated. Nevertheless, if spectral characteristics or other properties are known to be dominant in one waveform and not in the others, one may design a signal process analogous to a matched filter to detect those characteristics and suppress them if associated with the interference, or enhance them if associated with the desired signal. For the PN BPSK communications in the presence of sinusoidal CW interference and AWGN as previously described, we have

$$v(t) = \mathcal{D}[\frac{1}{2}As(t) + \frac{1}{2}\alpha\cos(\Delta\omega_k t + \Delta\theta_k) + n(t)] b(t). \quad (1-74)$$

Comparing $v(t)$ and $v(t)$, the individual contributions of the signal, the interference and the random noise can no longer be explicitly decoupled since in general \mathcal{D} may be non-linear, *i.e.* for any two real functions,

$$\mathcal{D}[f(t) + g(t)] \neq \mathcal{D}[f(t)] + \mathcal{D}[g(t)]. \quad (1-75)$$

Noting, however, that the desired signal is wideband PN coded with characteristics similar to random noise, interference rejection filters have been designed by others to notch out strong spectral components which are non-random. In this dissertation we investigate the use of the randomness

properties as a complementary technique to detect and suppress non-random interference.

The problem may now be stated more concisely as follows:

Given a basic PN system, find \mathcal{D} such that when $\alpha^2 \gg \sigma_N^2$,

$$PG|_{\mathcal{D}} > PG|_{\mathcal{I}} \quad (1-76)$$

or equivalently

$$P_e|_{\mathcal{D}} < P_e|_{\mathcal{I}}. \quad (1-77)$$

When $\alpha^2 \ll \sigma_N^2$,

$$PG|_{\mathcal{D}} \approx PG|_{\mathcal{I}}, \quad (1-78)$$

or equivalently

$$P_e|_{\mathcal{D}} \approx P_e|_{\mathcal{I}}, \quad (1-79)$$

where \mathcal{I} is the identity operator.

In this dissertation, the discussion of the PDSP \mathcal{D} will be limited in its application to the basic DS PN coherent BPSK system depicted in Figure 1.4-2. Extensions to other types of implementations of PN systems are suggested for future work.

1.5 Dissertation Outline

The problem of searching for possible enhancements to the receiver structure of spread spectrum systems is by no means complete. In this thesis, however, the results achieved thus far which show significant improvements in robustness and simplicity of design are motivated and analyzed. Given the complexity of spread spectrum systems and the non-linear nature of the algorithms which were derived from the structure of PN signals, most of the analysis was carried out using a computer-aided-design tool developed as part of this effort. Initially, a primitive spread spectrum simulator was built, including modules for data sources, PN modulators, noise generators, interference generators and PN receiver correlators. Subsequently, experiments were conducted to check various hypotheses, some of which are discussed in the appropriate sections.

Chapter I provides the introduction and background to the statement of the problem. Following a discussion of basic spread-spectrum techniques, the key issue of interference suppression beyond the capability of basic spread-spectrum systems is formulated mathematically, motivating the importance of pre-detection signal processing.

Chapter II introduces the randomness properties most often associated with PN coded signals. First we investigate the probabilistic manifestation of the randomness properties of random noise and prove their invariance to certain signal processes which are said to preserve the randomness properties.

Subsequently, we investigate the deterministic manifestations of the randomness properties of pseudonoise and show their transparency to the same randomness invariant transformations.

Chapter III is devoted to detection in the presence of Gaussian noise. Many communication systems utilize optimum binary detection for decisions in the presence of identically and independently distributed noise. Using signal space formulation, we derive the form of the sufficient statistic for optimum binary detection in the presence of non-identical and independently distributed noise and show that, as expected, the means and variances are identical for conventional and spread-spectrum communication systems. Since *a priori* knowledge of the noise parameters is most likely not available at the receiver, a decision rule which conditions the optimum binary decision on the outcome of many suboptimum decisions is described.

Chapter IV is concerned with the impact of interference on the optimum binary decision. Here we introduce the vector processing gain using signal space formulation and use it to evaluate the relative performance of four PDSPs which are also described. It is observed that improvements in processing gain of the PDSPs are not directly related to improvement in the overall processing gain of the spread-spectrum system. Numerous simulation results are shown to provide significant insight into the dependence of the processing gain upon both signal and interference waveform structure.

Appendix A includes some useful relations for the normal curve. In this

dissertation, probabilities of error are given in terms of the Q-function. The Q-function is plotted to provide the reader with an appreciation for its convexity. Moreover, the Q-function is related to other error functions which are often found in the literature.

Appendix B includes a listing of a program which may be used to demonstrate the utility of a selected PDSP in reducing a specific interference vector. Typical sample waveform inputs and outputs are shown in Appendix D corresponding to the narrative statistical inputs and outputs provided in Appendix C.

The contribution of this paper to communications includes:

- (a) a novel class of pre-detection signal processes useful in reducing the impact of often encountered interference phenomena;
- (b) a hybrid analog/digital decision logic which may be optimized for the binary error-erasure channel in the presence of non-identically and independently distributed noise phenomena;
- (c) a greater insight into the randomness properties of random noise in contrast to the "randomness" properties of pseudonoise;
- (d) a framework for designing and evaluating many other interference suppression techniques.

CHAPTER II

THE RANDOMNESS PROPERTIES

2.1 Introduction

Random variables $n_i, i = 1 \dots N$, may be obtained by uniformly sampling a random process $n(t)$. The random variables may be fully characterized by a probability function for discrete random variables or by a density function for continuous random variables. When sampling a random process, one may define as many properties of the process as the number of different types of experiments which one may choose to conduct. Typically, one may perform a series of Bernoulli experiments or trials to determine whether or not a given observation is consistent with or matches a given criterion or a set of criteria. The random variables associated with the sequence of such outcomes are known to be binomially distributed [43]. Many random experiments identify the set of outcomes Ω , the events of interest ξ , and the probability P on ξ . Consider, however, the design of an experiment in which a success outcome $A = A$ means that the observations are random and a failure outcome $A = \text{Not}(A)$ means that the observations are non-random. The random variables associated with such an experiment are also binomially distributed and after one or more iterations should converge to the degree of randomness inherent in the initial property for which randomness was of interest. If $R(A)$ is the relative frequency of observing random events A , according to Borel's law of large numbers [43],

$$P(|R(A) - P(A)| > \epsilon) \rightarrow 0, \text{ as } N \rightarrow \infty \quad (2-1)$$

where N is the number of independent and identical trials undertaken for the experiment and $P(A)$ is the probability that ξ is indeed random. Thus, if ξ is random, $P(A = A) = 1$.

For any random variable n , if one has *a priori* knowledge of its distribution, then one may design an experiment which tests for the degree to which any sequence of observations $\{n_i, i=1 \dots N\}$ is consistent with any one of the moments of n . The easiest moment to test for consistency is the zeroth moment whose test statistic is simply the sample mean. Sufficient statistics associated with other moments may also be used but involve more computational complexity. Another type of test is to consider the ratio of the autocorrelation to the crosscorrelation. If the samples are independent, this ratio should be large relative to the sample mean. Given a mean, one may consider testing for the various distribution of runs in sample sequences where a run is defined as a sequence of consecutive samples which do not cross the mean. Many different run properties may be defined, such as the length of runs, the number of runs, and the number of runs of a particular length and a given polarity. In addition, one may test for similar properties associated with the mean level crossing.

In communication systems, the key to performing a test is to keep it simple yet reliable and robust. In this chapter, we discuss several properties of random processes which may be effective in testing for randomness of a wide variety of random processes found in communication systems.

2.2 Properties of Noise Vectors

Pseudonoise (PN) is a set of multi-dimensional, deterministic vectors which simulate statistical properties of multi-dimensional random variable noise vectors. In this section the properties of real random noise are explored. Consider a time interval of duration T consisting of N subintervals, called chips, of duration T_c . For each chip, define a unit rectangular function,

$$q_i(t), \quad i=1 \dots N \quad (2-2)$$

given by

$$q_i(t) = q(t-(i-1)T), \quad i=1 \dots N \quad (2-3)$$

where,

$$q(t) = 1, \quad \text{when} \quad 0 \leq t \leq T_c \quad (2-4a)$$

and

$$q(t) = 0, \quad \text{otherwise.} \quad (2-4b)$$

The set of functions $\{q_i(t), i=1 \dots N\}$ form a complete set of time-orthogonal basis functions spanning the entire time interval T with scalar products given by

$$\langle q_i(t), q_j(t) \rangle = \int_{-\infty}^{+\infty} q_i(t) q_j(t) dt = T_c \delta_{ij} \quad \text{for } i, j=1 \dots N. \quad (2-5)$$

Consider the stochastic noise process $n(t)$ over the entire time interval T . The sample function of $n(t)$ within the i^{th} chip interval is denoted

$$n_i(t), \quad i=1 \dots N \quad (2-6)$$

and is given by

$$n_i(t) = n(t)q_i(t), \quad i=1 \dots N. \quad (2-7)$$

Equivalently,

$$n_i(t) = n(t), \quad \text{when } (i-1)T_c \leq t \leq iT_c \quad (2-8a)$$

and

$$n_i(t) = 0, \quad \text{otherwise.} \quad (2-8b)$$

Thus,

$$n(t) = \sum_{i=1}^N n_i(t). \quad (2-9)$$

The projection sample of $n(t)$ unto the i^{th} chip interval is given by

$$n_i = \langle n(t), q_i(t) \rangle = \sum_{j=1}^N \int_{-\infty}^{+\infty} n_j(t) q_i(t) dt \quad \text{for } i=1 \dots N. \quad (2-10)$$

Using (2-3), (2-5), and (2-7),

$$n_i = \int_{(i-1)T_c}^{iT_c} n(t) dt \quad \text{for } i=1 \dots N. \quad (2-11)$$

The samples n_i , $i=1 \dots N$, are, therefore, real random events, corresponding to N jointly distributed continuous random variables n_i , $i=1 \dots N$, with individual probability density function (pdf) $P_{n_i}(n_i)$, $i=1 \dots N$. For an arbitrary function $g(n_i)$ the expected value of $g(n_i)$ is evaluated by

$$E[g(n_i)] = \int_{-\infty}^{+\infty} g(n_i) P_{n_i}(n_i) dn_i . \quad (2-12)$$

The random variables n_i , $i = 1 \dots N$, form an N-dimensional random variable noise vector \underline{n} given by

$$\underline{n} = (n_1 , n_2 , n_3 , \dots , n_i , \dots , n_N) . \quad (2-13)$$

It is assumed that random variables n_i , $i=1 \dots N$, are mutually independent but not necessarily identically distributed random variables derived from $n_i(t)$ which is produced by the stochastic noise process. Denote the mean of n_i by μ_i . It is defined as the expected value of n_i and hence it is also the first moment of n_i with respect to the zero. Thus,

$$\mu_i = E[n_i] = \int_{-\infty}^{+\infty} n_i P_{n_i}(n_i) dn_i . \quad (2-14)$$

The variance or the square of the standard deviation of n_i is the second moment of n_i with respect to the mean and is given by

$$(\sigma_i)^2 = \text{Var}[n_i] = E[(n_i - \mu_n)^2] = E[(n_i)^2] . \quad (2-15)$$

Consider a random sample noise vector of \underline{n} given by

$$\underline{n} = (n_1 , n_2 , n_3 , \dots , n_i , \dots , n_N) . \quad (2-16)$$

The random sample noise vector sum, given by

$$L_N = n_1 + n_2 + n_3 + \dots + n_i + \dots + n_N \quad (2-17)$$

is also a random event of the random variable given by

$$L_N = n_1 + n_2 + n_3 + \dots + n_i + \dots + n_N . \quad (2-18)$$

The expected value of L_N is given by

$$\mu_N = E[L_N] = \sum_{i=1}^N E[n_i] = \sum_{i=1}^N \mu_i. \quad (2-19)$$

Since $n_i, i = 1 \dots N$ are mutually independent, they are uncorrelated and, therefore, the variance of L_N can be easily shown to be given by

$$(\sigma_N)^2 = \text{Var}[L_N] = E[(L_N - \mu_N)^2] = \sum_{i=1}^N E[(n_i - \mu_i)^2] = \sum_{i=1}^N (\sigma_i)^2. \quad (2-20)$$

In the remainder of this thesis, it is assumed that $n_i, i = 1 \dots N$ are symmetrically distributed about $\mu_i = 0$. Therefore, $\mu_N = 0$.

2.2.1 The polarity property. Given Ω , the outcome defined by the sign of a sample may be considered to result from a Bernoulli experiment in which the outcomes are either positive or negative. The N-dimensional sample vector may be considered as a sequence of N consecutive Bernoulli trials. By grouping all combinations of outcomes of the same sign into a single sample set, we obtain two sample sets of outcomes,

$$\mathcal{P} = \{ n_i \mid n_i > 0 \text{ and } i=0 \dots N^+ \} \quad (2-21a)$$

and

$$\mathcal{M} = \{ n_i \mid n_i < 0 \text{ and } i=0 \dots N^- \}. \quad (2-21b)$$

\mathcal{P} and \mathcal{M} are subsets of the set of N outcomes given by

$$\mathcal{N} = \mathcal{P} + \mathcal{M} \text{ such that } N = N^- + N^+. \quad (2-22)$$

N^+ and N^- are sample events of the integer random variables \mathbf{N}^+ and \mathbf{N}^- , respectively. N^+ may range from 0 to N while N^- may range from N to 0 correspondingly. Observations of many such sets for large N reveal the important property that there exists a high probability for an approximate balance in the outcome of the sign of the samples, *i.e.*

$$N^+ \approx N^- \approx \frac{1}{2}N \quad (2-23)$$

or equivalently,

$$E[\mathbf{N}^+] = E[\mathbf{N}^-] = \frac{1}{2}N. \quad (2-24)$$

Proof. The elements n_i , $i = 1 \dots N$ are zero mean continuous random variates with probability density functions given by $P_{n_i}[n_i]$. Since a positive event $n_i = n_i > 0$ is as equally likely as a negative event $n_i = n_i < 0$, the probability of a given sign event is given by

$$P_{n_i}[n_i > 0] = P_{n_i}[n_i < 0] = \frac{1}{2}. \quad (2-25)$$

This is analogous to the probabilities of tossing an ideal coin repeatedly and independently N times where N^+ / N^- may represent the number of heads/tails respectively. Therefore, N^+ and N^- are binomially distributed with probabilities that $N^\pm = k$, $k = 0 \dots N$ given by

$$P[N^\pm = k] = \left(\frac{1}{2}\right)^N C_{N,k} \quad (2-26)$$

where $C_{N,k}$ denotes the binomial coefficient given by

$$C_{N,k} = n! / [k!(n-k)!]. \quad (2-27)$$

Using (2-25), the means of N^+ and N^- are identical and are given by

$$E[N^+] = E[N^-] = N P_{n_i}[n_i > 0] = N P_{n_i}[n_i < 0] = \frac{1}{2}N, \quad (2-28)$$

and the variances of N^+ and N^- are also identical and are given by

$$[\sigma_N]^2 = \text{Var}[N^+] = \text{Var}[N^-] = \frac{1}{4}N. \quad (2-29)$$

For any given noise vector of length N , therefore, there is a high probability that

$$N^+ \approx N^-. \quad \text{QED}$$

For large N , it can be easily shown that $P[N^\pm = k]$ becomes Gaussian about the mean given in (2-28). As a numerical interpretation, when

$$N = 100, E[N^\pm] = 50, \sigma_N = 5. \quad (2-30)$$

Based upon the Gaussian approximation, approximately 68% of the noise vectors will have

$$|N^+ - N^-| < 10. \quad (2-31)$$

The equality of means given by (2-28) may be also derived without the use of the binomial distribution. In a sequence of N samples, consider the random variable given by

$$b_i = b \text{sgn}(n_i), \quad (2-32)$$

i.e.

$$b_i = +b \text{ if the } i^{\text{th}} \text{ trial is positive,} \quad (2-33a)$$

and

$$b_i = -b \text{ if the } i^{\text{th}} \text{ trial is negative.} \quad (2-33b)$$

Using (2-25)

$$E[b_i] = (b) P_{n_i} [n_i > 0] + (-b) P_{n_i} [n_i < 0] = 0. \quad (2-34)$$

Let

$$S_b = \sum_{i=1}^N b_i. \quad (2-35)$$

Equivalently,

$$S_b = (b)N^+ + (-b)N^- = b(N^+ - N^-). \quad (2-36)$$

Using (2-34) and (2-35), and the equivalency of interchanging the mean of the sum of random variables with the sum of the means of the random variables,

$$E[S_b] = E[\sum_{i=1}^N b_i] = \sum_{i=1}^N E[b_i] = 0. \quad (2-37)$$

Similarly, using (2-36),

$$E[S_b] = E[b(N^+ - N^-)] = b(E[N^+] - E[N^-]) = 0. \quad (2-38)$$

Therefore, in agreement with (2-28), $E[N^+] = E[N^-] = \frac{1}{2}N$.

2.2.2 The zero-crossing property. Consider the transformation of the N-dimensional random variable vector \underline{n} into an N-dimensional random variable vector \underline{x} which depicts the change in sign from one sample of n_i to next sample of n_{i+1} . A non-transition in sign is referred to as a non-zero crossing, an upward transition is referred to as a positive zero crossing, and a downward transition is referred to as a negative zero crossing. The resulting

zero crossings random variable vector is given by

$$\underline{x} = (x_1, x_2, x_3, \dots, x_i, \dots, x_N).$$
 (2-39)

The random variables x_i are functions of the random variables n_i and n_{i+1} and are given by

$$x_i = \frac{1}{2}[\text{sgn}(n_{i+1}) - \text{sgn}(n_i)] \quad \text{for } i=1 \dots N-1$$
 (2-40a)

and

$$x_N = \frac{1}{2}[\text{sgn}(n_1) - \text{sgn}(n_N)]$$
 (2-40b)

where

$$\text{sgn}(n_i) = n_i / |n_i|.$$
 (2-41)

Note that the random variables x_i are ternary variables of a symmetrically distributed zero mean noise process. A random sample zero-crossings vector of \underline{x} is given by

$$\underline{x} = (x_1, x_2, x_3, \dots, x_i, \dots, x_N).$$
 (2-42)

Consider any pair of consecutive samples n_i and n_{i+1} . The outcome x_i can be either a $-1, 0$, or $+1$. By grouping all the same outcomes in \underline{x} into a single sample set, three sample sets of outcomes result corresponding to each type of outcome. Namely,

$$\mathcal{X}^+ = \{ x_i \mid x_i > 0 \text{ and } i=1 \dots X^+ \}, \quad (2-43a)$$

$$\mathcal{X}^- = \{ x_i \mid x_i < 0 \text{ and } i=1 \dots X^- \} \quad (2-43b)$$

and

$$\mathcal{X}^0 = \{ x_i \mid x_i = 0 \text{ and } i=1 \dots X^0 \}. \quad (2-43c)$$

\mathcal{X}^+ , \mathcal{X}^- , and \mathcal{X}^0 are subsets of the set of N outcomes given by

$$\mathcal{X} = \mathcal{X}^- + \mathcal{X}^0 + \mathcal{X}^+ \quad \text{such that} \quad N = X^- + X^0 + X^+. \quad (2-44)$$

The integers X^- , X^0 and X^+ are sample events of the interdependent integer random variables X^- , X^0 and X^+ , respectively. X^0 may range from 0 to N while $X^\pm = X^+ + X^-$ may range from N to 0 for even N or from $N-1$ to 0 for odd N , correspondingly, X^\pm must be even. Observations of many such sets with large N reveal the important property that there exists a high probability for an approximate balance between the number of outcomes of non-zero crossings X^0 and the number of outcomes of zero crossings X^\pm , and in addition, there exists an exact balance between the number of outcomes of positive zero crossings X^+ and the number of outcomes of negative zero crossings X^- , *i.e.*

$$X^+ = X^- = \frac{1}{2}X^\pm \approx \frac{1}{2}X^0. \quad (2-45)$$

Proof. The elements n_i are zero mean continuous random variates with

probability density functions given by $P_{n_i}(n_i)$. The elements x_i are also zero mean but discrete random variates with probability functions given by $P_{x_i}(x_i)$. The probability that a non zero crossing will occur is given by

$$P_{x_i}(0) = P[x_i = 0] = P[n_i < 0, n_{i+1} < 0] + P[n_i > 0, n_{i+1} > 0], \quad (2-46)$$

the probability that a negative zero crossing will occur is given by

$$P_{x_i}(-1) = P[x_i = -1] = P[n_i > 0, n_{i+1} < 0], \quad (2-47)$$

and the probability that a positive zero crossing will occur is given by

$$P_{x_i}(+1) = P[x_i = +1] = P[n_i < 0, n_{i+1} > 0]. \quad (2-48)$$

The probability that any zero crossing will occur is given by

$$P_{x_i}(\pm 1) = P[x_i = +1] + P[x_i = -1] = P[x_i = \pm 1] = P[x_i \neq 0]. \quad (2-49)$$

Since n_i is independent of n_j for all $i \neq j$ and using (2-25)

$$P[n_i < 0, n_{i+1} < 0] = P[n_i < 0] P[n_{i+1} < 0] = \frac{1}{4}. \quad (2-50a)$$

Similarly,

$$P[n_i > 0, n_{i+1} > 0] = P[n_i > 0] P[n_{i+1} > 0] = \frac{1}{4}. \quad (2-50b)$$

$$P[n_i > 0, n_{i+1} < 0] = P[n_i > 0] P[n_{i+1} < 0] = \frac{1}{4} , \quad (2-50c)$$

and

$$P[n_i < 0, n_{i+1} > 0] = P[n_i < 0] P[n_{i+1} > 0] = \frac{1}{4} , \quad (2-50d)$$

A zero crossing event, $x_i \neq 0$, therefore, is as equally likely as a non-zero crossing event, $x_i = 0$. Thus ,

$$P[x_i \neq 0] = P[x_i = 0] = \frac{1}{2}. \quad (2-51)$$

The zero crossing N dimensional random variable sample vector may, therefore, be considered as a sequence of N consecutive Bernoulli trials for which the outcome is either a zero crossing or a non-zero crossing with probabilities given by (50). Similarly to N^+ and N^- , X^\pm and X^0 are also binomially distributed with probabilities that $X = k$, $k = 0..N$ given by

$$P[X = k] = \left(\frac{1}{2}\right)^N C_{N,k} , \quad \text{for } X = X^\pm \text{ or } X^0 \quad (2-52)$$

where $C_{N,k}$ is the binomial coefficient defined in (2-27). Since the means of X^\pm and X^0 are identical, we obtain

$$E[X^\pm] = E[X^0] = N P[x \neq 0] = N P[x_i = 0] = \frac{1}{2}N \quad (2-53)$$

and the variances of X^\pm and X^0 are also identical and are given by

$$[\sigma_{\mathbf{x}}]^2 = \text{Var}[\mathbf{X}^{\pm}] = \text{Var}[\mathbf{X}^0] = \frac{1}{4}N. \quad (2-54)$$

For any given noise vector of length N , therefore, there is a high probability that

$$\mathbf{X}^{\pm} \approx \mathbf{X}^0. \quad \text{QED}$$

It remains to prove that $X^+ = X^-$. Consider the i^{th} chip. A transition in sign will affect both the $i-1$ and the $i+1$ zero crossing samples. Only two types of effects are possible: a) when a transition occurs within a run, a pair of equal and opposite zero crossings are generated or b) when a transition occurs at a boundary of a run, the zero crossing propagates without creating any additional ones. For each positive zero crossing created, therefore, there must exist a corresponding negative zero crossing. QED

For large N , it can be easily shown that $P[\mathbf{X} = k]$ becomes Gaussian about the mean given in (2-53). As a numerical interpretation, when

$$N = 100, E[\mathbf{X}] = 50, \sigma_{\mathbf{x}} = 5. \quad (2-55)$$

Based upon the Gaussian approximation, approximately 68% of the noise vectors, will have

$$|\mathbf{X}^{\pm} - \mathbf{X}^0| < 10. \quad (2-56)$$

2.2.3 The autocorrelation property. Consider a sample N-dimensional random noise vector \underline{n} . The sample periodic/cyclic autocorrelation is given by

$$R_{N,k} = \sum_{i=1}^N n_i n_{i+k} \quad i,k = 1 \dots N. \quad (2-57)$$

$R_{N,k}$ are samples of the independent random variables $R_{N,k}$ which form an N-dimensional random variable vector given by

$$\underline{R}_N = \{ R_{N,0}, R_{N,1}, \dots, R_{N,k}, \dots, R_{N,N-1} \}. \quad (2-58)$$

Observations of many samples of \underline{R}_N for large N display the interrelationship among the sample vector elements given by

$$R_{N,0} \gg R_{N,k} \quad \text{for } k = 1 \dots N-1 \quad (2-59a)$$

where

$$R_{N,0} = E_N = (\underline{n} \cdot \underline{n}). \quad (2-59b)$$

$R_{N,0}$ is called the (self) energy of the sample vector \underline{n} since it is equal, within a constant of proportionality, to the energy of the stochastic process $n(t)$ given by

$$E_n = \int_0^T [n(t)]^2 dt. \quad (2-60)$$

$R_{N,k}$, for $k = 1 \dots N-1$, is called the cross energy of the sample vector \underline{n} .

Proof. Consider an arbitrary stochastic noise process $n(t)$ specified by the joint density function of the random variable vector $\underline{n}(t)$ obtained by sampling $n(t)$ at any finite set of time instants $\{t_i\}$, $i = 1 \dots N$. The samples $n(t_i)$, from the ensemble of waveforms which correspond to t_i , are defined as the samples of the random variable $n(t_i)$. Thus,

$$\underline{n}(t) = \{ n(t_1), n(t_2), \dots, n(t_i), \dots, n(t_N) \}. \quad (2-61)$$

The instantaneous samples also form an N-fold sample process vector given by

$$\underline{n}(t) = \{ n(t_1), n(t_2), \dots, n(t_i), \dots, n(t_N) \}. \quad (2-62)$$

The autocorrelation of the stochastic process $n(t)$ is defined as

$$R_n(t_1, t_2) = E[n(t_1) n(t_2)]. \quad (2-63)$$

For a wide-sense stationary white noise process, the double-sided power spectral density is given by

$$S_w(f) = \frac{1}{2} N_0. \quad (2-64)$$

Using the Wiener-Khintchine relations, the autocorrelation is given by the inverse Fourier transform of $S_w(f)$, *i.e.*

$$R_n(t_1, t_2) = \int_{-\infty}^{+\infty} S_w(f) \exp[-j2\pi f(t_2 - t_1)] df = \frac{1}{2} N_0 \delta(t_1 - t_2). \quad (2-65)$$

The two-valued nature of $R_n(t_1, t_2)$ portrays the ultimate in randomness which is, obviously, only approximated by $R_{N,k}$. If we subdivide the time interval T into N equal subintervals T_c , then the i^{th} sample may be associated with the stochastic process within the i^{th} subinterval such that the sample waveform is given by

$$n(t_i) = n(t) \quad \text{when} \quad (i-1)T_c \leq t_i = t \leq iT_c \quad (2-66a)$$

and

$$n(t_i) = 0 \quad \text{otherwise.} \quad (2-66b)$$

Comparing (2-66) with (2-8), $n_i(t) = n(t_i)$ and n_i may be obtained by integration of $n(t_i)$. Interchanging summation and averaging

$$E[R_{N,k}] = E\left[\sum_{i=1}^N n_i n_{i+k}\right] = \sum_{i=1}^N E[n_i n_{i+k}]. \quad (2-67)$$

Using (2-11) and (2-66),

$$E[R_{N,k}] = \sum_{i=1}^N \left\{ E\left[\int_{(i-1)T_c}^{iT_c} n(t_i) dt_i \int_{(i+k-1)T_c}^{(i+k)T_c} n(t_{i+k}) dt_{i+k} \right] \right\}. \quad (2-68)$$

Interchanging averaging and integration,

$$E[R_{N,k}] = \sum_{i=1}^N \left[\int_{(i-1)T_c}^{iT_c} dt_i \int_{(i+k-1)T_c}^{(i+k)T_c} dt_{i+k} E[n(t_i)n(t_{i+k})] \right]. \quad (2-69)$$

Using (2-65)

$$E[R_{N,k}] = \sum_{i=1}^N \left[\int_{(i-1)T_c}^{iT_c} dt_i \int_{(i+k-1)T_c}^{(i+k)T_c} dt_{i+k} \frac{1}{2} N_0 \delta(t_i - t_{i+k}) \right]. \quad (2-70)$$

Integrating over dt_{i+k} , yields

$$E[R_{N,k}] = \sum_{i=1}^N \left[\int_{(i-1)T_c}^{iT_c} dt_i \left(\frac{1}{2} N_0 \right) \right] = \frac{1}{2} N_0 T \quad \text{when } k=0 \quad (2-71a)$$

and

$$E[R_{N,k}] = 0 \quad \text{when } k \neq 0. \quad \text{QED.} \quad (2-71b)$$

Recall from (2-59) that $R_{N,0}$ is the scalar product of \underline{n} with itself. Moreover, n_i are independent zero mean Gaussian variates with non-identical standard deviations σ_i . Therefore, the mean of $R_{N,0}$ is given by

$$\mu_{R_{N,0}} = E[R_{N,0}] = E[E_N] = E[\underline{n} \cdot \underline{n}] \quad (2-72)$$

$$\mu_{R_{N,0}} = E[\sum_{i=1}^N (n_i)^2] = \sum_{i=1}^N E[(n_i)^2] = \sum_{i=1}^N (\sigma_i)^2 = (\sigma_N)^2 \quad (2-73)$$

Comparing (2-73) with (2-71a)

$$\sigma_N = \sqrt{\frac{1}{2} N_0 T} \quad (2-74)$$

and for n_i which are independently and identically distributed (i.i.d.),

$$\sigma_i = \sigma = \sqrt{\frac{1}{2} N_0 T_0}. \quad (2-75)$$

It is interesting to evaluate the variance of $R_{N,0}$. It is given by

$$(\sigma_{R_{N,0}})^2 = E[(R_{N,0} - \mu_{R_{N,0}})^2] = E[(R_{N,0})^2] - (\mu_{R_{N,0}})^2. \quad (2-76)$$

But

$$E[(R_{N,0})^2] = E\left[\left(\sum_{i=1}^N (n_i)^2\right)\left(\sum_{i=1}^N (n_i)^2\right)\right]. \quad (2-77)$$

Interchanging summation and averaging,

$$E[(R_{N,0})^2] = \sum_{i=1}^N E[(n_i)^4] + 2 \sum_{i \neq j}^N E[(n_i)^2 (n_j)^2]. \quad (2-78)$$

Since

$$E[(n_i)^4] = 3(\sigma_i)^4 \quad \text{and} \quad E[(n_i)^2] = (\sigma_i)^2, \quad (2-79)$$

and

$$(\mu_{R_{N,0}})^2 = \sum_{i=1}^N (\sigma_i)^4 + 2 \sum_{i \neq j}^N (\sigma_i)^2 (\sigma_j)^2, \quad (2-80)$$

we obtain

$$(\sigma_{R_{N,0}})^2 = 2 \sum_{i=1}^N (\sigma_i)^4. \quad (2-81)$$

Similarly, the standard deviation of $R_{N,k}$ for $k \neq 0$ is given by

$$(\sigma_{R_{N,k}})^2 = 2 \sum_{i=1}^N (\sigma_i)^4. \quad (2-82)$$

Using (2-71b), (2-80), (2-81) and (2-82), the ratio of the difference between

$\mu_{R_{N,0}}$ and $\mu_{R_{N,k}}$ and the spread about $\mu_{R_{N,0}}$ or $\mu_{R_{N,k}}$ is given by

$$\mu_{R_{N,0}} / \sigma_{R_{N,k}} = (\sigma_N)^2 / [2 \sum_{i=1}^N (\sigma_i)^4]^{1/2}. \quad (2-83)$$

The fact that this ratio is monotonically increasing with N is readily manifested when n_i are i.i.d., *i.e.* $\sigma = \sigma_i$ for $i = 1 \dots N$. For this case,

$$\mu_{R_{N,0}} / \sigma_{R_{N,k}} = [\frac{1}{2} N]^{1/2}. \quad (2-84)$$

2.2.4 The run properties. Consider an N-fold random variable noise vector \underline{n} . It is a sequence of random variables n_i , $i=1 \dots N$, whose conditional probability is given by

$$P_{n_i}(n_i | n_j) = P_{n_i, n_j}(n_i, n_j) / P_{n_j}(n_j). \quad (2-85)$$

Assuming that the random variables n_i , $i=1 \dots N$ are mutually independent, then (2-85) reduces to

$$P_{n_i, n_j}(n_i, n_j) = P_{n_i}(n_i) P_{n_j}(n_j). \quad (2-86)$$

In sampling \underline{n} , we observe groupings of consecutive samples of the same polarity. Such groupings are referred to as "runs". Runs may develop as a purely random phenomenon, as a result of filtering which introduces correlation between the output samples, or as a result of mixing or superposition of the random variable noise vector with a deterministic signal vector. We shall refer to runs resulting from correlated samples as "bursts". In this section, however, we consider runs which evolve in a purely random fashion. A run of length k , $k = 1 \dots N$ may start with n_i and persist with sample n_{i+j-1} , $i = 1 \dots N$ and $j = 1 \dots k$. By definition, the samples n_{i+j-1} and n_{i+k} are of opposite polarity.

Given a random noise vector \underline{n} , as in (2-13), consider its transformation into an L -dimensional vector \underline{l} such that the elements l_i represent the sign and length of run i . Note that \underline{l} is an integer vector such that

$$\sum_{i=1}^L |l_i| = N. \quad (2-87)$$

Let m_{+k} , $k = 1 \dots N$ denote the number of positive runs of length k which may be found in \underline{n} . Let m_{-k} , $k = 1 \dots N$ denote the number of negative runs of length k which may be found in \underline{n} . The total number of runs of either sign and of length k is given by

$$m_{\pm k} = m_{+k} + m_{-k}. \quad (2-88)$$

Consider all runs in any given sample sequence \underline{n} . Observations of many samples of \underline{n} for large N reveal the following important randomness properties:

a) The number of runs of length k and same polarity is approximately equal to the number of runs of the same length and opposite polarity. This approximation is expressed by

$$m_{+k} \approx m_{-k} . \quad (2-89)$$

b) The number of runs of length k to be found with high probability is given by

$$m_{\pm k} \approx N \left(\frac{1}{2}\right)^k . \quad (2-90)$$

c) The longest run to be found with high probability is of length n given by

$$n \approx \log_2(N) . \quad (2-91)$$

Proof. The run properties of the random noise vector \mathbf{n} are preserved by its transformation into the N -dimensional random vector \mathbf{b} given by

$$\mathbf{b} = (b_1, b_2, b_3, \dots, b_i, \dots, b_N) \quad (2-92)$$

where b_i was defined in (2-32). Without loss in generality, let $b = 1$, then

$$b_i = \text{sgn}(n_i) . \quad (2-93)$$

Since n_i is a zero mean random variable, the probability that $b_i = +1$ is given by

$$P[\mathbf{b}_i = +1] = P[\mathbf{n}_i > 0] = \frac{1}{2} . \quad (2-94)$$

Similarly, the probability that $\mathbf{b}_i = -1$ is given by

$$P[\mathbf{b}_i = -1] = P[\mathbf{n}_i < 0] = \frac{1}{2} . \quad (2-95)$$

It follows that in general for $\mathbf{b}_i = \pm 1$,

$$P[\mathbf{b}_i = \mathbf{b}_i] = \frac{1}{2} . \quad (2-96)$$

Since \mathbf{n}_i is statistically independent of \mathbf{n}_j for $i \neq j$, \mathbf{b}_i is also statistically independent of \mathbf{b}_j for $i \neq j$. Hence, the pair-wise joint probability function for \mathbf{b}_i and \mathbf{b}_j is given by

$$P[\mathbf{b}_i = \mathbf{b}_i , \mathbf{b}_j = \mathbf{b}_j] = P[\mathbf{b}_i = \mathbf{b}_i] P[\mathbf{b}_j = \mathbf{b}_j] = \frac{1}{4} . \quad (2-97)$$

A new run may start with any \mathbf{n}_i , $i = 1 \dots N$. The probability that a run which starts at \mathbf{n}_i will terminate at \mathbf{n}_i is given by

$$P[\mathbf{b}_i \neq \mathbf{b}_{i+1}] = P[\mathbf{b}_i = -1 | \mathbf{b}_{i+1} = +1] P[\mathbf{b}_{i+1} = +1] + P[\mathbf{b}_i = +1 | \mathbf{b}_{i+1} = -1] P[\mathbf{b}_{i+1} = -1] . \quad (2-98)$$

Using Bayes' rule, the independence property given in (2-96) and (2-95)

$$P[\mathbf{b}_i = \mathbf{b}_i | \mathbf{b}_j = \mathbf{b}_j] = P[\mathbf{b}_i = \mathbf{b}_i] = \frac{1}{2} . \quad (2-99)$$

Using (2-99) in (2-98)

$$P[\mathbf{b}_i \neq \mathbf{b}_{i+1}] = \frac{1}{2}. \quad (2-100)$$

The N-dimensional random variable noise vector may be considered as a Bernoulli experiment in which the two outcomes are $\mathbf{b}_i \neq \mathbf{b}_{i+1}$ or $\mathbf{b}_i = \mathbf{b}_{i+1}$. In N trials the expected number of runs of unit length is given by

$$E[m_{\pm 1}] = \frac{1}{2}N. \quad (2-101)$$

The probability that a unit run will occur and be positive is given by

$$P[l_i = +1] = P[\mathbf{b}_i = +1 | \mathbf{b}_{i+1} = -1] P[\mathbf{b}_{i+1} = -1] = \frac{1}{4}. \quad (2-102)$$

Similarly, the probability that a unit run will occur and be negative is given by

$$P[l_i = -1] = P[\mathbf{b}_i = -1 | \mathbf{b}_{i+1} = +1] P[\mathbf{b}_{i+1} = +1] = \frac{1}{4}. \quad (2-103)$$

Thus, since $P[l_i = +1] = P[l_i = -1]$,

$$E[m_{+1}] = E[m_{-1}] = \frac{1}{2}E[m_{\pm 1}] = \frac{1}{4}N. \quad (2-104)$$

A run which starts with any n_i may continue with n_{i+1} with probability given by

$$P[\mathbf{b}_i = \mathbf{b}_{i+1}] = 1 - P[\mathbf{b}_i \neq \mathbf{b}_{i+1}] = \frac{1}{2}. \quad (2-105)$$

Again, consider the random variable \mathbf{b}_i as a Bernoulli experiment in which the two outcomes are $\mathbf{b}_i \neq \mathbf{b}_{i+1}$ or $\mathbf{b}_i = \mathbf{b}_{i+1}$. Given N trials, the expected number of runs of length $k > 1$ is

$$E\left[\sum_{k=2}^N m_{+k}\right] = P[\mathbf{b}_i = \mathbf{b}_{i+1}]N = \frac{1}{2}N. \quad (2-106)$$

The probability that a multi-unit run will occur and be positive is given by

$$P[\mathbf{l}_i > +1] = P[\mathbf{b}_i = +1 | \mathbf{b}_{i+k} = -1]P[\mathbf{b}_{i+k} = -1] = \frac{1}{4}. \quad (2-107)$$

Similarly, the probability that a multi-unit run will occur and be negative is given by

$$P[\mathbf{l}_i < -1] = P[\mathbf{b}_i = -1 | \mathbf{b}_{i+k} = +1]P[\mathbf{b}_{i+k} = +1] = \frac{1}{4}. \quad (2-108)$$

Thus, since $P[\mathbf{l}_i > +1] = P[\mathbf{l}_i < -1]$,

$$E\left[\sum_{k=2}^N m_{+k}\right] = E\left[\sum_{k=2}^N m_{-k}\right] = \frac{1}{2}E\left[\sum_{k=2}^N m_{\pm k}\right] = \frac{1}{4}N. \quad (2-109)$$

In (2-104) it was proved that a balance exists between m_{+1} and m_{-1} . In (2-109) such a balance is seen to exist with respect to the sum of all the multi-unit

runs. To prove that in general $E[m_{+k}] = E[m_{-k}]$, consider the Bernoulli experiment with the two outcomes given by $l_i = \pm k$ and $l_i \neq \pm k$. Using Bayes' rule and the statistical independence of b_i and b_j for $i \neq j$, the probability $l_i = \pm k$ is given by

$$P[l_i = \pm k] = P[b_i = b_{i+1} = b_{i+2} = \dots = b_{i+k-1} \neq b_{i+k}] = P[b_{i+k} = -1] \prod_{j=1}^k P[b_{i+j-1} = +1] +$$

$$P[b_{i+k} = +1] \prod_{j=1}^k P[b_{i+j-1} = -1]. \quad (2-110)$$

Recall that $P[b_i = b_j] = \frac{1}{2}$ for $i = 1 \dots N$. Using this fact in (2-110) yields

$$P[l_i = \pm k] = \left(\frac{1}{2}\right)^k. \quad (2-111)$$

Note that

$$P[l_i = \pm k] = P[l_i = +k] + P[l_i = -k]. \quad (2-112)$$

Comparing (2-110) with (2-112),

$$P[l_i = +k] = P[b_{i+k} = -1] \prod_{j=1}^k P[b_{i+j-1} = +1] = \left(\frac{1}{2}\right)^{k-1} \quad (2-113)$$

and

$$P[l_i = -k] = P[b_{i+k} = +1] \prod_{j=1}^k P[b_{i+j-1} = -1] = \left(\frac{1}{2}\right)^{k-1}. \quad (2-114)$$

Note that this result is independent of N . A run may start anywhere in the sample noise vector \underline{n} . If it starts or persists at $i = N$, it may continue cyclically with $i = 1$.

