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DECENTRALIZED OPTIMAL CONTROL WITH APPLICATION TO DYNAMIC ROUTING IN COMPUTER COMMUNICATION NETWORKS

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DECENTRALIZED OPTIMAL CONTROL

WITH APPLICATION TO

DYNAMIC ROUTING IN COMPUTER COMMUNICATION NETWORKS

by

Ning Wang

Dissertation submitted to the Faculty of the Graduate School
of the New Jersey Institute of Technology in partial fulfillment
of the requirements for the degree of
Doctor of Engineering Science
1984

APPROVAL SHEET

Title of Thesis: Decentralized Optimal Control with Application to Dynamic Routing in Computer Communication Networks

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Doctor of Engineering Science, 1984

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ABSTRACT

Title of Thesis:

Decentralized Optimal Control with Application to

Dynamic Routing in Computer Communication Networks

Ning Wang, Doctor of Engineering Science, 1984

Thesis directed by: Professor of Electrical Engineering, Marshall C. Kuo

This research considers the dynamic routing problem of computer communication networks in the framework of decentralized control theory. The routing dynamics are modeled in terms of a state equation with multiple controllers. Routing, or control of message flow, is formulated as an optimal control problem with multiple decision makers. Each decision maker may have access to different set of information and work cooperatively to optimize a common system performance index.

Necessary and sufficient conditions for optimality are derived for a system with a deterministic and a stochastic traffic patterns under a linear information structure and a quadratic performance index. The resultant control strategies are examined with two extreme information cases: (1) complete information where all state information are available to every local controller through measurement or perfect communication, and (2) partial information where there is no communication among controllers. A three node network is used as a numerical example to interpret the results.

Acknowledgement

I wish to express my sincere gratitude to Professor Marshall C. Kuo for the time, guidance, and assistance given to me during the course of this research. Thanks also due to Professor Roy Plastock who introduced me the LINPACK programming package which made the numerical computation significantly easier. I would also like to thank professor Robert De Lucia and professor Stanley S. Reisman for serving on my doctoral committee.

I wish to dedicate this work to my wife Reenie and my son Michael for their patient understanding and support for so many years. Finally I like to thank my parents who have supported me when I most needed encouragement or help.

CONTENTS

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1. INTRODUCTION

In the formulation of traditional control problems it is generally assumed that the decision making process is centralized. That is, a single controller or decision maker has access to *all* sensor measurements and generates *all* control commands for the entire system as shown in **Figure 1-1**. This assumption of centralization in traditional control represents a special case of *classical information pattern* or *nested-information structure*. This implies that in the mathematical formulation of the problem one implicitly assumes that the central controller has access to all past measurements and controls and has instant recall of them. This assumption suggests, furthermore, the processing or computation for the control decision is taking place at a unique location. The notion of *centrality* is inherited by both the classical servomechanism and modern control and estimation theory which have been the dominating subjects for control engineering during the last three or four decades. In spite of many successful theories and applications, numerous researchers have come to realize that many systems with complexity cannot be handled within the existing framework.

When considering a large-scale system with several controllers or decision makers, the presupposition of centrality is no longer valid [1] [2]. Consider in **Figure 1-2**, each local controller receives only a subset of the total measurements and generates only a subset of the total decisions. Such decentralized control problems are characterized by so-called *non-classical information patterns* or *non-nested information structure*. What this means is that each local controller does not have instantaneous access to the other's measurements and decisions. The restriction on information transfer between certain groups of sensors and actuators is one of the basic characteristics in decentralized control.

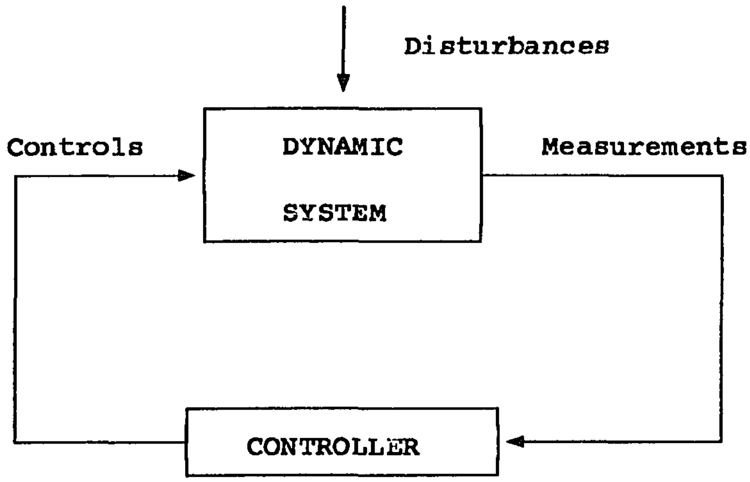

Figure 1-1. Structure of Centralized Control System

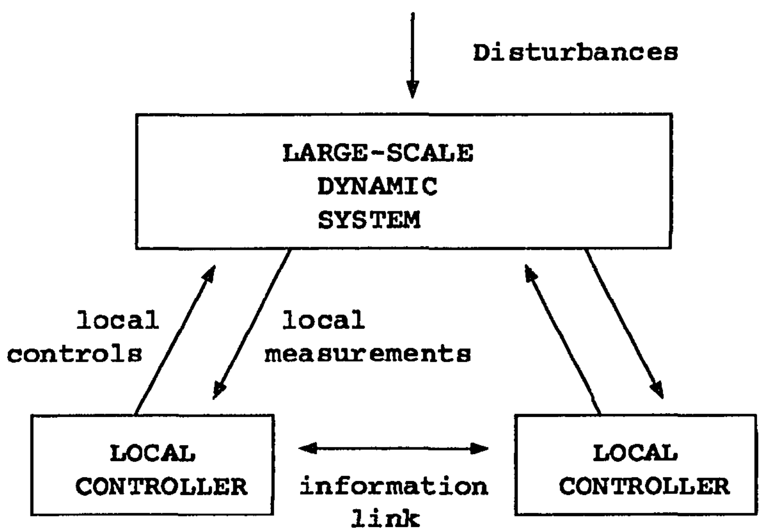

Figure 1-2. Structure of Decentralized Control System

1.1 Decentralized Control of Large-Scale Systems

1.1.1 Characteristics of Large-Scale Systems

The control of large-scale systems has been an active topic of research for the last few years. Recent progress have been reported in urban traffic control and transportation systems, water resources management and river pollution control, socio-economic systems, power systems load dispatch and frequency control, and communications networks routing and congestion control.

These large-scale systems often share one or more of the following common attributes:

- topologically configured as a network
- consisted of more than one controller
- characterized by geographical separation
- featured correlated but different information to different controllers.

The control problems in most of the large-scale systems such as economic and social systems are of great complexity and at present have no general theory to solve them. However, the computer communication networks are relatively simple in the sense that many problems in computer communication can be described by simple models. It seems therefore that joining decentralized control theory and problems in computer communication forms a good basis for new research.

1.1.2 Information Structure

The issue of information plays a central role in large-scale decentralized control problem. Information can be informally defined as a commodity that improves decisions [3]. In study of decentralized control, two kinds of information are often involved:

- Information about the system model; this includes system dynamics, performance index, and statistics of environmental disturbance. This type of information will be given in advance in our discussions and hence is called *a priori* information.

- Sensor (or state) information about the system response; this includes measurements and possible communication from other controllers. This class of information is also referred to as the *coupling information* for it represents the effects of external actions that are coupled into the local process.

As described in previous sections the information aspect is one of the most important factors in control of a large-scale system with multiple controllers or decision makers.

Information in a multi-controller environment was first studied by Rander [4] with a static team problem*. He has proved that with static information structures, a unique globally optimal solution exists for the linear quadratic Gaussian team problems. Ho and Chu [5] extended these results to dynamic Linear Quadratic Gaussian (LQG) team problem with the concept of nested information structure. This extension, however, is only for this special class of information structure, and it is well known that in the general case the optimal solutions are non-linear and are too difficult if not impossible to determine [6].

We shall formalize the above discussion in terms of a general model for a multiperson optimal control problem. Let us consider a decentralized system composed of N controllers indexed by $i=1,2,\cdots,N$ with output y obtained in some processes given by

$$y = g(u,\xi)$$

where u is the vector of manipulated input (control) or action, ξ is a random vector which

* A team can be visualized as the nodes of a network working together to optimize a common performance index.

represents all the environmental disturbance affecting the output. Each controller receives certain information z_i and controls the decision variable u_i. It is clear that the controller's success in optimizing a selected system performance index J will depend on the quality of the coupling information it receives. In the most general case, the set of state information $z=[z_1, z_2, \cdots, z_N]$ is a function of controls including actions taken by other controllers and a random vector ξ which represents all the uncertainties of the external world that are not controlled by any of the members.

$$z_i = h_i(u, \xi)$$

where $h_i(\cdot)$ is defined as the *information function*, and the collection of $h = [h_1, h_2, \cdots, h_N]$ is called the *information structure* of the system. The problem may have multiple goals (performance index) such as the case of an N-person differential game. However, in this research we consider the system with only one criterion. We denote the performance index common to all the members as

$$J = J(\gamma_1, \gamma_2, \cdots, \gamma_N)$$

where γ_i is the control strategy,

$$u_i = \gamma_i(z_i), \qquad \gamma_i \in \Gamma_i$$

and Γ_i is the class of admissible control strategies for controller i.

In this model, the controllers require *a priori* information on process $g(\cdot)$ (e.g. system dynamics), goal J (performance), and disturbance ξ as well as the sensor information z_i. Some concepts and definitions on information structure must be introduced at this point:

- A system has *perfect recall* or *perfect memory* if at time t, the controller remembers perfectly what it has known and what it has done before. That is, any information the controller had at some time remains available to it at any later time.

- An information structure is said to be *classical* if all controllers receive the same information and have perfect recall [7].

- An information structure is called *static* if there is no explicit relation between the control and information of different members.

  $z_i = h_i(\xi)$.

  In other words, z_i is only the function of ξ but is independent of what other controllers have done. It is *dynamic* otherwise.

  $z_i = h_i(\xi, u)$

- An information structure is called *partially nested* (PN) [5] if each controller is informationally superior to all of its precedents. That is, for each controller i and all its precedent j (which act preceding to i) the information z_j can be generated from z_i in the sense that knowing z_i implies knowing z_j.

- A system is *causal* if what happens in the future cannot affect what is observed now. In other words, if the control action of j affects the information z_i, then, in a causal system, u_i cannot affect the information z_j.

- A class of information structure is called *linear* if z_i can be expressed in terms of a linear function of ξ and some of the control actions other members have taken, i.e.

  $$z_i = H_i\xi + \sum_j D_{ij}\mu_j, \qquad i=1,2,\cdots,N$$

  where H_i and D_{ij} are matrices of appropriate dimensions and are known to all the members. Let the dimensions of ξ and h_i be n and m respectively. The dimension of H_i is, in general, $m \times n$, $0 \le m \le n$ and all rows of H_i are independent. The number m is called the rank of the information structure. If H_i has a full rank n for all i, the information is *complete*; when $m=0$, the information is *null*. In general, the information structure lies between these two extremes and is classified as *incomplete* or *partial information*.

1.1.3 Value of Information

Intuitively, we can state that better information for all controllers implies better performance for the entire system. Therefore, the achievable optimal performance depends upon the specific information structure chosen. In order to compare alternative information structures, it would be useful to introduce the concept of *value of information*. The value of information is defined as the improvement in performance due to the information, or

Value of Information = The best the controller can do with the information

− The best the controller can do without the information

Translated into mathematical form, this is

$$VI = \underset{\gamma \in \Gamma}{Min}\, \mathbf{E}\{J(\xi,\gamma(z))\} - \underset{\gamma \in \Gamma_c}{Min}\, \mathbf{E}\{J(\xi,\gamma(z))\}$$

where

Γ is the class of admissible control strategies with z

Γ_c is the class of admissible control strategies which are independent of z

$\mathbf{E}$ is the expectation over ξ.

We have briefly described the optimal control problem with multiple controllers and associated issues on information structure. In the following section, we shall introduce the computer communication networks and formulate the routing problem into the framework of decentralized control.

1.2 Introduction to Computer Communication Networks

During the last decade and half, we have witnessed the merging of the two most important technologies in this *information age*. A new discipline of **computer communication networks** was started as result of the integration of telecommunication engineering and computers. A computer network is a facility for interconnection of autonomous computer systems and terminals for the purpose of data transmission among them. In terms of geographic scope, there exist two kinds of computer communication networks: the *long-haul networks* which often span tens to thousands of miles for intercontinental transmission and the *local area networks* which span distances of a couple of miles within a building or cluster of buildings. Since the major function of computer communication networks is to provide data transport capability among computers and peripherals they are also referred to as data communication networks and transport networks in some of the literature.

The motivations for having computer networks and distributed systems are many. Computer networks allow geographically dispersed users to share expensive computing power, databases, software, and specialized hardware devices. Computer networks also provide high reliability of services to the user by offering alternative sources of supply. Another reason to have computer networks is the superior price/performance ratio of smaller mini/micro computers over a large main frame.

Installed in 1969, the Advanced Research Projects Agency Network (ARPANET) of the U. S. Department of Defence is generally recognized as one of the pioneering efforts in computer communication networking. Many computer networks of similar type have been developed and deployed since then and even more are being planned around the world. Public networks such as TELENET [8], TYMNET [9] and BPSS [10] in the United

States, DATAPAC [11] of Canada, TRANSPAC [12] and CYCLADES [13] in France, EDS [14] in German, and DDX [15] in Japan are examples of many networks in operation.

1.2.1 Computer Communication Network Elements

A computer communication network may be partitioned into a *communication subnetwork* or (subnet) and a *user-resource* subnetwork [16] as shown in **Figure 1-3**. The communication subnet is a collection of switching *nodes* interconnected by a set of high speed communication channels or *links*. The switching nodes consist of special communication computers and interface for the *host* and communication *concentrator* for low speed terminals. The major function of the subnet is data transportation. All user oriented processing and storage are handled by the user-resource subnetwork. The user-resource subnetwork includes computers and terminals as well as applications which provide services to the users of the network.

Three switching techniques are used for the construction of computer communication networks. *Circuit switching* originates from the telephone (voice) network where a complete end-to-end path must be established prior to commencement of data transmission. The path is held for the duration of the call. One of the alternative switching technique is called *message switching*. In message switching networks, messages to be transferred are first stored in the source node and then forwarded later to the destination node one hop at a time through intermediate nodes. The third technique is *packet switching*. With packet switching, a message is divided (packetized) into smaller units called packets. Packet switching also utilizes store-and-forward concept but each packet may be transferred to the destination via different routes. At the destination node packets of the same message must be reassembled (depacketized) before being sent to the receiver.

Each node represents a decision point and it has to make real time decisions on how to direct the different classes of messages and/or packets over the available links to their desired destinations. *Routing* is the selection of the particular path which data will take while traveling through the network to its destination node.

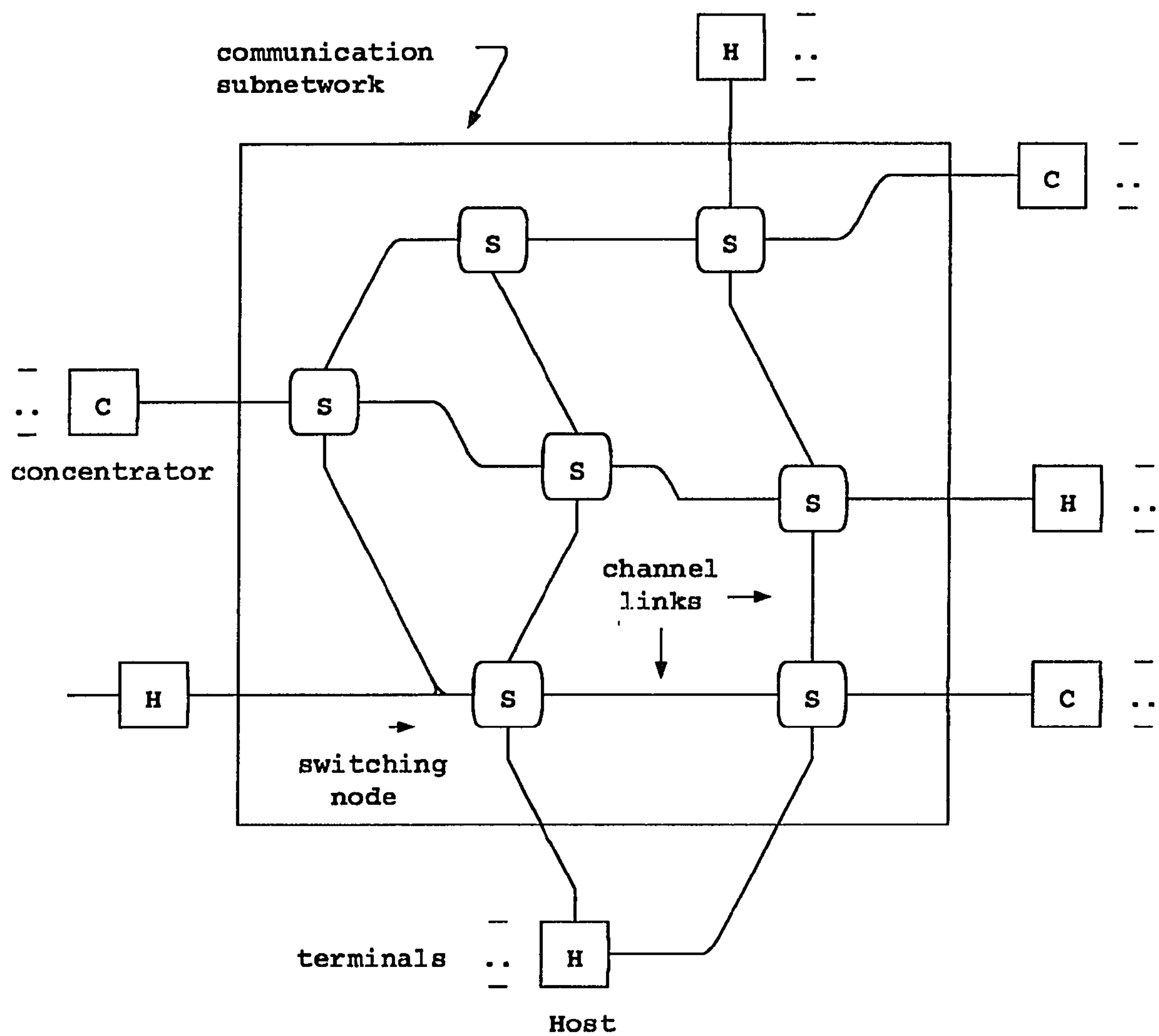

Figure 1-3. Elements of a Computer Communication Network

1.2.2 Classification of Routing Strategy

In the design and operation of computer communication networks, efficient routing strategies must be employed in order to achieve optimal network performance. The importance of routing in computer communication can best be described by the following quotes [17].

