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MATHEMATICAL MODELING AND ANALYTICAL SOLUTIONS OF DISSOLVED OXYGEN AND BIOCHEMICAL OXYGEN DEMAND PROFILES UNDER THREE-DIMENSIONAL, UNSTEADY-STATE CONDITIONS

New Jersey Institute of Technology   D.ENG.SC.   1981

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MATHEMATICAL MODELING AND ANALYTICAL SOLUTIONS

OF DISSOLVED OXYGEN AND BIOCHEMICAL OXYGEN DEMAND PROFILES

UNDER THREE-DIMENSIONAL, UNSTEADY-STATE CONDITIONS

BY

UING WOONG KIM

A DISSERTATION

PRESENTED IN PARTIAL FULFILLMENT OF

THE REQUIREMENTS FOR THE DEGREE

OF

DOCTOR OF ENGINEERING SCIENCE

AT

NEW JERSEY INSTITUTE OF TECHNOLOGY

Newark, New Jersey

1981

ABSTRACT

Mathematical models describing the temporal and spatial distribution of Dissolved Oxygen (DO) and Biochemical Oxygen Demand (BOD) for both pervious and impervious channel conditions in 3-dimensions have been constructed, and their analytical solutions were obtained. The solutions were implemented through the application of a digital computer, and their results are presented.

It has been found that there are significant differences in terms of DO and BOD distribution with respect to time and space in relatively deep and wide channels, thus necessitating the use of a 3-D, unsteady-state model.

APPROVAL OF DISSERTATION

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OF DISSOLVED OXYGEN AND BIOCHEMICAL OXYGEN DEMAND PROFILES
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BY

UING WOONG KIM

FOR

DEPARTMENT OF CIVIL AND ENVIRONMENTAL ENGINEERING
NEW JERSEY INSTITUTE OF TECHNOLOGY

BY

FACULTY COMMITTEE

APPROVED: CHAIRMAN

NEWARK, NEW JERSEY

1981

ACKNOWLEDGEMENT

The author expresses his heartfelt appreciation to his academic adviser, Dr. Robert Dresnack, P.E., Professor of Civil and Environmental Engineering of New Jersey Institute of Technology, for his continuous and invaluable guidance, encouragement and support throughout the research. Without his guidance, the research would never have become fruitful, for which the author will always be grateful.

Dr. James R. Pfafflin, MInst, CEng, a scholar and teacher of Civil and Environmental Engineering, provided the author with invaluable references and insight relating to the current research, as well as mental strength which the author greatly needed.

The author is grateful to Dr. Roman I. Andrushkiw, professor of Mathematics, who critically reviewed the mathematical development of the research and gave the author indispensable help during the formulation and solution of the mathematical model.

Dr. Paul Chan, professor of Civil and Environmental Engineering, also critically reviewed the manuscript from both a mathematical and engineering standpoint. His insight and understanding regarding the current research greatly enhanced the quality of the research, for which the author is greatly appreciative.

The author is also greatly indebted to the other members of the committee, Dr. Eugene Golub, P.E., professor and Chairman of Civil and Environmental Engineering, professor S. Cheng, professor T. Olenik,

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I. INTRODUCTION

An accurate prediction of the concentration levels of water quality parameters in the receiving body of water (river, lakes, etc.) has been one of the central topics in the water-quality engineering field for many reasons. This is particularly true of dissolved oxygen (DO) in the receiving water because it sustains all aquatic life (hence the most important parameter). In order for the receiving body of water to support aquatic life, it must first and foremost have an adequate supply of DO. If there were no organic pollutants being introduced into the receiving water from wastewater treatment plants and other outside sources, the DO concentration would always be at the saturation level, thus supplying more than enough oxygen to the aquatic environment.

All man-made pollutants eventually end up in the ocean via streams and rivers. Domestic sewage and industrial wastes are being treated in order to reduce their oxygen-consuming capacity before they are discharged into the receiving water. The more treatment they receive (hence more costly), the less oxygen they will deplete in the river, and vice versa.

The most cost-effective treatment is for the waste to be treated to such a degree that the depletion of oxygen due to the decomposition of treated effluents in the river is the maximum allowed by the regulatory agencies. That is to say that the wastes are being treated just enough to maintain the minimum DO standards which have been imposed by the regulatory agencies. In order for the optimum degree of treatment to be determined, an accurate prediction of DO concentrations along the river under the given river geometry,

hydraulic regime and oxygen depleting capacity (called Biochemical Oxygen Demand, abbreviated BOD and denoted L hereafter) must be made.

In the past the determination of DO (denoted C hereafter) and BOD (L) concentrations along the river relied mostly on the 1-dimensional mathematical model (either steady state or unsteady state). Because of the difficulties in obtaining analytical solutions of 1-D, unsteady state mathematical models, numerical solutions were widely used in predicting DO and BOD distribution along the river.

One-dimensional mathematical models are not adequate when river geometry (such as deep and wide) and/or hydraulic regimes (low velocity, for instance) are both not favorable to the 1-dimensional assumptions.

Furthermore, numerical solutions have suffered various maladies, such as instability, computation time, etc. The exact check of numerical solutions with respect to their stability has been difficult without having analytical solutions. In recent years the modeling of water quality in rivers has advanced from simple, one-dimensional analysis to the more accurate and also more complicated two- and three-dimensional approaches (Cleary, 1976). It has been shown that the distribution of dye concentrations along the river is considerably different in three directions, hence justifying the use of 3-D models (Cleary, 1973). Analytical solutions of 3-D, convective-dispersive equations with instantaneous discharges (Dirac Delta function) describing the distribution of dye along the river was obtained for the first time by the integral transform method (Cleary, 1973). Later (1975) this method was used to solve 3-D, convective-dispersive equations having generalized discharge

conditions (not instantaneous) with generalized boundary conditions (Dirichlet, Neumann and Robin or mixed).

The purpose of this research is to first construct the mathematical models then find the analytical solutions of both BOD and DO equations under 3-D (both convection and dispersion), unsteady-state flow regimes having appropriate boundary conditions.

Complete derivation of 3-D unsteady-state BOD and DO equations based on physical and kinetic principles has not yet been achieved. Furthermore, the current analytical solutions of the BOD equation do not give due regard to the channel conditions (i.e., pervious and impervious); hence the effect of channel conditions to the levels of BOD (hence DO), which are very signficant, is ignored.

The analytical solutions of general DO equations are not available at the present time (Cleary, personal communication, 1978).

It is believed that deriving general BOD and DO equations based on the physical principles and their analytical solutions will advance the art and science of stream modeling.

II. LITERATURE REVIEW

Literature on multi-dimensional unsteady-state BOD and DO models are notably lacking. There is, however, ample literature on multi-dimensional dye equations (without the first-order term) with analytical solutions. Because of the inclusion of first-order terms in the BOD and DO equations, multi-dimensional dye equations and their solutions are not useful to the current investigation and hence are excluded from the current literature review. Also excluded were numerical solutions on this subject.

The most representative work on the subject was that of Cleary's (1). In his paper (1) Cleary solved 3-D, unsteady-state BOD equations using integral transformation without due regard to the channel conditions.

Cleary, as others, did not investigate the solution of general DO equations.

The classical work of Carslaw and Jaeger (2) in their text entitled "Conduction of Heat in Solids" contains the most ample reference on the transfer of heat. However, again, since the heat equation does not contain a first-order term, their work offered little help to the current subject.

The excellent work of Crank (3) on the "Mathematics of Diffusion" was a similar case of the above. However, the above two references shed great insight on the mechanism of diffusion.

III. SPECIFIC STATEMENT OF THE PROBLEM

The specific objectives of the current investigations are:

1. Derive equations describing temporal and spatial distribution of Biochemical Oxygen Demand (BOD) and Dissolved Oxygen (DO) in three dimensions based upon physical principles,

2. Find their analytical solutions, and

3. Implement the solutions using digital computer.

IV. DERIVATION OF BOD AND DO EQUATIONS

1. Assumptions

   (1) PRINCIPLES OF LINEAR SYSTEMS

   The transport of any pollutant in the environment is caused by two major mechanisms--that is, advection (convection) and dispersion. Advection is transport due to the fluid motion, whereas dispersion is due to the random motion of the pollutants' molecules (molecular diffusion) and the turbulent action of the fluid carrying the pollutants (turbulent diffusion). Molecular diffusion occurs where there is a concentration gradient of the pollutants in the fluid whether the fluid is gas or liquid. Strictly speaking, the rate of molecular diffusion is not constant, but it depends upon the concentration of pollutants existing in the fluid.

   For instance, diffusion in high polymers depends markedly on concentration. However, molecular diffusion in a diluted realm such as mixing of pollutants with receiving waters (lake, stream, etc.) can reasonably be taken as constant. However, molecular diffusion in the receiving waters due to the discharge of pollutants is quite small compared to turbulent diffusion; thus it can be safely ignored.

   Turbulent diffusion is due to velocity gradients. Because of the boundary layer in the fluid flow, there exists velocity gradients which causes turbulent diffusion. Turbulent diffusion, also frequently referred to as eddy diffusion, is the major diffusion phenomena in the receiving waters because of its dominance in magnitude over molecular diffusion.

When conservative pollutants, such as chloride ion, is discharged into the receiving waters, only advection and diffusion govern the distribution in the receiving stream. However, when non-conservative pollutants, typified by Biochemical Oxygen Demand (BOD), are introduced into the receiving stream, they not only transport by advection and diffusion but also decay with time. Thus the distribution of such pollutants must take into account this decay phenomena. Therefore, the distribution of any non-conservative pollutants in the stream is due to three major mechanisms--that is, advection, turbulent diffusion, and decay.

It is known that the above three mechanisms do not interact with each other, i.e. they are independent of each other. Hence the principle of superposition of the linear systems can be applied in evaluating the distribution of pollutants in the receiving waters.

Under this assumption the distribution of pollutants in the receiving waters is merely a combined effect of advection, diffusion, and decay. There will be no product effect. Strictly speaking, no natural phenomena is linear. If we human beings understand them as being linear, it is due to insensitivities from our part, whether they are insensitive instruments or insensitive perceptions. Our basic understanding is that even though the actual transport phenomena of the pollutants in the receiving waters are non-linear, it is sufficient enough for us to assume them as linear for the sake of engineering approximation.

(2) IRREVERSIBLE FIRST-ORDER KINETICS

When pollutants decay in the receiving waters, it is generally believed that the rate of decay with respect to time is proportional to the first power of the amount of pollutant present at the time. This is called first-order reaction kinetics. It is known that Biochemical-Oxygen-Demanding material closely follows first-order reaction kinetics in their decay process. The decay of pollutants in the receiving waters is also irreversible. Simply speaking, this means that once the pollutants degrade to form other products, the products hence formed will not react again to form original substances.

Mathematically, the expression for decay is as follows:

$$\frac{\partial P}{\partial t} = - K_1 P \qquad \text{(first-order reaction)}$$

$$P \longrightarrow P_1 + P_2 + \cdots + P_n, \quad \text{(irreversible reaction)}$$
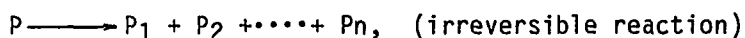
Where:

P = concentration of pollutants

K_1 = reaction constant

$P_1, P_2, \cdots, P_n$ = Products formed

Negative sign indicates P is decreasing with time.

2. DERIVATION OF BIOCHEMICAL OXYGEN DEMAND (BOD) EQUATION

Before deriving the BOD and DO equations, it is assumed that, in the interior of the river channel, BOD and DO are a continuous function of x, y, z, and t and that this holds also for the first differential coefficient with regard to t and for the first and second order differential coefficients with regard to x, y, and z. The principle of linear systems is the basis of our derivation for the BOD and DO equation. First, convection terms are derived assuming that this is the only transport mechanism.

In doing so, plug-flow conditions are assumed. Secondly, diffusion terms are derived under the assumption that this is the only transport mechanism in the receiving waters. Finally, the irreversible first-order reaction term is derived assuming that there is no convection and diffusion in the receiving waters. After deriving all the three terms, they are added up (or superimposed) assuming that the distribution of pollutants in the receiving waters follows the principle of linear systems.

(1) Convection Terms

It is assumed that convection is the only transport mechanism, that is, a plug flow condition.

$$Q,P \longrightarrow \qquad Q,P + \frac{\partial P}{\partial X} \Delta X$$

$$\longleftarrow \Delta X \longrightarrow$$
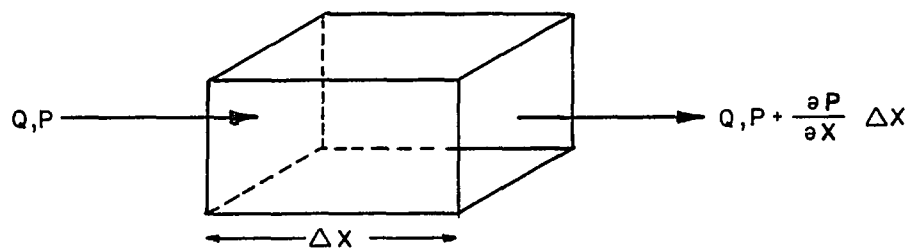
FIGURE 1 - ELEMENT OF VOLUME

Suppose that the above figure is an exaggerated infinitesimal element of a receiving water. Q denotes the rate of flow, and P denotes pollutant concentration. Applying the continuity equation for an incompressible fluid for an infinitesmal period of time (Δt),

Input-Output = Change in storage

$$Q \cdot P \cdot \Delta t - Q(P + \frac{\partial P}{\partial X} \cdot \Delta X)\Delta t = V \cdot \Delta P$$

Where V denotes the volume of the element.

Upon simplifying,

$$- \frac{\partial P}{\partial X} \cdot Q \cdot \Delta X \cdot \Delta t = V \cdot \Delta P$$

$$\frac{\Delta P}{\Delta t} = - \frac{Q}{V} \cdot \Delta X \cdot \left(\frac{\partial P}{\partial X}\right)$$

Let A denote the cross-sectional area of the element perpendicular to the direction of flow,

$$A \cdot \Delta X = V$$

$$\frac{\Delta P}{\Delta t} = - \frac{Q}{A \cdot \Delta X} \cdot \Delta X \left(\frac{\partial P}{\partial X}\right) = - \frac{Q}{A} \left(\frac{\partial P}{\partial X}\right)$$

Define mean velocity of the channel in the X direction,

$$\bar{u} = \frac{Q}{A}$$

It is of great importance to note that although u appears as a constant, the variations of velocity within the cross section are not ignored but are accounted for in the magnitudes of the diffusion coefficients (D_1, D_2, D_3), whose discussion will appear in the next section.

By using mean velocity, we write

$$\frac{\partial P}{\partial t} = - \bar{u} \frac{\partial P}{\partial X} \qquad \text{for X direction}$$

By the same fashion, we write

$$\frac{\partial P}{\partial t} = - \bar{v} \frac{\partial P}{\partial Y} \qquad \text{for Y direction}$$

$$\frac{\partial P}{\partial t} = - \bar{w} \frac{\partial P}{\partial Z} \qquad \text{for Z direction}$$

Where

$\bar{v}$ and $\bar{w}$ denote mean velocity in the Y and Z directions, respectively. For simplicity, hereafter the bar (-) is dropped in describing mean velocity.

Combining the above,

$$\left(\frac{\partial P}{\partial t}\right)_{convection} = -u\ \frac{\partial P}{\partial X} - v\ \frac{\partial P}{\partial Y} - w\ \frac{\partial P}{\partial Z} \qquad (1)$$

(2) Diffusion terms

$$4\,dy\,dz\left(Fx-\frac{\partial Fx}{\partial x}dx\right) \qquad\qquad 4\,dy\,dz\left(Fx+\frac{\partial Fx}{\partial x}dx\right)$$
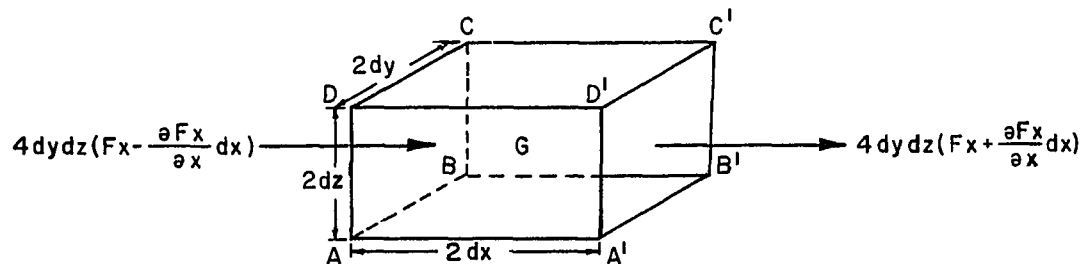
FIGURE 2 - DIFFUSION THROUGH CHANNEL

It is assumed that diffusion is the only transport mechanism. Again, consider an element of volume in the form of a rectangular parallelepiped whose sides are parallel to the axis of coordinates and are of length 2dx, 2dy, 2dz. Let the concentration of diffusing substance be P.

According to Fick's law, the rate of transfer of diffusing substance through a unit area of a section is proportional to the concentration gradient measured normal to the section, i.e.,

$$F = -Dx\ \frac{\partial P}{\partial X}$$

Where

D_x is a proportionality constant.

The negative sign indicates that the diffusion occurs in the direction opposite to that of increasing concentration. The rate at which the diffusing substance enters the element through the face ABDC is given by

$$4dydz \left(Fx - \frac{\partial Fx}{\partial X} \, dx\right)$$

Where Fx is the rate of transfer through the unit area of the corresponding plane G (x, y, z) where G is the center of the element. Similarly, the rate of loss of diffusing substance through the face A'B'C'D' is given by

$$4dydz \left(Fx + \frac{\partial Fx}{\partial X} \, dx\right)$$

Thus, the contribution to the rate of increase of diffusing substance in the element from these two faces is,

$$4dydz \left(Fx - \frac{\partial Fx}{\partial x} \, dx\right) - 4dydz \left(Fx + \frac{\partial Fx}{\partial x} \, dx\right) = -8dxdydz \, \frac{\partial Fx}{\partial x}$$

Similarly from other faces we obtain

$$-8dxdydz \, \frac{\partial Fy}{\partial y} \quad \text{and} \quad -8dxdydz \, \frac{\partial Fz}{\partial z}$$

But the rate at which the amount of diffusing substance in the element increases is also given by

$$8dxdydz \, \frac{\partial P}{\partial t}$$

Hence we must have,

$$8dxdydz \, \frac{\partial P}{\partial t} = -8dxdydz \, \frac{\partial Fx}{\partial x} - 8dxdydz \, \frac{\partial Fy}{\partial y} - 8dxdydz \, \frac{\partial Fz}{\partial z}$$

or

$$\frac{\partial P}{\partial t} + \frac{\partial Fx}{\partial x} + \frac{\partial Fy}{\partial y} + \frac{\partial Fz}{\partial z} = 0$$

If the diffusion coefficient D is constant, which is true for a diluted realm such as receiving waters, we have

$$Fx = -D \, \frac{\partial P}{\partial x} \quad Fy = -D \, \frac{\partial P}{\partial y} \quad \text{and} \quad Fz = -D \, \frac{\partial P}{\partial z}$$

Thus

$$\frac{\partial P}{\partial t} + \frac{\partial}{\partial x}\left(-D\ \frac{\partial P}{\partial x}\right) + \frac{\partial}{\partial y}\left(-D\ \frac{\partial P}{\partial y}\right) + \frac{\partial}{\partial z}\left(-D\ \frac{\partial P}{\partial z}\right) = 0$$

or

$$\frac{\partial P}{\partial t} = D\left(\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} + \frac{\partial^2 P}{\partial z^2}\right) = D\ \nabla^2 P \qquad (2)$$

Where   (Del) is a Laplacian operator.

The above is for molecular diffusion.

