ABSTRACT

Title of Thesis: Stability of Bicomponent Polymeric Liquids in Poiseuille Flow

Chin-Chang Jeng, Master of Science, 1984

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Polymer processing, involving two or more components, has become more popular recently in industrial plants. However, product quality is affected very much by the fluid stability. Some theoretical and experimental results concerning Newtonian and non-Newtonian flow in rectangular coordinates, have been published, and the fluid-fluid interfaces were observed to be unstable by some researchers.

In this paper, the linear stability of bicomponent non-Newtonian fluids flowing in a cylindrical tube was investigated by using the Ellis model. Only the very long wave and the axisymmetric disturbances were considered. The Ellis zero-shear-rate viscosity ratio, m($=\eta_{02}/\eta_{01}$), was found to be destabilizing. The half-zero-shear-rate-viscosity stress ratio, γ ($=\tau_{01}/\tau_{02}$), was shown to have a stabilizing effect. The power factor, α_1 and α_2 , have monotonous destabilizing effects. Surface tension, in general, will play a stabilizing role at the fluid-fluid interfaces.

STABILITY OF BICOMPONENT POLYMERIC LIQUIDS IN POISEUILLE FLOW

By

CHIN-CHANG JENG

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IN POISEUILLE FLOW

By

CHIN-CHANG JENG

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NOMENCLATURE

a	Position of interface (dimensionless)
b	density ratio
c	wave velocity
Fr	Froude number
m	zero-shear-rate viscosity ratio
R_e	Reynolds number
V_i	interfacial velocity
We	Weber number
α	wave number
γ	shear stress ratio
σ	interfacial tension
η	viscosity
τ	time (dimensionless)
δ	deviation of the interface
ρ	density

(I) Introduction

Polymer processing involving two or more different polymers has become the subject of considerable interest in recent years. Examples of such flow are numerous. In plastic processing the conbination of two melt streams in coextrusion process has become a very economical method of producing materials with unique properties which can not be achieved by using the individual polymer alone. In pratical problems, scientists made a lot effort trying to optimize the products by using compositive materials instead of simple component system. In polymer processing, involving two or more components, fluid-fluid interface has been observed to be unstable and some theoretical and experimental results have also been published, though in much less details than those for Newtonian fluids.

By using a hydrodynamic stability analysis, Yih (1) has found that for simple plane couette flow, viscosity stratification alone is sufficient to cause instability no matter how small Reynolds number is. KHAN and HAN [2,3], by studying stratified two-phase poiseuille flow between two parallel plates, pointed out that viscosity ratio and elasticity ratio of two super imposed fluids are important in determining the occurrence of interfacial instability, with the viscosity ratio predominant over the elasticity ratio. Schrenk and Bradley [4] confirmed that a wavelike distortion of the interface could arise under certain coextrusion conditions, implying the onset of instability. Li [5] has found that the presence of elasticity can not only destabilize simple flows but stabilize them for certain values of the parameters involved. Waters [6] studies two power-law fluids in plane couette flow and pointed out that the ratios of the power-law parameters for each layer can stabilize and destabilize the flow.

In 1971, HICKOX [7] studied the stability of a steady, axisymmetric, laminar, primary flow composed of two newtonian fluids flowing concentrically in a straight circular tube by using the method of small perturbations. He demonstrated that, regardless of the size of the Reynolds number, no situations are encountered for which the primary flow is stable to the asymmetric and axisymmetric disturbances, simultaneously. The primary cause of instability is found to be the difference in viscosities of the two fluids.

None of these analyses (or experiments) considered the concentric flow of bicomponent polymer melts in a cylindrical tube. This process is frequently observed in industrial plants like fiber spinning, extrusion (pipes forming) or injection molding. One of the main problems arises in this process is that the flow could become unstable, resulting in a product with irregular interface.

The rheological models most often used by experimentalists which predicts a shear-dependent viscosity is the so-called " Ellis-model " liquid. In this paper, the flow of concentric bicomponent polymer melts in circular pipe will be investigated by using this model. Only viscosity stratifications will be concerned.

(II) Time Independent Flow

In this investigation, the stability of an axi-symmetric, non-newtonian flow composed of two fluids flowing concertrically in a straight circular tube is considered. The fluids have different densities and viscosities and are incompressible and nondiffusive. An interface between the two fluids exists at some prescribed radial distance from the axis of symmetry.

The fluids with the interface perturbed is illustrated by the sketch in Fig 2-1.

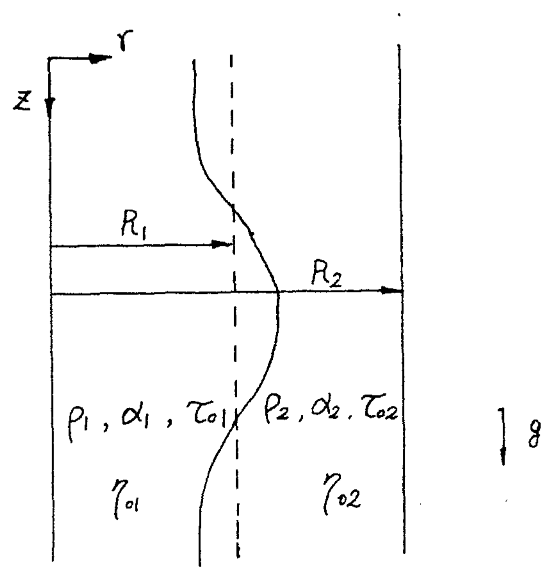


Fig 2-1: Definition Sketch

At steady state, the only nonzero velocity in the flow is the axial velocity, V_z , which is a function only of the radial position r. The flow system should satisfy the Cauchy's equation which will reduce to

$$\frac{\partial \bar{p}}{\partial r} = 0 \qquad (2-1)$$

$$\left[- \frac{\partial p}{\partial z} + \rho \cdot g \right]_{1,2} = \left[\frac{1}{r} \frac{\partial}{\partial r} (r \tau_{rz}) \right]_{1,2} \qquad (2-2)$$

The subscripts 1,2 refer to fluid 1 (inner) and fluid 2 (outer) respectively. If the left side of Eq (2-2) is kept constant and is represented by $(\Delta \bar{p})_{1,2}$, the solution of Eq (2-2) is

$$[\tau_{r,z} = \Delta \bar{p} \, r / 2 + c/r]_{1,2} \qquad (2-3)$$

If each fluid can be approximated by the Ellis model, then:

$$\frac{\eta_{0,i}}{\eta} = 1 + (\tau / \tau_{oi})^{\alpha_i - 1} \qquad i = 1,2 \quad (2-4)$$

Application of Eq (2-3) and Eq (2-4) to the inner and outer fluid regions seperately along with the reguirements of zero velocity on the rigid boundary, finite velocity and shear stress on the axis of symmetry, and continuity of velocity and shear stress across the interface will provide the complete solutions which are listed in Table 2-1.

The nondimensional forms for Table 2-1 can be derived by using the characteristic units as following:

length : R_1

density : ρ_1

time : $t^* = R_1 / V_i$

velocity : $V^* = V_i$

viscosity : η_{o1}

The results are shown in Table 2-2. The details of derivation were listed on Appendix I.

Table 2-1 : Steady state solution for two fluids

Fluid 1	Fluid 2
$\bar{P}_1 = (\rho_1 g - \Delta\bar{P}_1)\cdot z$	$\bar{P}_2 = (\rho_2 g - \Delta\bar{P}_2) z - \frac{\sigma}{R_1}$
$\bar{V}_{1z} = \frac{1}{\eta_{01}}\left[\frac{\Delta\bar{P}_1 R_1^2}{4}\left(1-\left(\frac{r}{R_1}\right)^2\right) + \frac{\tau_0 R_1}{\alpha_1+1}\cdot \left(\frac{\Delta\bar{P}_1 R}{2\tau_{01}}\right)^{\alpha_1}\cdot\left(1-\left(\frac{r}{R}\right)^{\alpha_1+1}\right)\right] + V_i$	$\bar{V}_{2z} = \frac{1}{\eta_{02}}\left[\frac{\Delta\bar{P}_2 R_2^2}{4}\left(1-\left(\frac{r}{R_2}\right)^2\right) - C_2 \ln\left(\frac{r}{R_2}\right)\right] + \frac{1}{\tau_{02}^{\alpha_2-1}}\int_r^{R_2}\left(\frac{\Delta\bar{P}_2}{2} r + \frac{C_2}{r}\right)^{\alpha_2} dr$
$\underset{\approx}{\bar{\tau}}_1 = \begin{pmatrix} 0 & 0 & \frac{\Delta\bar{P}_1}{2} r \\ 0 & 0 & 0 \\ \frac{\Delta\bar{P}_1}{2} r & 0 & 0 \end{pmatrix}$	$\underset{\approx}{\bar{\tau}}_2 = \begin{pmatrix} 0 & 0 & \frac{\Delta\bar{P}_2}{2} r + \frac{C_2}{r} \\ 0 & 0 & 0 \\ \frac{\Delta\bar{P}_2}{2} r + \frac{C_2}{r} & 0 & 0 \end{pmatrix}$
$\bar{\eta}_1 = \frac{\eta_{01}}{1 + \left(\lvert\tau_{1rz}\rvert/\tau_{01}\right)^{\alpha_1-1}} = \frac{\eta_{01}}{1 + \left(\frac{\Delta\bar{P}_1}{2\tau_{01}} r\right)^{\alpha_1-1}}$	$\bar{\eta}_2 = \frac{\eta_{02}}{1 + \left(\lvert\bar{\tau}_{2rz}\rvert/\tau_{02}\right)^{\alpha_2-1}} = \frac{\eta_{02}}{1 + \frac{1}{\tau_{02}^{\alpha_2-1}}\left(\frac{\Delta\bar{P}_2}{2} r + \frac{C_2}{r}\right)^{\alpha_2-1}}$
$\underset{\approx}{\bar{\Delta}}_1 = \begin{pmatrix} 0 & 0 & -\frac{\bar{\tau}_{1rz}}{\bar{\eta}_1} \\ 0 & 0 & 0 \\ -\frac{\bar{\tau}_{1rz}}{\bar{\eta}_1} & 0 & 0 \end{pmatrix}$	$\underset{\approx}{\bar{\Delta}}_2 = \begin{pmatrix} 0 & 0 & -\frac{\bar{\tau}_{2rz}}{\bar{\eta}_2} \\ 0 & 0 & 0 \\ -\frac{\bar{\tau}_{2rz}}{\bar{\eta}_2} & 0 & 0 \end{pmatrix}$

Table 2-2 : Nondimensional forms of steady state solutions

Fluid 1	Fluid 2
$\bar{P}_1 = \left(\frac{1}{Fr} - \beta_1\right)\cdot z$ $Fr = \frac{V_i^2}{g R_1}$; $\beta_1 = \frac{\Delta\bar{P}_1 \cdot R_1}{\rho_1 V_i^2}$	$\bar{P}_2 = \left(\frac{b}{Fr} - \beta_2\right) z - \frac{1}{We} = \bar{P}_1 - \frac{1}{We}$ $b = \frac{\rho_2}{\rho_1}$; $\beta_2 = \frac{\Delta\bar{P}_2 R_1}{\rho_1 V_i^2}$; $We = \frac{\rho_1 V_i^2 R_1}{\sigma}$
$\bar{v}_{1z} = \frac{\beta_1 \cdot Re}{4}\cdot\left[1 - r^2 + \frac{2D_1}{\alpha_1+1}\left(1 - r^{\alpha_1+1}\right)\right] + 1.$ $D_1 = \left(\frac{\Delta\bar{P}_1 \cdot R_1}{2\,\tau_{01}}\right)^{\alpha_1 - 1}.$	$\bar{v}_{2z} = \frac{\beta_2 Re}{4m}\cdot\left[(a^2 - r^2) - 2\hat{C}_2 \cdot \ln\left(\frac{r}{a}\right) + 2D_2\int_r^a \left(r + \frac{\hat{C}_2}{r}\right)^{\alpha_2}\cdot dr\right.$ $m = \frac{\eta_{02}}{\eta_{01}}$; $\hat{C}_2 = \frac{2C_2}{\Delta\bar{P}_2 R_1^2}$; $D_2 = \left(\frac{\Delta\bar{P}_2 R_1}{2\tau_{02}}\right)^{\alpha_2-1}$
$\bar{\underline{\underline{\tau}}}_1 = \begin{pmatrix} 0 & 0 & \frac{\beta_1}{2} r \\ 0 & 0 & 0 \\ \frac{\beta_1}{2} r & 0 & 0 \end{pmatrix}$	$\bar{\underline{\underline{\tau}}}_2 = \begin{pmatrix} 0 & 0 & \frac{\beta_2}{2}\left(r + \frac{\hat{C}_2}{r}\right) \\ 0 & 0 & 0 \\ \frac{\beta_2}{2}\left(r + \frac{\hat{C}_2}{r}\right) & 0 & 0 \end{pmatrix}$
$\bar{\eta}_1 = \frac{1}{1 + D_1 r^{\alpha_1 - 1}}$	$\bar{\eta}_2 = \frac{m}{1 + D_2\left(r + \frac{\hat{C}_2}{r}\right)^{\alpha_2 - 1}}$
$\bar{\underline{\underline{\Delta}}}_1 = \frac{Re}{\bar{\eta}_1}\begin{pmatrix} 0 & 0 & -\frac{\beta_1}{2} r \\ 0 & 0 & 0 \\ -\frac{\beta_1}{2} r & 0 & 0 \end{pmatrix}$	$\bar{\underline{\underline{\Delta}}}_2 = \frac{Re}{\bar{\eta}_2}\begin{pmatrix} 0 & 0 & \frac{-\beta_2}{2}\left(r + \frac{\hat{C}_2}{r}\right) \\ 0 & 0 & 0 \\ \frac{-\beta_2}{2}\left(r + \frac{\hat{C}_2}{r}\right) & 0 & 0 \end{pmatrix}$

(III) Differential System Governing Stability

The stability of the fluids described in the previous section is to be investigated through use of the method of small perturbations. This method which was rigorously formulated by Yih [1] was simple and straightforward. Following Yih [1], we seek solutions which have the forms

$$\psi = \psi_0 + \alpha \cdot \psi_1 + \alpha^2 \cdot \psi_2 + \ldots \ldots \ldots \ldots \ldots \ldots \qquad (3\text{-}1)$$

which is a non-singular perturbation around $\alpha = 0$ which corresponds to very long waves. "$\alpha \cdot R_e$" is assumed small compared with unity and, as pointed out by Yih [1], no matter how large R_e is, there is a range of α for which the perturbation procedure is valid.

The complete cauchy's equations for each fluid are

$$\frac{Dv_2}{Dt} = -\nabla \bar{p}_1 + \nabla \cdot \underset{\sim}{\tau_1} + g \qquad (3\text{-}2)$$

for fluid 1, and

$$\frac{Dv_2}{Dt} = -b \nabla \bar{p}_2 + b \nabla \cdot \underset{\sim}{\tau_2} + g \qquad (3\text{-}3)$$

for fluid 2.

The continuity equation is

$$\nabla \cdot v = 0 \qquad (3\text{-}4)$$

for both fluid.

It should be noted that Eq (3-2) & Eq (3-3) were written in nondimensional forms, where b is defined as the ratio of density (ρ_2 / ρ_1).

It is now assumed that the flow system is disturbed slightly so that the velocities and pressure and relevant non-zero stress consist of their steady state valued in the main flow plus a small perturbation. Thus, they can be expressed as

$$v_i = \bar{v}_i + v_i^* \qquad (3\text{-}5)$$

$$\tau_{ij} = \bar{\tau}_{ij} + \tau_{ij}^* \qquad i,j = r, \quad , z \qquad (3\text{-}5)$$

$$p_i = \bar{p}_i + p_i^* \qquad (3\text{-}7)$$

The barred quantities are steady values. The quantities with astericks represent perturbations to the steady state flow and are assumed to be small enough so that second or higher order product of these perturbed quantities are negligible. Remember that only the axial velocity and pressure have initial values different from zero. Thus, the shear stress tensor can be written as

$$\underset{\approx}{\tau}_1 = \bar{\underset{\approx}{\tau}}_1 + \underset{\approx}{\tau}_1^*$$

$$= \begin{pmatrix} \tau_{1rr}^* & \tau_{1r\theta}^* & \tau_{1rz}^* + \bar{\tau}_{rz} \\ \tau_{1r\theta}^* & \tau_{1\theta\theta}^* & \tau_{1\theta z}^* \\ \tau_{1rz}^* + \bar{\tau}_{rz} & \tau_{1\theta z}^* & \tau_{1zz}^* \end{pmatrix} \qquad (3\text{-}8)$$

for fluid 1. The corresponding second invariant is

$$II_{\underset{\approx}{\tau}_1} = \sum_i \sum_j \tau_{1ij}^2 = 2 (\bar{\tau}_{1rz} + \tau_{1rz}^*)^2 \quad (3-9)$$

The shear rate tensor can be expressed as

$$\underset{\approx}{\Delta}_1 = \bar{\underset{\approx}{\Delta}}_1 + \underset{\approx}{\Delta}_1^*$$

$$= - R_e \frac{\underset{\approx}{\tau}_1}{\eta_1}$$

$$= - R_e \left[1 + \left(\frac{ (\frac{1}{2} II_{\underset{\approx}{\tau}_1})^{1/2} }{ \tau_{01} / \rho_1 V_i^2 } \right)^{\alpha_1 - 1} \right] \underset{\approx}{\tau}_1$$

$$= - R_e \left[1 + \left(\frac{ \bar{\tau}_{1rz} + \tau_{1rz}^* }{ \tau_{01} / \rho_1 V_i^2 } \right)^{\alpha_1 - 1} \right] \underset{\approx}{\tau}_1 \quad (3-10)$$

for fluid 1. Since $\bar{\tau}_{1rz} > 0$ for $0 \leq r \leq R_1$ and $|\tau_{1rz}^*| << \bar{\tau}_{1rz}$, the absolute sign could be taken off from $|\bar{\tau}_{1rz} + \tau_{1rz}^*|$. Thus,

$$\Delta_{1rz} = - R_e \left[1 + \left(\frac{ \bar{\tau}_{1rz} + \tau_{1rz}^* }{ \tau_{01} / \rho_1 V_i^2 } \right)^{\alpha_1 - 1} \right] \tau_{1rz}^* \quad (3-11)$$

$$= - \frac{R_e}{\bar{\eta}_1} \tau_{1rr}^*$$

Similarity[**] ,

$$\Delta_{1\theta\theta} = - R_e \cdot \tau_{1\theta\theta}^{\ *} \ / \ \bar{\eta}_1 \qquad (3\text{-}12)$$

$$\Delta_{1zz} = - R_e \cdot \tau_{1zz}^{\ *} \ / \ \bar{\eta}_1 \qquad (3\text{-}13)$$

$$\Delta_{1r\theta} = - R_e \cdot \tau_{1r\theta}^{\ *} \ / \ \bar{\eta}_1 \qquad (3\text{-}14)$$

$$\Delta_{1\theta z} = - R_e \cdot \tau_{1\theta z}^{\ *} \ / \ \bar{\eta}_1 \qquad (3\text{-}15)$$

$$\Delta_{1rz} = - R_e \left[\bar{\tau}_{1rz} + \frac{\tau_{1rz}^{\ \alpha_1}}{(\quad / \ V_i^2)} + \tau_{1rz}^{\ *} + \right.$$

$$\left. \alpha_1 \cdot \left(\frac{\tau_{1rz}}{\tau_{01} / \rho_1 V_i^2} \right)^{\alpha_1 - 1} \cdot \tau_{1rz}^{\ *} \right] \qquad (3\text{-}16)$$

Since,

$$\underset{\sim}{\Delta} = \begin{pmatrix} 2\dfrac{\partial v_r^*}{\partial r} & r\dfrac{\partial}{\partial r}\left(\dfrac{v_\theta^*}{r}\right) + \dfrac{1}{r}\dfrac{\partial v_r^*}{\partial \theta} & \dfrac{\partial(\bar{v}_z + v_z^*)}{\partial r} + \dfrac{\partial v_r^*}{\partial z} \\ r\dfrac{\partial}{\partial r}\left(\dfrac{v_\theta^*}{r}\right) + \dfrac{1}{r}\dfrac{\partial v_r^*}{\partial \theta} & 2\left(\dfrac{1}{r}\dfrac{\partial v_\theta^*}{\partial \theta} + \dfrac{v_r^*}{r}\right) & \dfrac{\partial v_\theta^*}{\partial z} + \dfrac{1}{r}\dfrac{\partial v_z^*}{\partial \theta} \\ \dfrac{\partial(\bar{v}_z + v_z^*)}{\partial r} + \dfrac{\partial v_r^*}{\partial z} & \dfrac{\partial v_\theta^*}{\partial z} + \dfrac{1}{r}\dfrac{\partial v_z^*}{\partial \theta} & 2\cdot\dfrac{\partial v_z^*}{\partial z} \end{pmatrix} \qquad (3\text{-}17)$$