To finalize the proof of (2-89) and (2-90), denote the probability of the outcome $I_i = \pm k$ as p and the probability of the outcome $I_i \neq \pm k$ as $q = 1 - p$. Therefore,

$$p = \left(\frac{1}{2}\right)^k, \text{ and } q = 1 - \left(\frac{1}{2}\right)^k. \quad (2-115)$$

The random variable $m_{\pm k}$ is binomially distributed with a mean given by

$$E[m_{\pm k}] = Np = N \left(\frac{1}{2}\right)^k. \quad (2-116)$$

Similarly, we obtain

$$E[m_{+k}] = E[m_{-k}] = \frac{1}{2}Np = N \left(\frac{1}{2}\right)^{k-1}. \quad (2-117)$$

The variance of $m_{\pm k}$ is given by

$$\text{Var}[m_{\pm k}] = Npq = N \left[\left(\frac{1}{2}\right)^k - \left(\frac{1}{2}\right)^{2k} \right]. \quad (2-118)$$

For $k > 4$,

$$\text{Var}[m_{\pm k}] \approx N \left(\frac{1}{2}\right)^k. \quad (2-119)$$

Since $m_{\pm k} \geq 0$, note that both $E[m_{\pm k}]$ and $\text{Var}[m_{\pm k}]$ decrease with increasing k . To derive the longest k which may be found with high probability in an N -fold

sample of a random noise vector, we require that

$$E[m_{2k}] = 1 = N(\frac{1}{2})^k. \quad (2-120)$$

Taking the logarithm to the base 2, (2-91) is readily obtained.

QED

2.3 Noise Invariance Transformations

Consider functions or transformations of the N-dimensional random noise vector \underline{n} . Denote such transformations by $\mathcal{N}(\underline{n})$. Define \mathcal{N}_i as a member of a class of \mathcal{N} , such that \mathcal{N}_i preserves the randomness properties discussed above. The new M-dimensional random noise vector is given by

$$\mathcal{N}(\underline{n}) = (n^1, n^2, n^3, \dots, n^i, \dots, n^M). \quad (2-121)$$

Consider the transformations defined below:

$\mathcal{N}_0 \Rightarrow$ every sample of \underline{n} is multiplied by an arbitrary constant.

Proof. \mathcal{N}_0 is a linear operation. The output of any linear operation upon a Gaussian process is always Gaussian. Since \mathcal{N}_0 preserves the independence between samples and the zero-mean of the process, the randomness properties are preserved.

QED

$\mathcal{N}_1 \Rightarrow$ every sample of \underline{n} is reduced by the sample average of \underline{n} .

Proof. The sample average of \underline{n} is a random variable given by

$$\mathbf{A}_N = (1/N) \sum_{i=1}^N n_i \quad (2-122)$$

Since \mathbf{A}_N is the linear sum of zero-mean Gaussian variables, it is also a zero-mean Gaussian variable. The transformed random variable

$$\mathbf{n}^i = n_i - \mathbf{A}_N \quad (2-123)$$

is the difference between two zero-mean Gaussian variables. Hence \mathcal{N}_1 also retains the Gaussian nature of the input process. The extent to which the transformed samples are independent is obtained by considering

$$E[\mathbf{n}^i \mathbf{n}^j] = E[(n_i - \mathbf{A}_N)(n_j - \mathbf{A}_N)] = E[n_i n_j] - 2E[n_i \mathbf{A}_N] - E[(\mathbf{A}_N)^2]. \quad (2-124)$$

Recall that $E[n_i n_j] = (\sigma_i)^2 \delta_{ij}$, therefore,

$$E[\mathbf{n}^i \mathbf{n}^j] = (\sigma_i)^2 \delta_{ij} - (2/N) (\sigma_i)^2 + (1/N)^2 \sum (\sigma_i)^2. \quad (2-125)$$

For n_i which are i.i.d.,

$$E[\mathbf{n}^i \mathbf{n}^j] = E[\mathbf{n}_i \mathbf{n}_j] - (1/N) (\sigma_i)^2 . \quad (126)$$

For large N , therefore, the samples transformed by \mathcal{N}_1 become independent as required. QED

$\mathcal{N}_2 \Rightarrow$ every run of length $l_i > n = \log_2 N$ is nullified or reduced by the sample average of the run.

Proof. The sample average of a run which starts with n_i and persists for k samples is given by

$$\mathbf{A}_k = (1/k) \sum_{j=1}^k \mathbf{n}_{i+j-1} . \quad (2-127)$$

\mathbf{A}_k is the linear sum of independent random variables which are all of the same polarity. The expected value of \mathbf{A}_k is given by

$$E[\mathbf{A}_k] = (1/k) \sum_{j=1}^k E[\mathbf{n}_{i+j-1}] \quad (2-128)$$

where, depending upon the polarity of the run,

$$E[\mathbf{n}_{i+j-1}] = \pm \int_0^{+\infty} n P_n(n) dn = (\sigma_i) / \sqrt{1/2\pi} , \quad (2-129)$$

and

$$\text{Var}[n_{i+j-1}] = \int_0^{+\infty} (n - E[n_{i+j-1}])^2 P_n(n) dn = \frac{1}{2}(\sigma_i)^2 (1 - 1/\pi). \quad (2-130)$$

Only the random samples which belong to runs of length $k > n$ will be affected.

The samples which are averaged are given by

$$n^{i+j-1} = n_{i+j-1} - A_k \quad \text{for } j=1 \dots k, i=1 \dots N, |i| > n. \quad (2-131)$$

The samples which are nullified are given by

$$n^{i+j-1} = 0 \quad \text{for } j=1 \dots k, i=1 \dots N, |i| > n. \quad (2-132)$$

Otherwise,

$$n^{i+j-1} = n_{i+j-1} \quad \text{for } j=1 \dots k, i=1 \dots N, |i| \leq n. \quad (2-133)$$

Recall that the probability for a run of length k to occur in \underline{n} is given by

$$E[m_{\pm k}] = N \left(\frac{1}{2}\right)^k \quad \text{with} \quad \text{Var}[m_{\pm k}] = N \left\{ \left(\frac{1}{2}\right)^k - \left(\frac{1}{2}\right)^{2k} \right\}. \quad (2-134)$$

When a long run does occur, it only affects k/N of the samples. The remainder of the samples remain zero-mean, uncorrelated and Gaussian. The polarity and zero-crossing properties are unaffected since $E[m_{+k}] = E[m_{-k}]$. Furthermore, since $m_{\pm k}$ is binomially distributed, the tail end of the distribution will be cut off and redistributed within previous uncertainties in $E[m_{\pm k}]$ derived for $k < n$.

Hence to a good approximation, the run properties will also be preserved by \mathcal{N}_2 . The periodic autocorrelation property will also be preserved on the average since any correlation introduced by \mathcal{N}_2 will only affect the runs which occur with low probability. QED

$\mathcal{N}_3 \Rightarrow \underline{n}$ is shortened by removing a single sample from every run.

Proof. Since $E[m_{+k}] = E[m_{-k}]$, the removal of a single sample from every run results in a new random variable vector which on the average has $M = \frac{1}{2}N$ dimensions. Since the start of a new run within \underline{n} is totally random, the samples eliminated by \mathcal{N}_3 are also random, thereby preserving all of the randomness properties which are based upon the independence between zero-mean samples of the input \underline{n} . QED

2.4 Properties of PN Vectors

Having investigated the salient properties of random noise in the previous section, the extent to which these properties are simulated by pseudo-random noise is discussed in this section. PN has been the subject of extensive investigations by Golomb [44], Golay [45], Gold [46], Welti [47] and many others. The most widely known and used N-dimensional PN vectors are binary unipolar PN sequences of length N

$$\underline{a} = (a_1, a_2, a_3, \dots, a_i, \dots, a_N), \quad (2-135)$$

where

$$a_i = 0 \text{ or } a_i = 1. \quad (2-136)$$

A sample PN vector of \underline{a} is given by

$$\underline{a} = (a_1, a_2, a_3, \dots, a_i, \dots, a_N) \quad (2-137)$$

which may be generated by a variety of feedback shift registers (FBSR). The general structure of an n -stage FBSR is shown in Figure 2.4-1.

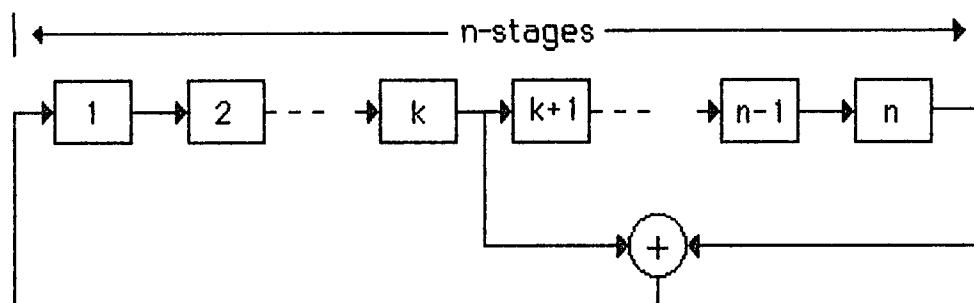


Figure 2.4-1. An n -Stage Feedback Shift Register (FBSR)

For the purpose of this dissertation, the advantages and disadvantages of various ways of generating PN are irrelevant. What is important, however, is the degree to which the random noise properties are satisfied. We shall consider the simplest class of PN generated by maximal length FBSRs. Two implementations

are illustrated. A 3-stage FBSR which corresponds to the generating polynomial given by $G(x) = x^3 + x + 1$ is shown in Figure 2.4-2. A 7-stage FBSR which corresponds to the generating polynomial given by $G(x) = x^7 + x^6 + x^4 + x + 1$ is shown in Figure 2.4-3.

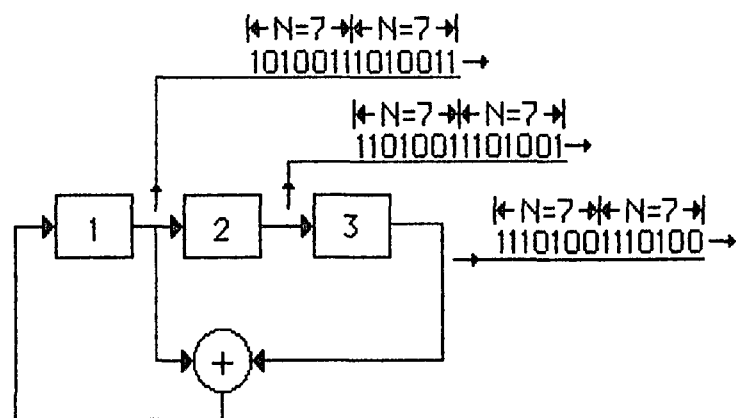


Figure 2.4-2. A 3-Stage Binary Maximal Length FBSR Corresponding to the Generating Polynomial $G(x) = x^3 + x + 1$

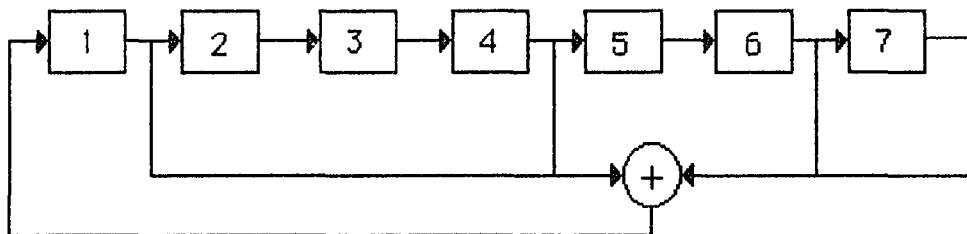


Figure 2.4-3. A 7-Stage Binary Maximal Length FBSR Corresponding to the Generating Polynomial $G(x) = x^7 + x^6 + x^4 + x + 1$

Consider an n-stage FBSR in a non-zero initial state. When clocked at a rate

$$R_c = (T_c)^{-1}, \quad (2-138)$$

there are at most

$$N = 2^n - 1 \quad (2-139)$$

unique states which may be found over a time interval $T = N T_c$. A PN sample vector is generated by tapping at the output of any of the n stages as shown in Figure 2.4-2. At $t \geq T + T_c$, the states repeat in the same order and hence the sequence formed by the elements of the PN vector is periodic every N elements.

2.4.1 The polarity property. In binary notation, let the variable $a_i = a$ denote a '1', and the variable $a_i = 0$ denote a '0'. Using the transformation

$$b_i = 2a_i - a \quad (2-140)$$

we may remove most of the DC component inherent in \underline{a} , to obtain a binary bipolar PN vector given by

$$\underline{b} = (b_1, b_2, b_3, \dots, b_i, \dots, b_N) \quad (2-141)$$

where

$$b_i = -a \text{ when } a_i = 0 \quad (2-142)$$

and

$$b_i = +a \text{ when } a_i = a. \quad (2-143)$$

Thus, the sample PN Vector of \underline{b} is denoted by

$$\underline{b} = (b_1, b_2, b_3, \dots, b_i, \dots, b_N). \quad (2-144)$$

By grouping all the same outcomes b_i , $i = 1 \dots N$ of a maximal length PN sequence into a single sample set, we obtain two sample sets of outcomes

$$\mathcal{P} = \{ b_i \mid b_i > 0 \text{ and } i=1 \dots N^+ \} \quad (2-145)$$

and

$$\mathcal{M} = \{ b_i \mid b_i < 0 \text{ and } i=1 \dots N^- \}. \quad (2-146)$$

\mathcal{P} and \mathcal{M} are subsets of the set of N outcomes given by

$$\mathcal{N} = \mathcal{P} + \mathcal{M} \quad \text{such that} \quad N = N^- + N^+. \quad (2-147)$$

For any \underline{b} generated by a PN generator of maximal length N

$$N^+ = N^- + 1 \quad (2-148)$$

and for large N

$$N^+ \approx N^- \approx \frac{1}{2}N. \quad (2-149)$$

Note that comparing (2-149) with (2-23), the polarity property of PN is in agreement with the polarity property defined for purely random noise.

Proof. Consider the sample PN vector of \mathbf{b} obtained by tapping the last stage of the BFSR. The variable elements b_i represent the last binary digit of odd or even states with $b_i = +a$ denoting a '1' and $b_i = -a$ denoting a '0'. In a maximal length PN sequence, the generator generates each of the possible states exactly once within each period N . The unique set of states of the BFSR, therefore, represent the set of integers from 1 to N . Since the all-zero state is excluded, the number of odd states always exceeds the number of even states by exactly 1. Since N is odd and $N^+ + N^- = N$, $N^+ = N^- + 1$. QED

2.4.2 The Zero Crossing Property. Consider the transformation of the N -dimensional PN variable vector \mathbf{b} into an N -dimensional PN variable vector \mathbf{x} depicting the change in sign from one sample of b_i to next sample of b_{i+1} . Recall that a non-transition in sign is referred to as a non-zero crossing, an upward transition is referred to as a positive zero crossing, and a downward transition is referred to as a negative zero crossing. The resulting zero crossings PN variable vector is given by

$$\mathbf{x} = (x_1, x_2, x_3, \dots, x_i, \dots, x_N). \quad (2-150)$$

The PN variables x_i are functions of the PN variables b_i and b_{i+1} and are given by

$$x_i = \frac{1}{2}[\text{sgn}(b_{i+1}) - \text{sgn}(b_i)] \quad \text{for } i=1 \dots N-1 \quad (2-151)$$

and

$$\mathbf{x}_N = \frac{1}{2}[\text{sgn}(\mathbf{b}_1) - \text{sgn}(\mathbf{b}_N)]. \quad (2-152)$$

Note that the PN variables x_i are ternary variables of a symmetrically distributed zero mean PN process. A PN sample zero crossings vector of \mathbf{x} is given by

$$\mathbf{x} = (x_1, x_2, x_3, \dots, x_i, \dots, x_N). \quad (2-153)$$

Consider any pair of consecutive samples b_i and b_{i+1} . The outcome x_i can be either -1 , 0 , or $+1$. By grouping all the same outcomes in \mathbf{x} into a single sample set, three sample sets of outcomes result, corresponding to each type of outcome. Namely,

$$\mathcal{X}^+ = \{x_i \mid x_i > 0 \text{ and } i=1 \dots X^+\}, \quad (2-154)$$

$$\mathcal{X}^- = \{x_i \mid x_i < 0 \text{ and } i=1 \dots X^-\}, \quad (2-155)$$

and

$$\mathcal{X}^0 = \{x_i \mid x_i = 0 \text{ and } i=1 \dots X^0\}. \quad (2-156)$$

\mathcal{X}^+ , \mathcal{X}^- , and \mathcal{X}^0 are subsets of the set of N outcomes given by

$$\mathcal{X} = \mathcal{X}^- + \mathcal{X}^0 + \mathcal{X}^+ \quad \text{such that} \quad N = X^- + X^0 + X^+. \quad (2-157)$$

The integers, X^- , X^0 and X^+ , are completely determined by the PN generator. Observations of any N-dimensional vector output from a maximal length PN generator result in the important randomness property that there exists an almost exact balance between the number of outcomes of non-zero crossings X^0 and the number of outcomes of zero crossings $X^\pm = X^- + X^+$. Since there must be an exact balance between the number of outcomes of positive zero crossings X^+ and the number of outcomes of negative zero crossings X^- , it follows that

$$X^+ = X^- = \frac{1}{2}X^\pm = \frac{1}{2}(X^0 + 1). \quad (2-158)$$

Proof. The sample PN vector \mathbf{b}_i and the N-1 vectors \mathbf{b}_j , $i \neq j$ formed by cyclically shifting the elements of \mathbf{b}_i , form a multiplicative Abelian group [44, p. 44] with respect to binary vector multiplication. Consider the vector

$$\mathbf{b}_k = (\mathbf{b}_i) (\mathbf{b}_j). \quad (2-159)$$

The vector \mathbf{b}_k is therefore also a sample PN vector in which the element $b_k > 0$ indicates that the corresponding elements b_i and b_j are of the same polarity and the element $b_k < 0$ indicates that the corresponding elements b_i and b_j are of different polarity. If \mathbf{b}_j is a single cyclic shift of \mathbf{b}_i , then the element $b_k > 0$, indicates a non-zero crossing and the element $b_k < 0$, indicates a zero crossing between elements b_i and b_{i+1} .

Let X^0 denote the number of non-zero crossings in \mathbf{b}_i ; let X^+ denote the

number of positive zero crossings in \underline{b}_i ; and let X^- denote the number of negative zero crossings in \underline{b}_i . Recall, however, that from the polarity property of any maximal length PN sample vector defined by (2-141), $N^+ = N^- + 1$. The non-zero crossings belong to \mathcal{P} and the zero crossings belong to \mathcal{M} . For $\{\underline{b}_i, i = 1 \dots N\}$ to form an Abelian group under multiplication, all elements must be multiplied by -1 . Hence, $N^- = N^+ + 1$. This is a result of the isomorphism with the corresponding Abelian group under addition in which the identity element under addition, a '0', corresponds to the identity element under multiplication which is a '1'. Therefore, $X^\pm = X^0 + 1$. Since X^\pm is even, each zero crossing of one polarity must be followed by a zero crossing of the opposite polarity. Therefore, $X^+ = X^- = \frac{1}{2} X^\pm = \frac{1}{2} (X^0 + 1)$. QED

2.4.3 The autocorrelation property. Consider a sample N-dimensional PN binary vector \underline{b}_i with elements b_i and a new sample N-dimensional PN binary vector \underline{b}_{i+k} with elements b_{i+k} , obtained by cyclically shifting the elements of \underline{b}_i by kT_c , $k=1 \dots N$. The periodic/cyclic autocorrelation of \underline{b}_i is given by

$$R_{N,k} = (\underline{b}_i \cdot \underline{b}_{i+k}) = \sum_{i=1}^N b_i b_{i+k} \quad i,k = 1 \dots N \quad (2-160)$$

Observations of any N-dimensional vector output from a maximal length PN generator result in a two-valued autocorrelation given by

$$R_{N,0} \gg R_{N,k} \quad \text{for } k = 1 \dots N-1 \quad (2-161)$$

where

$$R_{N,0} = E_b = (\underline{b}_i \cdot \underline{b}_i) \quad (2-162)$$

and

$$R_{N,k} = -a^2 \quad \text{for } k = 1 \dots N-1. \quad (2-163)$$

Note that E_b is called the energy of \underline{b} .

Proof. When b_i and b_{i+k} are of the same polarity, the summation element $b_j = b_i b_{i+k}$ represents an agreement. When b_i and b_{i+k} are of the opposite polarity, the summation element $b_j = b_i b_{i+k}$ represents a disagreement. The autocorrelation, therefore, is simply proportional to the difference between the number of disagreements N_D and the number of agreements N_A , *i.e.*

$$R_{N,k} = -a^2(N_D - N_A). \quad (2-164)$$

When $k=0$, $N_D = 0$ and $N_A = N$. Therefore,

$$R_{N,0} = E_b = (\underline{b}_i \cdot \underline{b}_i) = a^2 N. \quad (2-165)$$

When $k \neq 0$, the product element $b_j = b_i b_{i+k}$ in (2-160) is also elements of an Abelian sample N -dimensional PN binary vector \underline{b}_j with $N^+ = N_A$ and $N^- = N_D$. Recall that for an Abelian group under multiplication $N^- = N^+ + 1$ which yields,

$$N_D + N_A = N \quad \text{and} \quad N_D - N_A = 1. \quad (2-166)$$

Therefore,

$$R_{N,k} = -a^2 \quad \text{for } k = 1 \dots N. \quad (2-167)$$

For large N , $Na^2 \gg -a^2$ and hence, $R_{N,0} \gg R_{N,k}$ for $k = 1 \dots N-1$. QED

From the point of view that PN has a two-valued autocorrelation property, it is said to be consistent with the autocorrelation property of random noise.

2.4.4 The run properties. Consider a sample N -dimensional PN vector \underline{b} . It can be shown by enumerating all the possibilities [44, p. 43] that the total number of successive occurrences of a given polarity called runs, is given by

$$B = \sum_{k=1}^n m_{\pm k} = 2^{(n-1)} = \frac{1}{2}(N+1) \quad (2-168)$$

where $m_{\pm k}$ is the total number of runs of any polarity and length k , and n is the number of stages in the generator FBSR. For $k \leq n-2$,

$$m_{\pm k} = 2^{(n-1-k)} = \left(\frac{1}{2}\right)^k \left[\frac{1}{2}(N+1)\right] \approx \left(\frac{1}{2}\right)^k \frac{1}{2}N. \quad (2-169)$$

Recall that for random noise, $E[m_{\pm k}] = \left(\frac{1}{2}\right)^k N$. Comparing with (2-169), random noise is expected to have twice as many runs of length k as PN. From that point of view, the run distribution of PN is inconsistent with random noise. Consider, however, the ratio $(m_{\pm k})/(m_{\pm k \pm 1}) = 2$. This ratio is identical to the comparable ratio for random noise, *i.e.* $E[m_{\pm k}]/E[m_{\pm k \pm 1}]$. For large N , in both random

noise and PN there exist approximately two runs of length k for each run of length $k+1$. For random noise, the exceptions are stochastic, but for PN the exception exists on a regular basis only for $k = n$, in which case only one run of the opposite polarity and length $k = n-1$ is generated. As indicated by the limit of the summation in (2-168), the maximum run length is given by n . Recall that for a maximal length sequence, $N = 2^n - 1$ where n is the degree of the PN generator. Therefore,

$$n = \log_2 (N + 1), \quad (2-170)$$

which for large N is in good agreement with the longest likely run previously derived for random noise. In this regard, another important distinction between random noise and PN should be noted. In PN, the probability that a run of length $k > n$ will occur is zero whereas in random noise such probabilities are not high but may be significant.

2.5 PN Invariance Transformations

Consider functions or transformations of the N -dimensional PN vector \underline{b} . Denote such transformations by $\mathcal{N}(\underline{b})$. Define \mathcal{N}_i as a member of a class of \mathcal{N} such that \mathcal{N}_i preserves the randomness properties discussed in the preceding sections. The new M -dimensional random noise vector is given by

$$\mathcal{N}(\underline{b}) = (b^1, b^2, b^3, \dots, b^i, \dots, b^M). \quad (2-171)$$

Consider the transformations identical to those defined for random noise. The preservation of the randomness properties is proved for the following transformations:

$\mathcal{N}_0 \Rightarrow$ every sample of \underline{b} is multiplied by an arbitrary constant.

Proof. \mathcal{N}_0 is a linear operation performed independently on each PN variable element such that $b^i = Ab_i$. For $A > 0$, the polarities of the samples are not changed and therefore their zero crossings and run distributions remain unaffected. For $A < 0$, the polarities of the samples are all inverted thereby also preserving the original balance between N^+ and N^- . Inverting the polarities of individual samples inverts the runs and thereby inverts the zero crossings. Since there are equal numbers of positive and negative runs, the run distribution remains unaffected with the minor exception of the two longest runs. The output autocorrelation is now amplified by A^2 for $k=0 \dots N$ regardless of the sign of A . The two-valued nature of the output autocorrelation, therefore, remains unchanged. QED

$\mathcal{N}_1 \Rightarrow$ every sample of \underline{b} is reduced by the sample average of \underline{b} .

Proof. The sample average of \underline{b} is a random variable given by

$$A_N = (1/N) \sum_{i=1}^N b_i = a/N. \quad (2-172)$$

The transformed random variable is given by

$$\mathbf{b}^i = \mathbf{b}_i - A_N. \quad (2-173)$$

Since $A_N \ll a$, $\text{sgn}(\mathbf{b}^i) = \text{sgn}(\mathbf{b}_i)$. Thus, each variable element of the output vector retains the polarity of the input variable element. Therefore, the polarity, zero crossing, and run properties remain unchanged. The effect of \mathcal{N}_1 upon the autocorrelation property can be seen by evaluating the autocorrelation of the output vector given by

$$R'_{N,k} = \sum_{i=1}^N \mathbf{b}^i \mathbf{b}^{i+k} = \sum_{i=1}^N (\mathbf{b}_i - a/N)(\mathbf{b}_{i+k} - a/N) = R_{N,k} - a^2/N. \quad (2-174)$$

For large N the two-valued nature of the autocorrelation of the input PN vector given by $R_{N,k}$ is preserved. QED

$\mathcal{N}_2 \Rightarrow$ Every run of length $l_i > n = \log_2 N$ is nullified or reduced by the sample average of the run.

Proof. Since there are no runs of length $k > n$ in a sample of a maximal length PN vector $\mathbf{b}^i = \mathbf{b}_i$. Since the input vector satisfies the randomness properties, the output vector which is identical to the input vector must also satisfy the same randomness properties. QED

$\mathcal{N}_3 \Rightarrow \underline{b}$ is shortened by removing a single sample from every run.

Proof. Since there are equal number of positive and negative runs, the removal of a single sample from every run preserves the polarity property. The required run distribution is retained since all one-unit runs are eliminated, all two-unit runs become one-unit runs, all three-unit runs become two-unit runs and so on until the n -unit run becomes an $(n-1)$ -unit run. Hence, the new run distribution satisfies the requirement of a PN sequence generated by a maximal length FBSR with $n-1$ stages. Since there are an equal number of positive and negative runs in the output vector, it is easy to rearrange the runs without losing their zero crossings as shown in Figure 2.5-1. The output vector with shifted runs has the same run distribution as required for a PN sequence. Since it is a PN sequence which may have been generated by a FBSR with $n-1$ stages, the autocorrelation property is also preserved. QED

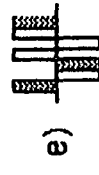
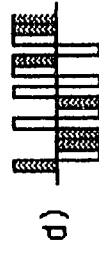
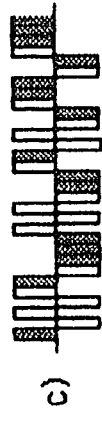
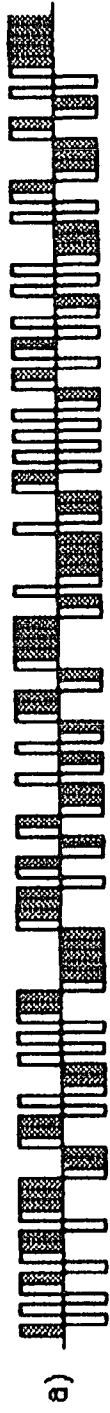


Figure 2.5-1
 Transforming Higher Order PN Vectors
 into Lower Order Vectors
 (a) \rightarrow b) \rightarrow c) \rightarrow d) \rightarrow e)

CHAPTER III

DETECTION IN THE PRESENCE OF AWGN

3.1 Introduction

To understand the performance of a communication system in the presence of both noise and interference and how it is affected by the introduction of additional signal processing, it is important to investigate detection in the presence of noise without interference as a baseline. In this chapter we investigate a variety of decision rules applicable to Gaussian noise which we will be able to modify or complement when non-random interference is also present. When communicating in AWGN environments, it is well known that the optimum detection of any independent symbol signal structure which is of duration T may be achieved by receivers of the matched filter type, or equivalently of the correlator type [39]. We assume that the symbols are binary, equiprobable and represented by any simplex waveform such as Binary Phase Shift Keying (BPSK), or bipolar. Optimum detection in more general communication systems requires more complexity such as additional matched filters or correlators and biases [4, p. 212]. Nevertheless, given the same ratio of signal energy to noise power spectral density (E_b/N_0), performance of the optimum detection depends only upon the relative distance of the signals in signal space and is independent of the symbol signal structure. Different modulation forms, *e.g.* orthogonal, and different receivers, *e.g.* non-coherent or

partially coherent, will produce different expressions for their dependency upon E_b/N_o , but no other parameter need be considered. We shall, therefore, restrict our discussion to the binary, equiprobable, simplex waveforms because in a high noise environment they provide the best performance for given E_b/N_o [39, p. 245]. More importantly, however, binary, equiprobable, simplex waveforms provide the simplest point of departure for discussing the signal processing concepts and algorithms under investigation. Generalizing these concepts and algorithms in the context of more complex communication systems may only obscure the fundamental nature of the research and is considered beyond the scope of any rigorous discussion in this dissertation.

The problem of optimum detection is simplified when analysis is performed in signal space, where signal waveforms are represented by signal vectors using an orthogonal or orthonormal basis [39, p.225]. We investigated, therefore, signal processing algorithms which may be considered as operators in signal space. As such, the signal space operators are independent of the modulation scheme and if found suitable in one modulation scheme these operators should be applied to other modulation schemes. It should be noted that we would not expect any new signal processing algorithm, linear or non-linear, which precedes or replaces the optimum receiver to result in any processing gain in AWGN. On the contrary, since the optimum receiver yields the best performance in AWGN, we would expect some performance degradation to result if the optimum receiver is altered in any structural way. It is important to note that AWGN may not be the only mechanism to corrupt the symbol signal. Since

AWGN is omni-present, however, any processing loss in an environment of only AWGN should be minor and more than compensated by the processing gain to be derived in the presence of non-AWGN environments or mixed noise-interference environments.

3.2 Signal Space Formulation

Consider the received wideband baseband waveform $r(t)$ which is the sum of the PN-coded source data waveform $\frac{1}{2}As(t)$ and the random noise process time sample $n(t)$ given by (1-29) and (1-31), respectively. Each symbol in a message may be represented in signal space using a complete set of time-orthogonal unit-rectangular basis function $\{q_i(t), i = 1 \dots N\}$ defined in (1-3). Generally the message alphabet symbol set is given by $\{\mathbf{a} \mid \mathbf{a} = a_0, a_1, \dots, a_j, \dots, a_{M-1}\}$. For a binary symbol set $M = 2$, and we might have $a_0 = '0'$ and $a_1 = '1'$. For a message of J bits, the j^{th} bit signal variable vector in signal space at the receiver is denoted \underline{s}_j . When a '0' is assumed to be received, then the j^{th} bit signal vector received is $\underline{s}_j = \underline{s}_0$, otherwise, $\underline{s}_j = \underline{s}_1$. Since the discussion in this chapter applies to any one of the bits in the message, we drop the first index which designates which bit in the message is received, for example, $\underline{s}_j = \underline{s}$. In an N -dimensional signal space, we project the received bit waveform onto the signal space coordinates to obtain the received signal vector given by

$$\underline{s} = (b_1, b_2, \dots, b_j, \dots, b_N) \quad (3-1)$$

where \mathbf{b}_i denotes the received signal variable projection in the i^{th} chip time interval T_c . Thus, when the source \mathbf{a} generates the message α_0 , the received signal sample vector is given by

$$\underline{\mathbf{s}}_0 = (b_{01}, b_{02}, \dots, b_{01}, \dots, b_{0N}). \quad (3-2)$$

In contrast, when the source \mathbf{a} generates the message α_1 , the received signal sample vector received is given by

$$\underline{\mathbf{s}}_1 = (b_{11}, b_{12}, \dots, b_{11}, \dots, b_{1N}). \quad (3-3)$$

We assume that $\{\underline{\mathbf{s}}_0, \underline{\mathbf{s}}_1\}$ is a simplex signal set. To attain minimum energy signals the components of $\underline{\mathbf{s}}_0$ should be antipodal to corresponding components of $\underline{\mathbf{s}}_1$, therefore, we require that

$$b_{0i} = -b_{1i} = \pm b \quad (3-4)$$

and

$$(b_{0i})^2 = (b_{1i})^2 = b^2, \quad (3-5)$$

where

$$b > 0 \text{ and } i = 1 \dots N. \quad (3-6)$$

We shall refer to a communication system as a DC system when the symbol data is not modulated by a PN generator. In contrast, we shall refer to a

communication system as a PN system when the symbol data is modulated by a PN generator. For a PN system, we shall also assume that the number of dimensions in signal space N also corresponds to the period of the PN sequence. Without loss of generality, given a DC system we have

$$b_{0i} = -b \text{ and } b_{1i} = +b, \quad i = 1 \dots N, \quad (3-7)$$

and given a PN system we have

$$b_i = \pm b, \quad (3-8)$$

depending upon the source a and the phase interval l of the PN generator.

In an AWGN environment, projecting the noise process onto the signal space coordinates, we obtain the relevant noise random variable vector given by

$$\mathbf{n} = (n_1, n_2, \dots, n_i, \dots, n_N), \quad (3-9)$$

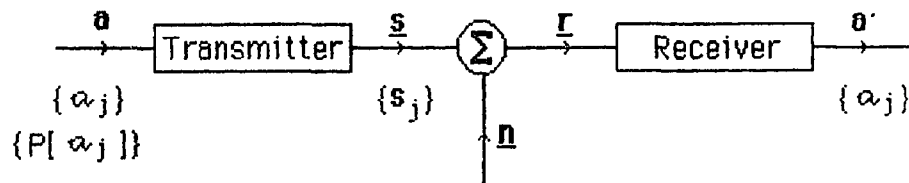
where n_i are independent, zero mean Gaussian random variables with probability density function (pdf)

$$p(n_i = n_i) = N(0, \sigma_i) = (1/\sqrt{2\pi} \sigma_i) \exp(-\frac{1}{2}n_i^2 / \sigma_i^2). \quad (3-10)$$

For the j^{th} bit interval the relevant noise sample vector $\mathbf{n} = \mathbf{n}_j$ and for the i^{th} chip in that bit $n_i = n_{ji}$.

Hypothesize that a '0' was transmitted many times using the basic vector communication system shown in Figure 3.2-1. This hypothesis is denoted H_0 .

(a) Vector Representation



(b) N-Dimensional Signal Space Decomposition

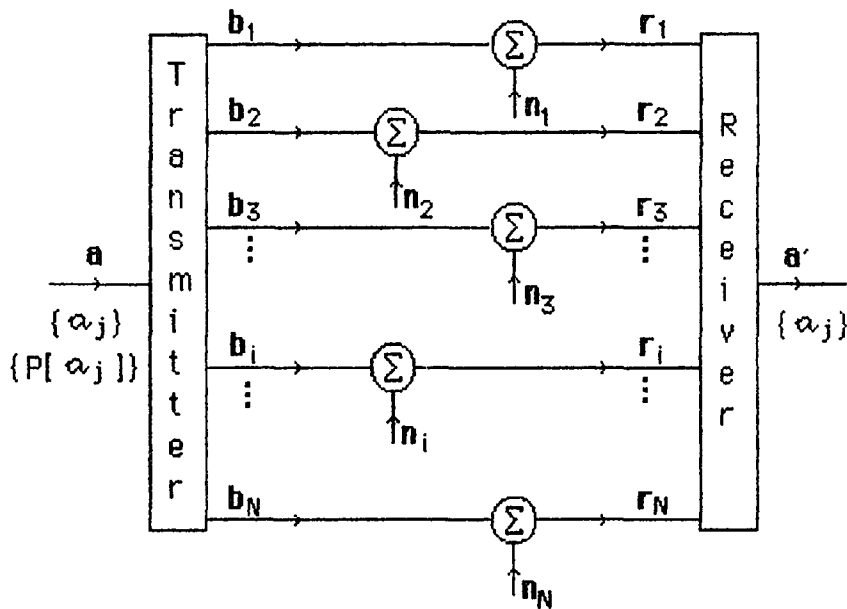


Figure 3.2-1. A Basic Vector Communication System in the Presence of Noise

The received sampled data random variable vector is given by

$$\mathbf{r} = (r_1, r_2, \dots, r_1, \dots, r_N), \quad (3-11)$$

where

$$\mathbf{r} = \mathbf{s}_0 + \mathbf{n} \quad (3-12)$$

and equivalently,

$$r_i = b_{0i} + n_i. \quad (3-13)$$

Hypothesizing that instead of a '0', a '1' was also transmitted many times and denoting this hypothesis as H_1 , the received data random variable vector is now given by

$$\mathbf{r} = \mathbf{s}_1 + \mathbf{n} \quad (3-14)$$

and equivalently,

$$r_i = b_{1i} + n_i. \quad (3-15)$$

The vector communication system described above is applicable to either DC or PN systems and may be considered as an N-dimensional diversity system as shown in Figure 3.2-1b.