"... *In the area of complex data communication networks, such as the ARPANET, only about 30 percent of the network resources are used to transmit real information, while the remaining 70 percent are used to transmit protocol (control) information. Sudden changes in demand and failures can set up dynamic instabilities ...*"

The routing policies can be classified as *static routing*, *quasi-static routing*,*and dynamic routing*. A pure static routing policy is time invariant, easy to analyze and simple to represent. Therefore, it is often used in the network planning stage. However, under heavy traffic fluctuation or component failures as is often the case in operation, a dynamic policy is required to cope with congestion and failure of the network. A quasi-static routing policy which may have some of the desired properties of both static and dynamic is also being used in many practical situations. With quasi-static routing policy, the routes are allowed to change only at given intervals of time or whenever the extreme conditions occur.

Based on whether the routes can be established centrally or locally, routing policies can also be distinguished as distributed and centralized policies. A distributed policy can respond quickly to a local disturbance and need not rely solely on the availability of the control node. However because of the complexity of the problem, very little analytical work has been done so far.

1.2.3 Routing as a Control Problem

This section discusses the routing problem from a high-level control-theoretic point of view. Details of model and formulation are postponed until Chapter 2.

As described earlier, a computer communication subnetwork consists of a large number of *nodes* interconnected by communication *links*. Data, in the form of small packets, are entered into the source node by a user according to mutually agreed upon procedures called protocols [18]. These data packets are placed in temporary storage (queue) and wait to be directed to their destinations via the intermediate nodes. At each node i, some routing decision must be made based upon the available local information so that the predefined network performance such as cost, delay, etc. can be minimized.

In the state space modeling of routing dynamics *state* variables are defined as the queue lengths at all nodes throughout the network, *routing* or decision variables are defined as the portion of the link capacity assigned for data transfer. From a decentralized control viewpoint, we let $\boldsymbol{x}$ represent the state of the network, and $\boldsymbol{u}$ denote the decision variables. The local information at the i^{th} node has the pattern described by

$$z_i = h_i(\boldsymbol{x},\boldsymbol{u}).$$

The collection of $\{h_1,h_2,\cdots,h_N\}$ is the information structure of the system. Because of the non-classical information pattern, the admissible routing strategies at the i^{th} node are assumed to be of the form

$$u_i = \gamma_i[h_i(\boldsymbol{x})].$$

The problem of finding the optimal routing strategies γ_i, may be loosely formulated as follows:

To find the routing strategies $\gamma_i^* \epsilon \Gamma_i$ for all i which minimizes the performance index of the network $J(\gamma_1,\gamma_2,\cdots,\gamma_N)$ subject to constraints imposed by routing dynamics, information structure and the form of the routing strategies.

In this research, we study the following two questions from the above formulation.

- for a given information structure, what is the optimal routing strategy?
- what is the impact of various information structure to a chosen network performance index?

1.3 Summary of Previous Work

1.3.1 Queuing Model

The best known existing analytical model for routing problems in computer communication networks is based on queuing theory [19]. In this model, several assumptions are made on network status and traffic characteristics; the messages enter the node at a Poisson rate, the packet length is exponentially distributed, the storage at a node is infinite, and the network components are perfectly reliable. Under such assumptions, the average packet delay can be expressed as a function of average flow rates in the channels. The optimal routing policy is defined as the policy that minimizes the total delay. The problem of finding optimal routes in a packet-switched computer communication network can be formulated as a nonlinear multicommodity flow problem [20]. This technique minimizes the total delay in the network in a steady-state sense and therefore the resultant optimum routing strategy is static and open-loop. In other words, the routing decision is constant in time and a function only of perfectly known average values of system and user parameters.

1.3.2 State Space Model

Segall [21] has introduced a model for message routing which is capable of rectifying the drawbacks of the queuing theory approach mentioned in the previous section. He suggested a state space model to represent the flow of messages in a store-and-forward data communication network. The minimum delay dynamic routing problem is then formulated as a centralized optimal control problem. The solution for this centralized optimal control problem has been obtained in [22]

A discrete version of the state-space model has been suggested by Meditch [23] and a minimum-variance dynamic routing policy has been developed based on a stochastic control formulation. Such policies have the property of regulating queue lengths at all nodes throughout the network. Implementation of this algorithm requires instantaneous knowledge of the queuing error to a central routing authority. The routing strategy proposed is basically a centralized routing policy.

1.4 Thesis Overview

1.4.1 Objective of thesis

The primary objective of this research is to study a class of linear decentralized control problems with a quadratic performance index and a forcing function. This problem can be viewed as the decentralized version of the classical servomechanism problem. We shall investigate the optimal control strategy for both deterministic and stochastic forcing functions with constraints on the information set.

The secondary objective is to consider the dynamic routing problem of computer communication networks in the framework of decentralized control theory. Based upon the concept of information structure, we shall exploit the optimal routing strategies.

1.4.2 Approach

Before getting to the discussion of our approach, it is necessary to give a high-level description of the model of the data flow in the network. (A detailed discussion will be given later in Chapter 2). It should be pointed out that the model introduced here is a variation of the original one suggested by Segall [21]. By following his definitions and notations, the *state variables* represent the quantity of data message (or queue length) stored at the nodes, distinguished according to the node of ultimate destination and the node of current residence. The *control variables* represent the flow of messages in the link, where each control represents that portion of a given link's rate capacity which is denoted to transmission messages of a given destination. The *inputs* are the flow rates of messages entering the network from a user host or terminal. In this research we consider two types of traffic inputs; the first kind is a deterministic function of time and the second is a stochastic function with known statistics.

The dynamic equation of data flow can be represented by

$$\dot{\boldsymbol{x}}(t) = \sum_{i=1}^{N} B_i(t)u_i(t) + \boldsymbol{r}(t), \qquad i=1,2,..,N$$

where $\boldsymbol{x}(t)$, $u_i(t)$, and $\boldsymbol{r}(t)$ are the vectors of the state variables, control variables and inputs respectively. In this context, the routing policy is the assignment of values for the $u_i(t)$ which generally may be a function of external user demands $\boldsymbol{r}(t)$ and queue length $\boldsymbol{x}(t)$.

It is clear that all of the information required to make the routing decisions may not be available to the controllers due to lack of information channels or possible engineering or economic constraints.

Instead of the specific computer communication routing dynamics, in this research we shall consider a more general class of linear time-varying system which supersedes the original representation.

$$\dot{\boldsymbol{x}}(t) = A(t)\boldsymbol{x}(t) + \sum_{i=1}^{N} B_i(t)u_i(t) + \boldsymbol{r}(t), \qquad i=1,2,..,N \tag{1.1}$$

Our approach is outlined as follows:

For the deterministic case, our approach is similar to that of Levine and Athans [24] which was used for solving a centralized linear quadratic problem. We first transform the original performance, a function of both initial states and feedback controls (gain matrix), into a new performance criterion independent of initial states. The problem is therefore converted into a parameter optimization problem. Sufficient conditions for optimality can be acquired by solving the Hamilton-Jacobi-Bellman equation. Necessary conditions for optimality are derived with the Matrix Minimum Principle [25].

Optimal control of network flow with stochastic input traffic can be posed as a decentralized servomechanism problem. We follow the approach used by Chong and Athans [26] who considered the control of a linear stochastic system with two controllers using a single performance index. The technical aspects of the extension is not straightforward. We have, in this research, a servomechanism problem with stochastic inputs. This additional input term causes a lot of complexity. We first transform the stochastic servomechanism problem into a deterministic problem with structure constraints on controls. Sufficient conditions for optimality are derived from the Hamilton-Jacobi-Bellman equation.

1.4.3 Summary of Results

In this section we describe the highlights of our major results in this research.

Based upon the state space model of computer communication networks, we formulate the dynamic routing problem in the framework of decentralized control theory. The system dynamics is a continuous linear-time-varying differential equation as shown in (1.1). We have restricted ourselves to the special case that the information structure is static and a linear combination of the state variables (queue lengthes) and possible observation noise.

$$z_i = H_i \boldsymbol{x}(t) + \boldsymbol{\theta}_i(t), \qquad i=1,2,\cdots,N \tag{1.2}$$

where $\boldsymbol{\theta}_i$ is a random noise with given statistics. The network performance is assumed to be a quadratic functional of $\boldsymbol{u}$ and $\boldsymbol{x}$.

$$J = \mathbf{E}\left\{ <\boldsymbol{x}(T),S\boldsymbol{x}(T)> + \int_{t_0}^{T} [<\boldsymbol{x}(t),Q(t)\boldsymbol{x}(t)> + \sum_{i=1}^{N} <u_i(t),R_i(t)u_i(t)>]dt \right\} \tag{1.3}$$

The necessary and sufficient conditions for optimal routing strategy are derived in Chapter 3 for an N-node network with deterministic traffic inputs. The conditions are summarized as Theorem 3.1 and Theorem 3.2. The optimal routing strategy for the i^{th} controller requires the solution of a two point boundary value problem involving matrix differential equations. We demonstrate the computation of routing strategy by a simple three node example.

In Chapter 4, the routing problem under consideration is extended for the stochastic traffic inputs. Here, we impose the dimensionality and linear structure of the compensator. The necessary and sufficient conditions for the optimal routing strategy are summarized as Theorem 4.1 and Theorem 4.2 respectively. The same network is used as an example to interpret the results with stochastic traffic inputs.

1.5 Synopsis of the Thesis

The remainder of the thesis is organized as follows. In chapter 2 the state space model of routing dynamics for a data communication network is described. The dynamic routing problem is then formulated in the framework of decentralized optimal control theory.

In chapter 3 we give the solution to this optimal control problem for a system with deterministic input under a linear information structure. We begin by deriving the sufficient conditions of optimality by the Hamilton-Jacobi-Bellman equation and claim that they are also necessary. The resultant control strategies are examined with two extreme information cases: (1) complete information where all state information are available to every controller through measurement or perfect communication, and (2) partial information where there is no communication among controllers. A three node network of ring structure is used as a numerical example to interpret the results.

In chapter 4 we investigate a system with stochastic input. Once again we derive the sufficient conditions of the optimal control and prove that they are also necessary as the case with deterministic input. The three node network example in chapter 3 is used to demonstrate the computation result.

In chapter 5 we summarize our results, present the specific contributions on our research and make suggestions for further work in the related area.

2. MODEL AND PROBLEM FORMULATION

2.1 Introduction

In this chapter, the state-space model of data flow in a stored-and-forward computer communication network is described in detail. This model for dynamic routing was first introduced by Segall in [21]. The model is based on a fundamental principle of *Conservation of Data Flow* which can be stated as follows:

At any point of time, the rate of change of data accumulation at a node is the difference of data flow entering to and departuring from that node.

The model does not consider individual pieces of data but rather considers the storage of data at each node broken down by destinations. The principle elements of the model are states, controls, and inputs which represent mathematically the three fundamentals of network operation: buffer storage, flow assignment and message input respectively. It has been shown that after appropriate normalization, the storage state (queue length) of the network at any time can be represented by an ordinary linear vector differential equation.

In this research, we consider two types of traffic inputs. The first type is a deterministic function of time representing a scheduled rate of demand. The second type of traffic is a random process with mean and covariance given. However, the knowledge of distribution is not required.

We begin with an introduction on the definitions and notations used in the following sections. Since we are discussing the network routing problem from the control system point of view, we shall use the term *controller* as a synonym of *router*. Our decentralized model can be seen as a direct modification of the model pioneered by Segall and Moss [22].

2.2 Basic Elements of the Decentralized Control Model

We visualize communication networks graphically to consist of a collection of nodes connected by a set of links between various pairs of the nodes. The general functions of nodes and links have been discussed in Chapter 1. We shall formalize the definitions in this section.

Definitions

A *network* is a collection of nodes represented by $\mathbf{N} = \{1,2,\cdots,N\}$ together with a set of links between ordered pairs * of nodes. The link connecting node i to node j is denoted by the symbol (i,j). We assume that link (i,j) is a directed link and is thus different from link (j,i). Hence, a link is a simplex channel. We shall use two links for a bidirectional duplex transmission. **Figure 2-1** illustrates the relationship of nodes and links in our model.

2 An ordered pair (a,b) is an ordered arrangement of two elements a and b. (a,b) and (b,a) are two different ordered pairs.

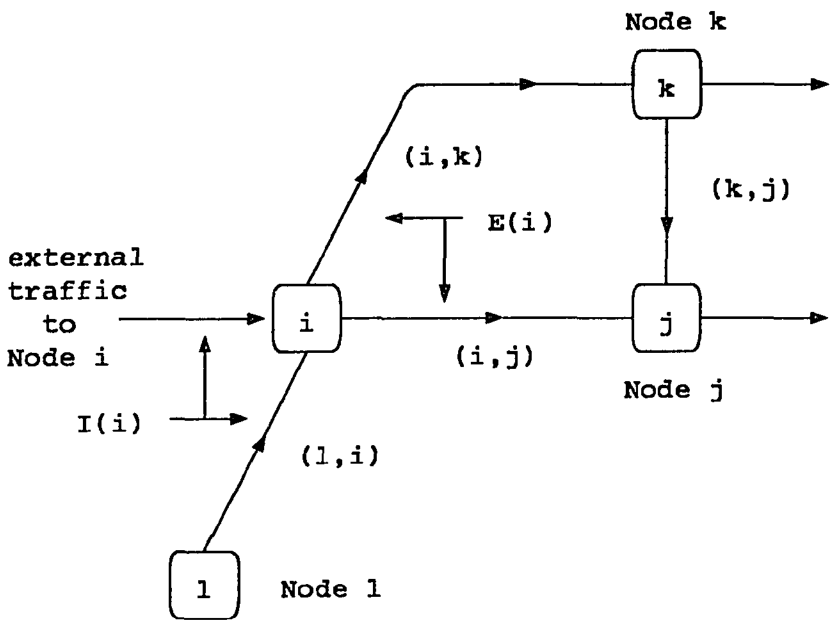

Figure 2-1. Model of Traffic Flow

Let the collection of all links actually existing in the network be

$$L \equiv \{(i,j) \text{ , such that } i,j \in \mathbf{N} \text{ and there exists a directed link connecting } i \text{ to } j\}$$

With respect to L, we need for each $i \in N$ two index sets. Let $E(i)$ be the collection of exit links from node i, and $I(i)$ be the collection of incoming links to node i.

Symbolically,

$$E(i) \equiv \{j \in \mathbf{N} : (i,j) \in L\}$$
$$I(i) \equiv \{l \in \mathbf{N} : (l,i) \in L\}$$

The user at each node in the network may input messages whose ultimate destination is any of the other nodes in the network. We characterize this message traffic as *external* traffic or *inputs* of the network.

The message traffic once entered into the associated node will be either immediately transmitted on an outgoing link or stored for eventual transmission. Each node in the network may serve as an intermediate storage area for message entering on incoming links enroute to their destinations. Once a message reaches its destination node, it is immediately forwarded to the appropriate user without further storage. Hence at each node $i \in N$ of the network at any point in time we may have message in residence whose destinations are all nodes other than i.

Let us imagine that at each node $i \in \mathbf{N}$, we have (N-1) distinct storage space. In each of these places we place messages whose destination is a particular node in the network regardless of its origin. We also select the bit to be the unit of data stored at each space. However, the model presented in this chapter and the analysis followed in next two chapters are equally valid for other units such as bytes, messages, or packets.

We consider a situation where the capacity of the link is unlimited. That is, no rate capacity constraint on each transmission link is concerned. We shall also assume the storage areas containing messages corresponding to the state variables are infinite in capacity.

We define the basic network variables

$x_i^j(t)$ = number of message bits at node i whose destination is node j.

$r_i^j(t)$ = external message arrival rate at node i whose destination is node j.

$u_{ik}^j(t)$ = messages departure rate at node i through link (i,k) whose destination is node j.

2.2.1 System Dynamics

We are now ready to describe the state-space dynamic equation for data flow within a network.

From the conservation of flow, at any node i the time rate of change of the number of the messages, with destination to node j, can be expressed as follows:

$$\dot{x}_i^j(t) = r_i^j(t) - \sum_{k \in E(i)} u_{ik}^j(t) + \sum_{l \in I(i)} u_{li}^j(t) \tag{2.1}$$

for all nodes i and destinations j, where

$E(i)$ and $I(i)$ have been defined in the last section.

