ANALYSIS OF UNSTEADY-STATE HEAT
CONDUCTION IN A CYLINDER WITH
THE FINITE ELEMENT METHOD

BY

RICHARD F. BERNARD

A THESIS
PRESENTED IN PARTIAL FULFILLMENT OF
THE REQUIREMENTS FOR THE DEGREE
OF
MASTER OF SCIENCE IN CHEMICAL ENGINEERING
AT
NEW JERSEY INSTITUTE OF TECHNOLOGY

Newark, New Jersey

1980

ABSTRACT

Unsteady-state heat conduction in a cylinder of finite dimensions with constant physical and thermal properties was analyzed with the finite element method. Symmetry of the cylinder and application of Newman's method reduced the problem to two independent one-dimensional problems. Finite element equations for the axial and radial dimensions were developed utilizing Galerkin's method of weighted residuals. Crank-Nicholson approximations were used for time derivatives. A computer program was written for solution of the finite element equations.

Solutions obtained were conditionally stable; dependent on the value of the ratio $K\Delta t/\rho Cpl^2$. For values of the ratio less than 1/3, errors in solutions at small times result. For values of the ratio greater than 2.0, very large errors result. The magnitude of the errors increase with increasing values of the ratio. For proper values of the ratio $K\Delta t/\rho Cpl^2$, finite element solutions converge to the analytical solutions by increasing the number of finite elements in the problem.

APPROVAL OF THESIS

ANALYSIS OF UNSTEADY-STATE HEAT CONDUCTION IN A CYLINDER WITH THE FINITE ELEMENT METHOD

BY

RICHARD F. BERNARD

FOR

DEPARTMENT OF CHEMICAL ENGINEERING

NEW JERSEY INSTITUTE OF TECHNOLOGY

BY

FACULTY COMMITTEE

APPROVED: ______________________

____

____

NEWARK, NEW JERSEY

MAY, 1980

To my wife, Michele

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

1.0 INTRODUCTION

1.1 Background

The analytical solution to an unsteady-state heat conduction problem requires the development of explicit functions containing infinite series for the dependent variable. Evaluation of the explicit function at a given time and position may be tedious and time-consuming. The finite element method of analysis is a numerical method for the solution of differential equations that is simpler than the analytical route and yet permits material heterogenities and more than one boundary condition within a given problem.

1.2 Problem Definition

The subject of this thesis is the solution of a three-dimensional unsteady-state heat conduction problem using the finite element method of analysis.

The physical object under examination is a cylinder of finite dimensions. Cylindrical coordinates, as shown in Figure 1.2-1 are employed. Any point within the cylinder may be identified by coordinates (x, r, θ). The cylinder may be considered to be of length L and radius R. In this development, the cylinder is assumed to be of a homogeneous material, with a constant thermal conductivity, density, and heat capacity throughout.

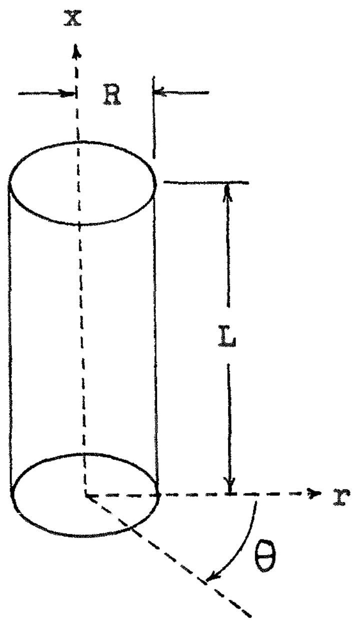


Figure 1.2-1

Cylinder and Coordinate System

The unsteady-state heat conduction problem to be examined is that of a cylinder at an initial temperature T^0 throughout whose surface is instantaneously brought to the temperature T^1 at time zero and thereafter maintained at temperature T^1. It is then desired to find the temperature distribution throughout the cylinder, or the temperature at a specific location within the cylinder, at a specific time.

1.3 Approach

The problem may be readily reduced to a two-dimensional problem in terms of x and r. Due to the symmetry of the cylinder about the x-axis, there is no variation of temperature with θ, angular position. That is, $\partial T/\partial \theta = 0$.

Application of Newman's method further reduces the two-dimensional problem to that of two one-dimensional problems. In Newman's method, the dimensionaless solution at any point within the object is the product of the dimensionaless solutions of that point in each dimension. Therefore, the differential equations for unsteady-state heat conduction in the x dimension and the r dimension,

$$K \frac{\partial^2 T}{\partial x^2} - \rho Cp \frac{\partial T}{\partial t} = 0 \qquad (1.3\text{-}1)$$

and

$$K \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T}{\partial r} \right) \right] - \rho Cp \frac{\partial T}{\partial t} = 0, \qquad (1.3\text{-}2)$$

must be solved independently using the finite element method of analysis.

2.0 FUNDAMENTALS OF THE FINITE ELEMENT METHOD

2.1 General

The concept underlying the finite element method of analysis is that of discretization. By discretization, it is meant that the extent or domain of a problem is broken down into smaller, more manageable pieces. Each piece of the problem may be termed a finite element. The solution to the problem may be formulated within each element easier than in the larger composite problem. Assembly of the solutions in each element and the application of the appropriate boundary conditions allows solution of the larger composite problem. It is the formulation of the solution to the problem within the individual finite elements that is the power of the finite element method in engineering analysis. Irregular geometries, heterogeneities in material properties, and several different boundary conditions in the same problem may be accomodated with the use of finite elements.

2.2 Discretization

In the one-dimensional heat conduction problem, the extent, or domain, of the problem, L, may be divided into n finite elements, each of length L/n. The end of an element, or intersection of two elements, is called a node. A problem having n elements therefore has n+1 nodes. The discretization of a one-dimensional problem into four finite elements is shown in Figure 2.2-1.