In the case of turbulent diffusion the same approach is used; however, the diffusion coefficients are no longer the same in the x, y, and z directions.   In this sense turbulent dispersion is viewed as molecular diffusion processes, i.e., dispersion by velocity gradients (turbulent dispersion) is governed by concentration gradients.   This view has been confirmed by many researchers in the past, including Taylor (1954), Elder (1959), Vanoni (1953) and Glover (1964).   Taylor and Elder noticed that the diffusion of momentum by turbulence is of fundamental importance in streamflow.   Through a fluid mechanics approach Taylor and Elder showed that the mechanism of turbulence can be described by a virtual coefficient of diffusion.

Introducing D_1, D_2, and D_3 as turbulent diffusion coefficients in the x, y, and z directions, we write,

$$\frac{\partial P}{\partial t} = D_1\ \frac{\partial^2 P}{\partial x^2} + D_2\ \frac{\partial^2 P}{\partial y^2} + D_3\ \frac{\partial^2 P}{\partial x^2} \qquad (3)$$

Since EQ. (3) is much greater than EQ. (2) in magnitude as illustrated previously, we write,

$$\left(\frac{\partial P}{\partial t}\right)_{\text{diffusion}} = D_1\ \frac{\partial^2 P}{\partial x^2} + D_2\ \frac{\partial^2 P}{\partial y^2} + D_3\ \frac{\partial^2 P}{\partial z^2} \qquad (4)$$

(3) IRREVERSIBLE FIRST-ORDER DECAY

Assume that there is only decay in the receiving water and the pollutants' decay is proportional to the concentration present.

$$P \longrightarrow P_1 + P_2 \cdots + P_n$$
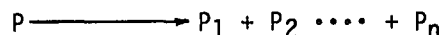
Where,

P is the pollutant concentration and $P_1, \cdots, P_n$ are the final products formed from decomposed pollutants.

Then

$$\left(\frac{\partial P}{\partial t}\right) = K_1 \ P^n$$

Here the superscript n is positive. The term, n, is not necessarily an integer but generally is a positive real number. If n is a positive real number, the reaction is called fractional-order.

Furthermore, if the reaction is first-order,

$$n = 1$$

Thus, an irreversible first-order reaction which describes a pollutants' decay in receiving waters is expressed as,

$$\left(\frac{\partial P}{\partial t}\right)_{decay} = -K_1 \ P \tag{5}$$

The above discussions related to the derivation of individual terms—namely, convection, diffusion, and decay.

If convection, diffusion, and decay occur simultaneously, one can write in accordance with the superposition principle the following equation:

$$\left(\frac{\partial P}{\partial t}\right)_{Total} = \left(\frac{\partial P}{\partial t}\right)_{convection} + \left(\frac{\partial P}{\partial t}\right)_{diffusion} + \left(\frac{\partial P}{\partial t}\right)_{decay}$$

Combining EQS. (1), (4), and (5) one obtains the following:

$$\frac{\partial P}{\partial t} = -u \frac{\partial P}{\partial x} -v \frac{\partial P}{\partial y} -w \frac{\partial P}{\partial z} + D_1 \frac{\partial^2 P}{\partial x^2} + D_2 \frac{\partial^2 P}{\partial y^2} + D_3 \frac{\partial^2 P}{\partial z^2} - K_1 P \tag{6}$$

The above is a general 3-dimensional, unsteady state model for the distribution of nonconservative pollutants, such as BOD-material in receiving waters (lakes, streams, etc.) with irreversible first order decay. EQ. (6) is a linear, second order, parabolic partial differential equation. Generally u, v, w, D_1, D_2, and D_3 are not constants, but rather a function of both time and space (x, y, z, t). Diffusion coefficients in particular are expressed as diffusion tensors in general. By judiciously dividing the receiving water into several segments, the terms u, v, w, D_1, D_2, and D_3 can be made sufficiently close to constant values.

It must be emphasized that EQ. (6) is not the BOD (Biochemical Oxygen Demand) equation yet. Since BOD is measured by the oxygen consumed or exerted by the decaying pollutants, EQ. (6) must accordingly be modified. If M denotes the total amount of pollutants discharged at a given time into the receiving water, the amount already exerted after a certain time has elapsed can be expressed as,

$$M - P(x,y,z,t) \qquad (7)$$

Since M is the total amount of pollutant present at time 0 and P is the amount of pollutants left (or present) at time t, EQ. (7) indicates the amount already decomposed (or exerted), i.e., BOD. Traditionally, BOD is expressed not as the concentration of pollutants itself already decomposed but as the "equivalent amount of oxygen" consumed to decompose the pollutants. In this sense, M is interpreted as the total amount of oxygen to completely decompose M. This total amount of oxygen is known as the ultimate BOD (BODu) and generally expressed as Lo. Thus the general 3-D, unsteady state BOD equation is,

$$L (x,y,z,t) = M - P(x,y,z,t) = Lo -P(x,y,z,t) \qquad (8)$$

Where,

$$\frac{\partial P}{\partial t} = -u \frac{\partial P}{\partial x} -v \frac{\partial P}{\partial y} -w \frac{\partial P}{\partial z} + D_1 \frac{\partial^2 P}{\partial x^2} + D_2 \frac{\partial^2 P}{\partial y^2} + D_3 \frac{\partial^2 P}{\partial z^2} - K_1 P$$

and L is the BOD as a function of both time and space. Since L is usually expressed as an equivalent amount of oxygen needed to degrade pollutants, M and P must also be expressed in this manner.

3. DERIVATION OF THE DISSOLVED OXYGEN (DO) EQUATION

All the assumptions used to develop the BOD equation apply in the derivation of the DO equation. In order to derive the DO equation in receiving waters, it is necessary to understand the general mechanisms of DO as to its depletion (sink) and replenishment (source).

(1) The Sources of DO

DO is supplied to the receiving water in two ways--from the atmosphere through diffusion (reaeration) and from the photosynthetic activity of algae. DO supply by reaeration is chiefly governed by the driving force at the interface of gas (air) and liquid (water). This driving force from the atmosphere is proportional to the DO deficit of the receiving water. DO deficit is defined as the difference between the saturation value of the DO (C_s) and the actual DO concentration present in the water body. Mathematically,

$$\left(\frac{\partial C}{\partial t}\right)_{reaeration} = K_2 (C_s - C)$$

Where,

C = DO concentration

C_s = Saturation value of DO, constant at a given temperature, pressure and salinity.

K_2 = a proportionality constant, known as the reaeration coefficient

DO is also supplied by algal activity (photosynthesis) in the receiving water. Since algal activity is only possible during sunlight radiation, DO supplied by algal activity is diurnal in nature. Mathematically,

$$(\frac{\partial C}{\partial t})_{algae} = A(t)$$

Where $A(t)$ denotes algal activity as a function of time.

(2) The Sinks of DO

DO is chiefly depleted by the decomposition of biochemically degradable organic material. Other sinks are benthal oxygen demand and algal respiration. Generally, these two are very small in magnitude compared to oxygen depletion by biodegradable organic material; thus, the sinks of DO can be written as, -

$$\frac{\partial C}{\partial t} = - K_1 P(x,y,z,t)$$

The negative sign indicates that BOD-material depletes DO. K_1 has been previously defined. Combining above with the assumption that linearity still holds, we write

$$(\frac{\partial C}{\partial t})_{Total} = (\frac{\partial C}{\partial t})_{reaeration} + (\frac{\partial C}{\partial t})_{algae} + (\frac{\partial C}{\partial t})_{BOD-material}$$

$$(\frac{\partial C}{\partial t}) = K_2(C_S - C) + A(t) - K_1 P(x,y,z,t)$$

Since

$$P(x,y,z,t) = Lo - L(x,y,z,t)$$

It follows that,

$$\frac{\partial C}{\partial t} = K_2(C_S - C) + A(t) - K_1[Lo - L(x,y,z,t)] \qquad (9)$$

Upon examining EQS. (8) and (9) it is found that the BOD equation must be solved in order to obtain an analytical solution of DO.

V. ANALYTICAL SOLUTIONS OF UNSTEADY-STATE, 3-D EQUATIONS

1. Analytical solution of BOD equation

One must first solve Equation (6) in order to solve Equation (8).

Rewriting EQ. (6),

$$\frac{\partial P}{\partial t} = -u \frac{\partial P}{\partial x} -v \frac{\partial P}{\partial y} -w \frac{\partial P}{\partial z} + D_1 \frac{\partial^2 P}{\partial x^2} + D_2 \frac{\partial^2 P}{\partial y^2} + D_3 \frac{\partial^2 P}{\partial z^2} - K_1 P \qquad (6)$$

Here P is the pollutant's concentration (not BOD) in the receiving water as a function of time and space. In order to solve the equation above, the boundary conditions and the coordinates system chosen are stated below.
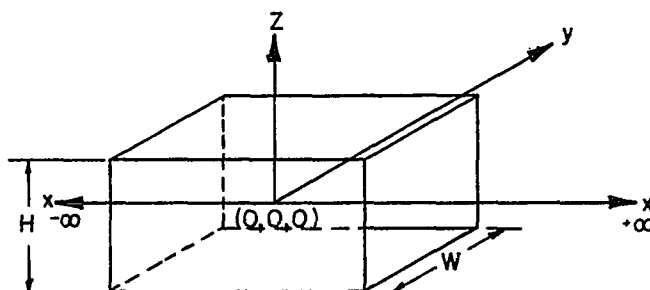
FIGURE 3 - COORDINATES SYSTEM OF RIVER CHANNEL

Suppose the above figure is an exaggerated infintesimal element of a river channel. W is the width of channel, and H is the depth of flow. The center of coordinates is located at the center of the parallelepiped. Equation (6) is solved first by assuming the pollutant is discharged at location (0,0,0) at time zero. The solution for a discharge point other than (0,0,0) can be determined from this solution by simply transforming the coordinates system.

The dimensions of x,y,z, and t are as follows:

$$-\infty < x < \infty$$

$$-\frac{W}{2} \le y \le \frac{W}{2}$$

$$-\frac{H}{2} \le z \le \frac{H}{2}$$

and

$$t \ge 0$$

(1) INITIAL AND BOUNDARY CONDITIONS

1. The pollutants' concentration must be zero at all points except (0,0,0) where pollutants are discharged at t=0. If pollutants are discharged at (X_0, Y_0, Z_0), they must be zero at all points but (X_0, Y_0, Z_0) at t=0. Mathematically, this can be written as,

$$P = 0 \quad \text{as} \quad t \longrightarrow 0, \text{ for } (x,y,z) \ne (0,0,0)$$

$$P = \infty \quad \text{as} \quad t \longrightarrow 0, \text{ if } (x,y,z) = (0,0,0)$$

$$(10)$$

2. No flux across the surface - Since no pollutants are introduced across the surface of the river channel, i.e., from the vertical (normal) direction across the surface, we write

$$\frac{\partial P}{\partial n} = 0 \quad \text{@ all points of the surface } (x,y,z) \qquad (11)$$

where $\partial/\partial n$ denotes the differentiation in the direction of the outward normal to the surface.

3. Since there is no concentration gradient of pollutants at the boundaries of the river channel, we write,

$$\frac{\partial P}{\partial x} = 0 \qquad \text{as } t \longrightarrow \infty$$

$$\frac{\partial P}{\partial y} = 0 \qquad @ \quad y \longrightarrow \pm \frac{W}{2}, \quad \text{and} \qquad\qquad (12)$$

$$\frac{\partial P}{\partial z} = 0 \qquad @ \quad z \longrightarrow \pm \frac{H}{2}$$

PERVIOUS STREAM CHANNELS

Organic, biodegradable pollutants are usually absorbed through the bottom and side banks of the channel which in turn travel further through the porous medium of the ground. This is especially true for BOD-material which contains substantial amounts of dissolved solids. In such cases, the stream geometry is stretched from a limited geometry into that of unlimited (unbounded) geometry. Let M equal the total amount of pollutants introduced, then

$$M = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P(x,y,z,t) \, dxdydz \qquad\qquad (13)$$

for an instantaneous point source. For continuous sources, we write,

$$\int_{0}^{t} m(t)dt = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P(x,y,z,t) \, dxdydz \qquad\qquad (14)$$

where m(t) is continuous sources mentioned above discharged within time dt. m(t) can be constant or periodic, such as a sinusoidal input.

The solutions will be investigated on both cases of EQS. (13) and (14). Note that the left hand side of EQ. (14) indicates the total amount of pollutants introduced during a time period of dt for a duration of t.

IMPERVIOUS STREAM CHANNELS

Suppose the bottom and banks of the stream channels are impervious due to man-made construction. An increasing number of streams are made impervious in order for the pollutants not to reach groundwater sources as well as to prevent erosion of channels. In such cases, EQS. (13) and (14) are modified as follows:

$$M = \int_{-\infty}^{\infty} \int_{-w/2}^{w/2} \int_{-H/2}^{H/2} P(x,y,z,t)\,dxdydz \qquad (15)$$

and

$$\int_{0}^{t} m(t)dt = \int_{-\infty}^{\infty} \int_{-w/2}^{w/2} \int_{-H/2}^{H/2} P(x,y,z,t)\,dxdydz \qquad (16)$$

Note that x is integrated from $-\infty$ to ∞ in both cases.

(2) SOLUTION PROCEDURES

Equation (6) will be solved first. Rewriting here,

$$\frac{\partial P}{\partial t} = -u \frac{\partial P}{\partial x} -v \frac{\partial P}{\partial y} -w \frac{\partial P}{\partial z} + D_1 \frac{\partial^2 P}{\partial x^2} + D_2 \frac{\partial^2 P}{\partial y^2} + D_3 \frac{\partial^2 P}{\partial z^2} - K_1 P \qquad (6)$$

CHANGE OF VARIABLES

Let $P(x,y,z,t) = \exp(ax + by + cz) \cdot \widetilde{P}(x,y,z,t)$, where a, b, and c are arbitrary constants. Denote further that $E=E(x,y,z) = \exp(ax + by + cz)$ in order to make the notation simple. Note that both P and $\widetilde{P}$ are a function of both time and space but $\exp(ax + by + cz)$ is only a function of space. Then,

$$\frac{\partial P}{\partial t} = E \widetilde{P}_t$$

$$u \frac{\partial P}{\partial x} = u (E\widetilde{P}_x + aE\widetilde{P})$$

$$v \frac{\partial P}{\partial y} = v (E\widetilde{P}_y + bE\widetilde{P})$$

$$w \frac{\partial P}{\partial z} = w (E\widetilde{P}_z + cE\widetilde{P})$$

and

$$D_1 \frac{\partial^2 P}{\partial x^2} = D_1 (E\widetilde{P}_{xx} + 2aE\widetilde{P}_x + a^2 E\widetilde{P}) \text{ and so on.}$$

The subscripts denote the differentials with respect to P. Inserting above to EQ. (6), dividing both sides of the equation by $E(\neq 0)$ and upon simplifying, we obtain

$$\widetilde{P}_t - D_1 \widetilde{P}_{xx} - D_2 \widetilde{P}_{yy} - D_3 \widetilde{P}_{zz} = -(au+bv+cw-a^2 D_1 - b^2 D_2 - c^2 D_3 + K_1) \widetilde{P} + \widetilde{P}_x(-u+2aD_1) + \widetilde{P}_y$$

$$(-v+2bD_2) + \widetilde{P}_z(-w+2cD_3)$$

The following equations are required in order to eliminate new convection terms, i.e., $\widetilde{P}_x$, $\widetilde{P}_y$, and $\widetilde{P}_z$.
Hence,

$$-u + 2aD_1 = 0$$

$$-v + 2bD_2 = 0$$

$$-w + 2cD_3 = 0$$

It follows that,

$$a = u/2D_1$$

$$b = v/2D_2, \text{ and}$$

$$c = w/2D_3$$

Note that since a, b, c, D_1, D_2, and D_3 are all constants there is no difficulty whatsoever in selecting a, b, and c to eliminate convection terms. With the values a, b, and c defined above, we write,

$$\widetilde{P}_t = D_1\widetilde{P}_{xx} + D_2\widetilde{P}_{yy} + D_3\widetilde{P}_{zz} - (u^2/4D_1 + v^2/4D_2 + w^2/4D_3)\ \widetilde{P}$$

Since the coefficient of P is constant, we let,

$$\widetilde{K}_1 = u^2/4D_1 + v^2/4D_2 + w^2/4D_3 + K_1$$

It follows that,

$$\widetilde{P}_t = D_1\widetilde{P}_{xx} + D_2\widetilde{P}_{yy} + D_3\widetilde{P}_{zz} - \widetilde{K}_1\widetilde{P}$$

By further letting,

$$x = \sqrt{D_1}\ \widetilde{x}$$
$$y = \sqrt{D_2}\ \widetilde{y}, \text{ and}$$
$$z = \sqrt{D_3}\ \widetilde{z}$$

Thus,

$$D_1\widetilde{P}_{xx} = D_1\ \frac{\partial}{\partial x} \times \left(\frac{\partial \widetilde{P}}{\partial x}\right) = D_1\ \frac{\partial}{\partial(\sqrt{D_1}\ \widetilde{x})}\ \left(\frac{\partial \widetilde{P}}{\partial(\sqrt{D_1}\ \widetilde{x})}\right) = D_1\ \left(\frac{1}{\sqrt{D_1}\ \sqrt{D_1}}\right)\ \frac{\partial^2 \widetilde{P}}{\partial \widetilde{x}^2}$$

It follows that,

$$D_1\widetilde{P}_{xx} = \widetilde{P}_{\widetilde{x}\widetilde{x}}, \quad D_2\widetilde{P}_{yy} = \widetilde{P}_{\widetilde{y}\widetilde{y}}, \quad \text{and so on}$$

Summarizing the above results,

$$\widetilde{P}_t = \widetilde{P}_{\widetilde{x}\widetilde{x}} + \widetilde{P}_{\widetilde{y}\widetilde{y}} + \widetilde{P}_{\widetilde{z}\widetilde{z}} - \widetilde{K}_1\widetilde{P}\ (\widetilde{x},\widetilde{y},\widetilde{z},t) \qquad (17)$$

Note that the coordinate system is now changed from

$$(x,y,z,) \text{ to } (\widetilde{x},\widetilde{y},\widetilde{z})$$

TREATMENT OF DECAY TERM

Let

$$\widetilde{P}(x,y,z,t) = \exp(\widetilde{K}_1 t)\ \widetilde{\widetilde{P}}(\widetilde{x},\widetilde{y},\widetilde{z},t)$$

Then

$$\frac{\partial \overline{\widetilde{p}}}{\partial t} = \overline{\widetilde{p}}(-\widetilde{K}_1 \exp(-\widetilde{K}_1 t)) + \exp(-\widetilde{K}_1 t) \frac{\partial \overline{\overline{p}}}{\partial t} \quad ,$$

$$\frac{\partial^2 \overline{\widetilde{p}}}{\partial \widetilde{x}^2} = (-\widetilde{K}_1 t) \frac{\partial^2 \overline{\overline{p}}}{\partial \widetilde{x}^2} \quad , \quad \text{and so on.}$$

Inserting above to EQ. (17), we get,

$$-\widetilde{K}_1 \exp(-\widetilde{K}_1 t) \overline{\widetilde{p}} + \exp(-\widetilde{K}_1 t) \frac{\partial \overline{\overline{p}}}{\partial t} = \exp(-\widetilde{K}_1 t)(\frac{\partial^2 \overline{\overline{p}}}{\partial \widetilde{x}^2} + \frac{\partial^2 \overline{\overline{p}}}{\partial \overline{y}^2} + \frac{\partial^2 \overline{\overline{p}}}{\partial \widetilde{z}^2}) - \widetilde{K}_1 \exp(-\widetilde{K}_1 t) \overline{\widetilde{p}}$$

Dividing both sides of the above equation by $\exp(-\widetilde{K}_1 t)$ $(\neq 0)$, we get,

$$\frac{\partial \overline{\widetilde{P}}}{\partial t} = \frac{\partial^2 \overline{\widetilde{P}}}{\partial \overline{x}^2} + \frac{\partial^2 \overline{\widetilde{P}}}{\partial \overline{y}^2} + \frac{\partial^2 \overline{\widetilde{P}}}{\partial \overline{z}^2} \tag{18}$$

Equation (18) can be solved by a number of ways, including integral transforms. Equation (18) will be solved by the method of separation of variables. Let the solution be of the form,

$$\overline{\widetilde{P}}(\overline{x},\overline{y},\overline{z},t) = f_1(\overline{x},t) \, f_2(\overline{y},t) \, f_3(\overline{z},t) = f_1 f_2 f_3 \tag{19}$$

Note that $\overline{\widetilde{P}}$ is expresed as a product of solution.