The state of the network is represented by x_i^j which is the amount of data (queue length) at each node waiting to be transferred. The queue lengths are then the solution to the differential equations (2.1).

Through a simple transformation, the system dynamics can be rewritten in vector forms as

$$\dot{\boldsymbol{x}}(t) = \sum_{i=1}^{N} B_i(t) u_i(t) + \boldsymbol{r}(t) \tag{2.2}$$

where, vector $\boldsymbol{r}$ represents all the external traffic from network users. It is clear that the system equation (2.2) contains multiple controllers u_i.

2.2.2 *Information Structure*

The routing decision to be made at node i for a packet with destination $j \neq i$ is an assignment of the packet to one of the links exit from i. The routing decision generally depends upon the states of the network $\boldsymbol{x}$ which represent the amount of traffic being held at all nodes. However, because our network is geographically distributed, to obtain a complete set of states is not always possible. In this research, we consider a situation where a local router has access to only a linear combination of a partial state information.

Taking into consideration possible noise corruption in the measurement process, we have

$$z_i(t) = C_i(t)\mathbf{x}(t) + \theta_i(t) \tag{2.3}$$

where z_i represents the information that the controller i can access and θ_i is the stochastic process for measurement noise. The dimension of θ_i is equivalent to that of z_i and the matrix C_i may not be full rank. Here, we have assumed implicitly the instantaneous availability of the state information to the measurement process. In designing many of the large-scale systems, additional information requires additional communication cost and likely also additional structure complexity.

2.2.3 Performance Index

Many criteria have been proposed for measuring the performance of a computer communication network. Typical performance criteria can be expressed in terms of average packet time delay, network throughput, queuing error, shortest path or, in some case, the cost of transmission.

The specific cost criterion used differs among the networks. Some networks used a fixed cost for each link in the network. We shall introduce a number of performance criteria for demonstrating the concept and models. However, our major concern in this research is a quadratic form of cost functional as discussed below in C.

A. Minimum Time Delay -

If $\mathbf{x}(t)$ is the amount of traffic in some buffer at time t, then

$$d = \int_{t_0}^{T} \mathbf{x}(t)dt \tag{2.5}$$

denotes the total time spent in this buffer by the traffic that passed through it during some period of interest $[t_0,T]$ where T is the final time such that the buffer is empty.

The total delay across the network during $[t_0,T]$ is given by

$$D = \int_{t_0}^{T} \left[\sum_{i \neq j}^{N} x_i^j(t) \right] dt \tag{2.6}$$

where T is the time at which all buffers are empty. This performance index has been used in Segall's formulation.

B. Minimum Queue Length -

Another performance index used is the mean-square error of the queue length against a pre-determined reference queue length. It may be stated as follows:

Let

$$e(t) \equiv x(t) - \bar{x} \tag{2.7}$$

where $\mathbf{x}(t)$ denotes the collection of x_i^j which is the amount of data (queue length) at node i whose destination is j. $\bar{x}$ is the desired queue length vector under a given input traffic conditions. The criterion of performance to be minimized is the covariance of $\{e(T)\}$

$$P(t) = \mathbf{E}[e(t)e'(t)] \tag{2.8}$$

where $\mathbf{E}$ is the expectation.

C. Minimum Cost - The performance index functional in this research is assumed to be in quadratic form as follows:

$$J = \mathbf{E}\left\{ <\mathbf{x}(T), S\mathbf{x}(T)> + \int_{t_0}^{T} [<\mathbf{x}(t), Q(t)\mathbf{x}(t)> + \sum_{i=1}^{N} <u_i(t), R_i(t)u_i(t)>] dt \right\} \tag{2.9}$$

although we can argue that the cost for operating such a network is tariffed through the regulation agency. The quadratic functional is not necessarily exclusive. However, we must admit the choice is purely of academic interest.

2.3 Problem Formulation

We now have all the elements to state our decentralized optimal control problem. The dynamic routing problem of computer communication networks can be expressed as:

Find the routing strategies $\gamma_i^* \in \Gamma_i$ for all i which minimizes the performance index of the network (2.9) subject to constraints imposed by traffic dynamics (2.2), information structure (2.3) and the form of the routing strategies.

We shall assume a linear structure for admissible control strategies. Details will be given in Chapter 3 and Chapter 4.

3. DECENTRALIZED OPTIMAL CONTROL WITH DETERMINISTIC TRAFFIC

3.1 Introduction

The dynamic routing problem of computer communication networks has been formulated in the framework of decentralized optimal control in Chapter 2. In this chapter, we derive the optimal routing strategies for each controller and discuss the information structure under which the routing strategy can be synthesized.

We consider a specific situation that the traffic entering to the network is a deterministic function of time. The routing decision is then based on *a priori* information on network topology, the performance index, the statistics of initial state of congestion at a reference time t_0 as well as the measurements obtained on-line. The information structure under study is assumed to be linear and static. Therefore, information available to a local controller is a linear combination of all or some of the network states. We further assume that the measurement can be obtained by each controller free of noise corruption. The network performance considered is a quadratic functional of states and controls with appropriate weight assigned.

We are concerned with the case that the control signal of each controller is generated via feedback from the local measurements. The structure of the feedback compensator is assumed to be linear and time-varying. With these assumptions, we are now ready to solve the deterministic decentralized control problem.

We begin by transferring the system dynamic equation to a closed-loop representation in section 3.2. The associated performance index is then modified accordingly to an explicit function of the control strategies. Thus, the original control problem is converted into a constrained deterministic optimization problem. In section 3.3, this constrained optimization problem is solved via the Matrix Minimum Principle and results are given in

Theorem 3.1 as the necessary conditions. We then derive the sufficient conditions for optimality by the Hamilton-Jacobi-Bellman equation in section 3.4.

The subsequent discussion is restricted to the special situation that there is no external traffic arriving to the network. This is a linear quadratic regulator problem with multiple controllers. We intend to provide the answer to the question of what control actions should be employed to clear a congested network? We examine the control strategy for two information structures: with full information and with only partial information available to a local controller. A numerical example is given to demonstrate the computation and interpret the results.

3.2 Problem Statement

Consider a special case of linear routing decision and a network with N routers which are represented by a set of differential equations

$$\dot{x}(t) = A(t)x(t) + \sum_{i=1}^{N} B_i(t)u_i(t) + r(t) \tag{3.1}$$

$$z_i(t) = C_i(t)x(t) \tag{3.2}$$

with states $x(\cdot) \in R^n$, controls $u_i(\cdot) \in R^{m_i}$, and outputs $z_i(\cdot) \in R^{k_i}$. If $A(t)=0$ in (3.1) then the routing dynamics reduce to the form provided by [21]. The components of $A(t)$ and $C_i(t)$ are continuous function of time t and the components of $B_i(t)$ are continuously differentiable functions of time t. The vector $r(t) \in R^n$ is a deterministic traffic pattern entered to the network. It is assumed here that the i^{th} controller has access only to the information z_i through measurement and possibly communication from other nodes regarding to the state x at time t. The information matrix, $C_i \neq C_j$ when $i \neq j$, may not be full rank.

The controller i makes decisions based on the available information by a linear time-varying feedback control law.

$$u_i(t) = G_i(t)z_i(t), \qquad i=1,2,\dots,N \tag{3.3}$$

where the gain $G_i(t)$ is a $m_i \times k_i$ matrix.

The performance index for all the controllers is assumed to be

$$J = \mathbf{E}\left\{\int_{t_0}^{T} [<\boldsymbol{x}(t),Q(t)\boldsymbol{x}(t)> + \sum_{i=1}^{N} <u_i(t),R_i(t)u_i(t)>]dt\right\} \tag{3.4}$$

where $t\epsilon[t_0,T]$, the initial state $\boldsymbol{x}(t_0)=\boldsymbol{x}_0$ is a random variable with Gaussian distribution whose mean and variance are given by $\mathbf{E}\{\boldsymbol{x}_0\}=\boldsymbol{m}$ and $\mathbf{E}\{\boldsymbol{x}_0'\boldsymbol{x}_0\}=\Sigma_0$, respectively; $\mathbf{E}$ is the expectation; and $<a,b>$ is the inner product of two vectors of a and b. Without loss of generality, we assume that the matrices $Q(t)$ and $R_i(t)$ are symmetric and semi-positive definite. However, $R_i(t)$ matrix may not be full rank.

The decentralized control problem can be stated as follows:

Given the dynamic constraint (3.1), information structure (3.2), and the mean and variance of the random variable $\boldsymbol{x}_0$, find the time-varying feedback gain $G_i^*(t)$ such that the network performance index J specified in (3.4) is minimum.

Using equation (3.1) to (3.3), the closed-loop decentralized system is governed by

$$\dot{\boldsymbol{x}}(t) = A(t)\boldsymbol{x}(t) + \sum_{i=1}^{N} B_i(t)G_i(t)C_i(t)\boldsymbol{x}(t) + \boldsymbol{r}(t) \tag{3.5}$$

The solution of $\boldsymbol{x}(t)$ is given as

$$\boldsymbol{x}(t) = \Phi(t,t_0)\boldsymbol{x}_0 + \int_{t_0}^{t} \Phi(t,\tau)\boldsymbol{r}(\tau)d\tau \tag{3.6}$$

or

$$\boldsymbol{x}(t) = \Phi(t,t_0)\boldsymbol{x}_0 + F'\boldsymbol{r}(t)$$

where

$$F'\boldsymbol{r}(t) = \int_{t_0}^{t} \Phi(t,\tau)\boldsymbol{r}(\tau)d\tau \tag{3.7}$$

$\Phi(t,t_0)$ is the fundamental transition matrix which satisfies

$$\dot{\Phi}(t,\tau) = (A(t)+\sum_{i=1}^{N} B_i(t)G_i(t)C_i(t))\Phi(t,\tau), \qquad \Phi(\tau,\tau)=I.$$

The performance index can be rewritten as

$$J = \mathbf{E}\left\{\int_{t_0}^{T} < (\Phi \boldsymbol{x}_0+F'\boldsymbol{r}),(D\Phi \boldsymbol{x}_0+DF'\boldsymbol{r}) > dt\right\} \tag{3.8}$$

where

$$\Phi = \Phi(t,t_0)$$

$$D = (Q+\sum_{i=1}^{N} C_i'G_i'R_iG_iC_i). \tag{3.9}$$

After taking the expectation (3.8) is reduced to the following form:

$$J = \mathrm{tr}\left\{\int_{t_0}^{T}\left[\Phi' D\Phi\Sigma_0+D\Phi\mathbf{m}(F\boldsymbol{r})'+D\Phi'\mathbf{m}'(F\boldsymbol{r})+D(F\boldsymbol{r})(F\boldsymbol{r})'\right]dt\right\} \tag{3.10}$$

where $\mathrm{tr}\{\cdot\}$ denotes the trace of $\{\ \}$. Substituting (3.9) into (3.10), the performance index can be expressed as a function of $G_i(t)$

$$J(G_i(t)) = \mathrm{tr}\left\{\int_{t_0}^{T}\left[\Phi'(t,t_0)(Q(t)+\sum_{i=1}^{N} C_i'(t)G_i'(t)R_i(t)G_i(t)C_i(t))\Phi(t,t_0)\Sigma_0\right.\right. \tag{3.11}$$

$$+(Q(t)+\sum_{i=1}^{N} C_i'(t)G_i'(t)R_i(t)G_i(t)C_i(t))\Phi(t,t_0)\mathbf{m}(F'\boldsymbol{r})'$$

$$+(Q(t)+\sum_{i=1}^{N} C_i'(t)G_i'(t)R_i(t)G_i(t)C_i(t))\Phi'(t,t_0)\mathbf{m}'(F'\boldsymbol{r})$$

$$\left.\left.+(Q(t)+\sum_{i=1}^{N} C_i'(t)G_i'(t)R_i(t)G_i(t)C_i(t))(F'\boldsymbol{r})(F'\boldsymbol{r})'\right]\right\}dt$$

where $F'\boldsymbol{r}(t)$ has been defined in (3.7).

The original problem may be restated as follows:

Given the dynamic constraint,

$$\dot{\Phi}(t,\tau) = (A(t)+\sum_{i=1}^{N}B_i(t)G_i(t)C_i(t))\Phi(t,\tau), \qquad \Phi(\tau,\tau)=I \tag{3.12}$$

find the time-varying feedback gain $G_i^*(t)$ such that the network performance index J specified in (3.11) is minimum.

3.3 Necessary Conditions for Optimality

We shall derive the necessary conditions of optimality by employing the Matrix Minimum Principle. Let $\Lambda(t)$ be an $n \times n$ costate (or Lagrange multiplier) matrix associated with $\Phi(t,t_0)$. Then the scalar Hamiltonian function H for the optimization problem is given by

$$H = \mathrm{E}\{< (\Phi x_0+F^t r),(D\Phi x_0+DF^t r) >\} + \mathrm{tr}\{< E\Phi,\Lambda' >\} \tag{3.13}$$

where D is defined in (3.9), and

$$E = A + \sum_{i=1}^{N}B_iG_iC_i \tag{3.14}$$

Since $\mathrm{E}\{x_0\}=m$ and $\mathrm{E}\{x_0'x_0\}=\Sigma_0$, respectively; using the trace operator $\mathrm{tr}\{\cdot\}$, the Hamiltonian H can be expanded into the following expression.

$$H = \mathrm{tr}\left\{\left[\Phi' D\Phi\Sigma_0 + D\Phi m(Fr)' + D\Phi' m'(Fr) + D(Fr)(Fr)'\right]\right\} + \mathrm{tr}\left\{E\Phi\Lambda'\right\} \tag{3.15}$$

We are interesting in determining the optimal control strategy $G_i^*(t)$ for all i. Equation (3.15) must be expressed as the function of G_i. Substituting (3.7), (3.9) and (3.14) into equation (3.15) yields

$$H = \mathrm{tr}\left\{\Phi'(Q+\sum_{i=1}^{N} C_i'G_i'R_iG_iC_i)\Phi\Sigma_0 + (Q+\sum_{i=1}^{N} C_i'G_i'R_iG_iC_i)\Phi \boldsymbol{m}(F^{\tau}\boldsymbol{r})' + (Q+\sum_{i=1}^{N} C_i'G_i'R_iG_iC_i)\Phi'\boldsymbol{m}'(F^{\tau}\boldsymbol{r}) + (Q+\sum_{i=1}^{N} C_i'G_i'R_iG_iC_i)(F^{\tau}\boldsymbol{r})(F^{\tau}\boldsymbol{r})'\right\} + \mathrm{tr}\left\{\Lambda'(A+\sum_{i=1}^{N} B_iG_iC_i)\Phi\right\} \tag{3.16}$$

A necessary condition that $G_i(t)$ minimizes H is that the following gradient matrix vanishes:

$$\frac{\partial H}{\partial G_i}\Big|_* = R_iG_i^*(C_i\Phi\Sigma_0\Phi'C_i')+R_iG_i^*(C_i(F\boldsymbol{r})(\Phi\boldsymbol{m})'C_i') + R_iG_i^*(C_i(\Phi\boldsymbol{m})(F\boldsymbol{r})'C_i')+R_iG_i^*(C_i(F\boldsymbol{r})(F\boldsymbol{r})'C_i') + {}^{u}B_i'\Lambda'\Phi'C_i' = \mathbf{0}. \tag{3.17}$$

The optimal control strategy for the i^{th} controller is

$$G_i^*(t) = -{}^{u}R_i(t)^{-1}B_i'(t)\Lambda(t)\Phi'(t,t_0)C_i'(t)\Psi_i^{-1}(t) \tag{3.18}$$

where,

$$\Psi_i(t) = C_i\Phi\Sigma_0\Phi'C_i'+C_i(F^{\tau}\boldsymbol{r})(\Phi\boldsymbol{m})'C_i' + C_i(\Phi\boldsymbol{m})(F^{\tau}\boldsymbol{r})'C_i'+C_i(F^{\tau}\boldsymbol{r})(F^{\tau}\boldsymbol{r})'C_i'. \tag{3.19}$$

The existence of Ψ^{-1} will be discussed in section 3.5. Using the necessary conditions of the Matrix Minimum Principle we obtain

$$\frac{\partial H}{\partial \Phi}\Big|_* = -\dot{\Lambda}(t) = -\frac{\partial H}{\partial \Phi}[\,\mathrm{tr}\{\Phi'D^*\Sigma_0\Phi + D^*\Phi\boldsymbol{m}(Fr)' + \Phi D^{*\prime}(Fr)\boldsymbol{m} + D^*(Fr)(Fr)' + \Lambda'E^*\Phi\}\,] \tag{3.20}$$

In order to attempt to determine a closed-loop control, we assume

$$\Lambda(t) = [P\Phi+P'\Phi+\eta]. \tag{3.21}$$

Then,

$$\dot{\Lambda}(t) = [\dot{P}\Phi+\dot{P}'\Phi+\dot{\eta}]. \tag{3.22}$$

We substitute this relation into canonical equations (3.20) and determine the requirements for a solution. The resultant equation is a second order equation of Φ. By setting the coefficient of the same degree equivalent, we obtain

$$-\dot{P}(t) = D^*(t)\Sigma_0+P(t)E^*(t)+E^*(t)P(t), \qquad P(T)=0 \tag{3.23}$$

$$-\dot{\eta}(t)=\boldsymbol{m}(F\boldsymbol{r})'D^*(t)+D^{*\prime}(t)(F\boldsymbol{r})'\boldsymbol{m}+\eta(t)E^*(t), \qquad \eta(T)=0 \tag{3.24}$$

where

$$D^*(t)=\left(Q(t)+\sum_{i=1}^{N}C_i'(t)G_i^{*\prime}(t)R_i(t)G_i^*(t)C_i(t)\right) \tag{3.25}$$

$$E^*(t)=\left(A(t)+\sum_{i=1}^{N}B_i(t)G_i^*(t)C_i(t)\right) \tag{3.26}$$

and

$$\dot{\Phi}(t,\tau) = (A(t)+\sum_{i=1}^{N}B_i(t)G_i(t)C_i(t))\ \Phi(t,\tau), \qquad \Phi(\tau,\tau)=I. \tag{3.27}$$

We shall summarize the conditions for optimality in the following Theorem.