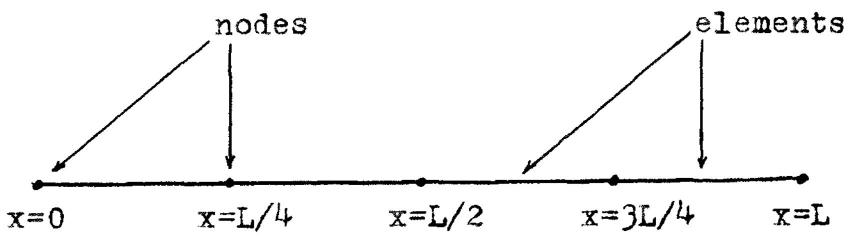


Figure 2.2-1

One-dimensional Finite Elements

2.3 Localized Coordinates

It is useful to define a non-dimensional, localized coordinate system within an element. A localized coordinate system allows analysis of a finite element as an entity in itself without the influence of neighboring elements; the non-dimensional nature of the coordinate system greatly facilitates the calculus in the derivation of the finite element equations. A one-dimensional element of length l is shown in Figure 2.3-1. Within this element, there are two reference points, node 1 and node 2, which may act as the basis for the coordinate system. With respect to node 1, the coordinate, s_1, of a point within the node may be defined as $s_1 = (x_2-x) / (x_2-x_1)$ or $s_1 = (x_2-x) / l$. Similarly, with respect to node 2,

$s_2 = (x-x_1) / (x_2-x_1)$ or $s_2 = (x-x_1) / l$. Expressed in this manner, the localized coordinate is non-dimensional and always has a value between 0 and 1.

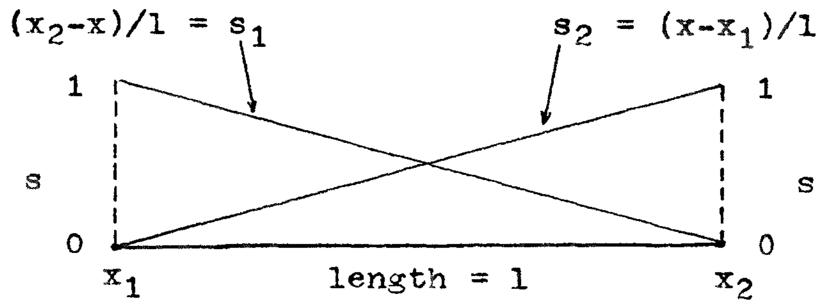


Figure 2.3-1

Localized Coordinates

2.4 Approximation Model and Interpolation Functions

An approximation model which represents the shape or form of the temperature profile within the element must be selected. The approximation model must be continuous within an element and may be linear, parabolic, or of higher order. The approximation model is defined in terms of two, or more, unknown nodal temperatures. For the one-dimensional heat conduction problem, a linear approximation may be used.

Interpolation functions, derived from the non-dimensional localized coordinate system, are used in the construction of the approximation model. An interpolation function is associated with a particular node and is defined only in the two adjacent elements on either side of the node. At all other locations within the domain of the problem, the interpolation function is equal to zero. Consider the two elements, each of length l, shown in Figure 2.4-1.

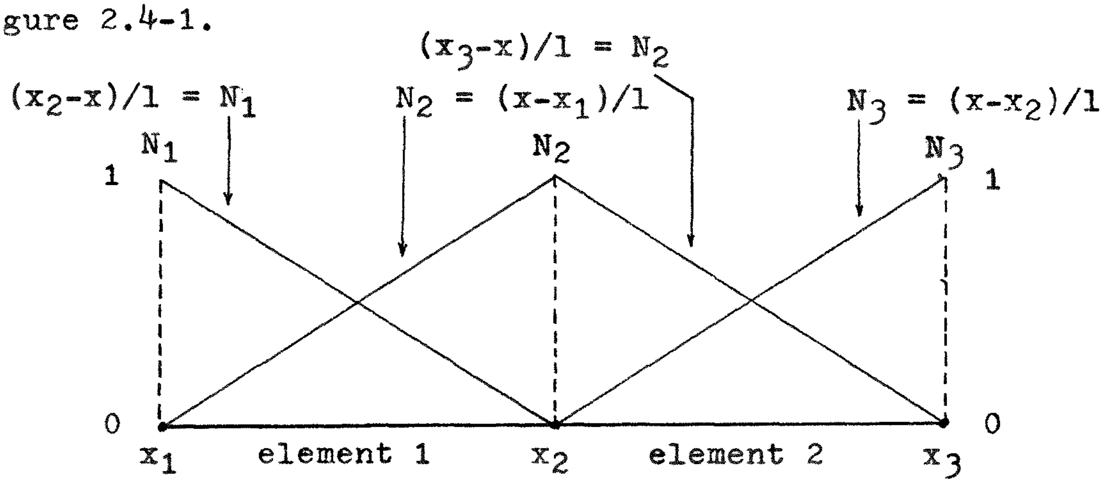


Figure 2.4-1

Interpolation Functions

Within element 1, N_1 and N_2 are defined and vary between 0 and 1. However, N_3 is not defined in element 1 and therefore equal to zero in element 1. N_2 is defined in both elements 1 and 2 and also ranges between 0 and 1.

A linear approximation model for temperatures within element 1 of Figure 2.4-1 may be expressed as

$$T = N_1T_1 + N_2T_2 \qquad (2.4\text{-}1)$$

where N_1 and N_2 are the interpolation functions defined within the element, and T_1 and T_2 are the unknown temperatures at nodes 1 and 2, respectively. Extending this model to a problem with n elements in its domain gives

$$T = N_1T_1 + N_2T_2 + N_3T_3 \dots + N_{n+1}T_{n+1} \qquad (2.4\text{-}2)$$

or

$$T = \sum_{i=1}^{n+1} N_iT_i \qquad (2.4\text{-}3)$$

Since at any given location only two interpolation functions are non-zero, Eqs. (2.4-2) and (2.4-3) reduce to a two term expression similar to Eq. (2.4-1).