3.3 The Binary Likelihood Ratio Test (BLRT)

Assuming that the binary symbols are equiprobable and that the cost of

making a mistake in favor of either symbol is independent of the symbol, the binary decisions are typically based upon the result of the maximum likelihood statistic $K(T)$ or $\mathcal{L}(T)$ obtained at the output of the matched filter or correlator as shown in Figures 1.4-1 and 1.4-2. In this section we derive the functional form of the sufficient statistic which is applicable to either a DC or a PN system. We assume that the relevant noise is zero-mean, Gaussian, independent from chip to chip, and with variances which may not be necessarily identical. Such situations may exist in practical systems where Gaussian-like broad-band interference may be turned on and off. The joint pdf of the received vector components under H_1 is defined by

$$p(\underline{r} = \underline{r} | H_1) \triangleq p(r_1 = r_1, r_2 = r_2, \dots, r_i = r_i, \dots, r_N = r_N | H_1). \quad (3-16)$$

The joint pdf of the noise vector components under H_1 is given by

$$p(\underline{n} = \underline{n} | H_1) = p(n_1 = r_1 - b_{11}, n_2 = r_2 - b_{12}, \dots, n_i = r_i - b_{1i}, \dots, n_N = r_N - b_{1N}). \quad (3-17)$$

We observe that

$$p(\underline{r} = \underline{r} | H_1) = p(\underline{n} = \underline{n} | H_1). \quad (3-18)$$

Similarly, the joint pdf of the received vector components under H_0 is defined as

$$p(\underline{r} = \underline{r} | H_0) \triangleq p(r_1 = r_1, r_2 = r_2, \dots, r_i = r_i, \dots, r_N = r_N | H_0). \quad (3-19)$$

The joint pdf of the noise vector components under H_0 is given by

$$p(\underline{n} = \underline{r} - \underline{s}_0) = p(n_1 = r_1 - b_{01}, n_2 = r_2 - b_{02}, \dots, n_i = r_i - b_{0i}, \dots, n_N = r_N - b_{0N}), \quad (3-20)$$

and, therefore,

$$p(\underline{r} = \underline{r} | H_0) = p(\underline{n} = \underline{r} - \underline{s}_0). \quad (3-21)$$

Since n_i is independent of n_j for $i \neq j$ and any $i, j = 1 \dots N$, under hypothesis H_k (where $k=0,1$), the joint pdf of all r_i is simply the product of the marginal pdfs of individual r_i . Thus,

$$p(\underline{r} = \underline{r} | H_k) \triangleq p(r_1 = r_1, r_2 = r_2, \dots, r_i = r_i, \dots, r_N = r_N | H_k) = \quad (3-22)$$

$$p(n_1 = r_1 - b_{k1}) p(n_2 = r_2 - b_{k2}) \dots p(n_i = r_i - b_{ki}) \dots p(n_N = r_N - b_{kN}),$$

or more compactly

$$p(\underline{r} = \underline{r} | H_k) = \prod_{i=1}^N p(n_i = r_i - b_{ki}), \quad (3-23)$$

where

$$p(\mathbf{n}_i = r_i - b_{ki}) = (1/\sqrt{2\pi} \sigma_i) \exp\{-\frac{1}{2}(r_i - b_{ki})^2 / \sigma_i^2\}. \quad (3-24)$$

The likelihood ratio, denoted by $\Lambda(\mathcal{L})$, provides for a numerical comparison between the two hypotheses in the form of the ratio of the pdfs defined as follows

$$\Lambda(\mathcal{L}) \triangleq p(\mathcal{L} = \mathcal{L} | H_1) / p(\mathcal{L} = \mathcal{L} | H_0). \quad (3-25)$$

Invoking the AWGN assumption,

$$\Lambda(\mathcal{L}) = \left(\prod_{i=1}^N p(\mathbf{n}_i = r_i - b_{1i}) \right) / \left(\prod_{i=1}^N p(\mathbf{n}_i = r_i - b_{0i}) \right). \quad (3-26)$$

Using the minimum probability of error criterion [48, p. 48], likelihood-ratio processors (LRPs) will compute $\Lambda(\mathcal{L})$ obtained during each symbol decision time interval T and compare it to a likelihood ratio test (LRT) threshold denoted by μ . Thus

$$\text{if } \Lambda(\mathcal{L}) > \mu \text{ then } H_1, \text{ otherwise } H_0. \quad (3-27)$$

The output of the LRP, a_k , will depend upon the decision rule designed into the processor logic. The LRT threshold for a MAP LRP is given by

$$\mu_{\text{MAP}} = P[a_0] / P[a_1]. \quad (3-28)$$

The LRT threshold for a ML LRP is obtained when

$$P[a_0] = P[a_1] = \frac{1}{2}. \quad (3-29)$$

Therefore,

$$\mu_{ML} = 1. \quad (3-30)$$

Thus, the MAP LRP is the optimum processor which minimizes the average overall probability of error P_e , i.e. if a_m is sent, it will set

$$a_k = a_m \text{ if and only if } P[a_m] \bullet p(\underline{r} = \underline{r} | a_m) \text{ is a maximum for } m = k. \quad (3-31)$$

In the discussion that follows we assume that

$$P[a_m] = \frac{1}{2} \text{ for } m = 0, 1, \quad (3-32)$$

consequently, we will consider only the ML LRP.

Substitution of (3-24) into (3-26) results in a LR which is applicable to both DC and PN systems as follows:

$$\Lambda(\underline{r}) = \left[\prod_{i=1}^N \exp\left\{-\frac{1}{2}(r_i - b_{1i})^2 / \sigma_i^2\right\} \right] / \left[\prod_{i=1}^N \exp\left\{-\frac{1}{2}(r_i - b_{0i})^2 / \sigma_i^2\right\} \right]. \quad (3-33)$$

Multiplying the numerator and denominator of (3-33) each by the factor

$$\left(\prod_{i=1}^N \exp\left\{-\frac{1}{2} (r_i - b_{0i})^2 / \sigma_i^2\right\}, \right. \quad (3-34)$$

we obtain

$$\Lambda(\underline{r}) = \left(\prod_{i=1}^N \exp\left\{\frac{1}{2} [(r_i - b_{0i})^2 - (r_i - b_{1i})^2] / \sigma_i^2\right\}. \right. \quad (3-35)$$

Using (3-5), the numerator in each of the exponential factors in (3-35) becomes linear and is given by

$$(r_i - b_{0i})^2 - (r_i - b_{1i})^2 = 2 r_i (b_{1i} - b_{0i}). \quad (3-36)$$

Due to the monotonicity of logarithmic operations, taking the natural logarithm of $\Lambda(\underline{r})$ cannot affect the outcome of the LRT. We, therefore, define the logarithmic LR (LLR) given by

$$\ell(\underline{r}) \triangleq \ln \Lambda(\underline{r}), \quad (3-37)$$

and a LRT equivalent to (3-27) given by

$$\text{if } \ell(\underline{r}) > 0 \text{ then } H_1, \text{ otherwise } H_0. \quad (3-38)$$

As a consequence of (3-37), the noise vector components imbedded in the received vector components contribute to the binary decision problem in an additive fashion given by

$$\ell(\mathcal{D}) = \sum_{i=1}^N a_i r_i \quad (3-39)$$

where

$$a_i = (b_{1i} - b_{0i}) / (\sigma_i^2).$$

Note that $\ell(\mathcal{D})$ constitutes a sufficient statistic since it includes all of the relevant data needed for a decision. Note that $\ell(\mathcal{D})$ is a linear combination of N samples r_i of the random variables r_i which are jointly distributed and mutually independent. Hence, $\ell(\mathcal{D})$ is also a sample of the random variable $\ell(\mathcal{D})$ whose mean is given by

$$E[\ell(\mathcal{D})] = E\left[\sum_{i=1}^N a_i r_i\right]. \quad (3-40)$$

Assuming that the noise is stationary, we may interchange the summation and averaging in (3-40) to obtain

$$E[\ell(\mathcal{D})] = \sum_{i=1}^N a_i E[r_i]. \quad (3-41)$$

Similarly we find the variance of $\ell(\mathcal{D})$ to be given by

$$\text{Var}[\ell(\mathcal{D})] = E[(\ell(\mathcal{D}) - E[\ell(\mathcal{D})])^2] = \sum_{i=1}^N a_i^2 \text{Var}[r_i] = \sum_{i=1}^N a_i^2 \sigma_i^2. \quad (3-42)$$

But

$$a_i^2 = (b_{1i} - b_{0i})^2 \sigma_i^{-4} = 4b^2 \sigma_i^{-4},$$

and therefore,

$$\text{Var}[\ell(\underline{r})] = 4b^2 \sum_{i=1}^N \sigma_i^{-2} = \sigma_N^2. \quad (3-43)$$

Thus, the variance of $\ell(\underline{r})$ is independent of the hypotheses H_k and is the same for a DC or a PN system.

For DC systems, let $\ell(\underline{r}) = \ell_A(\underline{r})$. we may now simplify (3-39) since using (3-7),

$$(b_{1i} - b_{0i}) = +2b. \quad (3-44)$$

Hence

$$\ell_A(\underline{r}) = 2b \left[\sum_{i=1}^N r_i / (\sigma_i)^2 \right]. \quad (3-45)$$

Given the hypothesis H_1 in which a '1' is assumed to have been transmitted, then $E[r_i] = b$, and the expected value of $\ell_A(\underline{r})$ is given by

$$E[\ell_A(\underline{r})] = 2b^2 \sum_{i=1}^N \sigma_i^{-2} = a_N. \quad (3-46)$$

For PN systems, let $\ell(\underline{r}) = \ell_B(\underline{r})$. We cannot simplify (3-39), as for DC systems, since a_i is now pseudorandom, *i.e.* from (3-8)

$$a_i = (b_{1i} - b_{0i}) / \sigma_i^2 = \pm 2b / \sigma_i^2 \text{ depending on } i. \quad (3-47)$$

Hence,

$$\ell_B(\mathcal{L}) = \sum_{i=1}^N r_i (b_{1i} - b_{0i}) / (\sigma_i)^2. \quad (3-48)$$

Given H_1 , we may nevertheless obtain $E[\ell_B(\mathcal{L})]$. Note that $E[r_i] = -b$, when $b_{1i} = -b$ and $E[r_i] = +b$, when $b_{1i} = +b$. Hence the product $a_i E[r_i]$ is always positive resulting in

$$E[\ell_B(\mathcal{L})] = 2b^2 \sum_{i=1}^N \sigma_i^{-2} = a_N = E[\ell_A(\mathcal{L})]. \quad (3-49)$$

Similarly, given H_0 we find that $E[\ell_A(\mathcal{L})] = E[\ell_B(\mathcal{L})] = -a_N$.

For noise which is i.i.d. $\sigma_i = \sigma$ for $i = 1 \dots N$. Therefore, (3-49) simplifies to

$$a_N = 2b^2 N / \sigma^2, \quad (3-50)$$

and similarly

$$\sigma_N^2 = 4b^2 N / \sigma^2. \quad (3-51)$$

The equality of the expected values and variances of $\ell_A(\mathcal{L})$ and $\ell_B(\mathcal{L})$ is significant in that it proves that at least from a decision theoretic point of view the decision regions of the DC and PN systems are equivalent and hence they should perform equally well. In addition we note that for random noise which is i.i.d. as well as non-i.i.d., $\sigma_N^2 = 2a_N$ independently of the power of either the signal or the noise. The dependence upon the power of the signal and the noise

occurs in both σ_N^2 and a_N as the same ratio of the signal and noise parameters. Therefore, by measuring a_N we automatically obtain its standard deviation σ_N . In practical systems, one may not have *a priori* knowledge of the σ_i . In spread spectrum systems, however, it may be possible to estimate σ_i by assuming that the chips in the neighborhood of 1 are i.i.d. and by using the sample variance in place of σ_i .

3.3.1 The logarithmic BLRT in a low noise environment Consider the transmission of a symbol through either a DC or a PN system. Given the same noise vector, in general, $\ell_A(\mathbf{r}) \neq \ell_B(\mathbf{r})$ since different weights are applied to the same noise sample by each type of system. In a low noise environment, however, we would expect their difference to be negligible. More concisely we state the following theorem:

THEOREM 1: In the limit as the noise becomes negligible in the mean square sense, the difference between the LLR of a DC system and that of a PN system also becomes negligible in the mean square sense, *i.e.*

$$\ell_A(\mathbf{r}) \approx \ell_B(\mathbf{r}). \quad (3-52)$$

Proof. Consider the ratio given by

$$\Gamma = \sqrt{(\text{Var}[\ell(\mathbf{r})])} / |E[\ell(\mathbf{r})]| \quad (3-53)$$

Using (3-43) and (3-49),

$$\Gamma = \sigma_N / a_N = (b^2 \sum_{i=1}^N \sigma_i^{-2})^{-1/2} . \quad (3-54)$$

The dependence on the key variables may be seen more readily for noise which is i.i.d. where we use (3-50) and (3-51) to obtain

$$\Gamma = \sigma / \sqrt{Nb^2} . \quad (3-55)$$

Since in the limit as the noise becomes negligible in the mean square sense, $\text{Var}[n_i] = \sigma_i^2 \rightarrow 0$ forcing $\Gamma \rightarrow 0$. But from (3-53) we must also have that $\text{Var}[\ell(\underline{r})] \rightarrow 0$ and since $\ell(\underline{r}) = \ell_A(\underline{r})$ or $\ell(\underline{r}) = \ell_B(\underline{r})$, $\ell_A(\underline{r}) \approx \ell_B(\underline{r})$. QED

Intuitively, THEOREM 1 may be motivated also by the fact that as the signal-to-noise ratio increases, the LLR begins to lose its randomness and becomes more and more deterministic within experimental error.

3.3.2 The logarithmic BLRT in a high noise environment. Consider the same transmission experiment as above. Namely, given the same noise vector, a symbol is transmitted through either a DC or a PN system. In a high noise environment, on the average, we would expect the difference between the two possible LLRs to be significant. This fact is expressed by the following theorem:

THEOREM 2: In the limit as the noise becomes dominant in the mean square sense, the difference between the LLR of a DC system and that of a PN system also becomes significant in the mean square sense increasing in proportion to the variance of the LLR.

Proof. The difference LLR is defined by $\ell_D(\mathcal{L}) = \ell_B(\mathcal{L}) - \ell_A(\mathcal{L})$. Consider the ratio given by

$$\Gamma_D = \sqrt{(\text{Var}[\ell_D(\mathcal{L})]) / \mathbb{E}[\ell_D(\mathcal{L})]} . \quad (3-56)$$

Using (3-45) and (3-48),

$$\ell_D(\mathcal{L}) = \sum_{i=1}^N (a_B - a_A) r_i \quad (3-57)$$

where

$$a_{Bi} = (b_{1i} - b_{0i}) / \sigma_i^2 \quad \text{and} \quad a_{Ai} = 2b / \sigma_i^2 .$$

Since $\ell_D(\mathcal{L})$ is a linear combination of the r_i , $i = 1 \dots N$, samples of the random variables r_i which are jointly distributed and mutually independent,

$$\text{Var}[\ell_D(\mathcal{L})] = \sum_{i=1}^N (a_B - a_A)^2 \text{Var}[r_i] \quad (3-58)$$

Expanding the coefficient in the summation,

$$(a_B - a_A)^2 = \sigma_i^{-4} (b_{1i} - b_{0i} - 2b)^2 = 8b^2 (1 - \text{sgn } b_{1i}) \sigma_i^{-4} ,$$

and since $\text{Var}[r_i] = \sigma_i^2$,

$$\text{Var}[\ell_D(\mathcal{L})] = \sum_{i=1}^N 8b^2 (1 - \text{sgn } b_{1i}) \sigma_i^{-2} . \quad (3-59)$$

Note that $(1 - \text{sgn } b_{1i})$ is 0 when b_{1i} is positive and 2 when b_{1i} is negative. Hence only the negative signal chips in the PN sequence contribute to $\text{Var}[\ell_D(\mathcal{L})]$. Since

the functional dependence of $\text{Var}[\ell_D(\underline{r})]$ is analogous to that of $\text{Var}[\ell(\underline{r})]$, Γ_D is proportional to σ_i and hence for high power noise, $\sigma_i \rightarrow \infty$, $\Gamma_D \rightarrow \infty$ and therefore $\text{Var}[\ell_D(\underline{r})] \rightarrow \infty$. QED

The dependence on the key variables may be seen more readily for noise which is i.i.d. Since for one period of the PN sequence, b_{1i} is positive for $(N+1)/2$ chips and negative for $(N-1)/2$ chips, we may simplify (3-59) to obtain

$$\text{Var}[\ell_D(\underline{r})] = 4b^2(N-1)\sigma^2 \quad (3-60)$$

$$\Gamma_D = \sigma \sqrt{N-1}/(bN) \approx \Gamma \quad (3-61)$$

3.4 Errors in Additive White Gaussian Noise (AWGN).

In making binary decisions, four transition probabilities are possible as shown in Figure 3.4-1. The probability $P\{1|1\}$ corresponds to the probability of correctly choosing hypothesis H_1 when a '1' was indeed transmitted. The probability $P\{0|1\}$ corresponds to the probability of incorrectly choosing hypothesis H_0 when a '1' was transmitted. Similarly, the probability $P\{0|0\}$ corresponds to the probability of correctly choosing hypothesis H_0 when a '0' was indeed transmitted. Finally, the probability $P\{1|0\}$ corresponds to the probability of incorrectly choosing hypothesis H_1 when a '0' was transmitted. Using the maximum likelihood ratio processor, the optimum decision was shown

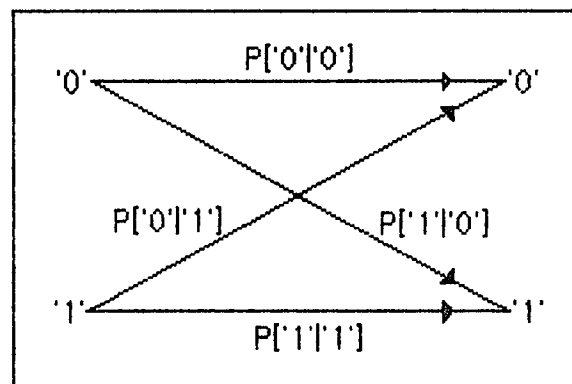


Figure 3.4-1. The Binary Error Channel Model

to depend upon the outcome of the sufficient statistic $\mathcal{L}(r)$. Since $\mathcal{L}(r)$ is a linear combination of Gaussian variates, it is also a Gaussian variate. If a '1' is transmitted, the mean is given by $a_{1N} = a_N$ and the variance $\sigma_{1N} = \sigma_N$. If a '0' is transmitted, the mean is given by $a_{0N} = -a_N$ and the variance $\sigma_{0N} = \sigma_N$. The two Gaussian variates corresponding to the transmission of a '1' and a '0' are shown in Figure 3.4-2. Using the definition of the Q-function discussed in Appendix A,

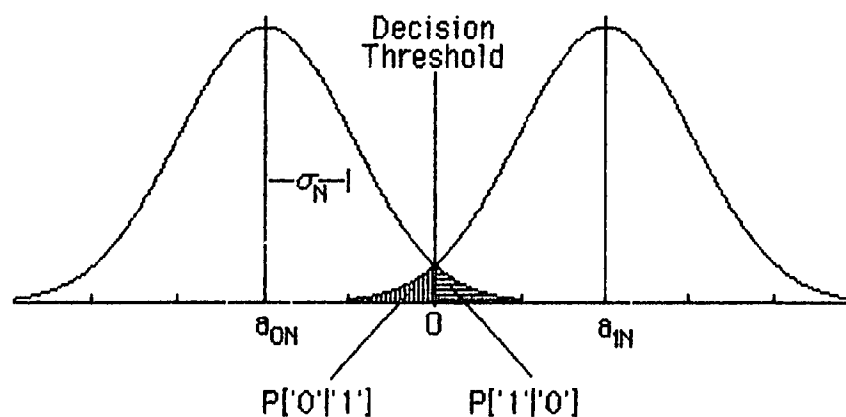


Figure 3.4-2. The Sufficient Statistic Probability Density Functions in the Presence of Gaussian Noise

the transition probabilities are given by

$$P[1|1] = P[\ell(r) > 0|1] = 1 - Q(a_{1N} / \sigma_{1N}), \quad (3-62)$$

$$P[0|1] = P[\ell(r) < 0|1] = Q(a_{1N} / \sigma_{1N}), \quad (3-63)$$

$$P[0|0] = P[\ell(r) < 0|0] = 1 - Q(a_{0N} / \sigma_{0N}), \quad (3-64)$$

$$P[1|0] = P[\ell(r) > 0|0] = Q(a_{0N} / \sigma_{0N}). \quad (3-65)$$

For a binary symmetric channel $a_{1N} = -a_{0N} = a_N$ and $\sigma_{1N} = \sigma_{0N} = \sigma_N$ and therefore,

$$P[1|1] = P[0|0] = 1 - Q(a_N / \sigma_N), \quad (3-66)$$

$$P[1|0] = P[0|1] = Q(a_N / \sigma_N). \quad (3-67)$$

The total probability of error is given by

$$P_e = P[0] P[1|0] + P[1] P[0|1] \quad (3-68)$$

For equiprobable *a priori* transmission probabilities, *i.e.* $P[1] = P[0] = \frac{1}{2}$, we obtain

$$P_e = Q(a_N / \sigma_N). \quad (3-69)$$

For noise which is i.i.d.,

$$P_e = Q(\sqrt{Nb^2 / \sigma^2}). \quad (3-70)$$

Assuming a BPSK signal of amplitude b , the energy per chip is given by $E_c = \frac{1}{2}b^2T_c$ and the energy per bit $E_b = NE_c$. For band limited Gaussian noise $\sigma^2 = N_0T_c$. Hence, we also arrive at the well known result [39, p.250]

$$P_e = Q(\sqrt{2E_b / N_0}). \quad (3-71)$$

In the previous section it was shown that in general, the LLR, $\ell(r)$, will differ for the same symbol when the same noise sample vector is added to either a DC or a PN system. As stated previously, the difference is due to the different weights assigned to the same noise components by each type of system. It is therefore possible for one noise vector to cause an error in one type of system and not in the other. We illustrate this graphically in Figure 3.4-3. Although typically the number of chips in a PN sequence is $N \gg 1$, graphically, this is seen in a two-dimensional abstraction as well. For the DC system, a '1' is represented by the dotted circle located at $(+b, +b)$ and a '0' by a blank circle located at $(-b, -b)$. In the PN system a '1' is represented by the dotted square located at $(+b, -b)$, and a '0' by a blank square located at $(-b, +b)$. Note that an error results for the DC system transmitting a '1' when the noise vector is found in the region defined by the \square -wedge. That same noise vector would also result

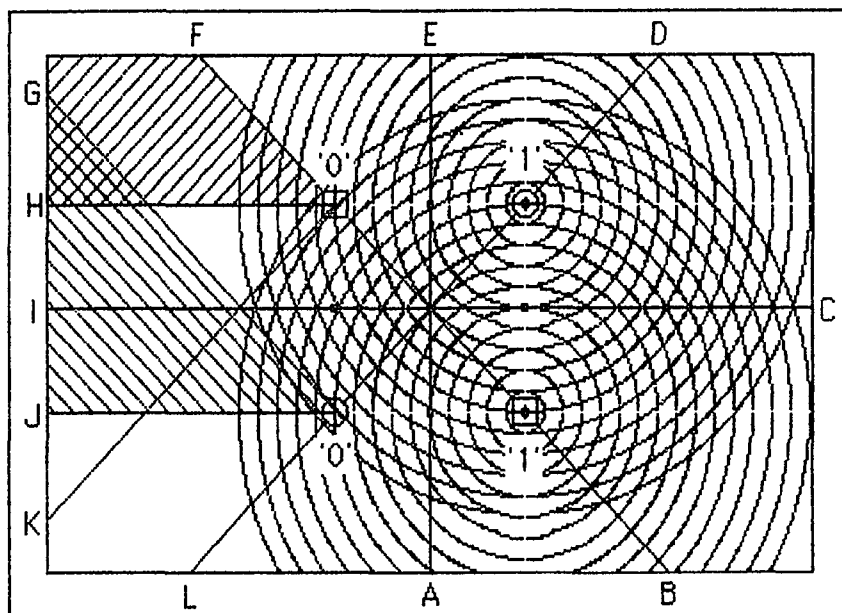


Figure 3.4-3. A 2-D Abstraction of Error Regions in DC and PN Systems.

in a PN system error since the noise vector would be found in the region defined by the GOJ-wedge. If, on the other hand, the noise vector in the DC (PN) system transmitting a '1' is found in the vicinity of $(-b,0)$, it would cause an error in the DC (PN) system but not in the PN (DC) system.

To reconcile the fact that it is possible for one noise vector to cause an error in one type of system and not in the other with the fact that both types of systems have identical performance, we state the following:

COROLLARY: For each noise vector which causes an error in a DC system and not in the PN system, there exists an equiprobable noise vector which causes an error in the PN system and not in the DC system.

Proof. Consider the noise vector consisting of a set of discrete GRVs. (A similar argument can be made using continuous GRVs, replacing discrete probabilities by the probability density function). Assume that there exists a noise vector sample which causes an error with probability P_{A_i} in the PN system. This same noise vector sample, however, does not cause an error in the DC system. Similarly there exists a noise vector sample which causes an error with probability P_{B_i} in the DC system and not in the PN system. In addition, there are noise vector samples which cause errors in both types of systems with probability P_{C_i} . Since the overall performance is independent of which system is utilized, the total probability of error in each system must be equal. Thus,

$$P_e = \sum (P_{A_i} + P_{C_i}) = \sum (P_{B_i} + P_{C_i}). \quad (3-72)$$

Therefore,

$$\sum P_{A_i} = \sum P_{B_i}. \quad (3-73)$$

Since individual P_{A_i} and P_{B_i} must uniquely be consistent with the same Gaussian probability distribution function, (3-72) can be satisfied if and only if P_{A_i} and P_{B_i} can be arranged in a one-to-one mapping as ordered pairs (P_{A_i}, P_{B_i}) such that $(P_{A_i} = P_{B_i})$. QED

3.5 Erasures in AWGN

Given an AWGN vector, the optimum detection requires correlation over the entire received vector to obtain a sufficient statistic with the most signal energy. This can be readily seen from (3-70). Unfortunately, the requisite integration (or summation) process removes vital information concerning the quality of the signal. In a high noise environment, it may be more costly to make a wrong decision than to postpone making a decision. The action of postponing a decision or ignoring available data in the detection and decision process defines an erasure. In this section we investigate the performance of the binary error-erasure channel (BE²C), as shown in Figure 3.5-1, subject to different decision methods. Since erasures are detected at the expense of both optimum binary correct and incorrect decisions, the optimum decision rule will depend upon the relative costs of errors and erasures, utilizing Bayes' criterion.

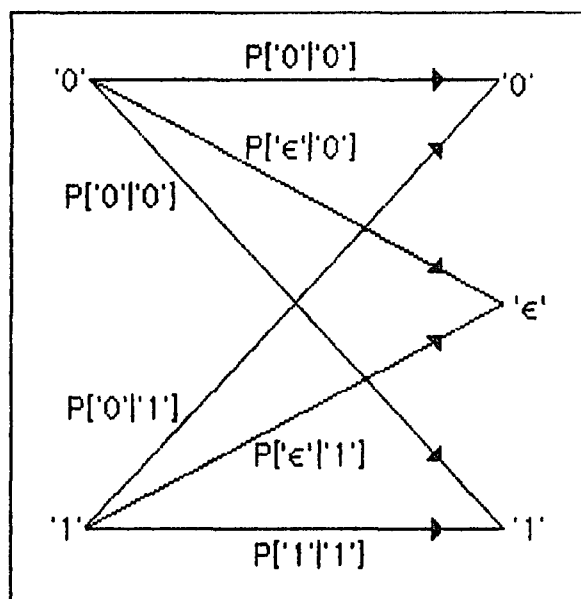


Figure 3.5-1. The Binary Error-Erasure Channel Model

There are many ways to detect erasures. While Bayes' criterion provides for the optimization of the three decision regions, the method by which we may parameterize their boundary is, to a large extent, at our disposal. In AWGN, one may categorize the methods of detecting erasures into two major classes. Erasures of class I are analog and typically are parameterized by an analog threshold. Erasures of class II are digital and are typically parameterized by the number of subdecisions used to make the final decision. A hybrid decision logic is also possible in which case both analog and digital logics are used to arrive the final decision. Typically, class I erasures are declared if the output of the correlator does not exceed a minimum erasure threshold above or below the optimum binary detection threshold, depending on which symbol is being detected. Bayes' rule may then be used to determine the optimum erasure threshold [49]. Such threshold-dependent schemes, however, are not robust since their performance is critically sensitive to small perturbations in the threshold calibration. Alternative schemes which produce class II erasures should be considered. If we segment the received signal vector into two or more subvectors which then undergo a separate correlation detection, also known as a hard decision, class II erasures may be declared depending upon the decision logic which utilizes the outcome of the optimum binary decision as well as the suboptimal sub-binary decisions and their weights. The number of subvectors considered will be limited, depending upon the feasibility of added correlation circuitry. As an illustration, consider segmenting the signal space into two subspaces. Since N is odd for a PN sequence, one subspace has $\frac{1}{2}(N+1)$ dimensions and the other has $\frac{1}{2}(N-1)$ dimensions. For large N , this difference becomes insignificant. A 2-D abstraction of such a subdivision is shown in Figure 3.5-2.

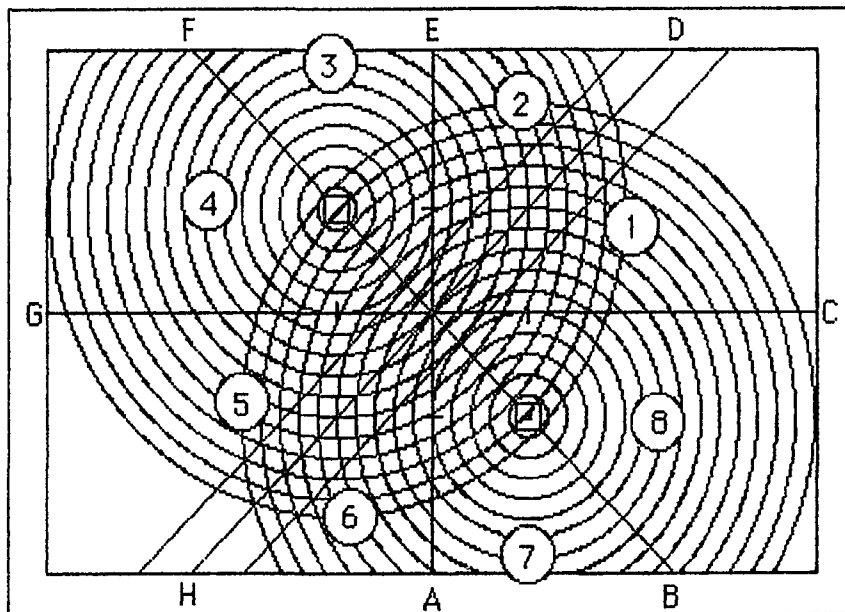


Figure 3.5-2. A 2-D Abstraction of Errors and Erasure Regions in PN Systems

If we assume that a '1' was transmitted, using the optimum binary decision rule, any \underline{r} detected to the "right" of the multi-dimensional optimum binary threshold represented by the DH-line is correctly received as a '1'. Otherwise, it is incorrectly received as a '0'. Class I erasure detection schemes provide for a multi-dimensional optimum erasure threshold represented by a line parallel to the DH-line either to the right or to the left of the DH-line depending upon whether a '1' or a '0' was transmitted. Class I erasures therefore, occur for any \underline{r} detected in the multi-dimensional region depicted between the DH-line and the parallel line representing the multi-dimensional optimum erasure threshold. Class II erasure detection schemes provide for an optimum multi-dimensional erasure region depicted by octants 1, 2, 5 and 6. Any \underline{r} falling into the region depicted by octant 7 and 8 is correctly received as a '1' and any \underline{r} detected in the region depicted by octant 3 and 4 is incorrectly received as a '0'. Since PN systems are N-dimensional, where typically $N \gg 1$, one

may optimize the erasure decision using the metric defined as the number of independent dimensions D which may disagree with the overall optimum binary decision. By definition $D \neq N$. Such a complete disagreement is impossible. This is easily illustrated by considering Figure 3.5-2. Moreover, when $D = 0$, the BE²C is reduced to the BEC. Therefore,

$$1 \leq D \leq N-1. \quad (3-74)$$

We note that as D increases, less erasures are detected resulting in more binary decisions.

How performance of various decision logics is enhanced by detecting erasures as well as errors may be evaluated in the context of the binary error-erasure symmetric channel (BE²SC). Let P_e denote the probability of error, P_ϵ denote the probability of erasure, and P_o denote the probability of correct reception. Referring to Figure 3.5-1 and assuming that the channel is symmetric, we have

$$P_e = P[1|0] = P[0|1], \quad (3-75)$$

$$P_\epsilon = P[1|\epsilon] = P[0|\epsilon] \quad (3-76)$$

and

$$P_o = P[1|1] = P[0|0]. \quad (3-77)$$

Since these disjoint events span the entire probability space, no other

independent event is possible so we must have

$$P_e + P_\epsilon + P_c = 1. \quad (3-78)$$

3.5.1 Analog signal detection. For a PN sequence of length N , let ϵ denote the analog erasure threshold factor. Let $\ell_N(\underline{r})$ represent the sample sufficient statistic obtained by correlating the received vector \underline{r} with the replica of \underline{s}_1 . Therefore, assuming H_1 , $E[\ell_N(\underline{r})] = a_{1N} = a_N$, and assuming H_0 , $E[\ell_N(\underline{r})] = a_{0N} = -a_N$. Irrespective of either hypotheses, recall that σ_N denotes the standard deviation of $\ell_N(\underline{r})$. To parameterize the erasure decision region, define ϵ as a small positive constant given by $0 < \epsilon \leq 1$. In the simplest case there exist eight possible outcomes in a transmission experiment. These are delineated as follows:

When

$$\text{transmitting a '1' and } 0 < \ell_N(\underline{r}) < \epsilon a_{1N}, \quad (3-79)$$

and when

$$\text{transmitting a '0' and } 0 > \ell_N(\underline{r}) > \epsilon a_{0N}, \quad (3-80)$$

the otherwise optimum binary correct decision is converted into an ' ϵ '.

When

$$\text{transmitting a '1' and } \ell_N(\underline{r}) > \epsilon a_{1N}, \quad (3-81)$$

the correct decision is in favor of a '1' .

When

$$\text{transmitting a '0' and } \ell_N(\Gamma) < \epsilon_{\text{BON}}, \quad (3-82)$$

the correct decision is in favor of a '0' .

When

$$\text{transmitting a '0' and } 0 < \ell_N(\Gamma) < \epsilon_{\text{BIN}}, \quad (3-83)$$

and when

$$\text{transmitting a '1' and } 0 > \ell_N(\Gamma) > \epsilon_{\text{BON}}, \quad (3-84)$$

the otherwise optimum binary error is converted into an 'ε' .

When

$$\text{transmitting a '0' and } \ell_N(\Gamma) > \epsilon_{\text{BIN}}, \quad (3-85)$$

as in the BSC case, an error is made in deciding a '1' , and when

$$\text{transmitting a '1' and } \ell_N(\Gamma) < \epsilon_{\text{BON}}, \quad (3-86)$$

also as in the BEC case, the decision is made incorrectly in favor of a '0' .

Using the above decision rule, since the LLRs are GRVs, it is straight forward to evaluate the probabilities identified in (75)-(77). Namely,

$$P_o = P[0|1] = P[\ell_N(\mathcal{L}) < \epsilon a_{0N}|1] \quad (3-87)$$

$$= Q(a_N(1+\epsilon)/\sigma_N),$$

$$P_\epsilon = P[\epsilon|1] = P[\epsilon a_{0N} \leq \ell_N(\mathcal{L}) \leq \epsilon a_{1N}|1] \quad (3-88)$$

$$= Q(a_N(1-\epsilon)/\sigma_N) - Q(a_N(1+\epsilon)/\sigma_N),$$

and

$$P_o = P[1|1] = P[\ell_N(\mathcal{L}) > \epsilon a_{1N}|1] \quad (3-89)$$

$$= 1 - P_o - P_\epsilon = 1 - Q(a_N(1-\epsilon)/\sigma_N).$$

Note that in the limit as $\epsilon \rightarrow 0$, the BE²SC is reduced to the BESC as a special case in which

$$P_o = Q(a_N/\sigma_N), \quad (3-90)$$

$$P_\epsilon = 0, \quad (3-91)$$

and

$$P_o = 1 - P_o = 1 - Q(a_N/\sigma_N). \quad (3-92)$$

3.5.2 Analog/digital hybrid signal detection. Given a PN sequence of length N which undergoes digital signal processing, we may choose to make hard decisions with respect to groups of chips which span the PN sequence and consider various decision rules which utilize the subdecisions to obtain an optimum decision for the channel under consideration. As a first example, consider subdecisions with respect to each chip in the sequence. We must obtain the sufficient statistic for individual chips, *i.e.* $\ell_{1i}(\mathcal{L})$ where $i = 1 \dots N$. In addition to the N chip LLRs, we also compute the bit LLR $\ell_N(\mathcal{L})$. Let $a_{11i} = E[\ell_{1i}(\mathcal{L})] = a_{1i}$ when a '1' is transmitted and let $a_{01i} = E[\ell_{1i}(\mathcal{L})] = -a_{1i}$ when a '0' is transmitted. The variance of $\ell_{1i}(\mathcal{L})$ denoted as σ_{1i}^2 is also computed analogously σ_N . Thus, analogously to (3-43) and (3-49), we find that

$$a_{11i} = +2b^2 (1/\sigma_1)^2 = a_{1i} , \quad (3-93)$$

$$a_{01i} = - a_{11i} = a_{1i} \quad (3-94)$$

and

$$\sigma_{1i}^2 = (2b)^2 (1/\sigma_1)^2. \quad (3-95)$$

To parameterize the erasure region, define δ as a small positive constant given by $0 < \delta \leq 1$. In addition, define N_+ and N_- as positive integers constrained by $N_+ + N_- = N$, where when correlating with a '1', N_+ is the number of $\ell_{1i}(\mathcal{L})$ in a sign agreement with $\ell_N(\mathcal{L})$, and N_- is the number of $\ell_{1i}(\mathcal{L})$ in a sign disagreement with $\ell_N(\mathcal{L})$. In the simplest case, therefore, there exist ten possible outcomes in a transmission experiment. These are delineated as follows:

When

$$\text{transmitting a '1' and } \ell_N(\mathcal{L}) > 0 \text{ and } 0 < N_+ < \delta N, \quad (3-96)$$

the otherwise optimum binary correct decision is converted into an 'e'. When

$$\text{transmitting a '1' and } \ell_N(\mathcal{L}) > 0 \text{ and } N_+ \geq \delta N, \quad (3-97)$$

the correct decision is the same correct decision which would result from the optimum binary minimum error criterion.