Theorem 3.1 (Necessary)

The matrices $G_i(t)$, $\Phi(t,t_0)$, and $P(t)$, $\eta(t)$, ζ, where $t\in[t_0,T]$ defined by equations (3.18), (3.27), and (3.23) to (3.24) satisfy the necessary conditions for optimality provided by the Matrix Minimum Principle for the decentralized control problem under consideration.

It must be noted that the conditions listed here are only necessary to the optimal control. It is entirely possible that there are several solutions of these necessary conditions which are not optimal. We shall next prove that the conditions given here satisfy the

Hamilton-Jacobi-Bellman sufficient conditions for optimality.

3.4 Sufficient Conditions for Optimality

Define

$$V^*(\Phi(t,\tau),t) = \mathrm{tr}\left\{\int_t^T \Big[\Phi^{*\prime}(\tau,t)(Q(\tau)+\sum_{i=1}^{N} C_i'(\tau)G_i'(\tau)R_i(\tau)G_i(\tau)C_i(\tau))\Phi(\tau,t)\Sigma_0\right. \tag{3.28}$$

$$+(Q(\tau)+\sum_{i=1}^{N} C_i'(\tau)G_i'(\tau)R_i(\tau)G_i(\tau)C_i(\tau))\Phi^*(\tau,t)\boldsymbol{m}(F^{\tau}\boldsymbol{r})'$$

$$+(Q(\tau)+\sum_{i=1}^{N} C_i'(\tau)G_i'(\tau)R_i(\tau)G_i(\tau)C_i(\tau))\Phi^{*\prime}(\tau,t)\boldsymbol{m}'(F^{\tau}\boldsymbol{r})$$

$$\left.+(Q(\tau)+\sum_{i=1}^{N} C_i'(\tau)G_i'(\tau)R_i(\tau)G_i(\tau)C_i(\tau))(F^{\tau}\boldsymbol{r})(F^{\tau}\boldsymbol{r})'\Big]\right\}dt.$$

Applying the Hamilton-Jacobi-Bellman equation gives

$$\frac{\partial V^*(\Phi(t,\tau),t)}{\partial t} = -\min_{G_i} \mathrm{tr}\left\{L+\left(\frac{\partial V^*}{\partial \Phi}\right)'(A+\sum_{i=1}^{N} B_iG_iC_i)\Phi\right\} \tag{3.29}$$

$$= -\min_{G_i} \mathrm{tr}\Big\{\Phi'(Q+\sum_{i=1}^{N} C_i'G_i'R_iG_iC_i)\Phi\Sigma_0$$

$$+(Q+\sum_{i=1}^{N} C_i'G_i'R_iG_iC_i)\Phi\boldsymbol{m}(F^{\tau}\boldsymbol{r})'$$

$$+(Q+\sum_{i=1}^{N} C_i'G_i'R_iG_iC_i)\Phi'\boldsymbol{m}'(F^{\tau}\boldsymbol{r})$$

$$+(Q+\sum_{i=1}^{N} C_i'G_i'R_iG_iC_i)(F^{\tau}\boldsymbol{r})(F^{\tau}\boldsymbol{r})'$$

$$+\left(\frac{\partial V^*}{\partial \Phi}\right)'(A+\sum_{i=1}^{N} B_iG_iC_i)\Phi\Big\}$$

with the terminal condition

$$V^*(\Phi(T,\tau),T) = 0. \tag{3.30}$$

Since we have assumed that $G_i^*(t)$ achieves the minimum in Hamilton-Jacobi-Bellman equation, we obtain from equation (3.29)

$$R_i G_i^*(C_i \Phi \Sigma_0 \Phi' C_i') + R_i G_i^*(C_i(F\boldsymbol{r})(\Phi \boldsymbol{m})' C_i') \tag{3.31}$$

$$+ R_i G_i^*(C_i(\Phi \boldsymbol{m})(F\boldsymbol{r})' C_i') + R_i G_i^*(C_i(F\boldsymbol{r})(F\boldsymbol{r})' C_i')$$

$$+ u B_i' \frac{\partial V^*}{\partial \Phi} \Phi' C_i' = 0.$$

The optimal gain for the i^{th} local controller is

$$G_i^*(t) = -u R_i^{-1} B_i' \frac{\partial V^*}{\partial \Phi} \Phi' C_i' \Psi_i^{-1} \tag{3.32}$$

where

$$\Psi_i = C_i \Phi \Sigma_0 \Phi' C_i' + C_i(F^\tau \boldsymbol{r})(\Phi \boldsymbol{m})' C_i' \tag{3.33}$$

$$+ C_i(\Phi \boldsymbol{m})(F^\tau \boldsymbol{r})' C_i' + C_i(F^\tau \boldsymbol{r})(F^\tau \boldsymbol{r})' C_i'.$$

To solve for (3.29) with G_i^* gain in (3.32) we assume the solution has the form

$$V^*(\Phi(t,\tau),t) = \mathrm{tr}[\Phi' P \Phi + \Phi \eta + \zeta] \tag{3.34}$$

Then, the optimal gain in (3.32) can be expressed as

$$G_i^* = -u R_i^{-1} B_i' (P\Phi + P'\Phi + \eta) \Phi' C_i' \Psi_i^{-1} \tag{3.35}$$

It follows from (3.7) and (3.33) the matrix $\Psi_i(t)$ in integral form is given by

$$\Psi_i(t) = C_i(t) \Phi^*(t,t_0) \Sigma_0 \Phi^{*\prime}(t,t_0) C_i'(t) \tag{3.36}$$

$$+ \int_{t_0}^{t} C_i(t) \Phi^*(t,s) \boldsymbol{r}(s) \boldsymbol{m}' \Phi^*(t,t_0) C_i'(t) ds$$

$$+ \int_{t_0}^{t} C_i(t) \Phi^*(t,t_0) \boldsymbol{m} \boldsymbol{r}'(s) \Phi^*(t,s) C_i'(t) ds$$

$$+ \int_{t_0}^{t} \int_{t_0}^{t} C_i(t) \Phi^*(t,s) \boldsymbol{r}(s) \boldsymbol{r}'(z) \Phi^{*\prime}(t,z) C_i'(t) ds dz$$

Substituting (3.34) into (3.29) and equating the coefficient of Φ of the same degree, it implies that $P(t)$, $\eta(t)$ and $\zeta(t)$ satisfy the following equations.

$$-\dot{P}(t) = D^*(t)\Sigma_0 + P(t)E^*(t) + E^*(t)P(t), \qquad P(T)=0 \tag{3.37}$$

$$-\dot{\eta}(t) = \boldsymbol{m}(F\boldsymbol{r})' D^*(t) + D^{*\prime}(F\boldsymbol{r})' \boldsymbol{m} + \eta(t) E^*(t), \qquad \eta(T)=0 \tag{3.38}$$

$$-\dot{\zeta}(t)=D^*(t)(Fr)\,(Fr)', \qquad \zeta(T)=0 \tag{3.39}$$

where $D^*(t)=\left(Q(t)+\sum_{i=1}^{N}C_i'(t)G_i^{*\prime}(t)R_i(t)G_i^*(t)C_i(t)\right)$

and $E^*(t)=\left(A(t)+\sum_{i=1}^{N}B_i(t)G_i^*(t)C_i(t)\right)$

Hence we can make the following conclusion:

Theorem 3.2 (Sufficient)

The sufficient conditions for optimality for the decentralized deterministic optimal control problem defined in equation (3.1) through (3.4) are given by (3.35) and (3.36) where $\Phi(t,t_0)$, $P(t)$, $\eta(t)$, and $\zeta(t)$ are the solution of equations (3.27), and (3.37) to (3.39).

It is noted that the necessary conditions for optimality given in Theorem 3.1 are also sufficient. However, equation (3.39) is only sufficient but not necessary.

3.5 Discussions

In this section, we consider the decentralized optimal control with different information structures. If the traffic input $r=\mathbf{0}$, then the original problem becomes a decentralized Linear Quadratic Regulator problem. G_i^* is reduced to

$$G_i^*(t) = -\tfrac{1}{2}R_i^{-1}(t)B_i'(t)\left(P(t)\Phi^*(t,t_0)+P'(t)\Phi^*(t,t_0)\right)\cdot\Phi^{*\prime}(t,t_0)C_i'(t)\cdot \left[C_i(t)\Phi^*(t,t_0)\Sigma_0\Phi^{*\prime}(t,t_0)C_i'(t)\right]^{-1} \tag{3.40}$$

where P must satisfy the Riccati equation (3.37). The results agree with the previous work by Levine and Athans [24]. Conditions (3.38) and (3.39) are not required. The value of optimal performance index can be evaluated by

$$V^*(\Phi^*(t,t_0),t) = \mathrm{tr}[\Phi^{*'}(t,t_0)P(t)\Phi^*(t,t_0)] \tag{3.41}$$

where $P(t)$ is computed from (3.34).

1. For the complete information case, every controller has access to all the state information at time t. The information matrix C_i has full rank k_i for all i and therefore the inverse matrix C_i^{-1} exists. The matrix G_i^* is further reduced to

$$G_i^*(t) = -{}^{1\!/\!}{}_{4}R_i^{-1}(t)B_i'(t)P(t)\Phi^{*'}(t,t_0)C_i^{-1} \tag{3.42}$$

2. For the partial information case, the information structure C_i has a rank $<k_i$. Furthermore if R_i is not full rank, the gain matrix can be expressed as

$$G_i^*(t) = -{}^{1\!/\!}{}_{4}R_i^{+}(t)B_i'(t)\Big(P(t)\Phi^*(t,t_0) + P'(t)\Phi^*(t,t_0) + \eta(t)\Big)\cdot \tag{3.43}$$

$$\Phi^{*'}(t,t_0)C_i'(t)\Psi_i^{+}(t) + \Big(W - R_i(t)^{+}R_i(t)W\Psi_i(t)\Psi_i^{+}(t)\Big)$$

where Ψ_i^{+} and R_i^{+} are the Moore-Penrose pseudoinverse of the matrices Ψ_i and R_i respectively and W is an arbitrary matrix with appropriate rank [27].

We have demonstrated here that the computation requirements for the optimal gain G_i^* can be greatly reduced if every controller has complete information over the entire system. It is expected that, in general, a system of full information structure will achieve a better system performance than that of partial information structure. We will verify this claim by the following example.

3.6 Numerical Example

A simple three node network given in [21] is used here. The network, as shown in the **Figure 3-1**, has a single destination, node 2. We consider a special situation that there is no external traffic involved. For simplicity of the notation, we drop the superscript 2. The dynamic equation in decentralized form is:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_{12} \\ u_{13} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} u_{31} \\ u_{32} \end{bmatrix} \tag{3.44}$$

$$z_i = C_i \cdot \begin{bmatrix} x_1 \\ x_3 \end{bmatrix}, \quad i = 1,3. \tag{3.45}$$

Assume the cost of routing or the performance index for all the controllers to be minimized is

$$J = \mathbf{E}\left\{ \int_{t_0}^{T} (x_1^2 + x_3^2 + u_1^2 + u_3^2) dt \right\} \tag{3.46}$$

The initial conditions $\mathbf{E}\{\boldsymbol{x}_0\}=0$ and $\mathbf{E}\{\boldsymbol{x}_0'\boldsymbol{x}_0\}=1$ are given.

The algorithm developed for the output feedback centralized problem in [24] could be extended for the computation of G_i^*. This algorithm has the property that the cost decreases at each interaction. Part of the computation has been performed by using the LINPACK software package [28]. The procedures are summarized as follows:

Step 0: obtain an initial value of $G_0(t)$ by setting $P_0(t)=0$ and $\Phi_0(t,t_0)=I$ in equation (3.37).

Step 1: compute $P_{n+1}(t)$ by integrating the Riccati equation (3.37) backwards in time with the terminal condition $P_{n+1}(t)=0$ and $G_n(t)$ from Step 0.

Step 2: compute $\Phi_{n+1}(t,t_0)$ by integrating equation (3.27) forward in time with initial condition $\Phi_{n+1}(t_0,t_0)=I$.

Step 3: compute $G_n(t)$ as in Step 0.

A. Full Information Case -

$$z_i = I \cdot \boldsymbol{x}$$

In this case, $C_i = I$, the available information are same at node 1 and node 3. This is a centralized or *classical* information structure.

The computer results of optimal gain are given by

$$G_1^* = -\begin{bmatrix} 0.444 & 0.543 \\ 0.543 & 0.592 \end{bmatrix} \tag{3.47}$$

and

$$G_3^* = -\begin{bmatrix} 0.155 & 0.690 \\ 0.690 & 0.172 \end{bmatrix} \tag{3.48}$$

where the optimal control are

$$u_1 = \begin{bmatrix} u_{12} \\ u_{13} \end{bmatrix} = G_1^* \cdot \begin{bmatrix} x_1 \\ x_3 \end{bmatrix} \tag{3.49}$$

and

$$u_3 = \begin{bmatrix} u_{31} \\ u_{32} \end{bmatrix} = G_3^* \cdot \begin{bmatrix} x_1 \\ x_3 \end{bmatrix} \tag{3.50}$$

The associated optimal performance $J^* = 0.402$.

B. A Partial Information Case - No communication on the state variables is allowed in this case. The information available to the node i is its own state value. That is

$$z_1 = [\,1 \;\; 0\,]\begin{bmatrix} x_1 \\ x_3 \end{bmatrix} \tag{3.51}$$

$$z_3 = [\,0 \;\; 1\,]\begin{bmatrix} x_1 \\ x_3 \end{bmatrix}$$

where $C_1 = [\,1 \;\; 0\,]$ and $C_3 = [\,0 \;\; 1\,]$.

The computer results of optimal gain for the partial information case are given by

$$G_1^* = -\begin{bmatrix} 0.099 & 0.032 \\ 0.032 & 0.097 \end{bmatrix} \tag{3.52}$$

and

$$G_3^* = -\begin{bmatrix} 0.046 & 0.025 \\ 0.025 & 0.047 \end{bmatrix} \tag{3.53}$$

where the optimal control are

$$u_1 = \begin{bmatrix} u_{12} \\ u_{13} \end{bmatrix} = G_1^* \cdot x_1 \tag{3.54}$$

and

$$u_3 = \begin{bmatrix} u_{31} \\ u_{32} \end{bmatrix} = G_3^* \cdot x_2. \tag{3.55}$$

The optimal performance for this case is $J^* = 0.857$.

It is interesting to note from **Figure 3-2** that the performance of the network is improved as more information available to every controllers.

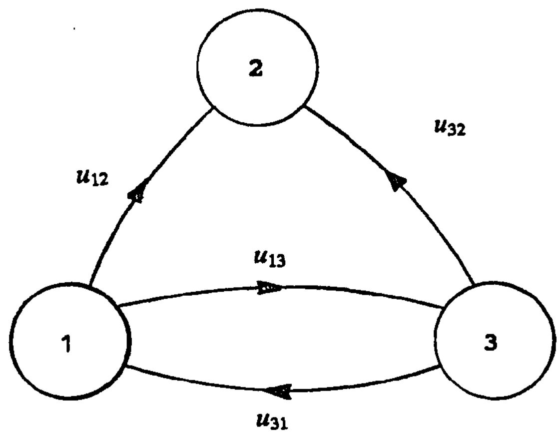

Figure 3-1. A Three-Node Network with Single Destination

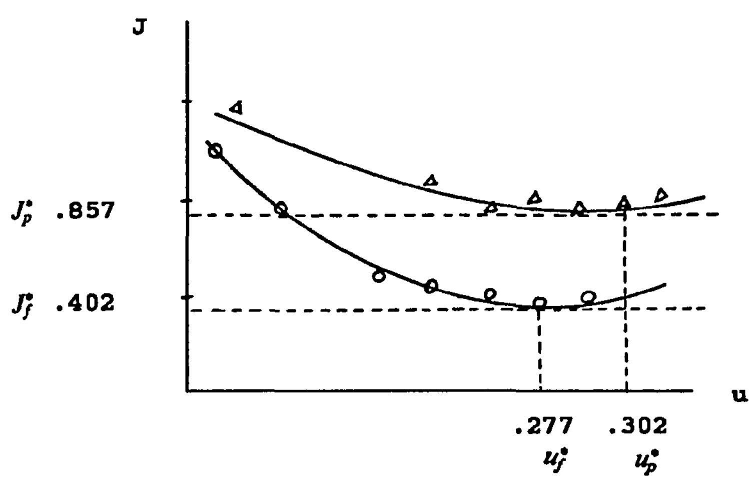

Figure 3-2. Plot of the Performance vs. Control variables with full information (o) and partial information (Δ) structures.