2.5 Galerkin's Method of Weighted Residuals

Galerkin's method of weighted residuals is one of the most commonly used methods for formulation of the finite element equations. Galerkin's method of weighted residuals is based on the concept of minimization of the residual error remaining after an approximate solution, as represented by Eq. (2.4-3), is substituted into the differential equation describing the problem. In Galerkin's method, the residual error is weighted with the interpolation function N_i. The weighted residual error then is summed over the domain of the problem and an approximate solution is found which minimizes the total residual error.

The differential equation for one-dimensional unsteady-state heat conduction in rectangular coordinates is

$$K \frac{\partial^2 T}{\partial x^2} - \rho C_p \frac{\partial T}{\partial t} = 0 \qquad (2.5\text{-}1)$$

Substitution of an approximate solution into Eq. (2.5-1) yields the residual error

$$R(x) = K\frac{\partial^2\left(\sum T_i N_i\right)}{\partial\ x^2} - \rho Cp\frac{\partial\left(\sum T_i N_i\right)}{\partial t} \quad i=1,2,3 \qquad (2.5\text{-}2)$$

The residual error R(x) will be equal to zero if the approximate solution equals the exact solution. Weighting the residual error with N_i over the domain of the problem results in the expression

$$\int_{x=D} R(x)\quad N_i \quad dx = 0 \qquad (2.5\text{-}3)$$

Performing the required differential calculus to evaluate the residual errors in Eq. (2.5-2) and integral calculus to minimize the total weighted residual errors in Eq. (2.5-3) results in a system of linear simultaneous equations of order n+1 where n is the number of elements comprising the domain of the problem.

2.6 Time Domain

In Eq. (2.5-1) the time derivative of temperature is approximated using a finite difference method such as a Euler or Crank-Nicholson procedure. The derived system of element equations will contain a vector of nodal temperatures at time t and a vector of nodal temperatures at time $t+\Delta t$.

2.7 Initial and Boundary Conditions

Initial conditions are set by specifying the vector of nodal temperatures at time t = 0.

Boundary conditions of the problem are easily imposed by simple modifications of the system of simultaneous equations. Both Dirichlet boundary conditions, specification of temperatures, and Neumann boundary conditions, specification of temperature gradients, may be accomodated in this manner.

2.8 Solution

The solution to the problem is obtained by solving the simultaneous finite element equations for the values of the unknown temperatures at the nodes of the finite elements. The simultaneous equations may be solved using direct or iterative numerical methods. As with finite difference methods, the approximate solution generally converges to the exact, or analytical, solution by increasing the number of nodes. However, for a given problem, the stability and convergence of the approximate solution is influenced by the size of the elements and the size of the time intervals at which a solution is obtained.

3.0 DEVELOPMENT AND SOLUTION OF FINITE ELEMENT EQUATIONS

The finite element equations for unsteady-state heat conduction in the axial and radial dimensions will be developed for the discretization of the problem into two elements, in each dimension, using Galerkin's method of weighted residuals. The results are easily extended to a greater number of finite elements.

3.1 Axial Dimension

The differential equation for one-dimensional heat conduction in rectangular coordinates is

$$K\frac{\partial^2 T}{\partial x^2} - \rho Cp \frac{\partial T}{\partial t} = 0 \qquad (3.1\text{-}1)$$

After substituting a linear approximation model for temperature, as in Eq. (2.4-3), into Eq. (3.1-1) for T, the residual error becomes:

$$R(x) = K\frac{\partial^2 \left(\sum N_i T_i\right)}{\partial x^2} - \rho\ Cp \frac{\partial \left(\sum N_i T_i\right)}{\partial t} \quad i=1,2,3 \qquad (3.1\text{-}2)$$

As the domain of the problem in the x dimension has been broken down into two elements, there are three nodes and three unknown nodal temperatures. Therefore the summation in the approximation model for T in Eq. (3.1-2) has three terms.

In Galerkin's method of weighted residuals, the residual error as defined by Eq. (3.1-2) is weighted with the approximation function N_i. The total weighted residual error over the domain of the problem is then minimized. This can be expressed mathematically as:

$$\int_{x_1}^{x_3} \left[K \frac{\partial^2 T}{\partial x^2} - \rho Cp \frac{\partial T}{\partial t} \right] N_j \, dx = 0 \qquad (3.1\text{-}3)$$

where $j = 1,2,3$ and $T = \sum_{i=1}^{3} N_i T_i$. Weighting the residual error with N_1, N_2, and N_3 produces three equations with T_1, T_2, and T_3 as unknowns.

The first term on the left side of Eq. (3.1-3)

$$\int_{x_1}^{x_3} K \frac{\partial^2 T}{\partial x^2} N_j \, dx \qquad j=1,2,3 \qquad (3.1\text{-}4)$$

may be integrated by parts to give

$$K \frac{\partial T}{\partial x} N_j \Big|_{x_1}^{x_3} - K \int_{x_1}^{x_3} \frac{\partial T}{\partial x} \frac{\partial N_j}{\partial x} \, dx \quad j=1,2,3 \qquad (3.1\text{-}5)$$

The term on the left side of Eq. (3.1-5) may be expressed as the column vector

$$\begin{bmatrix} K \dfrac{\partial T}{\partial x} N_1 \Big|_{x_1}^{x_3} \\[2ex] K \dfrac{\partial T}{\partial x} N_2 \Big|_{x_1}^{x_3} \\[2ex] K \dfrac{\partial T}{\partial x} N_3 \Big|_{x_1}^{x_3} \end{bmatrix} \qquad (3.1\text{-}6)$$

Since N_2 equals zero at x_3 and x_1, N_1 equals zero at x_3, and N_3 equals zero at x_1, Eq. (3.1-6) becomes

$$\begin{bmatrix} -K \dfrac{\partial T}{\partial x}\Big|_{x_1} \\ 0 \\ K \dfrac{\partial T}{\partial x}\Big|_{x_3} \end{bmatrix} \qquad (3.1\text{-}7)$$

Later in the formulation of the finite element equations, Neumann boundary conditions may be imposed on the problem by specifying temperature gradients at the two end nodes, x_1 and x_3, in Eq. (3.1-7).