Then,

$$\frac{\partial \overline{\widetilde{P}}}{\partial t} = f_2 f_3 \frac{\partial f_1}{\partial t} + f_1 f_3 \frac{\partial f_2}{\partial t} + f_1 f_2 \frac{\partial f_3}{\partial t}, \text{ and so on}$$

Inserting above to EQ. (18),

$$f_2 f_3 (\frac{\partial f_1}{\partial t} - \frac{\partial^2 f_1}{\partial \overline{x}^2}) + f_1 f_3 (\frac{\partial f_2}{\partial t} - \frac{\partial^2 f_2}{\partial \overline{y}^2}) + f_1 f_2 (\frac{\partial f_3}{\partial t} - \frac{\partial^2 f_3}{\partial \widetilde{z}^2}) = 0 \tag{20}$$

Since,

$$f_2 f_3 \neq 0, \quad f_1 f_3 \neq 0, \text{ and } f_1 f_2 \neq 0$$

We must have,

$$\frac{\partial f_1}{\partial t} - \frac{\partial^2 f_1}{\partial \tilde{x}^2} = 0 \tag{21}$$

$$\frac{\partial f_2}{\partial t} - \frac{\partial^2 f_2}{\partial \tilde{y}^2} = 0, \quad \text{and} \tag{22}$$

$$\frac{\partial f_3}{\partial t} - \frac{\partial^2 f_3}{\partial \tilde{z}^2} = 0 \tag{23}$$

in order for the left hand side of EQ. (20) to be zero. Note that since f_1 is only a function of $(\tilde{x}, t)$, f_2 a function of $(\tilde{y}, t)$, and f_3 a function of $(\tilde{z}, t)$ there is no other combination among EQS. (21), (22), and (23) which can make the left hand side of EQ. (20) zero.

The solutions of EQS. (21), (22), and (23) are,

$$f_1 = B_1 t^{-1/2} \exp(-\tilde{x}^2/4t)$$

$$f_2 = B_2 t^{-1/2} \exp(-\tilde{y}^2/4t), \quad \text{and}$$

$$f_3 = B_3 t^{-1/2} \exp(-\tilde{z}^2/4t)$$

where B_1, B_2, and B_3 are arbitrary integral constants.
Inserting the above to EQ. (19),

$$\tilde{\tilde{P}}(\tilde{x}, \tilde{y}, \tilde{z}, t) = Bt^{-3/2} \exp(-\tilde{x}^2/4t - \tilde{y}^2/4t - \tilde{z}^2/4t)$$

where

$$B = B_1 B_2 B_3$$

Transforming from $\tilde{\tilde{P}}$ to $\tilde{P}$ we write,

$$\tilde{P}(\tilde{x}, \tilde{y}, \tilde{z}, t) = Bt^{-3/2} \exp(-\tilde{K}_1 t) \exp(-\tilde{x}^2/4t - \tilde{y}^2/4t - \tilde{z}^2/4t)$$

Transforming back to the original coordinates (x, y, z),

$$\tilde{P}(x, y, z, t) = Bt^{-3/2} \exp(-\tilde{K}_1 t) \exp(-x^2/4D_1 t - y^2/4D_2 t - z^2/4D_3 t)$$

Finally converting from $\tilde{P}$ to P,

$$P(x,y,z,t) = B t^{-3/2} \exp(-\tilde{K}_1 t) \exp(ax+by+cz) \exp(-x^2/4D_1 t - y^2/4D_2 t - z^2/4D_3 t) \dots\dots \quad (24)$$

where,

B = arbitrary integral constant

$\tilde{K}_1 = K_1 + u^2/4D_1 + v^2/4D_2 + w^2/4D_3$

$a = u/2D_1$, $b = v/2D_2$, and $C = w/2D_3$

Note that EQ. (24) is of product form and each term weighs the others in order to take into account specific effects, such as first-order decay, convection, etc.

EQ. (24) is further simplified as follows.

First collecting exponents of the above equation for those related to x,

$$-u^2 t/4D_1 + ux/2D_1 - x^2/4D_1 t = -1/4D_1 t (x^2 - 2utx + u^2 t^2) = -1/4D_1 t (x-ut)^2$$

By the same fashion, we obtain expressions for the exponents related to y and z as

$$- 1/4D_2 t (y - vt)^2 \text{ and } - 1/4D_3 t (z - wt)^2$$

EQ. (24) is now rewritten as,

$$P(x,y,z,t) = B t^{-3/2} \exp(-K_1 t) \exp(-(x-ut)^2/4D_1 t -(y-vt)^2/4D_2 t -(z-wt)^2/4D_3 t)$$

DETERMINATION OF INTEGRAL CONSTANT
AND SOLUTIONS FOR BOD EQUATION FOR
PERVIOUS STREAM CHANNEL UNDER
INSTANTANEOUS POINT SOURCE

Since the total pollutants introduced must be contained somewhere in the system where there is no decay or where the decay constant is small, we write,

$$M = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P(x,y,z,t) \Bigg|_{\substack{\lim \\ K_1 \to 0}} dxdydz \qquad (25)$$

The above must hold for an instantaneous point source.
Rewriting EQ. (25),

$$B\left[\int_{-\infty}^{\infty} t^{-1/2} \exp(-(x-ut)^2/4D_1t)dx \cdot \int_{-\infty}^{\infty} t^{-1/2}\exp(-(y-vt)^2/4D_2t)dy \cdot \int_{-\infty}^{\infty} t^{-1/2}\exp(-(z-wt)^2/4D_3t)dz\right] = M \qquad ....(26)$$

The first integral is

$$\int_{-\infty}^{\infty} t^{-1/2} \exp(-(x-ut)^2/4D_1t)dx \qquad (27)$$

Since EQ. (26) must hold for zero or infinitely small convection term (or in general for all values of u), we write from EQ. (27),

$$\lim_{u \to o} \int_{-\infty}^{\infty} t^{1/2}\exp(-(x-ut)^2/4D_1t) \, dx = \int_{-\infty}^{\infty} t^{1/2}\exp(-x^2/4D_1t) \, dx.... \qquad (28)$$

Introducing the error function,

$$erf\, x = \frac{2}{\sqrt{\pi}} \int_{o}^{x} \exp(-\xi^2) \, d\xi \qquad (29)$$

so that,

$$\text{erf } \infty = 1$$

and

$$\text{erf}(-x) = -\text{erf}(x) \text{ (odd function)} \tag{30}$$

For small values of x we use the series for $\exp(-\xi^2)$ in EQ. (29) and thus obtain,

$$\text{erf } x = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-\xi^2) \, d\xi = \frac{2}{\sqrt{\pi}} \int_0^x \left(\sum_{n=0}^{\infty} \frac{(-1)^n \xi^{2n}}{n!} \right) d\xi \tag{31}$$

The above series is uniformly convergent. It may be integrated term by term and therefore,

$$\text{erf } x = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)n!} \tag{32}$$

Going back to EQ. (28) and letting,

$$-x^2/4D_1 t = \xi^2$$

then

$$\xi \longrightarrow \pm \infty \qquad \text{as} \quad x \longrightarrow \pm \infty$$

and

$$(1/4D_1 t) \, 2x dx - 2\xi d\xi = x(D_1 t)^{-1/2} d\xi$$

$$dx = 2D_1 t \cdot (D_1 t)^{1/2} d\xi = 2 (D_1 t)^{1/2} d\xi$$

Thus, EQ. (28) becomes,

$$\int_{-\infty}^{\infty} t^{-1/2} \exp(-x^2/4D_1 t) \, dx = \int_{-\infty}^{\infty} t^{-1/2} \exp(-\xi^2) \cdot 2(D_1 t)^{1/2} d\xi = 2D_1^{1/2} \int_{-\infty}^{\infty} \exp(-\xi^2) \, d\xi$$

The right hand side of the above equation is now the error function defined previously.

Since,

$$\int_{-\infty}^{\infty} \exp(-\xi^2)\, d\xi = -\int_{0}^{-\infty} \exp(-\xi^2)\, d\xi + \int_{0}^{\infty} \exp(-\xi^2)\, d\xi$$

It follows that,

$$\int_{-\infty}^{\infty} \exp(-\xi^2)\, d\xi = 2\int_{0}^{\infty} \exp(-\xi^2)\, d\xi$$

Recalling,

$$\operatorname{erf}\infty = 1 = 2/\sqrt{\pi}\int_{0}^{\infty} \exp(-\xi^2)\, d\xi$$

Hence,

$$\int_{0}^{\infty} \exp(-\xi^2)\, d\xi = \sqrt{\pi}/2$$

Thus

$$\int_{-\infty}^{\infty} \exp(-\xi^2)\, d\xi = \sqrt{\pi}$$

It follows that,

$$\int_{-\infty}^{\infty} t^{-1/2} \exp(-x^2/4D_1 t)\, dx = 2 D_1^{1/2}\int_{-\infty}^{\infty} \exp(-\xi^2)\, d\xi = 2\sqrt{D_1 \pi}$$

Second and third integrals of EQ. (26) are evaluated by the same fashion.

Thus EQ. (25) now becomes,

$$B\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P(x,y,z,t)\, dx\, dy\, dz \Bigg|_{\substack{\lim \\ K_1 \to 0}}, \;_{\substack{\lim \\ u,v,w \to 0}}\right] = M = 8\sqrt{D_1 D_2 D_3}\, \pi^{3/2}\cdot B$$

It follows that,

$$B = \frac{M}{\sqrt{8 \quad D_1D_2D_3 \; \pi}^{3/2}}$$

Thus the solution for instantaneous point source becomes,

$$P(x,y,z,t) = \frac{M \cdot t^{-3/2}}{\sqrt{8D_1D_2D_3\pi}^{3/2}} \; \exp(-K_1 t)\exp(-(x-ut)^2/4D_1t - (y-vt)^2/4D_2t - (z-wt)^2/4D_3t) \quad (33)$$

Since BOD (L) is related with pollutant concentration by EQ. (8), the solution of the general, 3-D, unsteady state equation under instantaneous point discharge is,

$$L(x,y,z,t) = M - P(x,y,z,t) = Lo - P(x,y,z,t) \quad (8)$$

thus,

$$L(x,y,z,t) = Lo\left[1 - \frac{t^{-3/2}}{\sqrt{8 \, D_1 D_2 D_3 \pi}^{3/2}} \; e^{-K_1 t} \cdot e^{-(x-ut)^2/4D_1t - (y-vt)^2/4D_2t - (z-wt)^2/4D_3t}\right]$$

It is of great importance to note that the second term of the above equation becomes zero as $t \longrightarrow \infty$--that is, pollutants' concentration becomes zero as $t \longrightarrow \infty$. This is true from a biochemical point of view in that all the biodegradable pollutants will eventually be degraded, leaving no pollutants in the stream. However, the BOD must approach Lo (ultimate BOD) as $t \longrightarrow \infty$-- that is, the oxygen required to degrade all the pollutants (ultimate BOD) will reach the maximum value, i.e., Lo. This is so since the second term of EQ. (33) becomes zero as $t \longrightarrow \infty$, making $L \longrightarrow Lo$ as $t \longrightarrow \infty$.

SOLUTION FOR IMPERVIOUS CHANNEL
UNDER INSTANTANEOUS POINT SOURCE

If the stream channel is impervious, the solution must be modified accordingly. Increasing numbers of stream channels are made impervious in order to prevent seepage and leaking of pollutants as well as for erosion control. Under this condition the total pollutants must be contained within the bounded geometry. Accordingly, the integral constant B must be changed. Thus we must have,

$$M = B \int_{-\infty}^{\infty} \int_{-w/2}^{w/2} \int_{-H/2}^{H/2} P(x,y,z,t) \Bigg|_{\substack{\lim \\ K_1 \to 0}} , \substack{\lim \\ u,v,w \to 0}} dxdydz$$

Using a previous evaluation on the error function, the second integral of the above equation is written as follows:

$$\int_{-w/2}^{w/2} t^{-1/2} \exp(-y^2/4D_2 t)\, dy = 4\sqrt{D_2} \int_{0}^{w/4\sqrt{D_2 t}} \exp(-\xi^2)\, d\xi \tag{34}$$

There are two cases that must be considered in evaluating the definite integral of EQ. (34). The reason of this will be made clear in the subsequent developments.

(1) CASE 1 - The upper limit of integral $(w/4\sqrt{D_2 t})$ is small $(\leq .48)$. Using EQ. (32) we get,

$$\text{erf } x = \frac{2}{\sqrt{\pi}} \int_{0}^{x} e^{-\xi^2}\, d\xi = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)n!}$$

$$\int_{-w/2}^{w/2} t^{-1/2} \exp(-y^2/4D_2 t)\, dy = 4\sqrt{D_2} \int_{0}^{w/4\sqrt{D_2 t}} e^{-\xi^2}\, d\xi = 4\sqrt{D_2} \sum_{n=0}^{\infty} \frac{(-1)^n (w/4\sqrt{D_2 t})^{2n+1}}{(2n+1)\, n!}$$

(2) CASE 2 - The upper limit of integral is large (>.48).

As seen from the above, if the quantity, $w/4\sqrt{D_2 t}$, is greater than .48, the infinite series does not converge because the numerator of the series increases much faster than the denominator as $n \longrightarrow \infty$.

That is,

$$\left(\frac{w}{4\sqrt{D_2 t}}\right)^{2n+1} \gg (2n+1)n! \ , \ as \ n \longrightarrow \infty$$

Any infinite series used in any engineering or mathematical evaluation must not diverge but converge very rapidly.

For large values of $w/4\sqrt{D_2 t}$, we proceed as follows;

Recalling,

$$erf \ x = \frac{2}{\sqrt{\pi}} \int_0^x e^{-\xi^2} d\xi$$

For large values of x we write,

$$\int_0^x e^{-\xi^2} d\xi = \int_x^\infty e^{-\xi^2} d\xi$$

A single integration by parts of

$$\int_x^\infty e^{-\xi^2} d\xi$$

gives

$$\int_x^\infty e^{-\xi^2} d\xi = \frac{1}{2x} e^{-x^2} - \frac{1}{2} \int_x^\infty \frac{e^{-\xi^2}}{\xi} d\xi$$

and repeating the process n times we find

$$\int_x^\infty e^{-\xi^2} d\xi = \frac{1}{2} e^{-\xi^2} \left(\frac{1}{x} - \frac{1}{2x^3} + \frac{1\cdot3}{2^2 x^5} - \cdots + (-1)^{n-1} \frac{1\cdot3\cdots(2n-3)}{2^{n-1} x^{2n-1}} + (-1)^n \frac{1\cdot3\cdots(2n-1)}{2^n} \right.$$

$$\left. \int_x^\infty \frac{e^{-\xi^2}}{\xi^{2n}} d\xi \right)$$

This series does not converge since the ratio of the nth term to the (n-1)th does not remain less than unity, as n increases.

However, if we take n terms of the series, the remainder--namely,

$$\frac{1 \cdot 3 \cdots (2n-1)}{2^n} \int_x^\infty \frac{e^{-\xi^2}}{\xi^{2n}} \, d\xi$$

is less than the nth term, since

$$\int_x^\infty \frac{e^{-\xi^2}}{\xi^{2n}} \, d\xi < e^{-x^2} \int_x^\infty \frac{d\xi}{\xi^{2n}}$$

We can thus stop at any term and take the sum of the terms up to this term as an approximation for the function, the error being less than in absolute value than the last term we have retained.

Thus it follows that,

$$\text{erf } x = \frac{1}{\sqrt{\pi}} e^{-x^2} \left(\frac{1}{x} - \frac{1}{2x^3} + \cdots + \frac{1 \cdot 3}{2^2 \cdot x^5} - \cdots \right)$$

The above series rapidly converge for large x.

Finally, we write for large x

$$\int_0^x e^{-\xi^2} \, d\xi = \frac{1}{2} e^{-x^2} \left(\frac{1}{x} - \frac{1}{2x^3} + \cdots + \frac{1 \cdot 3}{2^2 x^5} - \cdots \right)$$

For large $W/4\sqrt{D_2 t}$ we then write

$$\int_{-w/2}^{w/2} t^{-1/2} \exp(-y^2/4D_2 t) \, dt = 4\sqrt{D_2} \left[\frac{1}{2} \exp(-w/4\sqrt{D_2 t})\left(\frac{1}{w/4\sqrt{D_2 t}} + \frac{1}{2(w/4\sqrt{D_2 t})^2} + \cdots \right) \right]$$

The third integral of EQ. (26) is evaluated by the same fashion.

A great deal of care must be exercised in using the infinite series discussed so far in evaluating the actual integral constant. Since the infinite series evaluated above become <u>denominators</u>, in the actual determination of integral constant Case 1 and Case 2 must be <u>reversed</u>. The problem is further complicated from the fact that there are double infinite series involved in determining the integral constant. Our objective is that <u>both</u> series used in the evaluation of integral constant must converge. If only one series converge while other series diverge, we cannot guarantee as to the convergence of their product. We will examine this in detail in the following developments.

1. Both $W/4\sqrt{D_2 t}$ and $H/4\sqrt{D_3 t}$ are small;

We write,

$$M = B\left\{ 2\sqrt{D_1 \pi}\left[4\sqrt{D_2}(\frac{1}{2}\ \exp(-w/4\sqrt{D_2 t})^2\ (\frac{1}{w/4\sqrt{D_2 t}} - \frac{1}{2(w/4\sqrt{D_2 t})^3} + \cdots))\right]\left[\ 4\sqrt{D_3}\ (\frac{1}{2}\ \exp(-H/4\sqrt{D_3 t})^2\ (\frac{1}{H/4\sqrt{D_3 t}} - \frac{1}{2(H/4\sqrt{D_3 t})^3} \cdots))\right]\right\}$$

Upon simplifying,

$$M = B\left[8\sqrt{D_1 D_2 D_3 \pi}\ \exp(-(w/4\sqrt{D_2 t})^2 - (H/4\sqrt{D_3 t})^2)(\frac{1}{w/4\sqrt{D_2 t}} - \cdots)(\frac{1}{H/4\sqrt{D_3 t}} - \cdots)\right]$$

It follows that,

$$B = M \left[\frac{1}{8\sqrt{D_1 D_2 D_3 \pi}} \quad \exp\left(\left(w/4\sqrt{D_2 t}\right)^2 + \left(H/4\sqrt{D_3 t}\right)^2\right)^{-1} \left(\frac{1}{w/4\sqrt{D_2 t}} - \cdots\right)^{-1} \left(\frac{1}{H/4\sqrt{D_3 t}} - \cdots\right)^{-1} \right] \tag{34-1}$$

It is now clear that the above double infinite series are uniformly convergent for values of both $(W/4\sqrt{D_2 t})$ and $(H/4\sqrt{D_3 t})$ which are large.

It must be clearly understood that in the above process we used the infinite series of Case 2 ($W/4\sqrt{D_2 t}$ and $H/4\sqrt{D_3 t}$ are both large). Again, the reason is that those double infinite series become denominators in evaluating integral constants; thus we must reverse Case 1 and Case 2.

2. For both $W/4\sqrt{D_2 t}$ and $H/4\sqrt{D_3 t}$ are large;

Now we use Case 1 in this case instead of Case 2 for the reason repeated previously.