In contrast, when

$$\text{transmitting a '1' and } \ell_N(\mathcal{L}) < 0 \text{ and } 0 \leq N_+ \leq \delta N, \quad (3-98)$$

the otherwise optimum binary error is converted into an 'e', and when

$$\text{transmitting a '1' and } \ell_N(\mathcal{L}) > 0 \text{ and } N_+ \geq \delta N, \quad (3-99)$$

as in the $\overset{E}{BSC}$ case, an error is made in deciding a '1'. Finally, when

$$\text{transmitting a '1' and } \ell_N(\mathcal{L}) < 0 \text{ and } N_+ \geq \delta N, \quad (3-100)$$

also as in the BESC case, a '0' is decided upon incorrectly. The other five outcomes are obtained in a similar fashion when transmitting a '0'.

Using the above decision rule, the following joint probability distribution function must be evaluated to obtain its performance as a function of δ :

$$P_e = P\{\text{'0' | '1'}\} = P\{\ell_N(\Gamma) < 0, N_+ \geq \delta N \text{ | '1'}\} \quad (3-101)$$

$$= P\{\text{'1' | '0'}\} = P\{\ell_N(\Gamma) > 0, N_+ \geq \delta N \text{ | '0'}\}$$

$$P_e = P\{\text{'\epsilon' | '1'}\} = P\{\ell_N(\Gamma) < 0, N_+ \leq \delta N \text{ | '1'}\} \quad (3-102)$$

$$= P\{\ell_N(\Gamma) > 0, N_+ \leq \delta N \text{ | '1'}\}$$

$$= P\{\text{'\epsilon' | '0'}\} = P\{\ell_N(\Gamma) > 0, N_+ \leq \delta N \text{ | '0'}\}$$

$$= P\{\ell_N(\Gamma) < 0, N_+ \leq \delta N \text{ | '0'}\}$$

$$P_e = P\{\text{'1' | '1'}\} = P\{\ell_N(\Gamma) > 0, N_+ \geq \delta N \text{ | '1'}\} \quad (3-103)$$

$$= P\{\text{'0' | '0'}\} = P\{\ell_N(\Gamma) < 0, N_+ \geq \delta N \text{ | '0'}\}$$

Note that in the limit as $\delta \rightarrow \frac{1}{2}$, the BE²SC characterized by (101)-(103) is again reduced to the BESC as a special case in which (90)-(92) becomes applicable. To obtain numerical results for the above expressions, note that it is necessary to compute the integration of the joint density function of the two partially correlated LLRs, $\ell_{11}(\Gamma)$ and $\ell_N(\Gamma)$.

Optimum binary detection (OBD) and single chip decisions (SCD). Given a PN sequence of length N received in an i.i.d. AWGN environment, the errors produced in making hard decisions on individual chips of constant energy E_c are binomially distributed. The probability of exactly N^- errors in the sequence of N chips is given by

$$P[N^-|N] = C(N, N^-) P[1^-|1^-]^{N^-} (1 - P[1^-|1^-])^{N-N^-} \quad (3-104)$$

The probability of exactly N^+ correct chip decisions in the sequence of N chips is given by

$$P[N^+|N] = C(N, N^+) P[1^+|1^+]^{N^+} (1 - P[1^+|1^+])^{N-N^+} \quad (3-105)$$

where $C(N, N^\pm)$ is the binomial coefficient given by

$$C(N, N^\pm) = (1/N^\pm!) N! / (N - N^\pm)!$$

$P[1^-|1^-]$ is the optimum binary probability of a single chip error given by

$$P[1^-|1^-] = Q(a_{111}/\sigma_{11}) = Q(b/\sigma), \quad (3-106)$$

and $P[1^+|1^+]$ is the optimum binary probability of a single chip being correct given by

$$P[1^+ | 1^+] = 1 - Q(b/\sigma). \quad (3-107)$$

Note that

$$N = N^- + N^+, \quad (3-108)$$

$$N^+ = N_+ \text{ when } \ell_N(\mathcal{L}) > 0, \quad (3-109)$$

$$N^+ = N_- \text{ when } \ell_N(\mathcal{L}) < 0, \quad (3-110)$$

$$N^- = N_+ \text{ when } \ell_N(\mathcal{L}) < 0 \quad (3-111)$$

and

$$N^- = N_- \text{ when } \ell_N(\mathcal{L}) > 0. \quad (3-112)$$

Therefore,

$$P[N^\pm \geq \delta N | 1^+] = \sum_{N^\pm = \delta N}^N P[N^\pm | N] \quad (3-113)$$

and

$$P[N_\pm \geq \delta N | 1^+] = \sum_{N^\pm = \delta N}^N P[N_\pm | N]. \quad (3-114)$$

The probabilities given by (3-113)-(3-114), however, include a mix of errors and erasures as well as binary correct decisions. The only degenerate cases in which performance for the optimum binary decision based upon $\ell_N(\mathcal{L})$ will always agree with (3-113)-(3-114) occur when $N = 1$ or $N_\pm = N$, *i.e.* $\delta = 1$. The former situation

does not apply to PN sequences. In the latter case, where N is large, the correlation between $\ell_N(\underline{r})$ and N_{\pm} is expected to be reduced as N_{\pm} is reduced since there is more opportunity for the amplitudes of the samples to counter any imbalance of agreements with $\ell_N(\underline{r})$. The most conservative decision which can be made using the N subdecisions in conjunction with the optimum binary decision is to accept only unanimous decisions. Using (3-101)-(3-103) and the fact that the N subdecisions are independent of each other, we obtain the following result:

$$P_e = P[\ell_N(\underline{r}) < 0, N_+ = N] \quad (3-115)$$

$$= P[N^- = N | '1'] = \prod_{i=1}^N P[1^- | '1'] = \prod_{i=1}^N Q(b/\sigma_i)$$

$$P_o = P[\ell_N(\underline{r}) > 0, N_+ = N] \quad (3-116)$$

$$= P[N^+ = N | '1'] = \prod_{i=1}^N P[1^+ | '1'] = \prod_{i=1}^N [1 - Q(b/\sigma_i)]$$

$$P_e = 1 - P_e - P_o = 1 - \prod_{i=1}^N Q(b/\sigma_i) - \prod_{i=1}^N [1 - Q(b/\sigma_i)]. \quad (3-117)$$

To illustrate the conservative nature of this decision rule, assume that the noise is i.i.d.. The fraction of optimum binary errors discarded is given by

$$F_e = 1 - [Q(b/\sigma_i)]^N / Q(\sqrt{Nb}/\sigma_i) \quad (3-118)$$

Similarly, the fraction of optimum binary correct decisions discarded is given by

$$F_o = 1 - [1 - Q\{b/\sigma_i\}]^N / [1 - Q\{\sqrt{N}b/\sigma_i\}]. \quad (3-119)$$

In general, unfortunately, less conservative decisions, *i.e.* $\delta < 1$, cannot be computed analytically because $\ell_N(\underline{\Gamma})$ is only partially correlated to N_{\pm} , requiring the computation of multiply convolved Gaussian tail functions which must then be used to calculate the joint probability functions identified in (3-101)-(3-103).

OBD and multi-chip decisions (MCD). Given a PN sequence of length N , we may choose to subdivide it into independent groups of chips, called n -tuples, which will be used to make optimum binary subdecisions. These subdecisions may then be used in the decision rule to determine if the signal quality is good enough to make a binary decision. Otherwise, there is sufficient disagreement among the subdecisions and the optimum binary decision to justify an erasure. Let n denote the length of the n -tuples, and let N_n denote the number of independent n -tuples considered. For convenience, let $n = n$. As n increases, N_n decreases. If n does not divide evenly into N , we are left with an m -tuple such that $m < n$. We may choose to ignore the subdecision based upon the m -tuple, or appropriately weigh the m -tuple decision relative to other n -tuple decisions. In the following discussion, we choose to ignore the odd m -tuple. Therefore, we have

$$2 \leq N_n \leq N, \quad (3-120)$$

$$1 \leq n \leq \frac{1}{2}(N-1) \quad (3-121)$$

and

$$N_- + N_+ = N_n. \quad (3-122)$$

Note that N_n is the integer portion of N/n , *i.e.* $N_n = \lfloor N/n \rfloor$.

Let $P[n^- | '1']$ denote the optimum binary probability of an n -tuple error, and let $P[n^+ | '1']$ denote the optimum binary probability of an n -tuple being correct. These probabilities are given by

$$P[n^- | '1'] = Q(a_n / \sigma_n) \quad (3-123)$$

$$P[n^+ | '1'] = 1 - Q(a_n / \sigma_n). \quad (3-124)$$

Analogously to the discussion in the previous section for $n=1$, in the more general case for n given by (3-121), the most conservative decision which can be made using the N_n subdecisions in conjunction with the optimum binary decision is to accept only unanimous decisions, *i.e.* $N_+ = N_n$. Using (3-101)-(3-103) and the fact that the N subdecisions are independent of each other, we obtain the following result:

$$P_e = P[\ell_N(\Gamma) < 0, N_+ = N_n] \quad (3-125)$$

$$\cong P[N^- = N_n | '1'] = \prod_{i=1}^{N_n} P[n^- | '1'] = \prod_{i=1}^{N_n} Q(a_n / \sigma_{ni})$$

$$P_o = P[\ell_N(\Gamma) > 0, N_+ = N_n] \quad (3-126)$$

$$\cong P[N^+ = N_n | '1'] = \prod_{i=1}^{N_n} P[n^+ | '1'] = \prod_{i=1}^{N_n} [1 - Q(a_n / \sigma_{ni})]$$

$$P_\epsilon = 1 - P_e - P_o \cong 1 - \prod_{i=1}^{N_n} Q(a_n / \sigma_{ni}) - \prod_{i=1}^{N_n} [1 - Q(a_n / \sigma_{ni})]. \quad (3-127)$$

Assuming that the AWGN is i.i.d., the fraction of optimum binary errors discarded is given by

$$F_e \cong 1 - [Q(a_n / \sigma_n)]^N / Q(a_N / \sigma_N). \quad (3-128)$$

Similarly, the fraction of optimum binary correct decisions discarded is given by

$$F_o \cong 1 - [1 - Q(a_n / \sigma_n)]^N / [1 - Q(a_N / \sigma_N)]. \quad (3-129)$$

The approximate equality in the expressions above is due to the possibility that $m \neq 0$ in which case the m -tuple will contribute to a negligible decorrelation

between the $\ell_N(\Gamma)$ and $N_s = N_n$. For this general hybrid signal detection, as for the case $n=1$, less conservative decisions, *i.e.* $\delta < 1$, cannot be computed analytically because $\ell_N(\Gamma)$ is only partially correlated to N_s . This requires the computation of multiply convolved Gaussian tail functions which must then be used to calculate the joint probability functions identified in (3-101)-(3-103).

CHAPTER IV

DETECTION IN THE PRESENCE OF INTERFERENCE AND NOISE

4.1 Introduction

Interference may be distinguished from noise by its non-random properties. For example, as we have seen, the power of the noise is typically continuous in time and uniformly distributed over the band of interest. The power of interference, on the other hand, may be bursty and/or concentrated or restricted to selective spectral components or bands [50]. The on-off or high/low pulsing and spectral structuring of interference into one or more tones or narrowbands in comb-like arrangements, may cause severe degradations in performance with much less interference power than would be required if the interference were reduced to a broadband noise-like process. This may be easily verified by considering the signal space. The concentration of interference power in a few time-orthogonal or frequency-orthogonal coordinates in which the signal may be found may significantly affect the matched filter or correlator output and thereby result in a high probability of error. Such a coincidence of the interference projection into a few signal space coordinates need only occur at a rate slightly higher than the minimum bit error rate requirement in order to result in unacceptable performance. This is especially true in analog systems which integrate the contribution of each signal coordinate in an analog fashion. Such interference effects may be mitigated by digital techniques which limit the amount of received signal energy to be correlated. We see that in the presence of

high power interference the fact that digital detection techniques are suboptimal in the presence of only AWGN becomes less important.

The process of detection of digital data in the presence of interference and noise may be divided into three major parts: a) the detection of the presence of interference, b) the rejection or suppression of the interference and c) the detection of digital data based upon the residual waveform. Depending upon the interference and the source data waveform, several situations may arise. In one situation, it may be possible to detect the presence of interference but not be able to isolate and reject it without significant distortion of the source data waveform. In such a situation it may be better to erase the affected portion of the received waveform and perform the detection on the basis of any unerased segments. Such erasures may be applicable either in the time domain in the presence of a strong but short pulse interference, or in the frequency domain in the presence of strong but narrowband CW interference. In another situation, it may be possible to detect the presence of interference and to isolate and reject it without significant distortion of the source data waveform. In such a situation it may be better to perform the detection on the basis of the entire received waveform. Such rejection or suppression may also be applicable either in the time domain in the presence of a strong but short pulse interference or in the frequency domain in the presence of strong but narrowband CW interference. The case in which the interference is noise-like, *i.e.* constant in the time and frequency domains, is best handled by optimum binary detection as for AWGN, since the noise present in every signal coordinate is added non-coherently whereas the source data variables are added coherently.

In this chapter we discuss, in general, the processing gain of pre-detection signal processing algorithms designed to detect, erase and/or suppress or reject non-random interference in the presence of desired PN modulated source data waveforms. More specifically, we also describe a novel class of pre-detection signal processing algorithms \mathcal{D} , which is based upon the randomness properties of both random noise and PN waveforms. The algorithms are first demonstrated in the presence of interference only and subsequently in the presence of both interference and AWGN. The main concept underlying this class of interference suppression techniques is to test for a given randomness property. If that property is violated, then an interference burst is detected and isolated. Subsequently, the non-random content of the burst is then estimated and suppressed to yield a processing gain which will vary with the interference, noise and algorithmic parameters.

4.2 The Vector Communication System with Interference and Noise

Consider the received wideband baseband waveform $r(t)$ which is the sum of the PN-coded source data waveform $\frac{1}{2}As(t)$, the interference waveform $i(t)$, and the random noise process time sample $n(t)$ given by (1-29), (1-30) and (1-31), respectively. For a message of J bits, these waveforms may be represented as vectors in message space. Let the complete set of time-orthogonal unit-rectangular basis functions $\{p_j(t), j = 1 \dots J\}$ be defined for a message as in (1-8). Similarly, for the j^{th} bit transmission of duration T , these waveforms may be represented as vectors in signal space. Let the complete set of time-orthogonal unit-rectangular basis functions $\{q_i(t), i = 1 \dots N\}$

be defined for a bit as in (1-3). The relevant bit sample vector is given by

$$\underline{s}_j = (b_{j1}, b_{j2}, b_{j3}, \dots, b_{ji}, \dots, b_{jN}). \quad (4-1)$$

The relevant interference sample vector is given by

$$\underline{i}_j = (i_{j1}, i_{j2}, i_{j3}, \dots, i_{ji}, \dots, i_{jN}). \quad (4-2)$$

The relevant noise sample vector is given by

$$\underline{n}_j = (n_{j1}, n_{j2}, n_{j3}, \dots, n_{ji}, \dots, n_{jN}). \quad (4-3)$$

The relevant received sample vector is given by

$$\underline{r}_j = (r_{j1}, r_{j2}, r_{j3}, \dots, r_{ji}, \dots, r_{jN}), \quad (4-4)$$

where

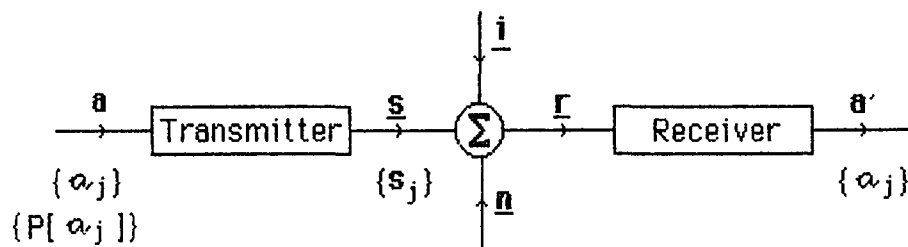
$$\underline{r}_j = \underline{s}_j + \underline{i}_j + \underline{n}_j \quad (4-5)$$

or equivalently

$$r_{ji} = b_{ji} + i_{ji} + n_{ji}. \quad (4-6)$$

The resulting vector communication system of interest is shown in Figure 4-1.

(a) Vector Representation



(b) N-Dimensional Signal Space Decomposition

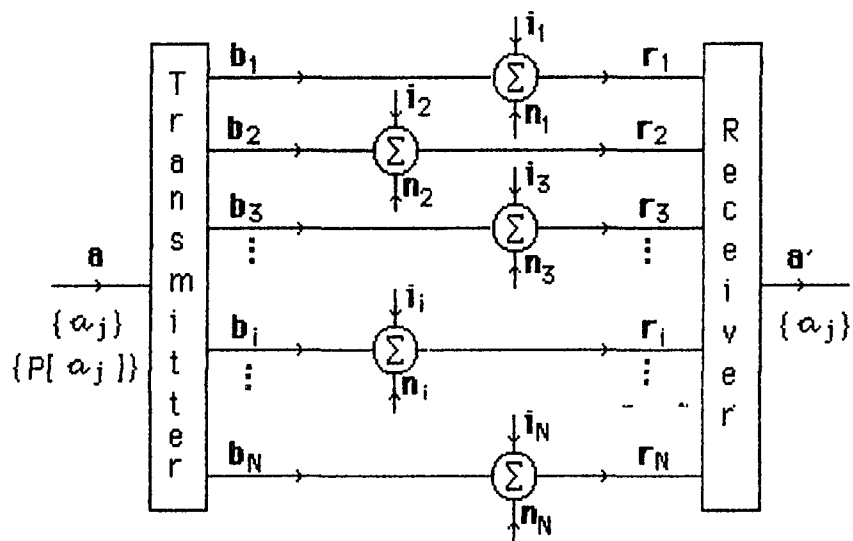


Figure 4.2-1. A Basic Vector Communication System in the Presence of Noise and Interference.

The received signal projection in the i^{th} chip interval T_c of the j^{th} bit interval T is given by

$$b_{ji} = \frac{1}{2}A \int_{-\infty}^{+\infty} s(t) p_j(t) q_i(t) dt = \frac{1}{2}A \int_{t_{j,i-1}}^{t_{j,i}} u_j b_i dt = \pm \frac{1}{2}AT_c \quad (4-7)$$

where

$$t_{j,i} = (j-1)T + iT_c \quad \text{and} \quad t_{j,i-1} = (j-1)T + (i-1)T_c. \quad (4-8)$$

In Chapter III, recall that $b_{ji} = \pm b$. Hence, $b = \frac{1}{2}AT_c$. Moreover, since $s(t) = u(t)b(t)$ where $u(t)$ is constant during each j^{th} time interval T , and assuming that $T = T_b = N T_c$, then the sample sequence $\{ b_{ji} \}$ is a complete (positive or negative) PN sequence (depending upon whether a '1' or a '0' is transmitted) which obeys the randomness properties discussed in chapter II.

The received interference sample which is relevant to the i^{th} signal coordinate of the j^{th} bit is given by the projection of the interference waveform onto the same signal coordinate given by

$$i_{ji} = \int_{-\infty}^{+\infty} i(t) p_j(t) q_i(t) dt = \int_{t_{j,i-1}}^{t_{j,i}} i(t) dt \quad (4-9)$$

Similarly, the received noise sample which is relevant to the i^{th} signal coordinate of the j^{th} bit is given by the projection of the interference waveform onto the same signal coordinate given by

$$n_{ji} = \int_{-\infty}^{+\infty} n(t) p_j(t) q_i(t) dt = \int_{t_{j,i-1}}^{t_{j,i}} n(t) dt . \quad (4-10)$$

To differentiate between the hypothesis H_0 that a '0' was transmitted and the hypothesis H_1 that a '1' was transmitted let $\underline{s}_j = \underline{s}_{j0}$ given H_0 and $\underline{s}_j = \underline{s}_{j1}$ given H_1 . Therefore, $\underline{s}_{j0} = -\underline{s}_{j1}$. Similarly, $b_{ji} = b_{j0i}$ given H_0 and $b_{ji} = b_{j1i}$ given H_1 , consequently, $b_{j0i} = -b_{j1i}$.

We may now apply the above signal space formulation to assess the impact on performance given a specific interference waveform. As an example, consider a CW interference given by (1-41) with interference-to-signal amplitude ratio of $(\alpha/A) = 30$, interference-to-signal frequency deviation given by $\Delta f_k = 1/T$ where $T = N T_c$ is the period of the PN signal consisting of $N = 127$ chips of unit amplitude. The duration of each chip is also assumed to be normalized, *i.e.* $T_c = 1$. For random noise which is i.i.d. with $\sigma = (\frac{1}{2})^{\frac{1}{2}}$, using (3-50), the expected value of the sufficient statistic $l(T)$ is simply given by the autocorrelation of the PN sequence *i.e.*,

$$a_N = E[l(T)] = [\underline{p}, \underline{p}] = 127.$$

In the presence of interference $l(T)$ is biased by an additional term given by $l_I(T) = [\underline{i}, \underline{p}]$ which is shown in Figure 4.2-1 to contribute constructively as well as destructively depending upon the interference-to-signal phase difference $\Delta\theta_k$. In the presence of interference and noise, different error-rate performance

will result depending upon $\Delta\theta_k$. For a given $\Delta\theta_k$, the interference bias is fixed and the shifted distributions of the sufficient statistics (for H_1 and H_0) are shown in Figure 4.2-2. Using (3-68) and the Q-function defined in Appendix A, we readily obtain

$$P_e = \frac{1}{2}Q\left[\frac{a_N + I(T, \Delta\theta_k)}{\sigma_N}\right] + \frac{1}{2}Q\left[\frac{a_N - I(T, \Delta\theta_k)}{\sigma_N}\right]. \quad (4-11)$$

Because of the convexity of the Q-function, for large interference,

$$P_e \approx \frac{1}{2}Q\left[\frac{a_N - I(T, \Delta\theta_k)}{\sigma_N}\right]. \quad (4-12)$$

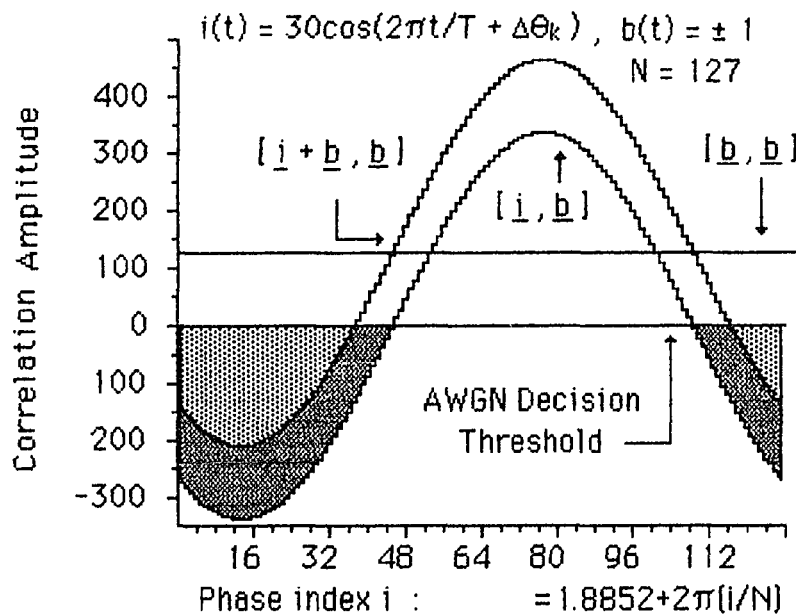


Figure 4.2-2. Example Mean Sufficient Statistic $I(T) = [i + \underline{b}, \underline{b}]$ in the Presence of Off-tone ($\Delta f_k = 1/T$) CW Interference as a Function of $\Delta\theta_k$

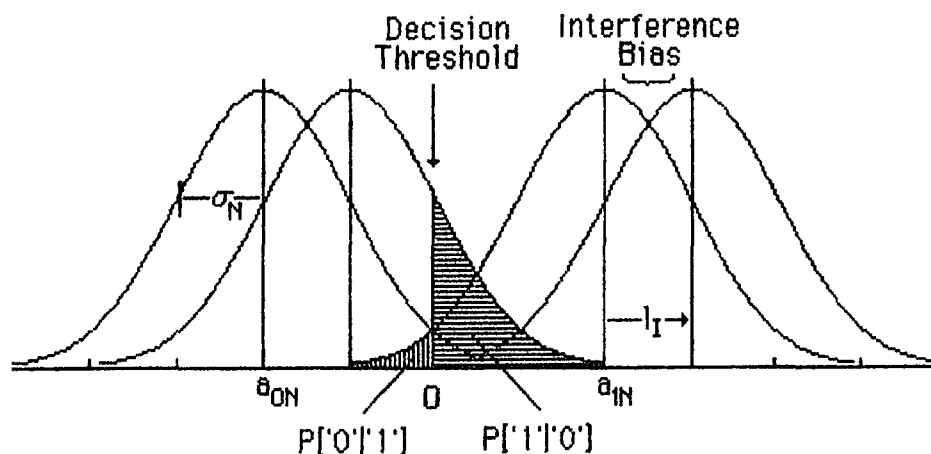


Figure 4.2-3. The Impact of Interference on the Sufficient Statistic for Binary Decisions in a Gaussian Noise Environment

4.3 The Processing Gain of Pre-detection Signal Processes

The processing gain is a figure of merit often used to compare the relative performance of different processes for a given input. It is the ratio of the signal-to-noise ratio SNR_o at the output of a given process to the signal to noise ratio SNR_i at the input of the process. In general, given a time varying input, the processing gain will also be time dependent. Typically, given a waveform $r(t)$ which includes a signal waveform $r_s(t)$, an interference waveform $r_i(t)$, and a stationary noise process sample waveform $r_n(t)$, the SNR is computed in terms of the time averaged signal power, $S[r(t)] = \langle r_s^2(t) \rangle$, the time-averaged interference power, $I[r(t)] = \langle r_i^2(t) \rangle$ and the time averaged noise power, $\eta[r(t)] = \langle r_n^2(t) \rangle$. Thus, $SNR[r(t)] = S[r(t)] / (I[r(t)] + \eta[r(t)])$. Note that this assumes that the interference and noise are uncorrelated. Generally, even if the noise and interference were uncorrelated, due to the finite time over which averaging is

performed, some correlation will result. When averaging over a finite time duration, and when the interference and noise are inherently correlated we define $I_{\eta}[r(t)] = \langle [r(t) - r_I(t)]^2 \rangle = \langle [r_I(t) + r_N(t)]^2 \rangle$. More generally, therefore, $SNR[r(t)] = S[r(t)] / (I_{\eta}[r(t)])$.

Similarly, if $\rho(t)$ is the output waveform to a process $\mathcal{D}[\]$ for which $r(t)$ was input, *i.e.* $\rho(t) = \mathcal{D}[r(t)]$, then the SNR is computed as above with $\rho(t)$ replacing $r(t)$. The processing gain of $\mathcal{D}[\]$, therefore, is given by

$$\begin{aligned} PG\{ \rho(t) = \mathcal{D}[r(t)] \} &= SNR\{\rho(t)\} / SNR\{r(t)\} \\ &= [I_{\eta}[r(t)] / I_{\eta}[\rho(t)]] [S\{\rho(t)\} / S\{r(t)\}]. \end{aligned} \quad (4-13)$$

Consider $SNR[r(t)]$, for the j^{th} bit interval T . Let

$$S = S[r(t)] = e_j[r_S(t)] / T, \quad (4-14)$$

where $e_j[r_S(t)]$ is the received signal waveform energy which also denotes the bit energy E_b . For the j^{th} bit, therefore,

$$e_j[r_S(t)] = \left(\frac{1}{2}A\right)^2 \int_{(j-1)T}^{jT} s_j^2(t) dt = \left(\frac{1}{2}A\right)^2 \sum_{i=1}^N \int_{t_{ji-1}}^{t_{ji}} (u_j b_i)^2 dt = \left(\frac{1}{2}A\right)^2 NT_0 = Nb^2 / T_0. \quad (4-15)$$

Since $T = NT_0$,

$$S = (\frac{1}{2}A)^2 . \quad (4-16)$$

As an alternative, consider the formulation of signal power in vector space. The vector product which defines the signal sample energy for the j^{th} bit is given by

$$e_j(\underline{s}) = (\underline{s} \cdot \underline{s})_j = \sum_{i=1}^N (b_{ji})^2 = Nb^2 = (\frac{1}{2}A)^2 TT_c = e_j(r_S(t))T_c = STT_c . \quad (4-17)$$

Recall that the signal contribution to the sufficient statistic is given by $l_S(T) = \pm \frac{1}{2}AT$. For the j^{th} bit, it is the output of the correlator at time $t = jT$. If this output is maintained for $jT \leq t < (j+1)T$, then $l_S(T)$ is a constant signal for the duration of the $j+1^{\text{st}}$ bit. Its energy, therefore, is given by

$$e_j(l_S) = \int_{jT}^{(j+1)T} l_S^2(T) dt = l_S^2(T)T = \left(\int_{(j-1)T}^{jT} v_S(t) dt \right)^2 T = (\frac{1}{2}A)^2 T^3 = Ne_j(\underline{s}) T = S T^3 . \quad (4-18)$$

Thus, by measuring the signal sample energy we also obtain the average signal power and the contribution of the signal to the sufficient statistic.

Averaging the noise over the j^{th} bit interval T , let

$$\eta(r(t)) = e_j(r_N(t))/T, \quad (4-19)$$

where $e_j(r_N(t))$ is the received noise sample waveform energy. For the j^{th} the bit, therefore,

$$e_j(r_N(t)) = \int_{(j-1)T}^j n_j(t) dt = \sum_{i=1}^N \int_{t_{ji-1}}^{t_{ji}} (n_{j,i}(t))^2 dt. \quad (4-20)$$

The expected value of $e_j(r_N(t))$ is given by

$$E[e_j(r_N(t))] = E\left[\int_{(j-1)T}^j n_j^2(t) dt\right] = E\left[\sum_{i=1}^N \int_{t_{ji-1}}^{t_{ji}} (n_{j,i}(t))^2 dt\right] = \sum_{i=1}^N \int_{t_{ji-1}}^{t_{ji}} E[(n_{j,i}(t))^2] dt. \quad (4-21)$$

For unlimited AWGN, $E[(n_{j,i}(t))^2] = \frac{1}{2}N_0\delta(t-t')$. Therefore,

$$E[e_j(r_N(t))] = \sum_{i=1}^N \frac{1}{2}N_0 = \frac{1}{2}N_0 N. \quad (4-22)$$

For band limited AWGN, however, $E[(n_{j,i}(t))^2] = R_n(0) = N_0 / T$. Therefore,

$$E[e_j(r_N(t))] = \sum_{i=1}^N N_0 / N = N_0. \quad (4-23)$$

Therefore, the expected value of the noise power is given by

$$E[\eta(r(t))] = E[e_j(r_N(t))]/T = \frac{1}{2}N_0 / T_0 \quad \text{for AWGN.} \quad (4-24a)$$

and

$$E[\eta(r(t))] = E[e_j(r_N(t))]/T = N_0 / T \quad \text{for LPF[AWGN].} \quad (4-24b)$$

In the absence of interference, for AWGN

$$E[\text{SNR}(r(t))] = S / E[\eta(r(t))] = S T_c / (\frac{1}{2} N_0) = 2e_j(r_S(t)) / (N N_0). \quad (4-25a)$$

Similarly, for band limited AWGN

$$E[\text{SNR}(r(t))] = S / E[\eta(r(t))] = S T / N_0 = e_j(r_S(t)) / N_0. \quad (4-25b)$$

As an alternative to the waveform SNR, consider the formulation of noise power in vector space. The vector product which defines the noise sample energy for the j^{th} bit is given by

$$e_j(\mathbf{n}) = (\mathbf{n} \cdot \mathbf{n})_j = \sum_{i=1}^N (n_{ji})^2 = R_{jN_0}. \quad (4-26)$$

For unlimited AWGN, $E[(n_{j,i}(t))^2] = \frac{1}{2} N_0 \delta(t-t')$. Using (2-69)

$$E[e_j(\mathbf{n})] = \frac{1}{2} N_0 T. \quad (4-27)$$

For band limited AWGN, $E[(n_{j,i}(t))^2] = R_n(0) = N_0 / T$. Using (2-69) again

$$E[e_j(\mathbf{n})] = N_0 T_c. \quad (4-28)$$

Recall that the noise contribution to the sufficient statistic is given by $l_N(T)$ as defined by (1-38). For the j^{th} bit, it is the output of the correlator at time $t = jT$. If this output is maintained for $jT \leq t < (j+1)T$, then $l_N(T)$ is a constant signal for the duration of the $(j+1)^{\text{st}}$ bit. Its energy, therefore, is given by

$$e_j(l_N) = \int_{jT}^{(j+1)T} l_N^2(T) dt = l_N^2(T)T = \left(\int_{(j-1)T}^{jT} v_N(t) dt \right)^2 = \left(\int_{(j-1)T}^{jT} n(t)b(t) dt \right)^2. \quad (4-29)$$

The expected value of $e_j(l_N)$ is given by

$$E[e_j(l_N)] = TE \left[\left(\sum_{i=1}^N b_i \int_{t_{ji-1}}^{t_{ji}} n_{j,i}(t) dt \right)^2 \right] = TE \left[\left(\sum_{i=1}^N b_i n_{j,i} \right)^2 \right] = TE \left[\left(\sum_{i=1}^N n'_{j,i} \right)^2 \right]. \quad (4-30)$$

Assuming that $n_{j,i}$ is a zero mean Gaussian with variance σ_i^2 and $b = \pm 1$, $n'_{j,i}$ is also a zero mean Gaussian with the same variance. Since $n'_{j,i}$ is independent of $n'_{j,k}$ for any $i \neq k$,

$$E[e_j(l_N)] = T \sum_{i=1}^N E[(n'_{j,i})^2] = TE[R'_{N,0}] = T \sum_{i=1}^N \sigma_i^2. \quad (4-31)$$

Assuming that $n'_{j,i}$ are i.i.d. with variance $\sigma^2 = N_0 T_c$,

$$E[e_j(l_N)] = N_0 T^2. \quad (4-32)$$

Then in the absence of interference we have at the output of the correlator

$$E[\text{SNR}(I_N)] = e_j(I_S) / E[e_j(I_N)] = S T / N_b = e_j(r_S(t)) / N_b. \quad (4-33)$$

Thus, by measuring the noise sample energy, we also obtain the average noise power and the contribution of the noise to the sufficient statistic.

Finally, consider averaging the interference over the j^{th} bit interval T .

Thus

$$I(r(t)) = e_j(r_I(t)) / T, \quad (4-34)$$

where $e_j(r_I(t))$ is the received interference waveform energy. For the j^{th} the bit, therefore,

$$e_j(r_I(t)) = \int_{(j-1)T}^{jT} i_j^2(t) dt = \sum_{i=1}^N \int_{t_{ji-1}}^{t_{ji}} i_{ji}^2(t) dt. \quad (4-35)$$

In signal space, the vector product which defines the interference sample energy for the j^{th} bit is given by

$$e_j(\underline{I}) = (\underline{I} \cdot \underline{I})_j = \sum_{i=1}^N (i_{ji})^2 = \sum_{i=1}^N \left(\int_{t_{ji-1}}^{t_{ji}} i_{ji}(t) dt \right)^2. \quad (4-36)$$

Note that unlike the signal and noise case, there is no general relationship between $e_j(r_I(t))$ and $e_j(\underline{I})$. Depending upon the interference waveform, $e_j(r_I(t))$ may be larger or smaller than $e_j(\underline{I})$. Note also that it is possible for $e_j(\underline{I}) = 0$

and $\underline{i} \neq 0$, whereas $e_j[r_I(t)] = 0$ only when there is no interference, *i.e.* $\underline{i} = 0$.

Recall that the interference contribution to the sufficient statistic is given by $l_I(T)$. For the j^{th} bit, it is the output of the correlator at time $t = jT$. If this output is maintained for $jT \leq t < (j+1)T$, then $l_I(T)$ is a constant interference waveform for the duration of the $j+1^{\text{st}}$ bit. Its energy is given by

$$e_j\{l_I\} = \int_{jT}^{(j+1)T} l_I^2(T) dt = l_I^2(T)T = \left(\int_{(j-1)T}^{jT} v_I(t) dt \right)^2. \quad (4-37)$$

But $v_I(t) = i(t)b(t)$, therefore,

$$e_j\{l_I\} = \left(\int_{(j-1)T}^{jT} i(t)b(t) dt \right)^2 = \left(\sum_{i=1}^N b_i i_{ji} \right)^2. \quad (4-38)$$

Given the three types of measures of signal energy identified above, we may formulate three types of sample SNRs and PGs which may be of interest when processing the j^{th} bit. The waveform sample SNR of $r(t)$ is given by

$$\text{SNR}_{j1}[r(t)] = e_j[r_S(t)] / e_j[r(t) - r_S(t)]. \quad (4-39)$$

The vector sample SNR of \underline{r} is given by

$$\text{SNR}_{j2}[\underline{r}] = e_j[\underline{s}] / e_j[\underline{r} - \underline{s}]. \quad (4-40)$$

The statistic sample SNR of r is given by

$$\text{SNR}_{j3}(r) = e_j(r_s) / e_j(r-r_s). \quad (4-41)$$

The waveform sample PG of $\rho(t) = \mathcal{D}[r(t)]$ was previously defined in (4-13). To distinguish it from other PGs let $\text{PG}_{j1}[\rho(t) = \mathcal{D}[r(t)]] = \text{PG}[\rho(t) = \mathcal{D}[r(t)]]$. It may also be rewritten as

$$\begin{aligned} \text{PG}_{j1}[\rho(t) = \mathcal{D}[r(t)]] &= \text{SNR}_{j1}(\rho(t)) / \text{SNR}_{j1}(r(t)) \\ &= [e_j[\mathcal{D}(s(t))]/e_j(s)][e_j(r(t)-s(t)) / e_j[\mathcal{D}(r(t)-s(t))]]. \end{aligned} \quad (4-42)$$

Analogously, the vector sample PG is given by

$$\begin{aligned} \text{PG}_{j2}(\rho = \mathcal{D}[r]) &= \text{SNR}_{j2}(\rho) / \text{SNR}_{j2}(r) \\ &= [e_j[\mathcal{D}(s)]/e_j(s)][e_j(r-s) / e_j[\mathcal{D}(r-s)]]. \end{aligned} \quad (4-43)$$

Similarly, the statistic sample PG is given by

$$\begin{aligned} \text{PG}_{j3}(\rho = \mathcal{D}[r]) &= \text{SNR}_{j3}(\rho) / \text{SNR}_{j3}(r) \\ &= [e_j[\mathcal{D}(s)]/e_j(s)][e_j(r-s) / e_j[\mathcal{D}(r-s)]]. \end{aligned} \quad (4-44)$$

The overall sample PG of a spread-spectrum system is most correctly obtained using PG_{β} . This is evident from the functional form of the SNRs, *i.e.* the functional form of the SNR prior to the PN correlator is the only one which corresponds to that of the SNR at the output of the PN correlator. If we insert a PDSP, to assess the overall PG we should calculate $PG_{\beta}[1(T) = \mathcal{C}[r]]$, where $\mathcal{C}[\]$ is the pre-detection correlation process. As an example, consider a basic PN DS BPSK communications, *i.e.* without $\mathcal{D}[\]$ as shown in Figure 1.4-1, in the presence of a single tone CW interference which is on-tune ($\Delta\omega_k = 0$) and in phase ($\Delta\phi_k = 0$) with respect to the signal carrier. The sample PG is given by

$$\begin{aligned} PG_{\beta}[1 = \mathcal{C}[r]] &= SNR_{\beta}(1) / SNR_{\beta}(r) \\ &= [e_j[\mathcal{C}[s]]/e_j[s]][e_j[i]/e_j[\mathcal{C}[i]]]. \end{aligned} \quad (4-45)$$

Since the PN correlator is assumed not to distort the signal, $e_j[\mathcal{C}[s]] = N^2 e_j[s]$. For this case of interference $i(t) = \frac{1}{2}\alpha$, therefore, $e_j[i] = T(\frac{1}{2}\alpha T)^2$. Moreover, since $e_j[\mathcal{C}[i]] = e_j[i_1]$ and from (1-54), $i_1 = (\frac{1}{2}\alpha T/N)$, we have $e_j[\mathcal{C}[i]] = T(\frac{1}{2}\alpha T/N)^2$. Therefore, $PG_{\beta}[1 = \mathcal{C}[r]] = N^4$. The fact that this result shows a processing gain which is a factor of N^2 higher than in (1-54) is easily explained by recalling that when the signal is spread at the transmitter we have a processing loss of N^2 . Thus, the overall processing gain is only N^2 .