Substituting the linear approximation model for T, the term on the left side of Eq. (3.1-5) becomes

$$K\int_{x_1}^{x_3} \frac{\partial\left(\sum N_i T_i\right)}{\partial x} \quad \frac{\partial N_j}{\partial x}\, dx \quad i=1,2,3; \quad j=1,2,3 \qquad (3.1\text{-}8)$$

Since the nodal unknown temperatures T_i are constant in the solution of the problem, Eq. (3.1-8) may be rearranged to

$$K \sum_{i=1}^{3} \int_{x_1}^{x_3} \frac{\partial N_i}{\partial x}\,\frac{\partial N_j}{\partial x}\, dx\ T_i \quad j = 1,2,3 \qquad (3.1\text{-}9)$$

For $j = 1$, Eq. (3.1-9) may be written as

$$K\int_{x_1}^{x_3} \frac{\partial N_1}{\partial x}\frac{\partial N_1}{\partial x}\, dx\ T_1 \ + \ K\int_{x_1}^{x_3} \frac{\partial N_2}{\partial x}\frac{\partial N_1}{\partial x}\, dx\ T_2 \ +$$

$$K\int_{x_1}^{x_3} \frac{\partial N_3}{\partial x}\frac{\partial N_1}{\partial x}\, dx\ T_3 \qquad (3.1\text{-}10)$$

Similar expressions may be written for Eq. (3.1-9) for j = 2 and j = 3 and assembled to produce the matrix and column vector

$$K\int_{x_1}^{x_3}\begin{bmatrix}\frac{\partial N_1}{\partial x}\frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial x}\frac{\partial N_1}{\partial x} & \frac{\partial N_3}{\partial x}\frac{\partial N_1}{\partial x}\\ \frac{\partial N_1}{\partial x}\frac{\partial N_2}{\partial x} & \frac{\partial N_2}{\partial x}\frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial x}\frac{\partial N_2}{\partial x}\\ \frac{\partial N_1}{\partial x}\frac{\partial N_3}{\partial x} & \frac{\partial N_2}{\partial x}\frac{\partial N_3}{\partial x} & \frac{\partial N_3}{\partial x}\frac{\partial N_3}{\partial x}\end{bmatrix}dx\begin{bmatrix}T_1\\ T_2\\ T_3\end{bmatrix} \qquad (3.1\text{-}11)$$

Elements of the matrix in Eq. (3.1-11) that are a function both of N_1 and N_3 are equal to zero. Since N_1 is only defined in the first finite element and N_3 is only defined in the second finite element, nowhere from x_1 to x_3 are both N_1 and N_3 simultaneously non-zero. Therefore Eq. (3.1-11) may be rewritten as

$$K\int_{x_1}^{x_3}\begin{bmatrix}\frac{\partial N_1}{\partial x}\frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial x}\frac{\partial N_1}{\partial x} & 0\\ \frac{\partial N_1}{\partial x}\frac{\partial N_2}{\partial x} & \frac{\partial N_2}{\partial x}\frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial x}\frac{\partial N_2}{\partial x}\\ 0 & \frac{\partial N_2}{\partial x}\frac{\partial N_3}{\partial x} & \frac{\partial N_3}{\partial x}\frac{\partial N_3}{\partial x}\end{bmatrix}dx\begin{bmatrix}T_1\\ T_2\\ T_3\end{bmatrix} \qquad (3.1\text{-}12)$$

Other elements in the matrix of Eq. (3.1-12) must be evaluated individually by inserting the proper expression for the interpolation functions N_i. For example,

$$\int_{x_1}^{x_3} \frac{\partial N_1}{\partial x} \frac{\partial N_1}{\partial x}\, dx = \int_{x_1}^{x_2} \frac{\partial N_1}{\partial x} \frac{\partial N_1}{\partial x}\, dx \tag{3.1-13}$$

$$N_1 = (x_2 - x)/(x_2 - x_1) = (x_2 - x)/l \tag{3.1-14}$$

$$\frac{\partial N_1}{\partial x} = \frac{\partial\,(x_2 - x)/l}{\partial x} = -\frac{1}{l} \tag{3.1-15}$$

$$\int_{x_1}^{x_2} \left(\frac{-1}{l}\right)\left(\frac{-1}{l}\right) dx = \frac{1}{l^2}\int_{x_1}^{x_2} dx = \frac{x_2 - x_1}{l^2} = \frac{1}{l} \tag{3.1-16}$$

Similarly,

$$\int_{x_1}^{x_3} \frac{\partial N_2}{\partial x} \frac{\partial N_2}{\partial x}\, dx = \int_{x_1}^{x_2} \frac{\partial N_2}{\partial x} \frac{\partial N_2}{\partial x}\, dx + \int_{x_2}^{x_3} \frac{\partial N_2}{\partial x} \frac{\partial N_2}{\partial x}\, dx \tag{3.1-17}$$