J_p^*= optimal performance for partial information
J_f^*= optimal performance for full information

u_p^*= total control for partial information
u_f^*= total control for full information

$$(u=\sqrt{u_1^2+u_3^2})$$

4. DECENTRALIZED OPTIMAL CONTROL WITH STOCHASTIC TRAFFIC

4.1 Introduction

It is well known that in centralized deterministic optimal control there is almost no difference between a control program and control strategy. A control program represents an open loop system where the control signal is determined only from a priori data irrespective of how the process develops. However, a control strategy represents a closed loop system where the control signal at time t depends on the state of the process at time t. They are equivalent in the sense that they will give the same value to the system performance index. This is mainly due to the fact that disturbances are approximated by a deterministic function which is known a priori in modeling the systems. The optimal feedback in this case is simply a function which maps the state space into the space of control variables as we have developed in Chapter 3. There is no dynamics involved in the feedback.

When considering the problem of controlling of stochastic linear systems, the dynamics of the feedback arise because the state is not known and must be reconstructed from measurements of output signals. The process of reconstruction of state variables from noise-corrupted output measurements is called *filtering*. The problem of optimal linear filtering for linear dynamic system was first solved by Kalman and Bucy [29] and was applied to find an optimal feedback control for stochastic linear systems with a quadratic performance criterion. The optimal control strategy under above conditions has been shown containing two distinct procedures: (1) find a state estimator which produces the best estimate of the state vector of the system from the measured outputs, and (2) find the optimal feedback law which gives the control signal as a linear function of the estimated state. This is referred to as the *certainty equivalence principle*, which emphasizes the fact that the optimal feedback will treat the estimated state as the true state, or the *separation theorem*, which indicates that the control problem is solved via two separate procedures: estimation and control.

The deterministic decentralized control problem in Chapter 3 can be generalized to have a stochastic forcing input and noisy information measurements. As the case in decentralized deterministic formulation, the routing of a computer communication network may be characterized by multiple controllers each controlling different routing decisions and having access to different congestion information of the network.

Our problem is to find the optimal decision rules for all controllers such that the chosen network performance index is minimized. Due to the limited information exchange in a large-scale network, we cannot expect the optimal control strategy is linear as has been shown by Witsenhausen. However, we must constrain the admissible control strategy to be linear in order to obtain an answer to the optimization problem.

We now consider the network routing problem and include the following conditions in our formulation.

- Traffic entering to the network from each node is a random process with known statistics. In many network problem formulations, it is often assumed that the incoming traffic to a node under consideration is Poisson. However, in this research, we relax the requirement such that the distribution of the process is not assumed.
- In chapter 3, the information structure is defined to be a linear combination of network states which perfectly describe the traffic congestion of the network. However, the measurement of on-line information considered in this chapter is subjected to noise corruption.

- We further impose constraints on the structure of the compensators that can be used by each controller. The reason is that the optimal compensator may be infinite dimensional under the limited information structure. We shall use an n^{th} order linear dynamic compensator driven by its own measurements for each controller H.

In section 4.2 the original problem of selecting the optimal parameters for the compensator is transformed into a deterministic optimal control problem involving matrix differential equations. The necessary conditions for optimality are deduced from the Matrix Minimum Principle in section 4.3. We shall derive the sufficient conditions of optimality through the stochastic Hamilton-Jacobi-Bellman partial differential equation in section 4.4.

It is noted that when the controllers have different information sets, the resultant optimal controls are not given by the separation theorem as pointed out by many researchers. Hence the controls are not obtained by applying the deterministic optimal control strategies on the state estimates. This *inseparable* property of the optimal control problem from optimal estimation can also be verified from the coupling of state and costate equations as indicated in Section 4.3 and Section 4.4.

4.2 Problem Statement

Consider a particular N-Node network which is described by

$$\dot{x}(t) = A(t)x(t) + \sum_{i=1}^{N} B_i(t)u_i(t) + r(t) \tag{4.1}$$

$$z_i(t) = C_i(t)x(t) + \theta_i(t) \tag{4.2}$$

where state variables $x(\cdot) \in R^n$, control variables $u_i(\cdot) \in R^{m_i}$, and output or information variables $z_i(\cdot) \in R^{k_i}$. The continuity requirement on matrices $A(t)$, $B_i(t)$ and $C_i(t)$ are same as that in (3-1) and (3-2). It is assumed that the traffic entering to the network $r(t) \in R^n$ now is a random process with zero mean and variance given by

$$cov\{r(t),r(\tau)\} = \Psi_r(t)\delta(t-\tau) \tag{4.3}$$

$\theta_i(t) \in R^{k_i}, i=1,2,\cdots,N$ are mutually independent noise processes with zero mean and covariances given by

$$cov\{\theta_i(t),\theta_i(\tau)\} = \Theta_i(t)\delta(t-\tau) \tag{4.4}$$

The matrix Θ_i is assumed to be symmetric and positive definite. Assume that the initial state $x(t_0) \equiv x_0$ is Gaussian with mean

$$E\{x(t_0)\} = m \tag{4.5}$$

and covariance

$$cov\{x(t_0),x(t_0)\} = \Sigma_0(t) \tag{4.6}$$

The information structure for i^{th} controller consists of *a priori* information as well as the on-line measurements. That is,

$$Y_i(t) = Z_i(t) \bigcup \{ \text{ a priori information } \}$$

where

$$Z_i(t) = \{z_i(s) \; ; \; t_0 \leq s \leq t\} \tag{4.7}$$

This equation indicates that the controller i has perfect recall on its measurement from t_0 up to time t. We furthermore assume that the system is causal. We claim a control function is admissible if

$$u_i(t) = \gamma_i[t \; , Z_i(t)] \tag{4.8}$$

and γ_i's satisfy the Lipshitz condition*. Our problem is to determine an admissible control $u_i^*(t)$ of the form (4.8) for all i such that the following performance criterion is minimized.

* The function $\gamma_i(t,\cdot)$ satisfies the Lipschitz condition if $\|\gamma(t,f)-\gamma(t,g)\| \leq \alpha \|f-g\|$ where $f, g \in C_r$, the class of continuous functions defined in [t0 , t], and α is a constant.

$$J = \mathbf{E}\left\{ <x(T),Sx(T)> + \int_{t_0}^{T} [<x(t),Q(t)x(t)> + \sum_{i=1}^{N} <u_i(t),R_i(t)u_i(t)>]dt \right\} \quad (4.9)$$

We assume that the matrices S, $Q(t)$ and $R_i(t)$ are symmetric and semi-positive definite Matrix $R_i(t)$ may still not be full rank.

The dynamic routing problem is now formulated as a stochastic decentralized control problem where each controller makes a routing decision based upon its information on traffic congestion together with *a priori* information such as traffic statistics, performance criterion, network topology $(A(t),B(t))$ etc..

We shall constrain the control functions of i to a linear transformation of the outputs of linear filters driven by the measurements.

$$u_i(t) = D_i(t)\hat{x}_i(t), \qquad i=1,2,\ldots,N \quad (4.10)$$

The filter equation is assumed to be the form of

$$\dot{\hat{x}}_i(t) = E_i(t)\hat{x}_i(t) + G_i(t)z_i(t) + H_i(t)u_i(t) \quad (4.11)$$

where $\hat{x}_i$ is the state of n-dimensional filters and $D_i(t)$, $E_i(t)$, $G_i(t)$, $H_i(t)$ are matrices of appropriate dimensions that are to be determined. Our problem is to determine the gain matrices $D_i(t)$, $E_i(t)$, $G_i(t)$, $H_i(t)$, and initial filter states $x_i(t_0)$, where $i=1,2,\cdots,N$, such that the controls given by these constraints are optimal. We shall reformulate this problem in terms of a deterministic problem. Substituting equation (4.2) and (4.10) into the state equation (4.1) and filter equation (4.11), we obtain

$$\dot{x}(t) = A(t)x(t) + \sum_{i=1}^{N} B_i(t)D_i(t)\hat{x}_i(t) + r(t) \quad (4.12)$$

$$\dot{\hat{x}}_i(t) = E_i(t)\hat{x}_i(t) + G_i(t)C_i(t)x(t) + H_i(t)D_i(t)\hat{x}_i(t) + G_i(t)\theta_i(t). \quad (4.13)$$

The estimation error is then given by

$$\hat{x}_i(t) = E_i(t)\hat{x}_i(t) + G_i(t)C_i(t)x(t) + H_i(t)D_i(t)\hat{x}_i(t) + G_i(t)\theta_i(t).$$

The estimation error is then given by

$$\dot{\boldsymbol{x}}-\dot{\hat{\boldsymbol{x}}}_i = (A-E_i+\sum_{\substack{j=1\\j\neq i}}^{N} B_jD_j-G_iC_i)\boldsymbol{x} - \sum_{\substack{j=1\\j\neq i}}^{N} B_jD_j(\boldsymbol{x}-\hat{\boldsymbol{x}}_j) + E_i(\boldsymbol{x}-\hat{\boldsymbol{x}}_i) + (B_i-H_i)D_i\hat{\boldsymbol{x}}_i - G_i\theta_i + \boldsymbol{r}, \quad i=1,2,\ldots,N \tag{4.14}$$

We further require $\hat{\boldsymbol{x}}_i(t)$ to be unbiased estimates of $\boldsymbol{x}(t)$ for all $u_i(t)$, i.e.,

$$\mathbf{E}\{\boldsymbol{x}(t)-\hat{\boldsymbol{x}}_i(t)|Z_i(t)\} = \mathbf{0}, \quad i=1,2,\ldots,N \tag{4.15}$$

$$\mathbf{E}\{\dot{\boldsymbol{x}}(t)-\dot{\hat{\boldsymbol{x}}}_i(t)|Z_i(t)\} = \mathbf{0}, \quad i=1,2,\ldots,N \tag{4.16}$$

for all t. Taking the conditional expectation of (4.14) yields

$$E_i(t) = A(t) + \sum_{\substack{j=1\\j\neq i}}^{N} B_j(t)D_j(t) - G_i(t)C_i(t) \tag{4.17}$$

$$H_i(t) = B_i(t), \qquad i=1,2,\cdots,N \quad i\neq j \tag{4.18}$$

Also the initial conditions become

$$\hat{\boldsymbol{x}}_i(t_0) = \boldsymbol{m}, \qquad i=1,2,\cdots,N.$$

Thus equation (4.14) has the form

$$\begin{aligned}\dot{\boldsymbol{x}}-\dot{\hat{\boldsymbol{x}}}_i &= -\sum_{\substack{j=1\\j\neq i}}^{N} B_jD_j(\boldsymbol{x}-\hat{\boldsymbol{x}}_j) + E_i(\boldsymbol{x}-\hat{\boldsymbol{x}}_i) - G_i\theta_i + \boldsymbol{r}\\ &= -\sum_{\substack{j=1\\j\neq i}}^{N} B_jD_j(\boldsymbol{x}-\hat{\boldsymbol{x}}_j) + (A+E_i+\sum_{\substack{j=1\\j\neq i}}^{N} B_jD_j-G_iC_i)(\boldsymbol{x}-\hat{\boldsymbol{x}}_i)\\ &\qquad - G_i\theta_i + \boldsymbol{r}, \quad i=1,2,\ldots,N\end{aligned} \tag{4.19}$$

Combining equations (4.1) and (4.19) into a matrix form, gives

$$\dot{\boldsymbol{l}}(t) = \hat{A}(t)\boldsymbol{l}(t) - \hat{B}(t)\boldsymbol{\theta}(t) + \boldsymbol{r}(t) \tag{4.20}$$

where

$$\dot{\boldsymbol{l}} = \begin{bmatrix} \dot{\boldsymbol{x}} \\ \dot{\boldsymbol{x}}-\dot{\hat{\boldsymbol{x}}}_1 \\ \dot{\boldsymbol{x}}-\dot{\hat{\boldsymbol{x}}}_2 \\ \dot{\boldsymbol{x}}-\dot{\hat{\boldsymbol{x}}}_3 \\ \cdot \\ \cdot \\ \cdot \\ \dot{\boldsymbol{x}}-\dot{\hat{\boldsymbol{x}}}_N \end{bmatrix}, \qquad \boldsymbol{l} = \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{x}-\hat{\boldsymbol{x}}_1 \\ \boldsymbol{x}-\hat{\boldsymbol{x}}_2 \\ \boldsymbol{x}-\hat{\boldsymbol{x}}_3 \\ \cdot \\ \cdot \\ \cdot \\ \boldsymbol{x}-\hat{\boldsymbol{x}}_N \end{bmatrix}, \qquad \boldsymbol{\theta} = \begin{bmatrix} \boldsymbol{\theta}_1 \\ \boldsymbol{\theta}_2 \\ \boldsymbol{\theta}_3 \\ \cdot \\ \cdot \\ \cdot \\ \boldsymbol{\theta}_N \end{bmatrix} \tag{4.21}$$

$$\hat{A} \equiv \begin{bmatrix} A_{11} & -B_1D_1 & -B_2D_2 & \cdots & \cdots & -B_ND_N \\ 0 & A_{22} & -B_2D_2 & \cdots & \cdots & -B_ND_N \\ 0 & -B_1D_1 & A_{33} & \cdots & \cdots & -B_ND_N \\ 0 & -B_1D_1 & -B_2D_2 & A_{44} & \cdots & \cdot \\ \cdot & -B_1D_1 & -B_2D_2 & -B_3D_3 & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdots & \cdot \\ 0 & -B_1D_1 & -B_2D_2 & \cdots & \cdots & A_{NN} \end{bmatrix} \tag{4.22}$$

where

$$A_{11} = A+\sum_{j=1}^{N} B_jD_j$$

$$A_{22} = A+\sum_{j=2}^{N} B_jD_j-G_1C_1$$

$$A_{33} = A+\sum_{\substack{j=1 \\ j\neq 2}}^{N} B_jD_j-G_2C_2$$

$$A_{44} = A+\sum_{\substack{j=1 \\ j\neq 3}}^{N} B_jD_j-G_3C_3$$

$\cdots$

$$A_{NN} = A+\sum_{j=1}^{N-1} B_jD_j-G_{N-1}C_{N-1}$$

$$\hat{B} \equiv \begin{bmatrix} 0 & 0 & \cdots & 0 \\ G_1 & 0 & \cdots & 0 \\ 0 & G_2 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & G_{N-1} \end{bmatrix} \tag{4.23}$$

Define the second moment matrix $\Psi_l(t)$ as $E\{l(t)l'(t)\}$. We shall derive $\Psi_l(t)$ by first solving $l(t)$ from equation (4.20).

$$l(t) = \Phi(t,t_0)l(t_0) + \int_{t_0}^{t}\Phi(t,\tau) + [r(\tau) - \hat{B}(\tau)\theta(\tau)]d\tau \tag{4.24}$$

where

$$\dot{\Phi}(t,t_0) = \hat{A}(t)\Phi(t,t_0), \qquad \Phi(t_0,t_0) = I. \tag{4.25}$$

Then

$$l(t)\cdot l'(t) = \left\{\Phi(t,t_0)l(t_0) + \int_{t_0}^{t}\Phi(t,\tau)[r(\tau_1) - \hat{B}(\tau_1)\theta(\tau_1)]d\tau_1\right\} \left\{\Phi(t,t_0)l(t_0) + \int_{t_0}^{t}\Phi(t,\tau_2)[\, r(\tau 2) - \hat{B}(\tau_2)\theta(\tau_2)\,]d\tau_2\right\}' \tag{4.26}$$

Expansion of $l(t)\cdot l'(t)$ is given in APPENDIX 4A.