4.4 Interference Suppression Using the Randomness Properties

The spreading of the interference energy by the PN mixer occurs independently of the interference waveform structure. If we could reduce (suppress or reject) the interference prior to the PN mixing, the PN mixer will simply spread whatever reduced interference energy is received by it. In the following sections we explore the processing gain of a class of pre-detection signal processes (PDSPs) which are placed prior to the PN mixer. This class of PDSPs is distinguished by invoking the noise and PN invariant properties discussed in Chapter II. The figure of merit used to evaluate the performance of such PDSPs is the sample vector PG, $\gamma = PG_{jT}[\rho = \mathcal{D}[L]]$. γ is the j th outcome of the random variable $\boldsymbol{\gamma}$ which is sampled periodically every T units of time. $\boldsymbol{\gamma} = \boldsymbol{\gamma}$ is random not only because of the random noise but also because the interference is likely to vary from symbol to symbol. It is suggested that it would be meaningless to calculate the overall PG of a spread spectrum system without first considering the density function of PG of which $\boldsymbol{\gamma}$ is a factor. The next level of processing gain, derived from the error-erasure detection correction processing, may then be computed or bounded utilizing the processing gain at pre-detection.

Consider the PDSP \mathcal{D} inserted at baseband immediately preceding the PN mixer as shown in Figure 1.4-2. Assuming that a '1' is transmitted, the input sample signal energy is given by

$$e_b = \sum_{i=1}^N b_i^2 \quad (4-46)$$

The input sample noise and interference combined energy is given by

$$e_{in} = \sum_{i=1}^N (r_i - b_i)^2 . \quad (4-47)$$

The input sample vector $SNR(\underline{r})$, therefore, may be written as

$$\xi_i = e_b / e_{in} . \quad (4-48)$$

At the output of \mathcal{D} , the output sample signal energy is given by

$$\mathcal{D}[e_b] = \sum_{i=1}^N (\mathcal{D}[b_i])^2 . \quad (4-49)$$

Let $\beta_i = \mathcal{D}[b_i]$ and $\rho_i = \mathcal{D}[r_i]$. The output sample interference and noise energy is given by

$$\mathcal{D}[e_{in}] = \sum_{i=1}^N (\rho_i - \beta_i)^2 \quad (4-50)$$

or alternatively, we could measure

$$\mathcal{D}'[e_{in}] = \sum_{i=1}^N (\mathcal{D}[r_i - b_i])^2 . \quad (4-51)$$

This alternative is stated only as a possibility which will not be used herein. In general $\mathcal{D}[e_{in}] \neq \mathcal{D}'[e_{in}]$. Since $\mathcal{D}[e_{in}]$ is most analogous to e_{in} , it is used to measure the residual interference-noise energy at the output of $\mathcal{D}[\]$. The

output sample vector SNR(ρ) may be written as

$$\xi_o = \mathcal{D}[e_b] / \mathcal{D}[e_n]. \quad (4-52)$$

From the above it follows that the sample vector processing gain for the PDSP \mathcal{D} is given by

$$\gamma = \xi_o / \xi_i = \delta e_n / \mathcal{D}[e_n] \quad (4-53)$$

where

$$\delta = \mathcal{D}[e_b] / e_b \quad (4-54)$$

defines the signal distortion factor.

4.4.1 The randomness invariant erasure algorithm. A strong non-random additive interference is expected to destroy the randomness introduced into the received vector by the AWGN and the PN coded signal. A randomness invariant erasure algorithm is defined as a signal process which tests for consistency of a given sample N-dimensional vector with the randomness properties. If any section of the vector exceeds a given threshold of a randomness test, that section is erased (zeroed) and thereby excluded from contributing to the PN correlation. Any one or combination of randomness properties may be invoked in a test for a given bit. One may test for excessive imbalance of the polarity or zero crossings. One may test for large deviations from the two-valued nature of the autocorrelation function. Finally, one may test for consistency with the run properties. When interference is known to be bursty, it may extend for a short

time T_1 relative to the bit interval T , *i.e.* $T_1 < T$. In cases where sufficient processing gain remains for the rest of the bit interval $T - T_1$, it may be more efficient to detect, locate and simply erase or ignore the received vector components located within T_1 . Of the many different tests for randomness which may be devised, the polarity, zero crossing and run lengths are found to be most useful. In particular, recall that in an N -dimensional PN sequence, the longest run expected is given by $n = \log_2(N+1)$. The probability that a run of length $k > n+1$ will occur in random noise was shown to be given simply by $P[l = k] = (\frac{1}{2})^k$. For $k > 7$, $P[l = 8] \approx 4 \times 10^{-3}$ and $P[l = 9] \approx 1 \times 10^{-6}$. Hence for a sequence with $N = 127$, as a rough estimate the signal-to-noise ratio used to calculate the optimum binary decision will be reduced by $8/127$ or approximately 6% once every 256 bits and by $20/127$ or 16% once every million bits. If, due to AWGN only, two runs occur with $k > 7$, they will occur with probabilities less than $(\frac{1}{2})^{15}$. When a run of length $k > 7$ occurs due to AWGN only, the burst detected is a false alarm. As may be ascertained from the numerical examples such a false alarm may have a negligible effect upon the bit error rate especially at low signal-to-noise ratio where the convexity of the error function changes negligibly given 5-10% loss in processing gain. Therefore, in this simple algorithm any sample which is associated with a run l_i of length $k \geq l = n+1$ will be first detected and then erased.

Consider as an example a bursty (on-off keyed) CW tone interference with carrier frequency at or near the signal carrier frequency, *i.e.* $\Delta \omega_k \leq 2\pi n/T$. The burst detection algorithm will detect bursts of duration $T_1 > nT_c$. If the

interference remains on for most or all of T , then the entire received vector will be erased unless an appropriate step is taken to inhibit erasures as required. The interference may not be detected when the frequency offset $\Delta\omega_k > 2\pi n/T$, since the interference may then force all runs to be shorter than n .

When calculating the sample processing gain, the received variable elements which are erased must be excluded in the summation of the output. If we group all received sample elements which belong to runs of length 1 or greater into one subset $B = \{r_k \mid r_k \in l_i \geq 1, k = 1 \dots N, i = 1 \dots N\}$, then the subset $\text{NOT } B = \{r_j \mid r_j \in l_i < 1, j = 1 \dots N, i = 1 \dots N\}$ includes only sample elements which belong to runs shorter than 1. Note that if $\text{NOT } B = \phi$, then ξ_o is indeterminate. Otherwise, since, $\rho_j = \mathcal{D}[r_j] = r_j$ and $\rho_k = \mathcal{D}[r_k] = 0$

$$\xi_o = \left[\sum_{i \neq k}^N (b_i)^2 \right] / \left[\sum_{i \neq k}^N (r_i - b_i)^2 \right]. \quad (4-55)$$

4.4.2 The randomness invariant average algorithm. When the interference is continuous and sinusoidal consisting of either a single tone or narrowband waveform whose highest frequency is offset from the source data signal carrier by $\Delta\omega_k < 2\pi n/T$, then the relatively high level of DC which results at baseband may be detected by the burst detection algorithm as described in the previous section. Recall that for PN sequences of maximal length N , the randomness properties are preserved by averaging over the entire sequence. As a first example let $\Delta\omega_k = 0$. The interference in this case is manifested as a DC term given by $i(t) = \frac{1}{2}\alpha \cos \Delta\theta_k$, where $\Delta\theta_k$ is the relative phase between the source

data carrier and the interference carrier. Assuming that $\frac{1}{2}\alpha\cos\Delta\theta_k \gg \frac{1}{2}A$, where A is the received signal amplitude, the interference may be detected over the entire bit interval T . Rather than declaring an erasure, as with the previous algorithm, it is possible to remove (reject) the entire DC bias of \underline{r} . Since most of the DC bias inherent in \underline{r} is due to the interference, a significant processing gain may result. The DC bias removed from every sample within this totally bursty symbol is given by

$$r_B = (1/N) \sum_{i=1}^N r_i. \quad (4-56)$$

Therefore, $\rho_i = r_i - r_B$. In general, every vector \underline{r} may be decomposed into two parts: a DC bias constant vector given by \underline{r}_B and a zero-mean vector \underline{r}_A . Thus $\underline{r} = \underline{r}_A + \underline{r}_B$. Similarly, we may decompose the signal vector $\underline{s} = \underline{d} = \underline{d}_A + \underline{d}_B$, the interference vector $\underline{i} = \underline{i}_A + \underline{i}_B$ and the noise vector $\underline{n} = \underline{n}_A + \underline{n}_B$. Note that $\underline{r}_B = \underline{d}_B + \underline{i}_B + \underline{n}_B$ and $\underline{r}_A = \underline{d}_A + \underline{i}_A + \underline{n}_A$. In the absence of noise, or equivalently when the noise is negligible, *i.e.* $\underline{n} \approx 0$, and when $i(t) = \frac{1}{2}\alpha\cos\Delta\theta_k$, then $\underline{i} = \underline{i}_B$ where the elements of \underline{i}_B , $i_{iB} = i_B = (T_c)\frac{1}{2}\alpha\cos\Delta\theta_k$ for $i = 1 \dots N$. If we assume that $\underline{d} \approx \underline{d}_A = \underline{d}$, then $\delta \approx 1$ and $\rho_i \approx \beta_i$. Therefore, $\mathcal{D}[e_{in}] \approx 0$ and $\gamma \approx \infty$.

More generally, when $1/T < \Delta f_k < n/T$, more than one burst is most likely to result. In this case we average independently over each detected burst. Averaging over short bursts of lengths $k > n$ may also result in significant processing gain achievable within the burst since there are many more short runs in a PN sequence than longer runs. Recall that in a PN sequence of maximal

length N and degree n , there is only one run of length n and one run of length $n-1$ whereas there are $\frac{1}{4}N$ runs of length one and $\frac{1}{8}N$ runs of length 2. By removing the average or DC bias within a burst we not only remove the DC bias corresponding to $i(t)$ but also to $s(t)$. The increase in the processing gain due to the removal of the DC bias of the interference, however, should more than compensate for the signal distortion expressed by δ .

To assess the possible losses due to short term averaging, *i.e.* within a burst, we introduce the notion of a burst processing gain γ_B . Assume that a burst has been detected beginning with chip $i = j$ and ending with chip $i = k > j$, where $k-j \geq 1$. Analogously to the sample vector processing gain, the processing gain for this burst is given by

$$\gamma_B = \delta_B \left[\sum_{i=j}^{k>j} (r_i - b_i)^2 \right] / \left[\sum_{i=j}^{k>j} (\rho_i - \beta_i)^2 \right] \quad (4-57)$$

where

$$\rho_i = r_i - (k-j+1)^{-1} \sum_{m=j}^{k>j} r_m = r_i - r_{iB} \quad (4-58)$$

and

$$\beta_i = b_i - (k-j+1)^{-1} \sum_{m=j}^{k>j} b_m = b_i - b_{iB} \quad (4-59)$$

and the signal distortion factor relevant to the burst is given by

$$\delta_B = \left[\sum_{i=j}^{k>j} (\beta_i)^2 \right] / \left[\sum_{i=j}^{k>j} (b_i)^2 \right]. \quad (4-60)$$

Note that within a burst, the averaging process is linear. If we define the zero-sample mean interference within the burst as

$$\iota_i = i_i - (k-j+1)^{-1} \sum_{m=j}^{k>j} i_m = i_i - i_{iB}, \quad (4-61)$$

and the zero-sample mean random noise within the burst as

$$\nu_i = n_i - (k-j+1)^{-1} \sum_{m=j}^{k>j} n_m = n_i - n_{iB}, \quad (4-62)$$

then, within the burst only, $\rho_i = \beta_i + \iota_i + \nu_i$ and $r_{iB} = b_{iB} + i_{iB} + n_{iB}$. Since

$$\rho_i - \beta_i = \iota_i + \nu_i \quad \text{and} \quad r_i - b_i = \iota_i + i_{iB} + n_i, \quad (4-63)$$

we obtain

$$\gamma_B = \delta_B \left[\sum_{i=j}^{k>j} (\iota_i + i_{iB} + n_i)^2 \right] / \left[\sum_{i=j}^{k>j} (\iota_i + \nu_i)^2 \right]. \quad (4-64)$$

If we can neglect the noise relative to the interference, then

$$\gamma_B = \delta_B \left[\sum_{i=j}^{k>j} (\iota_i + i_{iB})^2 \right] / \left[\sum_{i=j}^{k>j} (\iota_i)^2 \right]. \quad (4-65)$$

Since $\sum_{i=j}^{k>j} (\iota_i) = 0$,

$$\gamma_B = \delta_B \left[(k-j+1)(i_{iB})^2 + \sum_{i=j}^{k>j} (\iota_i)^2 \right] / \left[\sum_{i=j}^{k>j} (\iota_i)^2 \right]. \quad (4-66)$$

Note that corresponding to an on-tune, single tone CW interferer, *i.e.* when $k-j+1 = N$ and $\epsilon_i = 0$ for all $i = 1 \dots N$, the overall processing gain of the spread spectrum system results in $\gamma = \gamma_B = \infty$ for an arbitrary phase $\Delta\theta_k$. In practical situations, however, γ is finite due to the presence of finite noise. Nevertheless, we may conclude that with this algorithm the processing gain increases as the per bit DC bias in the received baseband interference waveform is increased.

4.4.3 The randomness invariant piece-wise average algorithm. In the previous section we have seen how short term averaging affects the processing gain. The burst processing gain is greater than unity provided the DC bias of the interference in the burst is higher than the DC bias of the PN sequence in the burst. This result may be easily generalized to smaller sections into which one may choose to partition the burst. Using the same burst detection algorithm, the burst is subdivided into equal or almost equal sections of length $l_s \approx l$ where l is the minimum burst threshold.

To assess the possible losses due to short term averaging, *i.e.* within a burst section, we introduce the notion of a burst section processing gain γ_L . Let us assume that a burst has been detected and subdivided into L sections where $1 \leq L \leq (k-j+1)/l$. Consider the section beginning with chip $i = j$ and ending with chip $i = k > j$, where $k-j \geq l$. In a manner analogous to the sample burst processing gain, we conclude that the processing gain for this burst section is given by

$$\gamma_L = \delta_L \left[\sum_{i=j}^{k>j} (\rho_i - \beta_i)^2 \right] / \left[\sum_{i=j}^{k>j} (\rho_i - \beta_i)^2 \right] \quad (4-70)$$

where

$$\rho_i = \rho_i - (k-j+1)^{-1} \sum_{m=j}^{k>j} \rho_m = \rho_i - \rho_{iL}, \quad (4-71)$$

and

$$\beta_i = \beta_i - (k-j+1)^{-1} \sum_{m=j}^{k>j} \beta_m = \beta_i - \beta_{iL}, \quad (4-72)$$

and the signal distortion factor relevant to the burst section is given by

$$\delta_L = \left[\sum_{i=j}^{k>j} (\beta_i)^2 \right] / \left[\sum_{i=j}^{k>j} (b_i)^2 \right]. \quad (4-73)$$

Note that now within a burst section the averaging process is linear. If we define the zero-sample mean interference within the burst section as

$$i_i = i_i - (k-j+1)^{-1} \sum_{m=j}^{k>j} i_m = i_i - i_{iL}, \quad (4-74)$$

and the zero-sample mean random noise within the burst section as

$$v_i = n_i - (k-j+1)^{-1} \sum_{m=j}^{k>j} n_m = n_i - n_{iL}, \quad (4-75)$$

then, within the burst section only, $\rho_i = \beta_i + i_i + v_i$ and $\rho_{iL} = \beta_{iL} + i_{iL} + n_{iL}$. Since

$$p_i - \beta_i = \iota_i + \nu_i \quad \text{and} \quad r_i - b_i = \iota_i + i_{iL} + n_i, \quad (4-76)$$

we obtain

$$\gamma_L = \delta_L \left[\sum_{i=j}^{k>j} (\iota_i + i_{iL} + n_i)^2 \right] / \left[\sum_{i=j}^{k>j} (\iota_i + \nu_i)^2 \right]. \quad (4-77)$$

If we can neglect the noise relative to the interference, then

$$\gamma_L = \delta_L \left[\sum_{i=j}^{k>j} (\iota_i + i_{iL})^2 \right] / \left[\sum_{i=j}^{k>j} (\iota_i)^2 \right]. \quad (4-78)$$

Since $\sum_{i=j}^{k>j} (\iota_i) = 0$,

$$\gamma_L = \delta_L \left[(k-j+1)(i_{iL})^2 + \sum_{i=j}^{k>j} (\iota_i)^2 \right] / \left[\sum_{i=j}^{k>j} (\iota_i)^2 \right]. \quad (4-79)$$

Similarly to burst averaging, burst section averaging will result in desirable processing gain, provided the DC content of the interference exceeds that of the PN waveform within the section. Overall this algorithm is expected to outperform the burst averaging algorithm since it provides a better approximation of the interference in the vicinity of the samples being averaged. This becomes more important as the frequency of the interference drifts away from the center frequency of the signal carrier. The greater distortion factor associated with this algorithm due to the shorter term averaging, when compared with the burst averaging algorithm, is mitigated by the fact that approximately half the runs in the PN vector are of lengths 1 and 2.

4.4.4 The randomness invariant piece-wise linear average correction algorithm.

Much of the need which may exist with the previous correction algorithms to adapt the burst averaging interval depending upon the interference, may be removed by interpolating the approximate shape of the interference waveform and removing the piece-wise smoothed contributions which are insensitive to the PN fluctuations. The previous correction algorithms were based upon the general zero order decomposition of any vector into two parts: a DC bias and a zero-mean vector or subvector. In general, however, any vector or subvector may be decomposed into the sum of a DC bias vector, a linear bias vector and a residual zero-mean vector. In low noise environments, quadratic and higher order decompositions may also be possible. Due to the zero crossing randomness, both the DC bias and the linear part of both random noise and PN-coded vectors are expected to be negligible over most burst sections. This algorithm, therefore, is based upon the general first order decomposition of the received vector carried out on a burst section by burst section. Again, as before, runs which exceed the minimum burst threshold l are declared to be bursts. The burst is then subdivided into sections for each of which a local average is obtained.

To assess the possible losses due to short term averaging, *i.e.* within a burst section, we use the burst section processing gain γ_L given by (4-57). Consider two adjacent burst sections L and M. Let r_{iL} and r_{iM} be the local averages associated with section L and M respectively. The slope obtained between these burst section averages is simply $\Delta r_{LM} = (r_{iM} - r_{iL})/l$, where l is the length of the burst section used. With section L starting at chip j in the PN

sequence, the DC bias is r_{jL} which is the linearly interpolated magnitude at the j^{th} chip. The output magnitude is therefore given by

$$\rho_i = r_i - \Delta_{LM}^r(i-j+1) - r_{jL} \quad (4-80)$$

and

$$\beta_i = b_i - \Delta_{LM}^b(i-j+1) - b_{jL} \quad (4-81)$$

where $\Delta_{LM}^b = (b_{iM} - b_{iL})/l$, and the signal distortion factor relevant to the burst section is given by (4-73).

Note that now within a burst section the averaging process is linear. If we define the zero-mean sample interference within the burst section as

$$\iota_i = i_i - \Delta_{LM}^i(i-j+1) - i_{jL}, \quad (4-82)$$

and the zero-mean sample random noise within the burst section as

$$\nu_i = n_i - \Delta_{LM}^n(i-j+1) - n_{jL}, \quad (4-83)$$

then, within the burst section only, $\rho_i = \beta_i + \iota_i + \nu_i$ and $r_{jL} = b_{jL} + i_{jL} + n_{jL}$ and $\Delta_{LM}^r = \Delta_{LM}^b + \Delta_{LM}^i + \Delta_{LM}^n$. Since

$$\rho_i - \beta_i = \iota_i + \nu_i \quad \text{and} \quad r_i - b_i = \iota_i + i_{jL} + \Delta_{LM}^i(i-j+1) + n_i, \quad (4-84)$$

we obtain

$$\gamma_L = \delta_L \left[\sum_{i=j}^{k>j} (\tau_i + i_{jL} + \Delta_{LM}^{i-j+1} + n_i)^2 \right] / \left[\sum_{i=j}^{k>j} (\tau_i + v_i)^2 \right]. \quad (4-85)$$

If we can neglect the noise relative to the interference, then

$$\gamma_L = \delta_L \left[\sum_{i=j}^{k>j} (\tau_i + i_{jL} + \Delta_{LM}^{i-j+1})^2 \right] / \left[\sum_{i=j}^{k>j} (\tau_i)^2 \right], \quad (4-86)$$

or

$$\gamma_L = \delta_L \left[I_L + \sum_{i=j}^{k>j} (\tau_i)^2 \right] / \left[\sum_{i=j}^{k>j} (\tau_i)^2 \right], \quad (4-87)$$

where

$$I_L \approx \sum_{i=j}^{k>j} (i_{jL} + \Delta_{LM}^{i-j+1})^2. \quad (4-88)$$

Similarly to burst section averaging, this algorithm will result in desirable processing gain provided the combined DC and linear content of the interference magnitude exceeds that of the PN waveform within the section. Overall, this algorithm is expected to outperform the burst section averaging algorithm since it provides a better approximation of the interference in the vicinity of the samples being processed. This becomes more important as the frequency of the interference drifts away from the center frequency of the signal carrier. The greater distortion factor associated with this algorithm due to the shorter term averaging when compared with the burst section averaging algorithm is mitigated by the fact that approximately half the zero crossings in the PN vector are of lengths 1 and 2.

4.5 Simulated PDSPs Performance Results

To test the performance of the PDSPs discussed, we undertook to simulate the algorithms and compare their absolute as well as relative performance given a specific PN coded signal $b(t)$, a specific interference wave form $i(t)$ and pseudorandomly generated Gaussian noise $n(t)$. Many more experiments may be conducted than can possibly be reported herein. In this section, therefore, we shall only provide illustrative results. More general analytical derivation of the results is impractical due to the inherently non-linear nature of the algorithms. For the results reported in this section, $b(t)$ is a bipolar PN waveform with amplitude $b = \pm 1$. The length of the bit interval is normalized to N chips where N is a complete period of the PN sequence generated by the 7-stage FBSR shown in Figure 2.4-3. The general form of the interference investigated is given by

$$i(t) = (\frac{1}{2}\alpha)\cos(2\pi f_a t)\cos[2\pi\Delta f_k t + \beta\sin(2\pi f_m t) + \Delta\theta_k].$$

This general waveform for the interference allows the investigation of AM, FM and AM/PM types of interferences where f_a is an AM modulation frequency, f_m is an FM modulation frequency, and β is the FM modulation index. Δf_k and $\Delta\theta_k$ are the frequency and phase difference between the interference CW tone and the signal carrier given by (1-46). For the specific results reported herein, we assume that f_a, β and f_m are all zero, *i.e.* we have a single interfering tone. The characteristic parameters measured for any sample vector are provided in Table 4-1 for the sample PN vector, off-tone ($\Delta f_k = 1/T, \Delta\theta_k = 1/T$) interference

sample and Gaussian noise sample ($\sigma = 10$). E_b is given by (4-46), E_{r-b} is given by (4-47), ξ is given by (4-48), δ is given by (4-54) and γ is given by (4-53). C_r is the vector correlation of the given vector with \underline{b} . μ and σ are the sample

Parameter	\underline{b}	\underline{i}	\underline{n}	$\underline{b} + \underline{i}$	$\underline{b} + \underline{n}$	$\underline{b} + \underline{i} + \underline{n}$
E_b	127	-	-	127	127	127
E_{r-b}	-	6449	13784	6449	13784	19067
ξ	-	-	-	.020	.0092	.0067
δ	-	-	-	1	1	1
γ	-	-	-	1	1	1
C_r	127	93	-172	220	-45	48
μ	0	0	-1	0	-1	-1
σ	1	7	10	7	10	12
N^+/N^-	64/63	65/62	55/72	67/60	57/70	67/60
C^+/C^-	32/32	1/1	28/29	3/3	31/32	20/20
B^+/B^-	32/32	1/1	29/29	3/3	32/32	20/20
'1'	16/16	0/0	14/10	0/2	19/14	10/8
'2'	8/8	0/0	7/6	1/0	5/6	2/5
'3'	4/4	0/0	5/6	1/0	5/7	5/2
'4'	2/2	0/0	3/4	0/0	2/3	1/1
'5'	1/1	0/0	0/2	0/0	1/1	0/1
'6'	0/1	0/0	0/1	0/0	0/1	0/1
'7'	1/0	0/0	0/0	0/0	0/0	0/0
'8'	0/0	0/0	0/0	0/0	0/0	0/0
'9'	0/0	0/0	0/0	0/0	0/0	0/1
'10'	0/0	0/0	0/0	0/0	1/0	1/0

Table 4-1. Characteristic Parameters of Example PN Vector \underline{b} , Off-tone CW Interference Vector \underline{i} , and Gaussian Noise Vector \underline{n} and their Linear Combinations

mean and standard deviation. The total number of positive /negative samples is given by N^+ / N^- , respectively. The total number of positive/negative zero-crossings is given by C^+ / C^- , respectively. The total number of positive /negative runs or bursts is given by B^+ / B^- , respectively. Similarly, the total numbers of positive /negative runs of length 1-10 are also shown in rows '1' - '10'.

A comparison of the four PDSPs described in Section 4.4 is provided in Table 4-2. PDSP F1 corresponds to the randomness invariant erasure algorithm; PDSP F2 corresponds to the randomness invariant average algorithm; PDSP F3 corresponds to the randomness invariant piece-wise average algorithm; and finally, PDSP F4 corresponds to the randomness invariant piece-wise linear-average algorithm. We compared these four algorithms as a function of the standard deviation of the noise σ using an off-tone CW interference with $\alpha/A = 30$, $\Delta f_k = 1/T$, and $\Delta \theta_k = 1.8852$ rads. As can be readily seen, PDSP F4 outperforms the other PDSPs at low Gaussian noise environments. As the power of the random noise increases, the performance advantage decreases. As expected, PDSP F1 does poorly even at low noise levels since the interference is not bursty.

Having determined the superiority of PDSP F4 to combat this type of interference, we continued to investigate its performance in the absence of random noise, or equivalently in a high interference-to-noise power ratio environment.

σ	$\gamma_{\mathcal{D}}$ (dB)	$\mathcal{E}\{\underline{b} + \underline{i} + \underline{n}, \underline{b}\} / \mathcal{E}\{\mathcal{D}[\underline{b} + \underline{i} + \underline{n}, \underline{b}]\}$	γ_e
1	13.7	-151/ 1	5
2	2.5	-129/ 9	5
3	.5	-125/ 3	4
4	1.8	-90/-53	1
5	2.1	-164/-23	4
6	.34	-48/-26	1
7	3.1	-167/-48	3
8	-2.7	-182/ -67	3
9	.46	-309/ -65	5
10	-0.1	-310/-36	7

Table 4-2a. The Sample PG of PDSPs F1 and the Impact upon the Sufficient Statistic as a Function of the Noise Standard Deviation σ

σ	$\gamma_{\mathcal{D}}$ (dB)	$\mathcal{E}\{\underline{b} + \underline{i} + \underline{n}, \underline{b}\} / \mathcal{E}\{\mathcal{D}[\underline{b} + \underline{i} + \underline{n}, \underline{b}]\}$	γ_e
1	8.0	-151/32	9
2	8.2	-129/36	8
3	7.7	-125/61	15
4	---	---	---
5	8.2	-164/52	15
6	7.6	-48/104	58
7	8.6	-167/66	23
8	6.6	-182/-77	2
9	6.5	-309/-32	8
10	6.2	-310/-87	4

Table 4-2b. The Sample PG of PDSPs F2 and the Impact upon the Sufficient Statistic as a Function of the Noise Standard Deviation σ

σ	$\gamma_{\mathcal{D}}(\text{dB})$	$\mathcal{E}(\underline{b} + \underline{i} + \underline{n}, \underline{b}) / \mathcal{E}(\mathcal{D}[\underline{b} + \underline{i} + \underline{n}, \underline{b}])$	γ_e
1	15.7	-151/37	10
2	16.5	-129/82	32
3	14.7	-125/65	17
4	11.3	-90/89	33
5	10.4	-164/13	7
6	10.4	-48/68	9
7	9.8	-167/26	8
8	7.3	-182/-38	4
9	7.1	-309/-90	4
10	6.1	-310/-100	7

Table 4-2c. The Sample PG of PDSPs F3 and the Impact upon the Sufficient Statistic as a Function of the Noise Standard Deviation σ

σ	$\gamma_{\mathcal{D}}(\text{dB})$	$\mathcal{E}(\underline{b} + \underline{i} + \underline{n}, \underline{b}) / \mathcal{E}(\mathcal{D}[\underline{b} + \underline{i} + \underline{n}, \underline{b}])$	γ_e
0	23.9	-137/88	46
1	21.3	-151/67	21
2	19.1	-129/95	64
3	16.5	-125/92	52
4	13.4	-90/107	118
5	11.4	-164/44	12
6	11.5	-48/118	378
7	10.6	-167/21	8
8	7.6	-182/-23	4
9	7.6	-309/-70	5
10	6.3	-310/-103	4

Table 4-2d. The Sample PG of PDSPs F4 and the Impact upon the Sufficient Statistic as a Function of the Noise Standard Deviation σ

Using the same off-tone interference waveform, we first varied α/A while keeping $\Delta f_k = 1/T$ and $\Delta \theta_k = 1.8852$ rads. These results are summarized in Table 4-3. Note that the processing gain of the PDSP denoted by $\gamma_{\mathcal{D}}$ saturates

rather quickly whereas the processing gain of the whole PN system denoted by

$$\gamma_{\mathcal{C}} = \{(\mathcal{C}[\underline{b} + \underline{i} + \underline{n}, \underline{b}] - \mathcal{C}[\underline{b}, \underline{b}]) / (\mathcal{C}[\mathcal{D}[\underline{b} + \underline{i} + \underline{n}], \underline{b}] - \mathcal{C}[\mathcal{D}[\underline{b}], \underline{b}])\}^2$$

continues to vary.

α/b	$\gamma_{\mathcal{D}}$ (dB)	$\mathcal{C}[\underline{b} + \underline{1}, \underline{b}] / \mathcal{C}[\mathcal{D}[\underline{b} + \underline{1}], \underline{b}]$	$\gamma_{\mathcal{C}}$
5	19.1	83/109	6
10	19.3	39/94	7
15	19.2	-5/185	5
20	23.0	-49/89	21
25	23.0	-93/83	25
30	23.9	-137/88	46
35	23.9	-182/85	54
40	23.9	-226/81	58
45	23.9	-270/78	66
50	23.9	-314/74	69
55	23.9	-358/70	72
60	23.9	-402/67	76
65	23.9	-446/63	80
70	23.9	-490/60	85
75	23.9	-534/56	87
80	23.9	-578/53	91
85	23.9	-622/49	92
90	23.9	-666/46	95
95	23.9	-910/42	149
100	23.9	-754/39	100
105	23.9	-799/35	102
115	23.9	-887/28	104
120	23.9	-931/24	106
125	23.9	-975/21	108
130	23.9	-1019/17	109
135	23.9	-1063/14	111
140	23.9	-1107/10	111
145	23.9	-1151/7	113
150	23.9	-1195/3	114

Table 4-3. The Sample PG of PDSP F4 and the Impact upon the Sufficient Statistic as a Function of the Amplitude Ratio α/b

Then we varied $\Delta\theta_k$ while keeping $\alpha/A = 30$ and $\Delta f_k = 1/T$. These results are summarized in Table 4-4.

Phase index i	$\gamma_{\mathcal{D}}$ (dB)	$\mathcal{C}\{\underline{b} + \underline{1}, \underline{b}\} / \mathcal{C}\{\mathcal{D}[\underline{b} + \underline{1}, \underline{b}]\}$	γ_e
0	23.9	-137/88	46
1	23.0	-145/79	32
2	23.4	-151/72	26
3	23.7	-158/82	40
4	22.8	-164/90	62
5	21.1	-170/85	50
6	20.3	-175/68	26
7	20.4	-180/68	27
8	19.5	-185/51	17
9	30.4	-189/101	148
10	29.5	-193/99	131
11	29.7	-196/101	154
12	29.7	-199/102	170
13	28.4	-202/112	481
14	29.2	-204/113	559
15	29.4	-206/135	1732
16	29.8	-208/125	28056
17	29.6	-209/130	12544
18	29.4	-209/133	3136
19	29.2	-210/132	4543
20	28.3	-209/128	112896
21	29.2	-209/99	144
22	29.5	-208/100	154
23	27.0	-206/91	86
24	27.7	-205/103	191
25	27.8	-202/105	224

Table 4-4. The Sample PG of PDSP F4 and the Impact upon the Sufficient Statistic as a Function of the Relative Phase Index $i = N(\Delta\theta_k - 1.8852)/2\pi$ ($i = 1$ corresponds to $\Delta\theta_k = 2^\circ$)

Finally, we vary Δf_k while keeping $\alpha/A = 30$ and $\Delta\theta_k = 1.8852$ rads. These results are summarized in Table 4-5.

Δf_k	$\gamma_{\mathcal{D}}$ (dB)	$\mathcal{C}[\underline{b} + \underline{L} \underline{b}] / \mathcal{C}[\mathcal{D}[\underline{b} + \underline{L} \underline{b}]]$	γ_e
.004	25.5	224/135	147
.008	23.9	-137/88	46
.012	18.1	154/175	0.3
.016	11.0	-124/106	143
.020	11.2	34/118	107
.024	5.9	-235/77	52
.028	6.4	233/101	17
.032	7.4	-137/7	5
.036	7.2	-51/245	2
.040	7.2	410/141	409
.044	7.1	225/141	49
.048	6.8	-234/-40	5
.052	6.0	63/245	0.3
.056	5.6	384/57	13
.060	1.1	439/411	1
.064	0	18/18	1
.068	0	228/228	1
.072	0	20/20	1
.076	0	-170/-170	1
.080	0	325/325	1

Table 4-5. The Sample PG of PDSP F4 and the Impact upon the Sufficient Statistic as a Function of the Relative Frequency Δf_k

4.6 Performance Degradation in AWGN due to PDSP

The probability of occurrence of a run of length k was derived in (2-111) to be $P[1_i = \pm k] = (\frac{1}{2})^k$. Using (2-116), the expected number of runs of length k in a sequence of N chips is given by $E[m_{\pm k}] = N (\frac{1}{2})^k$. Note that for $k > n = \log_2 N$, $E[m_{\pm k}] < 1$. Hence, for $k > n$, $E[m_{\pm k}]$ is also the probability of finding a single run of length k in a sequence of N chips, *i.e.*

$$P[k = k] = E[m_{\pm k}] = N (\frac{1}{2})^k \quad (4-89)$$

Since the probability of finding a second run of length k is independent of the probability of finding the first run of length k , the joint probability

$$P[k_1 = k_1, k_2 = k_2] = N (\frac{1}{2})^{k_1} (\frac{1}{2})^{k_2} \quad (4-90)$$

As a numerical example, let $N = 128$, $n = 7$, $k_1 = 8$ and $k_2 = 8$, then $P[k = k] = \frac{1}{2}$ and $P[k_1 = 8, k_2 = 8] = 1/512 < 5 \times 10^{-3}$. For $k_1 = 9$ and $k_2 = 8$, then $P[k = k] = \frac{1}{4}$ and $P[k_1 = 9, k_2 = 8] = 1/1024 < 1 \times 10^{-3}$. Since the burst detection algorithm looks for runs of length $k \geq l$ where l is the selected minimum burst length threshold, only runs of length k need be considered.

For an N -dimensional vector detected in AWGN only environment, the probability of error for optimum binary decisions is given by

$$P_e = P_e(N) = Q(a_N / \sigma_N). \quad (4-67)$$

If L is the number of chips erased due to the burst detection algorithm and the optimum binary decision is based upon $M = N - L$ remaining chips, then

$$P_e(M) = Q(a_M / \sigma_M). \quad (4-91)$$

Knowing the probability that L or M will occur, *i.e.* $P[L = L]$, the total probability that no burst will occur is given by

$$P[L = 0] = 1 - \sum_{L=1}^N P[L = L]. \quad (4-92)$$

Therefore, the total probability of error is given by

$$P'_e = P[L = 0] P_e + \sum_{L=k}^N P[L = L] Q(a_M / \sigma_M) \quad (4-93)$$

where $P[L = 0] = P[L < 1]$ is the probability that no runs of length $k > n$ are detected and $P[L = L]$ is the probability that any combination of one or more runs of length $k > n$ are detected and erased. Note that the summation represents performance degradation in the presence of AWGN. To estimate the degradation, note that

$$P[L = L] \approx P[K = L] = N \left(\frac{1}{2}\right)^L. \quad (4-94)$$

Let $L - n = m$, then the total probability of error with the PDSP inserted may be approximated by

$$P'_e \approx Q[a_N / \sigma_N] + \sum_{m=1}^{N-n} \left(\frac{1}{2}\right)^m Q[a_M / \sigma_M]. \quad (4-95)$$

Considering the numerical examples given above and the convexity of the Q-function, it is observed that (4-95) is a tight lower bound for P'_e . Note that this performance may be significantly improved by conducting a second test which checks for the consistency of the detected long bursts with the randomness properties. If found to be consistent, the test would inhibit erasures of such runs. The definition of an algorithm to implement such a test is suggested for future investigation.

CHAPTER V

CONCLUSIONS AND RECOMMENDATIONS

5.1 Introduction

Having investigated typical interference and noise phenomena in basic PN communication systems has lead to the definition and design of a novel class of non-parametric pre-detection signal processes which have been shown to have important interference rejection properties. The observed results and areas for future investigations are presented repectively in the following two sections.