We write

$$M = B \left[2\sqrt{D_1 \pi} \cdot 16\sqrt{D_2 D_3} \sum_{n=0}^{\infty} \left(\frac{(-1)^n \left(w/4\sqrt{D_2 t}\right)^{2n+1}}{(2n+1)n!} \right) \sum_{n=0}^{\infty} \left(\frac{(-1)^n \left(H/4\sqrt{D_3 t}\right)^{2n+1}}{(2n+1)n!} \right) \right]$$

Thus,

$$B = \frac{M}{32\sqrt{D_1 D_2 D_3 \pi}} \left[\sum_{n=0}^{\infty} \left(\frac{(-1)^n \left(w/4\sqrt{D_2 t}\right)^{2n+1}}{(2n+1)\, n!} \cdot \frac{(-1)^n \left(H/4\sqrt{D_3 t}\right)^{2n+1}}{(2n+1)\, n!} \right) \right]^{-1} \tag{34-2}$$

Again, it is evident that when $W/4\sqrt{D_2 t}$ and $H/4\sqrt{D_3 t}$ are large, the above series rapidly converges because denominators of the series increase much faster than the numerator.

There are cases where only $W/4\sqrt{D_2 t}$ is small, and $H/4\sqrt{D_3 t}$ is large or vice versa. In this case, we have to select the series appropriately so that the product of the two series converge.

3. $W/4\sqrt{D_2 t}$ is small and $H/4\sqrt{D_3 t}$ is large

We write

$$M = B \left[2\sqrt{D_1 \pi} \cdot 4\sqrt{D_2} \left(\frac{1}{2} \exp(-w/4\sqrt{D_2 t})^2 \left(\frac{1}{w/4\sqrt{D_2 t}} - \cdots\right) \cdot 4\sqrt{D_3} \sum_{n=0}^{\infty} \frac{(-1)^n (H/4\sqrt{D_3 t})^{2n+1}}{(2n+1)n!}\right) \right]$$

It follows that,

$$B = \frac{M}{16\sqrt{D_1 D_2 D_3 \pi}} \left[\exp(w/4\sqrt{D_2 t})^2 \left(\frac{1}{w/4\sqrt{D_2 t}} - \cdots\right) \sum_{n=0}^{\infty} \frac{(-1)^n (H/4\sqrt{D_3 t})^{2n+1}}{(2n+1)n!} \right]^{-1} \quad (34\text{-}3)$$

4. $W/4\sqrt{D_2 t}$ is large and $H/4\sqrt{D_3 t}$ is small

This is the reverse case of the previous one.

$$M = B \left[2\sqrt{D_1 \pi} \cdot 4\sqrt{D_2} \sum_{n=0}^{\infty} \frac{(-1)^n (w/4\sqrt{D t})^{2n+1}}{(2n+1)n!} \cdot 4\sqrt{D_3} \left(\frac{1}{2} \exp(-H/4\sqrt{D_3 t})^2 \cdot \left(\frac{1}{H/4\sqrt{D_3 t}} - \cdots\right)\right) \right]$$

Thus,

$$B = \frac{M}{16\sqrt{D_1 D_2 D_3 \pi}} \left[\exp(H/4\sqrt{D_3 t})^2 \sum_{n=0}^{\infty} \frac{(-1)^n (w/4\sqrt{D_2 t})^{2n+1}}{(2n+1)n!} \left(\frac{1}{H/4\sqrt{D_3 t}} - \cdots\right) \right]^{-1} \quad (34\text{-}4)$$

As it is seen above, in the case of impervious channels the problem is enormously complicat   coupled with the magnitude of traverse and vertical diffusion coefficients   , D_3). Thus, in the case of impervious channel the BOD (and thus DO) must be evaluated by different equations (1 out of 4 in the above) each time t varies. In other words, no one fixed formula can be used even under the same discharge condition and same original pollutant concentration.

The net effect of this is to allow use of greater t (time) in evaluating unsteady-state conditions, not limiting upper limits of the time scale used in the unsteady-state condition only to the time when the pollutant reaches the bank.

All of the above cases of different integral constants will be deliberately incorporated in the subsequent computer program to be used in the current model.

Finally, the solution for the general BOD equation under impervious channel conditions is written as;

$$L(x,y,z,t) = Lo \left[1 - B \cdot t^{-3/2} \cdot e^{-K_1 t} \cdot \exp(-(x-ut)^2/4D_1t - (y-vt)^2/4D_2t - (z-wt)^2/4D_3t) \right] \quad (35)$$

Where B is defined as previously shown (EQ. (34)). Again, B can change at different elapsed times as mentioned previously.

Solutions Under Continuous Sources

For the above case we can generally write the total pollutants contained in the system under input of m(t) per dt as,

$$\int_0^t m(t)dt \qquad (36)$$

Note that m(t) can be any function desired. One example would be a periodic function.

EQ. (36) will be used in place of M(=Lo) in determining the integral constant. Thus the solutions are as follows:

$$L(x,y,z,t) = \int_0^t m(t)dt \left[1 - \frac{t^{-3/2}}{8\sqrt{D_1 D_2 D_3}\,\pi^{3/2}} e^{K_1 t} e^{(-(x-ut)^2/4D_1 t - (y-vt)^2/4D_2 t - (z-wt)^2/4D_3 t)} \right] \qquad (37)$$

for a pervious stream channel and

$$L(x,y,z,t) = \int_0^t m(t)dt \left[1 - Bt^{-3/2} e^{-K_1 t} e^{(-(x-ut)/4D_1 t - (y-vt)/4D_2 t - (z-wt)/4D_3 t)} \right] \qquad (38)$$

for impervious stream channels.

B has been defined previously (EQS. (34-1, 2, 3, 4)). The general BOD solutions obtained so far are then inserted to the DO equation in order to get general solutions of DO.

For solutions of the BOD equation when pollutants discharge at points other than (0,0,0), one merely changes the coordinate system for solution purposes. For instance, if the discharge point is (Xo,Yo,Zo) at

t=to, we simply use (X-Xo), (Y-Yo), (Z-Zo) and (t-to) in place of x, y, z, and t in <u>all</u> of the above solutions.

2. Analytical Solution of the DO Equation

The general 3-D, unsteady-state DO (dissolved oxygen) equation is EQ. (9), as has been previously derived. Rewriting EQ. (9) here,

$$\frac{\partial c}{\partial t} = K_2(Cs-c) + A(t) - K_1[Lo-L(x,y,z,t)] \qquad (9)$$

Where

C = DO concentration

Cs = DO saturation constant

K_2 = reaeration constant

A(t) = DO supplied by algal activity

K_1 = Pollutants' oxidation rate

Lo = M = ultimate BOD or initial (total) pollutants' conc.

L(x,y,z,t) = BOD as a function of time and space

Note that all 3-dimensional space variables are carried by BOD as L(x,y,z,t). <u>This means that the DO concentration at a given location under a given time will be governed by BOD at that location.</u>

Let,

Cs-C = D

so that,

$$\frac{dD}{dt} = - \frac{dc}{dt}$$

Thus EQ. (9) is now rewritten in terms of D, where D denotes DO deficit as a function of time and space.

We write,

$$\frac{\partial D}{\partial t} = - K_2 D - A(t) + K_1[Lo - L(x,y,z,t)] \qquad (39)$$

Initial and boundary conditions are;

$$D \longrightarrow Do \ (i.e., \ C \longrightarrow Co) \ as \ t \longrightarrow o, \quad and$$
$$D \longrightarrow 0 \ (i.e., \ C \longrightarrow Cs) \ as \ t \longrightarrow \infty \qquad (40)$$

Where D and Co are initial DO deficit and initial DO concentration, respectively.

The above boundary condition holds only when $A(t) = 0$. However, when there is algal activity $(A(t) \neq 0)$, D (hence C) will be governed by $A(t)$ as a sole contributor of DO.

When $t \longrightarrow \infty$, there should be no DO deficit and the DO concentration will approach the saturation concentration, i.e., Cs.

This is so because the pollutants' concentration approaches 0 (or BOD approaches maximum value, Lo) when $t \longrightarrow \infty$. When pollutants' concentration reaches 0 due to degradation, DO is no more depleted, allowing it to approach saturation concentration.

SOLUTION STRATEGY

By observing the analytical solutions of BOD equations, it is seen that it is practically impossible to carry these "long terms" associated with BOD solutions into the DO equation in order to obtain an analytical solution of DO. It seems that the analytical solution of DO by directly carrying BOD solutions is either impossible or prohibitively complicated.

In order to alleviate this problem, associated BOD values are obtained first (depending upon channel conditions) from various BOD solutions under a given time, then these val⁻s are entered into the DO equation. Thus the third term in EQ. (39) becomes constant.

The same thing is done for DO supply by algal activity. After these have been accomplished, EQ. (39) becomes a simply ordinary differential equation.

Mathematically,

$$\frac{\partial D}{\partial t} = - K_2 D - A(t) + K_1[Lo - L(x,y,z,t)] \qquad (39)$$

Let $A(t) = Ac$ @ a given location and given time, and

$[Lo-L(x,y,z,t)] = Pc$ @ a given location and given time.

Where Ac and Pc are now constants.

Thus,

$$\frac{\partial D}{\partial t} = -K_2 D - Ac + K_1 Pc$$

or

$$\frac{dD}{dt} = -K_2 D - Ac + K_1 Pc \qquad (41)$$

The treatment of Ac (DO supply by algal activity at a given time and space) and Pc (oxygen-equivalent pollutants' concentration or BOD not exerted at a given time and space) as constants is justified from the mathematical point of view on the basis that DO supply by algal activity and DO depletion by pollutants' decomposition are not a function of the DO equation. That is, the functions describing the temporal and spatial distribution of algal activity and BOD stand by themselves.

STREETER-PHELPS EQUATION

In order to put the 3-D, unsteady-state DO model and its solution into the proper perspective, it is worthwhile to closely examine the classical 1-D, steady-state DO model known as the Streeter-Phelps equation. The following is the original form of the equation;

$$\frac{dD}{dt} = K_1 L - K_2 D \qquad (42)$$

EQ. (42) has been known as a 1-D, steady-state DO model.

STEADY-STATE vs. UNSTEADY-STATE

When it comes to steady-state or unsteady-state, _time_ is the criterion while _space_ is the criterion when it comes to uniform or non-uniform (flow).

Steady-state means that at a given location of a channel the change of any parameters under consideration with respect to time is constant. That is, when "space" is fixed, the variation of any parameters in terms of concentration is constant around the clock. Unsteady-state is exactly the opposite of this. If the given location (space is fixed) can be represented by a single value, whether it is the x, y, or z value in the cartesian coordinates, for instance, it is 1-dimensional. This is then expanded to 2-D and 3-D. By closely examining EQ. (42) it is noted that dD/dt appears in the left-hand side of the equation and it is not zero in this steady-state model. However, in almost all steady-state DO models, dD/dt is set to zero on the basis that the DO variation with respect to time is constant; hence dD/dt = 0. One typical case of this appears in Wastewater Engineering (6). The two entirely

opposite views on dD/dt between the original Streeter-Phelps equation (dD/dt $\neq$ 0) and all other work (dD/dt = 0) stems from gross misunderstandings of the subject in terms of steady-stateness and unsteady-stateness. In the original Streeter-Phelps equation, dD/dt was meant to be used "dD/dt in the direction of flow (usually longitudinal)," but not in the cross-section of the channel. This view originally implied in the Streeter-Phelps equation is correct, thus "dD/dt along the channel under a given pollutant discharge."

The following figure will help to precisely explain the above.
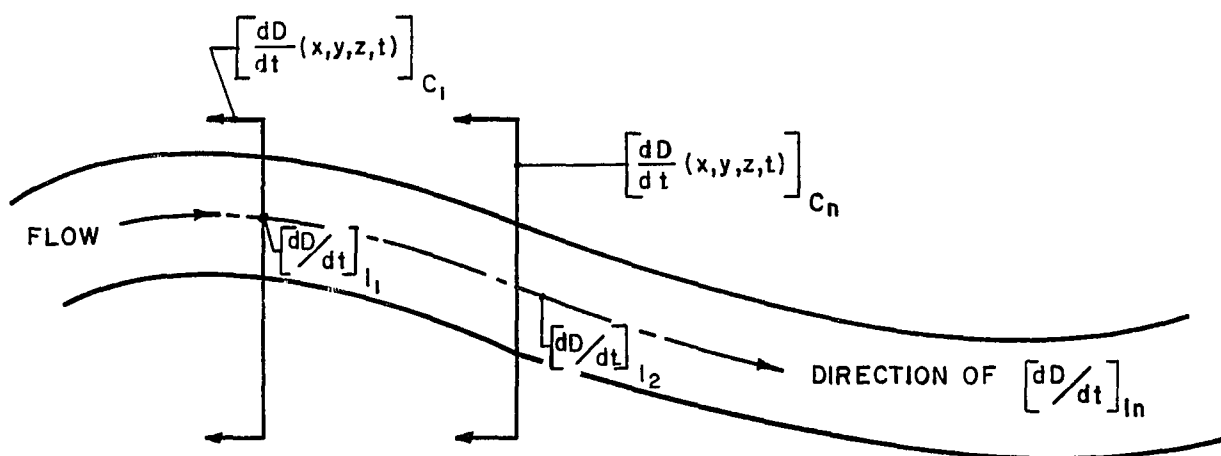
FIGURE 4

GRAPHICAL REPRESENTATION OF TEMPORAL AND SPATIAL
DISTRIBUTION OF DO IN THE RIVER CHANNEL

In the above Figure 4, dD/dt appeared in the Streeter-Phelps equation is $[dD/dt]_{ln}$, where subscript ln denotes "longitudinal" direction. However, $[dD/dt(x,y,z,t)]_{Cn}$, where Cn denotes cross-sectional areas of the channel is entirely different from $[dD/dt]_{ln}$. In effect, there are two different dD/dt, one along the river channel, $[dD/dt]_{ln}$, and the other through the cross-section of channel, $[dD/dt(x,y,z,t)]_{Cn}$.

(1)   Under steady-state conditions $[dD/dt\ (x,y,z,t)]_{Cn} = 0$, not $[dD/dt]_{1n} = 0$. In fact, $[dD/dt]_{1n}$ never becomes zero. If this is set to zero, it means that there is no DO variation along the river channel. That is, regardless of the location of the channel there is no variation of DO or just one value of DO for the entire channel. If it is so, why bother to do a DO analysis in the channel? Under a given pollutant discharge there _is_ DO variation along the channel, thus making $[dD/dt]_{1n}$ (which is commonly referred to as dD/dt) zero would not make any sense at all.

(2)   Under unsteady-state condition both $[dD/dt]_{1n}$ and $[dD/dt\ (x,y,z,t)]_{Cn}$ are not zero. Thus there is "DO variation not only along the channel (a different location of the channel will have different DO value), but also at a given location DO varies with time (unsteady-state) and space (2-D or 3-D)."

Summarizing the above results,

1-D, steady state DO model;

$$dD/dt = K_1L - K_2D \neq 0 \qquad \text{(Streeter-Phelps)} \qquad (42)$$

$$dD/dt = K_1L - K_2D - v\ dD/dx = 0 \qquad \text{(Others)} \qquad (43)$$

Note that in the Streeter-Phelps equation, $dD/dt \neq 0$, which is correct. In other works, $dD/dt = 0$ and further there is an additional term, $-vdD/dx$, where v is the fluid velocity along x axis.

Since $dv = dx/dt$ and $vdD/dx = dx/dt\ .\ dD/dx = dD/dt$, EQ. (43) happens to be identical to EQ. (42), hence is the solution. Convection term (v) must not enter into the DO formulation, as is the case of Streeter-Phelps equation. The fallacy of EQ. (43) is evident from two respects. Firstly, if the convection terms are carried into DO formulation as well, it results in

two convection terms in one DO equation because there are convection terms in the BOD equation already. Secondly, since K_1, K_2, and v are constants, L (residual BOD) and D (D deficit) are only a function of x only, which is not true. As known, $L = f(t)$ in the 1-D, steady-state model, i.e. $L = L_o - L_o e^{-K_1 t}$. Furthermore, since x is the only independent variable in EQ. (47), there is no way to obtain D as well as L as a function of $\underline{t}$ along the channel.

SOLUTION OF 3-D, UNSTEADY-STATE DO MODEL

Recalling EQ. (41),

$$dD/dt = -K_2 D + K_1 Pc - Ac$$

Note that Pc and Ac are constants under a given location and given time, as has previously been explained.

Direct integration between Do (initial deficit, i.e., $D = Do$ at $t = to$) and $D(t)$ yields;

$$\int_{Do}^{D} \frac{dD}{(-K_2 D + K_1 P_c - A_c)} = \int_{t_o}^{t} dt = \frac{1}{K_2} \left(\ln \frac{-K_2 D + K_1 P_c - A_c}{-K_2 D_o + K_1 P_c - A_c} \right) = t - to$$

Let $to = 0$ (initial time), then

$$D = (Do - K_1/K_2 Pc + Ac/K_2)e^{-K_2 t} + K_1/K_2 Pc - Ac/K_2$$

Since

$$D = C_s - C$$

and

$$D_o = C_s - C_o$$

It follows that,

$$C = C_s - [(C_s - C_o - K_1/K_2 P_c + A_c/K_2)e^{-K_2 t} + K_1/K_2 P_c - A_c/K_2]$$

or

$$C(x,y,z,t) = C_s - [(C_s - C_o - K_1/K_2 P_c(x,y,z,t) + A_c(t)/K_2)e^{-K_2 t}$$

$$+ K_1/K_2 P_c(x,y,z,t) - A_c(t)/K_2] \qquad (44)$$

EQ. (44) is a general solution for DO under 3-D, unsteady state conditions.

TREATMENT OF THE PHOTOSYNTHETIC TERM $(A_c(t))$

It is known that DO supply by photosynthetic activity of algae may be one of the major sources of DO in a receiving water. In many cases photosynthetic oxygen supply is the only major source of dissolved oxygen (DO) in the stream. In what is believed to be Camp's last published paper (7) among his many original and creative research papers published through his illustrious career, he concluded that "DO supply by photosynthesis is considerably higher than that by atmospheric reaeration."

This is especially true for a wide and deep stream channel where atmospheric reaeration is not significant in terms of DO supply. For this reason, many wastewater treatment plants (WWTPs) maintain "tertiary ponds" after regular treatment processes. The main function of these ponds is to raise DO levels through algal activity in the ponds.

By doing so, the DO from the pond effluent significantly increases, thus providing a better opportunity for the WWTPs to meet the DO criteria imposed by regulatory agencies.

Without photosynthesis the DO concentration in the water would never exceed the saturation level which is a constant value at a given temperature and pressure, obeying Dalton's and Henry's law on partial pressure. The author observed as high as 35 mg/l DO at 20°C and 1 atmosphere from the Chatham Township tertiary pond in New Jersey. For an average stream the DO level can rise to 15 mg/l during the maximum period of sunlight. Thus it is of great importance to closely examine the DO supply through photosynthetic activity of algae.

The findings of a study conducted by O'Connor and DiToro (8) on this subject is generally used for the calculation of DO supply by algal activity. Generally, the photosynthetic source of DO depends upon many factors such as sunlight, temperature, mass of algae and nutrients (especially phosphates and nitrogen in the form of ammonia and nitrate). If the photosynthetic rate is assumed to vary as the sunlight intensity during the day, and is further assumed to be zero at night, then this source according to O'Connor (8) may be defined by the following periodic function.

$$Ac(t) = Ac(t_s) = Pm[2P/\pi + 2\sum_{n=1}^{\infty} An \quad COS((t_s - P/2) \cdot 2\pi n)] \qquad (45)$$

where

p = period of sunlight

Pm = the maximum rate of $Ac(t)(=Ac(tm))$

ts = time from beginning of sunlight

Mathematically, EQ. (45) is a Fourier series used for Fourier analysis.