** : Detail derivation in Appendix II.

Application of Eqs (3-11) - (3-17), we can rewrite the shear rate tensor as following:

$$\Delta_{1rr}^{*} = 2\,\partial v_{1r}^{*} / \partial r = -R_e \cdot \tau_{1rr}^{*} / \bar{\eta}_1 \qquad (3\text{-}18)$$

$$\Delta_{1\theta\theta}^{*} = 2\left(\frac{1}{r}\frac{\partial v_1^{*}}{\partial \theta} + v_{1r}^{*} / r \right) = -R_e \cdot \tau_{1\theta\theta}^{*} / \bar{\eta}_1 \qquad (3\text{-}19)$$

$$\Delta_{1zz}^{*} = 2 \cdot \frac{\partial}{\partial z}(\bar{v}_{1z} + v_{1z}^{*}) = 2\,\frac{\partial v_{1z}^{*}}{\partial z}$$

$$= -R_e \cdot \tau_{1zz}^{*} / \bar{\eta}_1 \qquad (3\text{-}20)$$

$$\Delta_{1\theta r}^{*} = \Delta_{1r\theta}^{*} = r\frac{\partial}{\partial r}\left(\frac{v_{1\theta}}{r} \right)^{*} + \frac{1}{r}\frac{\partial v_{1r}^{*}}{\partial \theta} = -R_e \cdot \tau_{1r\theta}^{*} / \bar{\eta}_1$$

$$(3\text{-}21)$$

$$\Delta_{1z\theta}^{*} = \Delta_{1\theta z}^{*} = \frac{\partial v_{1\theta}^{*}}{\partial z} + \frac{1}{r}\frac{\partial}{\partial \theta}(\bar{v}_{1z} + v_{1z}^{*})$$

$$= \frac{\partial v_{1\theta}^{*}}{\partial z} + \frac{1}{r}\frac{\partial v_{1z}^{*}}{\partial \theta} = -R_e \cdot \tau_{1\theta z}^{*} / \bar{\eta}_1 \qquad (3\text{-}22)$$

$$\Delta_{1rz}^{*} = \Delta_{1zr}^{*} = \frac{\partial v_{1z}^{*}}{\partial r} + \frac{\partial v_{1r}^{*}}{\partial z} = -R_e\,[\,1 + \alpha_1 D_1 r\,]\,\tau_{1rz}^{*}$$

$$= -\frac{R_e}{\mu}\,\tau_{1rz}^{*} \qquad (3\text{-}23)$$

Application of Eq (3-2), the r-component of the equation of motion is

$$\frac{\partial v_{1r}^{*}}{\partial r} + v_{1r}^{*}\frac{\partial v_{1r}^{*}}{\partial r} + \frac{v_{1\theta}^{*}}{r}\frac{\partial v_{1r}^{*}}{\partial \theta} - \frac{v_{1\theta}^{*2}}{r} + (\bar{v}_{1z} + v_{1z}^{*})\frac{\partial v_{1r}^{*}}{\partial z}$$

$$= -\frac{\partial}{\partial r}(\bar{P}_1 + P_1^{*}) - \left[\frac{1}{r}\frac{\partial}{\partial r}(r\tau_{1rr}^{*}) + \frac{1}{r}\frac{\partial \tau_{1r\theta}^{*}}{\partial \theta} - \frac{\tau_{1\theta\theta}^{*}}{r} + \frac{\partial \tau_{1rz}^{*}}{\partial z}\right]$$

Neglecting the terms whose perturbed power greater than two, we get

$$\frac{\partial v_{1r}^{*}}{\partial t} + v_{1z}\frac{\partial v_{1r}^{*}}{\partial z} = -\frac{\partial P_1^{*}}{\partial r} - \left[\frac{1}{r}\frac{\partial}{\partial r}(r\tau_{1rr}^{*}) + \frac{1}{r}\frac{\partial \tau_{1r\theta}^{*}}{\partial \theta} - \frac{\tau_{1\theta\theta}^{*}}{r} + \frac{\partial \tau_{1rz}^{*}}{\partial z}\right]$$

for r-component. Similarity to θ , z components and continuity equation:

θ - component

$$\frac{\partial v_{1\theta}^{*}}{\partial t} + \bar{v}_{1z}\frac{\partial v_{1\theta}^{*}}{\partial z} = -\frac{1}{r}\frac{\partial P_1^{*}}{\partial \theta} - \left[\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\tau_{1r\theta}^{*}) + \frac{1}{r}\frac{\partial \tau_{1\theta\theta}^{*}}{\partial \theta} + \frac{\partial \tau_{1\theta z}^{*}}{\partial z}\right]$$

z-component

$$\frac{\partial v_{1z}^{*}}{\partial t} + v_{1z}' \cdot v_{1r}^{*} + \bar{v}_{1z}\frac{\partial v_{1z}^{*}}{\partial z} = -\frac{\partial P_1^{*}}{\partial z} - \left[\frac{1}{r}\frac{\partial}{\partial r}(r\tau_{1rz}^{*}) + \frac{1}{r}\frac{\partial \tau_{1\theta z}^{*}}{\partial \theta} + \frac{\partial \tau_{1zz}^{*}}{\partial z}\right]$$

Continuity equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r v_{1r}^{*} \right) + \frac{1}{r}\frac{\partial v_{1\theta}^{*}}{\partial \theta} + \frac{\partial v_{1z}^{*}}{\partial z} = 0 \qquad (3\text{-}28)$$

for fluid 1. It should be note that the starry sign indicated the purturbed values and the barred mean the steady (primary) values.

Following the procedure of Batchelor and Gill [3], the perturbation terms for the fluid are assumed to have the forms

$$v_r^{*} = iG(r)\ \mathrm{EXP} \qquad (3\text{-}29)$$

$$v^{*} = H(r)\ \mathrm{EXP} \qquad (3\text{-}30)$$

$$v_z^{*} = F(r)\ \mathrm{EXP} \qquad (3\text{-}31)$$

and

$$p^{*} = p(r)\ \mathrm{EXP} \qquad (3\text{-}32)$$

Where $\mathrm{EXP} = \exp\left[\, in\theta + i\alpha\cdot(z - c\tau\cdot) \,\right]$ and G, H, F, P are nondimensional functions of r,

$$r = R/R_1 \quad ; \quad z = z/R_1 \quad ; \quad \tau = tV_i/R_1 \qquad (3\text{-}33)$$

and and c are the nondimensional wavenumber and speed repectively. The parameter can be zero or any integer value and is the means by which the angular dependence of the purturbation terms is expressed. The i is the imaginery number $(-1)^{\frac{1}{2}}$. In general, the wave speed c can be complex. It is the sign of its imaginary part which will ultinately determine the stability or instability of the flow. If the imaginary part of c is positive, the perturbation terms will grow exponentially with time and the flow is considered unstable.

Application of Eqs (3-18) - (3-23) and Eqs (3-29) - (3-32), the perturbed terms of shear stress tensor were determined, i.e.

$$\tau_{1rr}^{*} = -2\,\bar{\eta}_1 / R_e \cdot \partial v_{1r}^{*} / \partial r = -\,i2\cdot\bar{\eta}\,G_1' \; EXP/R_e \qquad (3\text{-}34)$$

$$\tau_{1\theta\theta}^{*} = -\,i2\cdot\bar{\eta}_1 \left(\frac{nH_1 + G_1}{r} \right) EXP/R_e \qquad (3\text{-}35)$$

$$\tau_{1zz}^{*} = -\,i\alpha\cdot 2\,\bar{\eta}_1 F_1 EXP/R_e \qquad (3\text{-}36)$$

$$\tau_{1r\theta}^{*} = -\frac{\bar{\eta}_1}{R_e} \left[H_1' - \frac{H_1}{r} - \frac{nG_1}{r} \right] EXP \qquad (3\text{-}37)$$

$$\tau_{1\theta z}^{*} = -\,i\cdot\frac{\bar{\eta}_1}{R_e} \left[\alpha H_1 + \frac{n}{r} F_1 \right] EXP \qquad (3\text{-}38)$$

$$\tau_{1rz}^{*} = -\frac{\mu_1}{R_e} \left[F_1' - \alpha\cdot G_1 \right] EXP \qquad (3\text{-}38)$$

Substituting Eqs (3-34) - (3-39) into Eqs (3-25) - (3-28), the governing equations for fluid 1 are readily written as following **

r-component

$$\alpha\cdot(\bar{v}_{1z} - c)G_1 = p_1' - \frac{i\cdot\bar{\eta}_1}{R_e}\Big[2G_1'' + 2\Big(\frac{r\bar{\eta}_1'}{\bar{\eta}_1} + 1\Big)\frac{G_1'}{r} - \Big(\frac{n^2+2}{r^2} + \alpha^2\cdot\frac{\mu_1}{\bar{\eta}_1}\Big)G_1 + n\Big(\frac{H_1'}{r} - 3\frac{H_1}{r^2}\Big) + \alpha\cdot\frac{\mu_1}{\bar{\eta}_1}F_1' \Big] \tag{3-40}$$

θ -component

$$\alpha(\bar{v}_{1z} - c)H_1 = -\frac{n}{r}p_1 - \frac{i\bar{\eta}_1}{R_e}\Big[H_1'' + \Big(\frac{r\bar{\eta}_1'}{\bar{\eta}_1} + 1\Big)\frac{H_1'}{r} - \Big(\frac{r\bar{\eta}_1'/\bar{\eta}_1 + 1 + 2n^2}{r^2} + \alpha^2\Big)H_1 - n\Big(\frac{G_1'}{r} + \Big(\frac{r\cdot\bar{\eta}_1'}{\bar{\eta}_1} + 3\Big)\frac{G_1}{r^2}\Big) - \alpha n\frac{F_1}{r} \Big] \tag{3-41}$$

** Refer to Appendix III for detail.

z-component

$$\alpha(\bar{v}_{1z} - c)F_1 + v'_{1z}\cdot G_1 = -\alpha p_1 - \frac{i\cdot\bar{\eta}_1}{R_e}\Big[\frac{\mu_1}{\bar{\eta}_1}F''_1 + \frac{\mu_1}{\bar{\eta}_1}\Big(1+\frac{r\mu'_1}{\mu_1}\Big)\frac{F'_1}{r} - \Big(\frac{n^2}{r^2}+2\alpha^2\Big)F_1 - \frac{\mu_1}{\bar{\eta}_1}\cdot\alpha\cdot\Big(G'_1 + \Big(1+\frac{r\cdot\mu'_1}{\mu_1}\Big)\frac{G_1}{r}\Big) - \frac{n\cdot\alpha}{r}H_1\Big] \qquad (3\text{-}42)$$

Continuity equation

$$G'_1 + \frac{G_1}{r} + \frac{n}{r}H_1 + \alpha\cdot F_1 = 0 \qquad (3\text{-}43)$$

Applying the same procedure to fluid 2, we can get the similar equation of motion and continuity equation as following **

r-component

$$b\alpha(\bar{v}_{2z} - c)G_2 = p'_2 - \frac{i\bar{\eta}_2}{R_e}\Big[2G''_2 + 2\Big(\frac{r\bar{\eta}'_2}{\bar{\eta}_2}+1\Big)\Big(\frac{G_2}{r}\Big)' - \Big(\frac{n^2+2}{r^2} + m\alpha^2\frac{\mu_2}{\bar{\eta}_2}\Big)G_2 + n\Big(\frac{H'_2}{r} - 3\frac{H_2}{r^2}\Big) + m\alpha\cdot\frac{\mu_2}{\bar{\eta}_2}F'_2\Big] \qquad (3\text{-}44)$$

** Refer to Appendix III for detail

θ -component

$$b\alpha\cdot(\bar{v}_{2z} - c)H_2 = -\frac{n}{r}p_2 - \frac{i\cdot\bar{\eta}_2}{R_e}\Big[H_2'' + \Big(\frac{r\bar{\eta}_2'}{\bar{\eta}_2} + 1\Big)\frac{H_2'}{r}$$

$$-\Big(\frac{r\cdot\bar{\eta}_2'/\bar{\eta}_2 + 1 + 2n^2}{r^2} + \alpha^2\Big)H_2$$

$$-n\Big(\frac{G_2'}{r} + \Big(\frac{r\cdot\bar{\eta}_2'}{\bar{\eta}_2} + 3\Big)\frac{G_2}{r^2}\Big) - \alpha n\frac{F_2}{r}$$

(3-45)

z-component

$$b[\alpha\cdot(\bar{v}_{2z} - c)F_2 + v_{2z}'G_2] = -\alpha p_2 - \frac{i\cdot\bar{\eta}_2}{R_e}\Big[\frac{m\mu_2}{\bar{\eta}_2}F_2''$$

$$+\frac{m\mu_2}{\bar{\eta}_2}\Big(1 + \frac{r\cdot\mu_2'}{\mu_2}\Big)\frac{F_2'}{r}$$

$$-\Big(\frac{n^2}{r^2} + 2\cdot\alpha^2\Big)F_2 - \frac{m\mu_2}{\bar{\eta}_2}\alpha\cdot$$

$$\Big(G_2' + \Big(1 + \frac{r\mu_2'}{\mu_2}\Big)\frac{G_2}{r}$$

$$-\frac{n\alpha}{r}H_2\Big] \qquad (3\text{-}45)$$

Continuity equation

$$G_2' + \frac{G_2}{r} + \frac{n}{r}H_2 + \alpha\cdot F_2 = 0 \qquad (3\text{-}46)$$

Thus, except for the factor b and m, these equations of motion for fluid 2 have the same forms as those for fluid 1. The simultaneous solutions of these equations together with the appropriate boundary and interfacial conditions will provide information from which the instability of the bicomponent non-newtonian flow can be inferred.

(IV) Bounday and Interfacial Conditions

The boundary conditions expressing finiteness of velocity along the axis of symmetry and no slip at the rigid surface are

$$G_1(0), \quad H_1(0), \quad F_1(0) \qquad \text{----} \quad \text{Finite} \qquad (4\text{-}1)$$

and

$$G_2(a) = H_2(a) = F_2(a) = 0 \qquad (4\text{-}2)$$

Where $a = R_2/R_1$.

The interfacial conditions require continuity of velocities, shear stress and normal stress. These conditions must be evaluated carefully, becasuse, strictly speaking, they are to be applied at the interface of the disturbed flow, $r = 1 + \delta$, and not at the original interface, $r = 1$.

Because of the periodic disturbance, we can assume a wavy form described by the equation

$$r = 1 + \delta = 1 + \delta_0 \cdot \exp[\, in \cdot \theta + i\alpha(\, z - c\tau\,)] \qquad (4\text{-}3)$$

where δ_0 is the amplitude of the fluctuation of the interface from its mean position at $r = 1$ and is an infinitesimal quantity to be determined by the interface conditions. Thus, the substantial derivative of δ with respect to time must be equal to the radial component of the perturbed velocity, i.e.

$$\left(\frac{D\delta}{Dt} \right)_{r=1+\delta} = v_r^* = iG(1+\delta)EXP \tag{4-4}$$

rearrange above equation, we can find

$$\left(\frac{\partial}{\partial t} + v_{1z}\frac{\partial}{\partial z} \right)\cdot\delta = v_r^* = iG(1+\delta)EXP \tag{4-5}$$

or

$$-i\alpha c\delta + i\alpha(\bar{v}_{1z})_{r=1+\delta}\cdot\delta = iG(1+\delta)EXP \tag{4-6}$$

recalling that v_{1z} is equal to v_{2z} at the interface. Expanding Eq (4-6) in Taylor series around r=1

$$-i\alpha c\delta + i\alpha\bar{v}_{1z}(1)\cdot\delta + i\alpha(\bar{v}_{1z}')_{r=1}\cdot\delta^2$$

$$= i\left[G(1) + G'(1)\cdot\delta \right] EXP \tag{4-7}$$

and neglecting terms above second order in infinitesimal quantities, we have

$$\delta = \frac{G(1)}{\alpha\cdot[\bar{v}_{1z}(1) - c]} \quad EXP \tag{4-8}$$

Continuity of v_r across the interface requires that

$$v_{1r}(1+\delta, \theta, z, t) = v_{2r}(1+\delta, \theta, z, t) \quad (4\text{-}9)$$

i.e.

$$v_{1r}^{*}(1+\delta) = v_{2r}^{*}(1+\delta) \quad (4\text{-}10)$$

or

$$v_{1r}^{*}(1) + v_{1r}^{*\prime}(1)\cdot\delta = v_{2r}^{*}(1) + v_{2r}^{*\prime}(1)\cdot\delta \quad (4\text{-}11)$$

Since both v_{1r}^{*} and v_{2r}^{*} are infinitesimal quantities, we can get

$$v_{1r}^{*}(1) = v_{2r}^{*}(1) \quad (4\text{-}12)$$

by eliminating all the second order terms. Equation (4-12) is equivalent to

$$G_1(1) = G_2(1) \quad (4\text{-}13)$$

Similary, the continuity of v accross the interface will result in

$$v_1^{*}(1) = v_2^{*}(2) \quad (4\text{-}14)$$

or

$$H_1(1) = H_2(1) \quad (4\text{-}15)$$

Continuity of v_z requires a more careful formulation because there is a gradient of axial velocity in the mean flow which is discontinuous at the interface. The condition requires that

$$v_{1z}\,(1+\delta\ ,\ \theta\ ,\ z\ ,\ t\ ) = v_{2z}\ (\ 1+\delta\ ,\ \theta\ ,\ z\ ,\ t) \quad (4\text{-}16)$$

or

$$\bar{v}_{1z}\ (1+\delta\ ) + v_{1z}^{*}\ (1+\delta\ ,\theta\ ,\ z\ ,\ t) = \bar{v}_{2z}\ (1+\delta\ ) + v_{2z}^{*}\ (1+\delta\ ,\theta,\ z,\ t) \quad (4\text{-}17)$$

Expanding in Taylor's series around r=1

$$\bar{v}_{1z}(1) + \bar{v}_{1z}'(1)\cdot\delta + v_{1z}^{*}(1) + v_{1z}^{*}(1)\cdot\delta$$
$$= \bar{v}_{2z}(1) + \bar{v}_{2z}'\cdot\delta + v_{2z}^{*}(1) + v_{2z}^{*}(1) \quad (4\text{-}18)$$

and neglecting terms above second order, we have

$$\bar{v}_{1z}'(1)\cdot\delta + v_{1z}^{*}(1) = \bar{v}_{2z}'(1)\cdot\delta + v_{2z}^{*}(1) \quad (4\text{-}19)$$

Since $\bar{v}_{1z}(1) = \bar{v}_{2z}(1)$, we rearrange above equation by applying Eq (4-8) and Eq (3-18)

$$F + \frac{\overline{v}_{1z}' \cdot G_1}{\alpha \cdot (v_{z,1}(1)-c)} = f + \frac{\overline{v}_{2z}' \cdot G_2}{(v_{z,1}(1)-c)} \qquad (4\text{-}20)$$

at $r=1$

Continuity of stresses across the interface can be expressed as

$$T_{ij}^{1} \cdot \hat{n}_j = T_{ij}^{2} \cdot \hat{n}_j \qquad (4\text{-}21)$$

at $r = 1+\delta$ and $i = r, \theta, z$. Where $\hat{n}$ is the unit normal vector of the interface given by

$$\underset{\sim}{\hat{n}} = \frac{\nabla (r-1-\delta)}{|\nabla(r-1-\delta)|} = \frac{\nabla(r-\delta)}{|\nabla(r-\delta)|} \qquad (4\text{-}22)$$

where

$$\nabla(r-\delta) = [\frac{\partial}{\partial r} (r-\delta)]\cdot\hat{\mathcal{U}}_r + [\frac{1}{r} \frac{\partial}{\partial \theta} (r-\delta)]\cdot\hat{\mathcal{U}}_\theta$$

$$+ [\frac{\partial}{\partial z} (r-\delta)]\cdot\hat{\mathcal{U}}_z$$

$$= \hat{\mathcal{U}}_r + [- \frac{in}{r}\delta]\cdot\hat{\mathcal{U}}_\theta + [- i\alpha\delta]\cdot\hat{\mathcal{U}}_z \qquad (4\text{-}23)$$