5.2 Conclusions

The PDSPs discussed in this dissertation form a class of non-parametric interference reduction filters which are easily implementable in VLSI circuitry. Theoretically, the novel class of PDSPs described provide for an infinite processing gain in rejecting narrowband/CW interference which occurs at the carrier frequency with an arbitrary phase and arbitrary amplitude. The processing gain is reduced as the frequencies of the interferers deviate from the carrier. The PDSPs described are almost transparent to random noise and performance degradation for detection in the presence of random noise when compared to a matched filter reactor was shown to be insignificant. When compared to other non-parametric and parametric interference

rejection/suppression filters, the novel class of PDSPs described are computationally significantly less complex. Experimental simulation results show that the overall processing gain of the PN communication system is non-linearly dependent upon the processing gain of the PDSP inserted.

5.3 Recommendations

The theoretical framework, coupled with experimental simulation of the concepts and techniques described in this dissertation, provide the foundation for future investigations of similar nature. Due to the non-linear nature of the algorithms, many more experiments should be undertaken to optimize the parameters identified and decide when to invoke each algorithm. Additional enhancements may be undertaken to provide higher processing gain at higher signal-to-interference frequency deviations. Decision rules which make use of run properties other than the longest run property may prove to be useful. Additional investigations which compare the performance of this class of interference rejection filters with other filters identified in the references should also provide additional insight. The algorithms described may also be easily extended to applications in which long PN sequences are used. Finally, it is recommended that when the impact of interference rejection filters is investigated, in addition to processing gain and probability of error results, sample output signal waveforms be obtained and documented to provide additional insight into the interference rejection mechanism.

APPENDIX A

FUNCTIONS DEFINED FOR THE NORMAL CURVE

In this Appendix several definitions and relations in connection with Gaussian noise which are often referenced in the main body of this dissertation are provided for convenience. A univariate Gaussian random variable, n , is characterized by two parameters: a) its expected value, $E[n]$, known also as its mean μ and b) its variance, $\text{Var}[n]$, known also as the square of its standard deviation, σ . These parameters may be derived from the probability density function (pdf) of n , given by

$$P_n(n) = \{1/(\sqrt{2\pi}\sigma)\} \exp\{-\frac{1}{2}(n - \mu)^2/\sigma^2\} \quad -\infty < n < +\infty, \quad (\text{A-1})$$

where n is a real number sample event of n . Therefore, one can easily confirm that

$$E[n] \triangleq \int_{-\infty}^{+\infty} n P_n(n) dn = \mu \quad (\text{A-2})$$

and

$$\text{Var}[n] \triangleq E[(n - \mu)^2] = \int_{-\infty}^{+\infty} (n - \mu)^2 P_n(n) dn = \sigma^2. \quad (\text{A-3})$$

When checking for consistency of numerical results, it is useful to verify the accuracy of the relations given by

$$\sigma P_n(\mu) \approx 0.399, \quad (\text{A-4a})$$

$$\sigma P_n(\mu \pm \sigma) \approx 0.242 \quad (\text{A-4b})$$

and

$$\sigma P_n(\mu \pm 2\sigma) \approx 0.0539. \quad (\text{A-4c})$$

The probability distribution function (PDF) of n is given by

$$F_n(a) = P_n[n < a] = \int_{-\infty}^a P_n(n) \, dn. \quad (\text{A-5})$$

Similarly to (A-4), we may check for the consistency of parameters of an allegedly Gaussian random variable using the properties of (A-5) given by

$$F_n(\mu) = 0.5 \quad (\text{A-6a})$$

$$F_n(\mu - \sigma) \approx 0.159 \quad (\text{A-6b})$$

$$F_n(\mu + \sigma) \approx 0.841 \quad (\text{A-6c})$$

and

$$F_n(\mu + x) + F_n(\mu - x) = 1 \quad (\text{A-6d})$$

For α , a real number, it is often convenient to define and tabulate a universal function (Q-function), $Q(\alpha)$, given by

$$Q(\alpha) = (1/\sqrt{2\pi}) \int_{\alpha}^{+\infty} \exp\{-\frac{1}{2}\beta^2\} d\beta. \quad (\text{A-7})$$

It can be shown, therefore, that

$$F_n(a) = 1 - Q((a - \mu)/\sigma), \quad (\text{A-8})$$

and the probability that $a_1 \leq n \leq a_2$ is given by

$$P[a_1 \leq n \leq a_2] = F_n(a_2) - F_n(a_1) = Q((a_1 - \mu)/\sigma) - Q((a_2 - \mu)/\sigma). \quad (\text{A-9})$$

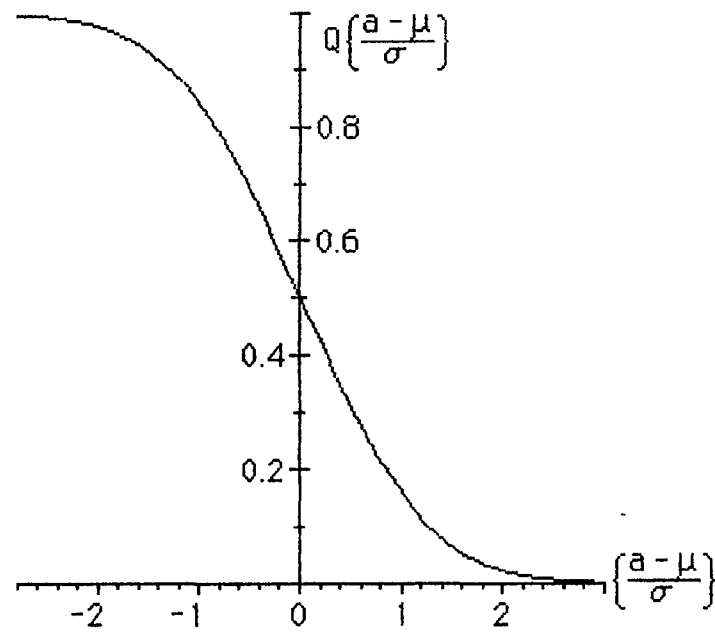
The Q-function is plotted in Figure A-1 for both small and large arguments using approximations provided by Abramowitz and Stegan [51, p. 297].

To complement $Q(a)$, two other universal functions and their complements have been defined and tabulated. Unfortunately, both of these functions have been dubbed as error functions. To avoid confusion, they are distinguished by a subscript. Many authors [52, p. 64] like to define the function

$$\text{erf}_1(\alpha) = \frac{1}{2} - \text{erfc}_1(\alpha) = (1/\sqrt{2\pi}) \int_0^{\alpha} \exp\{-\frac{1}{2}\beta^2\} d\beta. \quad (\text{A10})$$

Other authors[51, p. 297] find it convenient to define and tabulate the function

(a)



(b)

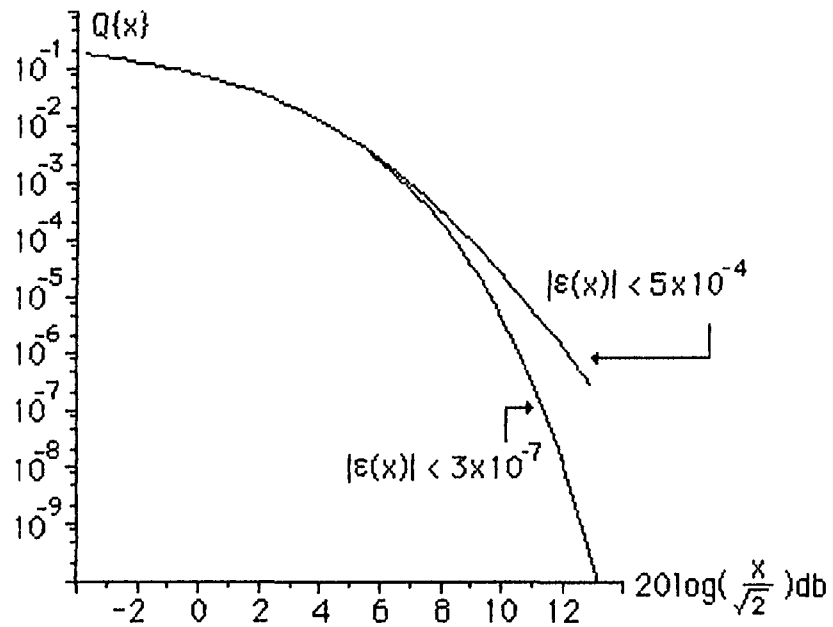


Figure A-1. The Q-Function: (a) Small Arguments; (b) Large Arguments

$$\operatorname{erf}_2(\alpha) = 1 - \operatorname{erfc}_2(\alpha) = (2/\sqrt{\pi}) \int_0^\alpha \exp\{-\beta^2\} d\beta. \quad (\text{A-11})$$

Note that

$$\operatorname{erf}_1(\infty) = \frac{1}{2}, \text{ where as } \operatorname{erf}_2(\infty) = 1. \quad (\text{A-12})$$

Using (A-5), (A-10) and (A-11), when $a \geq \mu$, we obtain

$$F_n(a) = \frac{1}{2} + \operatorname{erf}_1\{(a - \mu)/\sigma\} \quad (\text{A-13a})$$

or

$$= 1 - \operatorname{erfc}_1\{(a - \mu)/\sigma\}. \quad (\text{A-13b})$$

When $a < \mu$, we obtain

$$F_n(a) = \frac{1}{2} - \operatorname{erf}_1\{|a - \mu|/\sigma\} \quad (\text{A-13c})$$

$$= \operatorname{erfc}_1\{|a - \mu|/\sigma\}. \quad (\text{A-13d})$$

Similarly, when $a \geq \mu$, we obtain

$$F_n(a) = \frac{1}{2} + \frac{1}{2}\operatorname{erf}_2\{\sqrt{\frac{1}{2}}(a - \mu)/\sigma\}, \quad (\text{A-14a})$$

or

$$= 1 - \frac{1}{2}\operatorname{erfc}_2\{\sqrt{\frac{1}{2}}(a - \mu)/\sigma\}. \quad (\text{A-14b})$$

When $a < \mu$, we obtain

$$F_n(a) = \frac{1}{2} - \frac{1}{2}\operatorname{erf}_2\{\sqrt{\frac{1}{2}}|a - \mu|/\sigma\} \quad (\text{A-14c})$$

or

$$= \frac{1}{2}\operatorname{erfc}_2\{\sqrt{\frac{1}{2}}|a - \mu|/\sigma\}. \quad (\text{A-14d})$$

Using (A-7), (A-10), and (A-11), when $a \geq \mu$, we obtain

$$Q\left(\frac{a - \mu}{\sigma}\right) = \text{erfc}_1\left\{\frac{a - \mu}{\sigma}\right\} \quad (\text{A-15a})$$

$$= \frac{1}{2} \text{erfc}_2\left[\sqrt{\frac{1}{2}}\frac{a - \mu}{\sigma}\right]. \quad (\text{A-15b})$$

Similarly, when $a < \mu$, we obtain

$$Q\left(\frac{a - \mu}{\sigma}\right) = 1 - \text{erfc}_1\left\{\left|\frac{a - \mu}{\sigma}\right|\right\} \quad (\text{A-15c})$$

or

$$= 1 - \frac{1}{2} \text{erfc}_2\left[\sqrt{\frac{1}{2}}\left|\frac{a - \mu}{\sigma}\right|\right]. \quad (\text{A-15d})$$

The relationships among the various cumulative probabilities are shown graphically in Figure A-2.

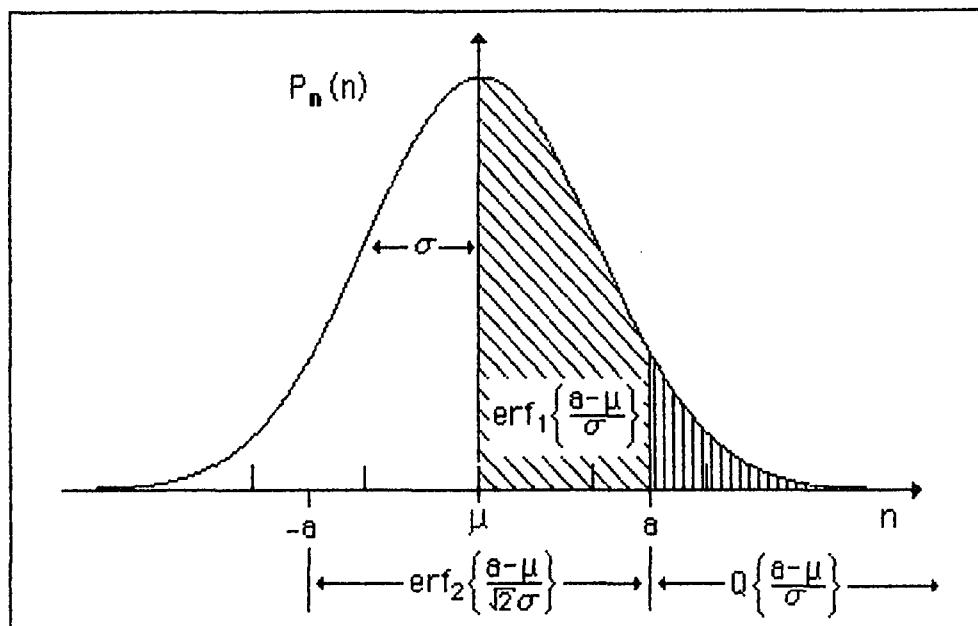


Figure A-2. The Normal Probability Density Function

APPENDIX B

RANDOMNESS INVARIANT ALGORITHMS DEMONSTRATION

COMPUTER PROGRAM LISTING

The algorithms simulating the PDSPs discussed in this dissertation are provided below in the context of a computer-aided design tool demonstration program. The program may, therefore, be used to undertake any further tests of the algorithms presented herein. In addition, it is intended to provide the framework for obtaining interactive results for various other waveforms which may be of interest and to compare and evaluate other interference suppression techniques which may be of interest as well. The listing is provided in Microsoft Basic Version 2.0 which has been coded and tested using the Apple Macintosh personal computer system.

Arrays used to generate the N-dimensional PN vector

$p(i)$ -Degree of the i th non-zero term of the irreducible primitive polynomial generating the PN vector

$f(i)$ -Binary state of the i th feedback tap of the n -stage FBSR

$s(i)$ -Binary state of the i th stage of the n -stage FBSR

`DIM p(20),f(20),s(20)`

$t(i)$ -time base,

$b(i)$ -N-dimensional PN mixed source signal vector,

$i(i)$ -Interference projection vector,

$n(i)$ -AWGN projection vector, $r(i)$ -Received vector,

$d(i)$ -Distorted (by PDSP) N-dimensional PN signal vector,

```
DIM t(128),b(128),i(128),n(128),r(128),x(128),y(128),cr(128),d(128)
```

Arrays used to accumulate run statistic

bu(i)- number of positive runs of length i

bd(i)- number of negative runs of length i

bt(i)- number of positive/negative runs of length i

```
DIM bu(10),bd(10),bt(10)
```

Clear the screen. Use n%-stage maximal length FBSR to generate PN vector of period m%

```
Start:CLS:n%=7:m%=2^n%-1
```

Set up screen plotting parameters. Screen is 512x342 pixels

```
xo=100:yo=150:xl=100:yl=40:xh=356:yh=260:lj=4:li=2:xj=8:yj=10:xi=4:yi=2
```

```
xd=(xh-xl)/xj:yd=(yh-yl)/yj:ex=1:ey=1
```

```
xb=0:xt=128:xs=(xt-xb)/xj:xg=xd/xs:x$="Time":xtab=26
```

Generate the time base in PN chip units, Tc

```
FOR i%=0 TO m%+1
```

```
  t(i%)=i%:x(i%)=t(i%)*xg
```

```
NEXT i%
```

Option to generate an N-dimensional PN mixed, binary '1' source signal vector

```
Gen.b:LOCATE 1,1:INPUT;"Generate PN signal? Enter y or n: ",ei$:GOSUB clear1
```

```
  IF ei$="n" THEN Xgen.b
```

```
  a=1:Signal$="b(t)"
```

```
  LOCATE 1,1:INPUT;"Change parameters? Enter y or n: ",ei$:GOSUB clear1
```

```
  IF ei$="n" THEN Calc.b
```

```
  LOCATE 1,1:INPUT;"Amplitude: ";a:INPUT;"Name PN Signal: ";Signal$
```

```
  GOSUB clear1:GOSUB clear2
```

```
Calc.b:RESTORE :
```

```
  LOCATE 1,1:LPRINT TAB(14);"Generating PN signal named "; Signal$
```

```
  LPRINT TAB(14);"PN signal amplitude is ";a
```

Initialize the PN generator irreducible primitive polynomial

```
DATA 0,7,6,4,1,0,0,0 ' g(x) = x7 + x6 + x4 + x + 1
```

Initialize the feedback taps and initial state of the n%-stage FBSR

```
FOR i%=0 TO n%
  READ p%(i%):f%(i%)=0:s%(i%)=1
NEXT i%
```

Generate the output unipolar PN vector

```
FOR i%=0 TO m%+1
  b(i%)=0
NEXT i%
FOR i%=1 TO n%
  IF p%(i%)<>0 THEN f%(p%(i%))=1
  IF p%(i%)=0 THEN f%(p%(i%))=0
NEXT i%
FOR i%=1 TO m%
  b(i%)=s%(1):k1%=0
  FOR k%=1 TO n%
    k1%=k1%+s%(k%)*f%(k%):IF k1%=2 THEN k1%=0
  NEXT k%
  FOR k%=1 TO n%-1
    s%(n%-k%+1)=s%(n%-k%)
  NEXT k%
  s%(1)=k1%
NEXT i%
```

Transform the PN vector from unipolar to bipolar.

Mix PN vector with '1' rectangular source signal vector
to generate an N-dimensional PN mixed, source signal vector

```

FOR i%=1 TO m%
  IF b(i%)=0 THEN b(i%)=-1
  b(i%)=a*b(i%)
NEXT i%
b(m%+1)=b(1):b(0)=b(m%+1)
Xgen.b:GOSUB clear1

```

Option to load an N-dimensional source signal vector
previously generated and stored

```

Load.b:LOCATE 1,1:INPUT;"Load signal? Enter y or n",ei$:GOSUB clear1
IF ei$="n" THEN Xload.b
LOCATE 1,1:INPUT;"Which signal file?",file$:GOSUB clear1
LPRINT TAB(14);"Loading PN signal named ";Signal$;" from file ";File$
OPEN "I",*1,file$,512
FOR i%=0 TO m%
  INPUT #1,b(i%)
NEXT i%
CLOSE #1
Xload.b:GOSUB clear1

```

Option to plot the current N-dimensional source signal vector

```

Plot.b:LOCATE 1,1:INPUT;"plot signal? Enter y or n",ei$:GOSUB clear1
IF ei$="n" THEN Xplot.b
LPRINT TAB(14);"Plotting the PN signal named ";Signal$
LOCATE 1,1:INPUT;"Clear screen? Enter y or n",ei$:GOSUB clear1
IF ei$="y" THEN CLS
yb=-10:yt=10
LOCATE 1,1:PRINT"Y max = ";yt:INPUT;"Change Y max? Enter #
(0 retains current value)",ei$:GOSUB clear1:GOSUB clear2
IF ei<> 0 THEN yt=ei
IF ei<> 0 THEN yb=-ei
ys=(yt-yb)/yj:yg=yd/ys:y$="Amplitude":ytab=4
FOR i%=1 TO m%
  y(i%)=b(i%)*yg
NEXT i%
IF ei$="y" THEN GOSUB plotxy
CALL MOVETO(xo+x(1)-1,yo)

```



```

CALL LINETO(xo+x(1)-1,yo-y(1))
FOR i%=1 TO m%-1
  CALL LINETO(xo+x(i%)+1,yo-y(i%))
  CALL LINETO(xo+x(i%)+1,yo-y(i%+1))
NEXT i%
LOCATE 1,1:INPUT;"Plot Figure # is",FigNo$:GOSUB clear1
LPRINT TAB(14);"Signal ";Signal$;" is Plotted in Figure D-";FigNo$
Xplot.b:

```

Option to store the current N-dimensional source signal vector

```

Store.b:LOCATE 1,1:INPUT;"Store signal? Enter y or n",ei$:GOSUB clear1
IF ei$="n" THEN Xstore.b
LOCATE 1,1:INPUT;"Which file?",file$:GOSUB clear1
LOCATE 1,1:PRINT"Storing the PN signal named";Signal$;" in ";File$
OPEN "o",#1,File$,512
FOR i%=0 TO m%
  WRITE#1,b(i%)
NEXT i%
CLOSE#1
Xstore.b:

```

Option to collect statistics on N-dimensional source signal vector

```

Stat.b:LOCATE 1,1:INPUT;"Signal Statistics? Enter y or n",ei$:GOSUB clear1
IF ei$="n" THEN Xstat.b
LPRINT TAB(14);"Calculating statistics for ";Signal$
CALL stat(b())
Xstat.b:

```

Option to generate or load a new N-dimensional source signal vector

```

LOCATE 1,1:INPUT;"Iterate signal generation? Enter y or n",ei$:GOSUB clear1
IF ei$="y" THEN Gen.b

```

End signal generation, loading, plotting, storing and statistic

Option to generate an N-dimensional projection of unmodulated/modulated CW/bursty interference vector.

Gen.i:

```

LOCATE 1,1:INPUT;"Generate interference? Enter y or n",ei$:GOSUB clear1
IF ei$="n" THEN Xgen.i
INPUT;"The interference name is ",Inter$
i1%=1:i2%=m%+1:ai=30:fa=0:fi=.008*:fmi=0:fm=.008*:pi=1.8852*
FOR i% = 0 TO m%:i(i%)=0:NEXT i%
LOCATE 2,1:PRINT " Burst start(1-126)";i1%;" Burst end(2-127)";i2%
PRINT"Interference amplitude ai=";ai
PRINT" frequency fi=";fi;" phase, pi=";pi
PRINT "AM frequency fa = ";fa
PRINT " FM modulation index fmi = ";fmi;" FM frequency fm = ";fm:GOSUB
clear1
LOCATE 1,1:INPUT;"Change parameters? Enter y or n",ei$:GOSUB clear1
IF ei$="n" THEN Calc.i
LOCATE 1,1:INPUT;"Burst start(1-126) i1% ";i1%:INPUT;"Burst end(2-127) i2%
";i2%:GOSUB clear1
LOCATE 1,1:INPUT;"Interference amplitude ai";ai
LOCATE 1,1:INPUT;"frequency fi";fi
LOCATE 1,1:INPUT;"phase";pi:GOSUB clear1
LOCATE 1,1:INPUT;"AM frequency fa";fa:GOSUB clear1
LOCATE 1,1:INPUT;"FM frequency fm ";fm:INPUT;"FM modulation index fmi
";fmi:GOSUB clear1

```

```

Calc.i:LOCATE 1,1:LPRINT TAB(14);"Generating interference named "; Inter$
LPRINT TAB(14);"Burst start(1-126)";i1%;" Burst end(2-127)";i2%
LPRINT TAB(14);"Interference amplitude ai=";ai;" frequency fi=";fi
LPRINT TAB(14);"phase pi=";pi;" AM frequency fa = ";fa
LPRINT TAB(14);"FM modulation index fmi = ";fmi;" FM frequency fm =
";fm:GOSUB clear1
LOCATE 1,1:PRINT"Generating interference":GOSUB clear2
p2=2*3.14159*
FOR i%=i1% TO i2%
i(i%)=ai*(COS(p2*fa*t(i%)))*COS(p2*fi*t(i%)+fmi*COS(p2*fm*t(i%))+pi)
NEXT i%

```

Xgen.i:

Option to load an N-dimensional interference vector,
previously generated and stored

```
Load.i:LOCATE 1,1:INPUT;"Load interference? Enter y or n",ei$:GOSUB clear1
```

```
  IF ei$="n" THEN Xload.i
  LOCATE 1,1:INPUT;"Which interference file?",file$:GOSUB clear1
  LOCATE 1,1:PRINT"Loading interference ";Inter$;" on file named ";file$
  OPEN "1",#1,file$,512
  FOR i%=0 TO m%
    INPUT #1,i(i%)
  NEXT i%
  CLOSE #1
```

```
Xload.i:
```

Option to plot the current N-Dimensional interference vector

```
Plot.i:LOCATE 1,1:INPUT;"Plot interference? Enter y or n",ei$:GOSUB clear1
```

```
  IF ei$="n" THEN Xplot.i
  LOCATE 1,1:LPRINT TAB(14);"Plotting the interference named ";Inter$
  LOCATE 1,1:INPUT;"Clear screen? Enter y or n",ei$:GOSUB clear1
  IF ei$="y" THEN CLS
  yb=-20:yt=20
  LOCATE 1,1:PRINT"Y max = ";yt:INPUT;"Change Y max? Enter #"
  (0 retains current value)",ei$:GOSUB clear1:GOSUB clear2
  IF ei<> 0 THEN yt=ei
  IF ei<> 0 THEN yb=-ei
  ys=(yt-yb)/yj:yg=yd/ys:y$="Amplitude":ytab=4
  FOR i%=1 TO m%
    y(i%)=i(i%)*yg
  NEXT i%
  IF ei$="y" THEN GOSUB Plotxy
  CALL MOVETO(xo+x(1),yo-y(1))
  FOR i%=1 TO m%
    CALL LINETO(xo+x(i%),yo-y(i%))
```

```

NEXT i%
LOCATE 1,1:INPUT;"Plot Figure # is",FigNo$:GOSUB clear1
LPRINT TAB(14);"Interference ";Inter$;" is Plotted in Figure D-";FigNo$
Xplot.i:

```

Option to store an N-dimensional interference vector,
previously generated or loaded

```

Store.i:LOCATE 1,1:INPUT;"Store interference? Enter y or n",ei$:GOSUB clear1
IF ei$="n" THEN Xstore.i
LOCATE 1,1:INPUT;"Which interference file?",file$:GOSUB clear1
LOCATE 1,1:PRINT"Storing interference named ";Inter$;" in file ";File$
OPEN "o",#1,File$,512
FOR i%=0 TO m%
  WRITE#1,i(i%)
NEXT i%
CLOSE#1

```

Xstore.i:

Option to collect statistics on N-dimensional interference vector

```

Stat.i:LOCATE 1,1:INPUT;"Interference Statistics? Enter y or n",ei$:GOSUB clear1
IF ei$="n" THEN Xstat.i
LOCATE 1,1:LPRINT TAB(14);"Calculating statistics for ";Inter$
CALL stat(i())

```

Xstat.i:

```

LOCATE 1,1:INPUT;"Iterate interference generation? Enter y or n",ei$:GOSUB
clear1
IF ei$="y" THEN Gen.i

```

Option to generate or load a new N-dimensional projection of the relevant
noise vector

```

Gen.n:LOCATE 1,1:INPUT;"Generate noise? Enter y or n",ei$:GOSUB clear1
Noise$="n(t)":IF ei$="n" THEN Xgen.n
n=0:sn=1:LOCATE 2,1:PRINT "Noise mean, n=";n;"Noise standard deviation";sn
LOCATE 1,1:INPUT;"Change parameters? Enter y or n",ei$:GOSUB clear1
IF ei$="n" THEN Calc.n

```

```
LOCATE 1,1:INPUT;"Mean";n:INPUT;"Standard deviation";sn:GOSUB clear1:GOSUB
clear2
INPUT;"The noise name is ";Noise$
```

```
Calc.n:LOCATE 1,1:LPRINT TAB(14);"Generating noise named ";Noise$:RANDOMIZE
TIMER
LPRINT TAB(14);"Noise mean, n=";n;" Noise standard deviation";sn
FOR i%=1 TO m%+1 STEP 2
  Gauss:uu=2*RND-1:vv=2*RND-1:ss=uu^2+vv^2
  IF ss>=1 THEN Gauss
  qq=uu*SQR(-2*LOG(ss)/ss):rr=vv*SQR(-2*LOG(ss)/ss)
  n(i%)=n+qq*sn:n(i%+1)=n+rr*sn
NEXT i%
```

Xgen.n:

Option to load an N-dimensional noise vector
previously generated and stored

```
Load.n:LOCATE 1,1:INPUT;"Load noise? Enter y or n",ei$:GOSUB clear1
IF ei$="n" THEN Xload.n
LOCATE 1,1:INPUT;"Which File?";file$:GOSUB clear1
LOCATE 1,1:PRINT"Loading file named ";file$
LPRINT TAB(14);"Loading file named ";file$;" for noise ";Noise$
OPEN "I",*1,file$,512
FOR i%=0 TO m%
  INPUT *1,n(i%)
NEXT i%
CLOSE *1
```

Xload.n:

Option to plot the current N-Dimensional noise vector

```
Plot.n:LOCATE 1,1:INPUT;"Plot noise? Enter y or n",ei$:GOSUB clear1
IF ei$="n" THEN Xplot.n
LOCATE 1,1:LPRINT TAB(14);"Plotting the noise named ";Noise$
LOCATE 1,1:INPUT;"Clear screen? Enter y or n",ei$:GOSUB clear1
IF ei$="y" THEN CLS
yb=-20:yt=20
```

```

LOCATE 1,1:PRINT"Y max = ";yt:INPUT;"Change Y max? Enter #
(O retains current value)",ei:GOSUB clear1:GOSUB clear2
IF ei<> 0 THEN yt=ei
IF ei<> 0 THEN yb=-ei
ys=(yt-yb)/yj:yg=yd/ys:y$="Amplitude":ytab=4
FOR i%=1 TO m%
    y(i%)=n(i%)*yg
NEXT i%
IF ei$="y" THEN GOSUB Plotxy
FOR i%=1 TO m%
    CALL MOVETO(xo+x(i%),yo)
    CALL LINETO(xo+x(i%),yo-y(i%))
NEXT i%
LOCATE 1,1:INPUT;"Plot Figure # is",FigNo$:GOSUB clear1
LPRINT TAB(14);"Noise ";Noise$;" is Plotted in Figure D-";FigNo$
Xplot.n:

```

Option to store the current N-dimensional noise vector

```

Store.n:LOCATE 1,1:INPUT;"Store noise? Enter y or n",ei$:GOSUB clear1
IF ei$="n" THEN Xstore.n
LOCATE 1,1:INPUT;"Which noise file?",file$:GOSUB clear1
LOCATE 1,1:PRINT"Storing Noise named ";Noise$;" on file ";File$
OPEN "o",*1,file$,512
FOR i%=0 TO m%
    WRITE#1,n(i%)
NEXT i%
CLOSE#1
Xstore.n:

```

Option to collect statistics on N-dimensional noise vector

```

Stat.n:LOCATE 1,1:INPUT;"Noise Statistics? Enter y or n",ei$:GOSUB clear1
IF ei$="n" THEN Xstat.n
LOCATE 1,1:LPRINT TAB(14);"Calculating statistics for ";Noise$
CALL stat(n())
Xstat.n:

```

```

LOCATE 1,1:INPUT;"Iterate noise generation? Enter y or n",ei$:GOSUB clear1

```

IF ei\$="y" THEN Gen.n

Restart:

Option to generate or load a new N-dimensional received vector at baseband

Option 1: Pure signal reception, free of interference and noise

Option 2: Pure interference reception, signal not transmitted

Option 3: Pure noise reception, signal not transmitted

Option 4: Signal and interference are free of noise

Option 5: Signal and Noise are received without interference

Option 6: Interference and noise, signal not transmitted

Option 7: Signal is received corrupted by interference and noise

Gen.r:LOCATE 1,1:INPUT;"Generate received vector? Enter y or n",ei\$:GOSUB clear1

IF ei\$="n" THEN Xgen.r

INPUT;"Receive 1:b, 2:i, 3:n, 4:b+i, 5:b+n, 6:i+n, 7:b+i+n? Enter 1-7",ei%:GOSUB

clear1

INPUT;"Received Signal Components Designation is",Total\$:GOSUB clear1

IF ei%<1 OR ei%>7 THEN Gen.r

Calc.r:LOCATE 1,1:PRINT "Generating the received PDSP input signal";

ei%,"";Total\$

LPRINT TAB(14);"Generating the received PDSP input signal ";ei%,"";Total\$

Newr=0

FOR i%=1 TO m%

d(i%)=b(i%)

IF ei%=1 THEN r(i%)=b(i%)

IF ei%=2 THEN r(i%)=i(i%)

IF ei%=3 THEN r(i%)=n(i%)

IF ei%=4 THEN r(i%)=b(i%)+i(i%)

IF ei%=5 THEN r(i%)=b(i%)+n(i%)

IF ei%=6 THEN r(i%)=i(i%)+n(i%)

IF ei%=7 THEN r(i%)=b(i%)+i(i%)+n(i%)

NEXT i%

Xgen.r:

Option to load an N-dimensional received PDSP input vector previously generated and stored

```
Load.r:LOCATE 1,1:INPUT;"Load received PDSP input vector?
Enter y or n",ei$:GOSUB clear1
IF ei$="n" THEN Xload.r
Newr=0
LOCATE 1,1:INPUT;"which file?";Total$:GOSUB clear1
LOCATE 1,1:PRINT"Loading file named ";Total$
OPEN "1",*1,Total$,512
FOR i%=0 TO m%
  INPUT #1,r(i%)
NEXT i%
CLOSE #1
```

Xload.r:

Option to plot the current N-Dimensional received PDSP input vector

```
Plot.r:LOCATE 1,1:INPUT;"plot received PDSP input vector?
Enter y or n",ei$:GOSUB clear1
IF ei$="n" THEN Xplot.r
LOCATE 1,1:LPRINT TAB(14);"Plotting the received PDSP input signal ";Total$
LOCATE 1,1:INPUT;"Clear screen? Enter y or n",ei$:GOSUB clear1
IF ei$="y" THEN CLS
yb=-30:yt=30
LOCATE 1,1:PRINT"Y max = ";yt:INPUT;"Change Y max? Enter #
(0 retains current value)",ei$:GOSUB clear1:GOSUB clear2
IF ei<> 0 THEN yt=ei
IF ei<> 0 THEN yb=-ei
ys=(yt-yb)/yj:yg=yd/ys:y$="Amplitude":ytab=4
FOR i%=1 TO m%
  y(i%)=r(i%)*yg
NEXT i%
IF ei$="y" THEN GOSUB Plotxy
CALL MOVETO(xo+x(1)-1,yo)
CALL LINETO(xo+x(1)-1,yo-y(1))
FOR i%=1 TO m%-1
  CALL LINETO(xo+x(i%)+1,yo-y(i%))
```



```

    CALL LINETO(xo+x(i%)+1,yo-y(i%+1))
NEXT i%
LOCATE 1,1:INPUT;"Plot Figure * is",FigNo$:GOSUB clear1
LPRINT TAB(14);"Received PDSP input ";Total$;" is plotted in Figure D-";FigNo$
Xplot.r:

```

Option to store the current N-dimensional PDSP input vector

```

Store.r:LOCATE 1,1:INPUT;"Store PDSP input signal?
Enter y or n",ei$:GOSUB clear1
IF ei$="n" THEN Xstore.r
LOCATE 1,1:INPUT;"Which PDSP input file?",File$:GOSUB clear1
LOCATE 1,1:PRINT"Storing PDSP received signal named ";Total$;" in file ";File$
OPEN "o",*1,File$,512
FOR i%=0 TO m%
    WRITE*1,r(i%)
NEXT i%
CLOSE*1
Xstore.r:

```

Option to collect statistics on N-dimensional received signal vector

```

Stat.r:LOCATE 1,1:INPUT;"Received signal Statistics?
Enter y or n",ei$:GOSUB clear1
IF ei$="n" THEN Xstat.r
LOCATE 1,1:LPRINT TAB(14);"Calculating statistics for ";Input "+Total$
CALL stat(r())
Xstat.r:

```

```
IF Newr>0 THEN Detect.b
```

Compute the sample vector signal-to-noise ratio (SNR) input,
before any Pre-detection signal processing (PDSP)

E_b - Energy of the source signal vector
 E_{rb} -Energy of the noise + interference vector.
 SNR_i -PDSP algorithm SNR input

```

SNRi:Eb=0:Erb=0
FOR i%=1 TO m%:Eb=Eb+b(i%)*b(i%):Erb=Erb+(r(i%)-b(i%))*(r(i%)-b(i%)):NEXTi%

```

```

IF Erb>0 THEN snri=Eb/Erb ELSE snri=0
LOCATE 2,1:PRINT"input bit energy ";Eb;" Other energy ";Erb;
" SNRi ";10*LOG(snri)/LOG(10)
LPRINT TAB(14);"input bit energy ";Eb;" Other energy ";Erb;
" SNRi ";10*LOG(snri)/LOG(10)
LOCATE 1,1:PRINT"Correlating ";Total$; "with signal replica"

```

Crb-Correlation of PDSP input with signal replica

```

crb=0
FOR i%=1 TO m%
  crb=crb+r(i%)*b(i%)
NEXT i%
Iter%=0 'Number of iterations through PDSP

```

Option to perform Pre-detection Signal Processing

```

Detect.b:LOCATE 1,1:INPUT;"Detect bursts? Enter y or n",ei$:GOSUB clear2
IF ei$="n" THEN Xdetect.b
Newr=1:Iter%=Iter%+1

```

Option to select any one of five PDSP algorithms which detect runs of length 1 or longer called anomalous runs or bursts

- Suboption 0: Keep received vector intact
- Suboption 1: Erase anomalous runs from signal
- Suboption 2: Remove the average component of anomalous runs
- Suboption 3: Remove the average component of sections of anomalous runs
- Suboption 4: Remove the section-linear moving average component of anomalous runs
- Suboption 5: Remove the subsection-linear moving average component of anomalous runs

```

Filter.b:LOCATE 1,1:INPUT;"Filter bursts? Enter filter# 0-5",f%:GOSUB clear1
IF f%<0 OR f%>5 THEN Filter.b
LOCATE 1,1:INPUT;"Clear screen? Enter y or n",ei$:GOSUB clear1
CLS:IF ei$="y" THEN CLS

```

Initialize the burst detection parameters

h- run polarity decision threshold

l%- maximum length of normal run expected

mb-number of bursts encountered in the received vector

nb-a flag set to indicate a burst (anomalous run)

```
LOCATE 1,1: PRINT"Signal processing using PDSP F";f%;" Iteration # ";lter%
```

```
LPRINT TAB(14);"Signal processing using PDSP F";f%;" Iteration # ";lter%
```

```
h=0:l%=8:mb=0:nb=0:fb=0
```

```
FOR i%=1 TO m%-1
```

```
  IF(r(i%)>h) AND (r(i%+1)>h) THEN nb=nb+1
```

```
  IF(r(i%)<h) AND (r(i%+1)<h) THEN nb=nb-1
```

```
  IF(r(i%)<h) AND (r(i%+1)>h) AND (fb=0)THEN nb=0
```

```
  IF(r(i%)>h) AND (r(i%+1)<h) AND (fb=0)THEN nb=0
```

```
  IF(nb>l%-1) OR (nb<-l%+1) AND (fb=0)THEN a%=i%-ABS(nb)+1
```

```
  IF(i%>1) AND (nb>l%-1) OR (nb<-l%+1)THEN fb=1
```

```
  IF(r(i%)<h) AND (r(i%+1)>h) AND (fb=1)THEN nb=0
```

```
  IF(r(i%)>h) AND (r(i%+1)<h) AND (fb=1)THEN nb=0
```

```
  IF(nb=0) AND (fb=1) THEN PRINT a%;"<---polarity burst--->";i%
```

```
  IF (nb=0) AND (fb=1) AND (f%>0) THEN CALL levela(d),(a%),(i%)
```

```
  IF (nb=0) AND (fb=1) AND (f%>0) THEN CALL levela(r),(a%),(i%)
```

```
  IF (nb=0) AND (fb=1) THEN mb=mb+1
```

```
  IF nb=0 THEN fb=0
```

```
NEXT i%
```

```
a%=m%-ABS(nb)
```

```
IF fb=1 THEN PRINT a%;"<---polarity burst--->";m%
```

```
IF fb=1 AND (f%>0)THEN CALL levela(d),(a%),(m%)
```

```
IF fb=1 AND (f%>0)THEN CALL levela(r),(a%),(m%)
```

```
IF fb=1 THEN mb=mb+1
```

Compute the sample vector signal-to-noise ratio (SNR) output, after
Pre-detection signal processing (PDSP)

Ed- Energy of the source signal vector distorted by the PDSP

Erd-Energy of the output noise + interference vector.