Usually, only the first few terms are enough to approximate EQ. (45). The speed of convergence is found by calculating explained variance

from the series obtained and comparing that with that found from observed data sets. It is of importance to note that the independent variable t(ts) in EQ. (45) is not the same as t in EQ. (44). The former is measured from the beginning of sunlight and the latter is just clock time measured from the beginning of pollutant discharge. Furthermore, ts becomes zero as soon as there is no sunlight radiation. If there is no algal activity in certain reaches of the stream, one merely sets this term to zero for that specific segment and so forth. In terms of calculational processes, Ac(t) is calculated separately with due regard to t, i.e. starting from the beginning of sunlight independent from pollutant discharge.

VI. DIGITAL COMPUTER IMPLEMENTATION

A digital computer program was written in order to implement the analytical solutions of the general BOD and DO equations.

1. PREPARATION OF COMPUTER IMPLEMENTATION

(1) Stream Geometry

According to the classical study of Leopold and Maddock, Jr. (9), there are certain relationships which correlate well between stream flow (Q) and various channel parameters related to geometry (such as width and depth) as well as velocity of flow. The following relationships were obtained by the authors after observing numerous collected data for many streams in the United States.

$$W = m_1 Q^a$$

$$H = m_2 Q^b$$

$$V = m_3 Q^c$$

where m_1, m_2, m_3, a, b, and c are arbitrary constants. W, H, and V are the width, depth and mean velocity of the channel.

Since, $W \cdot H \cdot V = Q = m_1 \cdot m_2 \cdot m_3 Q^{a+b+c}$

it follows that,

$$m_1 \cdot m_2 \cdot m_3 = 1, \quad \text{and}$$

$$a+b+c = 1$$

Thus if m_1 and m_2 are specified so is m_3. Also, if a and b are specified, so is c. Generally, if one knows any two constants out of three, the remaining one can be determined by the above relationships.

In the above cited reference the authors published data on channel geometry and velocity along with the flow rate on major rivers in the U.S. For the current computer implementation the author used data directly from this reference instead of creating hypothetical river geometry for meaningful results. Using the above mentioned relationships, one can apply any observed river data for computer simulation purpose.

(2) Other Input Data

All other input data was prepared in such a way that they are typical values reported in the literature.

In calculating dispersion coefficients the author consulted those references considered excellent ones in this field (10, 11, 12, 13).

In calculating D_1, D_2, D_3, hydraulic radius (R) and "C" (Chezy C), values were calculated from the given stream geometry with varying "n" (Manning's). Then shear velocity is obtained from this. After these have been done the following ratios were then obtained from appropriate references (10-13).

$$D_1/R_{u*} \quad \text{and} \quad D_2/R_{u*}$$

where

u* is the shear velocity

From this D_1 and D_2 are computed from the computer program provided herein. Assuming vertical dispersion is a function of the following (12), i.e.

$$D_3 = f(u_s/u, W/H, D_2)$$

where u_s is the horizontal velocity of the channel at the <u>location</u> of the sampling site.

The following relationship suggested by Beltaos (11) has been used to calculate D_3;

$$D_3 = D_2/(us/u)^2 (W/H)^2$$

(3) Planning for Computer Output

The author wrote the computer programs in such a way that it can be used and/or adopted easily for general purposes.

The input data consists of the following:

1. Mode of Discharge (instantaneous or continuous)

2. River geometry (width, depth, Q, cross-sectional area, u,v,w) and Manning's "n"

3. Ultimate BOD and K_1

4. Cs and Co

5. Data for photosynthesis evaluation (P, Pm, ts)

6. Time interval

7. X interval

8. Y interval

9. Z interval

The following information is provided to aid the users regarding the preparation of the input data to be used in the current computer program. The user information regarding the computer program and input data requirements is also fully explained in the program.

The unique feature of the program is that it exclusively uses "FUNCTION SUBPROGRAMS", and each term in the general BOD and DO solutions is

treated as a separate function subprogram. This substantially simplifies the program, reduces the computation time, and makes the expansion and modification of the program easy.

The program also uses a minimum size of arrays to save the memory requirement of the computer.

The input data is prepared in FREE FORMAT.

ORDER OF READ STATEMENTS	DESCRIPTION OF READ STATEMENTS
1	TITLE OF RIVER (up to 40 alphanumeric characters)
2	MODE OF DISCHARGE (INST for instantaneous discharge; CONT for continuous discharge)
3	WIDTH, DEPTH, CROSS-SECTIONAL AREA, FLOW in CFS and DURATION OF THE RECORD in four digits
4	Manning's "n"
5	U, Us, V, and W in ft/sec
6	RATIO for D_1/R_u* and D_2/R_u*
7	ULTIMATE BOD in lbs. and K_1 in DAY^{-1}
8	m(t) in lbs/sec, and amplitude of m(t)
9	Cs, Co, in mg/l and K_2 in DAY^{-1}
10	P, Pm in mg/l/day
11	NUMBER OF TIME INTERVAL
12	TIME INTERVAL in seconds
13	NUMBER OF X INTERVAL
14	X INTERVAL IN ft
15	NUMBER OF Y INTERVAL
16	Y INTERVAL IN ft
17	NUMBER OF Z INTERVAL
18	Z INTERVAL IN ft

In the current computer simulation, velocity of the flow in the Y and Z directions is assumed zero.

The output is designed in the following way to fully realize the 3-D, unsteady-state condition;

1. First the mode of discharge is selected (instantaneous or continuous).

2. At a fixed time,

   x (downstream distance) is first selected
   y (lateral location of channel) is selected next
   finally z (depth of channel) is selected

Thus at a given time and x location there can be several values of BOD and DO with varying width and depth.

Then x is changed followed by y and z values.

After the above cycle, time is then changed and the calculations repeat again.

Note that the above means of calculation is necessary in order to fully appreciate the meaning of 3-D, unsteady-state conditions.

2. INTERPRETATION OF THE COMPUTER OUTPUT

    (1) General Explanation of the Computer Output

As one can imagine, there exists a "deluge" of data from the 3-D, unsteady-state computer simulation. There are four independent variables (x, y,z,t) and two dependent variables (DO & BOD). The number of output data for a simple run is as follows:

   z - mid-depth, 1/4 of depth from surface or bottom (z=2)

   y - mid-width, 1/4 of width from left or right bank (y=2)

   x - say 10 locations (x=10)

   t - say 10 different elapsed times (t=10)
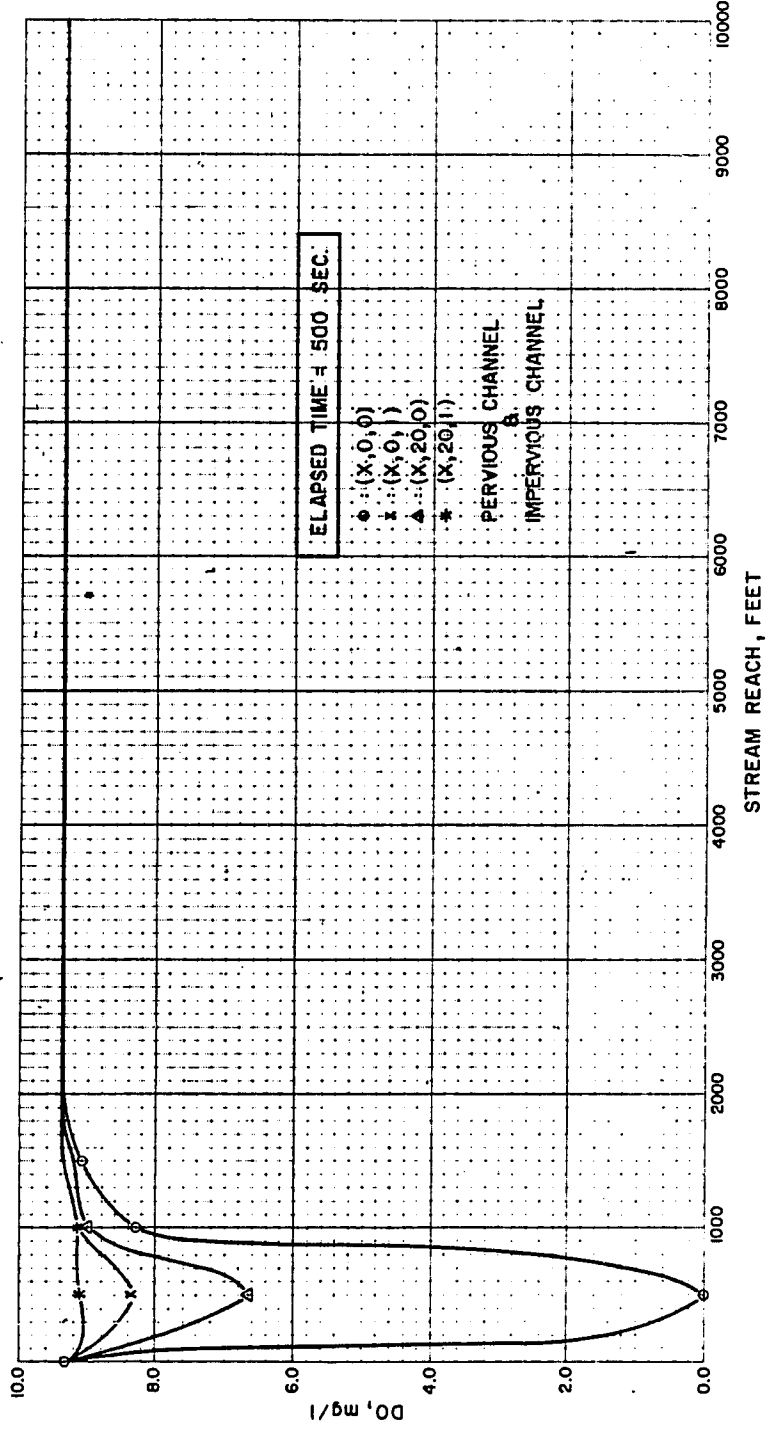
FIGURE 5 - DO @ ELAPSED TIME OF 500 SECONDS

55

Thus the total number of output will be 2x2x10x10 = 400 for one parameter (DO or BOD).

For the current computer simulation the author used 50 different x's and 50 different t's, resulting in a total of 10,000 different output.

TABLE 1 contains input and base-data for the current simulation. A total of eight graphs were constructed using only a part of the output data.

FIGURES 5 through 8 represent the relationship between elapsed time (t) and resultant DO under different channel conditions.

FIGURES 9 through 12 describe the level of DO under different channel conditions (pervious and impervious channel) for the _same_ discharge conditions.

Individual interpretation of each graph follows.

(2) DO vs. Time

FIGURE 5 - DO under elapsed time of 500 seconds is shown for both pervious and impervious channels. At a given elapsed time (500 seconds in this case) and x location, four different DO's were calculated. The coordinates of these four locations are as follows (these four coordinates have been used throughout the current simulation);

1.   (x,0,0)

2.   (x,0,1)

3.   (x,20,0)

4.   (x,20,1)

Coordinate No. 1 indicates the center of channel; No. 2, 1/4 depth from surface (or bottom) in the center of the channel; and so on.
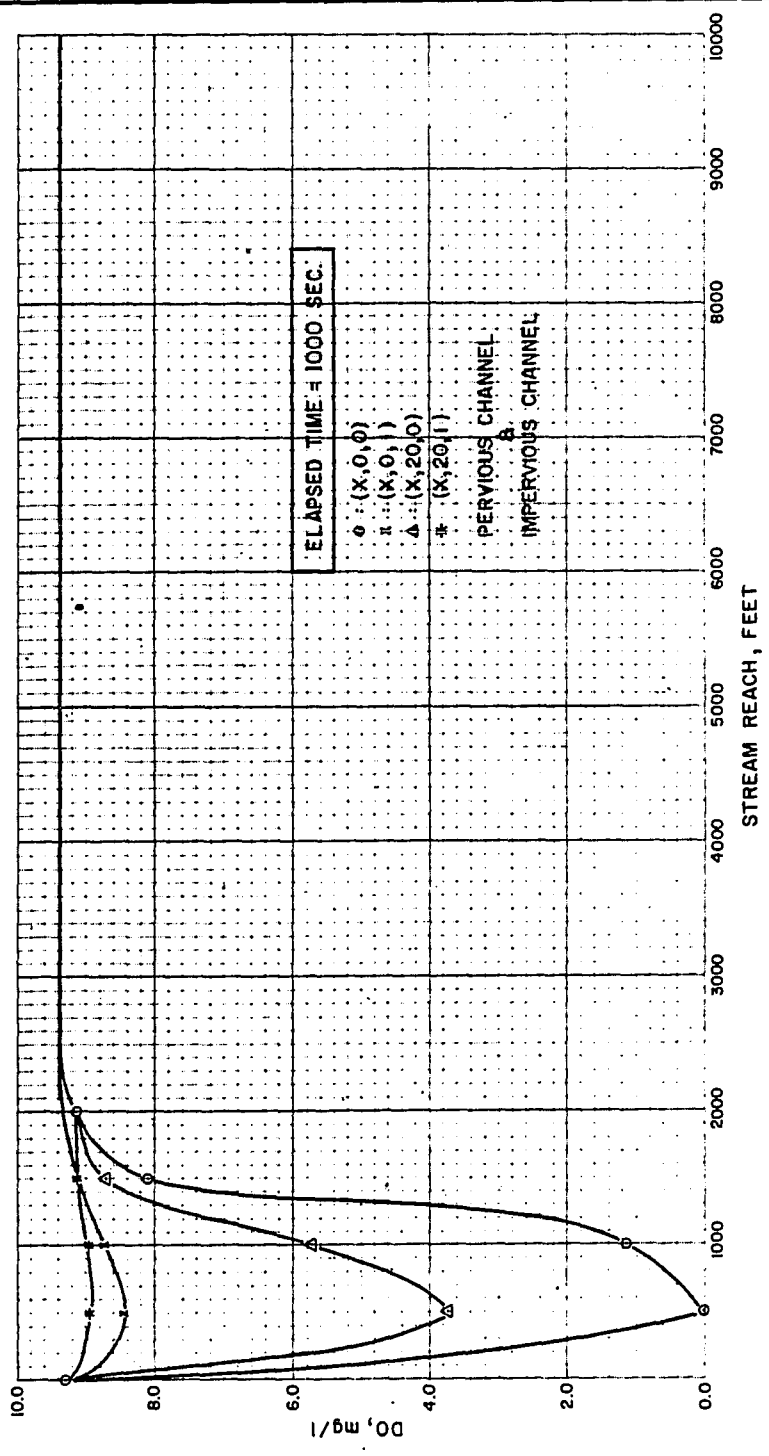
FIGURE 6-DO @ ELAPSED TIME OF 1000 SECONDS

Since the slug of pollutant moves fastest through the center of the channel, DO in the center of the channel will be the lowest. By similar reasoning, the pollutant will move, or reach the slowest through coordinate No. 4, i.e. $(x,20,1)$; hence resulting in the highest DO. This is shown in FIGURE 5.

At $(x,0,0)$ DO has been depleted immediately after the introduction of the pollutant, i.e, $x = 500$ feet. However, at $(x,20,1)$ there is almost no DO depletion at all, while $(x,20,0)$ shows the second lowest DO and $(x,0,1)$ exhibits the second highest DO.

This is expected due to the fact that traverse diffusion is greater in magnitude than that of vertical diffusion; hence DO at $(x,20,0)$ is less than DO @ $(x,0,1)$.

As is seen from the figure, there is more than 10 folds difference in DO under the same x location, a clear benefit of 3-D, unsteady-state model.

It is interesting to note that DO at an elapsed time of 500 seconds exhibits no difference between a pervious and impervious channel. This is mainly due to the short elapsed time. Physically this indicates that the loss of pollutants through the pervious layers of the channel is minimal because of the short elapsed time.

With longer elapsed time, the loss becomes significant enough to cause a difference in DO between pervious and impervious channels. In the current simulation the elapsed time which gives the difference in the DO level has been about 1000 seconds.

The mathematical treatment regarding this matter will follow later.
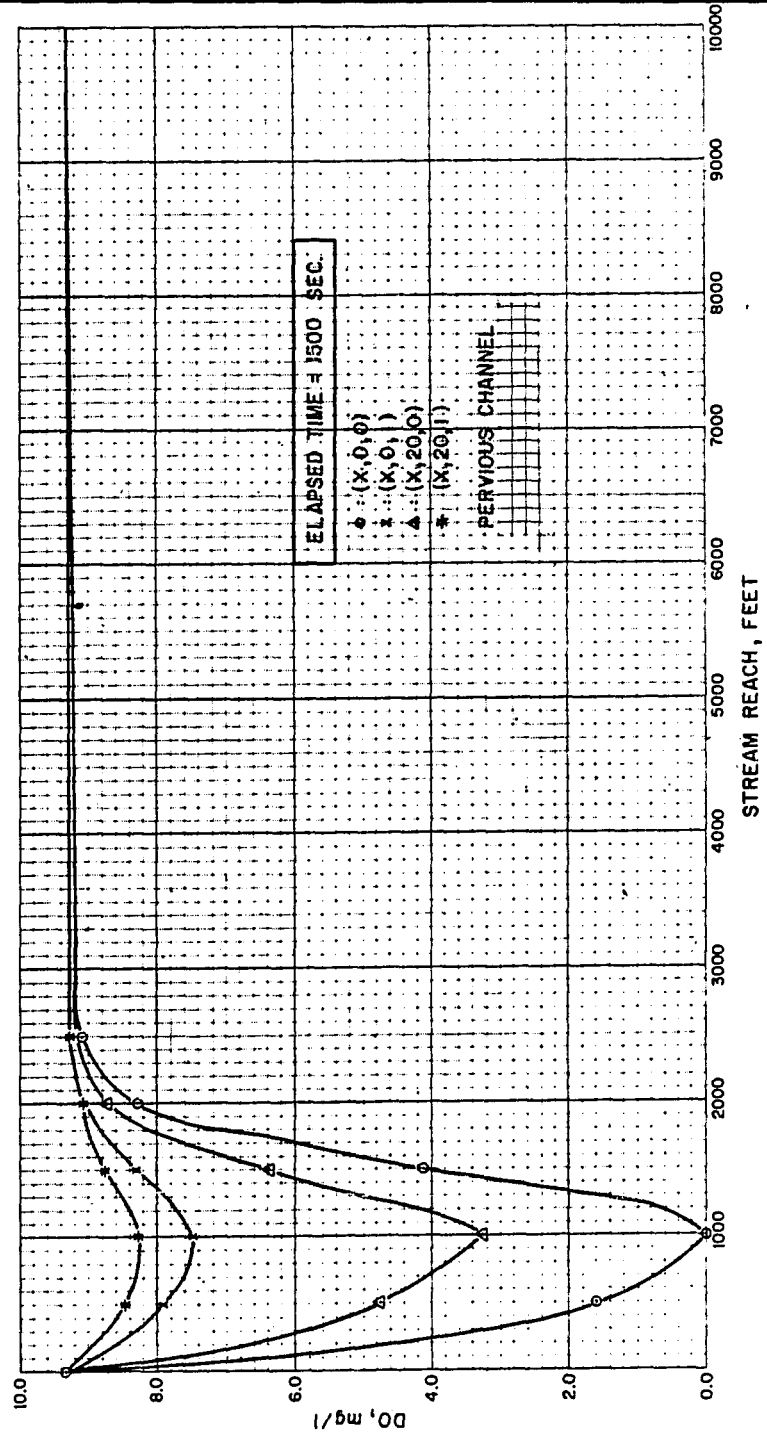
FIGURE 7 - DO. @ ELAPSED TIME OF 1500 SECONDS

59

ELAPSED TIME = 3500 SEC.

o : (X,0,0)
x : (X,0,)
▲ : (X,20,0)
+ : (X,20,1)