$\hat{\mathcal{U}}_r$, $\hat{\mathcal{U}}_\theta$, $\hat{\mathcal{U}}_z$ are unit vectors in the r, θ, z directions respectively. So,

$$n_r = \frac{1}{|\nabla(r-\delta)|} \; ; \quad n = \frac{-\frac{in}{r}\cdot\delta}{|\nabla(r-\delta)|} \; ;$$

$$n_z = \frac{-i\alpha\delta}{|\nabla(r-\delta)|} \tag{4-24}$$

Since δ is an infinitesimal quantity, the components of $\hat{n}$ in the θ and z directions are small compared to that in the r direction. Expanding Eq (4-21), we get

$$T_{rr}^{1}\cdot\hat{n}_r + T_{r\theta}^{1}\cdot\hat{n} + T_{rz}^{1}\cdot\hat{n}_z = T_{rr}^{2}\cdot\hat{n}_r + T_{r\theta}^{2}\cdot\hat{n}_\theta + T_{rz}^{2}\cdot\hat{n}_z \tag{4-25}$$

$$T_{\theta r}^{1}\cdot\hat{n}_r + T_{\theta\theta}^{1}\cdot\hat{n} + T_{\theta z}^{1}\cdot\hat{n}_z = T_{\theta r}^{2}\cdot\hat{n}_r + T_{\theta\theta}^{2}\cdot\hat{n}_\theta + T_{\theta z}^{2}\cdot\hat{n}_z \tag{4-26}$$

$$T_{zr}^{1}\cdot\hat{n}_r + T_{z\theta}^{1}\cdot\hat{n} + T_{zz}^{1}\cdot\hat{n}_z = T_{zr}^{2}\,\hat{n}_r + T_{z\theta}^{2}\cdot\hat{n}_\theta + T_{zz}^{2}\cdot\hat{n}_z \tag{4-27}$$

at $r=1+\delta$. Where 1 and 2 represent inner and outer fluid, respectively. However, for primary flow

$$\bar{\tau}_{rr}^{i} = 0$$

$$\bar{\tau}_{r\theta}^{i} = \bar{\tau}_{\theta r}^{i} = 0$$

$$\bar{\tau}_{\theta\theta}^{i} = 0$$

$$\bar{\tau}^{i}_{\theta z} = \bar{\tau}^{i}_{z\theta} = 0$$

$$\bar{\tau}^{i}_{zz} = 0 \qquad i = 1,\ 2 \tag{4-28}$$

everywhere and everytime. The equation of state tells us that

$$\underset{\sim}{T} = -\,p\underset{\sim}{I} - \underset{\sim}{\tau} \tag{4-29}$$

where p is a function of z direction only. Thus, equations (4-25) - (4-27) result in

$$\tau^{*,1}_{rr}\cdot\hat{n}_r + \tau^{*,1}_{r\theta}\cdot\hat{n}_\theta + (\bar{\tau}^{1}_{rz} + \tau^{*,1}_{rz})\,\hat{n}_z$$

$$= \tau^{*,2}_{rr}\cdot\hat{n}_r + \tau^{*,2}_{r\theta}\cdot\hat{n}_\theta + (\bar{\tau}^{2}_{rz} + \tau^{*,2}_{rz})\,\hat{n}_z \tag{4-30}$$

$$\tau^{*,1}_{\theta r}\cdot\hat{n}_r + \tau^{*,1}_{\theta\theta}\cdot\hat{n}_\theta + \tau^{*,1}_{\theta z}\cdot\hat{n}_z = \tau^{*,2}_{\theta r}\cdot\hat{n}_r + \tau^{*,2}_{\theta\theta}\cdot\hat{n}_\theta + \tau^{*,2}_{\theta z}\cdot\hat{n}_z \tag{4-31}$$

$$(\bar{\tau}^{1}_{zr} + \tau^{*,1}_{zr})\cdot\hat{n}_r + \tau^{*,1}_{z\theta}\cdot n + (p + \tau^{*,1}_{zz})\cdot\hat{n}_z$$

$$= (\bar{\tau}^{2}_{zr} + \tau^{*,2}_{zr})\cdot\hat{n}_r + \tau^{*,2}_{z\theta}\cdot\hat{n}_\theta + (\bar{p} + \tau^{*,2}_{zz})\cdot\hat{n}_z \tag{4-32}$$

at $r = 1+\delta$. Expanding $\bar{\tau}^{1}_{rz}$ and $\bar{\tau}^{2}_{rz}$ about 1 and neglecting all terms over second order, Eq (4-30) becomes

$$\tau_{rr}^{*,1} \cdot \hat{n}_r + [\, \bar{\tau}_{rz}^{1}(1) + \left(\frac{d\bar{\tau}_{rz}^{1}}{dr} \right)_{r=1} \cdot \delta \,] \cdot \hat{n}_z = \tau_{rr}^{*,2} \cdot \hat{n}_r$$

$$+ [\, \bar{\tau}_{rz}^{2}(1) + \left(\frac{d\bar{\tau}_{rz}^{2}}{dr} \right)_{r=1} \cdot \delta \,] \cdot \hat{n}_z \tag{4-33}$$

or

$$\tau_{rr}^{*,1} \cdot \hat{n}_r + \bar{\tau}_{rz}^{1}(1) \cdot \hat{n}_z = \tau_{rr}^{*,2} \cdot \hat{n}_r + \bar{\tau}_{rz}^{2}(1) \cdot \hat{n}_z \tag{4-34}$$

Using the fact that $\tau_{rz}^{1}(1) = \tau_{rz}^{2}(1)$, we have

$$\tau_{rr}^{*,1} = \tau_{rr}^{*,2} \qquad \text{at} \quad r=1 \tag{4-35}$$

Similar procedure applied to the second and third equation above yields

$$\tau_{\theta r}^{*,1} = \tau_{\theta r}^{*,2} \qquad \text{at} \quad r=1 \tag{4-36}$$

and

$$\left(\frac{d\bar{\tau}_{zr}^{1}}{dr} \right)_{r=1} \cdot \delta + \tau_{zr}^{*,1} = \left(\frac{d\bar{\tau}_{zr}^{2}}{dr} \right)_{r=1} \cdot \delta + \tau_{zr}^{*,2}$$

$$\text{at} \quad r=1 \tag{4-37}$$

Application of Eq (3-27), (3-41) and (4-8), Eq (4-37) reduces to

$$\frac{\beta_1}{2} \cdot \frac{G_1}{\alpha(\, \bar{v}_{1z}(1) - c \,)} - \frac{\mu_1}{R_e} (\, F_1' - \alpha \cdot G_1)$$

$$= \frac{\beta_2}{2} \left(1 - \frac{\hat{c}_2^{\,2}}{r^2} \right) \frac{G_2}{\alpha \cdot (\, \bar{v}_{2z}(1) - c)} - (\, F_2' - \alpha \cdot G_2 \,) \frac{m\,\mu_2}{R_e} \tag{4-38}$$

at r=1

Application of Eq (3-25), Eq (3-40) and Eq (4-36), Eq (4-36) reduces to

$$\bar{\eta}_1 \cdot \left(H_1' - \frac{H_1}{r} - \frac{nG_1}{r} \right) = \bar{\eta}_2 \cdot \left(H_2' - \frac{H_2}{r} - \frac{nG_2}{r} \right) \qquad (4\text{-}39)$$

at $r=1$.

The normal stress condition at the interface is the most complicated because the difference in normal stress across the interface is counterbalanced by the action of surface tension between the two fluids. It must also be remembered that the normal stress includes a derivative of the radial velocity, i.e., τ_{rr}^* in addition to the pressure. Hence, the difference of the quantity is evaluated for the inner and outer fluids by the following form

$$-\left(\bar{p} + P^* \right) - \tau_{rr}^* \qquad (4\text{-}40)$$

and this quantity must be equivalent to

$$-\frac{1}{W_e}\left(\frac{1}{R_{/\!/}} + \frac{1}{R_{\perp}} \right) \qquad (4\text{-}41)$$

where $\bar{p}$ is the mean pressure, W_e is the Weber number defined in section(II), and $R_{/\!/}$ and $R_{\perp}$ are the non-dimensional principal radii of curvature of the interface. A radius of curvature is positive if the center of curvature lies in region 1 (inner fluid). The radius of curvature $R_{/\!/}$ is evaluated in a plane which contains the axis of symmetry while $R_{\perp}$ is the radius of curvature in a plane taken perpendicular to the axis. The radius of curvature are given by

$$\frac{1}{R_{/\!/}} = \frac{\partial^2 \delta}{\partial z^2} = - \alpha^2 \delta \tag{4-42}$$

and

$$\frac{1}{R_{\perp}} = 1 + (n^2 - 1) \cdot \delta \tag{4-43}$$

Application of Eq (3-24), (3-37), (4-40), (4-42) and (4-43), the normal stress condition at the interface can be written as

$$(p_1 - i \frac{2 \bar{\eta}_1}{R_e} G_1') - (p_2 - i \frac{2 \cdot \bar{\eta}_2}{R_e} G_2') = (\frac{\alpha^2 + 1 - n^2}{W_e}) \cdot \frac{G_1}{\alpha (\bar{v}_{1z}(1) - c)} \tag{4-44}$$

The results of this section were summarized in the Table 4-1. They were used in conjunction with the governing differential equations to provide a solution to the stability problem. Since six constants arose in the solution of each set of governing equations, there were a total of tweleve constants to be determined from the tweleve boundary and interfacial conditions.

The differential system represents an eigenvalue since c must take an specific value in order that the solution not be identically zero. The flow will be unstable, neutrally stable, or stable accordingly as the imaginary part of c, c_i , is positive, zero, negative.

$G_1(0), \quad H_1(0), \quad F_1(0)$ finite

$$G_2(a) = H_2(a) = F_2(a) = 0$$

$$G_1(1) = G_2(1)$$

$$H_1(1) = H_2(1)$$

$$F_1 + \frac{\bar{v}'_{1z}\, G_1}{\alpha(\bar{v}_{1z} - c)} = F_2 + \frac{\bar{v}'_{2z}\, G_2}{\alpha(\bar{v}_{2z} - c)} \qquad \text{at } r=1$$

$$\frac{\beta_1}{2} \frac{G_1}{\alpha(\bar{v}_{1z} - c)} - \frac{\mu_1}{R_e} (F'_1 - \alpha G_1) = \frac{\beta_2}{2} \left(1 - \frac{\hat{c}_2}{r^2} \right) \frac{G_2}{\alpha(\bar{v}_{2z} - c)}$$

$$- \frac{m\mu_2}{R_e} (F'_2 - \alpha G_2) \quad \text{at } r=1$$

$$\bar{\eta}_1 \cdot \left(H'_1 - \frac{H_1}{r} - \frac{nG_1}{r} \right) = \bar{\eta}_2 \cdot \left(H'_2 - \frac{H_2}{r} - \frac{nG_2}{r} \right) \qquad \text{at } r=1$$

$$\left(P_1 - i \frac{2\bar{\eta}_1}{R_e} G'_1 \right) - \left(P_2 - i \frac{2\bar{\eta}_2}{R_e} G'_2 \right) = \left(\frac{\alpha^2 + 1 - n^2}{W_e} \right) \frac{G_1}{\alpha(\bar{v}_{1z} - c)}$$

at $r=1$

Table 4-1

(V) Solution for the Axisymmetric Case (n=0)

The differential governing equations in section III will now be solved by the regular perturbation procedure described in section III. The series expansions given in Eq (3-1) will be substituted into the governing differential equations and boundary conditions. Then terms of the same power of α will be equated separately in each equation. This procedure will allow a solution to be built up step-by-step from the first approximation to any degree of accuracy required. In order to determine the first approximation to the onset of instability, it will be necessary to proceed only as far as the second approximation.

When n=0 , the equation of motion associated with the θ coordinate, (3-26) and (3-45), express only a relationship governing circumferential velocity and may, if required, be solved after the other three equations in the differential system have been solved. In order to determine the stability of the flow, it will not be necessary to solve (3-26) at all. Thus, the order of the differential system is reduced by 2, and there will be a total of eight constants to be determined instead of tweleve.

Omitting H_1 and H_2 from consideration and taking n=0, yields

$$\alpha\cdot(\bar{v}_{1z}-c)G_1 = P_1' - \frac{i\cdot\bar{\eta}_1}{R_e}\left[2G_1'' + 2\left(\frac{r\cdot\bar{\eta}_1'}{\bar{\eta}_1}+1\right)\frac{G_1'}{r} - \left(\frac{2}{r^2}+\alpha^2\frac{\mu_1}{\bar{\eta}_1}\right)G_1 + \alpha\cdot\frac{\mu_1}{\bar{\eta}_1}F_1'\right] \qquad (5\text{-}1)$$

$$\alpha\cdot(\bar{v}_{1z}-c)F_1+\bar{v}'_{1z}G_1=-\alpha\cdot p_1-\frac{i\cdot\bar{\eta}_1}{R_e}\Big[\frac{\mu_1}{\bar{\eta}_1}F''_1+\frac{\mu_1}{\bar{\eta}_1}\cdot\left(1+\frac{r\cdot\mu'_1}{\mu_1}\right)\frac{F'_1}{r}-2\alpha^2\cdot F_1-\frac{\mu_1}{\bar{\eta}_1}\alpha\cdot\Big(G'_1+\left(1+\frac{r\cdot\mu'_1}{\mu_1}\right)\frac{G_1}{r}\Big)\Big] \qquad (5\text{-}2)$$

and

$$G'_1+G_1/r+\alpha\cdot F_1=0 \qquad (5\text{-}3)$$

Eliminating p between Eq (5-1) and (5-2), and combining Eq (5-3) provides the solution of F_1 and G_1. The procedure of solutions for fluid 2 is similar to that for fluid 1.

A. First Approximation

If Eq (3-1) is used in Eqs (5-1) - (5-3), we get the first approximation

$$F_{1,0}=A_1\left(\frac{r^2}{4}+\frac{\alpha_1 D_1}{2(\alpha_1+1)}r^{\alpha_1+1}\right)+A_2\left(\ln r+\frac{\alpha_1 D_1}{(\alpha_1-1)}r^{\alpha_1-1}\right)+A_3 \qquad (5\text{-}4)$$

$$G_{1,1}=-A_1\left(\frac{r^3}{16}+\frac{\alpha_1 D_1}{2(\alpha_1+1)\cdot(\alpha_1+3)}r^{\alpha_1+2}\right)-$$

$$A_2 \left(\frac{1}{2} r \ln r + \frac{\alpha_1 D_1}{(\alpha_1 - 1)(\alpha_1 + 1)} r^{\alpha_1} + \frac{r}{4} \right)$$

$$- \frac{A_3}{2} r + \frac{A_4}{r}$$

for fluid 1. And

$$F_{2,0} = B_1 \left(\frac{1}{4} r^2 - \frac{\alpha_2 D_2}{2} \phi_1(r) \right) + B_2 \left(\ln r - \alpha_2 D_2 \phi_2(r) \right)$$

$$+ B_3 \tag{5-6}$$

$$G_{2,1} = B_1 \left(\frac{-1}{16} r^3 + \frac{\alpha_2 D_2}{4} r \phi_1(r) - \frac{\alpha_2 D_2}{4} \cdot \frac{\phi_3(r)}{r} \right) + B_2$$

$$\left(- \frac{r}{2} \ln r + \frac{r}{4} + \frac{\alpha_2 D_2}{4} r \cdot \phi_2(r) - \frac{\alpha_2 D_2}{4} \frac{\phi_1(r)}{r} \right)$$

$$- \frac{r}{2} B_3 + B_4 / r \tag{5-7}$$

where the first subscript of F and G means fluid region and the second subscript indicate the degree of approximation. The all coefficients at right side of equation are integral constants. And

$$\phi_1(r) = \int_r^a r \left(r + \frac{\hat{c}_2}{r} \right)^{\alpha_2 - 1} \cdot dr$$

$$\phi_2(r) = \int_r^a \frac{1}{r} \left(r + \frac{\hat{c}_2}{r} \right)^{\alpha_2 - 1} dr$$

$$\phi_3(r) = \int_r^a r^3 \left(r + \frac{\hat{c}_2}{r} \right)^{\alpha_2 - 1} dr$$

Application of the boundary and interfacial conditions result in

$$A_2 = A_4 = 0 \tag{5-8}$$

$$\left(\frac{a^2}{4} \right) B_1 + (\ln a) B_2 = - B_3 \tag{5-9}$$

$$\left(\frac{a^4}{16} \right) B_1 + \left(\frac{a^2}{4} \right) B_2 = - B_4 \tag{5-10}$$

$$\begin{aligned} &\left[\frac{1}{16} + \frac{\alpha_1 D_1}{2(\ +1)(\ +3)} \right] A_1 + \frac{A_3}{2} + \left[\frac{-1}{16} + \frac{\alpha_2 D_2 \cdot \phi_1(1)}{4} \right. \\ &\quad \left. - \frac{\alpha_2 D_2 \cdot \phi_3(1)}{4} + \frac{a^2}{8} - \frac{a^4}{16} \right] B_1 + \left[\frac{1}{4} + \frac{\alpha_2 D_2 \cdot \phi_2(1)}{4} \right. \\ &\quad \left. - \frac{\alpha_2 D_2 \cdot \phi_1(1)}{4} + \frac{\ln a}{2} - \frac{a^2}{4} \right] B_2 = 0 \end{aligned} \tag{5-11}$$

$$\begin{aligned} &\left[\frac{(\beta_1 - \beta_2 + \beta_2 \hat{C}_2)}{2 \cdot \pi} \left(\frac{1}{2} + \frac{\alpha_1 D_1}{(\alpha_1 + 1)} \right) + 1 \right] A_1 + \left[\frac{\beta_1 - \beta_2 + \beta_2 \hat{C}_2}{\pi} \right] A_3 \\ &\quad + \left[\frac{m}{2} - \frac{(\beta_1 - \beta_2 + \beta_2 \hat{C}_2)}{2 \cdot \pi} \left(\frac{1}{2} - \alpha_2 D_2 \cdot \phi_1(1) + \frac{a^2}{2} \right) \right] B_1 \\ &\quad + \left[m + \frac{(\beta_1 - \beta_2 + \beta_2 \hat{C}_2)}{2 \cdot \pi} \left(2\alpha_2 D_2 \cdot \phi_2(1) + 2 \ln a \right) \right] B_2 = 0 \end{aligned} \tag{5-12}$$

and

$$A_1 + 0 \cdot A_3 - B_1 + 0 \cdot B_2 = 0 \qquad (5\text{-}13)$$

Taking $A_3=1$, the constants A_1 , B_1 , B_2 can be solved by solving Eqs (5-11) - (5-13) simultaneously. Then B_3 and B_4 can be readily determined by using Eqs (5-9) - (5-10). Thus, c_0 can be derived by using Eq (4-20) which will yield

$$c_0 = \overline{v}_{1z}\ (1) + \frac{(\overline{v}'_{z1} - \overline{v}'_{z2})}{(F_{10} - F_{20})} G_{11} \qquad (5\text{-}14)$$

at r=1. Since c_0 is real, no instability will be manifested at this stage of approximation. It is thus necessary to proceed to next approximation.