SNRo-PDSP algorithm SNR output

```
SNRo:Ed=0:Erd=0
```

```
FOR i%=1 TO m%:Ed=Ed+d(i%)*d(i%):Erd=Erd+(r(i%)-d(i%))*(r(i%)-d(i%)):NEXTi%
```

```
IF Erd>0 THEN snro=Ed/Erd ELSE snro=0
```

```

LOCATE 3,1:PRINT"output bit energy ";Ed;" other energy ";
Erd;" SNRo ";10*LOG(snro)/LOG(10)
LPRINT TAB(14);"Output bit energy ";Ed;" Other energy ";
Erd;" SNRo ";10*LOG(snro)/LOG(10)

```

Computing the sample processing distortion factor, processing gain of PDSP

Pd- Processing distortion factor of PDSP

Pg-Processing gain of the PDSP

```

pd=0:IF Eb>0 THEN pd=Ed/Eb
pg=0:IF snri>0 THEN pg=pd*snro/snri
PRINT"Processed";mb;" bursts. Sample PD = ";pd;
" Sample PG = ";10*LOG(pg)/LOG(10)
LPRINT TAB(14);"Processed";mb;" bursts. Sample PD = ";pd;" Sample PG =
";10*LOG(pg)/LOG(10)
LOCATE 1,1:PRINT"Correlating ";Total;"with signal replica"

```

Computing the overall Processing gain

Cfb- the PDSP total output sample correlation with PN replica

Cdb- the PDSP distorted signal output sample correlation with PN replica

Cd - Correlation distortion factor

Cg - Overall correlation processing gain

```

Cfb=0
FOR i%=1 TO m%
Cfb=Cfb+r(i%)*b(i%)
NEXT i%
LOCATE 1,1:PRINT"Correlating processed";Signal$;"with signal replica"
Cdb=0
FOR i%=1 TO m%
Cdb=Cdb+d(i%)*b(i%)
NEXT i%
Cd=0:IF Eb>0 THEN Cd=Cdb/Eb
Cg=0:IF ABS(Cfb-Cdb)>0 THEN Cg=(Cd*(Crb-Eb)/(Cfb-Cdb))^2
LOCATE 5,1:PRINT"Sample Crb = ";Crb;" Sample Cdb = ";Cdb;" Sample Cfb =
";Cfb
PRINT"Sample Cd = ";Cd;" Sample Cg = ";Cg
LPRINT TAB(14);"Sample Crb = ";Crb;" Sample Cdb = ";Cdb;" Sample Cfb = ";Cfb
LPRINT TAB(14);"Sample Cd = ";Cd;" Sample Cg = ";Cg

```

Option to plot the current N-Dimensional received PDSP output vector

```

Plot.f:LOCATE 1,1:INPUT;"plot filtered vector? Enter y or n",ei$:GOSUB clear1
IF ei$="n" THEN Xplot.f
LOCATE 1,1:LPRINT TAB(14);"Plotting the filtered received signal ";Total$
LOCATE 1,1:INPUT;"Clear screen? Enter y or n",ei$:GOSUB clear1
IF ei$="y" THEN CLS
yb=-30:yt=30
LOCATE 1,1:PRINT"Y max = ";yt:INPUT;"Change Y max? Enter *(0 retains
current value)",ei$:GOSUB clear1:GOSUB clear2
IF ei<> 0 THEN yt=ei
IF ei<> 0 THEN yb=-ei
ys=(yt-yb)/yj:yg=yd/ys:y$="Amplitude":ytab=4
FOR i%=1 TO m%
  y(i%)=r(i%)*yg
NEXT i%
IF ei$="y" THEN GOSUB Plotxy
CALL MOVETO(xo+x(1)-1,yo)
CALL LINETO(xo+x(1)-1,yo-y(1))
FOR i%=1 TO m%-1
  CALL LINETO(xo+x(i%)+1,yo-y(i%))
  CALL LINETO(xo+x(i%)+1,yo-y(i%+1))
NEXT i%
LOCATE 1,1:INPUT;"Plot Figure * is",FigNo$:GOSUB clear1
LPRINT TAB(14);"Received PDSP output ";Total$;
" is plotted in Figure D-";FigNo$
Xplot.f:

```

Option to plot the current N-Dimensional PDSP distorted signal vector

```

Plot.d:LOCATE 1,1:INPUT;"plot output signal vector? Enter y or n",ei$:GOSUB
clear1
IF ei$="n" THEN Xplot.d
LOCATE 1,1:LPRINT TAB(14);"Plotting the PN output signal named
";"processed "+Signal$
LOCATE 1,1:INPUT;"Clear screen? Enter y or n",ei$:GOSUB clear1

```

```

IF ei$="y" THEN CLS
yb=-30:yt=30
LOCATE 1,1:PRINT"Y max = ";yt:INPUT;"Change Y max? Enter #
(O retains current value)",ei:GOSUB clear1:GOSUB clear2
IF ei<> 0 THEN yt=ei
IF ei<> 0 THEN yb=-ei
ys=(yt-yb)/yj:yg=yd/ys:y$="Amplitude":ytab=4
FOR i%=1 TO m%
  y(i%)=d(i%)*yg
NEXT i%
IF ei$="y" THEN GOSUB Plotxy
CALL MOVETO(xo+x(1)-1,yo)
CALL LINETO(xo+x(1)-1,yo-y(1))
FOR i%=1 TO m%-1
  CALL LINETO(xo+x(i%)+1,yo-y(i%))
  CALL LINETO(xo+x(i%)+1,yo-y(i%+1))
NEXT i%
LOCATE 1,1:INPUT;"Plot Figure # is",FigNo$:GOSUB clear1
LPRINT TAB(14);"distorted signal ";Signal$;" is plotted in Figure D-";FigNo$
Xplot.d:

```

Option to store the current N-dimensional PDSP output vector

```

Store.f:LOCATE 1,1:INPUT;"Store PDSP output signal? Enter y or n",ei$
GOSUB clear1
IF ei$="n" THEN Xstore.f
LOCATE 1,1:INPUT;"Which PDSP output signal file?",file$:GOSUB clear1
LOCATE 1,1:PRINT"Storing PDSP output signal named ";Total$;" in file
";File$
OPEN "o",#1,file$,512
FOR i%=0 TO m%
  WRITE#1,r(i%)
NEXT i%
CLOSE#1
Xstore.f:

```

Option to collect statistics on N-dimensional PDSP output signal vector

```
Stat.f:LOCATE 1,1:INPUT;"PDSP output signal Statistics?
Enter y or n",ei$:GOSUB clear1
IF ei$="n" THEN Xstat.f
LOCATE 1,1:LPRINT TAB(14);"Calculating statistics for ";
"Processed "+Total$
CALL stat(r())
Xstat.f:
```

Option to collect statistics on N-dimensional PDSP distorted signal vector

```
Stat.d:LOCATE 1,1:INPUT;"Distorted signal Statistics?
Enter y or n",ei$:GOSUB clear1
IF ei$="n" THEN Xstat.d
LOCATE 1,1:LPRINT TAB(14);"Calculating statistics for ";
"Processed "+Signal$
CALL stat(d())
Xstat.d:
```

```
LOCATE 1,1:INPUT;"Iterate burst detection? Enter y or n",ei$:GOSUBclear1
LOCATE 1,1:LPRINT"End of PDSP F";f%;" Iteration # ";lter%
IF ei$="y" THEN Detect.b
Xdetect.b:GOSUB clear1
```

```
LOCATE 1,1:INPUT;"Iterate received vector generation? Enter y or
n",ei$:GOSUB clear1
IF ei$="y" THEN Gen.r
```

```
Cr.b:LOCATE 1,1:INPUT;"Correlation with shifted source replica? Enter y or
n",ei$:GOSUB clear1
IF ei$="n" THEN Xcr.b
LOCATE 1,1:INPUT;"Maximum source replica shift? Enter 0-126",l%:GOSUB
clear1
IF l%>m%-1 THEN Xcr.b
```

```

LOCATE 1,1: PRINT "Sit and relax, the correlation function is being generated"
FOR k%=0 TO 1%
  FOR i%=1 TO m%-k%
    y(i%+k%)=b(i%)
  NEXT i%
  FOR i%=m%-k%+1 TO m%
    j%=i%-(m%-k%)
    y(j%)=b(i%)
  NEXT i%
  cr(k%)=0
  FOR i%=1 TO m%
    cr(k%)=cr(k%)+r(i%)*y(i%)
  NEXT i%
NEXT k%
Xcr.b:GOSUB clear1

```

```

Cr.i:LOCATE 1,1:INPUT;"Correlation with shifted interference? Enter y or
n",ei$:GOSUB clear1
IF ei$="n" THEN Xcr.i
LOCATE 1,1:INPUT;"Maximum source replica shift? Enter 0-126",l%:GOSUB
clear1
IF l%>m%-1 THEN Xcr.i
LOCATE 1,1: PRINT "Sit and relax, the correlation function is being generated"
LOCATE 1,1:PRINT"Generating interference":GOSUB clear2
p2=2*3.14159#
FOR k%=0 TO m%
  pik = pi+(k%)*p2/m%
  FOR i%=1 TO m%
    y(i%)=ai*(COS(p2*fa*t(i%)))*COS(p2*fi*t(i%)+fmi*COS(p2*fm*t(i%))+pik)
  NEXT i%
  'FOR i%=1 TO m%
  'y(i%)=y(i%)+b(i%) include only if correlating with interference+signal
  'NEXT i%
  cr(k%)=0
  FOR i%=1 TO m%
    cr(k%)=cr(k%)+y(i%)*b(i%)
  NEXT i%
NEXT k%

```



```

    INPUT #1,cr(i%)
NEXT i%
CLOSE #1
Xload.cr:

```

Option to plot the current N-Dimensional correlation vector

```

Plot.cr:LOCATE 1,1:INPUT;"Plot correlation function? Enter y or n",ei$:GOSUB
clear1
IF ei$="n" THEN Xplot.cr
LOCATE 1,1:LPRINT TAB(14);"Plotting the correlation ";cor$
Minmax.cr:il=1:ih=1:crl=cr(0):crh=cr(0)
FOR i%=1 TO m%
    IF cr(i%)<crl THEN il=i%
    IF cr(i%)<crl THEN crl=cr(i%)
    IF cr(i%)>crh THEN ih=i%
    IF cr(i%)>crh THEN crh=cr(i%)
NEXT i%
IF ABS(crl)>ABS(crh) THEN yt=ABS(crl)
IF ABS(crl)<ABS(crh) THEN yt=ABS(crh)
yb=-yt
LOCATE 1,1:PRINT"Y max = ";yt:INPUT;"Change Y max? Enter *(0 retains
current value)",ei$:GOSUB clear1:GOSUB clear2
IF ei<> 0 THEN yt=ei
IF ei<> 0 THEN yb=-ei
LOCATE 1,1:INPUT;"Clear screen? Enter y or n",ei$:GOSUB clear1
ys=(yt-yb)/yj:yg=yd/ys:y$="Correlation":ytab=3
IF ei$="y" THEN CLS
FOR i%=1 TO m%
    y(i%)=cr(i%)*yg
NEXT i%
IF ei$="y" THEN GOSUB Plotxy
CALL MOVETO(xo+x(0),yo-y(0))
FOR i%=1 TO l%
    CALL LINETO(xo+x(i%),yo-y(i%))
NEXT i%
LOCATE 1,1:INPUT;"Plot Figure # is",FigNo$:GOSUB clear1
LPRINT TAB(14);"Correlation ";Cor$;" is plotted in Figure D-";FigNo$
Xplot.cr:

```

Option to store the current N-dimensional correlation vector

```
Store.cr:LOCATE 1,1:INPUT;"Store correlation? Enter y or n",ei$:GOSUB clear1
  IF ei$="n" THEN Xstore.cr
  LOCATE 1,1:INPUT;"which file?",file$:GOSUB clear1
  LOCATE 1,1:PRINT"Storing correlation in file named ";file$
  OPEN "o",*1,file$,512
  FOR i%=0 TO m%
    WRITE*1,cr(i%)
  NEXT i%
  CLOSE#1
Xstore.cr:
```

Option to collect statistics on correlation vector

```
Stat.cr:LOCATE 1,1:INPUT;"Correlation Statistics? Enter y or n",ei$:GOSUB clear1
  IF ei$="n" THEN Xstat.cr
  LOCATE 1,1:LPRINT TAB(14);"Calculating statistics for ";Cor$
  CALL stat(cr())
Xstat.cr:
```

```
Finish:LOCATE 1,1:INPUT;"Start, Transmit, Interfer,Noise,Receive,Filter, End, or
Quit? Enter s,r,f, e, or q",ei$:GOSUB clear1
  IF ei$="s" THEN Start
  IF ei$="t" THEN Gen.b
  IF ei$="i" THEN Gen.i
  IF ei$="n" THEN Gen.n
  IF ei$="r" THEN Gen.r
  IF ei$="f" THEN Detect.b
  IF ei$="e" THEN END
  IF ei$="q" THEN SYSTEM
END
```

Subroutine to clear the first line of the screen for a new prompt

```
clear1: LOCATE 1,1
  PRINT"
RETURN
```

Subroutine to clear the second line of the screen for a new prompt

```
clear2: LOCATE 2,1
        PRINT"
RETURN
```

Subroutine to perform Burst(Anomalous Run) Signal Processing

Level 1: Erase burst

Level 2: Remove the average component of the entire burst

Level 3: Remove the average component of each burst section

Level 4: Remove the piecewise linear smoothing fit between burst sections

level 5: Remove the piecewise linear smoothing fit between burst subsections

Note : A burst section is chosen here to correspond to the minimum length , $l\%$, of an anomalous run detected between time samples, $a\%$ and $b\%$.

```
SUB levela(r(1),a%,b%)STATIC
        SHARED f%,h,l%
        ON f% GOTO Level1,Level2,Level3,Level4,level5
```

Level1: Erase the entire burst

```
        FOR i%=a% TO b%
            r(i%)=h
        NEXT i%
EXIT SUB
```

Level2: Remove the Dc component of the entire burst

```
        r=0:n=b%-a%+1
        FOR i%=a% TO b%
            r=r+r(i%)
        NEXT i%
        r=r/n
        FOR i%=a% TO b%
            r(i%)=r(i%)-r
        NEXT i%
EXIT SUB
```

Level3: Remove the Dc component of each burst section

```

m=INT((b%-a%+1)/l%)
j1%=a%:           First m-1 sections
FOR i%=1 TO m-1
  j2%=j1%+l%-1
  r=0
  FOR j%=j1% TO j2%
    r=r+r(j%)
  NEXT j%
  r=r/l%
  FOR j%=j1% TO j2%
    r(j%)=r(j%)-r
  NEXT j%
  j1%=j2%+1
NEXT i%
r=0:           Final section
FOR j%=j1% TO b%
  r=r+r(j%)
NEXT j%
r=r/(b%-j1%+1)
FOR j%=j1% TO b%
  r(j%)=r(j%)-r
NEXT j%
EXIT SUB

```

Level4: Piecewise linear smoothing using l

```

m=INT((b%-a%+1)/l%)
IF m<2 THEN Level2
j1%=a%:           First section
j2%=j1%+l%-1
r1=0
FOR j%=j1% TO j2%

```

```

    r1=r1+r(j%)
  NEXT j%
  r1=r1/l%:ri=0
FOR i%=2 TO m
  j3%=j2%+1
  j4%=j3%+l%-1
  r2=0
  FOR j%=j3% TO j4%
    r2=r2+r(j%)
  NEXT j%
  r2=r2/l%:s12=(r2-r1)/l%
  FOR j%=j1% TO j2%
    r(j%)=r(j%)-s12*(j%-j1%)-ri
  NEXT j%
  j1%=j3%:j2%=j4%:r1=r2:ri=ri+s12*l%
NEXT i%

```

Final section

```

  FOR j%=j3% TO b%
    r(j%)=r(j%)-s12*(j%-j1%)-ri
  NEXT j%
GOTO level3
Level5: Piecewise linear smoothing over 1/2

```

```

l2=l%/2:m=INT((b%-a%+1)/l2)
IF m<2 THEN Level2
  j1%=a%: First section
  j2%=j1%+l2-1
  r1=0
  FOR j%=j1% TO j2%
    r1=r1+r(j%)
  NEXT j%
  r1=r1/l%:ri=0
FOR i%=2 TO m
  j3%=j2%+1
  j4%=j3%+l2-1
  r2=0
  FOR j%=j3% TO j4%

```

```

    r2=r2+r(j%)
NEXT j%
r2=r2/12:s12=(r2-r1)/12
FOR j%=j1% TO j2%
    r(j%)=r(j%)-s12*(j%-j1%)-ri
NEXT j%
j1%=j3%:j2%=j4%:r1=r2:ri=ri+s12*12
NEXT i%

```

Final section

```

FOR j%=j3% TO b%
    r(j%)=r(j%)-s12*(j%-j1%)-ri
NEXT j%
GOTO level3

```

END SUB

SUB stat(r(1))STATIC

 SHARED m%,n%,b(),bu(),bd(),bt()

Option to compute statistical properties of the PDSP received sample vector output.

il-location of sample lowest value of received vector.
ih-location of sample highest value of received vector.
rl-sample lowest value of received vector.
rh-sample highest value of received vector.
r- sample average value of received vector.
ar-sample average of absolute value of received vector.
er-sample energy of received vector.
vr- sample variance of received vector.
sr- sample standard deviation of received vector.
pr-sample power of received vector.

LOCATE 1,1:PRINT"calculating statistics"

Minmax.r:il=1:ih=1:rl=r(1):rh=r(1)

FOR i%=2 TO m%

 IF r(i%)<rl THEN il=i%

 IF r(i%)>rh THEN rh=r(i%)

```

    IF r(i%)>rh THEN ih=i%
    IF r(i%)>rh THEN rh=r(i%)
NEXT i%
PRINT "Highest r = ";rh;" at ";ih;". Lowest r = ";rl;" at ";il
LPRINT TAB(14);"Highest r = ";rh;" at ";ih;". Lowest r = ";rl;" at ";il
Average.r:r=0
Avrgabs.r:ar=0
FOR i%=1 TO m%
    r=r+r(i%):ar=ar+ABS(r(i%))
NEXT i%
    r=r/m%:ar=ar/m%
Energy.r:er=0
Variance.r:vr=0
FOR i%=1 TO m%
    er=er+r(i%)^2:vr=vr+(r(i%)-r)^2
NEXT i%
Power.r:pr=er/m%:vr=vr/m%:sr=SQR(vr)

```

nu-sample number of positive polarity
 nd-sample number of negative polarity
 cu-sample number of positive zero-crossing
 cd-sample number of negative zero-crossing
 bu-sample number of positive runs
 bd-sample number of negative runs
 bt-sample number of all runs
 fg- flag identifying the start and sign of the current run

```

Polarity.r:nu=0:nd=0
Crossing.r:cu=0:cd=0
Burst.r:bu=0:bd=0
IF r(1)>=0 THEN fg=1
IF r(1)<0 THEN fg=-1
FOR i%=1 TO 10
    bu(i%)=0:bd(i%)=0:bt(i%)=0
NEXT i%
'Count polarity and crossings
FOR i%=1 TO m%
    IF (fg=1) AND (r(i%)<0) THEN cd=cd+1
    IF (fg=-1) AND (r(i%)>0) THEN cu=cu+1

```

```

    IF r(i%)<0 THEN fg=-1
    IF r(i%)>0 THEN fg=1
    IF r(i%)<0 THEN nd=nd+1
    IF r(i%)>0 THEN nu=nu+1
NEXT i%
'Determine start of new burst
j=0
FOR i%=1 TO n%
    IF r(i%)>0 AND r(m%)<0 THEN j=-i%
    IF r(i%)<0 AND r(m%)>0 THEN j=i%
    IF j<>0 THEN k%=ABS(j)
    IF j<>0 THEN New.burst
NEXT i%
k%=1
New.burst:'Count bursts by polarity and length
IF r(k%)>0 THEN pb%=1
IF r(k%)<0 THEN nb%=1
FOR i%=k%+1 TO m%
    IF r(i%)>0 AND r(i%-1)>0 THEN pb%=pb%+1
    IF r(i%)<0 AND r(i%-1)<0 THEN nb%=nb%+1
    IF r(i%)>0 AND r(i%-1)<0 THEN bd=bd+1
    IF r(i%)>0 AND r(i%-1)<0 AND nb%<11 THEN bd(nb%)=bd(nb%)+1
    IF r(i%)>0 AND r(i%-1)<0 THEN pb%=1
    IF r(i%)<0 AND r(i%-1)>0 THEN bu=bu+1
    IF r(i%)<0 AND r(i%-1)>0 AND pb%<11 THEN bu(pb%)=bu(pb%)+1
    IF r(i%)<0 AND r(i%-1)>0 THEN nb%=1
NEXT i%
'Process last burst
IF r(1)>0 AND r(m%)>0 THEN pb%=pb%+k%-1
IF r(1)<0 AND r(m%)<0 THEN nb%=nb%+k%-1
IF r(k%)>0 AND r(m%)<0 THEN bd=bd+1
IF r(k%)>0 AND r(m%)<0 AND nb%<11 THEN bd(nb%)=bd(nb%)+1
IF r(k%)<0 AND r(m%)>0 THEN bu=bu+1
IF r(k%)<0 AND r(m%)>0 AND pb%<11 THEN bu(pb%)=bu(pb%)+1
FOR i%=1 TO 10
    bt(i%)=bu(i%)+bd(i%)
NEXT i%
bt=bu+bd
'Correlating with signal replica

```



```

cr=0
FOR i%=1 TO m%
  cr=cr+r(i%)*b(i%)
NEXT i%
LOCATE 1,1:INPUT;"Clear screen? Enter y or n",ei$:GOSUB clear1
IF ei$="y" THEN CLS
CALL TEXTFACE(4)
PRINT SPC(8);"      'N' 'C' 'B' '1' '2' '3' '4' '5' '6' '7' '8' '9' '10'"
CALL TEXTFACE(0):
PRINT SPC(8);" (+) ";
PRINT
USING"****";nu;cu;bu;
bu(1);bu(2);bu(3);bu(4);bu(5);bu(6);bu(7);bu(8);bu(9);bu(10)
CALL TEXTFACE(4)
PRINT SPC(8);" (-) ";
PRINT USING"****";nd;cd;bd;
bd(1);bd(2);bd(3);bd(4);bd(5);bd(6);bd(7);bd(8);bd(9);bd(10)
PRINT SPC(8);"Average      St. Dv.      Energy      Power      Correlation"
CALL TEXTFACE(0):
LOCATE 7,2:PRINT USING "*****";r;sr;er;pr;cr
LPRINT
LPRINT
TAB(18);"_____ "
LPRINT TAB(18);vector$;
LPRINT TAB(26);" 'N' 'C' 'B' '1' '2' '3' '4' '5' '6' '7' '8' '9' '10'"
LPRINT TAB(18);"(+) ";
LPRINT USING"****";nu;cu;bu;
bu(1);bu(2);bu(3);bu(4);bu(5);bu(6);bu(7);bu(8);bu(9);bu(10)
LPRINT TAB(18);"(-) ";
LPRINT USING"****";nd;cd;bd;
bd(1);bd(2);bd(3);bd(4);bd(5);bd(6);bd(7);bd(8);bd(9);bd(10)
LPRINT TAB(18);" Average      St. Dv.      Energy      Power      Correlation"
LPRINT TAB(19);
LPRINT USING "*****";r;sr;er;pr;cr
LPRINT
TAB(18);"_____ "
END SUB
END

```

APPENDIX C

INVARIANT ALGORITHMS DEMONSTRATION

STATISTICAL RESULTS

In this Appendix, we provide a sample output using the demonstration program listed in Appendix B. The definition of each of the parameters shown is given in Appendix B. The plotted Figures are given in Appendix D. In the following narrative, PDSP F1 refers to the randomness invariant erasure algorithm, PDSP F2 refers to the randomness invariant average algorithm, PDSP F3 refers to the randomness invariant piece-wise average algorithm, and finally, PDSP F4 refers to the randomness invariant piece-wise linear-average correction algorithm.

```
Generating PN signal named b(t)
PN signal amplitude is 1
Plotting the PN signal named b(t)
Signal b(t) is Plotted in Figure D-1
Calculating statistics for b(t)
Highest r = 1 at 1 . Lowest r = -1 at 2
```

	'N'	'C'	'B'	'1'	'2'	'3'	'4'	'5'	'6'	'7'	'8'	'9'	'10'
(+)	64	32	32	16	8	4	2	1	0	1	0	0	0
(-)	63	32	32	16	8	4	2	1	1	0	0	0	0
Average				St. Dv.		Energy			Power		Correlation		
	0			1		127			1		127		

Generating interference named $i(t)$
 Burst start(1-126) 1 Burst end(2-127) 128
 Interference amplitude $a_i = 30$ frequency $f_i = .008$
 phase $\pi_i = 1.8852$ AM frequency $f_a = 0$
 FM modulation index $f_{mi} = 0$ FM frequency $f_m = .008$
 Plotting the interference named $i(t)$
 Interference $i(t)$ is Plotted in Figure D-2
 Calculating statistics for $i(t)$
 Highest $r = 29.99071$ at 87 . Lowest $r = -30$ at 25

	'N'	'C'	'B'	'1'	'2'	'3'	'4'	'5'	'6'	'7'	'8'	'9'	'10'
(+)	62	1	1	0	0	0	0	0	0	0	0	0	0
(-)	65	1	1	0	0	0	0	0	0	0	0	0	0
Average				St. Dv.		Energy		Power		Correlation			
	-0			21		56511		445		-264			

Loading file named n01 for noise $n(t)$
 Plotting the noise named $n(t)$
 Noise $n(t)$ is Plotted in Figure D-3
 Calculating statistics for $n(t)$
 Highest $r = 2.155245$ at 75 . Lowest $r = -2.192679$ at 5

	'N'	'C'	'B'	'1'	'2'	'3'	'4'	'5'	'6'	'7'	'8'	'9'	'10'
(+)	71	34	35	18	7	5	2	2	1	0	0	0	0
(-)	56	35	35	20	11	3	0	1	0	0	0	0	0
Average				St. Dv.		Energy		Power		Correlation			
	0			1		104		1		-14			

Generating the received PDSP input signal 7 : $r(t) = b(t) + i(t) + n(t)$
 Plotting the received PDSP input signal $r(t) = b(t) + i(t) + n(t)$
 Received PDSP input $r(t) = b(t) + i(t) + n(t)$ is plotted in Figure D-4
 Calculating statistics for Input $r(t) = b(t) + i(t) + n(t)$
 Highest $r = 32.23127$ at 83 . Lowest $r = -32.39125$ at 26

	'N'	'C'	'B'	'1'	'2'	'3'	'4'	'5'	'6'	'7'	'8'	'9'	'10'
(+)	61	1	1	0	0	0	0	0	0	0	0	0	0
(-)	66	1	1	0	0	0	0	0	0	0	0	0	0
Average				St. Dv.		Energy		Power		Correlation			
	-0			21		56633		446		-151			

input bit energy 127 Other energy 57062.7 SNRI -26.52549

Signal processing using PDSP F 1 Iteration # 1
 Output bit energy 1 Other energy .1477754 SNRo 8.303979
 Processed 3 bursts. Sample PD = 7.874016E-03 Sample PG = 13.79143
 Sample Crb = -151.2583 Sample Cdb = 1 Sample Cfb = 1.384416
 Sample Cd = 7.874016E-03 Sample Cg = 32.4853
 Plotting the filtered received signal $r(t)=b(t)+i(t)+n(t)$
 Received PDSP output $r(t)=b(t)+i(t)+n(t)$ is plotted in Figure D-5
 Plotting the PN output signal named processed $b(t)$
 Processed signal $b(t)$ is plotted in Figure D-5
 Calculating statistics for Processed $r(t)=b(t)+i(t)+n(t)$
 Highest $r = 1.384416$ at 57. Lowest $r = 0$ at 1

	'N'	'C'	'B'	'1'	'2'	'3'	'4'	'5'	'6'	'7'	'8'	'9'	'10'
(+)	1	0	0	0	0	0	0	0	0	0	0	0	0
(-)	0	0	0	0	0	0	0	0	0	0	0	0	0
Average			St. Dv.		Energy		Power		Correlation				
	0		0		2		0		1				

Calculating statistics for Processed $b(t)$
 Highest $r = 1$ at 57. Lowest $r = 0$ at 1

	'N'	'C'	'B'	'1'	'2'	'3'	'4'	'5'	'6'	'7'	'8'	'9'	'10'
(+)	1	0	0	0	0	0	0	0	0	0	0	0	0
(-)	0	0	0	0	0	0	0	0	0	0	0	0	0
Average			St. Dv.		Energy		Power		Correlation				
	0		0		1		0		1				

End of PDSP F 1 Iteration # 1
 Generating interference named $I(t)$
 Burst start(1-126) 16 Burst end(2-127) 64
 Interference amplitude $a_i = 30$ frequency $f_i = .008$
 phase $\pi_i = 1.8852$ AM frequency $f_a = 0$
 FM modulation index $f_{m_i} = 0$ FM frequency $f_m = 0$
 Calculating statistics for $I(t)$
 Highest $r = 11.40008$ at 64. Lowest $r = -30$ at 25

	'N'	'C'	'B'	'1'	'2'	'3'	'4'	'5'	'6'	'7'	'8'	'9'	'10'
(+)	8	1	0	0	0	0	0	0	0	0	0	0	0
(-)	41	1	1	0	0	0	0	0	0	0	0	0	0
Average			St. Dv.		Energy		Power		Correlation				
	-6		12		22400		176		74				

Generating the received PDSP input signal 7 : $R(t)=b(t)+I(t)+n(t)$
 Calculating statistics for Input $R(t)=b(t)+I(t)+n(t)$
 Highest r = 11.85187 at 64 . Lowest r = -32.39125 at 26

	'N'	'C'	'B'	'1'	'2'	'3'	'4'	'5'	'6'	'7'	'8'	'9'	'10'
(+)	51	21	22	11	5	2	1	2	0	0	0	0	1
(-)	76	22	22	12	6	1	2	0	0	0	0	0	0
	Average		St. Dv.			Energy			Power		Correlation		
	-6		12			22493			177		187		

input bit energy 127 Other energy 22245.48 SNRi -22.43438
 Signal processing using PDSP F 1 Iteration # 1
 Output bit energy 77 Other energy 63.30408 SNRo .8505901
 Processed 2 bursts. Sample PD = .6062992 Sample PG = 21.11184
 Sample Crb = 187.1388 Sample Cdb = 77 Sample Cfb = 75.34483
 Sample Cd = .6062992 Sample Cg = 485.2838
 Plotting the filtered received signal $R(t)=b(t)+I(t)+n(t)$
 Received PDSP output $R(t)=b(t)+I(t)+n(t)$ is plotted in Figure D-6
 End of PDSP F 1 Iteration # 1
 Generating interference named $i(t)$
 Burst start(1-126) 1 Burst end(2-127) 128
 Interference amplitude ai= 30 frequency fi= .008
 phase pi= 1.8852 AM frequency fa = 0
 FM modulation index fmi = 0 FM frequency fm = .008
 Generating the received PDSP input signal 7 : $r(t)=b(t)+i(t)+n(t)$
 input bit energy 127 Other energy 57062.7 SNRi -26.52549
 Signal processing using PDSP F 2 Iteration # 1
 Output bit energy 124.7285 Other energy 8815.557 SNRo -18.49284
 Processed 3 bursts. Sample PD = .9821142 Sample PG = 7.954265
 Sample Crb = -151.2583 Sample Cdb = 124.7286 Sample Cfb = 32.26596
 Sample Cd = .9821147 Sample Cg = 8.735518
 Plotting the filtered received signal $r(t)=b(t)+i(t)+n(t)$
 Received PDSP output $r(t)=b(t)+i(t)+n(t)$ is plotted in Figure D-7
 Calculating statistics for Processed $r(t)=b(t)+i(t)+n(t)$
 Highest r = 19.57347 at 56 . Lowest r = -19.59801 at 58

	'N'	'C'	'B'	'1'	'2'	'3'	'4'	'5'	'6'	'7'	'8'	'9'	'10'
(+)	64	3	4	0	0	0	0	1	0	0	0	1	0
(-)	63	4	4	0	0	0	0	1	0	0	0	0	0
	Average		St. Dv.			Energy			Power		Correlation		
	0		8			8755			69		32		

End of PDSP F 2 Iteration # 1
 Signal processing using PDSP F 2 Iteration # 2
 Output bit energy 122.6108 Other energy 2545.659 SNRo -13.17271
 Processed 6 bursts. Sample PD = .9654397 Sample PG = 13.20002
 Sample Crb = -151.2583 Sample Cdb = 122.535 Sample Cfb = 22.50014
 Sample Cd = .9648424 Sample Cg = 7.202888
 Plotting the filtered received signal $r(t)=b(t)+i(t)+n(t)$
 Received PDSP output $r(t)=b(t)+i(t)+n(t)$ is plotted in Figure D-8
 Calculating statistics for Processed $r(t)=b(t)+i(t)+n(t)$
 Highest $r = 12.86227$ at 1. Lowest $r = -11.55891$ at 58

	'N'	'C'	'B'	'1'	'2'	'3'	'4'	'5'	'6'	'7'	'8'	'9'	'10'
(+)	65	10	11	2	1	0	1	1	2	0	3	0	0
(-)	62	11	11	2	1	1	0	2	1	1	2	0	0
	Average		St. Dv.			Energy		Power		Correlation			
	0		4			2467		19		23			

End of PDSP F 2 Iteration # 2
 Signal processing using PDSP F 2 Iteration # 3
 Output bit energy 120.5985 Other energy 2394.102 SNRo -12.97801
 Processed 2 bursts. Sample PD = .9495943 Sample PG = 13.32286
 Sample Crb = -151.2583 Sample Cdb = 121.3344 Sample Cfb = 10.05102
 Sample Cd = .9553894 Sample Cg = 5.706848
 Plotting the filtered received signal $r(t)=b(t)+i(t)+n(t)$
 Received PDSP output $r(t)=b(t)+i(t)+n(t)$ is plotted in Figure D-9
 Plotting the PN output signal named processed $b(t)$
 Processed signal $b(t)$ is plotted in Figure D-10
 Calculating statistics for Processed $r(t)=b(t)+i(t)+n(t)$
 Highest $r = 12.86227$ at 1. Lowest $r = -11.55891$ at 58

	'N'	'C'	'B'	'1'	'2'	'3'	'4'	'5'	'6'	'7'	'8'	'9'	'10'
(+)	67	18	19	7	3	2	1	1	1	0	3	1	0
(-)	60	19	19	10	1	1	0	3	1	0	3	0	0
	Average		St. Dv.			Energy		Power		Correlation			
	0		4			2284		18		10			

Calculating statistics for Processed $b(t)$
 Highest $r = 1.285714$ at 108. Lowest $r = -1.4$ at 118

	'N'	'C'	'B'	'1'	'2'	'3'	'4'	'5'	'6'	'7'	'8'	'9'	'10'
(+)	64	32	32	16	8	4	2	1	0	1	0	0	0
(-)	63	32	32	16	8	4	2	1	1	0	0	0	0
	Average		St. Dv.			Energy		Power		Correlation			
	-0		1			121		1		121			

End of PDSP F 2 Iteration # 3
 Generating the received PDSP input signal 7 : $r(t)=b(t)+i(t)+n(t)$
 input bit energy 127 Other energy 57062.7 SNRi -26.52549
 Signal processing using PDSP F 3 Iteration # 1
 Output bit energy 113.5667 Other energy 1208.656 SNRo -10.27052
 Processed 3 bursts. Sample PD = .8942258 Sample PG = 15.76944
 Sample Crb = -151.2583 Sample Cdb = 113.5667 Sample Cfb = 37.16476
 Sample Cd = .8942257 Sample Cg = 10.60675
 Plotting the filtered received signal $r(t)=b(t)+i(t)+n(t)$
 Received PDSP output $r(t)=b(t)+i(t)+n(t)$ is plotted in Figure D-11
 Plotting the PN output signal named processed $b(t)$
 Processed signal $b(t)$ is plotted in Figure D-12
 Calculating statistics for Processed $r(t)=b(t)+i(t)+n(t)$
 Highest $r = 7.753758$ at 1 . Lowest $r = -7.608033$ at 117

	'N'	'C'	'B'	'1'	'2'	'3'	'4'	'5'	'6'	'7'	'8'	'9'	'10'
(+)	66	22	23	7	5	2	3	5	1	0	0	0	0
(-)	61	23	23	10	2	3	3	4	1	0	0	0	0
Average			St. Dv.		Energy			Power		Correlation			
	0		3		1169			9		37			

Calculating statistics for Processed $b(t)$
 Highest $r = 1.75$ at 78 . Lowest $r = -1.5$ at 11

	'N'	'C'	'B'	'1'	'2'	'3'	'4'	'5'	'6'	'7'	'8'	'9'	'10'
(+)	64	32	32	16	8	4	2	1	0	1	0	0	0
(-)	63	32	32	16	8	4	2	1	1	0	0	0	0
Average			St. Dv.		Energy			Power		Correlation			
	0		1		114			1		114			

End of PDSP F 3 Iteration # 1
 Signal processing using PDSP F 3 Iteration # 2
 Output bit energy 113.5667 Other energy 1208.656 SNRo -10.27052
 Processed 0 bursts. Sample PD = .8942258 Sample PG = 15.76944
 Sample Crb = -151.2583 Sample Cdb = 113.5667 Sample Cfb = 37.16476
 Sample Cd = .8942257 Sample Cg = 10.60675
 End of PDSP F 3 Iteration # 2

Generating the received PDSP input signal 7 : $r(t)=b(t)+i(t)+n(t)$
input bit energy 127 Other energy 57062.7 SNRi -26.52549
Signal processing using PDSP F 4 Iteration # 1
Output bit energy 111.0686 Other energy 323.8286 SNRo -4.647239
Processed 3 bursts. Sample PD = .8745561 Sample PG = 21.29613
Sample Crb = -151.2583 Sample Cdb = 111.0667 Sample Cfb = 66.62285
Sample Cd = .8745407 Sample Cg = 29.98015
Plotting the filtered received signal $r(t)=b(t)+i(t)+n(t)$
Received PDSP output $r(t)=b(t)+i(t)+n(t)$ is plotted in Figure D-13
Plotting the PN output signal named processed $b(t)$
Processed signal $b(t)$ is plotted in Figure D-14
Calculating statistics for Processed $r(t)=b(t)+i(t)+n(t)$
Highest $r = 4.494033$ at 119 . Lowest $r = -7.33235$ at 127

	'N'	'C'	'B'	'1'	'2'	'3'	'4'	'5'	'6'	'7'	'8'	'9'	'10'
(+)	62	36	37	21	10	4	1	1	0	0	0	0	0
(-)	65	37	37	21	9	3	3	1	0	0	0	0	0
Average				St. Dv.		Energy			Power		Correlation		
	0			2		348			3		67		

Calculating statistics for Processed $b(t)$
Highest $r = 1.828125$ at 33 . Lowest $r = -1.59375$ at 11

	'N'	'C'	'B'	'1'	'2'	'3'	'4'	'5'	'6'	'7'	'8'	'9'	'10'
(+)	66	33	33	16	9	4	2	1	0	1	0	0	0
(-)	61	33	33	17	9	4	2	0	1	0	0	0	0
Average				St. Dv.		Energy			Power		Correlation		
	0			1		111			1		111		

End of PDSP F 4 Iteration # 1
Signal processing using PDSP F 4 Iteration # 2
Output bit energy 111.0686 Other energy 323.8286 SNRo -4.647239
Processed 0 bursts. Sample PD = .8745561 Sample PG = 21.29613
Sample Crb = -151.2583 Sample Cdb = 111.0667 Sample Cfb = 66.62285
Sample Cd = .8745407 Sample Cg = 29.98015
End of PDSP F 4 Iteration # 2

APPENDIX D

INVARIANT ALGORITHMS DEMONSTRATION

WAVEFORM RESULTS

In this Appendix, we provide sample waveforms which resulted from the run in Appendix C. The parameters pertaining to each waveform are also provided in Appendix C.