PERVIOUS CHANNEL

DO, mg/l

STREAM REACH, FEET
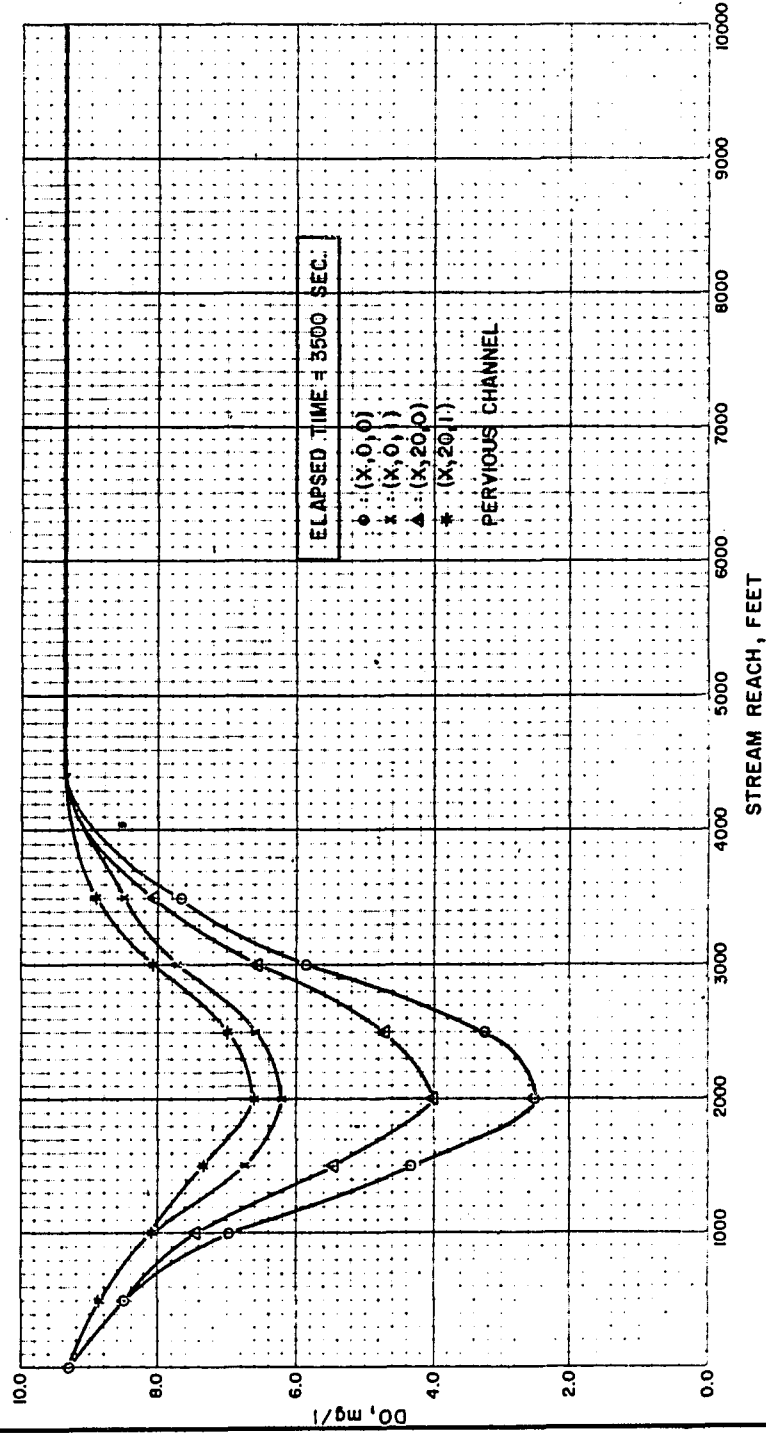
FIGURE 8-DO @ ELAPSED TIME OF 3500 SECONDS

FIGURE 6 through FIGURE 8 - the general configuration of the DO levels are similar to those of FIGURE 5. However, at location (x,0,0) there is a sign of DO recovery starting from FIGURE 6. This is evidenced by two facts. First, the <u>slope</u> of DO lines become less steep and, secondly the actual DO values become greater.

This is primarily due to the fact that the slug of the pollutant has just passed the specific location under our consideration. At location (x,20,0) DO levels do not recover until after 1500 seconds of elapsed time, indicating that the slug of pollutant still did not pass under that indicated period of time. This is attributable to the time required to carry the pollutants through diffusion.

By the same reasoning, DO levels at locations of (x,0,1) and (x,20,1) go down slowly, starting from FIGURE 6

Note that as time progresses DO levels at the above two locations decrease, intensifying DO sags. As seen from FIGURE 8 (at an elapsed time of 3500 seconds) the DO sag at (x,0,1) and (x,20,1) are quite noticeable. Again, this indicates the "lag" time required to disperse the pollutant to the adjacent layers. Finally, it is worthwhile to observe that DO levels in the channel are of the following order;

DO @ (x,0,0)<(x,20,0)<(x,0,1)<(x,20,1)

The above order of DO levels were expected and reflects the greater magnitude of traverse diffusion over that of vertical diffusion.

(3) DO Under Different Channel Conditions

FIGURE 9 through FIGURE 12 indicate <u>the effect of channel condition</u> on the DO levels.

As mentioned previously, DO level differences in pervious and impervious channels become significant approximately after 1000 seconds, indicating that the loss of pollutant through pervious layers of channel becomes noticeable. Generally, the loss is mainly affected by the elapsed time. However, the change of DO levels will not linearly increase/decrease with that of the loss. This is so because the DO levels are not only affected by the loss but also affected by the "redistribution" of pollutant from the adjacent layers which is a function of both time and the magnitude of diffusion coefficients.

This phenomena are indicated in FIGURES 9 through 12, where the difference in DO levels between pervious and impervious channels increases with increasing elapsed time to a certain extent. This can be examined from the mathematical point of view.

It is worthwhile to note that the DO sags under the 3-D, unsteady-state condition consists of multiple sags, (FIGURES 9-12). This is a drastic difference from those of 1-D, steady-state conditions under which there is only one sag.

The reason for this will be pursued from a mathematical point of view in the subsequent section.

      (4) Difference between Pervious and Impervious Channel

1. Pervious Channel

    From EQ. (25)

$$M = B \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} t^{-1/2} e^{-x^2/4D_1 t} \cdot t^{-1/2} e^{-y^2/4D_2 t} \cdot t^{-1/2} e^{-z^2/4D_3 t} \, dx \, dy \, dz \qquad (25)$$

Where,

$$B = \frac{M}{8 \sqrt{D_1 D_2 D_3} \, \pi^{3/2}}$$
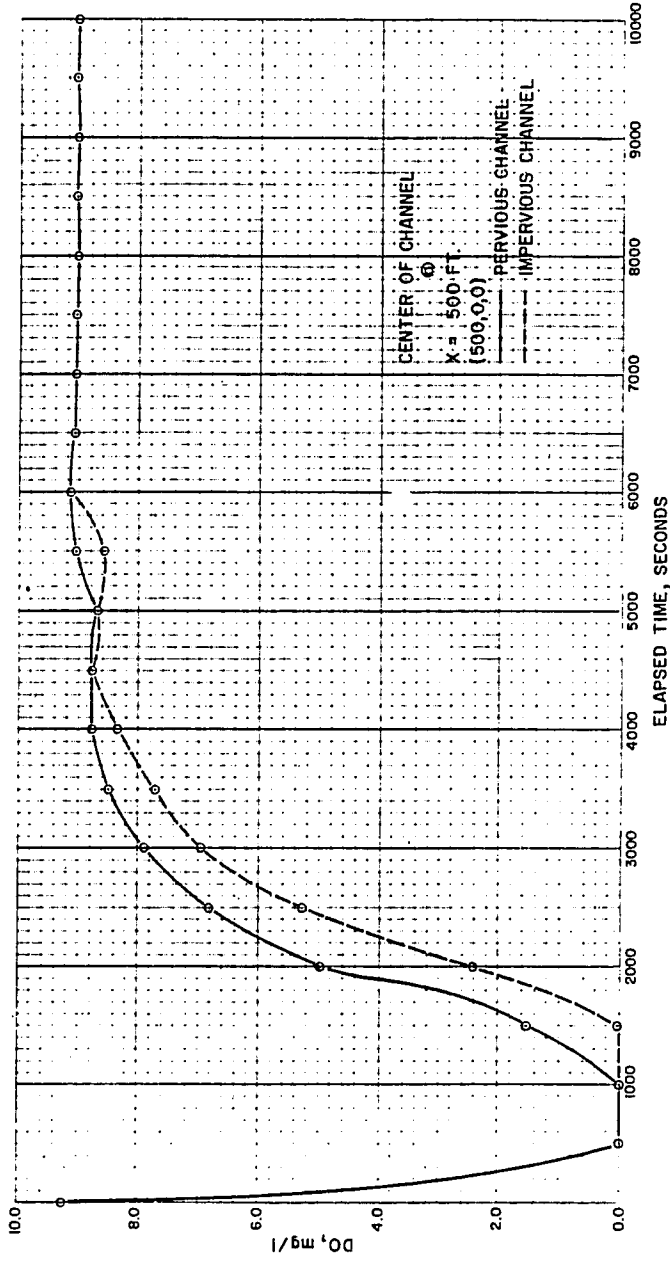
FIGURE 9 - DO UNDER DIFFERENT CHANNEL CONDITIONS @ LOCATION OF (500,0,0)

63

64

FIGURE IO - DO UNDER DIFFERENT CHANNEL CONDITIONS
@ LOCATION OF (500,20,1)

ELAPSED TIME, SECONDS

DO, mg/l

10.0   8.0   6.0   4.0   2.0   0.0

1000   2000   3000   4000   5000   6000   7000   8000   9000   10000

X = 500 FT
Y = 20 FT
Z = 1 FT
(500,20,1)
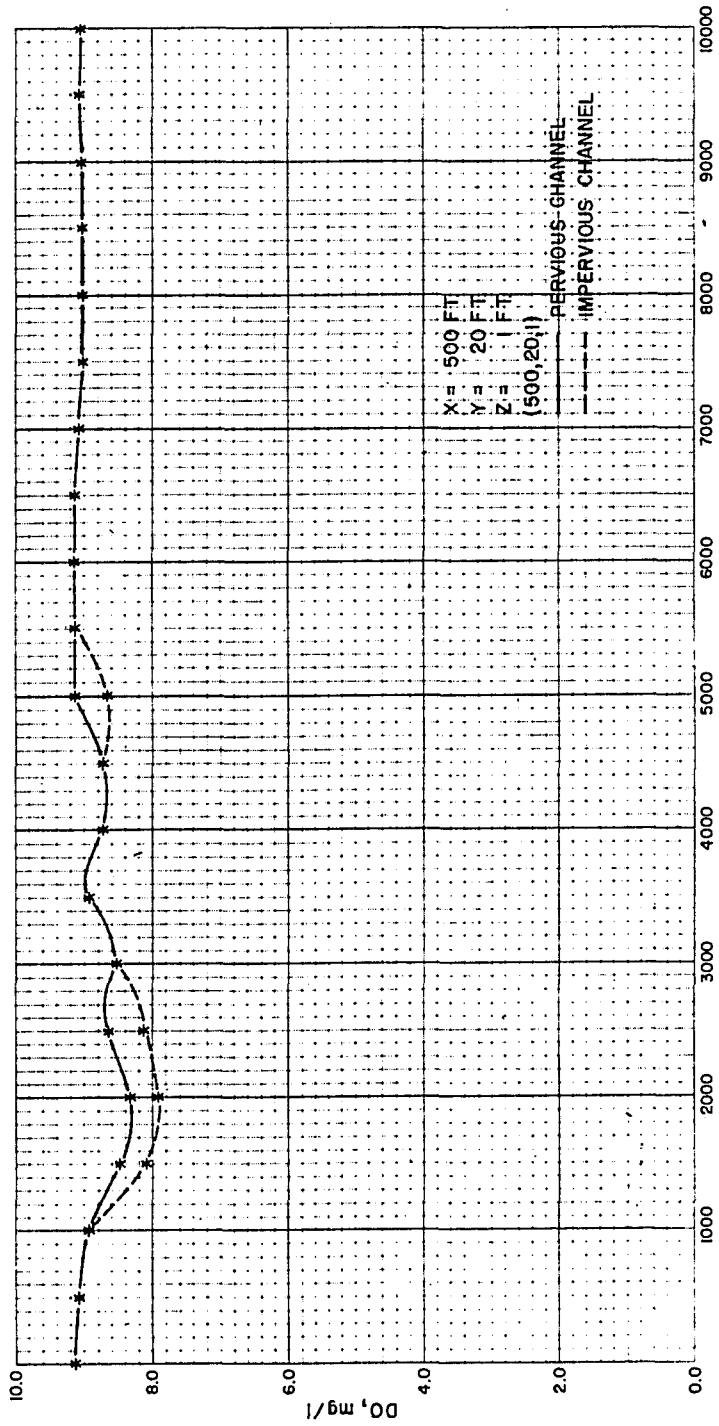
————— PERVIOUS CHANNEL
- - - - - IMPERVIOUS CHANNEL
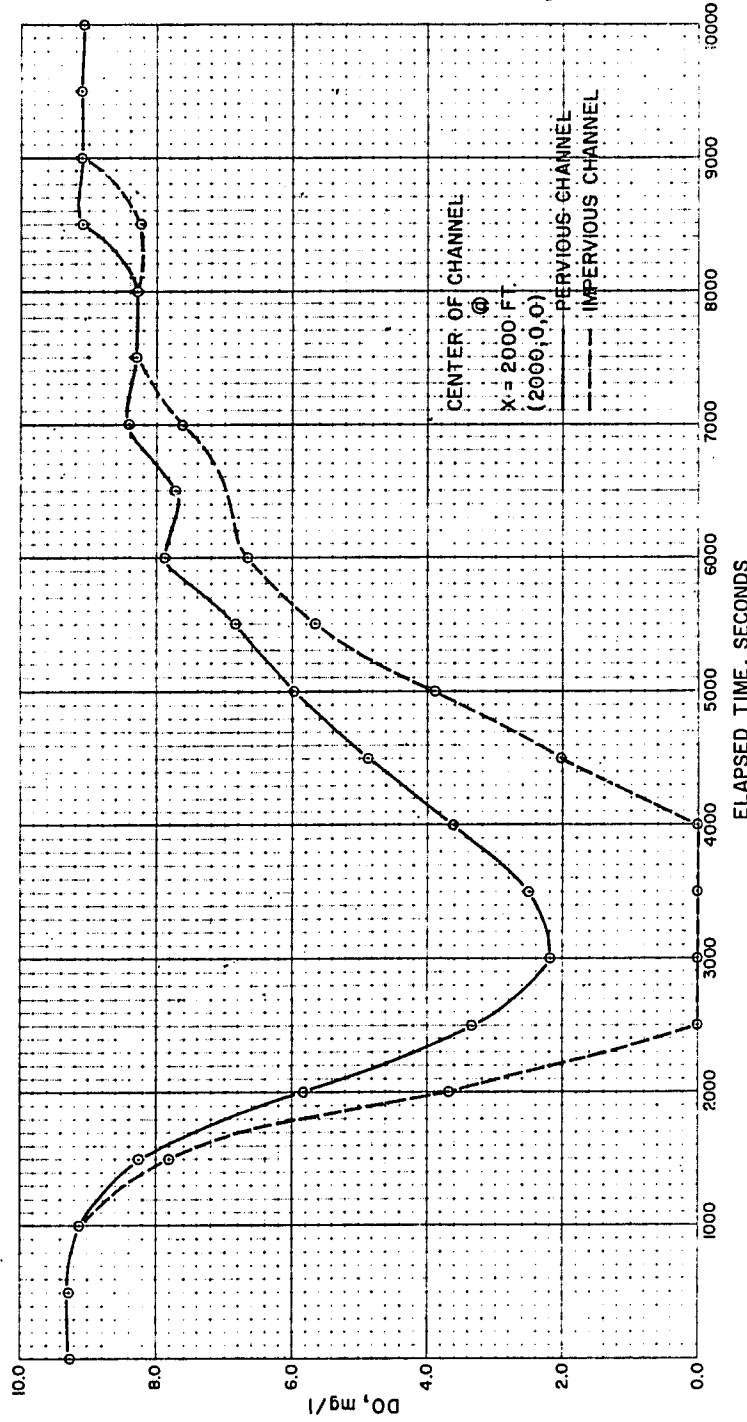
FIGURE 11 - DO UNDER DIFFERENT CHANNEL CONDITIONS
@ LOCATION OF (2000,0,0)

65
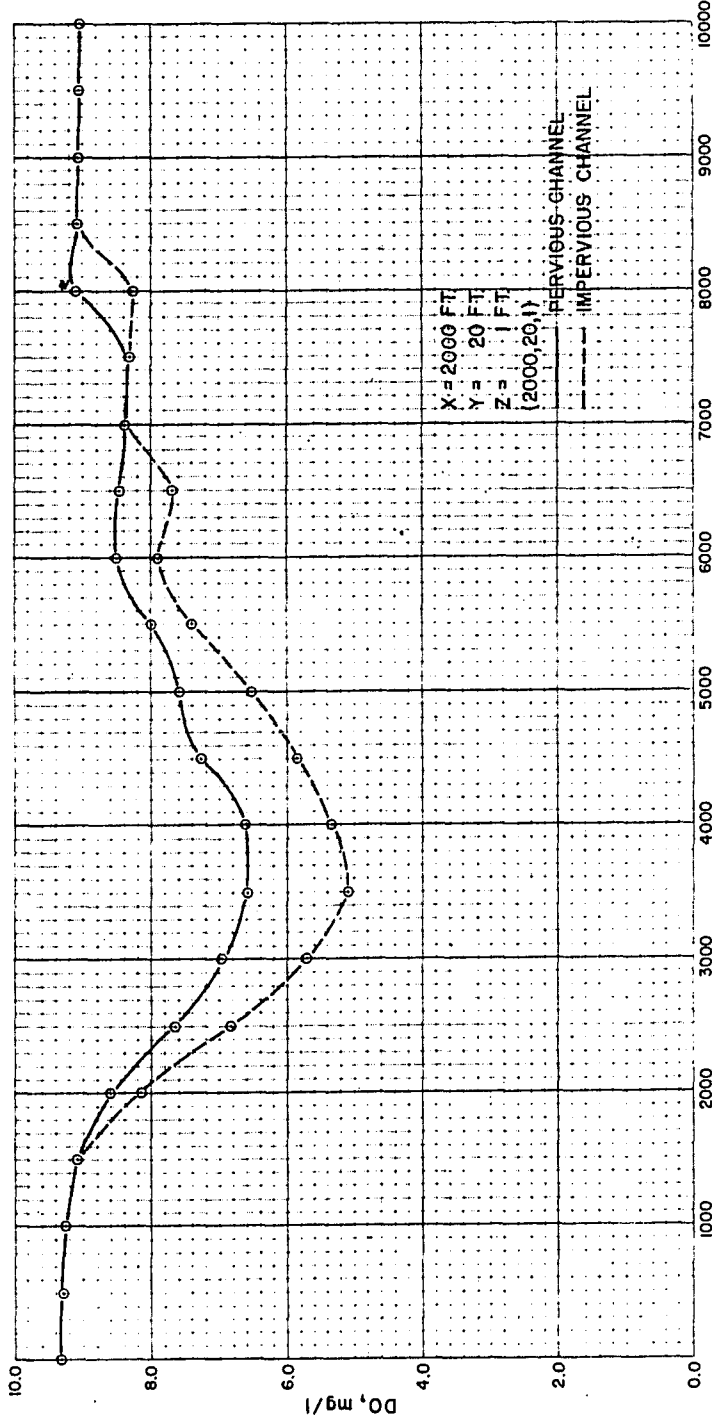
FIGURE 12 – DO UNDER DIFFERENT CHANNEL CONDITIONS
@ LOCATION OF (2000,20,1)

66

2. Impervious Channel

For impervious channel we must have,

$$M = B \int_{-\infty}^{\infty} \int_{-W/2}^{W/2} \int_{-H/2}^{H/2} t^{-1/2} e^{-x^2/4D_1 t} \cdot t^{-1/2} e^{-y^2/4D_2 t} \cdot t^{-1/2} e^{-z^2/4D_3 t} \ dx \ dy \ dz$$

$$M = B \left(2\sqrt{D_1 \pi}\right) \left[\frac{\sqrt{\pi}}{2} \cdot \frac{2}{\sqrt{\pi}} \int_{-W/2}^{W/2} t^{-1/2} e^{-y^2/4D_2 t} dy \right]\left[\frac{\sqrt{\pi}}{2} \cdot \frac{2}{\sqrt{\pi}} \int_{-H/2}^{H/2} t^{-1/2} e^{-z^2/4D_3 t} \ dz \right]$$

Since,

$$\int_{-W/2}^{W/2} t^{-1/2} e^{-y^2/4D_2 t} \, dy = 2 \int_{0}^{W/2} t^{-1/2} e^{-y^2/4D_2 t} \ dy \ , \quad \text{etc.}$$

It follows that,

$$M = B \left(2\sqrt{D_1 \pi}\right)\left(\sqrt{\pi}/2\right)\left(\sqrt{\pi}/2\right) \left[4/\sqrt{\pi} \int_{0}^{W/2} t^{-1/2} e^{-y^2/4D_2 t} \ dy \right]\left[4/\sqrt{\pi} \int_{0}^{H/2} t^{-1/2} e^{-z^2/4D_3 t} \ dz \right]$$

By letting,

$$\xi = y/\sqrt{4D_2 t}$$

We get,

$$dy = 2\sqrt{D_2 t} \ d\xi \quad , \quad \text{etc.}$$

It follows that,

$$M = B(2\sqrt{D_1\pi})(\pi/4) \cdot 4\sqrt{D_2} \cdot 4\sqrt{D_3}\left[2/\sqrt{\pi}\int_0^{w/4\sqrt{D_2 t}} e^{\xi^2} d\xi\right]\left[2/\sqrt{\pi}\int_0^{H/4\sqrt{D_3 t}} e^{\xi^2} d\xi\right]$$

$$M = B \cdot 8\sqrt{D_1 D_2 D_3 \pi}^{3/2}(erf(w/4\sqrt{D_2 t}))(erf(H/4\sqrt{D_3 t}))$$

Finally,

$$B = \frac{M}{8\sqrt{D_1 D_2 D_3}\,\pi^{3/2}} \qquad \frac{1}{erf(w/4\sqrt{D_2 t})\,erf(H/4\sqrt{D_3 t})}$$

Note that the first term at the right hand side of the above equation is the same as the integral constant of the pervious channel.