B. Second Approximation

Starting from Eq (5-1) to (5-3) with the same procedure described in first approximation, the solution for this stage can be redily written as following

$$F_{11} = i \cdot [\, S_1(r) + {}^*A_1^* \left(\frac{r^2}{4} + \frac{\alpha_1 D_1}{2(\alpha_1+1)} \cdot r^{\alpha_1+1} \right) + {}^*A_2^* \left[\ln r + \frac{\alpha_1 D_1}{(\alpha_1-1)} \; r^{\alpha_1+1} \right] + {}^*A_3^* \,] \qquad (5\text{-}15)$$

$$G_{12} = i \left[S_2(r) - {}^*A_1^* \left(\frac{r^3}{16} + \frac{\alpha_1 D_1}{2(\alpha_1+1)\cdot(\alpha_1+3)} r^{\alpha_1+2} \right) - {}^*A_2^* \left(\frac{r}{2} \ln r - \frac{r}{4} + \frac{\alpha_1 D_1}{(\alpha_1-1)(\alpha_1+1)} r^{\alpha_1} \right) - \frac{{}^*A_3^*}{2} r \right] \qquad (5\text{-}16)$$

where

$$S_1(r) = -\mathcal{E}_1\left[\frac{r^{2\alpha_1+4}}{(2\alpha_1+4)} + \frac{\alpha_1 D_1}{(3\alpha_1+3)} r^{3\alpha_1+3} \right] + \mathcal{E}_2\left[\frac{r^{\alpha_1+5}}{(\alpha_1+5)} + \frac{\alpha_1 D_1}{(2\alpha_1+4)} r^{2\alpha_1+4} \right] + \mathcal{E}_3\left[\frac{r^{\alpha_1+3}}{(\alpha_1+3)} + \frac{\alpha_1 D_1}{(2\alpha_1+2)} r^{2\alpha_1+2} \right] - \mathcal{E}_4\left[\frac{r^6}{6} + \frac{\alpha_1 D_1}{(\alpha_1+5)} r^{\alpha_1+5} \right] + \mathcal{E}_5\left[\frac{r^4}{4} + \frac{\alpha_1 D_1}{(\alpha_1+3)} r^{\alpha_1+3} \right]$$

$$\mathcal{E}_1 = 2\, Q_1 \cdot R_e \cdot \alpha_1 D_1^2\, A_1 \,/\, [\, (\alpha_1+1)^2(\alpha_1+1)\,(2\alpha_1+4)\,]$$

$$\mathcal{E}_2 = Q_1\, R_e\, D_1\, A_1 \left(\frac{\alpha_1}{(\alpha_1+1)\cdot(\alpha_1+3)} - \frac{3}{8} \right)$$

$$\mathcal{E}_3 = Q_1\, D_1\, A_3(\alpha_1-1) \,/\, (\alpha_1+1) + R_e \cdot A_1 \alpha_1 D_1 \left[Q_1\left(1 + \frac{2D_1}{\alpha_1+1} \right) + 1 - C_0 \right] / \left[\, 2(\alpha_1+1)\,\right]$$

$$\mathcal{E}_4 = Q_1 \cdot R_e \cdot A_1 \,/\, 8$$

$$\mathcal{E}_5 = R_e \cdot A_1 \cdot \left[Q_1\left(1 + \frac{2D_1}{\alpha_1+1} \right) + 1 - c_0 \right] / \, 4$$

$$Q_1 = \beta_1 \cdot R_e \; / \; 4$$

$$S_2(r) = -\frac{1}{r}\int r \; S_1(r)\, dr$$

for fluid 1. Similarity to fluid 2, the solutions are

$$F_{21} = i \; [\; S_3(r) + {}^*B_1^* \left(\frac{r^2}{4} - \frac{\alpha_2 D_2}{2} \phi_1(r) \right) + {}^*B_2^* \left(\ln r - \alpha_2 D_2 \phi_2(r) \right) + {}^*B_3^* \;] \tag{5-17}$$

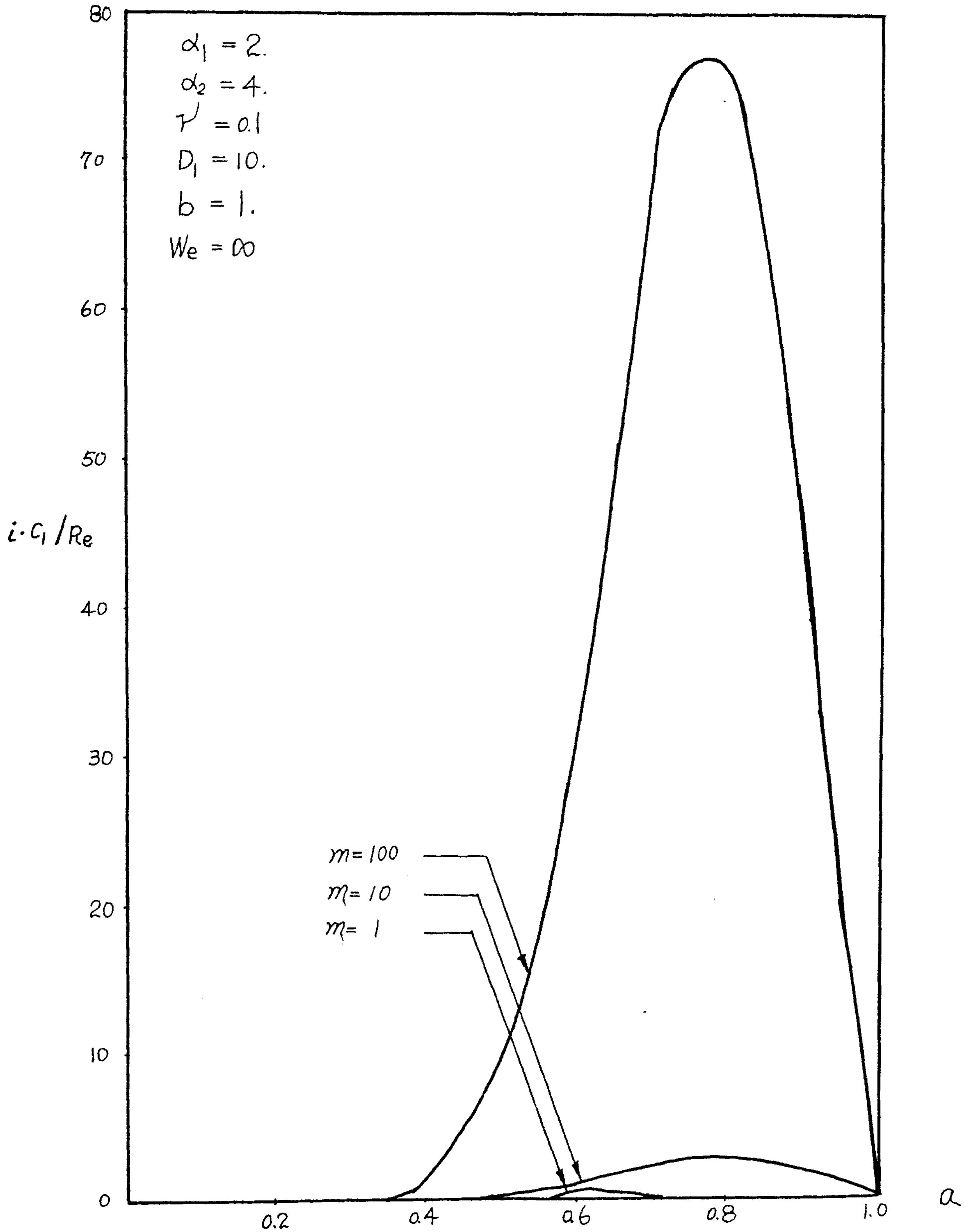
and

$$G_{22} = i \; [\; S_4(r) + {}^*B_1^* \left[\frac{-1}{16} r^3 + \frac{\alpha_2 D_2}{4} r \cdot \phi_1(r) - \frac{\alpha_2 D_2}{4} \cdot \frac{\phi_3(r)}{r} \right] + {}^*B_2^* \left[-\frac{r}{2} \ln r + \frac{r}{4} + \frac{\alpha_2 D_2}{4} r \phi_2(r) - \frac{\alpha_2 D_2}{4} \cdot \frac{\phi_1(r)}{r} \right] - \frac{r}{2} {}^*B_3^* + \frac{{}^*B_4^*}{r} \;] \;] \tag{5-18}$$

Those eight integral constants are determined by applying the boundary and interfacial conditions listed in section (IV). The eigen value, c_1 , is thus calculated by

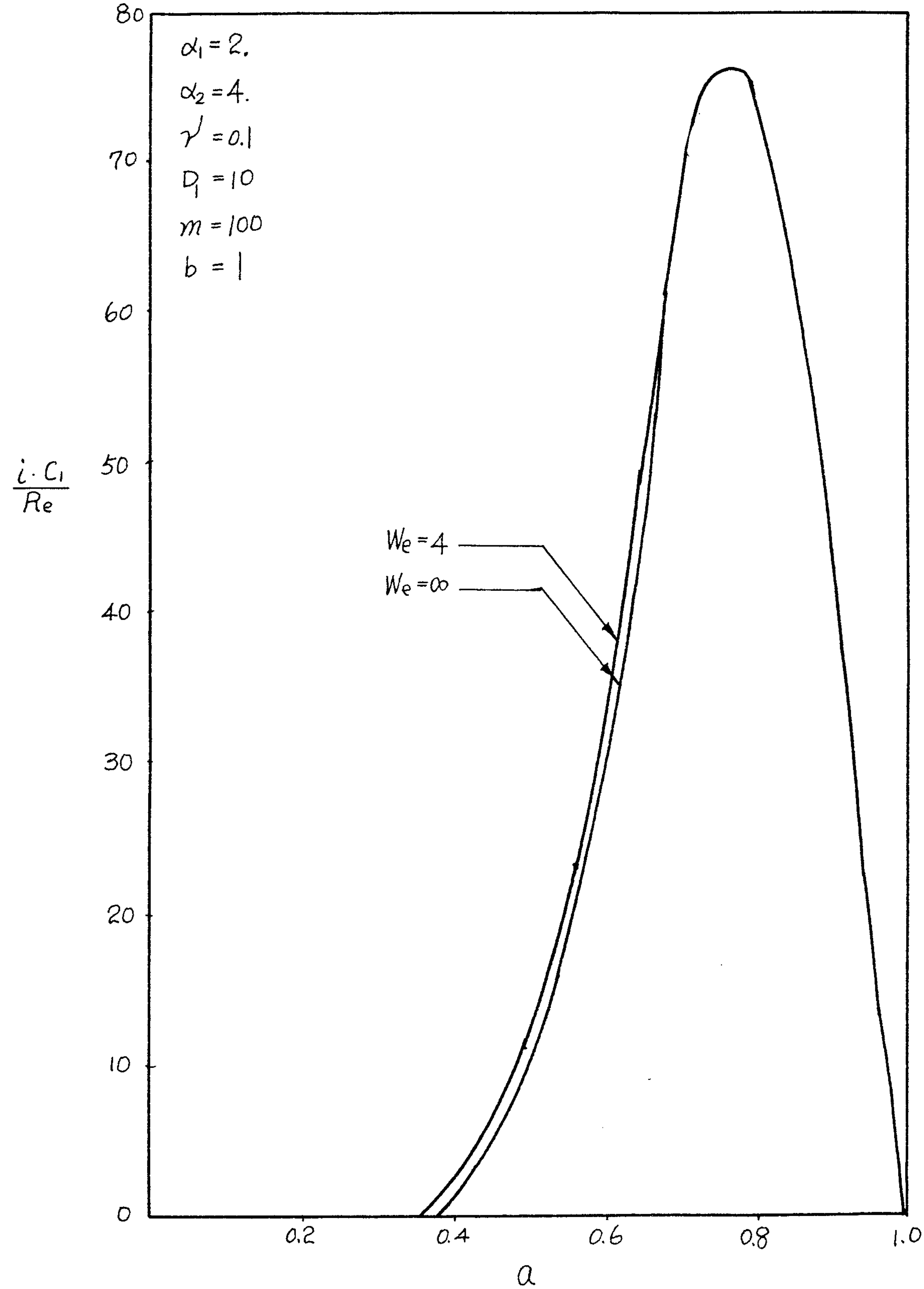
$$c_1 = \frac{(\overline{v}_{1z} - c_0)}{(F_{10} - F_{20})} \left(F_{11} - F_{21} \right) + \frac{(\overline{v}'_{1z} - \overline{v}'_{2z})}{(F_{10} - F_{20})} G_{12} \tag{5-19}$$

The results were carried out for a variety of situations by using the Univac 90/80-3 computer. The influences of zero-shear-rate viscosity ratio (m), shear-stress ratio (γ'), power parameter (α_1 , α_2) and surface tension on axisymmetric disturbances for unidirectional flow are exhibited in the graphs.



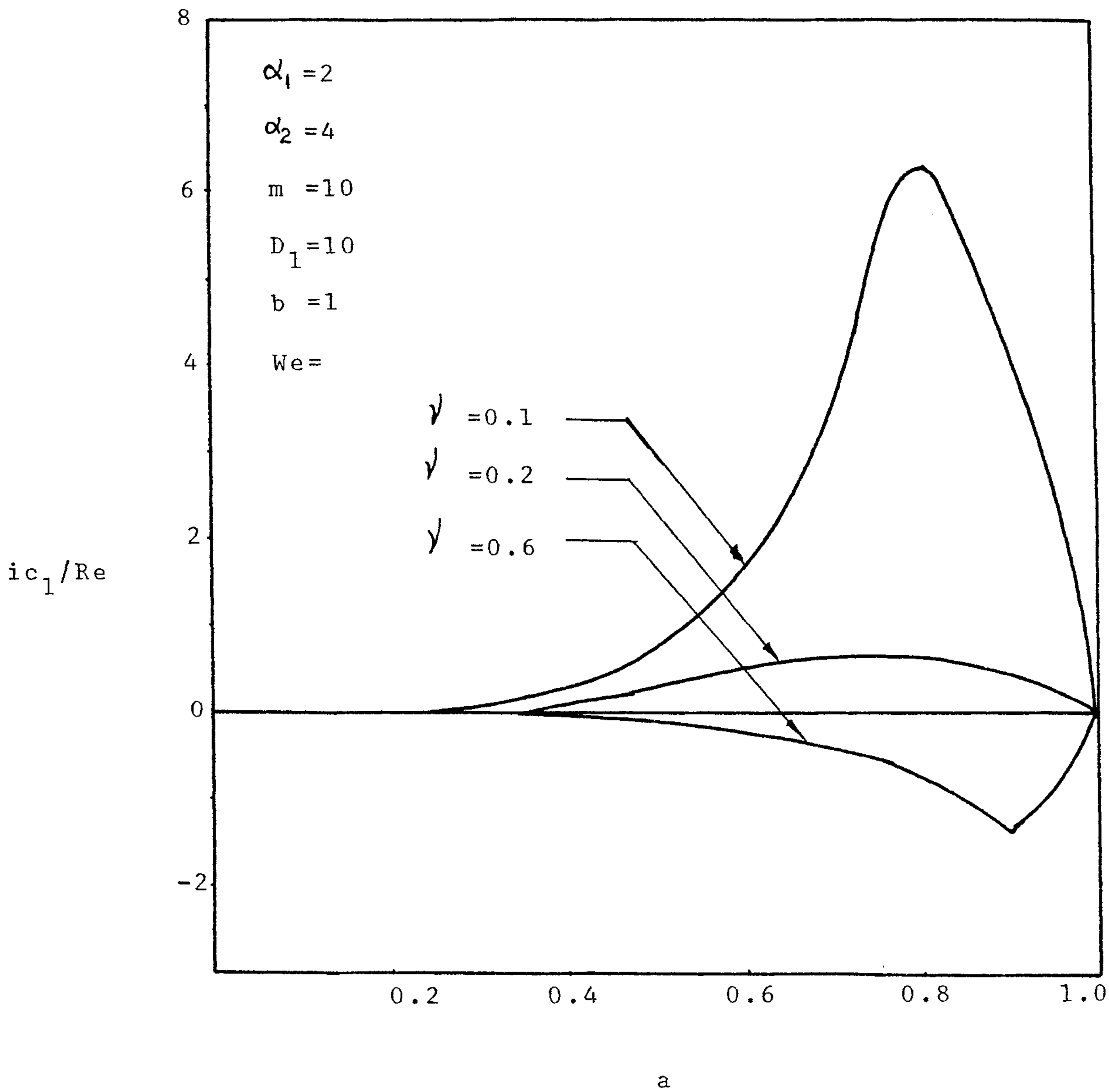
$\alpha_1 = 2.$
$\alpha_2 = 4.$
$\nu = 0.1$
$D_1 = 10.$
$b = 1.$
$W_e = \infty$
$i \cdot c_1 / R_e$
80
70
60
50
40
30
20
10
0
0.2
0.4
0.6
0.8
1.0
a
$m = 100$
$m = 10$
$m = 1$

Fig 5-1



80
70
60
50
40
30
20
10
0
$\alpha_1 = 2.$
$\alpha_2 = 4.$
$\gamma = 0.1$
$D_1 = 10$
$m = 100$
$b = 1$
$\frac{i \cdot C_1}{Re}$
We = 4
We = ∞
0.2
0.4
0.6
0.8
1.0
a

Fig 5-2



8
6
4
2
0
-2
α_1 =2
α_2 =4
m =10
D_1=10
b =1
We=
γ =0.1
γ =0.2
γ =0.6
ic_1/Re
0.2
0.4
0.6
0.8
1.0
a

Fig 5-3

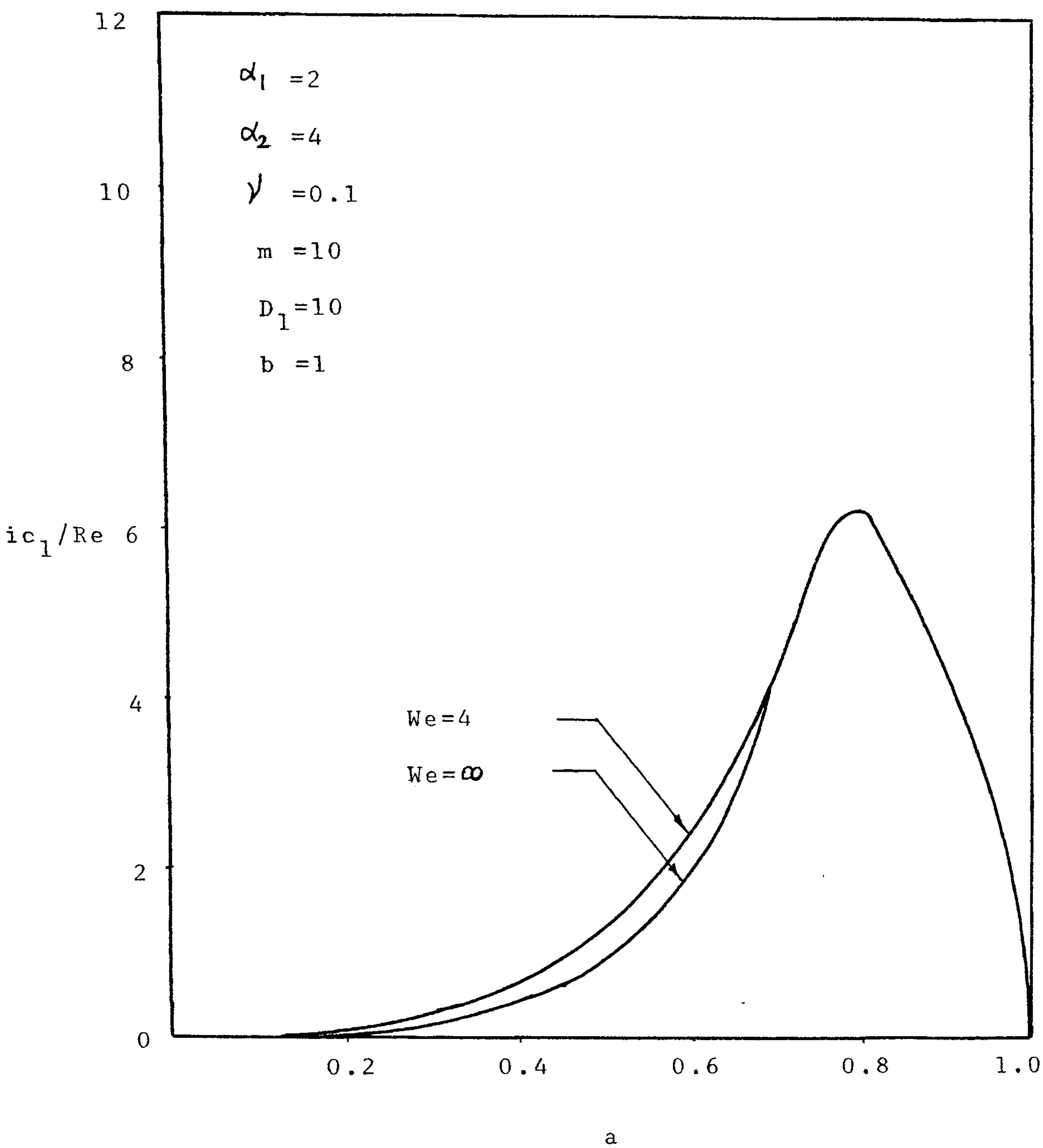


Fig 5-4

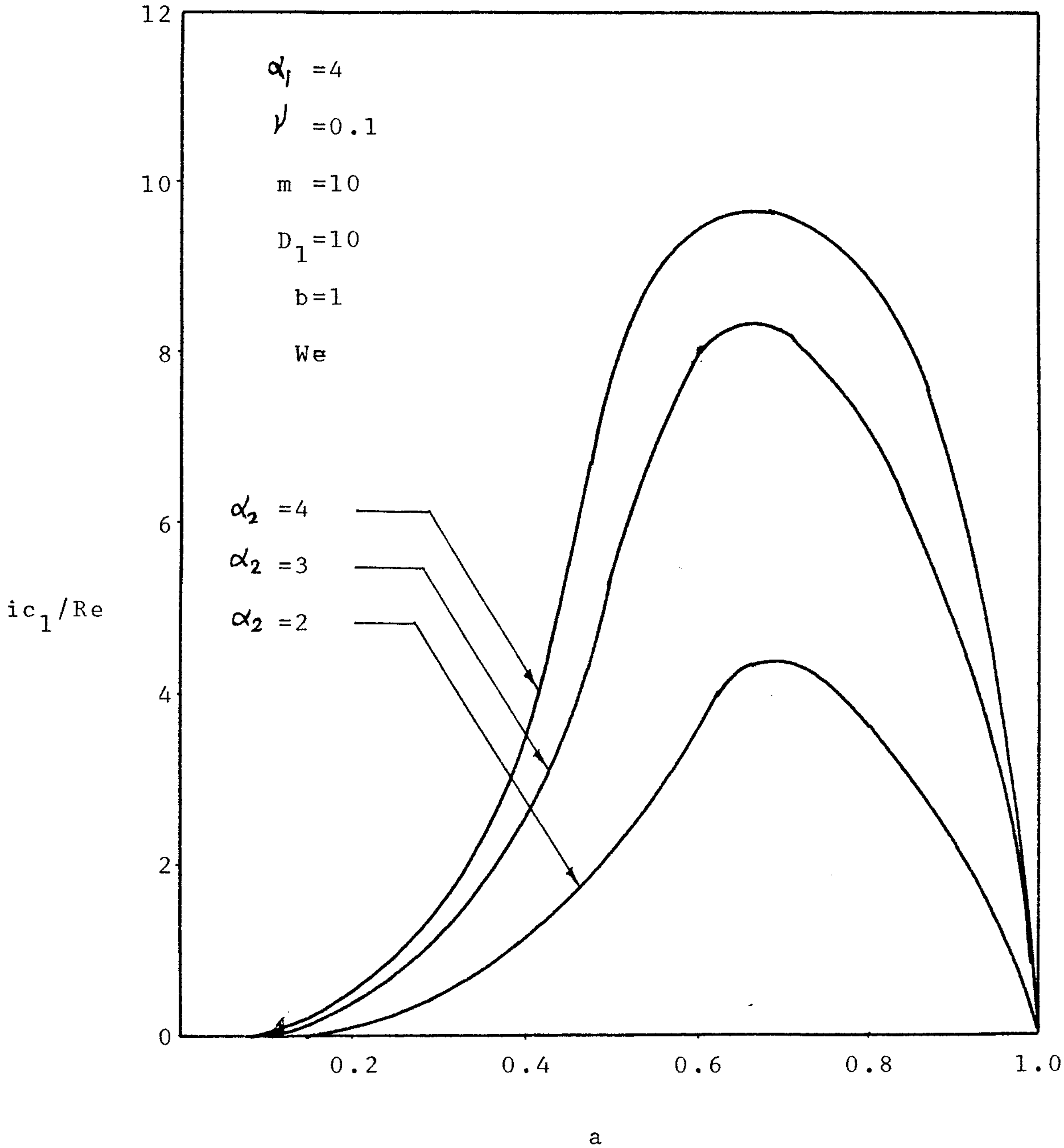

12
10
8
6
4
2
0
0.2
0.4
0.6
0.8
1.0
α_1 =4
ν =0.1
m =10
D_1=10
b=1
We
α_2 =4
α_2 =3
α_2 =2
ic_1/Re
a