First we generate a sample PN waveform $b(t)$ shown in Figure D-1 to which we add an interference waveform shown in Figure D-2. Finally, we add the zero-mean, $\sigma = 1$ Gaussian noise shown in Figure D-3 to form the received input waveform $r(t) = b(t) + i(t) + n(t)$ shown in Figure D-4. $r(t)$ is then processed by each of the four PDSPs F1 through F4. The PDSP F1 output when interference is continuous is provided in Figure D-5. Note the significant processing distortion. In Figure D-6, the PDSP F1 output shows the erasure caused by a bursty interference turned on during chip intervals 16-64. Regenerating the off-tone interference for the duration of the entire PN period, Figures D-7, D-8 and D-9 show the iterated waveform output of PDSP F2 for two iterations and Figure D-10 shows the final resulting signal distortion due to PDSP F2. With PDSP F3 and F4, no iterations are required and Figures D-11 and D-13 show the resulting PDSP output. The corresponding PN waveforms distorted by the PDSPs are shown in Figures D-12 and D-14.

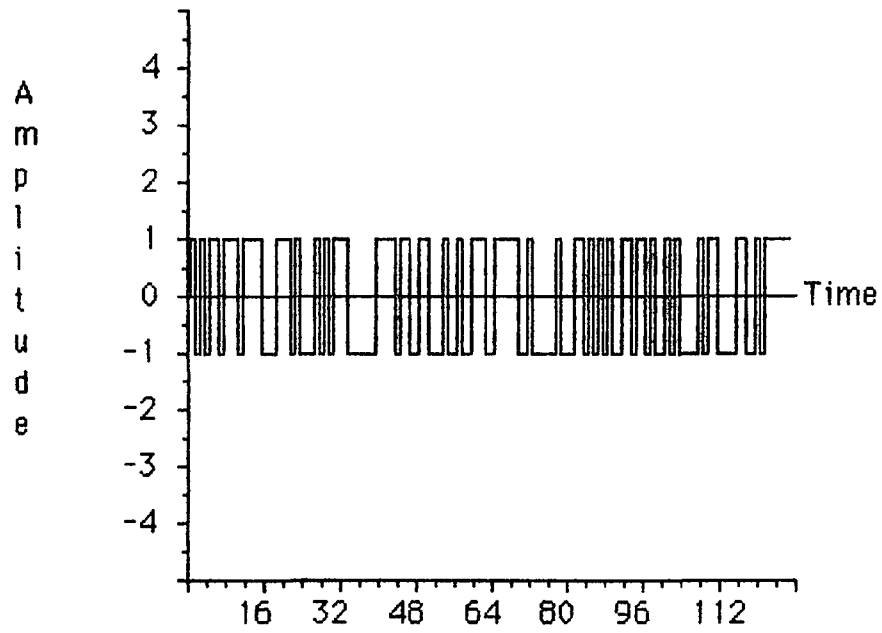


Figure D-1. PN-Coded Waveform at Input to PDSPs

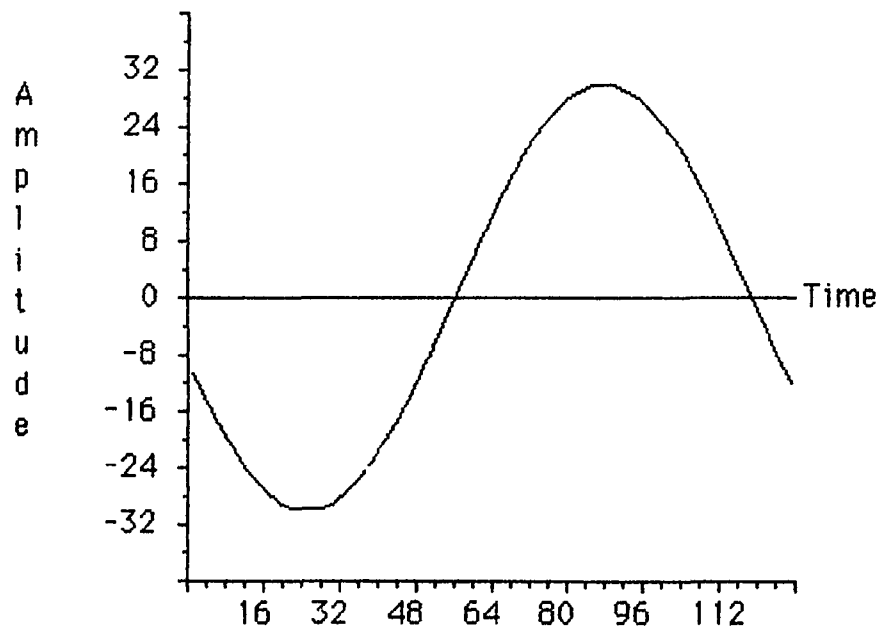


Figure D-2. Relevant Interference at Input of PDSPs

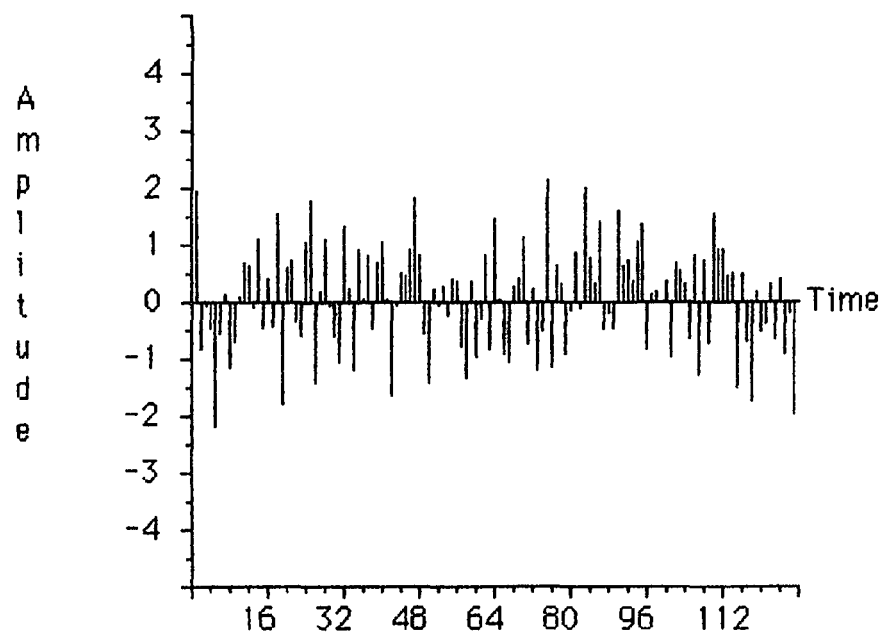


Figure D-3. Relevant Noise at Input of PDSPs

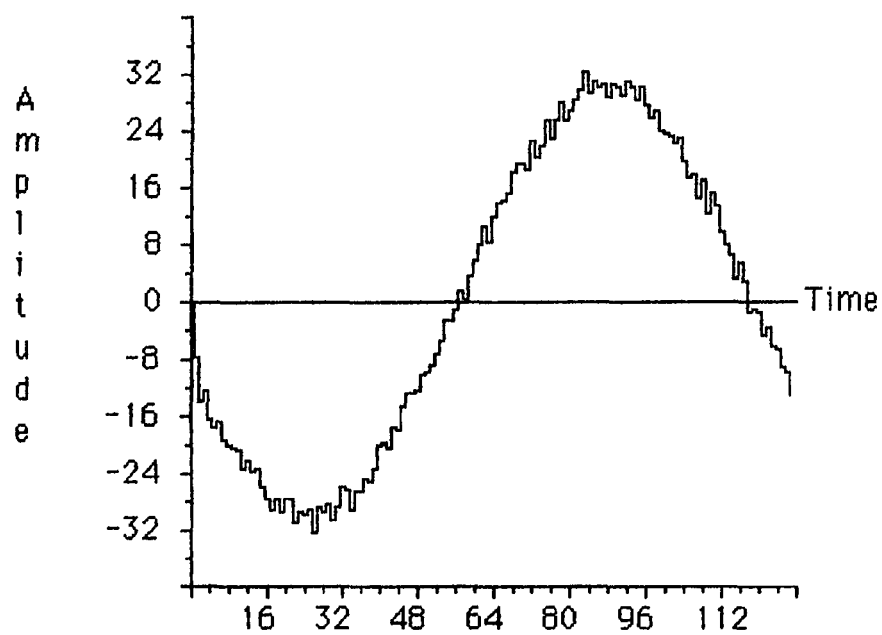


Figure D-4. Received Signal at Input of PDSPs

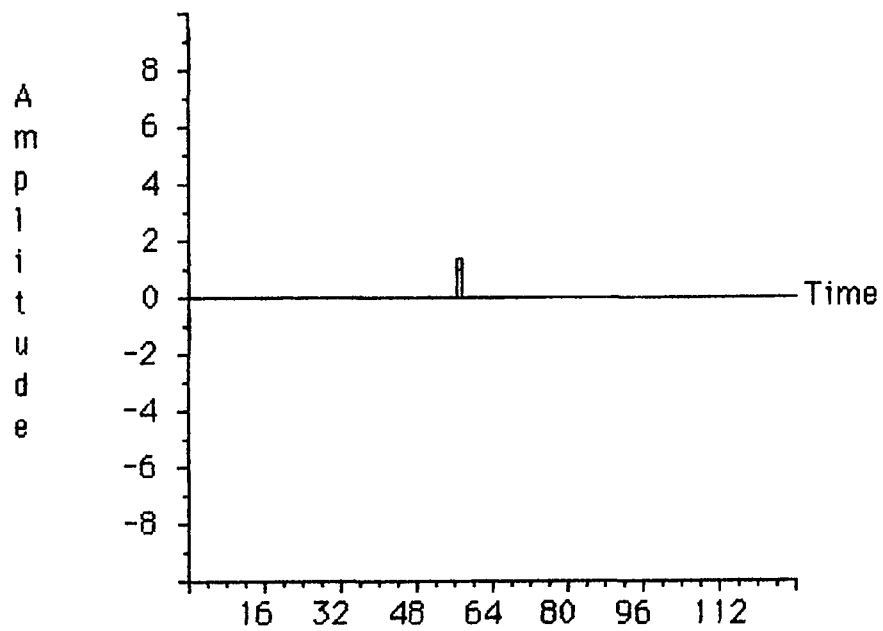


Figure D-5. Received and Distorted PN Waveforms at Output of PDSP F1

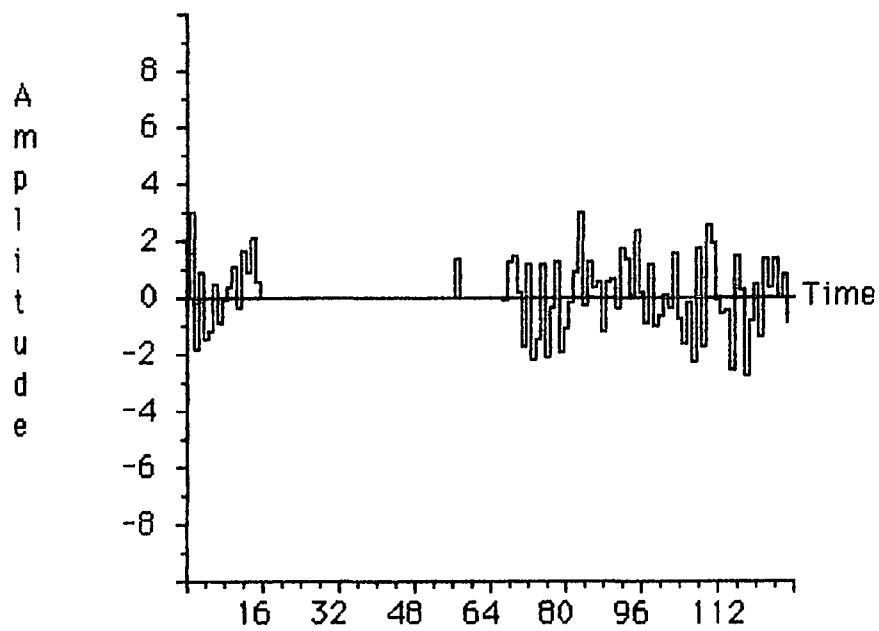


Figure D-6. Received Waveform at Output of PDSP F1 after Iteration #1

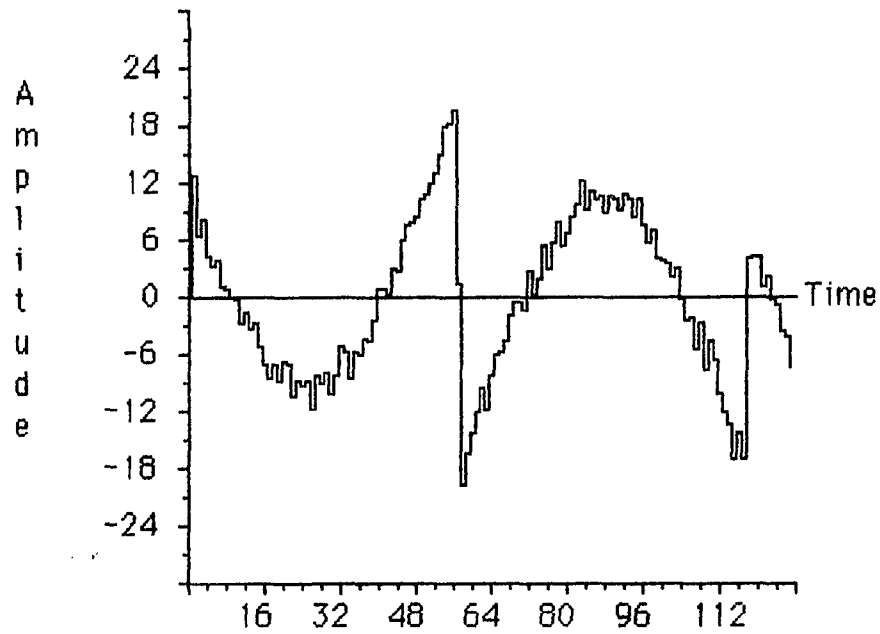


Figure D-7. Received Waveform at Output of PDSP F2 after Iteration #1

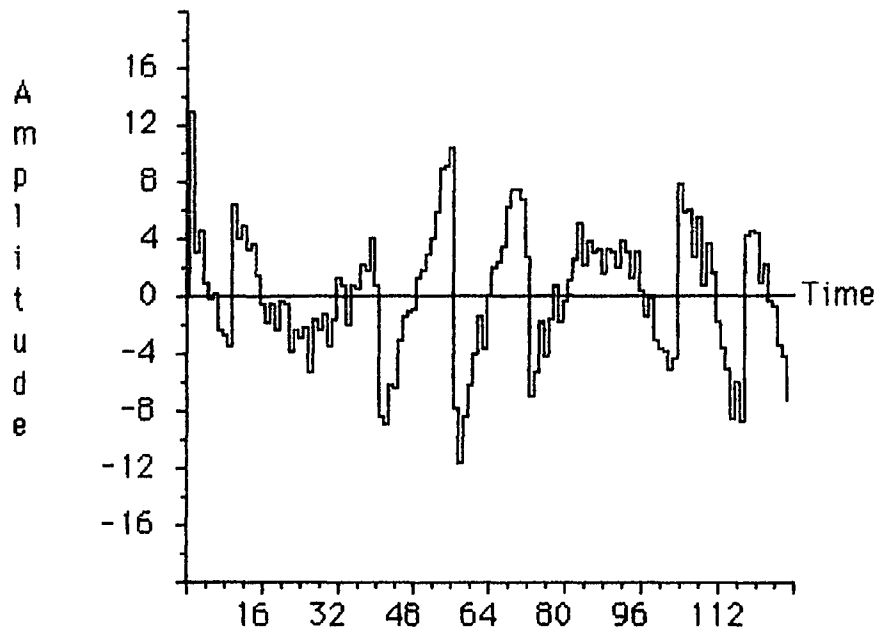


Figure D-8. Received Waveform at Output of PDSP F2 after Iteration #2

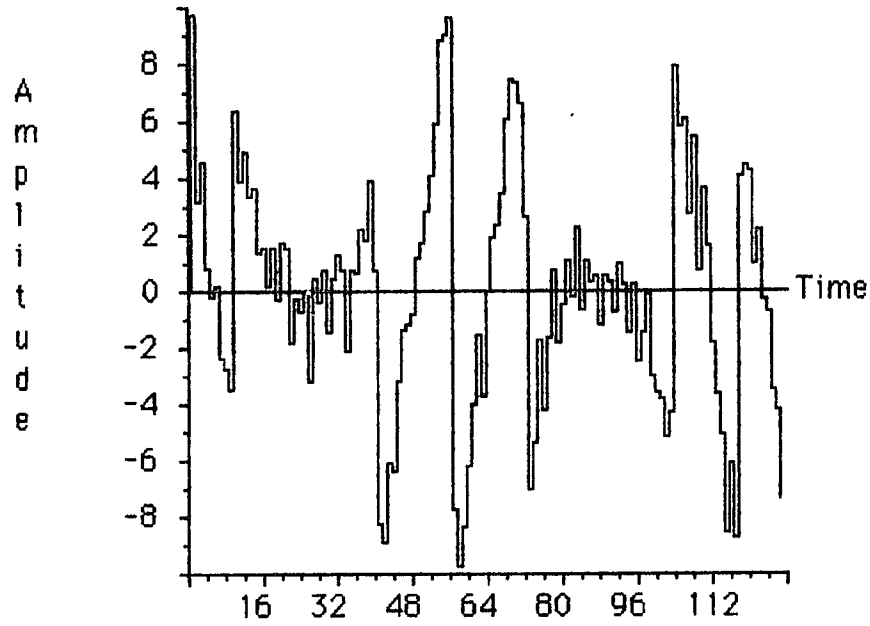


Figure D-9. Received Waveform at Output of PDSP F2 after Iteration #3

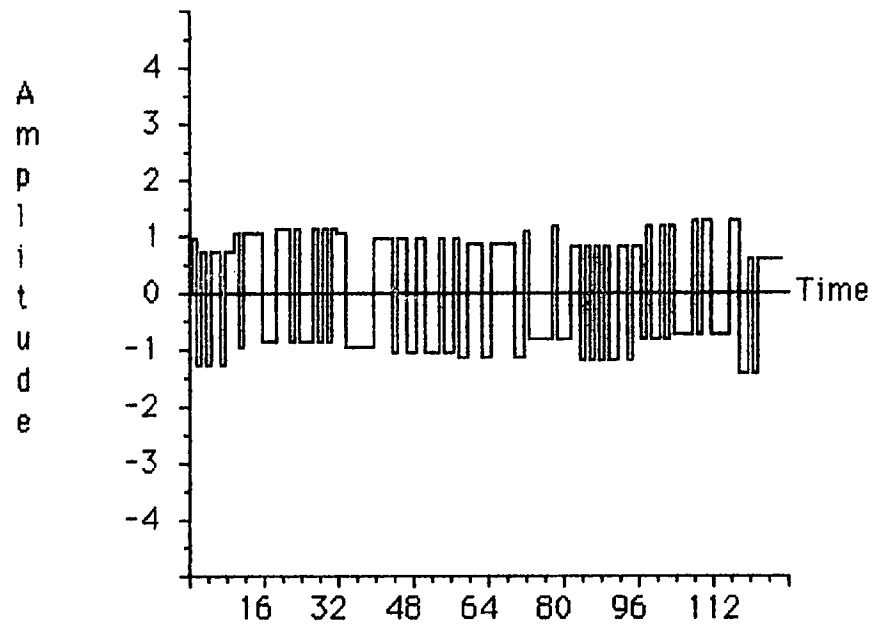


Figure D-10. Distorted PN Signal at Output of PDSP F2

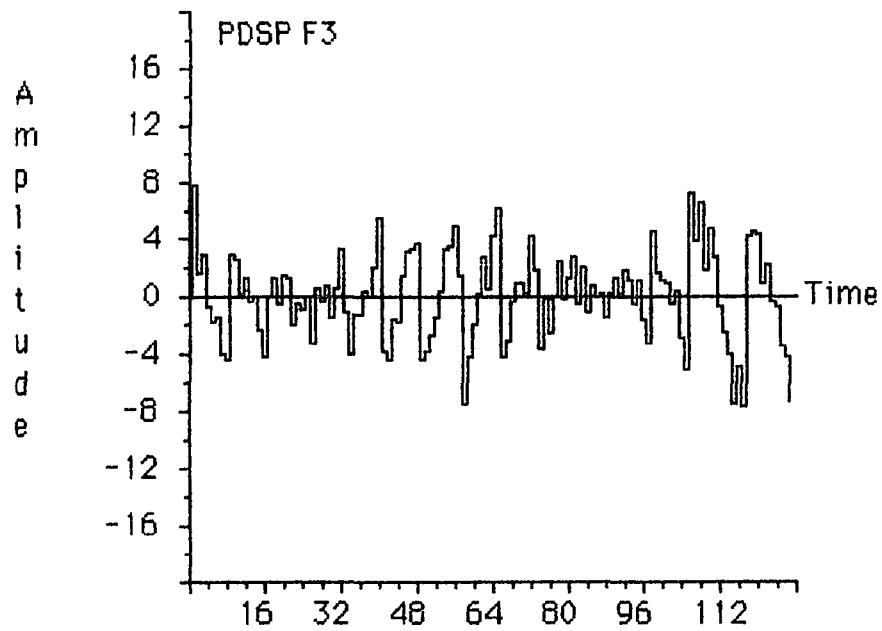


Figure D-11. Received Waveform at Output of PDSP F3 after Iteration #1

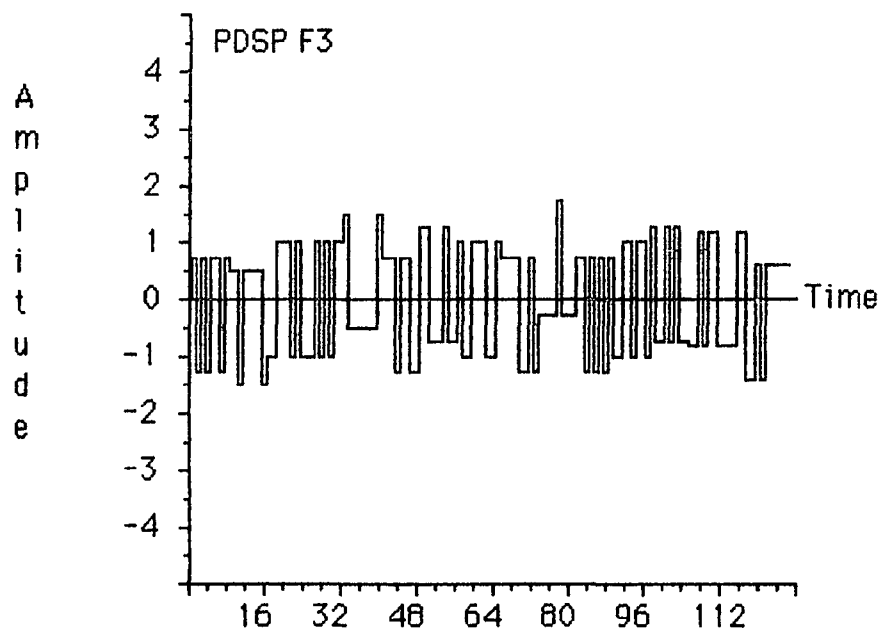


Figure D-12. Distorted PN Signal at Output of PDSP F3

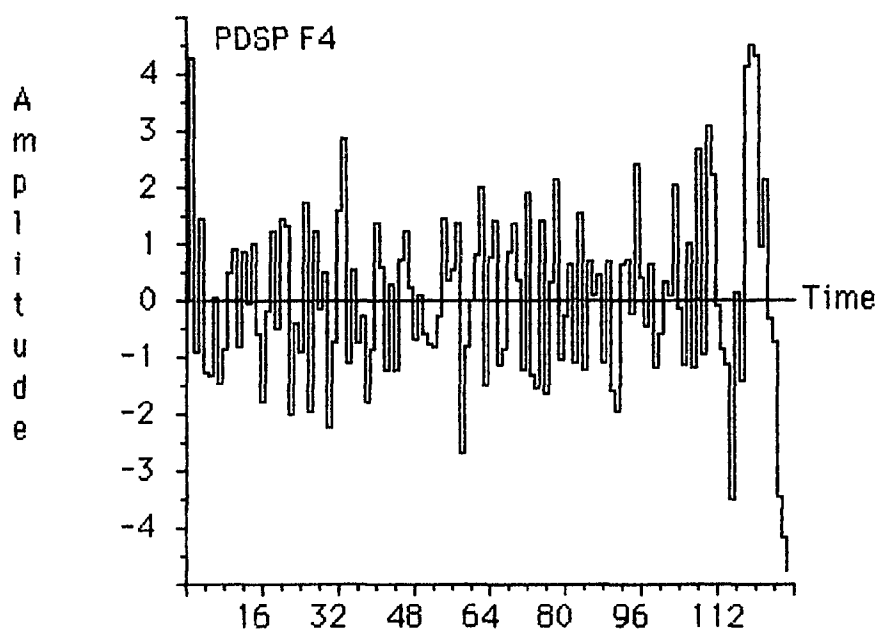


Figure D-13. Received Waveform at Output of PDSP F4 after Iteration #1

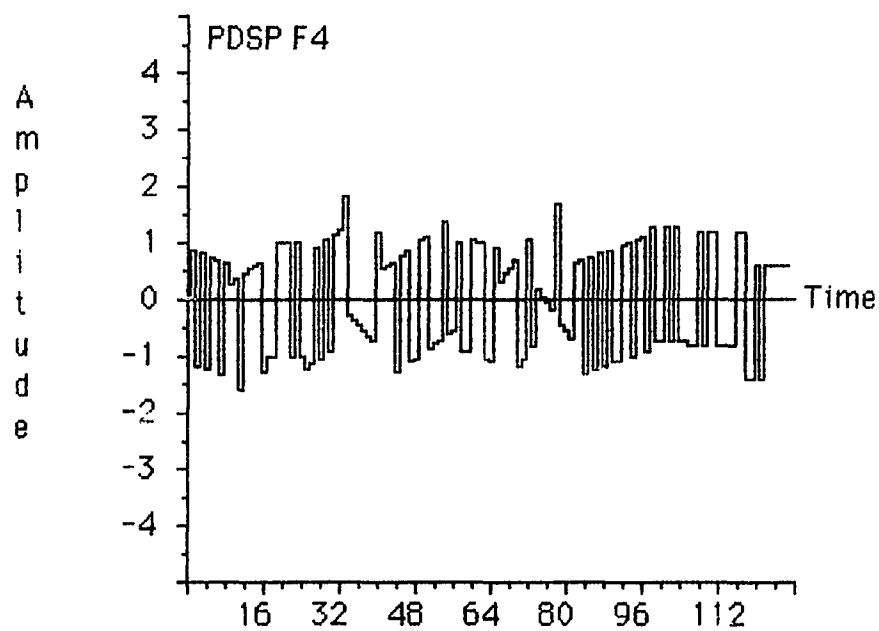


Figure D-14. Distorted PN Signal at Output of PDSP F4

REFERENCES

1. Engel, J.S., Digital Transmission in the Presence of Impulsive Noise , University Microfilms, Inc., Ann Arbor, Michigan, Ph.D., Dissertation, Polytechnic Institute of Brooklyn, 1964.
2. Pickholtz, R.L., Schilling, D.L., Millstein, L.B., "Theory of Spread-Spectrum Communications --A Tutorial", IEEE Transactions on Communications , Vol. COM-30, No. 5, May 1982, pp. 855-884.
3. Sholtz, R. A., "The Origin of Spread-Spectrum Communications", IEEE Transactions on Communications , Vol. COM-30, No. 5, May 1982, pp. 822-854.
4. Holmes, J. K. , Coherent Spread Spectrum Systems, Wiley-Interscience, New York, 1982.
5. Schilling, D.L, Millstein, L.B., Pickholtz, R.L., and Brown, R.W., "Optimization of the Processing Gain of an M-ary Direct Sequence Spread-Spectrum Communication System", IEEE Transactions on Communications , Vol. COM-28, No.8, August 1980, pp. 1389-1398.
6. Levitt, B.K., "Effect of Modulation Format and Jamming Spectrum on Performance of Direct Sequence Spread-Spectrum Systems", IEEE National Telecommunications Conference Record , Vol. 1, 1980, p.3.4.1.

7. Viterbi, A.J., "Spread Spectrum Communications--Myths and Realities", IEEE Communications Magazine, Vol. 17, No.3, May 1979, pp. 11-18.
8. Viterbi, A.J., "When Not to Spread Spectrum--a Sequel", IEEE Communications Magazine, Vol. 23, No. 4, April, 1985, pp. 12-17.
9. Ketchum, J.W., Proakis, J. G., "Adaptive Algorithms for Estimating and Suppressing Narrow-Band Interference in PN Spread-Spectrum Systems", IEEE Transaction on Communications , Vol. COM-30, No. 5, May 1982, pp. 913-924.
10. Li, L.M., Milstein, L.B., "Rejection of Narrow-Band Interference in PN Spread-Spectrum Systems Using Transversal Filters", IEEE Transactions on Communication , Vol. COM-30, No.5, May 1982, pp. 925-928.
11. Matsumoto, M., Cooper, G.R., "Performance of a Nonlinear FH-DPSK Spread Spectrum Receiver with Multiple Narrow-Band Interfering Signals", IEEE Transactions on Communications , Vol. COM-30, No. 5, May 1982, pp. 937-942.
12. Iltis, R.A., Milstein, L.B., "Performance Analysis of Narrow-Band Interference Rejection Techniques in DS Spread-Spectrum Systems", IEEE Transactions on Communications , Vol COM-32, No. 11, November 1984, pp. 1169-77.

13. Saulnier, G.J., Das, P., Milstein, L.B., "Suppression of Narrow-Band Interference in a PN Spread-Spectrum Receiver Using a CTD-Based Adaptive Filter", IEEE Transactions on Communications , Vol. COM-32, No. 11, November 1984, pp. 1227-32.
14. Sasaoka, H., Yoshimoto, S., Hamamoto, N., "Spread Spectrum Systems with Reduction Techniques of Co-Channel Interference", Rev. Radio Res. Lab., Vol. 29, No. 153, December 1983, pp. 539-550.
15. Masry, E., "Closed-Form Analytical Results for the Rejection of Narrow-Band Interference in PN Spread-Spectrum Systems. I. Linear Prediction Filters", IEEE Transactions on Communications , Vol. COM-32, No. 8, August 1984, pp. 888-896.
16. Panatik, C.M., Toplicar, J.R., "Adaptive Interference Suppression Using SAW Hybrid Programmable Transversal Filters", Ultrasonics Symposium Proceedings , Vol.1, November 1983, pp. 170-174.
17. Amoroso, F., "Adaptive A/D Converter to Suppress CW Interference in DSPN Spread-Spectrum Communications", IEEE Transactions on Communications, Vol.COM-31, No. 10, October 1983, pp. 1117-23.

18. Sulnier, G.J., Das, P., Iltis, R.A., Milstein, L.B., "A CCD Implemented Adaptive Filter for Estimation and Suppression of Narrowband Interference in a PNB Spectrum Receiver", IEEE Military Communications Conference , Vol. 3, November 1983, pp. 695-9.
19. Iltis, R.A., Milstein, L.B., "A Performance Analysis of Maximum Likelihood and Least Squares Estimation Receivers Designed for Narrowband Interference Rejection", IEEE Global Telecommunications Conference , Vol 3, December 1983, pp. 1304-8.
20. Saulnier, G.J., Das, P., "Suppression of Narrowband Interference in a P-N Spread Spectrum Receiver Using a CCD Based LMS Adaptive Filter", IEEE Global Telecommunications Conference, Vol. 3, December 1983, pp. 1289-93.
21. Iltis, R.A., Milstein, L.B., "Estimation Techniques for Narrowband Interference Rejection", IEEE International Conference on Communications: Integrating Communication for World Progress, Vol.3, June 1983, pp. 1582-6.
22. Bar-Ness, Y., Haber, F., "Self-Correcting Interference Cancelling Processor for Point-to-Point Communications", 24th Midwest Symposium on Circuits and Systems , June 1981, pp. 663-6.

23. Shepard, M.M., Gutman, L.L., "Adaptive Interference Suppression Filter for Spread Spectrum Signals", IEEE National Aerospace and Electronics Conference , Vol.1, May 1983, pp. 579-85.
24. Toplicar, J.R., "Adaptive Interference Suppression Using SAW Programmable Transversal Filters", IEEE National Aerospace and Electronics Conference, Vol. 1, May 1983, pp. 513-519.
25. Milstein, L.B., Das, P.K., "An Analysis of a real-Time Transform Domain Filtering Digital Communication System. II. Wide-Band Interference Rejection", IEEE Transactions on Communications , Vol. COM-31, No. 1, January 1983, pp.21-7.
26. Ketchum, J.W., Proakis, J.G., Anderson, P.H., Hsu, F.M., "Receiver Processing Techniques for PN Spread Spectrum Signals Transmitted through a Time-Dispersive Channel", IEEE Military Communications Conference, Progress in Spread Spectrum Communications , Vol. 2, October 1982, Chap. 29.4/pp.1-6.
27. Tou, C.P., Chang, B.C., "Interference and Noise Suppressions Using Spread Spectrum Systems", IEEE International Symposium on Electromagnetic Compatibility , September 1982, pp. 16-21.

27. Milstein, L.B., Das, P.K., Gevargiz, J., "Processing Gain Advantage of Transform Domain Filtering DS-Spread Spectrum Systems", IEEE Military Communications Conference. Progress in Spread Spectrum Communications, Vol. 1, October 1982, Chap. 21.2/pp.1-4.
28. Milstein, L.B., Das, P.K., "A SAW Implemented Wideband Interference Rejection Spread Spectrum System", IEEE International Conference on Communications. The Digital Revolution, Vol. 3, June 1982, Chap. 7E.5/pp.1-5.
29. Loh-Ming Li, Milstein, L.B., IEEE Transactions on Communications , Vol. 30, No. 5, May 1982, pp. 925-8.
30. Ketchum, J.W., Proakis, J.G., "Adaptive Algorithms for Estimating and Suppressing Narrow-Band Interference in PN Spread-Spectrum Systems", IEEE Transactions on Communications., Vol. 30, No. 5, Pt. 1, May 1982, pp. 913-24.
31. Loh-Ming Li, Milstein, L.B., "The Use of Adaptive Filters for Narrowband Interference Rejection", IEEE National Telecommunications Conference. Innovative Telecommunications-Key to the Future. Vol. 1., December 1981, Chap BB.4/pp.1-4.

32. Proakis, J.G., Ketchum, J.W., "Narrowband Interference Suppressions in Pseudo-Noise Spread Spectrum Systems", IEEE International Symposium on Information Theory, February 1981, No. 41.
33. Scholtz, R.A., "Centered CW Interference Rejection Using Spread Spectrum Techniques (Satellite Communications)", International Conference on Communications, June 1980, Chap. 53.6, pp.1-3.
34. Milstein, L.B., Das, P.K., Arsenault, D.R., "Narrowband Jammer Suppression in Spread Spectrum System Using Saw Devices", IEEE National Telecommunications Conference Part III, 43.2, pp.1-5, December 1978.
35. Grant, P.M., Kino, G.S., "Adaptive Filter Based on Saw Monolithic Storage Correlator", Electronics Letters, Vol. 14, No. 176, August 1978, pp. 562-64.
36. Bouvier, M.J. Jr., "The Rejection of Large CW Interferers in Spread Spectrum Systems", IEEE Transactions on Communications, Vol. COM-26, No.2, February 1978, pp. 254-6.
37. Hsu, F.M., Giordano, A., "Digital Whitening Techniques for Improving Spread Spectrum Communications Performance in the Presence of Narrow-Band Jamming and Interference", IEEE International Symposium on Information Theory, October 1977.

38. Baer, H.P., "Interference Effects of Hard Limiting in PN Spread-Spectrum Systems", IEEE Transactions on Communications , Vol. COM-30, No. 5, May 1982, pp.1010-1017.
39. Wozencraft, J. M. , Jacobs, I. M., Principles of Communication Engineering , Wiley-Interscience, New york, 1965.
40. Haykin, S., Communication Systems , John Wiley and Sons, New York, 1978.
41. Singh, R., "Performance of a Direct Sequence Spread Spectrum System with with Long Period and Short Period Code Sequences", IEEE International Communications Conference , Vol. 3, 1981, pp. 45.2.5-45.2.4.
42. Dixon, R. C., Spread Spectrum Systems , John Wiley and Sons, New York, 1976.
43. Rohatgi, V.K., Statistical Inference , John Wiley and Sons, New York, 1984.
44. Golomb, S. W. , Shift Register Sequences, Holden-Day, San Francisco, 1967.
45. Golay, M.J., Complementary Series, IRE Transactions on Information Theory, Vol. IT-7, April, 1961.
46. Gold, R., "Optimal Binary Sequences for Spread Spectrum Multiplexing", IEEE Transactions on Information Theory, Vol. IT-13, 1967, pp.619-621.

47. Welty, G. R., "Quaternary Codes for Pulsed Radar", IRE Transactions on Information Theory, June 1960, pp. 400-408.
48. Van Trees, H.L., Detection, Estimation, and Modulation Theory Part 1, John Wiley & Sons, New York, 1968.
49. Melsa, J. L., and Cohn, D. L., Decision and Estimation Theory, McGraw-Hill Book Company, New York, 1978.
50. Pawula, R. F., and Mathis, R. F., "A Spread Spectrum System with Frequency Hopping and Sequentially Balanced Modulation--Part II: Operation in Jamming and Multipath", IEEE Transactions on Communications, Vol. COM-28, No.10, October, 1980.
51. Abramowitz, M., and Stegun, I. A., Handbook of Mathematical Functions, U.S. Govt. Printing Office, Washington, D.C., June 1964.
52. Papoulis, A., Probability, Random Variables and Stochastic Processes, McGraw-Hill, New York, 1965.