Since $r(t)$ and $\theta(t)$ have zero mean, and $r(t)$, $l(t)$, and $\theta(t)$ are mutually independent, taking expectation of (4A-1) gives

$$\Psi_l(t) = \Phi(t,t_0)\Psi_l(t_0)\Phi'(t,t_0) + \int_{t_0}^{t}\int_{t_0}^{t}\Phi(t,\tau_1)\Psi_r(\tau_1)\delta(\tau_1-\tau_2)\Phi'(t,\tau_2)d\tau_1 d\tau_2 + \int_{t_0}^{t}\int_{t_0}^{t}\Phi(t,\tau_1)\hat{B}(\tau_1)\Psi_\theta(\tau_1)\delta(\tau_1-\tau_2)\hat{B}'(\tau_2)\Phi'(t,\tau_2)d\tau_1 d\tau_2 \tag{4.27}$$

where

$$\Psi_l(t) \equiv \mathbf{E}\{\boldsymbol{l}(t)\,\boldsymbol{l}'(t)\}$$
$$\Psi_r(t-\tau) \equiv \mathbf{E}\{\boldsymbol{r}(t)\,\boldsymbol{r}'(t)\} = cov\{\boldsymbol{r}(t)\boldsymbol{r}'(t)\}$$
$$\Psi_\theta(t) = \mathbf{E}\{\boldsymbol{\theta}(t)\boldsymbol{\theta}'(\tau)\} = cov\{\boldsymbol{\theta}(t)\boldsymbol{\theta}'(\tau)\}.$$

The components of Ψ_θ are

$$\Psi_\theta(\tau)(t-\tau) \equiv \begin{bmatrix} \Theta_1 & 0 & 0 & \cdots & 0 \\ 0 & \Theta_2 & 0 & \cdots & 0 \\ \cdot & \cdot & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & \Theta_N \end{bmatrix}$$

From the property of $\delta(t)$ functions,

$$\int_{t_1}^{t_2} f(\tau)\delta(\tau-c)d\tau = f(c), \qquad t_1<c<t_2 \tag{4.28}$$

the second term of (4.27) yields

$$\int_{t_0}^{t}\int_{t_0}^{t}\Phi(t,\tau_1)\Psi_r(\tau_1)\delta(\tau_1-\tau_2)\Phi'(t,\tau_2)d\tau_1 d\tau_2$$
$$= \int_{t_0}^{t}\Phi(t,\tau_2)\Psi_r(\tau_2)\Phi'(t,\tau_2)d\tau_2$$
$$= \int_{t_0}^{t}\Phi(t,\tau)\Psi_r(\tau)\Phi'(t,\tau)d\tau$$

Similar property is applied to the third term of (4.27). Thus (4.27) can be written as

$$\Psi_l(t) = \Phi_l(t,t_0)\Psi_l(t_0)\Phi_l'(t,t_0) + \int_{t_0}^{t}\Phi_l(t,\tau)\Psi_r(\tau)\Phi_l'(t,\tau)d\tau + \int_{t_0}^{t}\Phi_l(t,\tau)\hat{B}(\tau)\Phi_\theta(\tau)\hat{B}'(\tau)\Phi_l'(t,\tau)d\tau \tag{4.29}$$

Differentiation of (4-29) with respect to t (details see APPENDIX 4B) yields

$$\dot{\Psi}_l(t) = \hat{A}(t)\Psi_l(t) + \Psi_l(t)\hat{A}'(t) + \hat{B}(t)\Psi_\theta(t)\hat{B}'(t) + \Psi_r(t) \tag{4.30}$$

On the other hand $\Psi_l(t)$ may be expressed as

$$\Psi_l \equiv \mathrm{E}\{\boldsymbol{l}\cdot\boldsymbol{l}'\} = \mathrm{E}\left\{ \begin{bmatrix} \dot{\boldsymbol{x}} \\ \dot{\boldsymbol{x}}-\dot{\hat{\boldsymbol{x}}}_1 \\ \dot{\boldsymbol{x}}-\dot{\hat{\boldsymbol{x}}}_2 \\ \dot{\boldsymbol{x}}-\dot{\hat{\boldsymbol{x}}}_3 \\ \cdot \\ \cdot \\ \cdot \\ \dot{\boldsymbol{x}}-\dot{\hat{\boldsymbol{x}}}_N \end{bmatrix} \cdot \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{x}-\hat{\boldsymbol{x}}_1 \\ \boldsymbol{x}-\hat{\boldsymbol{x}}_2 \\ \boldsymbol{x}-\hat{\boldsymbol{x}}_3 \\ \cdot \\ \cdot \\ \cdot \\ \boldsymbol{x}-\hat{\boldsymbol{x}}_N \end{bmatrix}' \right\} \tag{4.31}$$

The outer product of (4.31) can be expanded into a matrix.

$$\begin{bmatrix} \boldsymbol{x}\boldsymbol{x}' & \boldsymbol{x}(\boldsymbol{x}'-\hat{\boldsymbol{x}}_1') & \cdots & \boldsymbol{x}(\boldsymbol{x}'-\hat{\boldsymbol{x}}_N') \\ (\boldsymbol{x}-\hat{\boldsymbol{x}}_1)\boldsymbol{x}' & (\boldsymbol{x}-\hat{\boldsymbol{x}}_1)(\boldsymbol{x}'-\hat{\boldsymbol{x}}_1') & \cdots & (\boldsymbol{x}-\hat{\boldsymbol{x}}_1)(\boldsymbol{x}'-\hat{\boldsymbol{x}}_N') \\ (\boldsymbol{x}-\hat{\boldsymbol{x}}_2)\boldsymbol{x}' & (\boldsymbol{x}-\hat{\boldsymbol{x}}_2)(\boldsymbol{x}'-\hat{\boldsymbol{x}}_1') & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ (\boldsymbol{x}-\hat{\boldsymbol{x}}_N)\boldsymbol{x}' & (\boldsymbol{x}-\hat{\boldsymbol{x}}_N)(\boldsymbol{x}'-\hat{\boldsymbol{x}}_1') & \cdots & (\boldsymbol{x}-\hat{\boldsymbol{x}}_N)(\boldsymbol{x}'-\hat{\boldsymbol{x}}_N') \end{bmatrix} \tag{4.32}$$

Taking the expectation of the matrix (4-32) gives

$$\Psi_l(t) \equiv \begin{bmatrix} m_{00} & m_{01} & m_{02} & m_{03} & \cdots & m_{0N} \\ m_{10} & m_{11} & m_{12} & m_{13} & \cdots & m_{1N} \\ m_{20} & m_{21} & m_{22} & m_{23} & \cdots & m_{2N} \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ m_{N0} & m_{N1} & m_{N2} & m_{N3} & \cdots & m_{NN} \end{bmatrix} \tag{4.33}$$

The initial condition of Ψ_l is then

$$\Psi_l(t_0) = \begin{bmatrix} \Sigma_0 + \boldsymbol{mm}' & \Sigma_0 & \Sigma_0 & \Sigma_0 & \cdots & \Sigma_0 \\ \Sigma_0 & \Sigma_0 & \Sigma_0 & \Sigma_0 & \cdots & \Sigma_0 \\ \Sigma_0 & \Sigma_0 & \Sigma_0 & \Sigma_0 & \cdots & \Sigma_0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ \Sigma_0 & \Sigma_0 & \Sigma_0 & \Sigma_0 & \cdots & \Sigma_0 \end{bmatrix} \tag{4.34}$$

Here we note that

$$\begin{aligned} m_{00} + m_{11} - m_{10} - m_{01} &= \hat{x}_1 \hat{x}_1' \\ m_{00} + m_{22} - m_{20} - m_{02} &= \hat{x}_2 \hat{x}_2' \\ &\cdots \\ m_{00} + m_{ii} - m_{i0} - m_{0i} &= \hat{x}_i \hat{x}_i' \\ &\cdots \\ m_{00} + m_{NN} - m_{N0} - m_{0N} &= \hat{x}_N \hat{x}_N' \end{aligned} \tag{4.35}$$

Let us introduce a new matrix $\hat{Q}(t)$

$$\hat{Q} = \begin{bmatrix} Q + \sum_{i=1}^{N} D_i' R_i D_i & -D_1' R_1 D_1 & -D_2' R_2 D_2 & \cdots & -D_N' R_N D_N \\ -D_1' R_1 D_1 & -D_1' R_1 D_1 & 0 & \cdots & 0 \\ -D_2' R_2 D_2 & 0 & -D_2' R_2 D_2 & \cdots & 0 \\ -D_3' R_3 D_3 & 0 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & 0 \\ -D_N' R_N D_N & 0 & 0 & \cdots & -D_N' R_N D_N \end{bmatrix} \tag{4.36}$$

Multiplying matrix $\hat{Q}$ with Ψ_l and taking the trace, we have

$$\begin{aligned}
&\mathrm{tr}\{\hat{Q}\Psi_l\} \qquad (4.37)\\
&= \mathrm{tr}\{Qm_{00}+\sum_{i=1}^{N}\Big(D_i'R_iD_im_{00}+D_1'R_1D_1m_{11}+D_2'R_2D_2m_{22}+\cdots+D_N'R_ND_Nm_{NN}\Big)\\
&\quad -D_1'R_1D_1m_{10}-D_2'R_2D_2m_{20}-D_3'R_3D_3m_{30}-\cdots-D_N'R_Nm_{N0}\\
&\quad -D_1'R_1D_1m_{01}-D_2'R_2D_2m_{02}-D_3'R_3D_3m_{03}-\cdots-D_N'R_Nm_{0N}\}\\
&= \mathrm{tr}\{Qm_{00}+\sum_{i=1}^{N}D_i'R_iD_i[m_{00}+m_{ii}-m_{i0}-m_{0i}]\}
\end{aligned}$$

Multiplying matrix $\hat{S}$ with Ψ_l and taking the trace, we have

$$\begin{aligned}
\mathrm{tr}\{\hat{S}\ \Psi_l(\tau)\} &= \mathrm{tr}\{S\ m_{00}\} \qquad (4.38)\\
&= \mathrm{tr}\{S\ \mathrm{E}[\boldsymbol{x}(\tau)\boldsymbol{x}'(\tau)]\}\\
&= \mathrm{E}\{\boldsymbol{x}'(\tau)\ S\ \boldsymbol{x}(\tau)\}
\end{aligned}$$

We have shown that J defined in (4.9) can be written in the following form:

$$J = \mathrm{tr}[\hat{S}\Psi_l(T)] + \mathrm{tr}\int_{t_0}^{T}[\hat{Q}(t)\Psi_l(t)]dt \qquad (4.39)$$

where

$$\hat{S} = \begin{bmatrix} S & 0 & 0 & \cdots & 0\\ 0 & 0 & 0 & \cdots & 0\\ 0 & 0 & 0 & \cdots & 0\\ 0 & 0 & 0 & \cdots & 0\\ \cdot & \cdot & \cdot & \cdots & \cdot\\ \cdot & \cdot & \cdot & \cdots & \cdot\\ \cdot & \cdot & \cdot & \cdots & \cdot\\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix},$$

Details of the proof is given in APPENDIX 4C.

The original problem has been transformed into a deterministic problem and may be stated as follows:

Find the matrices $D_i^*(t)$, and $G_i^*(t)$, where $i=1,2,\ldots,N$ subject to the constraints

$$\dot{\Psi}_l(t) = \hat{A}(t)\Psi_l(t) + \Psi_l(t)\hat{A}'(t) + \hat{B}(t)\Psi_\theta(t)\hat{B}'(t) + \Psi_r(t) \qquad (4.30)$$

with $\Psi_l(t_0)$ defined in (4.34) such that the performance criterion (4.39) is minimized.

4.3 Necessary Conditions for Optimality

The necessary conditions for optimality and associated two-point boundary value differential equations for a stochastic linear quadratic regulator with two controllers have been given by Chong and Athans [26]. However, the problem considered in this chapter includes an additional forcing term $\boldsymbol{r}$ (input traffic) in the system equation as seen in (4.1) and the total number of controllers is generalized to N. Thus, in this chapter we are investigating an N-person servomechanism problem.

Let $\Lambda(t)$ be the $3n \times 3n$ costate matrix associated with $\Psi_l(t)$. The Hamiltonian function for this problem is

$$H = \mathrm{tr}[\hat{Q}\Psi_l] + \mathrm{tr}[\ (\hat{A}\Psi_l + \Psi_l\hat{A}' + \hat{B}\Psi_\theta\hat{B}' + \Psi_r)\Lambda'\ ]. \tag{4.41}$$

Using the Matrix Minimum Principle [25] the necessary conditions for D_i and G_i to be optimal can be derived as follows:

$$\dot{\Psi}_l(t) = \frac{\partial H}{\partial \Lambda}|_* = \hat{A}(t)\Psi_l(t) + \Psi_l(t)\hat{A}'(t) + \hat{B}(t)\Psi_\theta(t)\hat{B}'(t) + \Psi_r(t) \tag{4.42}$$
$$= \Psi_l(t_0).$$

Assume that the Lagrange Multiplier has the form of

$$\Lambda(t) = P(t)\Psi_l(t) + \rho(t)$$

Then,

$$-\dot{P}(t) = \hat{Q}(t) + P(t)\hat{A}(t) + \hat{A}(t)'P(t), \qquad P(T) = \hat{S} \tag{4.43}$$
$$-\dot{\rho}(t) = P'(t)\hat{B}(t)\Psi_\theta(t)\hat{B}'(t) + P'(t)\Psi_r(t), \qquad \rho(T) = \mathbf{0} \tag{4.44}$$

The conditions for optimality are:

Condition 1:

$$0 = \frac{\partial H}{\partial G_i}|_* = -2\, I_i'P\Psi_l I_i C_i' + 2\, I_i'\Psi_l I_i G_i \Theta_i, \qquad i=1,2,\cdots,N \tag{4.45}$$

Condition 2:

$$0 = \frac{\partial H}{\partial D_i}|_* = R_i D_i (I_0 - I_i)' \Psi_l (I_j - I_i) + \sum_{\substack{j=0 \\ j \neq i}}^{N} B_i' I_j' P \Psi_l (I_j - I_i), \qquad i=1,2,\cdots,N \tag{4.46}$$

We shall summarize the results into the following Theorem.

Theorem 4.1 (Necessary)

The optimal control matrices to the optimization problem specified by equations (4.30), (4.34), and (4-39) are given by the following expressions:

$$D_i(t) = - R_i^{-1}(t) \sum_{\substack{j=0 \\ j \neq i}}^{N} B_i'(t) I_j' P(t) \Psi_l(t)(I_j - I_i) \cdot \left[(I_0 - I_i)' \Psi_l(t)(I_j - I_i) \right]^{-1}, \qquad i=1,2,\cdots,N \tag{4.47}$$

$$G_i(t) = (I_i' \Psi_l(t) I_i)^{-1} I_i' P(t) \Psi_l(t) I_i C_i'(t) \Theta_i^{-1}(t), \qquad i=1,2,\ldots,N \tag{4.48}$$

where $\Psi_l(t)$ is the solution of equation (4.30) with $\Psi_l(t_0)$ given

$$\dot{\Psi}_l(t) = \hat{A}(t)\Psi_l(t) + \Psi_l(t)\hat{A}'(t) + \hat{B}(t)\Psi_\theta(t)\hat{B}'(t) + \Psi_r(t)$$

and $P(t)$ is the solution of equation (4.49) with terminal condition $P(T)$

$$-\dot{P}(t) = \hat{Q}(t) + P'(t)\hat{A}(t) + \hat{A}'(t)P'(t), \qquad P(T) = \hat{S} \tag{4.49}$$

$$-\dot{\rho}(t) = P'(t)\hat{B}(t)\Psi_\theta(t)\hat{B}'(t) + P'(t)\Psi_r(t), \qquad \rho(T) = \mathbf{0}. \tag{4.50}$$

These are only necessary conditions, and it is possible that there are several solutions of these necessary conditions which are not optimal. We shall derive the sufficient conditions for optimality in next section.

4.4 Sufficient Conditions for Optimality

Let

$$V(\Psi_l(t),t) = J(\Psi_l(t),t) = \mathrm{tr}[\hat{S}\Psi_l(T)] + \mathrm{tr}\int_t^T [\hat{Q}(t)\Psi_l(t)]dt \tag{4.51}$$

The Hamilton-Jacobi-Bellman equation then implies that

$$\frac{\partial V}{\partial t} + \min_{D_l, G_l} \mathrm{tr}\left\{ \hat{Q}(t)\Psi_l(t) + \left(\frac{\partial V}{\partial \Psi_l}\right)' \left[\hat{A}(t)\Psi_l(t) + \Psi_l(t)\hat{A}'(t) + \hat{B}(t)\Theta(t)\hat{B}'(t) + \Psi_r(t)\right] \right\} = 0 \tag{4.52}$$

with $V(\Psi_l(T),T) = \mathrm{tr}[\hat{S}\Psi_l(T)]$.

Expression of $\hat{A}$, $\hat{Q}$, and $\hat{B}$ in terms of identity matrices. Define

$$I_0 = \begin{bmatrix} I \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}, \quad I_1 = \begin{bmatrix} 0 \\ I \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}, \quad I_2 = \begin{bmatrix} 0 \\ 0 \\ I \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}, \quad \cdots, \quad I_N = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ I \end{bmatrix} \tag{4.53}$$

Then the matrix

$$\begin{bmatrix} A & 0 & 0 & \cdots & 0 \\ 0 & A & 0 & \cdots & 0 \\ 0 & 0 & A & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & A \end{bmatrix}$$

can be expressed as

$$
= \begin{bmatrix} I & 0 & 0 & \cdots & 0 \\ 0 & I & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & I \end{bmatrix} A \begin{bmatrix} I & 0 & 0 & \cdots & 0 \\ 0 & I & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & I \end{bmatrix}
$$

$$
= (I_0 + I_1 + I_2 + \cdots + I_N)\, A\, (I_0 + I_1 + I2 + \cdots + I_N)' \qquad (4.54)
$$

The matrix $\hat{A}$ of (4.22) can be expressed as

$$
\begin{aligned}
\hat{A} = {} & (I_0 + I_1 + I_2 + \cdots + I_N)\, A\, (I_0 + I_1 + I_2 + \cdots + I_N)' \\
& + I_0B_1D_1[I_0 - I_1]' + I_2B_1D_1[I_2 - I_1]' + I_3B_1D_1[I_3 - I_1]' \\
& \quad + I_4B_1D_1[I_4 - I_1]' + \cdots + I_NB_1D_1[I_N - I_1]' \\
& + I_0B_2D_2[I_0 - I_2]' + I_1B_2D_2[I_1 - I_2]' + I_3B_2D_2[I_3 - I_2]' \\
& \quad + I_4B_2D_2[I_4 - I_2]' + \cdots + I_NB_2D_2[I_N - I_2]' \\
& + I_0B_3D_3[I_0 - I_3]' + I_1B_3D_3[I_1 - I_3]' + I_2B_3D_3[I_2 - I_3]' \\
& \quad + I_4B_3D_3[I_4 - I_3]' + \cdots + I_NB_3D_3[I_N - I_3]' \\
\cdots \\
& + I_0B_ND_N[I_0 - I_N]' + I_1B_ND_N[I_1 - I_N]' + I_2B_ND_N[I_2 - I_N]' \\
& \quad + I_4B_ND_N[I_4 - I_N]' + \cdots + I_{N-1}B_ND_N[I_{N-1} - I_N]' \\
& - I_1G_1C_1I_1' - I_2G_2C_2I_2' - I_NG_NC_NI_N'
\end{aligned}
$$

$$
\begin{aligned}
\hat{A} = {} & (I_0 + I_1 + I_2 + \cdots + I_N)\, A\, (I_0 + I_1 + I_2 + \cdots + I_N)' \\
& + \sum_{\substack{j=0 \\ j\neq 1}}^{N} I_jB_1D_1[I_j - I_1]' + \sum_{\substack{j=0 \\ j\neq 2}}^{N} I_jB_2D_2[I_j - I_2]' + \cdots \\
& + \sum_{\substack{j=0 \\ j\neq 2}}^{N} I_jB_ND_N[I_j - I_N]' - \sum_{j=1}^{N} I_jG_jC_jI_j'
\end{aligned}
$$

$$
\begin{aligned}
\hat{A} = {} & (I_0 + I_1 + I_2 + \cdots + I_N)\, A\, (I_0 + I_1 + I_2 + \cdots + I_N)' \\
& + \sum_{\substack{j=0 \\ j\neq i}}^{N} \sum_{i=1}^{N} I_jB_iD_i[I_j - I_i]' - \sum_{j=1}^{N} I_jG_jC_jI_j'
\end{aligned} \qquad (4.55)
$$