Fig 5-5

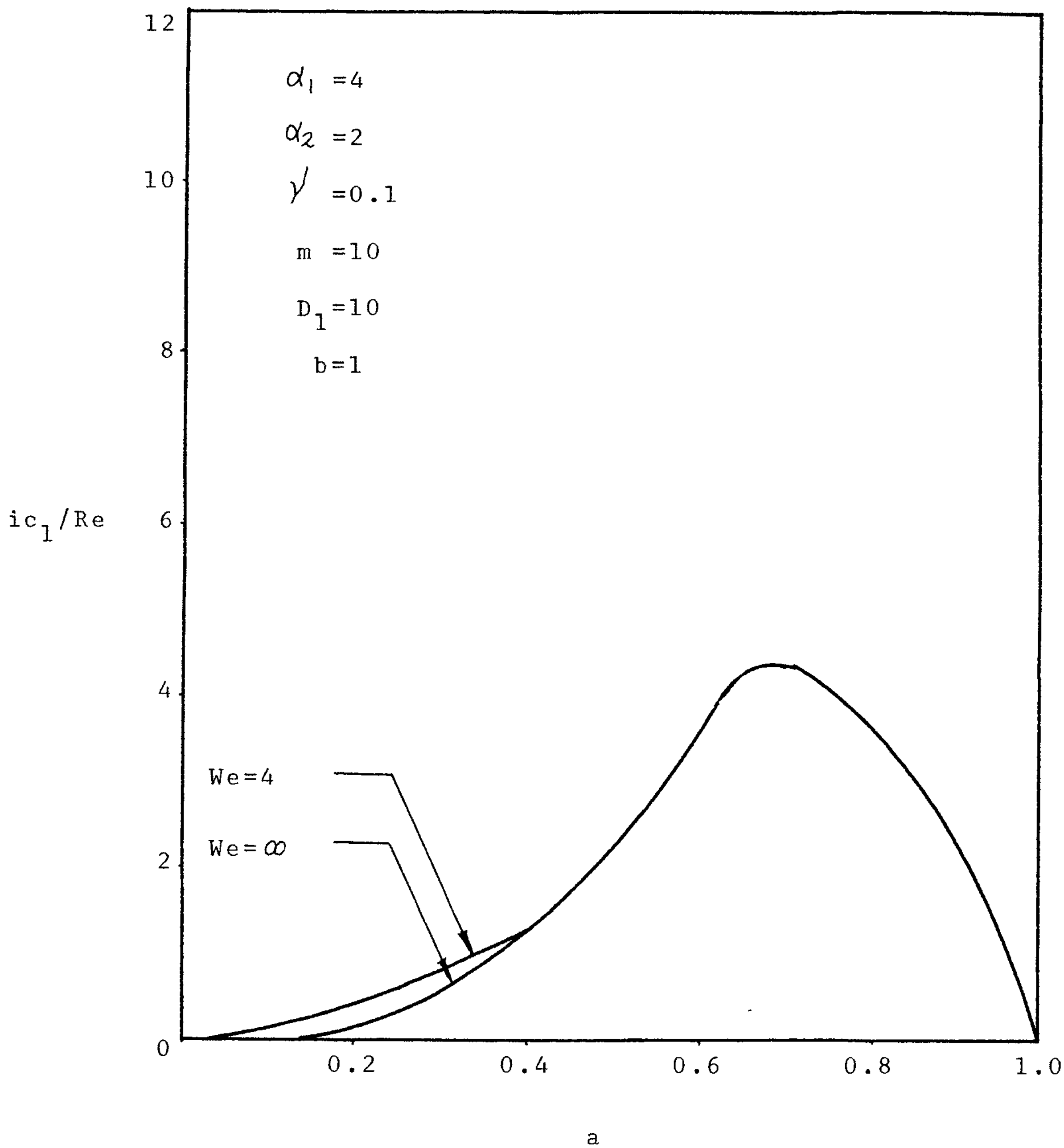

α_1 =4
α_2 =2
γ =0.1
m =10
D_1=10
b=1
We=4
We=∞
ic_1/Re
a
0
2
4
6
8
10
12
0.2
0.4
0.6
0.8
1.0

Fig 5-6

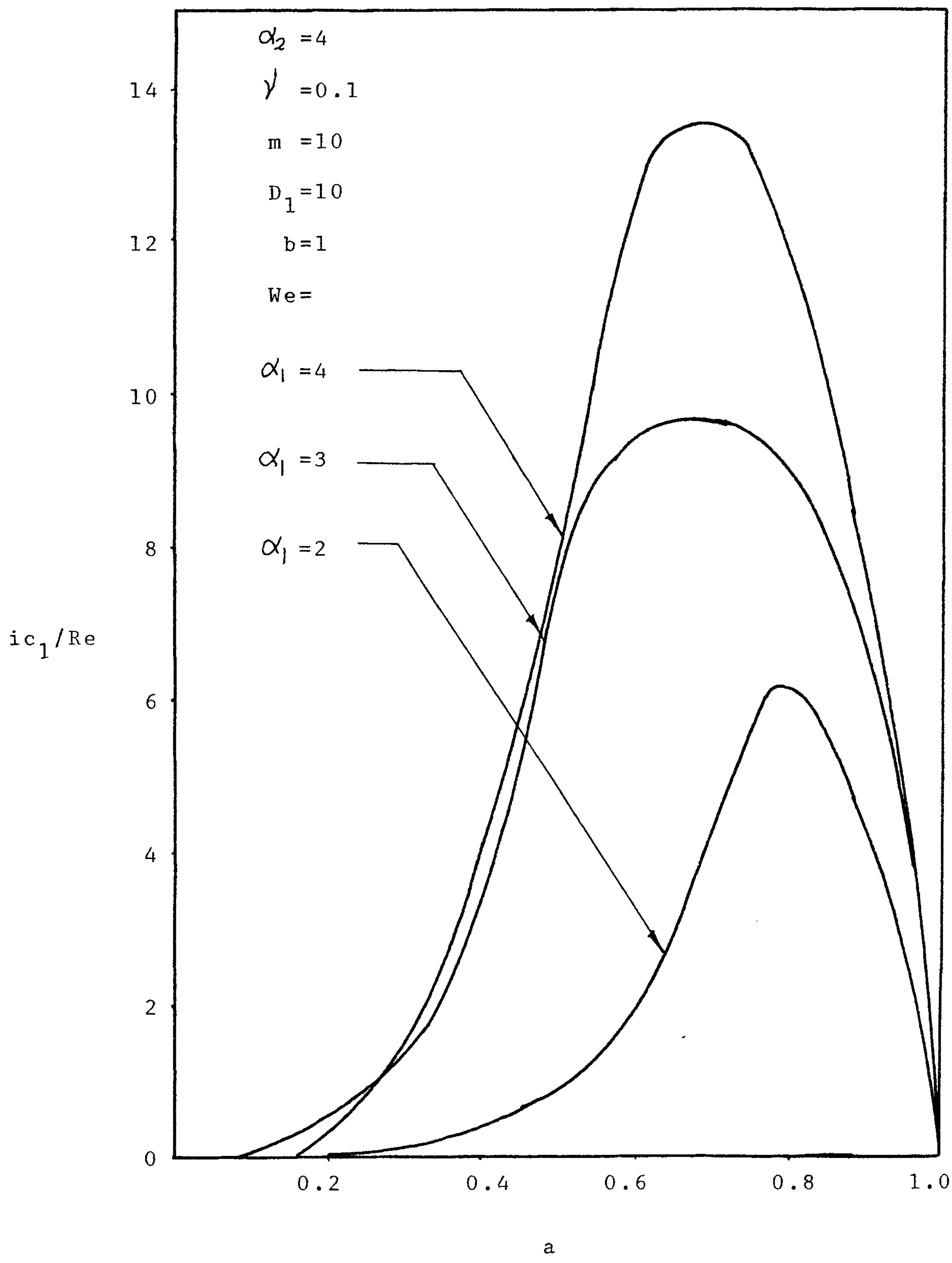
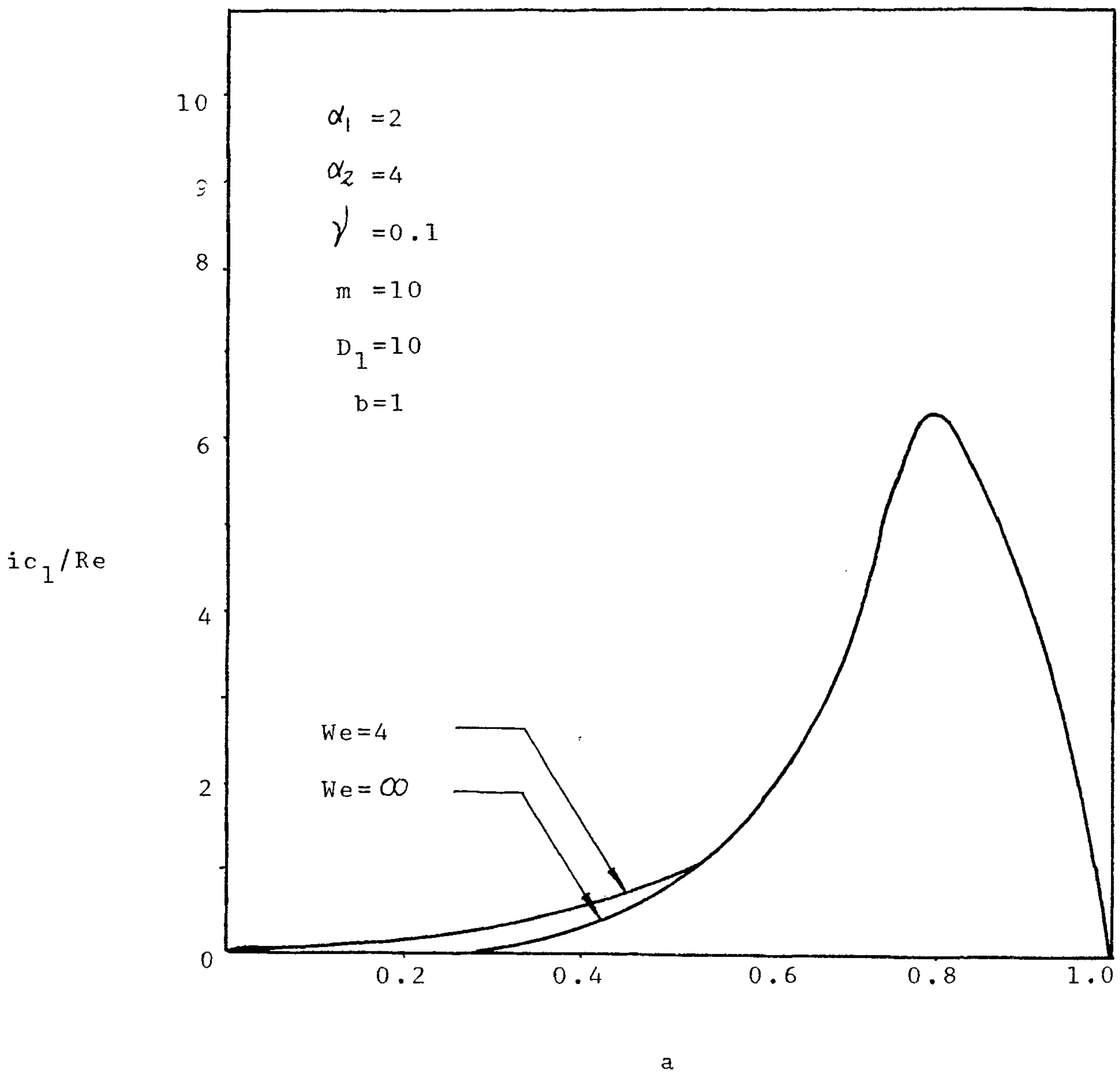

α2 =4
γ =0.1
m =10
D1=10
b=1
We=
α1 =4
α1 =3
α1 =2
14
12
10
8
6
4
2
0
0.2
0.4
0.6
0.8
1.0
ic1/Re
a

Fig 5-7



10
9
8
6
4
2
0
α_1 =2
α_2 =4
γ =0.1
m =10
D_1=10
b=1
ic_1/Re
We=4
We=∞
0.2
0.4
0.6
0.8
1.0
a

Fig 5-8

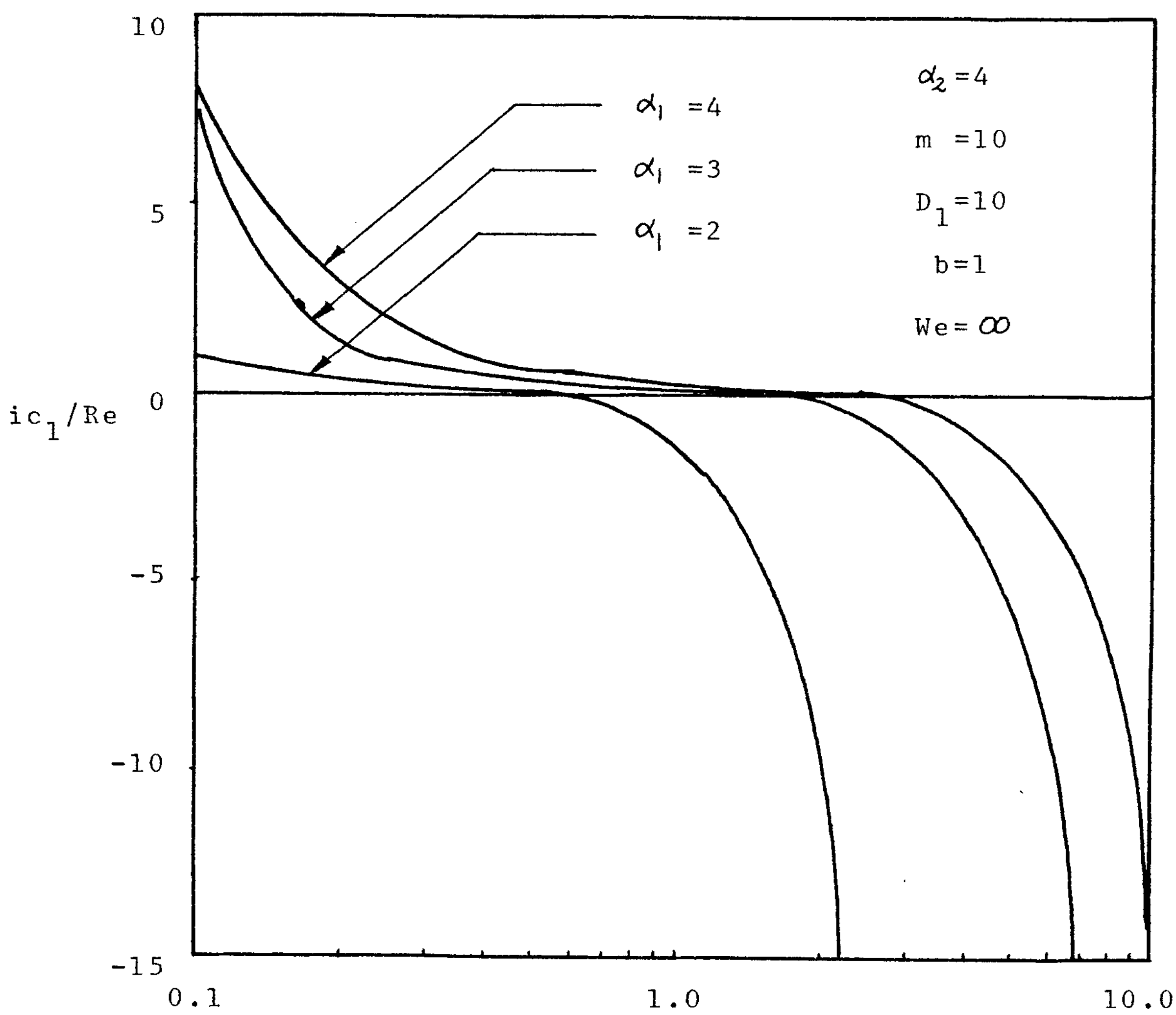

10
5
0
-5
-10
-15
ic_1/Re
0.1
1.0
10.0
α_1 =4
α_1 =3
α_1 =2
α_2=4
m =10
D_1=10
b=1
We=∞