Since,

$$erf\ x \leq 1$$

B(in impervious channel) = B(in pervious channel) x greater than 1

The net effect of this is that remaining pollutant concentration in the impervious channel is greater than that of the pervious channel under the same conditions, or, BOD becomes smaller, the difference of which is,

$$\frac{1}{erf(w/4\sqrt{D_2 t})\,erf(H/4\sqrt{D_3 t})}\quad (\geq 1)\quad \text{times that of pervious channel}$$

This factor is a function of W, H, D_2, D_3, and t.

Now, if

$$W/4\sqrt{D_2 t}\quad or\quad H/4\sqrt{D_3 t}\ \geq 2$$

then the factor becomes unity, hence no difference between channel conditions.

Since W, H, D_2, and D_3 are constants, the deciding factor is time.

Now, if t increases the quantity $W/4\sqrt{D_2 t}$ and $H/4\sqrt{D_3 t}$ decreases; hence erf $(W/4\sqrt{D_2 t})$ and erf $(H/4\sqrt{D_3 t})$ decrease.

This will in turn increase the factor, i.e., the difference between the pervious and the impervious channel, or vice versa.

It is interesting to note that the difference between the pervious and impervious channel in terms of DO and BOD is related to the error function.

In the following section the DO sags under 3-D, unsteady state conditions will be examined and will see how they are different from those of 1-D, steady-state.

(5) DO Sag in 3-D, Unsteady-State Conditions

In the 1-D, steady-state model the DO sag curve is a smiple form, i.e., DO level goes down right after the discharge of pollutant. Then after reaching the lowest level of DO (critical deficit) the DO level goes up, finally reaching the highest point (saturation level). Functionally, the DO sag curve resembles a simple quadratic form.

There is only one sag in this case.

However, in the 3-D, unsteady-state model the DO sag curve consists of multiple sags, and it no longer resembles a quadratic form from a functional standpoint. The DO sag curve is closer to a multi-degree polynomial form. This is so since DO is a function of many independent variables, i.e.,

$$DO(x,y,z,t) = f(t^{-3/2}, e^{-x^2/4\sqrt{D_1 t}}, \dots, erf(w/4\sqrt{D_2 t}), erf(H/4\sqrt{D_3 t}))$$

Physically, this is due to the pollutant's redistribution or secondary migration from adjacent depths and widths. In other words, the distinct difference in pollutants' distribution at a given cross-section later smoothes

<u>out</u> due to the dispersion in 3-D (caused mainly by velocity difference at a given cross-section due to the shear layer). Due to this "smoothing" action, the DO sag after the first "major" sag is not distinct in the second and subsequent sags. As the time goes by, the sags become less and less. In a physical sense, this phenomena can be described as a <u>transient disturbance</u> caused by starting the oscillations by introducing the pollutants.

VII. CONCLUSIONS

The following conclusions were derived during the current research:

1. Generally, there is a distinct difference in the DO levels with different locations of the channel (both longitudinally and cross-sectionally) under a given elapsed time. The magnitude of the difference can be significant, and the magnitude depends upon the location, elapsed time and initial pollutant concentration.

   Accordingly the representative DO value at a given cross-seciton of the channel must be width, as well as depth, averaged instead of taking the DO value calculated only at a single point of the channel.

2. Generally, DO levels can markedly differ with different channel conditions i.e., pervious and impervious. DO levels under the pervious channel conditions can be significantly higher than those of impervious channels under the same conditions, the magnitude of which is primarily a function of time.

3. There are multiple DO sags in the 3-D, unsteady-state conditions.

VIII RECOMMENDATIONS FOR FUTURE RESEARCH

The current mathematical models on DO and BOD under 3-D, unsteady state
conditions can be applied to other areas, such as predicting the concentra-
tions of radioactive material in the receiving waters.
Future applications in these areas are highly recommended.

The current models can also be applied as to the determination of the
validity of stream improvements, i.e., whether or not the stream improvements
through the channel reconstruction (impervious) can be justified from the
standpoint of DO levels.

```
              ANALYSIS OF DO & BOD
              ••••••••••••••••••••
                       IN
            3-D,UNSTEADY-STATE CONDITIONS
            ••••••••••••••••••••••••••••••


                 SYSTEM INPUT DATA
                 -----------------


DISPERSION COEFFICIENTS(SQ. FT/DAY) :

        LONGITUDINAL = 7035517.00
          TRAVERSL   =   10131.14
          VERTICAL   =      18.06

MEAN VELOCITY(FT/DAY)            :

        HORIZONTAL = 51840.0
          LATERAL  =     0.0
         VERTICAL  =     0.0


BOD VELOCITY CONSTANT(PER DAY,BASE E) =2.30
REAERATION CONSTANT(PER DAY,BASE E) =0.05


POLLUTANT LOADING(LBS./CU. FT) :

     INSTANTANEOUS = 2000.00
       CONTINUOUS  = 2000.00

DO SATURATION VALUE(MG/L) =9.27
     INITIAL DO(MG/L) =9.27


RIVER GEOMETRY - SCIOTO RIVER AT LARUE,OHIO    :

             WIDTH(FT) =    90.0
         MEAN DEPTH(FT) =    3.8
  CROSS-SECTIONAL AREA(SQ. FT) =  342.0
   MEAN ANNUAL DISCHARGE(CFS) =  198.0
         YEARS OF RECORD =    16
```

TABLE 1-INPUT & DATA FOR COMPUTER SIMULATION

IX. REFERENCES

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SCIOTO RIVER AT LARUE,OHIO

```
SCIOTO RIVER AT LARUE,OHIO
*INST*
90.0,3.0,342.0,198.0,0.18
0.025
51840.0,51840.0,0.0,0.0,2.0
500.0,0.72
2000.0,2.3
2000.0,5.0
9.27,9.27,0.05
0.5,0.0
50
500.,1000.,1500.,2000.,2500.,3000.,3500.,4000.,4500.,5000.,5500.,6000.,6500.,7000.,7500.,8000.,8500.,9000.,9500
10000.,10500.,11000.,11500.,12000.,12500.,13000.,13500.,14000.,14500.,15000.,15500.,16000.,16500.,17000.,17500.,18000.,18500.,19000.,19500
20000.,20500.,21000.,21500.,22000.,22500.,23000.,23500.,24000.,24500
50
500.,1000.,1500.,2000.,2500.,3000.,3500.,4000.,4500.,5000.,5500.,6000.,6500.,7000.,7500.,8000.,8500.,9000.,9500
12000.,10500.,11000.,11500.,12000.,12500.,13000.,13500.,14000.,14500.,15000.,15500.,16000.,16500.,17000.,17500.,18000.,18500.,19000.,19500
20000.,20500.,21000.,21500.,22000.,22500.,23000.,23500.,24000.,24500
2
0.0,20.0
2
0.0,1.5
```

TABLE 1-CONTD.(DIRECT INPUT DATA)

APPENDIX I-FORTRAN IV COMPUTER PROGRAM
TO SOLVE 3-D,UNSTEADY STATE DO & BOD EQUATIONS

```
C..........................................
(00001) C..........................................
(00002) C..........................................
(00003) C..... PROGRAM TO ANALYZE DISSOLVED OXYGEN (DO) AND
(00004) C..... BIOCHEMICAL OXYGEN DEMAND (BOD) UNDER
(00005) C..... 3-DIMENSIONAL,UNSTEADY STATE CONDITIONS
(00006) C.....
(00007) C.....
(00008) C..... THE PROGRAM UTILIZE ANATICAL SOLUTIONS OF
(00009) C..... DO & BOD IN 3-D,UNSTEADY STATE CONDITIONS
(00010) C.....
(00011) C.....
(00012) C..... WRITTEN IN FORTRAN IV LANGUAGE
(00013) C..... WRITTEN BY LING L. KIM
(00014) C.....
(00015) C.....
(00016) C..... DOCTORAL CANDIDATE
(00017) C..... DEPARTMENT OF CIVIL & ENVIRONMENTAL ENGINEERING
(00018) C..... NEW JERSEY INSTITUTE OF TECHNOLOGY
(00019) C..... 323 HIGH STREET
(00020) C..... NEWARK,NEW JERSEY 07102
(00021) C.....
(00022) C.....
(00023) C..... NOTE :
(00024) C.....
(00025) C.....
(00026) C..... ANY PORTION OF THIS PROGRAM IS NOT TO BE COPIED,SOLD, OR
(00027) C..... TRANSFERRED WITHOUT THE WRITTEN PERMISSION OF
(00028) C..... THE NEW JERSEY INSTITUTE OF TECHNOLOGY AND THE AUTHOR
(00029) C..... THE PROGRAM IS THE SOLE PROPERTY OF THE INSTITUTE AND THE AUTHOR
(00030) C.....
(00031) C..... THE CAPACITY OF THE PROGRAM :
(00032) C.....
(00033) C.....
(00034) C.....
(00035) C..... 1. THE PROGRAM CAN TAKE ANY KIND OF DISCHARGE CONDITIONS
(00036) C..... WHETHER THEY ARE INSTANTANEOUS OR CONTINUOUS.
(00037) C..... 2. THE PROGRAM CAN ANALYZE DO AND BOD OF ANY LOCATIONS(X,Y,Z)
(00038) C..... OF THE CHANNEL AT DIFFERENT ELAPSED TIME.
(00039) C..... 3. THERE ARE NO LIMITATIONS IN THE PROGRAM
(00040) C..... IN TERMS OF THE MAGNITUDE OF SPACE(X,Y,Z), AND TIME
(00041) C.....
(00042) C.....
(00043) C..... SPECIAL NOTE :
(00044) C.....
(00045) C.....
(00046) C..... INPUT DATA ARE ENTERED IN *FREE* FORMAT TO EASE DATA ENTRY
(00047) C..... ALL THE INPUT DATA ARE *ECHO* CHECKED BY THE PROGRAM
(00048) C..... IN ORDER TO ELIMINATE ANY ERRORS IN THE INPUT DATA
(00049) C..... BASICALLY ALL THE *INDEPENDENT* VARIABLES
(00050) C..... SPECIFIED IN THE ANALYTICAL SOLUTIONS BECOME INPUT DATA
(00051) C..... THIS INCLUDE INITIAL POLLUTANTS* CONCENTRATIONS ALONG WITH
(00052) C..... VARIOUS REACTION COEFFICIENTS,POLLUTANTS* DISCHARGE MODE
(00053) C..... STREAM GEOMETRY,VELOCITY,ETC.
(00054) C.....
(00055) C.....
(00056) C..... THE USERS MUST REFER TO THE ANALYTICAL SOLUTIONS OF
(00057) C..... THE GENERAL DO & BOD EQUATIONS IN ORDER TO FULLY UNDERSTAND
(00058) C..... THE *FLOW* OF THE PROGRAM
```