In the element between x_1 and x_2

$$N_2 = (x - x_1)/(x_2 - x_1) = (x - x_1)/l \tag{3.1-18}$$

$$\frac{\partial N_2}{\partial x} = \frac{\partial\,(x - x_1)/l}{\partial x} = \frac{1}{l} \tag{3.1-19}$$

In the element between x_2 and x_3

$$N_2 = (x_3 - x)/(x_3 - x_2) = (x_3 - x)/l \tag{3.1-20}$$

$$\frac{\partial N_2}{\partial x} = \frac{\partial\,(x_3 - x)/l}{\partial x} = -\frac{1}{l} \tag{3.1-21}$$

Therefore

$$\int_{x_1}^{x_3} \frac{\partial N_2}{\partial x} \frac{\partial N_2}{\partial x}\, dx = \int_{x_1}^{x_2} \left(\frac{1}{l}\right)\left(\frac{1}{l}\right) dx + \int_{x_2}^{x_3} \left(\frac{-1}{l}\right)\left(\frac{-1}{l}\right) dx =$$

$$\frac{(x_2 - x_1)}{1^2} + \frac{(x_3 - x_2)}{1^2} = \frac{21}{1^2} = \frac{2}{1} \qquad (3.1\text{-}22)$$

Evaluated in this manner, Eq. (3.1-12) becomes

$$\frac{K}{1}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} \qquad (3.1\text{-}23)$$

Inserting the linear approximation model for T, the second term on the left side of Eq. (3.1-3) becomes

$$\int_{x_1}^{x_3} \rho\, Cp\, \frac{\partial\left(\sum N_i T_i\right)}{\partial t}\, N_j\, dx \quad i = 1,2,3; \quad j = 1,2,3 \qquad (3.1\text{-}24)$$

For j = 1, Eq. (3.1-24) becomes

$$\rho Cp \int_{x_1}^{x_3} N_1 N_1\, dx\, \frac{\partial T_1}{\partial t} + \rho Cp \int_{x_1}^{x_3} N_2 N_1\, dx\, \frac{\partial T_2}{\partial t} +$$

$$\rho Cp \int_{x_1}^{x_3} N_3 N_1\, dx\, \frac{\partial T_3}{\partial t} \qquad (3.1\text{-}25)$$

Similar expressions may be written for j = 2 and j = 3 and assembled to produce the matrix and column vector

$$\rho\, Cp \int_{x_1}^{x_3} \begin{bmatrix} N_1N_1 & N_2N_1 & N_3N_1 \\ N_1N_2 & N_2N_2 & N_3N_2 \\ N_1N_3 & N_2N_3 & N_3N_3 \end{bmatrix} dx \begin{bmatrix} \frac{\partial T_1}{\partial t} \\ \frac{\partial T_2}{\partial t} \\ \frac{\partial T_3}{\partial t} \end{bmatrix} \qquad (3.1\text{-}26)$$

The elements of the matrix in Eq. (3.1-26) are evaluated as before.

$$\int_{x_1}^{x_3} N_1N_1\ dx = \int_{x_1}^{x_2} \left(\frac{x_2-x}{x_2-x_1}\right)^2 dx = \frac{-1}{1^2}\ \frac{(x_2-x)^3}{3}\Bigg|_{x_1}^{x_2}$$

$$= \frac{1}{1^2}\ \frac{1^3}{3} = \frac{1}{3} \qquad (3.1\text{-}27)$$

Since $N_1 + N_2 = 1$ within the first finite element;

$$\int_{x_1}^{x_3} N_1N_2\ dx = \int_{x_1}^{x_2} N_1\ (1-N_1)\ dx = \int_{x_1}^{x_2} N_1\ dx - \int_{x_1}^{x_2} N_1N_1\ dx \qquad (3.1\text{-}28)$$

$$\int_{x_1}^{x_2} N_1\ dx = \int_{x_1}^{x_2}\left(\frac{x_2-x}{x_2-x_1}\right) dx = \frac{1}{1}\left(x_2x-\frac{x^2}{2}\right)\Bigg|_{x_1}^{x_2} =$$

$$\frac{1}{1}\left(\frac{x_2}{2} - \frac{x_1}{2}\right)(x_2-x_1) = \frac{1}{2} \qquad (3.1\text{-}29)$$

Therefore,

$$\int_{x_1}^{x_3} N_1N_2\ dx = \frac{1}{2} - \frac{1}{3} = \frac{1}{6} \qquad (3.1\text{-}30)$$

Similarly,

$$\int_{x_1}^{x_3} N_2N_2\ dx = \int_{x_1}^{x_2}\left(\frac{x-x_1}{x_2-x_1}\right)^2 dx + \int_{x_2}^{x_3}\left(\frac{x_3-x}{x_3-x_2}\right)^2 dx =$$

$$\frac{1}{1^2}\ \frac{(x-x_1)^3}{3}\Bigg|_{x_1}^{x_2} + \frac{-1}{1^2}\ \frac{(x_3-x)^3}{3}\Bigg|_{x_2}^{x_3} = \frac{1}{1^2}\ \frac{1^3}{3} + \frac{1}{1^2}\ \frac{1^3}{3} =$$

$$\frac{2\ 1}{3} = \frac{4\ 1}{6} \qquad (3.1\text{-}31)$$

After evaluating all the elements of the matrix, Eq. (3.1-26) then becomes

$$\frac{\rho \, Cp \, l}{6} \begin{bmatrix} 2 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} \frac{\partial T_1}{\partial t} \\ \frac{\partial T_2}{\partial t} \\ \frac{\partial T_3}{\partial t} \end{bmatrix} \qquad (3.1\text{-}32)$$