Fig 5-9

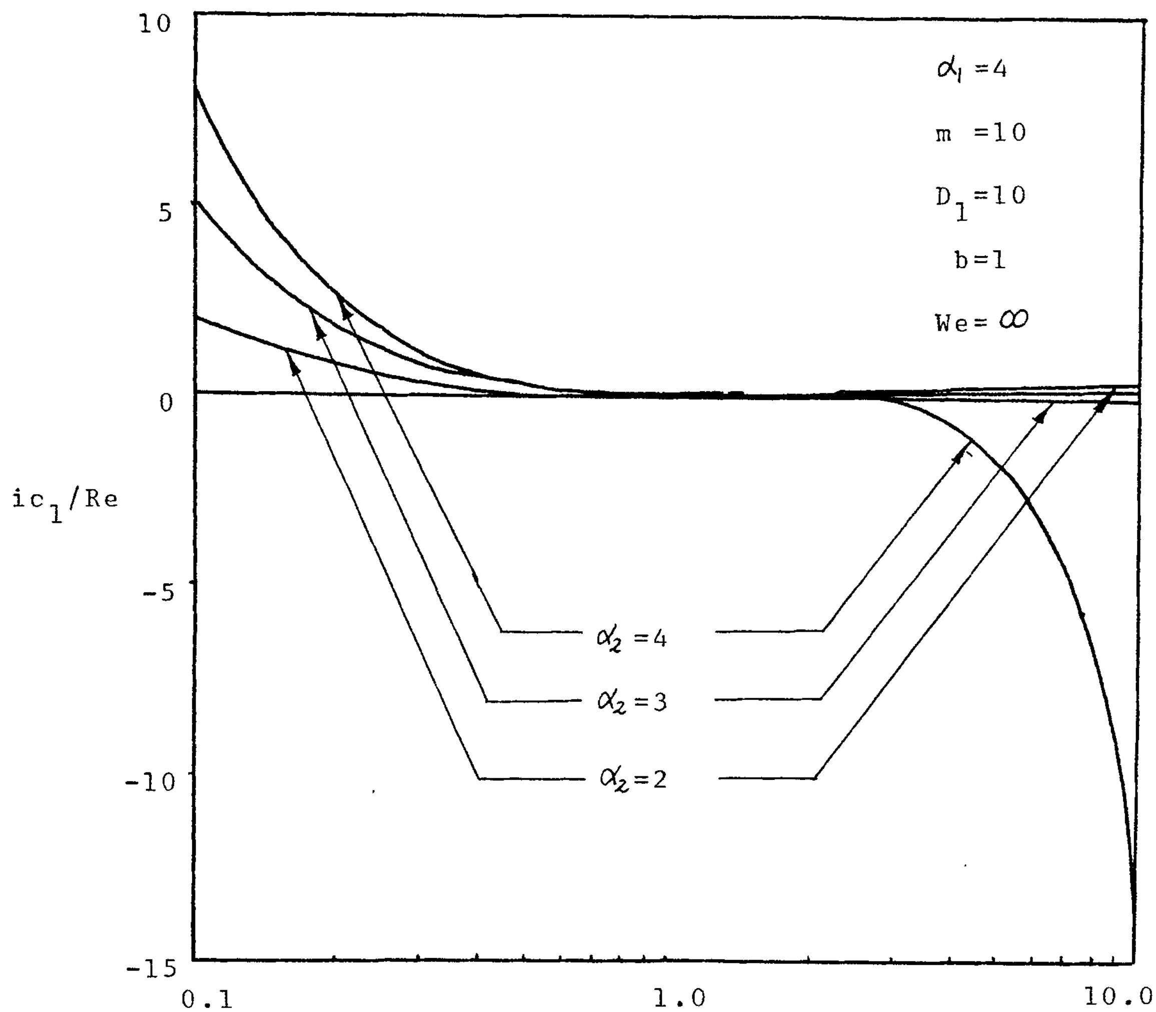


Fig 5-10

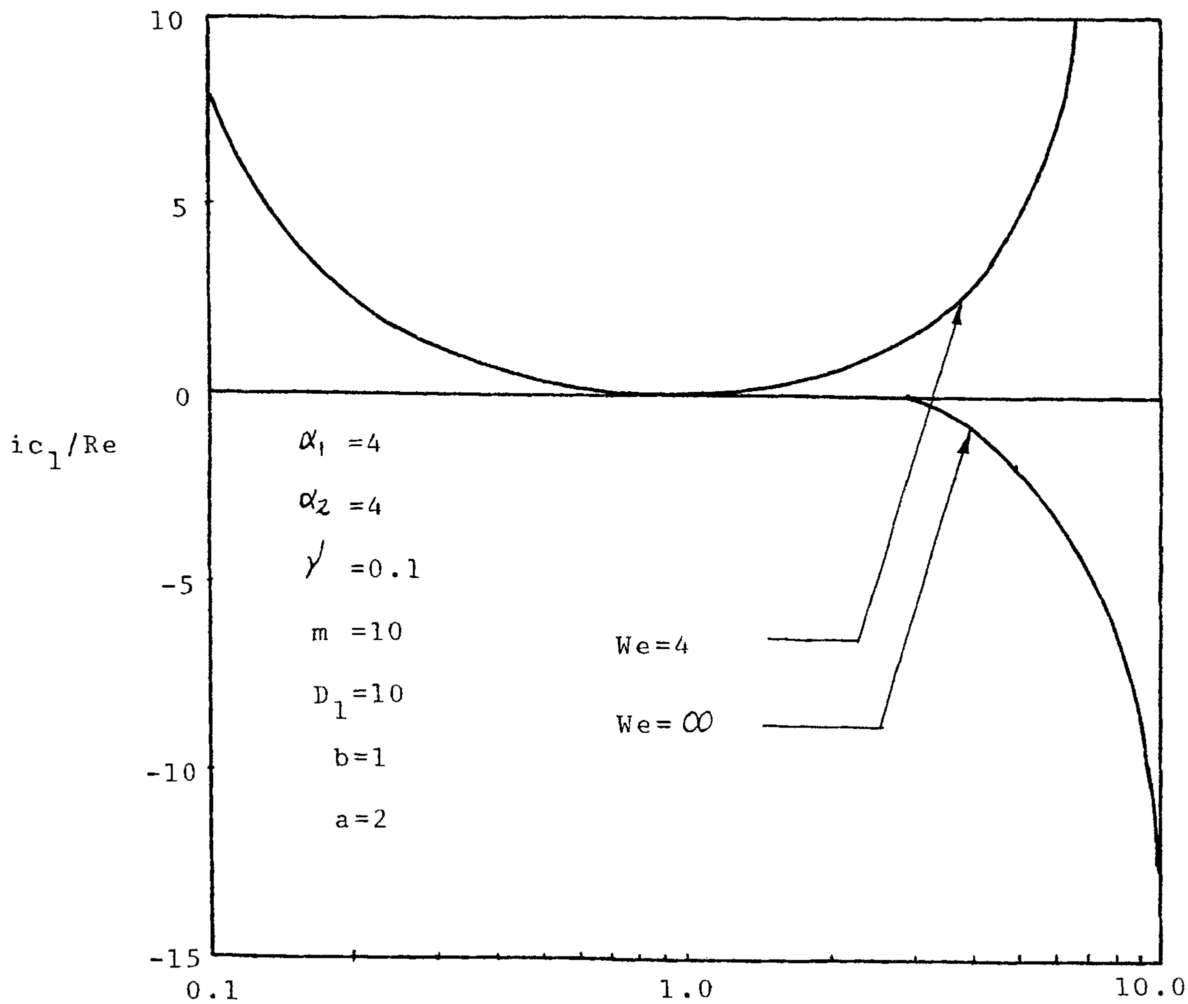
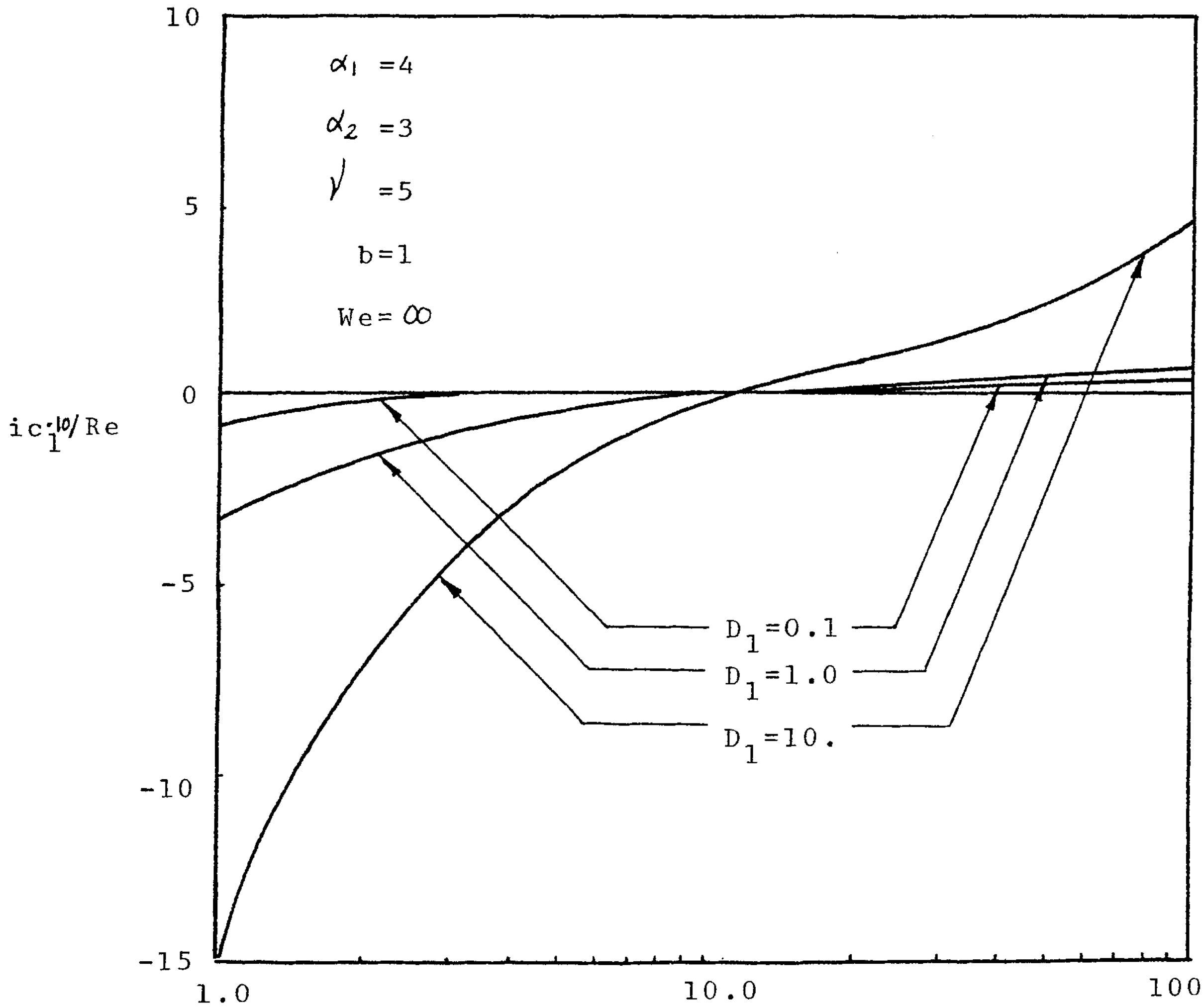

10
5
0
-5
-10
-15
ic_1/Re
α_1 =4
α_2 =4
γ =0.1
m =10
D_1=10
b=1
a=2
We=4
We=∞
0.1
1.0
10.0

Fig 5-11



10
5
0
-5
-10
-15
$ic_1^{\cdot 10}/Re$
$\alpha_1 = 4$
$\alpha_2 = 3$
$\nu = 5$
b=1
We= ∞
D_1=0.1
D_1=1.0
D_1=10.
1.0
10.0
100

Fig 5-12

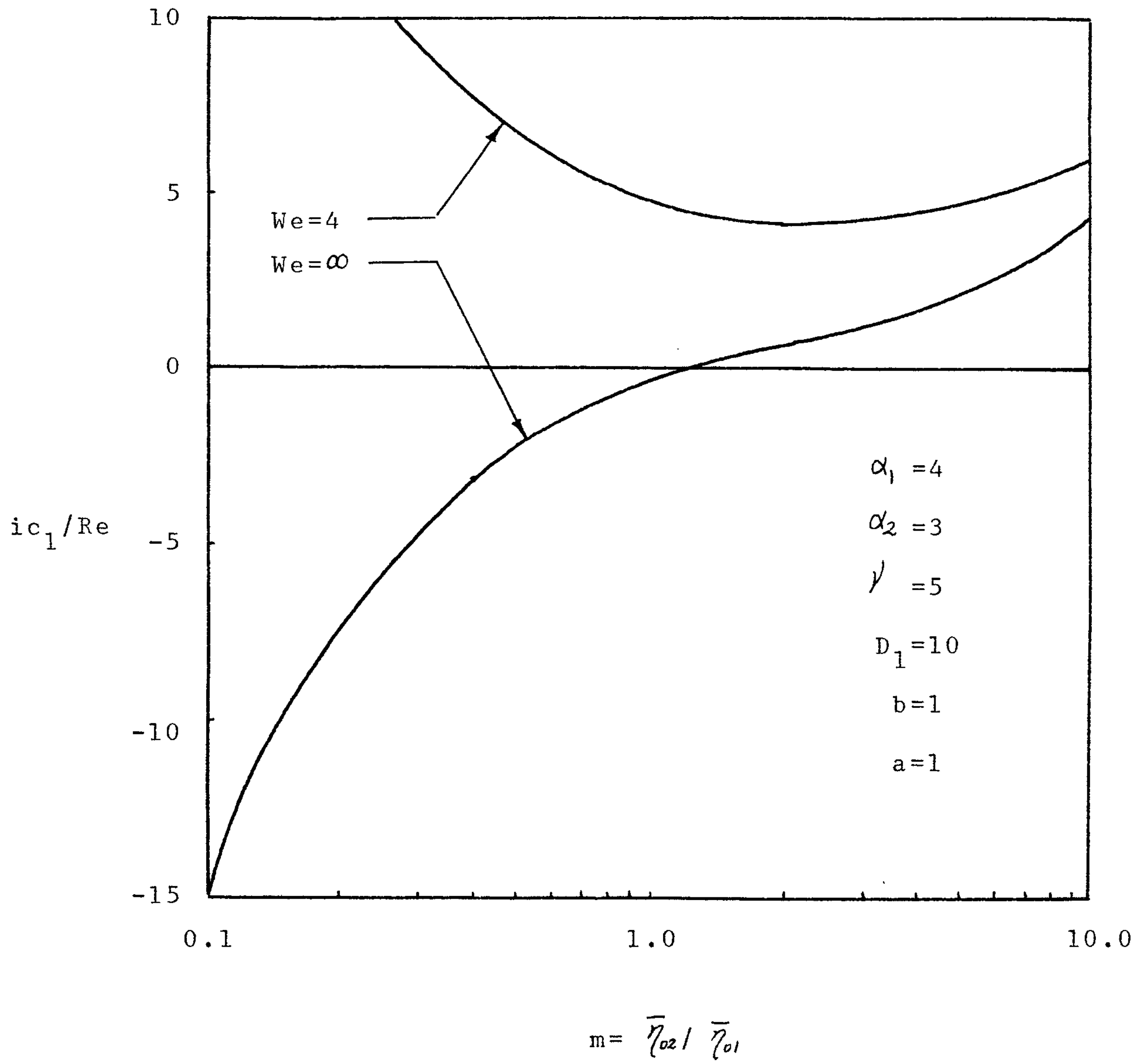

10
5
0
-5
-10
-15
ic$_1$/Re
We=4
We=∞
α_1 =4
α_2 =3
γ =5
D_1=10
b=1
a=1
0.1
1.0
10.0
m= $\bar{\eta}_{02}$ / $\bar{\eta}_{01}$

Fig 5-13

(VI) Discussion

In the present work, numerical analyses were performed for the axisymmetric case. The parameters were chosen in ranges typically found for some common non-Newtonian fluids. Owing to the large number of parameter combinations, the actual eigenvalue, c_1 , of each particular case must be found by using the computer program listed on the Appendix IV.

From Fig (5-1) and Fig (5-12), the viscosity ratio is shown to be destabilizing, i.e. the larger the value of m, the larger the wave growth rate. On the contrary, the shear rate ratio was found to stabilize the flow as its value increased (Fig (5-3) and Fig (5-9)). From Figs (5-5) and (5-9), the factor α_1 seems to have monotonous destabilizing effects. The same monotonous destabilizing effect of α_2 could be seen from Figs (5-7), and (5-10). For ν larger than 1, the surface tension would play a stabilizing role as seen from Fig (5-11), while its effect is negligible for $\nu < 1$, as seen from Figs (5-2), (5-4), (5-8) & (5-11). From Fig (5-12), the effect of D_1 is seen to be stabilizing for lower value of m (<10) and destabilizing for higher value of m (> 10). For m smaller than 10, the surface tension will play a stabilizing role, as shown by comparing Figs (5-4), (5-6) and (5-13).

The most important conclusion to be drawn from the numerical results of the previous section is that the cause of instability is the difference in zero-shear-rate viscosity (m), shear streee (ν), and power parameter (α_1 ; α_2). Surface tension, in general has a stabilizing effects.

Hickox (7) studied the stability of both axisymmetric and asymmetric disturbance for Newtonian fluids with the same geometry. He pointed out that the surface tension would have a negligible effect for m=20 (which was also found for non-Newtonian fluids). He also indicated that an increasing viscosity ratio has a stabilizing influence on asymmetric disturbances, but has a destabilizing effects on axisymmetric disturbances. From our work, the increasing zero-shear-rate viscosity ratio was also found to have a destabilizing effect in the axisymmetric case. Comparing the results for Newtonian and non-Newtonian fluids with axisymmetric disturbances, we find the there is a range of interfacial stability for Newtonian fluids which can not be seen in non-Newtonian systems.

Since only long waves are considered, and since instability is manifested for any Reynolds number however small, turbulence is not expected as an end result of the instability. The long waves considered in this analysis will experience an initial growth rate which is exponential in time. But once the wave amplitude becomes finite, nonlinear effects will become important and must be accounted for.

In the analyses we have assumed the fluid to be non-diffusive. From the physical point of view, this is not unrealistic since, for example, there are many polymers which are not mixed together.

APPENDIX I

The only nonzero velocity of steady state flow is the axial velocity, $\bar{v}_z$, which is a function of r. The Cauchy's equation will be reduced to

$$0 = -\partial\bar{p}_1/\partial r$$

$$0 = -\partial\bar{p}_1/\partial\theta$$

$$0 = -\partial\bar{p}_1/\partial z - \frac{1}{r}\frac{\partial}{\partial r}\left(r\cdot\bar{\tau}_{rz} \right) + \rho_1 g \qquad \text{(AI-1)}$$

for fluid 1, and

$$0 = -\partial\bar{p}_2/\partial r$$

$$0 = -\partial\bar{p}_2/\partial\theta$$

$$0 = -\partial\bar{p}_2/\partial z - \frac{1}{r}\frac{\partial}{\partial r}\left(r\cdot\bar{\tau}_{2rz} \right) + \rho_2\cdot g \qquad \text{(AI-2)}$$

for fluid 2. The axial velocity, thus, can be found by applying Eqs (AI-1), (AI-2) with two boundary conditions and one interfacial condition which are

$$\bar{v}_{1z}(0) \quad \text{finite} \qquad \text{(AI-3)}$$

$$\bar{v}_{2z}(R_2) = 0 \qquad \text{(AI-4)}$$

$$\bar{v}_{1z}(R_1) = \bar{V}_{2z}(R_2) \qquad \text{(AI-5)}$$

Equation (AI-1) result in

$$\bar{\tau}_{1rz} = \frac{\Delta \bar{p}_1}{2} r + \frac{c_1}{r} \tag{AI-6}$$

where $\Delta \bar{p}_1 = \rho_1 g - \partial \bar{p}_1 / \partial z$. The integral constant, c_1, must be zero for fittness of Eq (AI-3).