By using the same technique, we have

$$\hat{Q} = I_0QI_0' + I_0D_1'R_1D_1[I_0 - I_1]' + I_1D_1'R_1D_1[I_1 - I_0]'$$
$$+ I_0D_2'R_2D_2[I_0 - I_2]' + I_2D_2'R_2D_2[I_2 - I_0]'$$
$$+ I_0D_3'R_3D_3[I_0 - I_3]' + I_3D_3'R_3D_3[I_3 - I_0]'$$
$$+ \cdots$$
$$+ I_0D_N'R_ND_N[I_0 - I_N]' + I_ND_N'R_ND_N[I_N - I_0]'$$

$$\hat{Q} = I_0QI_0' + I_0\sum_{j=1}^{N} D_j'R_jD_j[I_0 - I_j]' - \sum_{j=1}^{N} I_jD_j'R_jD_j[I_0 - I_j]'$$

or

$$\hat{Q} = I_0QI_0' + \sum_{j=1}^{N}(I_0 - I_j)D_j'R_jD_j(I_0 - I_j)'. \tag{4.56}$$

We can express $\Psi_\theta(t)$ in the following form:

$$\Psi_\theta(t) \equiv \mathbf{E}\{\boldsymbol{\theta}\cdot\boldsymbol{\theta}'\} = \mathbf{E}\left\{ \begin{bmatrix} \boldsymbol{\theta}_1 \\ \boldsymbol{\theta}_2 \\ \boldsymbol{\theta}_3 \\ \cdot \\ \cdot \\ \cdot \\ \boldsymbol{\theta} \end{bmatrix} \cdot \begin{bmatrix} \boldsymbol{\theta}_1 \\ \boldsymbol{\theta}_2 \\ \boldsymbol{\theta}_3 \\ \cdot \\ \cdot \\ \cdot \\ \boldsymbol{\theta} \end{bmatrix}' \right\} \tag{4.57}$$

$$= \begin{bmatrix} \mathbf{E}(\boldsymbol{\theta}_1\boldsymbol{\theta}_1') & 0 & 0 & \cdots & 0 \\ 0 & \mathbf{E}(\boldsymbol{\theta}_1\boldsymbol{\theta}_2') & 0 & \cdots & 0 \\ 0 & 0 & \mathbf{E}(\boldsymbol{\theta}_1\boldsymbol{\theta}_3') & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & \mathbf{E}(\boldsymbol{\theta}_1\boldsymbol{\theta}_N') \end{bmatrix}$$

where $\boldsymbol{\theta}_1$, $\boldsymbol{\theta}_2$, $\cdots$, $\boldsymbol{\theta}_N$ are independent.

$$\equiv \begin{bmatrix} \Theta_1 & 0 & 0 & \cdots & 0 \\ 0 & \Theta_2 & 0 & \cdots & 0 \\ 0 & 0 & \Theta_3 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & \Theta_N \end{bmatrix} .$$

Then,

$$\hat{B}(t)\Psi_\theta(t)\hat{B}'(t) = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & G_2\Theta_2G_2' & 0 & \cdots & 0 \\ 0 & 0 & G_3\Theta_3G_3' & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & G_N\Theta_NG_N' \end{bmatrix}$$

$$= I_1G_1\Theta_1G_1'I_1' + I_2G_2\Theta_2G_2'I_2' + \cdots + I_NG_N\Theta_NG_N'I_N' = \sum_{j=1}^{N} I_jG_j\Theta_jG_j'I_j' \tag{4.58}$$

In order to get a closed-loop control strategy, we let

$$V(\Psi_l(t),t) = \mathrm{tr}\Big[P(t)\Psi_l(t) + \rho(t)\Big] \tag{4.59}$$

where P is symmetrical. Then

$$\left(\frac{\partial V}{\partial \Psi_l}\right)' = P(t) \tag{4.60}$$

From equation (4.52)

$$\frac{\partial V}{\partial t} + \min_{D_i,G_i} \mathrm{tr}\Big\{\hat{Q}(t)\Psi_l(t) + P'(t)[\hat{A}(t)\Psi_l(t) + \Psi_l(t)\hat{A}'(t) + \hat{B}(t)\Psi_\theta(t)\hat{B}'(t) + \Psi_r(t)]\Big\} = \mathbf{0} \tag{4.61}$$

To find G_i^*, we let $\frac{\partial \mathrm{tr}\{\cdot\}}{\partial G_i} = \mathbf{0}$

$$\frac{\partial}{\partial G_i}\operatorname{tr}\{P'(-\sum_{j=1}^{N} I_jG_jC_jI_j')\Psi_l - \Psi_l(\sum_{j=1}^{N} I_jC_j'G_j'I_j') + \sum_{j=1}^{N} I_jG_j\Theta_jG_j'I_j'\} = 0$$

or

$$-I_i'P\Psi_l'I_iC_i' - I_i'P'\Psi_lI_iC_i' + I_i'\Psi_lI_iG_i\Theta_i' + I_i'\Psi_l'I_iG_i\Theta_i = 0 \tag{4.62}$$

Since P, Ψ_l and Θ_i are symmetric matrices, equation (4.62) reduces to

$$-2\,I_i'P\Psi_lI_iC_i' + 2\,I_i'\Psi_lI_iG_i\Theta_i = 0 \tag{4.63}$$

From which we get

Condition 1.

$$G_i(t) = (I_i'\Psi_l(t)I_i)^{-1}I_i'P(t)\Psi_l(t)I_iC_i'(t)\Theta_i^{-1}(t), \qquad i=1,2,\ldots,N. \tag{4.64}$$

To find D_i^*,

$$\frac{\partial}{\partial D_i}\operatorname{tr}\left\{ \sum_{j=1}^{N}(I_0 - I_j)D_j'R_jD_j(I_0 - I_j)'\Psi_l + P'[\sum_{\substack{j=0\\ j\neq i}}^{N}\sum_{i=1}^{N} I_jB_iD_i(I_j - I_i)'\Psi_l + P'\Psi_l\sum_{\substack{j=0\\ j\neq i}}^{N}\sum_{i=1}^{N}(I_j - I_i)D_i'B_i'I_j'] \right\} = 0$$

$$R_iD_i(I_0 - I_i)'\Psi_l(I_0 - I_i) + R_i'D_i(I_0 - I_i)'\Psi_l'(I_0 - I_i) + \sum_{\substack{j=0\\ j\neq i}}^{N} B_i'I_j'P\Psi_l'(I_j - I_i) + \sum_{\substack{j=0\\ j\neq i}}^{N} B_i'I_j'P'\Psi_l(I_j - I_i) = 0. \tag{4.65}$$

Since R_i, Ψ_l and P are symmetric, we have

$$2\,R_iD_i(I_0 - I_i)'\Psi_l(I_j - I_i) + 2\sum_{\substack{j=0\\ j\neq i}}^{N} B_i'I_j'P\Psi_l(I_j - I_i) = 0, \qquad i=1,2,\cdots,N. \tag{4.66}$$

Condition 2

$$D_i(t) = -R_i^{-1}(t)\sum_{\substack{j=0\\ j\neq i}}^{N} B_i'(t)I_j'P(t)\Psi_l(t)(I_j-I_i)\cdot\left[(I_0-I_i)'\Psi_l(t)(I_j-I_i)\right]^{-1}, \qquad i=1,2,\cdots,N. \tag{4.67}$$

Taking the partial derivative of (4.59) with respect to t, we have

$$\frac{\partial V}{\partial t} = \mathrm{tr}\left[\dot{P}\Psi_l + \dot{\rho}\right]. \qquad (4.68)$$

The right hand side of (4.68) is equal to the right hand side of (4.45). This is a first order equation of $\Psi_l(t)$. By utilizing the property of trace operation

$$\mathrm{tr}[P'(t)\Psi_l(t)\hat{A}'(t)] = \mathrm{tr}[\hat{A}'(t)P'(t)\Psi_l(t)] \qquad (4.69)$$

The optimal control must satisfy the following conditions:

$$-\dot{P}(t) = \hat{Q}(t) + P(t)\hat{A}(t) + \hat{A}(t)'P(t), \qquad P(T) = \hat{S} \qquad (4.70)$$

$$-\dot{\rho}(t) = P'(t)\hat{B}(t)\Psi_\theta(t)\hat{B}'(t) + P'(t)\Psi_r(t), \qquad \rho(T) = \mathbf{0} \qquad (4.71)$$

The sufficient conditions for optimality are summarized as Theorem 4.2.

Theorem 4.2 (Sufficient)

Equations (4.64), (4.67) and (4.70), (4.71) are also sufficient conditions for the decentralized optimal control problem defined by equations (4.1) to (4.9).

We have proved that the conditions for optimality are necessary and sufficient.

4.5 Discussions

In this chapter the dynamic routing problem is considered in the framework of linear stochastic decentralized optimal control of cooperative type. That is, all controllers (routers) in the network work together as a team to optimize the same network performance.

We have assumed that the on-line communication between routers is prohibited so that the information structure is limited to the static type. However, we must emphasis the fact that the information sets of any two controllers are different, i.e.

$$z_i(t) \neq z_j(t), \qquad i \neq j$$

This is true even when the C_is are identity matrices (or states are fully observable) and the measurement noises have the same statistics. z_i and z_j are in general different, since

$\boldsymbol{\theta}_1$ and $\boldsymbol{\theta}_2$ are assumed independent. Thus, the problem we consider here has a decentralized and non-classical information structure.

As proved in [26], the matrices $(I_i'\Psi_l I_i)$ and $[(I_0-I_i)'\Psi_l(I_0-I_i)]$ are both positive definite* hence they have inverses. However, if the matrix R_i is not full rank, the optimal D_i is expressed as

$$D_i(t) = -R_i^+(t)\sum_{\substack{j=0\\ j\neq i}}^{N} B_i'(t)I_j'P(t)\Psi_l(t)(I_j-I_i)\cdot\left[(I_0-I_i)'\Psi_l(t)(I_j-I_i)\right]^{-1}, \quad i=1,2,\cdots,N$$

where R_i^+ is once again the pseudoinverse of matrix R_i.

It should be noted that the implementation of the optimal routing strategies can be carried out in two phases: the parameter optimization (to find D_i^*, G_i^*) can be performed off-line in advance for all routers in a centralized manner, but the transformation of measurements into control signals (to compute u_i^*) must be done on-line locally in real time.

4.6 Numerical Example

We use the same network given in chapter 3 to demonstrate the computation and numerical procedures. The network under consideration is shown in **Figure 4-1**, has a single destination node 2. We once again remove the superscript 2 for simplicity of the notation. The stochastic dynamic equation in decentralized form is:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix}-1 & -1\\ 0 & 1\end{bmatrix}\begin{bmatrix}u_{12}\\ u_{13}\end{bmatrix} + \begin{bmatrix}1 & 0\\ -1 & -1\end{bmatrix}\begin{bmatrix}u_{31}\\ u_{32}\end{bmatrix} + \begin{bmatrix}r_1\\ r_2\end{bmatrix} \qquad (4.73)$$

* A real, symmetric matrix Ψ_l is *positive definite* if for all real vectors $x\neq 0$ we have $x'\Psi_l x>0$.

$$z_i = C_i \cdot \begin{bmatrix} x_1 \\ x_3 \end{bmatrix} + \begin{bmatrix} \theta_1 \\ \theta_3 \end{bmatrix}, \quad i = 1,3. \tag{4.74}$$

We assume additional information on incoming traffic and measurement noise.

$$cov\{\boldsymbol{r}(t),\boldsymbol{r}(\tau)\} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \delta(t-\tau) \tag{4.75}$$

$$cov\{\theta_1(t),\theta_3(\tau)\} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \delta(t-\tau) \tag{4.76}$$

The initial state $\boldsymbol{x}(t_0)$ is Gaussian with

$$\mathbf{E}\{\boldsymbol{x}(t_0)\} = [1,\ 0]' \tag{4.77}$$

$$cov\{\boldsymbol{x}(t_0),\boldsymbol{x}(t_0)\} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \delta(t-\tau). \tag{4.78}$$

Assume the cost of routing or the performance index for all the controllers to be minimized is

$$J = \mathbf{E}\left\{ x_1^2(T)+x_3^2(T) + \int_{t_0}^{T}(x_1^2(t)+x_3^2(t)+u_1^2(t)+u_3^2(t))dt \right\} \tag{4.79}$$

We can obtain the optimal gain matrices D_1, D_2 and G_1, G_2 by solving the two-point boundary value differential equations using the method as the one in Chapter 3. The procedures are summarized as follows:

Step 0: To obtain an initial value of $G_{i0}(t)$, $D_{i0}(t)$ by setting $P_0(t)=\hat{S}$ and substitute $\Psi_{l0}(t_0)$ in equations (4.64) and (4.67).

Knowing G_{i0},D_{i0}, we compute $\hat{A}$, $\hat{Q}$, $\hat{B}$ and then find P_{n+1} by Step 1.

Step 1: To compute $P_{n+1}(t)$ by integrating the equation (4.70) backwards in time with the terminal condition $P_{n+1}(t)=\hat{S}$ and $G_n(t)$ from

Step 0.

Matrix $\rho_{n+1}(t)$ can be solved by equation (4.71) with $\rho_0(t)=0$.

Step 2: To compute $\Psi_{l_{n+1}}(t)$ by integrating equation (4.40) forward in time with initial condition $\Psi_{l_{n+1}}(t_0)$.

Step 3: compute $G_n(t)$ and $D_n(t)$ as in Step 0.

We once again examine the cases with full information and partial information. The computer results of optimal gain are given by

A. For Full Information Case, $C_i=I$, then $z_i=\theta_i$. Each controller can access to all the noise corrupted state variables. The optimal control matrices are:

$$G_1^* = -\begin{bmatrix} 2.371 & 1.148 \\ 1.148 & 2.272 \end{bmatrix} \tag{4.80}$$

and

$$G_3^* = -\begin{bmatrix} 2.202 & 1.095 \\ 1.094 & 2.331 \end{bmatrix} \tag{4.81}$$

$$D_1^* = -\begin{bmatrix} 1.777 & 1.841 \\ 1.842 & 2.016 \end{bmatrix} \tag{4.82}$$

and

$$D_3^* = -\begin{bmatrix} 2.155 & 1.690 \\ 1.690 & 2.172 \end{bmatrix} \tag{4.83}$$

where the optimal control are

$$u_1^* = \begin{bmatrix} u_{12} \\ u_{13} \end{bmatrix} = D_1^* \cdot \begin{bmatrix} \hat{x}_1 \\ \hat{x}_3 \end{bmatrix} \tag{4.84}$$

and

$$u_3^* = \begin{bmatrix} u_{31} \\ u_{32} \end{bmatrix} = D_3^* \cdot \begin{bmatrix} \hat{x}_1 \\ \hat{x}_3 \end{bmatrix} \tag{4.85}$$

The associated optimal performance $J^* = 1.47$.

B. Partial Information Case - We let

$$z_1 = [\,1 \;\; 0\,]\begin{bmatrix} x_1 \\ x_3 \end{bmatrix} + \begin{bmatrix} \theta_1 \\ \theta_3 \end{bmatrix} \tag{4.86}$$

$$z_3 = [\,0 \;\; 1\,]\begin{bmatrix} x_1 \\ x_3 \end{bmatrix} + \begin{bmatrix} \theta_1 \\ \theta_3 \end{bmatrix} \tag{4.87}$$

where $C_1 = [\,1 \;\; 0\,]$ and $C_3 = [\,0 \;\; 1\,]$. The optimal control matrices are:

$$G_1^* = -\begin{bmatrix} 0.552 & 0.188 \\ 0.188 & 0.793 \end{bmatrix} \tag{4.88}$$

and

$$G_3^* = -\begin{bmatrix} 0.702 & 0.195 \\ 0.194 & 0.831 \end{bmatrix} \tag{4.89}$$

$$D_1^* = -\begin{bmatrix} 1.012 & 0.236 \\ 0.237 & 0.473 \end{bmatrix} \tag{4.90}$$

and

$$D_3^* = -\begin{bmatrix} 0.098 & 0.334 \\ 0.334 & 0.797 \end{bmatrix} \tag{4.91}$$

where the optimal control are

$$u_1^* = \begin{bmatrix} u_{12} \\ u_{13} \end{bmatrix} = D_1^* \cdot \begin{bmatrix} \hat{x}_1 \\ \hat{x}_3 \end{bmatrix} \tag{4.92}$$

and

$$u_3^* = \begin{bmatrix} u_{31} \\ u_{32} \end{bmatrix} = D_3^* \cdot \begin{bmatrix} \hat{x}_1 \\ \hat{x}_3 \end{bmatrix} \tag{4.93}$$

The associated optimal performance $J^* = 2.66$.

The performance depends critically on the amount of information available at the time the value of the control signal should be determined. We claim that better information make better decisions. This is proved to be true even in a stochastic environment.