Inserting Eqs. (3.1-7), (3.1-23), and (3.1-32) in Eq. (3.1-3) gives the assembled finite element equations for the axial dimension in terms of the unknown axial nodal temperatures and their time derivatives:

$$\begin{bmatrix} -K \left.\frac{\partial T}{\partial x}\right|_{x_1} \\ 0 \\ K \left.\frac{\partial T}{\partial x}\right|_{x_3} \end{bmatrix} - \frac{K}{l} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} \quad -$$

$$\frac{\rho\, Cp\ l}{6} \begin{bmatrix} 2 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} \frac{\partial T_1}{\partial t} \\ \frac{\partial T_2}{\partial t} \\ \frac{\partial T_3}{\partial t} \end{bmatrix} = 0 \qquad (3.1\text{-}33)$$

or, in simplified finite element method notation,

$$\left[RX\right] - \left[A\right] * \left[TX\right] - \left[B\right] * \left[\dot{T}\right] = 0 \qquad (3.1\text{-}34)$$

where

$\left[RX\right]$ = column vector of axial Neumann boundary conditions

$\left[A\right]$ = assemblage matrix of axial element conductive properties

$\left[B\right]$ = assemblage matrix of axial element capacitance properties

$\left[TX\right]$ = column vector of unknown axial nodal temperatures

$\left[\dot{T}\right]$ = column vector of time derivatives of unknown axial nodal temperatures

Eq. (3.1-34) may be rearranged to

$$\left[A\right] * \left[TX\right] + \left[B\right] * \left[\dot{T}\right] = \left[RX\right] \qquad (3.1\text{-}35)$$

This expression is the mathematical statement of Galerkin's method of weighted residuals applied to a one-dimensional unsteady-state heat conduction problem in rectangular coordinates.

To solve Eq. (3.1-35) for the nodal temperatures, the time derivatives of the nodal temperatures must be expressed as a function of the nodal temperatures. In finite element analysis this is usually done with a finite difference approximation. Eq. (3.1-35) can be expressed in finite difference form as

$$[A] * \left(\theta [TX]_{t+\Delta t} + (1-\theta) \; [TX]_t \right) +$$
$$[B] * \left(\theta [\dot{T}]_{t+\Delta t} + (1-\theta) \, [\dot{T}]_t \right) = [RX]_{t+\Delta t} \quad (3.1\text{-}36)$$

If $\theta = \frac{1}{2}$, the Crank-Nicholson method is obtained. The time derivatives in Eq. (3.1-36) may be approximated by

$$\frac{1}{2} \left([\dot{T}]_{t+\Delta t} + [\dot{T}]_t \right) = \frac{[TX]_{t+\Delta t} - [TX]_t}{\Delta t} \quad (3.1\text{-}37)$$

Eq. (3.1-36) then becomes

$$\left([A] + \frac{2}{\Delta t} [B] \right) * [TX]_{t+\Delta t} = 2 \; [RX]_{t+\Delta t} -$$
$$\left([A] - \frac{2}{\Delta t} [B] \right) * [TX]_t \quad (3.1\text{-}38)$$

This equation relates the nodal temperatures at time $t + \Delta t$ to the nodal temperatures at t and the boundary conditions of the problem.

3.2 Radial Dimension

The development of the finite element equations for unsteady-state heat conduction in the radial direction in a cylindrical coordinate system proceeds in a similar fashion as that in the axial dimension. The governing differential equation is

$$K\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)\right] - \rho Cp\frac{\partial T}{\partial t} = 0 \qquad (3.2\text{-}1)$$

After rearranging and performing the differentiation, Eq. (3.2-1) becomes

$$K\frac{\partial T}{\partial r} + Kr\frac{\partial^2 T}{\partial r^2} - \rho Cp\, r\frac{\partial T}{\partial t} = 0 \qquad (3.2\text{-}2)$$

Application of Galerkin's method of weighted residuals to minimize the residual error over the domain of the problem in the radial dimension yields

$$\int_{r_1}^{r_3}\left(K\frac{\partial T}{\partial r} + Kr\frac{\partial^2 T}{\partial r^2} - \rho\, Cpr\frac{\partial T}{\partial t}\right)N_j\, dr = 0 \qquad (3.2\text{-}3)$$

where $j = 1,2,3$ and $T = \sum_{i=1}^{3} N_i T_i$. Here the interpolation function N_i and unknown nodal temperatures T_i apply to finite elements in the radial dimension from $r = 0$ to $r = R$, the radius of the cylinder; and are distinct from those in the axial dimension.

The second term in the integrand of Eq. (3.2-3) may be integrated by parts,

$$\int_{r_1}^{r_3} u \, dv = u \, v \Big|_{r_1}^{r_3} - \int_{r_1}^{r_3} v \, du \tag{3.2-4}$$

with

$$u = r \, N_j \tag{3.2-5}$$

and

$$dv = \frac{\partial^2 T}{\partial \, r^2} \; dr \tag{3.2-6}$$

Therefore,

$$\frac{\partial u}{\partial r} = N_j + r \, \frac{\partial N_j}{\partial r} \tag{3.2-7}$$

$$v = \frac{\partial T}{\partial r} \tag{3.2-8}$$

and Eq. (3.2-4) becomes

$$\int_{r_1}^{r_3} Kr \; \frac{\partial^2 T}{\partial r^2} \; N_j \, dr =$$

$$KrN_j \; \frac{\partial T}{\partial r}\Bigg|_{r_1}^{r_3} - \int_{r_1}^{r_3} K \, \frac{\partial T}{\partial r} \left(N_j + r \, \frac{\partial N_j}{\partial r} \right) dr =$$

$$KrN_j \; \frac{\partial T}{\partial r}\Bigg|_{r_1}^{r_3} - \int_{r_1}^{r_3} K \; \frac{\partial T}{\partial r} \; N_j \, dr - \int_{r_1}^{r_3} Kr \; \frac{\partial T}{\partial r} \; \frac{\partial N_j}{\partial r} \; dr \tag{3.2-9}$$

It is apparent that the second term on the right side of Eq. (3.2-9) cancels out the first term of Eq. (3.2-3).