Now

$$\bar{v}_{1z}' = - \frac{\bar{\tau}_{1rz}}{\bar{\eta}_1}$$

$$= - \frac{1}{\eta_{01}} \bar{\tau}_{1rz} \left[1 + \left(\frac{|\tau_{1rz}|}{\tau_{01}} \right)^{\alpha_1 - 1} \right] \tag{AI-7}$$

Since

$$v_{1z}' < 0 \quad , \quad \text{for} \quad 0 < r < R_1$$

i.e.

$$\bar{\tau}_{1rz} > 0 \quad , \qquad \text{for} \qquad 0 < r < R_1$$

Thus

$$\bar{v}_{1rz}' = - \frac{1}{\eta_{01}} \left[\bar{\tau}_{1rz} + \frac{\tau_{1rz}^{\alpha_1}}{\tau_{01}^{\alpha_1 - 1}} \right]$$

$$= - \frac{1}{\eta_{01}} \left[\frac{\Delta \bar{p}_1}{2} r + \left(\frac{\Delta \bar{p}_1}{2 \tau_{01}} \right)^{\alpha_1} \cdot \tau_{01} \, r^{\alpha_1} \right]$$

$\Rightarrow$

$$\bar{v}_{1z} = - \frac{1}{\eta_{01}} \left[\frac{\Delta \bar{p}_1}{4} r^2 + \left(\frac{\Delta \bar{p}_1}{2 \tau_{01}} \right)^{\alpha_1} \cdot \frac{\tau_{01}}{\alpha_1 + 1} r^{\alpha_1 + 1} + B_1 \right]$$

(AI-8)

Similarity, the equation (AI-2) will have the solution form as following:

$$\bar{v}_{2z} = -\frac{1}{\eta_{02}}\left[\frac{\Delta\bar{p}_2}{2} r^2 + c_2 \ln r + \frac{1}{\tau_{02}^{\alpha_2-1}} \int_*^r \left(\frac{\Delta\bar{p}_2}{2} r + \frac{c_2}{r}\right)^{\alpha_2} dr + B_2\right] \qquad \text{(AI-9)}$$

Application fo Eq (AI-4), (AI-5), (AI-8) and (AI-9), we can solve those three integral constants as

$$B_1 = -\frac{\Delta\bar{p}_1}{4} R_1^2 - \left(\frac{\Delta\bar{p}_1}{2\tau_{01}}\right)^{\alpha_1} \cdot \frac{\tau_{01}}{\alpha_1+1} R_1^{\alpha_1+1} - \eta_{01} V_i$$

$$B_2 = -\frac{\Delta\bar{p}_2}{4} R_2^2 - c_2 \cdot \ln R_2 - \frac{1}{\tau_{02}^{\alpha_2-1}} \int_*^{R_2} \left(\frac{\Delta\bar{p}_2}{2} r + \frac{c_2}{r}\right)^{\alpha_2} dr$$

and

$$c_2 = \left(\frac{\Delta\bar{p}_1 - \Delta\bar{p}_2}{2}\right) R_1^2$$

where V_i is the interfacial velocity which has the expression as

$$V_i = \frac{1}{\eta_{02}}\left[\frac{\Delta p_2 R_2^2}{4}\left(1 - \left(\frac{R_1}{R_2}\right)^2\right) - c_2 \ln\left(\frac{R_1}{R_2}\right) + \frac{1}{\tau_{02}^{\alpha_2-1}} \int_{R_1}^{R_2} \left(\frac{\Delta\bar{p}_2}{2} r + \frac{c_2}{r}\right)^{\alpha_2} dr\right]$$

The equations of steady - state flow could be non-dimensionalized by using the characteristic units as

length : R_1

time : R_1/V_i

velocity : V_i

stress : V_i^2

density : ρ_1

viscosity : η_{01}

Thus

$$\bar{p}_1 = \left(\frac{g}{V_i^2} - \frac{\Delta \bar{p}_1}{\rho_1 V_i^2} \right) R_1 z = \left(\frac{1}{Fr} - \beta_1 \right) z$$

and

$$\bar{p}_2 = \left(\frac{\rho_2}{\rho_1} \frac{g}{V_i^2} - \frac{\Delta \bar{p}_2}{\rho_1 V_i^2} \right) R_1 z - \frac{\sigma}{\rho_1 V_i^2 R_1}$$

$$= \left(\frac{b}{Fr} - \beta_2 \right) z - \frac{1}{W_e}$$

$$= \bar{p}_1 - \frac{1}{W_e}$$

where $Fr = V_i^2 / gR_1$; $\beta_1 = \Delta \bar{p}_1 R_1 / \rho_1 V_i^2$;

$\beta_2 = \Delta \bar{p}_2 R_1 / \rho_1 V_i^2$; $b = \rho_2 / \rho_1$; $W_e = \rho_1 \cdot V_i^2 R_1 / \sigma$

and $a = R_2/R_1$.

The axial velocities will be

$$\bar{v}_{1z} = \frac{1}{\eta_{01} V_i} \left[\frac{\Delta \bar{p}_1 R_1^2}{4} (1 - r^2) + \frac{\tau_{01} R_1}{\alpha_1 + 1} \left(\frac{\Delta \bar{p}_1 R_1}{2 \tau_{01}} \right)^{\alpha_1} \right.$$

$$\left. (1 - r^{\alpha_1 + 1}) \right] + 1$$

$$= \frac{\beta_1 R_e}{4} \left[1 - r^2 + \frac{2 D_1}{\alpha_1 + 1} (1 - r^{\alpha_1 + 1}) \right] + 1$$

and

$$\bar{v}_{2z} = \frac{1}{\eta_{02} V_i} \left[\frac{\Delta \bar{p}_2 R_2^2}{4} \left(1 - \left(\frac{r}{R_2} \right)^2 \right) - C_2 \ln \left(\frac{r}{R_2} \right) \right.$$

$$\left. + \frac{1}{\tau_{02}^{\alpha_2 - 1}} \int_r^{R_2} \left(\frac{\Delta \bar{p}_2}{2} r + \frac{c_2}{r} \right)^{\alpha_2} dr \right]$$

$$= \frac{1}{\eta_{02} V_i} \left[\frac{\Delta \bar{p}_2 R_1^2}{4} (a^2 - r^2) - c_2 \ln \left(\frac{r}{a} \right) \right.$$

$$\left. + \frac{R_1}{\tau_{02}^{\alpha_2 - 1}} \left(\frac{\Delta \bar{p}_2 R_1}{2} \right)^{\alpha_1} \cdot \int_r^a \left(r + \left(\frac{2 c_2}{\Delta \bar{p}_2 R_1^2} \right) \frac{1}{r} \right)^{\alpha_2} dr \right]$$

$$= \frac{R_e \beta_2}{4m} \left[(a^2 - r^2) - 2 \hat{c}_2 \ln \left(\frac{r}{a} \right) + 2 D_2 \int_r^a \left(r + \frac{\hat{c}_2}{r} \right)^{\alpha_2} dr \right]$$

where

$$D_1 = (\Delta \bar{p}_1 R_1 / 2 \tau_{01})^{\alpha_1 - 1} \; ; \quad D_2 = (\Delta \bar{p}_1 R_1 / 2 \tau_{02})^{\alpha_2 - 1} \; ;$$

$$m = \eta_{02} / \eta_{01} \quad \text{and} \quad \hat{c}_2 = 2 c_2 / \Delta \bar{p}_2 R_1^2$$

The viscosities of fluids will be

$$\bar{\eta}_1 = \frac{1}{1 + (D_1 r)^{\alpha_1 - 1}}$$

and

$$\bar{\eta}_2 = \frac{m}{1 + \left(\frac{\Delta \bar{p}_2 R_1}{2\, \tau_{02}} \right)^{\alpha_2 - 1} \cdot \left(r + \frac{\hat{c}_2}{r} \right)^{\alpha_2 - 1}}$$

$$= \frac{m}{1 + D_2 \left(r + \frac{c_2}{r} \right)^{\alpha_2 - 1}}$$

The shear stress tensors will be

$$\bar{\underset{\approx}{\tau}}_1 = \bar{\underset{\approx}{\tau}}_1 / \rho_1 V_i^2 = \begin{pmatrix} 0 & 0 & D_1 r \\ 0 & 0 & 0 \\ D_1 r & 0 & 0 \end{pmatrix}$$

and

$$\bar{\underset{\approx}{\tau}}_2 = \bar{\underset{\approx}{\tau}}_2 / \rho_1 V_i^2 = \begin{pmatrix} 0 & 0 & \frac{\beta_2}{2}\left(r + \frac{\hat{c}_2}{r}\right) \\ 0 & 0 & 0 \\ \frac{\beta_2}{2}\left(r + \frac{\hat{c}_2}{r}\right) & 0 & 0 \end{pmatrix}$$

The corresponding shear rate for each fluid will be

$$\bar{\underset{\approx}{\Delta}}_1 = - \bar{\underset{\approx}{\tau}}_1 / \bar{\eta}_1 = \frac{R_e}{\bar{\eta}_1} \begin{pmatrix} 0 & 0 & -\frac{\beta_1}{2} r \\ 0 & 0 & 0 \\ -\frac{\beta_1}{2} r & 0 & 0 \end{pmatrix}$$

and

$$\bar{\underset{\approx}{\Delta}}_2 = -\underset{\approx}{\bar{\tau}}_2 / \bar{\eta}_2 = \left(\frac{R_1}{V_i}\right) \cdot \frac{1}{\bar{\eta}_2} \begin{pmatrix} 0 & 0 & -\rho_1 V_i^2 \frac{\beta_2}{2}(r+\frac{\hat{c}_2}{r}) \\ 0 & 0 & 0 \\ -\rho_1 V_i^2 \frac{\beta_2}{2}(r+\frac{\hat{c}_2}{r}) & 0 & 0 \end{pmatrix}$$

$$= \frac{R_e}{\bar{\eta}_2} \begin{pmatrix} 0 & 0 & \frac{\beta_2}{2}(r+\frac{\hat{c}_2}{r}) \\ 0 & 0 & 0 \\ -\frac{\beta_2}{2}(r+\frac{\hat{c}_2}{r}) & 0 & 0 \end{pmatrix}$$

The following function groups can be rewritten as

$$\frac{\beta_1 R_e}{4} = Q_1 = \frac{1}{4} \frac{\Delta\bar{p}_1 R_1}{\rho_1 V_i^2} \cdot \frac{\rho_1 V_i R_1}{\eta_{01}}$$

$$= \frac{m}{a^2 k} \cdot \frac{1}{\left[1 - \frac{1}{a^2} + \frac{2\hat{c}_2}{a^2} \cdot \ln a + 2(\gamma k)^{\alpha_2 - 1} \cdot D_1^{\frac{\alpha_2-1}{\alpha_1-1}} \int_1^a (r+\frac{\hat{c}_2}{r})^{\alpha_2} \cdot dr\right]}$$

where $k = \Delta\bar{p}_2 / \Delta\bar{p}_1$ and $\gamma = \tau_{01} / \tau_{02}$

$$\frac{\beta_2 R_e}{4m} = Q_2 = \frac{1}{4m}\,\frac{\Delta\bar{p}_2 R_1}{\rho_1 V_i^2}\,\frac{\rho_1 V_i R_1}{\eta_{01}}$$

$$= \frac{1}{a^2}\,\frac{1}{\left[1-\frac{1}{a^2}+\frac{2\hat{c}_2}{a^2}\cdot\ln a + 2(\gamma k)^{\alpha_2-1}\cdot D_1^{\frac{\alpha_2-1}{\alpha_1-1}}\cdot\int_1^a (r+\frac{\hat{c}_2}{r})^{\alpha_2}\cdot dr\right]}$$

and

$$D_2 = \left(\frac{\Delta\bar{p}_2 R_1}{2\,\tau_{02}}\right)^{\alpha_2-1}$$

$$= (\gamma k)^{\alpha_2-1}\cdot D_1^{\frac{\alpha_2-1}{\alpha_1-1}}$$

APPENDIX II

The shear rate for fluid 1 can be calculated by

$$\underset{\approx}{\Delta}_1 = \underset{\approx}{\bar{\Delta}}_1 + \underset{\approx}{\overset{*}{\Delta}}_1 = - R_e \frac{\underset{\approx}{\tau}_1}{\eta_1}$$

Since

$$1/\bar{\eta}_1 = \left(\frac{1}{2} II_{\underset{\approx}{\tau}_1} \right)^{\frac{1}{2}} = \left| \bar{\tau}_{1rz} + \tau^{*}_{1rz} \right|$$

It should be noted that $\bar{\tau}_{1rz}$ is greater than zero for $0 \leq r \leq R_1$ and $\left|\tau_{1rz}^{*}\right|$ is negligible compared with $\bar{\tau}_{1rz}$. We can get

$$\left|\bar{\tau}_{1rz} + \tau_{1rz}^{*}\right| = \bar{\tau}_{1rz} + \tau_{1rz}^{*}$$

Thus

$$\underset{\approx}{\Delta}_1 = - R_e \frac{\underset{\approx}{\bar{\tau}}_1}{\eta_1}$$

$$= - R_e \left[1 + \left(\frac{\left(\frac{1}{2} II_{\underset{\approx}{\tau}_1} \right)^{\frac{1}{2}}}{\tau_{01} / \rho_1 V_i^2} \right)^{\alpha_1 - 1} \right] \cdot \underset{\approx}{\tau}_1$$

$$= - R_e \left[1 + \left(\frac{\bar{\tau}_{1rz} + \tau_{1rz}^{*}}{\tau_{01} / \rho_1 V_i^2} \right)^{\alpha_1 - 1} \right] \cdot \underset{\approx}{\tau}_1$$

i.e.

$$\Delta_{1rr} = - R_e \left[1 + \left(\frac{\bar{\tau}_{1rz} + \tau_{1rz}^{*}}{\tau_{01} / \rho_1 V_i^2} \right)^{\alpha_1 - 1} \right] \cdot \tau_{1rr}^{*}$$

$$= - \frac{R_e}{\bar{\eta}_1} \tau_{1rr}^{*}$$

$$_{1\theta\theta} = - R_e \left[1 + \left(\frac{\bar{\tau}_{1rz} + \tau_{1rz}^{*}}{_{01} / \; V_i^2} \right)^{\alpha_1 - 1} \right] \cdot \tau_{1\theta\theta}^{*}$$

$$= - \frac{R_e}{\bar{\eta}_1} \cdot \tau_{1\theta\theta}^{*}$$

$$\Delta_{1zz} = - R_e \left[1 + \left(\frac{\bar{\tau}_{1rz} + \tau_{1rz}^{*}}{\tau_{01} / \rho_1 V_i^2} \right)^{\alpha_1 - 1} \right] \tau_{1zz}^{*}$$

$$= - \frac{R_e}{\bar{\eta}_1} \tau_{1zz}^{*}$$

$$\Delta_{1r\theta} = - R_e \left[1 + \left(\frac{\tau_{1rz} + \tau_{1rz}^{*}}{\tau_{01} / \rho_1 V_i^2} \right) \right] \tau_{1r\theta}^{*}$$

$$= - R_e \frac{\tau_{1r\theta}^{*}}{\bar{\eta}_1}$$

$$\Delta_{1\theta z} = - R_e \left[1 + \left(\frac{\bar{\tau}_{1rz} + \tau_{1rz}^{*}}{\tau_{01} / \rho_1 V_i^2} \right)^{\alpha_1 - 1} \right] \; _{1\;z}^{*}$$

$$_{1rz} = - R_e \left[1 + \left(\frac{\bar{\tau}_{1rz} + \tau_{1rz}^{*}}{\tau_{01} / \rho_1 V_i^2} \right)^{\alpha_1 - 1} \right] \left(\bar{\tau}_{1rz} + \tau_{1rz}^{*} \right)$$

$$= - R_e \left[\bar{\tau}_{1rz} + \tau_{1rz}^{*} + \frac{1}{(\tau_{01} / \rho_1 V_i^2)^{\alpha_1 - 1}} (\bar{\tau}_{1rz} + \tau_{1rz}^{*})^{\alpha_1} \right]$$

$$= - R_e \left[\bar{\tau}_{1rz} + \tau_{1rz}^{*} + \frac{\bar{\tau}_{1rz}^{\alpha_1}}{(\tau_{01} / \rho_1 V_i^2)^{\alpha_1 - 1}} \left(1 + \frac{\tau_{1rz}^{*}}{\bar{\tau}_{1rz}} \right)^{\alpha_1} \right]$$

$$= - R_e \left[\bar{\tau}_{1rz} + \tau_{1rz}^{*} + \frac{\tau_{1rz}^{*}}{(\tau_{01} / \rho_1 V_i^2)^{\alpha_1 - 1}} \left(1 + \alpha_1 \frac{\tau_{1rz}^{*}}{\bar{\tau}_{1rz}} \right) \right]$$

$$= - R_e \left[\bar{\tau}_{1rz} + \frac{\bar{\tau}_{1rz}^{\alpha_1}}{(\tau_{01} / \rho_1 V_i^2)^{\alpha_1 - 1}} + \tau_{1rz}^{*} + \alpha_1 \left(\frac{\bar{\tau}_{1rz}}{\tau_{01} / \rho_1 V_i^2} \right)^{\alpha_1 - 1} \cdot \tau_{1rz}^{*} \right]$$

Similarity to fluid 2.

$$\underset{\approx}{\Delta}_2 = \bar{\underset{\approx}{\Delta}}_2 + \overset{*}{\underset{\approx}{\Delta}}_2 = - R_e \cdot \frac{\underset{\approx}{\tau}_2}{\eta_2}$$

Since

$$\frac{\eta_{02}}{\eta_2} = 1 + \left(\frac{\left(\frac{1}{2}\, II\, \underset{\approx}{\tau}_2 \right)^{\frac{1}{2}}}{\tau_{02}} \right)^{\alpha_2 - 1}$$

i.e.

$$\frac{m}{\eta_2} = 1 + \left(\frac{\left| \bar{\tau}_{2rz} + \tau_{2rz}^{*} \right|}{\tau_{02} / \rho_1 V_i^2} \right)^{\alpha_2 - 1}$$

So

$$\underset{\approx}{\Delta}_2 = - \frac{R_e}{m} \left[1 + \left(\frac{\bar{\tau}_{2rz} + \tau_{2rz}^{*}}{\tau_{02} / \rho_1 V_i^2} \right)^{\alpha_2 - 1} \right] \cdot \underset{\approx}{\tau}_2$$

Thus, the shear rate tensor for fluid 2 were shown as following

$$\Delta_{2rr}^{*} = - \frac{R_e}{\bar{\eta}_2} \tau_{2rr}^{*}$$

$$\Delta_{2}^{*} = - \frac{R_e}{\bar{\eta}_2} \tau_{2\theta\theta}^{*}$$

$$\Delta_{2zz}^{*} = - \frac{R_e}{\bar{\eta}_2} \tau_{2zz}^{*}$$

$$\Delta_{2r\theta}^{*} = - \frac{R_e}{\bar{\eta}_2} \tau_{2r\theta}^{*}$$

$$\Delta_{2\theta z}^{*} = - \frac{R_e}{\bar{\eta}_2} \cdot \tau_{2\theta z}^{*}$$

$$\Delta_{2rz}^{*} = - \frac{R_e}{m} \left[1 + \left(\frac{\bar{\tau}_{2rz} + \tau_{2rz}^{*}}{\tau_{02} / \rho_1 V_i^2} \right)^{\alpha_2 - 1} \right] \left(\bar{\tau}_{2rz} + \tau_{2rz}^{*} \right)$$

$$= - \frac{R_e}{m} \left[\bar{\tau}_{2rz} + \frac{\bar{\tau}_{2rz}^{\alpha_2}}{\left(\tau_{02} / \rho_1 V_i^2 \right)^{\alpha_2 - 1}} + \tau_{2rz}^{*} \right.$$

$$\left. + \alpha_2 \left(\frac{\bar{\tau}_{2rz}}{\tau_{02} / \rho_1 V_i^2} \right)^{\alpha_2 - 1} \cdot \tau_{2rz}^{*} \right]$$

APPENDIX III

The governing equation of fluid can be derived from Cauchy's equation by applying Eqs (3-34) - (3-39).