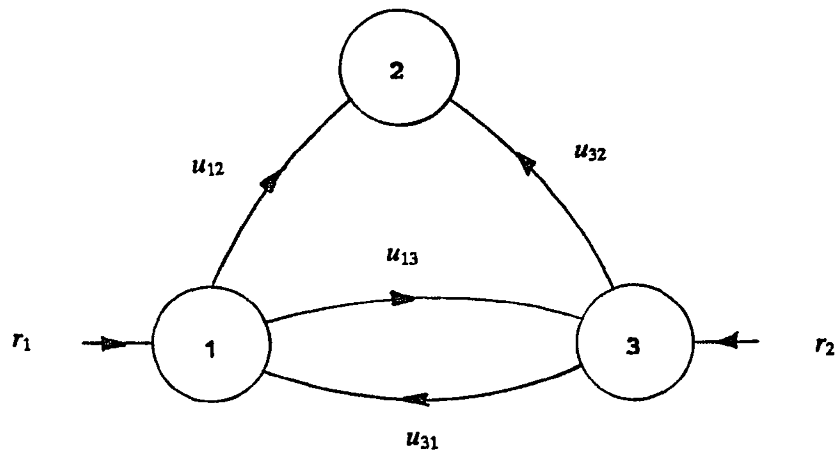

Figure 4-1. A Three-Node Network with Single Destinstion

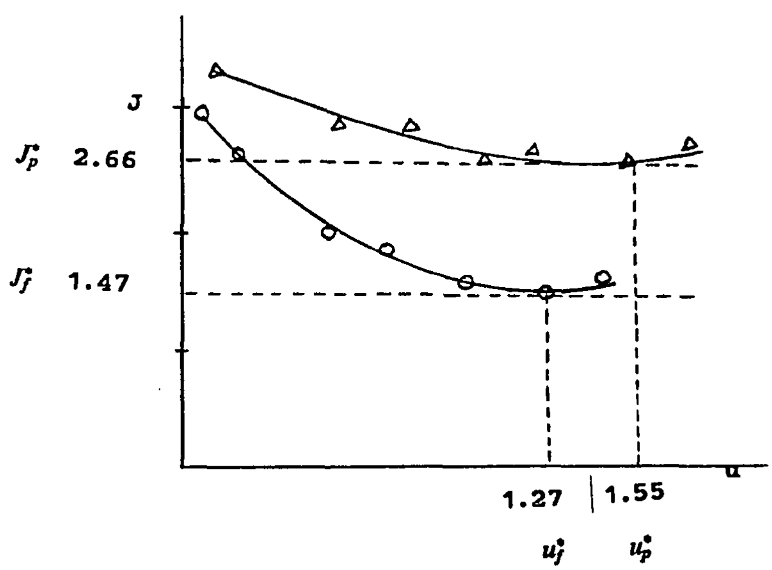

Figure 4-2. Plot of the Performance vs. Control variables with full information (o) and partial information (Δ) structures.

J_p^*= optimal performance for partial information
J_f^*= optimal performance for full information

u_p^*= total control for partial information
u_f^*= total control for full information

($u=\sqrt{u_1^2+u_2^2}$)

APPENDIX 4A

$$\boldsymbol{l}(t)\cdot\boldsymbol{l}'(t) = \left\{\Phi(t,t_0)\boldsymbol{l}(t_0) + \int_{t_0}^{t}\Phi(t,\tau)[\boldsymbol{r}(\tau_1) - \hat{B}(\tau_1)\boldsymbol{\theta}(\tau_1)]d\tau_1\right\} \left\{\Phi(t,t_0)\boldsymbol{l}(t_0) + \int_{t_0}^{t}\Phi(t,\tau_2)[\ \boldsymbol{r}(\tau 2) - \hat{B}(\tau_2)\boldsymbol{\theta}(\tau_2)\ ]d\tau_2\right\}' \tag{4A.1}$$

Multiplying (4A.1) item by item, yields

$$\begin{aligned} \boldsymbol{l}(t)\cdot\boldsymbol{l}'(t) &= \Phi(t,t_0)\boldsymbol{l}(t_0)\boldsymbol{l}'(t_0)\Phi'(t,t_0) \\ &+ \int_{t_0}^{t}[\Phi(t,t_0)\boldsymbol{l}(t_0)\boldsymbol{r}'(\tau)\Phi'(t,\tau_2) - \Phi(t,t_0)\boldsymbol{l}(t_0)\boldsymbol{\theta}'(\tau_2)\hat{B}'(\tau_2)\Phi'(t,\tau_2)]d\tau_2 \\ &+ \int_{t_0}^{t}[\Phi(t,\tau_1)\boldsymbol{r}(\tau_1)\boldsymbol{l}'(t_0)\Phi'(t,t_0) - \Phi(t,\tau_1)\hat{B}(\tau_1)\boldsymbol{\theta}(\tau_1)\boldsymbol{l}'(t_0)\Phi'(t,t_0)]d\tau_1 \\ &+ \int_{t_0}^{t}\int_{t_0}^{t}\Big\{\Phi(t,\tau_1)\hat{B}(\tau_1)\boldsymbol{\theta}(\tau_1)\boldsymbol{r}'(\tau_2)\Phi(t,\tau_2) - \Phi(t,\tau_1)\boldsymbol{r}(\tau_1)\boldsymbol{\theta}'(\tau_2)\hat{B}'(\tau_2)\Phi'(t,\tau_2) \\ &- \Phi(t,\tau_1)\boldsymbol{r}(\tau_1)\boldsymbol{\theta}'(\tau_2)\hat{B}'(\tau_2)\Phi'(t,\tau_2) + \Phi(t,\tau_1)\hat{B}(\tau_1)\boldsymbol{\theta}(\tau_1)\boldsymbol{\theta}'(\tau_2)\hat{B}'(\tau_2)\Phi'(t,\tau_2)\Big\}d\tau_1 d\tau_2 \end{aligned} \tag{4A.2}$$

APPENDIX 4B

$$\dot{\Psi}_l(t) = \dot{\Phi}(t,t_0)\Psi_l(t_0)\Phi'(t,t_0) + \Phi(t,t_0)\Psi_l(t_0)\dot{\Phi}'(t,t_0) + \int_{t_0}^{t}\dot{\Phi}(t,\tau)\Psi_r(\tau)\Phi'(t,\tau)d\tau + \int_{t_0}^{t}\Phi(t,\tau)\Psi_r(\tau)\dot{\Phi}'(t,\tau)d\tau + \Phi(t,t)\Psi_r(t)\Phi'(t,t) + \int_{t_0}^{t}\dot{\Phi}(t,\tau)\hat{B}(\tau)\Psi_\theta(\tau)\Phi'(t,\tau)d\tau + \int_{t_0}^{t}\Phi(t,\tau)\hat{B}(\tau)\Psi_\theta(\tau)\dot{\Phi}'(t,\tau)d\tau + \Phi(t,t)\hat{B}(t)\Psi_\theta(t)\hat{B}'(t)\Phi'(t,t) \tag{4B.1}$$

$$\dot{\Psi}_l(t) = \hat{A}(t)\left\{\Phi(t,t_0)\Psi_l(t_0)\Phi'(t,t_0) + \int_{t_0}^{t}\Phi(t,\tau)\hat{B}(\tau)\Psi_\theta(\tau)\Phi'(t,\tau)d\tau + \int_{t_0}^{t}\Phi(t,\tau)\Psi_r(\tau)\Phi'(t,\tau)d\tau\right\} + \left\{\Phi(t,t_0)\Psi_l(t_0)\Phi'(t,t_0) + \int_{t_0}^{t}\Phi(t,\tau)\hat{B}(\tau)\Psi_\theta(\tau)\Phi'(t,\tau)d\tau + \int_{t_0}^{t}\Phi(t,\tau)\Psi_r(\tau)\Phi'(t,\tau)d\tau\,\hat{A}'(t)\right\} + \hat{B}(t)\Psi_\theta(t)\hat{B}'(t) + \Psi_r(t) \tag{4B.2}$$

With the aid of (4.29), equation (4B.2) reduces to (4.30)

$$\dot{\Psi}_l(t) = \hat{A}(t)\Psi_l(t) + \Psi_l(t)\hat{A}'(t) + \hat{B}(t)\Psi_\theta(t)\hat{B}'(t) + \Psi_r(t) \tag{4.30}$$

APPENDIX 4C

$$u_i' R_i u_i = \mathrm{tr}[R_i u_i u_i'] = \mathrm{tr}[R_i D_i \hat{\boldsymbol{x}}_i \hat{\boldsymbol{x}}_i' D_i'] \tag{4C.1}$$

$$\begin{aligned} \mathrm{E}\{u_i' R_i u_i\} &= \mathrm{tr}[R_i D_i \mathrm{E}\{\hat{\boldsymbol{x}}_i \hat{\boldsymbol{x}}_i'\} D_i'] \\ &= \mathrm{tr}[D_i' R_i D_i \mathrm{E}\{\hat{\boldsymbol{x}}_i \hat{\boldsymbol{x}}_i'\}] \\ &= \mathrm{tr}[D_i' R_i D_i \{m_{00} + m_{ii} - m_{i0} - m_{0i}\}] \end{aligned} \tag{4C.2}$$

From the property of trace

$$\boldsymbol{x}' Q \boldsymbol{x} = \mathrm{tr}[Q \boldsymbol{x}\boldsymbol{x}'] \tag{4C.3}$$

$$\mathrm{E}\{\boldsymbol{x}' Q \boldsymbol{x}\} = \mathrm{tr}[Q \mathrm{E}\{\boldsymbol{x}\boldsymbol{x}'\}] = \mathrm{tr}[Q m_{00}] \tag{4C.4}$$

Then, from equation (4-28), we have

$$\begin{aligned} &\mathbf{E}\int_{t_0}^{T} [\boldsymbol{x}' Q \boldsymbol{x} + \sum_{i=1}^{N} u_i' R_i u_i] dt \\ &= \int_{t_0}^{T} \mathrm{tr}[Q m_{00} + \sum_{i=1}^{N} D_i' R_i D_i (m_{00} + m_{ii} - m_{i0} - m_{0i})] dt \\ &= \mathrm{tr}\{\hat{Q}(t) \Psi_l(t)\} \end{aligned} \tag{4C.5}$$

Follows from equations (4-28), (4-29) and (4-31) that

$$J = \mathrm{tr}\left[\hat{S} \Psi_l(T)\right] + \mathrm{tr}\int_{t_0}^{T} [\hat{Q}(t) \Psi_l(t)] dt$$

5. CONCLUSIONS

5.1 Discussion

As stated in Chapter 1, the objective of this research is to study the dynamic routing problem in the framework of decentralized control theory. The routing of a computer communication network can be modeled by a differential equation with multiple controllers. Each of the controllers has access to only a limited amount of traffic information and makes routing decisions so as to minimize a common network performance functional. This approach gives a distributed routing strategy which can adapt dynamically to the observed traffic situation. This formulation also suggests a methodology to analyze the traffic measurement and routing strategy design from information theoretic points of view.

In this chapter, we attempt to review in perspective the accomplishments of the preceding chapters. In Chapter 1, we provide a brief overview of the theory of decentralized control and an introduction to computer communication networks. We highlight the information structure which is the major difficulty encountered while solving a decentralized control problem. As the first step toward deriving the distributed optimal routing strategy a linear dynamic state space model is described in Chapter 2 which is a variation of Segall's model with multiple controllers. We associate with the model a quadratic performance functional which may be considered as the tariff artificially imposed to the network users. The traffic measurement seen by a specific controller is a linear combination of network states with possible noise corruption. But not all the states are available to every controller. The desired minimization of the performance functional results in a decentralized linear optimal problem with constraints on the form of feedback control strategy, information structure, and network dynamics. The model is sufficiently general to represent both a deterministic and a stochastic demand environment.

We now digress a moment to discuss the meaning of the so-called *optimal* solution of a control problem. In the context of centralized control, it is known that the optimal feedback control for a linear quadratic system can be constructed directly from a complete set of state variables. This represents the case of optimal state feedback. The control strategy that minimizes the performance functional is called the *optimal control*. Frequently the state variables of a linear system are not available for feedback purposes and one must generate the control signals from output measurements which may contain only a limited number of state variables and are often corrupted by measurement noise. This situation is referred to as the optimal output feedback and the resultant control strategy is called *suboptimal control* in some of the control literature. In general, the output feedback strategy causes a degraded yet acceptable system performance.

Having formulated the decentralized optimal problem, the remainder of the dissertation is devoted to developing a technique for finding the feedback solution for the cases in which the inputs are deterministic functions of time and stochastic processes representing the random demand of the network users.

The optimal control strategy under the deterministic formulation is developed in Chapter 3. We utilize the Matrix Minimum Principle to derive the necessary conditions for optimality. The procedure is complicated and non-trivial due to the appearance of the forcing term in the system dynamic equation. These conditions have been proved to be also sufficient by the Hamilton-Jacobi-Bellman partial differential equation.

We modify our formulation to the stochastic environment in Chapter 4. The traffic pattern now is modeled by a stochastic process with known mean and variance and the traffic measurement is subjected to the disturbance of Gaussian noise. We employ an n-

dimensional filter with a simple structure to generate the estimates of the state variables, which is then operated on by *gains* to produce the control variables. The stochastic control problem with multiple controllers is converted to a deterministic control problem and then solved by the Matrix Minimum Principle. We derive the sufficient conditions with specified information structure for stochastic setups. We also confirm that the optimal controls are not given by the separation theory.

Throughout the development we have exploited the linearity in system dynamics, information structure, admissible control and the structure of the filter to obtain many of our results. However, the conditions we have derived for both deterministic and stochastic formulations involve the solution of non-linear two point boundary value problems with matrix differential equations.

5.2 Summary of Results

In this section we describe briefly the primary results of the dissertation in the order in which they appear.

In Section 3.2 we transform the open-loop representation of the original decentralized control problem into a closed-loop form and pose the optimal control problem as the parametric optimization problem. In Section 3.3 we develop the necessary conditions of optimality associated with the decentralized optimal control problem with a general linear and static information structure. We also prove the important result that the necessary conditions are also sufficient. This is usually true only in a centralized environment with classical information structure but not in decentralized environment in general *. Section

* Another known exception is the system with partially nested information structure such as with one-step delay information.

3.5 discusses the issue of the information structure using a special case where no external traffic is entering to the network. This is corresponding to the mode of network operation where one desires to relieve the network congestion by inhibiting the external user traffic and getting rid of the current message backlogs stored in the nodes. We use a small network of ring structure to illustrate the computation results. This example has been used in Segall's work. This is considered to be a purely academic example and is not chosen to prove the utility of this approach to a real life design.

Chapter 4 is devoted for the solution of the stochastic decentralized case. The approach is very close to the work done by Chong and Athans. We have, however, generalized the two persons regulator problem into an N person decentralized servomechanism problem. We reformulate the stochastic control problem to a deterministic problem and look for the conditions for optimality. We present the necessary conditions in Section 4.3 and sufficient conditions in Section 4.4. We use the same network example to illustrate results for the stochastic case.

5.3 Contributions

In this dissertation we present a study of the dynamic routing problem of computer communication networks in the framework of decentralized control theory. Unlike the work done previously in state-space modeling and control theoretic solutions to the routing problems, our approach yields a distributed routing strategy. The resultant routing strategy in both deterministic and stochastic cases has *dynamic* and *closed-loop* properties. In addition, we suggest a methodology to study the area of information structures and the optimal routing strategies. This leads to a very important subject of signaling channel design for many communication networks particularly the integrated voice-data networks.

From the decentralized control point of view, it is generally believed that the sufficient conditions for optimality are not available or very difficult to find. This is particularly true for problems involving non-classical information structures. The difficulty arises, in the decentralized case, from the lack of precise understanding of the meaning of *principle of optimality* which is the basic tool for deriving the sufficient conditions. We find a complete set of feedback solutions to a meaningful linear decentralized control problem for both deterministic and stochastic cases. Of particular interest is that our solutions satisfy both necessary and sufficient conditions for optimality.

The essential contributions of this research may be summarized as follows:

- We have introduced a new analytical model for problems of dynamic routing in computer communication networks. It is a modification of Segall's with multiple routing controllers. This model allows us to formulate and investigate the routing problems in the framework of decentralized control theory.
- We have derived the optimal feedback for a class of decentralized optimal control problems with deterministic forcing inputs (decentralized servomechanism). This achievement was considered to be a significant contribution to the decentralized optimal control literature.
- We have extended our formulation to have stochastic forcing inputs. The conditions (necessary and sufficient) for optimality were established by extending the work of Chong and Athans to N controllers.

5.4 Suggestions for Future Work

We view this work as an initial investigation into the application of decentralized control theory to problems of dynamic routing in communication networks. The decentralized formulation is particularly useful in analyzing a large-scale network spanning nationwide and even worldwide. We have made a number of idealized assumptions in the process of formulation of the problem: for instance, quadratic performance index, instantaneous state information to reduce the complexity of the problem. The routing decisions at each node still require an extensive computation effort. Further simplifications must be made and efficient algorithms must be obtained for any practical applicability. However, the results we have obtained can provide valuable insight into the dynamic routing problems.

We can extend our work in two directions. In the context of computer communication networks, we may modify the model to include the transmission delay of state information. A more realistic cost functional such as the total packet delay can be used for the evaluation of network performance. Based upon our model, we can formulate a multi-level structure in terms of hierarchical decentralized control which will provide the so-called delta-routing strategy [30].

The possible extension in the decentralized control is seen as an open end. Although many successful stories have been reported in decentralized control with non-classical information structure, the problem with dynamic information structure generally is still unresolved. An associated problem arisen during our investigation is the issue of stability and controllability of the network and their relationship with various information structures.

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