The first term on the right side of Eq. (3.2-9) may be expressed as the column vector

$$\begin{bmatrix} KrN_1 \left.\dfrac{\partial T}{\partial r}\right|_0^R \\ \\ KrN_2 \left.\dfrac{\partial T}{\partial r}\right|_0^R \\ \\ KrN_3 \left.\dfrac{\partial T}{\partial r}\right|_0^R \end{bmatrix} \qquad (3.2\text{-}10)$$

where $r_1 = 0$ and $r_3 = R$. The first element of Eq. (3.2-10) equals zero since N_1 is undefined and equals zero at $r = R$. Similarly, the second element of Eq. (3.2-10) equals zero since N_2 is undefined and equals zero at $r = R$ and $r = 0$. With $N_3 = 1$ at $r = R$, Eq. (3.2-10) reduces to

$$\begin{bmatrix} 0 \\ \\ 0 \\ \\ KR \left.\dfrac{\partial T}{\partial r}\right|_R \end{bmatrix} \qquad (3.2\text{-}11)$$

Neumann boundary conditions may be specified by heat fluxes at the surface of the cylinder with Eq. (3.2-11).

To evaluate the third term on the right side of Eq. (3.2-9), the linear approximation model for T is substituted

$$\int_{r_1}^{r_3} Kr \frac{\partial\left(\sum N_i T_i\right)}{\partial r} \; \frac{\partial N_j}{\partial r} \, dr \quad i = 1,2,3; \quad j = 1,2,3 \qquad (3.2\text{-}12)$$

Three equations, each in terms of the three unknown nodal temperatures, result after expanding and integrating Eq. (3.2-12). For $j = 1$,

$$\int_{r_1}^{r_3} Kr \frac{\partial N_1}{\partial r} \frac{\partial N_1}{\partial r} \, dr \; T_1 + \int_{r_1}^{r_3} Kr \frac{\partial N_2}{\partial r} \frac{\partial N_1}{\partial r} \, dr \; T_2 +$$

$$\int_{r_1}^{r_3} Kr \frac{\partial N_3}{\partial r} \frac{\partial N_1}{\partial r} \, dr \; T_3 \qquad (3.2\text{-}13)$$

reduces to

$$\int_{r_1}^{r_2} Kr \left(\frac{-1}{l}\right)\left(\frac{-1}{l}\right) dr \; T_1 + \int_{r_1}^{r_2} Kr \left(\frac{1}{l}\right)\left(\frac{-1}{l}\right) dr \; T_2 \qquad (3.2\text{-}14)$$

where l is the length of the finite element. The limits of integration of the first and second terms in Eq. (3.2-14) are r_2 and r_1 since N_1 is undefined and equals zero from r_2 to r_3. The third term of Eq. (3.2-13) equals zero since nowhere from r_1 to r_3 are N_1 and N_3 simultaneously non-zero. Eq. (3.2-14) is integrated to

$$\frac{K}{l^2} \left(\frac{r_2^2 - r_1^2}{2}\right) T_1 - \frac{K}{l^2} \left(\frac{r_2^2 - r_1^2}{2}\right) T_2 \qquad (3.2\text{-}15)$$

In like manner, Eq. (3.2-12) becomes for $j = 2$

$$\frac{-K}{l^2}\left(\frac{r_2^2 - r_1^2}{2}\right)T_1 + \frac{K}{l^2}\left(\frac{r_2^2 - r_1^2}{2}\right)T_2 +$$

$$\frac{K}{l^2}\left(\frac{r_3^2 - r_2^2}{2}\right)T_2 - \frac{K}{l^2}\left(\frac{r_3^2 - r_2^2}{2}\right)T_3 \qquad (3.2\text{-}16)$$

and for j = 3

$$\frac{-K}{l^2}\left(\frac{r_3^2 - r_2^2}{2}\right)T_2 + \frac{K}{l^2}\left(\frac{r_3^2 - r_2^2}{2}\right)T_3 \qquad (3.2\text{-}17)$$

Eq. (3.2-12) can therefore be written in matrix notation by assembling Eqs. (3.2-15), (3.2-16), and (3.2-17) as

$$\frac{K}{l^2}\begin{bmatrix} \left(\frac{r_2^2 - r_1^2}{2}\right) & \left(\frac{r_2^2 - r_1^2}{2}\right) & 0 \\ -\left(\frac{r_2^2 - r_1^2}{2}\right) & \left(\frac{r_2^2 - r_1^2}{2} + \frac{r_3^2 - r_2^2}{2}\right) & -\left(\frac{r_3^2 - r_2^2}{2}\right) \\ 0 & -\left(\frac{r_3^2 - r_2^2}{2}\right) & \left(\frac{r_3^2 - r_2^2}{2}\right) \end{bmatrix}\begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} \qquad (3.2\text{-}18)$$

The third term in the integrand of Eq. (3.2-3) is now evaluated. The linear approximation model for T is inserted to yield

$$\int_{r_1}^{r_3} \rho\, Cpr \;\frac{\partial\left(\sum N_i T_i\right)}{\partial t}\; N_j\, dr \quad i = 1,2,3; \quad j = 1,2,3 \qquad (3.2\text{-}19)$$

Expanding Eq. (3.2-19) results in three equations with the time derivatives of the nodal temperatures as unknowns. Eq. (3.2-19) may be written in matrix notation as

$$\rho \, Cp \int_{r_1}^{r_3} \begin{bmatrix} N_1N_1r & N_2N_1r & N_3N_1r \\ N_1N_2r & N_2N_2r & N_2N_2r \\ N_1N_3r & N_2N_3r & N_3N_3r \end{bmatrix} dr \quad \begin{bmatrix} \dfrac{\partial T_1}{\partial t} \\ \dfrac{\partial T_2}{\partial t} \\ \dfrac{\partial T_3}{\partial t} \end{bmatrix} \qquad (3.2\text{-}20)$$