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(0059)  C***********************************************
(0060)  C-----|
(0061)  C-----|
(0062)  C****** THE PROGRAM EXCLUSIVELY UTILIZE *FORTRAN*S*
(0063)  C****** *USER-DEFINED* FUNCTION CAPABILITY
(0064)  C    *  IN ORDER TO MAXIMIZE THE PROGRAM EFFICIENCY
(0065)  C    *  |
(0066)  C-----|
(0067)  C-----|
(0068)  C-----|
(0069)  C***********************************************
(0070)        IMPLICIT REAL(A-H,K-Z)
(0071)        INTEGER TITLE(40),KOUNT,KOUNTT,KOUNTX,KOUNTY,KOUNTZ
(0072)        COMMON /CONST1/ L0,D1,D2,D3,K1,K2,U,USTATN,USHEAR,V,W,PI,
(0073)       1WIDTH,DEPTH
(0074)        COMMON /CONST2/ CS,C0,P,PM,M0
(0075)        COMMON /CONST3/ TINTVL(100),XINTVL(100),YINTVL(100),ZINTVL(100)
(0076)        COMMON /CONST4/ LOCNV1,LOCNV2
(0077)
(0078)        DATA IMODE1/*INST*/
(0079)        DATA IMODE2/*CONT*/
(0080)        CALL OPEAF(INTS(5),*DCHANNEL*,INTS(8),INTS(1),INTS(0),CODE)
(0081)        IF(CODE.NE.INTS(0)) CALL ERRPRS(0,37,0,C,0,0)
(0082)        CALL OPEAF(INTS(6),*CHANNELOUT*,INTS(10),INTS(2),INTS(0),CODE)
(0083)        IF(CODE.NE.INTS(0)) CALL ERRPRS(0,37,0,0,0,0)
(0084)        INP=5
(0085)        IOUT=6
(0086)  C------ READ IN CONSTANTS
(0087)        READ(INP,90) TITLE
(0088)   90   FORMAT(50A1)
(0089)        READ(INP,*) IMODE
(0090)        READ(INP,*) WIDTH,DEPTH,CROSSA,Q,IYEAR
(0091)        READ(INP,*) MANING
(0092)        READ(INP,*) U,USTATN,V,W
(0093)        READ(INP,*) D1RATO,D2RATO
(0094)        READ(INP,*) L0,K1
(0095)        READ(INP,*) M0,FACTOR
(0096)        READ(INP,*) CS,C0,K2
(0097)        READ(INP,*) P,PM
(0098)        READ(INP,*) INUMT
(0099)        READ(INP,*) (TINTVL(I),I=1,INUMT)
(0100)        READ(INP,*) INUMX
(0101)        READ(INP,*) (XINTVL(I),I=1,INUMX)
(0102)        READ(INP,*) INUMY
(0103)        READ(INP,*) (YINTVL(I),I=1,INUMY)
(0104)        READ(INP,*) INUMZ
(0105)        READ(INP,*) (ZINTVL(I),I=1,INUMZ)
(0106)        PI=3.14159
(0107)        LOCNV1=53.59*1000/28.314
(0108)        LOCNV2=LOCNV1
(0109)        R=CROSSA/(WIDTH+DEPTH*2)
(0110)        C=1.486/MANING*R**0.16667
(0111)        USHEAR=U*SQRT(32.2)/C
(0112)        D1=D1RATO*R*USHEAR
(0113)        D2=D2RATO*R*USHEAR
(0114)        D3=D2/((USTATN/U)*(WIDTH/DEPTH)**2)
(0115)        WRITE(IOUT,100)
(0116)  100   FORMAT(*1*,//////,T56,*ANALYSIS OF DO & BOD*,/T56,20(*.*),/
```

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```
C *************************************

(0117)          1T65,*IN*,/,
(0118)          2T51,*3-D,UNSTEADY-STATE CONDITIONS*,/T51,29(*,*),////)
(0119)          WRITE(IOUT,105)
(0120)    105   FORMAT(T58,*SYSTEM INPUT DATA*,//T58,17(*-*))
(0121)          WRITE(IOUT,110) D1,D2,D3
(0122)    110   FORMAT(//,*T48,*DISPERSION COEFFICIENTS(SG. FT/DAY) :*,
(0123)          1T48,//T59,*LONGITUDINAL =*,F11,2,/T63,*TRAVERSE =*,F11,2,/
(0124)          2T63,*VERTICAL =*,F11,2,//)
(0125)          WRITE(IOUT,115) U,V,W
(0126)    115   FORMAT(T57,*MEAN VELOCITY(FT/DAY)            :*,
(0127)          1//,T61,*HORIZONTAL =*,F8,1,/T64,*LATERAL =*,F8,1,
(0128)          2/T63,*VERTICAL =*,F8,1,///)
(0129)          WRITE(IOUT,120) K1,K2,L0,M0
(0130)    120   FORMAT(T47,*BOD VELOCITY CONSTANT(PER DAY,BASE E) =*,F4,2,
(0131)          1/T49,*REAERATION CONSTANT(PER DAY,BASE E) =*,F4,2,
(0132)          2//T56,*POLLUTANT LOADING(LBS,/CU. FT) :*,F8,2,/
(0133)          3T58,*INSTANTANEOUS =*,F8,2,/
(0134)          4T61,*CONTINUOUS =*,F8,2,//)
(0135)          WRITE(IOUT,122) CS,C0
(0136)    122   FORMAT(T53,*DO SATURATION VALUE(MG/L) =*,F4,2,
(0137)          1/T62,*INITIAL DO(MG/L) =*,F4,2,///)
(0138)          WRITE(IOUT,125) ITTLE,WIDTH,DEPTH,CRCSSA,Q,IYEAR
(0139)    125   FORMAT(T40,*RIVER GEOMETRY - *,40A1,*:*,
(0140)          1//T62,*WIDTH(FT) =*,F8,1,/
(0141)          2T57,*MEAN DEPTH(FT) =*,F8,1,/
(0142)          3T43,*CROSS-SECTIONAL AREA(SQ. FT) =*,F8,1,/
(0143)          4T45,*MEAN ANNUAL DISCHARGE(CFS) =*,F8,1,/
(0144)          5T56,*YEARS OF RECORD =*,I6)
(0145)          WRITE(IOUT,130)
(0146)    130   FORMAT(*1*,//////T59,*SYSTEM OUTPUT*,/T59,13(*,*),///
(0147)          1T67,*PERVIOUS*,T90,*IMPERVIOUS*,/
(0148)          2T10,*MODE*,T21,*ELAPSED*,T68,*CHANNEL*,T92,*CHANNEL*,/
(0149)          3T11,*OF*,T22,*TIME*,T35,*X*,T44,*Y*,T52,*Z*,
(0150)          4T66,*BOD*,T77,*DO*,T89,*BOD*,T101,*DO*,/
(0151)          5T7,*DISCHARGE*,T22,*(DAY)*,T33,*(FT)*,T43,*(FT)*,T51,*(FT)*,
(0152)          6T62,*(#/CU. FT)*,T75,*(MG/L)*,T86,*(#/CU. FT)*,T99,*(MG/L)*,
(0153)          7/T7,9(*-*),T22,5(*-*),T33,4(*-*),T43,4(*-*),T51,4(*-*),
(0154)          8T62,10(*-*),T75,6(*-*),T86,10(*-*),T99,6(*-*),////)
(0155)          IF(IMODE.EQ.IMODE1) WRITE(IOUT,135)
(0156)          IF(IMODE.EQ.IMODE2) WRITE(IOUT,140)
(0157)    135   FORMAT(* *,T5,*INSTANTANEOUS*)
(0158)    140   FORMAT(* *,T7,*CONTINUOUS*)
(0159)   C
(0160)   C
(0161)   C
(0162)   C
(0163)   C
(0164)   C----- PERFORM CALCULATIONS AND OUTPUT FOR ALL COMBINATIONS OF X,Y,AND Z.
(0165)   C-----BEGIN OUTMOST LOOP TO VARY T,@ A GIVEN X
(0166)   C----- THEN VARY Y @ A GIVEN X,THEN Z @ A GIVEN Z
(0167)   C----- THIS IS THE RIGHT PROCEDURES FOR 3-D,UNSTEADY-CONDITIONS
(0168)          KOUNT=0
(0169)          DO 330 IT=1,INUMT
(0170)          T=TINTVL(IT)/86400.
(0171)          KOUNT=KOUNT+1
(0172)          IF(KOUNT.EQ.1) WRITE(IOUT,145) T
(0173)          IF(KOUNT.GT.1) WRITE(IOUT,146) T
(0174)    145   FORMAT(*,*,T21,F7,5)
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C.............................................................

(0175)         146 FORMAT(' ',T21,F7.5)
(0176)             KOUNTX=0
(0177)             DO 500 IX=1,INUMX
(0178)             X=XIATVL(IX)
(0179)             KOUNTX=KOUNTX+1
(0180)             IF(KOUNTX.EG.1) WRITE(IOUT,150) X
(0181)             IF(KOUNTX.GT.1) WRITE(IOUT,151) X
(0182)         150 FORMAT(' ',T30,I7)
(0183)         151 FORMAT(' ',T30,I7)
(0184)   C----- BEGIN A INNER LOOP TO VARY Y
(0185)             KOUNTY=0
(0186)             DO 510 IY=1,INUMY
(0187)             Y=YIATVL(IY)
(0188)             KOUNTY=KOUNTY+1
(0189)             IF(KOUNTY.EG.1) WRITE(IOUT,155) Y
(0190)             IF(KOUNTY.GT.1) WRITE(IOUT,156) Y
(0191)         155 FORMAT(' ',T41,I6)
(0192)         156 FORMAT(' ',T41,I6)
(0193)   C----- BEGIN A INNERMOST LOOP TO VARY Z
(0194)             KOUNTZ=0
(0195)             DO 520 IZ=1,INUMZ
(0196)             Z=ZIATVL(IZ)
(0197)             KOUNTZ=KOUNTZ+1
(0198)             IF(KOUNTZ.EG.1) WRITE(IOUT,160) Z
(0199)             IF(KOUNTZ.GT.1) WRITE(IOUT,161) Z
(0200)         160 FORMAT(' ',T51,I4)
(0201)         161 FORMAT(' ',T51,I4)
(0202)   C----- PERFORM CALCULATIONS AND OUTPUT THE RESULTS
(0203)             IF(IMODE.EQ.IMODE2) GO TO 99
(0204)             CONTINUE
(0205)             L1P=L1PERV(X,Y,Z,T)
(0206)             L1P=L1P*28.314/453590
(0207)             C1P=C1PERV(X,Y,Z,T)
(0208)             L1I=L1IMP(X,Y,Z,T)
(0209)             L1I=L1I*28.314/453590
(0210)             C1I=C1IMP(X,Y,Z,T)
(0211)             GO TO 999
(0212)          99 L2P=L2PERV(X,Y,Z,T)
(0213)             L2P=L2P*28.314/453590
(0214)             C2P=C2PERV(X,Y,Z,T)
(0215)             L2I=L2IMP(X,Y,Z,T)
(0216)             L2I=L2I*28.314/453590
(0217)             C2I=C2IMP(X,Y,Z,T)
(0218)         999 IF(IMODE.EQ.IMODE1) WRITE(IOUT,200) L1P,C1P,L1I,C1I
(0219)             IF(IMODE.EQ.IMODE2) WRITE(IOUT,200) L2P,C2P,L2I,C2I
(0220)         200 FORMAT(' ',T60,F12.4,T76,F5.2,T84,F12.4,T100,F5.2)
(0221)         520 CONTINUE
(0222)         510 CONTINUE
(0223)         500 CONTINUE
(0224)         300 CONTINUE
(0225)             CALL CLOSEF(INTS(5))
(0226)             CALL CLOSEF(INTS(6))
(0227)             CALL EXIT
(0228)             END
(0229)             FUNCTION L1PROD(X,Y,Z,T)
(0230)   C----- FUNCTION TO COMPUTE THE PRODUCT COMMON TO BOD EQUATIONS
(0231)             IMPLICIT REAL(A-H,K-Z)
(0232)             COMMON /CONST1/ L0,D1,D2,D3,K1,K2,U,USTATM,USHEAR,V,W,PI,
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C ***************************************

(0233)        1WIDTH,DEPTH
(0234)        COMMON /CONST2/ CS,CO,P,PM,MO
(0235)        COMMON /CONST3/ TINTVL(100),XINTVL(100),YINTVL(100),ZINTVL(100)
(0236)        COMMON /CONST4/ LOCNV1,LOCNV2
(0237)
(0238)        TEP~1=T**(-3./2.)
(0239)        TERM2=EXP(-(*K1*T)
(0240)        TERM3=EXP(-1*((X-U*T)**2/(4*D1*T)))
(0241)        TERM4=EXP(-1*((Y-V*T)**2/(4*D2*T)))
(0242)        TERM5=EXP(-1*((Z-W*T)**2/(4*D3*T)))
(0243)        LPROG=TERM1*TERM2*TERM3*TERM4*TERM5
(0244)        RETURN
(0245)        END
(0246)        FUNCTION L1PERV(X,Y,Z,T)
(0247) C----- FUNCTION TO CALCULATF L FOR BOD WITH INSTANTANEOUS POINT SOURCE
(0248)        IMPLICIT REAL(A-H,K-Z)
(0249)        COMMC' /CONST1/ L0,D1,D2,D3,K1,K2,U,USTATN,USHEAR,V,W,PI,
(0250)        1WIDTH,DEPTH
(0251)        COMMON /CONST2/ CS,CO,F,PP,MO
(0252)        COMMON /CONST3/ TINTVL(100),XINTVL(100),YINTVL(100),ZINTVL(100)
(0253)        COMMON /CONST4/ LOCNV1,LOCNV2
(0254)
(0255)        DEN=2*SQRT(D1*D2*D3)*PI**(3./2.)
(0256)        L1PERV=L0*(1.*LOCNV1-LOCNV2*LPROD(X,Y,Z,T)/DEN)
(0257)        IF(L1PERV.LE.0) L1PERV=0.0
(0258)        RETURN
(0259)        END
(0260)        FUNCTION PC1PER(X,Y,Z,T)
(0261) C-----
(0262)        IMPLICIT REAL(A-Z,K-7)
(0263)        COMMON /CONST1/ L0,D1,D2,D3,K1,K2,U,USTATN,USHEAR,V,W,PI,
(0264)        1WIDTH,DEPTH
(0265)        COMMON /CONST2/ CS,CO,P,PM,MO
(0266)        COMMON /CONST3/ TINTVL(100),XINTVL(100),YINTVL(100),ZINTVL(100)
(0267)        COMMON /CONST4/ LOCNV2
(0268)
(0269)        PC1PER=L0*(1.*LOCNV1)-L1PERV(X,Y,Z,T)
(0270)        RETURN
(0271)        END
(0272)        FUNCTION C1PERV(X,Y,Z,T)
(0273) C----- FUNCTION TO COMPUTE C FOR BOD PERVIOUS STREAM CHANNEL INSTANTANEOUS POINT SOURCE
(0274)        IMPLICIT REAL(A-H,K-Z)
(0275)        COMMON /CONST1/ L0,D1,D2,D3,K1,K2,U,USTATN,USHEAR,V,W,PI,
(0276)        1WIDTH,DEPTH
(0277)        COMMON /CONST2/ CS,CO,P,PM,MO
(0278)        COMMON /CONST3/ TINTVL(100),XINTVL(100),YINTVL(100),ZINTVL(100)
(0279)        COMMON /CONST4/ LOCNV1,LOCNV2
(0280)
(0281)        SUM1=(CS-CO-K1/K2*PC1PER(X,Y,Z,T)+1/K2*AC(T))*EXP(-1*K2*T)
(0282)        SUM2=K1/K2*PC1PER(X,Y,Z,T)-1/K2*AC(T)
(0283)        C1PERV=CS-(SUM1+SUM2)
(0284)        IF(C1PERV.LT.0.0) C1PERV=0.0
(0285)        RETURN
(0286)        END
(0287)        FUNCTION AC(T)
(0288) C----- COMPUTE PHOTOSYNTHETIC OXYGEN SOURCE
(0289) C----- ALSO FIND THE NUMBER OF TERMS TO APPROXIMATE INFINITE FOURIER SERIES
(0290)        IMPLICIT REAL(A-H,K-Z)
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C.............................................

(0291)      INTEGER N
(0292)      COMMON /CONST1/ L0,D1,D2,D3,K1,K2,U,USTATN,USHEAR,V,W,PI,
(0293)     1WIDTH,DEPTH
(0294)      COMMON /CONST2/ CS,C0,P,PM,M0
(0295)      COMMON /CONST3/ TINTVL(100),XINTVL(100),YINTVL(100),ZINTVL(100)
(0296)      COMMON /CONST4/ LOCNV1,LOCNV2
(0297)
(0298)      IF(PM.NE.0.0) GO TO 99
(0299)      AC=0.0
(0300)      RETURN
(0301)   99 SUM=0.
(0302)      N=0
(0303)  100 CONTINUE
(0304)      N=N+1
(0305)      AN=2.*PI/P*COS(N*PI*P)/((PI/P)**2-(2.*N*PI)**2)
(0306)      TERM=2.*AN*COS(2.*PI*N*(T-P/2))
(0307)      SUM=SUM+TERM
(0308)      DELTA=ABS(0.05*SUM)
(0309)      IF(ABS(TERM).LE.DELTA) GO TO 200
(0310)      GO TO 100
(0311)  200 CONTINUE
(0312)      AC=PM*(2.*P/PI+2.*SUM)
(0313)      RETURN
(0314)      END
(0315)      FUNCTION INTM(T,FACTOR)
(0316) C----- TO COMPUTE THE IMPACT OF CONTINUOUS INPUT
(0317) C----- FOR PERVIOUS STREAM CHANNEL
(0318)      IMPLICIT REAL(A-H,K-Z)
(0319)      COMMON /CONST1/ L0,D1,D2,D3,K1,K2,U,USTATN,USHEAR,V,W,PI,
(0320)     1WIDTH,DEPTH
(0321)      COMMON /CONST2/ CS,C0,P,PM,M0
(0322)      COMMON /CONST3/ TINTVL(100),XINTVL(100),YINTVL(100),ZINTVL(100)
(0323)      COMMON /CONST4/ LOCNV1,LOCNV2
(0324)
(0325)      INTEGER N
(0326)      INTM=M0*SIN(T*FACTOR)
(0327)      RETURN
(0328)      END
(0329)      FUNCTION PC2PER(X,Y,Z,T)
(0330)      IMPLICIT REAL(A-H,K-Z)
(0331)      COMMON /CONST1/ L0,D1,D2,D3,K1,K2,U,USTATN,USHEAR,V,W,PI,
(0332)     1WIDTH,DEPTH
(0333)      COMMON /CONST2/ CS,C0,P,PM,M0
(0334)      COMMON /CONST3/ TINTVL(100),XINTVL(100),YINTVL(100),ZINTVL(100)
(0335)      COMMON /CONST4/ LOCNV1,LOCNV2
(0336)
(0337)      PC2PER=INTM(T,FACTOR)*1.*LOCNV1-L2PERV(X,Y,Z,T)
(0338)      RETURN
(0339)      END
(0340)      FUNCTION L2PERV(X,Y,Z,T)
(0341)
(0342) C-----
(0343) C-----
(0344)      IMPLICIT REAL(A-H,K-Z)
(0345)      COMMON /CONST1/ L0,D1,D2,D3,K1,K2,U,USTATN,USHEAR,V,W,PI,
(0346)     1WIDTH,DEPTH
(0347)      COMMON /CONST2/ CS,C0,P,PM,M0
(0348)      COMMON /CONST3/ TINTVL(100),XINTVL(100),YINTVL(100),ZINTVL(100)
(0348)      COMMON /CONST4/ LOCNV1,LOCNV2
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(03491)        DEN=8·SQRT(D1·D2·D3)·PI··(3./2.)
(03501)        L2PERV=IRTH(T·FACTOR)·(1.·LOCNV1-LOCNV2·LPROD(X·Y·Z·T)/DEN)
(03511)        IF(L2PERV·LT·0.0) L2PERV=0.0
(03521)        RETURN·
(03531)        END
(03541)
(03551)
(03561)
(03571)  C----  FUNCTION C2PERV(X·Y·Z·T)
(03581)  C----
(03591)  C----
(03601)
(03611)        IMPLICIT REAL(A-H·K-Z)
(03621)        COMMON /CONST1/ L0·D1·D2·D3·K1·K2·U·USTATN·USHEAR·V·W·PI·
(03631)       1WIDTH·DEPTH
(03641)        COMMON /CONST2/ CS·C0·P·PM·M0
(03651)        COMMON /CONST3/ TINTVL(100)·XINTVL(100)·YINTVL(100)·ZINTVL(100)
(03661)        COMMON /CONST4/ LOCNV1·LOCNV2
(03671)
(03681)        SUM1=(CS-C0-K1/K2·PC2PER(X·Y·Z·T)·1/K2·AC(T))·EXP(-1·K2·T)
(03691)        SUM2=K1/K2·PC2PER(X·Y·Z·T)-1/K2·AC(T)
(03701)        C2PERV=CS-(SUM1·SUM2)
(03711)        IF(C2PERV·LT·0·0) C2PERV=0·0
(03721)        RETURN
(03731)        END
(03741)
(03751)
(03761)
(03771)
(03781)  C----  FUNCTION L1IMP(X·Y·Z·T)
(03791)  C----
(03801)  C----
(03811)
(03821)        IMPLICIT REAL(A-H·K-Z)
(03831)        COMMON /CONST1/ L0·C1·D2·D3·K1·K2·U·USTATN·USHEAR·V·W·PI·
(03841)       1WIDTH·DEPTH
(03851)        COMMON /CONST2/ CS·C0·P·PM·M0
(03861)        COMMON /CONST3/ TINTVL(100)·XINTVL(100)·YINTVL(100)·ZINTVL(100)
(03871)        COMMON /CONST4/ LOCNV1·LOCNV2
(03881)
(03891)        CONSTW=WIDTH/(4.·SQRT(D2·T))
(03901)        CONSTD=DEPTH/(4.·SQRT(D3·T))
(03911)        IF(CONSTW·GE·2.0·AND·CONSTD·GE·2.0) GO TO 10
(03921)        PCONST=0·3275911
(03931)        A1=0·254829592
(03941)        A2=-0·284496736
(03951)        A3=1·421413741
(03961)        A4=-1·453152027
(03971)        A5=1·061405429
(03981)
(03991)  C----
(04001)  C----
(04011)        DELTW=1·/(1·+PCONST·CONSTW)
(04021)        DELTD=1·/(1·+PCONST·CONSTD)
(04031)        ERFW=1-(A1·DELTW·A2·DELTW··2·A3·DELTW··3·A4·DELTW··4·A5·DELTW··5)·
(04041)       1EXP(-1·DELTW··2)
(04051)        ERFD=1-(A1·DELTD·A2·DELTD··2·A3·DELTD··3·A4·DELTD··4·A5·DELTD··5)·
(04061)       1EXP(-1·DELTD··2)
(04071)        GO TO 20
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C******************************************
(0407)        10 EPFW=1.0
(0408)           ERFD=1.0
(0409)        20 PRCD1=1./(R..*SQRT(D1*D2*D3)*PI..1.5*ERFW*ERFD)
(0410)           L1IMP=L0+(1.*LOCNV1-LOCNV2*PROD1*LPROD(X,Y,Z,T))
(0411)           IF(L1IMP.LT.0.0) L1IMP=0.0
(0412)           RETURN
(0413)           END
(0414)
(0415)
(0416)
(0417)           FUNCTION PC1IMP(X,Y,Z,T)
(0418)
(0419)    C-----
(0420)    C-----
(0421)           IMPLICIT REAL(A-H,K-Z)
(0422)           COMMON /CONST1/ L0,D1,D2,D3,K1,K2,U,USTATN,USHEAR,V,W,PI,
(0423)          1WIDTH,DEPTH
(0424)           COMMON /CONST2/ CS,CO,P,PM,M0
(0425)           COMMON /CONST3/ TINTVL(100),XINTVL(100),YINTVL(100),ZINTVL(100)
(0426)           COMMON /CONST4/ LOCNV1,LOCNV2
(0427)           PC1IMP=L0*1.*LOCNV1-L1IMP(X,Y,Z,T)
(0428)           RETURN
(0429)           END
(0430)
(0431)
(0432)
(0433)           FUNCTION C1IMP(X,Y,Z,T)
(0434)
(0435)    C-----
(0436)    C-----
(0437)           IMPLICIT REAL(A-H,K-Z)
(0438)           COMMON /CONST1/ L0,D1,D2,D3,K1,K2,U,USTATN,USHEAR,V,W,PI,
(0439)          1WIDTH,DEPTH
(0440)           COMMON /CONST2/ CS,CO,P,PM,M0
(0441)           COMMON /CONST3/ TINTVL(100),XINTVL(100),YINTVL(100),ZINTVL(100)
(0442)           COMMON /CONST4/ LOCNV1,LOCNV2
(0443)           SUM1=(CS-CO-K1/K2*PC1IMP(X,Y,Z,T)*1/K2*AC(T))*EXP(-1*K2*T)
(0444)           SUM2=K1/K2*PC1IMP(X,Y,Z,T)-1/K2*AC(T)
(0445)           C1IMP=CS-(SUM1+SUM2)
(0446)           IF(C1IMP.LT.0.0) C1IMP=0.0
(0447)           RETURN
(0448)           END
(0449)
(0450)
(0451)
(0452)           FUNCTION L2IMP(X,Y,Z,T)
(0453)
(0454)    C-----
(0455)    C-----
(0456)           IMPLICIT REAL(A-H,K-Z)
(0457)           COMMON /CONST1/ L0,D1,D2,D3,K1,K2,U,USTATN,USHEAR,V,W,PI,
(0458)          1WIDTH,DEPTH
(0459)           COMMON /CONST2/ CS,CO,P,PM,M0
(0460)           COMMON /CONST3/ TINTVL(100),XINTVL(100),YINTVL(100),ZINTVL(100)
(0461)           COMMON /CONST4/ LOCNV1,LOCNV2
(0462)           CONSTW=WIDTH/(4.*SQRT(D2*T))
(0463)           CONSTD=DEPTH/(4.*SQRT(D3*T))
(0464)           IF(CONSTW.GE.2.0.AND.CONSTD.GE.2.0) GO TO 10
```