For fluid 1:

r-component

$$\frac{\partial v_{1r}^{*}}{\partial t} + \bar{v}_{1z} \frac{\partial v_{1r}^{*}}{\partial z} = -\frac{\partial p_1^{*}}{\partial r} - \left[\frac{1}{r} \frac{\partial}{\partial r}(r \tau_{1rr}^{*}) + \frac{1}{r} \frac{\partial \tau_{1r\theta}^{*}}{\partial \theta} - \frac{\tau_{1\theta\theta}^{*}}{r} + \frac{\partial \tau_{1rz}^{*}}{\partial z} \right]$$

$\Rightarrow$

$$-i\alpha c\ (iG_1) + \bar{v}_{1z}\ i\alpha(iG_1) = -\ p_1' + \frac{i}{R_e} \left[\frac{2}{r} \frac{\partial}{\partial r} (r \bar{\eta}_1 G_1') + \frac{n\bar{\eta}_1}{r} (H_1' - \frac{H_1}{r} - \frac{nG_1}{r}) - 2\bar{\eta}_1 \frac{(nH_1+G_1)}{r^2} + \alpha \cdot \mu_1 (F_1' - \alpha \cdot G_1) \right]$$

$\Rightarrow$

$$\alpha \cdot (\bar{v}_{1z} - c)G_1 = p_1' - \frac{i \cdot \bar{\eta}_1}{R_e} \left[2G_1'' + 2(\frac{r \cdot \bar{\eta}_1'}{\bar{\eta}_1} + 1) \frac{G_1'}{r} - (\frac{n^2+2}{r^2} + \alpha^2 \frac{\mu_1}{\bar{\eta}_1}) G_1 + n(\frac{H_1'}{r} - 3 \frac{H_1}{r^2}) + \alpha \frac{\mu_1}{\bar{\eta}_1} F_1' \right]$$

θ -component

$$\frac{\partial v_{1\theta}^*}{\partial t} + \bar{v}_{1z}\frac{\partial v_{1\theta}^*}{\partial z} = -\frac{1}{r}\frac{\partial p_1^*}{\partial \theta} - \left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \tau_{1r\theta}^* \right)\right.$$

$$\left. + \frac{1}{r}\frac{\partial \tau_{1\theta\theta}^*}{\partial \theta} + \frac{\partial \tau_{1\theta z}^*}{\partial z} \right]$$

$\Rightarrow$

$$- i\alpha c H_1 + i\alpha \bar{v}_{1z} H_1 = -\frac{in}{r} p_1 + \frac{i}{R_e}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \bar{\eta}_1 \left(r \frac{d}{dr}\left(\frac{H_1}{r}\right)\right.\right.\right.$$

$$\left.\left. - \frac{n}{r} G_1 \right)\right) + \frac{2n\cdot\bar{\eta}_1}{r}\left(\frac{n}{r} H_1 + \frac{G_1}{r} \right)$$

$$\left. + \alpha\bar{\eta}_1 \left(\alpha H_1 + \frac{n}{r} F_1 \right) \right]$$

$\Rightarrow$

$$\alpha\left(\bar{v}_{1z} - c \right) H_1 = -\frac{n}{r} p_1 + \frac{i\cdot\bar{\eta}_1}{R_e}\left[H_1'' + \left(\frac{r\,\bar{\eta}_1'}{\bar{\eta}_1} + 1 \right)\frac{H_1'}{r}\right.$$

$$- \left(\frac{r\,\bar{\eta}_1'/\bar{\eta}_1 + 1 + 2n^2}{r^2} + \alpha^2 \right) H_1 - n\left(\frac{G_1'}{r}\right.$$

$$\left.\left. + \left(\frac{r\cdot\bar{\eta}_1'}{\bar{\eta}_1} + 3 \right)\frac{G_1}{r^2} \right) - \alpha n \frac{F_1}{r} \right]$$

z-component

$$\frac{\partial v_{1z}^{*}}{\partial t} + \bar{v}_{1z}' \cdot v_{1r}^{*} + v_{1z} \cdot \frac{\partial v_{1z}^{*}}{\partial z} = - \frac{\partial p_1^{*}}{\partial z} - [\frac{1}{r} \frac{\partial}{\partial r}(r \tau_{1rz}^{*}) + \frac{1}{r} \frac{\partial \tau_{1\theta z}^{*}}{\partial \theta} + \frac{\partial \tau_{1zz}^{*}}{\partial z}]$$

$$\Rightarrow \quad -i\alpha c F_1 + i\bar{v}_{1z}' \cdot G_1 + i\alpha \bar{v}_{1z} \cdot F_1 = -i\alpha p_1 + \frac{1}{R_e} [\mu_1 (\frac{F_1'}{r} - \alpha \frac{G_1}{r}) + (\mu_1 (F_1' - \alpha G_1))' + n \bar{\eta}_1 (\frac{\alpha H_1}{r} + \frac{n}{r^2} F_1) + 2 \cdot \bar{\eta}_1 \cdot \alpha^2 \cdot F_1]$$

$$\Rightarrow \quad \alpha \cdot (\bar{v}_{1z} - c) F_1 + \bar{v}_{1z}' G_1 = -\alpha p_1 - \frac{i \cdot \bar{\eta}_1}{R_e} [\frac{\mu_1}{\bar{\eta}_1} F_1'' + \frac{\mu_1}{\bar{\eta}_1} (1 + \frac{r \mu_1'}{\mu_1}) \frac{F_1'}{r} - (\frac{n^2}{r^2} + 2\alpha^2) F_1 - \frac{\mu_1}{\bar{\eta}_1} \alpha (G_1' + (1 + \frac{r \mu_1'}{\mu_1}) \frac{G_1}{r} - \frac{n\alpha}{r} H_1]$$

Continuity equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r v_{1r}^{*} \right) + \frac{1}{r}\frac{\partial v_{1\theta}^{*}}{\partial \theta} + \frac{\partial v_{1z}^{*}}{\partial z} = 0$$

$$\Rightarrow \quad G_1' + \frac{G_1}{r} + \frac{n}{r} H_1 + \alpha F_1 = 0$$

The governing equations for fluid 2 are similar to those for fluid 1 except the density ratio, b, and characteristic viscosity ratio, $m = \eta_{02} / \eta_{01}$. We shall omit the derivation procedures for them.

APPENDIX IV

```
       SUBROUTINE SSS(S3,S4,AV,A,CO,B,B1,B2,B3,Q2,RF2,RM,
      1D2,B4)
       M=RM
       W0=B*Q2*RF2*D2/M
       W1=A**(2.*RF2+4.)/(2.*RF2+4.)+RF2*D2*A**
      1(3.*RF2+3.)/(3.*RF2+3.)
       W2=A**(2.*RF2+2.)/(2.*RF2+2.)+RF2*D2*A**
      1(3.*RF2+1.)/(3.*RF2+1.)
       W3=A**(RF2+5.)/(RF2+5.)+RF2*D2*A**(2.*RF2
      1+4.)/(2.*RF2+4.)
       W4=A**(RF2+3.)*ALOG(A)/(RF2+3.)-2.*A**(RF2+3.)/((
      1RF2+3.)**(2.))+RF2*D2*(A**(2.*RF2+2.)*ALOG(A)-A**
      1(2.*RF2+2.)/(2.*RF2+2.)-A**(2.*RF2+2.)/(RF2+3.))
      1/(2.*RF2+2.)
       W5=A**(RF2+3.)/(RF2+3.)+RF2*D2*A**(2.
      1*RF2+2.)/(2.*RF2+2.)
       W6=A**(RF2+1.)/(RF2+1.)+RF2*D2*A**(2.*RF2
      1)/(2.*RF2)
       W7=A*A*ALOG(A)/4.-A*A/4.+RF2*D2*(A**(RF2+1.)*ALOG
      1(A)-A**(RF2+1.)/(RF2+1.)-A**(RF2+1.)/2.)/(2.*RF2+2.)
       E21=-W0*D2*B1*W1/((RF2+1.)**(2.)*(2.*RF2+4.))
       E22=-W0*2.*D2*B2*W2/((RF2-1.)*(RF2+1.)*(2.*RF2+2.))
       E23=W0*B1/RF2*W3/(2.*(RF2+1.)*(RF2+5.))
       E24=2.*W0*B2/RF2*W4/((RF2+1.)*(RF2+3.))
       E25=2.*W0*B3/RF2*W5/((RF2+1.)*(RF2+3.))
       E26=W0*B1*(CO/Q2-A*A+B2/B1)*W5/(2.*(RF2+1.)*(RF2+3.))
       E28=-W0*D2*B1*W1/(2.*(RF2+1.)*(2.*RF2+4.))
       E30=W0*D2*B2*W2/(2.*(RF2+1.)*(2.*RF2+2.))
       E31=W0*B2*(CO/Q2-A*A)*W6/((RF2-1.)*(RF2+1.))
       E32=W0*B2*W5/(2.*(RF2-1.)*(RF2+3.))
       E33=-W0*D2*B2*W2/(2.*(RF2-1.)*(2.*RF2+2.))
       E34=W0*B1*W3/(2.*(RF2+3.)*(RF2+5.))
       E35=W0*D2*B1*W1/(2.*(RF2+3.)*(2.*RF2+4.))
       E36=W0*B2*(A*A-CO/Q2)*W7/(RF2*D2)
       E37=W0*B2/RF2*W4/(RF2+3.)
       E38=W0*B1/RF2*W3/(8.*(RF2+5.))
       E39=W0*(2.*B3-B2)/RF2*W5/(2*(RF2+3.))
       E40=-2.*W0*B4/RF2*W6/(RF2+1.)
       E41=-B*Q2*B1*(A**(6.)/6.+RF2*D2*A**(RF2+5.)/(RF2
      1+5.))/(48.*M)
       E42=B*Q2*B1*(A*A-CO/Q2-2.*B2/B1)*(A**(4.)/4.+RF2
      1*D2*A**(RF2+3.)/(RF2+3.))/(16.*M)
       S3=E21+E22+E23+E24+E25+E26+E28+E30+E31+E32+E33+E34
      1+E35+E36+E37+E38+E39+E40+E41+E42
       WW1=-A**(2.*RF2+5.)/((2.*RF2+4.)*(2.*RF2+6.))-RF2
      1*D2*A**(3.*RF2+4.)/((3.*RF2+3.)*(3.*RF2+5.))
       WW2=-A**(2.*RF2+3.)/((2.*RF2+2.)*(2.*RF2+4.))-RF2
      1*D2*A**(3.*RF2+2.)/((3.*RF2+1.)*(3.*RF2+3.))
       WW3=-A**(RF2+6.)/((RF2+5.)*(RF2+7.))-RF2*D2*A**(
      12.*RF2+5.)/((2.*RF2+4.)*(2.*RF2+6.))
       Z1=-A**(RF2+4.)*ALOG(A)/((RF2+3.)*(RF2+5.))+2.*A
      1**(RF2+4.)/((RF2+3.)**(2.)*(RF2+5.))+A**(RF2+4.)
      1/((RF2+3.)*(RF2+5.)**(2.))
       Z2=A**(2.*RF2+3.)*ALOG(A)/(2.*RF2+4.)-A**(2.*RF2
      1+3.)/((2.*RF2+4.)**(2.))-A**(2.*RF2+4.)/((2.*RF2+2.
      1)*(2.*RF2+5.))-A**(2.*RF2+4.)/((RF2+3.)*(2.*RF2+5.))
       WW4=Z1-RF2*D2*Z2/(2.*RF2+2.)
       WW5=-A**(RF2+4.)/((RF2+3.)*(RF2+5.))-RF2*D2*A**(
      12.*RF2+3.)/((2.*RF2+2.)*(2.*RF2+2.)*(2.*RF2+4.))
       WW6=-A**(RF2+2.)/((RF2+1.)*(RF2+3.))-D2*A**(2.*RF2
      1+1.)/(2.*(2.*RF2+2.))
       Z3=A**(RF2+2.)*ALOG(A)/(RF2+3.)-A**(RF2+2.)/((RF2
      1+3.)**(2.))-A**(RF2+2.)/((RF2+1.)*(RF2+3.))-A**(
      1RF2+2.)/(2.*(RF2+3.))
```

```
     1*(RF2+1.))
      S4=E21*WW1/W1+E22*WW2/W2+E23*WW3/W3+E24*WW4/W4+E25
     1*WW5/W5+E26*WW5/W5+E28*WW1/W1+E30*WW2/W2
      S4=S4+E31*WW6/W6+E32*WW5/W5+E33*WW2/W2+E34*WW3/W3
     1+E35*WW1/W1+E36*WW7/W7+E37*WW4/W4
      S4=S4+E38*WW3/W3+E39*WW5/W5+E40*WW6/W6
      E43=B*Q2*B1*(A**(7.)/48.+RF2*D2*A**(RF2+6.)
     1/((RF2+5.)*(RF2+7.)))/(48.*M)
      E44=-B*Q2*B1*(A*A-CO/Q2-2.*B2/B1)*(A**(5.)/24.+RF2
     1*D2*A**(RF2+4.)/((RF2+3.)*(RF2+5.)))/(16.*M)
      S4=S4+E43+E44
      AV=E21/W1+E22/W2+E23/W3+E24/W4*(-1./(RF2+3.))+E25
     1/W5+E26/W5+E28/W1+E30/W2+E31/W6+E32/W5+E33/W2+E34/
     1W3+E35/W1+E36/W7*(-1./4.)+E37/W4*(-1./(RF2+3.))+
     1E38/W3+E39/W5+E40/W6+B*Q2*B1*(-1./12.+(A*A-CO/Q2
     1-2.*B2/B1)/4.)/(4.*M)
      RETURN
      END
      SUBROUTINE  GAUSS(X,Y,N,EPS)
      DIMENSION X(5,5),Y(5)
      DO 1 I=1,N
      K=I
      IF(I-N)21,7,21
21    IF(ABS(X(I,I))-EPS)6,6,7
6     K=K+1
      Y(I)=Y(I)+Y(K)
      DO  23  J=1,N
23    X(I,J)=X(I,J)+X(K,J)
      GO TO 21
7     DIV=X(I,I)
      Y(I)=Y(I)/DIV
      DO 9 J=1,N
9     X(I,J)=X(I,J)/DIV
      DO 1 M=1,N
      DELT=X(M,I)
      IF(ABS(DELT)-EPS) 1,1,16
16    IF(M-I) 10,1,10
10    Y(M)=Y(M)-Y(I)*DELT
      DO 11  J=1,N
11    X(M,J)=X(M,J)-X(I,J)*DELT
1     CONTINUE
      RETURN
      END
C EIGEN VALUE,C1, FOR AXISYMMETRIC CASE BY USING
C ELLIS MODEL
C
C
      DIMENSION X(5,5),Y(5)
      REAL M,K
      RF1=2.
      RF2=4.
      GR=0.1
      M=100.
      D1=10.
      WRITE(2,301) RF1,RF2,GR,M,D1
301   FORMAT(//'      RF1='F5.1'   RF2='F5.1'   GR='F5.1
     1'   M='F5.1'   D1='F5.1)
      K=1.
      RF12=(RF2-1.)/(RF1-1.)
101   R1=0.0079
      R2=0.079
      RD=(R2-R1)/9.
410   A=R2/R1
      B=1.
```

```
     1*D1**(RF12)*A**(RF2+1.)/(RF2+1.))
      Q2=Q1*K/M
      V1V2=2.*(Q2*(1.+D2)-Q1-Q1*D1)
      X(1,1)=1./16.+RF1*D1/(2.*(RF1+1.)*(RF1+3.))
      X(1,2)=-(A**(4.)+1.)/16.+A*A/8.-RF2*D2*A**(RF2+3.)/(4.*
     1RF2+12.)+RF2*D2*A**(RF2+1.)/(4.*RF2+4.)-RF2*D2/(2.*(RF2
     1+1.)*(RF2+3.))
      X(1,3)=(1.-A*A)/4.+RF2*D2*A**(RF2-1.)/(RF2-1.)-RF2*D2
     1*A**(RF2+1.)/(2.*RF2+2.)-RF2*D2/(RF2*RF2-1.)+ALOG(A)/2.
      E1=-2.*(Q1-M*Q2)/V1V2
      X(2,1)=E1*(1./4.+RF1*D1/(2.*(RF1+1.)))-1./2.
      X(2,2)=M/2.+E1*((A*A-1.)/4.+RF2*D2*(A**(RF2+1.)-1.)/
     1(2.*(RF2+1.)))
      X(2,3)=M+E1*(ALOG(A)+RF2*D2*(A**(RF2-1.)-1.)/(RF2-1.))
      X(3,1)=1.
      X(3,2)=-M
      X(3,3)=0.
      A2=0.
      A3=1.
      Y(1)=-A3/2.
      Y(2)=-E1*A3
      Y(3)=0.
      EPS=0.000001
      N1=3
      CALL GAUSS(X,Y,N1,EPS)
      A1=Y(1)
      B1=Y(2)
      B2=Y(3)
      B3=-B1*(A*A/4.+RF2*D2*A**(RF2+1.)/(2.*(RF2+1.)))-B2
     1*(ALOG(A)+RF2*D2*A**(RF2-1.)/(RF2-1.))
      B4=-B1*(A**(4.)/16+RF2*D2*A**(RF2+3.)/(4.*(RF2+3.)))
     1-B2*(A*A/4.+RF2*D2*A**(RF2+1.)/(2.*(RF2+1.)))
      F1F2=A1*(1./4.+RF1*D1/(2.*(RF1+1.)))+A3+B1*((A*A
     1-1.)/4.+RF2*D2*(A**(RF2+1.)-1.)/(2.*RF2+2.))+B2*
     1(ALOG(A)+RF2*D2*(A**(RF2-1.)-1.)/(RF2-1.))
      C0=1.+V1V2*(-A1*(1./16.+RF1*D1/(2.*(RF1+1.)*(RF1
     1+3.)))-A3/2.)/F1F2
      SI1=Q1*2.*RF1*D1*D1*A1/((RF1+1.)*(RF1+1.)*(RF1+3.))
      SI2=Q1*D1*A1*(RF1/(RF1+1.)*1./(RF1+3.)-3./8.)
      SI3=Q1*D1*A3*(RF1-1.)/(RF1+1.)+A1*RF1*D1*(Q1*(1.
     1+2.*D1/(RF1+1.))-C0)/(2.*(RF1+1.))
      SI4=Q1*A1/8.
      SI5=A1/4.*(Q1*(1.+2.*D1/(RF1+1.))-C0)
      V1C0=1.-C0
      E4=SI1*(1./((2.*RF1+4.)*(2.*RF1+6.))+RF1*D1/((3.
     1*RF1+3.)*(3.*RF1+5.)))/(2.*RF1+4.)
      E5=SI2*(1./((RF1+5.)*(RF1+7.))+RF1*D1/((2.*RF1+4.
     1)*(2.*RF1+6.)))/(RF1+5.)
      E6=SI3*(1./((RF1+3.)*(RF1+5.))+RF1*D1/((2.*RF1+2.
     1)*(2.*RF1+4.)))/(RF1+3.)
      E7=SI4*(1./48.+RF1*D1/((RF1+5.)*(RF1+7.)))/6.
      E8=SI5*(1./24.+RF1*D1/((RF1+3.)*(RF1+5.)))/4.
      S2=E4-E5-E6+E7-E8
      E9=SI1*(1./(2.*RF1+4.)+RF1*D1/(3.*RF1+3.))/(2.*RF1
     1+4.)
      E10=SI2*(1./(RF1+5.)+RF1*D1/(2.*RF1+4.))/(RF1+5.)
      E11=SI3*(1./(RF1+3.)+RF1*D1/(2.*RF1+2.))/(RF1+3.)
      E12=SI4*(1./6.+RF1*D1/(RF1+5.))/6.
      E13=SI5*(1./4.+RF1*D1/(RF1+3.))/4.
      S1=-E9+E10+E11-E12+E13
      A3S=0.
      X(1,1)=0.
      X(1,2)=A*A/4.+RF2*D2*A**(RF2+1.)/(2.*(RF2+1.))
      X(1,3)=ALOG(A)+RF2*D2*A**(RF2-1.)/(RF2-1.)
      X(1,4)=1.
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      X(2,1)=0.
      X(2,2)=A*A*A/16.+RF2*D2*A**(RF2+2.)/(2.*(RF2+1.)
     1*(RF2+3.))
      X(2,3)=A*ALOG(A)/2.-A/4.+RF2*D2*A**(RF2)/((RF2
     1-1.)*(RF2+1.))
      X(2,4)=A/2.
      X(2,5)=-1./A
      X(3,1)=1./16.+RF1*D1/(2.*(RF1+1.)*(RF1+3.))
      X(3,2)=-1./16.-RF2*D2/(2.*(RF2+1.)*(RF2+3.))
      X(3,3)=1./4.-RF2*D2/((RF2-1.)*(RF2+1.))
      X(3,4)=-1./2.
      X(3,5)=1.
      X(4,1)=1.
      X(4,2)=-M
      X(4,3)=0.
      X(4,4)=0.
      X(4,5)=0.
      EE=A1/2.+A2-M*(B1/2.+B2)
      X(5,1)=(EE*(1./4.+RF1*D1/(2.*RF1+2.)))/F1F2-1./2.
     1+(1./16.+RF1*D1/(2.*(RF1+1.)*(RF1+3.)))*(-EE*V1V2
     1/(F1F2*V1V2))
      X(5,2)=M/2.-EE*(1./4.+RF2*D2/(2.*(RF2+1.)))/F1F2
      X(5,3)=M-EE*RF2*D2/(F1F2*(RF2-1.))
      X(5,4)=-EE/F1F2
      X(5,5)=0.
      RM=M
      CALL SSS(S3,S4,AV,A,CO,B,B1,B2,B3,Q2,RF2,RM,D2,B4)
      S3A=S3
      S4A=S4
      AA=1.
      CALL SSS(S3,S4,AV,AA,CO,B,B1,B2,B3,Q2,RF2,RM,D2,B4)
      S31=S3
      S41=S4
      Y(1)=-S3A
      Y(2)=-S4A
      Y(3)=S2-S41
      Y(4)=A3*(Q1*(1.+2.*D1/(RF1+1.))-CO)-B*Q2*(B3*(A*A
     1-CO/Q2-2.*B4))
      Y(5)=EE*(S31-S1-V1V2*S2/V1CO)/F1F2-M*AV-SI1/(2.
     1*RF1+4.)+SI2/(RF1+5.)+SI3/(RF1+3.)-SI4/6.+SI5/4.
      N2=5
      CALL GAUSS(X,Y,N2,EPS)
      A1S=Y(1)
      B1S=Y(2)
      B2S=Y(3)
      B3S=Y(4)
      B4S=Y(5)
      CC=V1CO*(S1+A1S*(1./4.+RF1*D1/(2.*RF1+2.))-S31-B1S
     1*(1./4.+RF2*D2/(2.*RF2+2.))-B2S*RF2*D2/(RF2-1.)-B3S)
      C1=(CC+V1V2*(S2-A1S*(1./16.+RF1*D1/(2.*(RF1+1.)*
     1(RF1+3.)))))/F1F2
      BB=R1/R2
      WRITE(2,4000) M,BB,C1
 4000 FORMAT('      M='F5.1'    R1/R2='F5.1'      C1/RE='
     1F30.6)
      IF(BB.GE.0.9) GO TO 5000
      R1=R1+RD
      GO TO 410
 5000 CALL EXIT
      END
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