The elements of the matrix in Eq. (3.2-20) are evaluated individually. For example,

$$\int_{r_1}^{r_3} N_1N_1r \; dr = \int_{r_1}^{r_2} \left(\frac{r_2 - r}{1}\right)\left(\frac{r_2 - r}{1}\right) r \; dr =$$

$$\frac{1}{1^2} \int_{r_1}^{r_2} r_2^2 \, r - 2r_2r^2 + r^3 \; dr = \frac{1}{1^2} \left(\frac{r_2^2r^2}{2} - \frac{2r_2r^3}{3} + \frac{r^4}{4}\right)\Bigg|_{r_1}^{r_2} =$$

$$\frac{1}{1^2}\left[\frac{r_2^2}{2}\left(r_2^2 - r_1^2\right) - \frac{2r_2}{3}\left(r_2^3 - r_1^3\right) + \frac{1}{4}\left(r_2^4 - r_1^4\right)\right] \qquad (3.2\text{-}21)$$

$$\int_{r_1}^{r_3} N_2N_2r \; dr =$$

$$\int_{r_1}^{r_2} \left(\frac{r - r_1}{1}\right)\left(\frac{r - r_1}{1}\right) r \; dr + \int_{r_2}^{r_3} \left(\frac{r_3 - r}{1}\right)\left(\frac{r_3 - r}{1}\right) r \; dr =$$

$$\int_{r_1}^{r_2} \left(\frac{r_1 - r}{l}\right)\left(\frac{r_1 - r}{l}\right) r\, dr + \int_{r_2}^{r_3} \left(\frac{r_3 - r}{l}\right)\left(\frac{r_3 - r}{l}\right) r\, dr =$$

$$\frac{1}{l^2}\left[\frac{r_1^2}{2}\left(r_2^2 - r_1^2\right) - \frac{2r_1}{3}\left(r_2^3 - r_1^3\right) + \frac{1}{4}\left(r_2^4 - r_1^4\right)\right] +$$

$$\frac{1}{l^2}\left[\frac{r_3^2}{2}\left(r_3^2 - r_2^2\right) - \frac{2r_3}{3}\left(r_3^3 - r_2^3\right) + \frac{1}{4}\left(r_3^4 - r_2^4\right)\right]$$

(3.2-22)

$$\int_{r_1}^{r_3} N_1N_2 r\, dr = \int_{r_1}^{r_2} \left(\frac{r_2 - r}{l}\right)\left(\frac{r - r_1}{l}\right) r\, dr =$$

$$-\int_{r_1}^{r_2} \left(\frac{r_2 - r}{l}\right)\left(\frac{r_1 - r}{l}\right) r\, dr =$$

$$\frac{-1}{l^2}\int_{r_1}^{r_2} r_2 r_1 r - r_1 r^2 - r_2 r^2 + r^3\, dr =$$

$$\frac{-1}{l^2}\left[\frac{r_2 r_1}{2}\left(r_2^2 - r_1^2\right) - \frac{r_1 + r_2}{3}\left(r_2^3 - r_1^3\right) + \frac{1}{4}\left(r_2^4 - r_1^4\right)\right]$$

(3.2-23)

$$\int_{r_1}^{r_3} N_2N_3 r\, dr = \int_{r_2}^{r_3} \left(\frac{r_3 - r}{l}\right)\left(\frac{r - r_2}{l}\right) r\, dr =$$

$$-\int_{r_2}^{r_3} \left(\frac{r_3 - r}{l}\right)\left(\frac{r_2 - r}{l}\right) r\, dr = \frac{-1}{l^2}\int_{r_2}^{r_3} r_3 r_2 r - r_2 r^2 - r_3 r^2 + r^3\, dr$$

$$= \frac{-1}{l^2}\left[\frac{r_3 r_2}{2}\left(r_3^2 - r_2^2\right) - \frac{r_2 + r_3}{3}\left(r_3^3 - r_2^3\right) + \frac{1}{4}\left(r_3^4 - r_2^4\right)\right]$$

(3.2-24)

Eqs. (3.2-21), (3.2-22), (3.2-23), and (3.2-24) are all the same function of, at most, three nodal coordinates. With specification of the nodal coordinates, the function is easily computed. For simplicity of notation the function will be denoted F_{ij} where i and j refer to the interpolation functions in the integrals of Eq. (3.2-20). Therefore Eq. (3.2-20) becomes

$$\frac{\rho C_p}{l^2} \begin{bmatrix} F_{11} & -F_{21} & 0 \\ -F_{12} & F_{22} & -F_{32} \\ 0 & -F_{23} & F_{33} \end{bmatrix} \begin{bmatrix} \frac{\partial T_1}{\partial t} \\ \frac{\partial T_2}{\partial t} \\ \frac{\partial T_3}{\partial t} \end{bmatrix} \qquad (3.2\text{-}25)$$

Inserting Eqs. (3.2-11), (3.2-18), and (3.2-25) into Eq. (3.2-3) gives the assembled finite element equations for the radial dimension in terms of the unknown radial nodal temperatures and their derivatives:

$$
\begin{bmatrix} 0 \\ 0 \\ KR \left.\dfrac{\partial T}{\partial r}\right|_R \end{bmatrix}
- \frac{K}{l^2}
\begin{bmatrix}
\left(\dfrac{r_2^2 - r_1^2}{2}\right) & -\left(\dfrac{r_2^2 - r_1^2}{2}\right) & 0 \\
-\left(\dfrac{r_2^2 - r_1^2}{2}\right) & \left(\dfrac{r_2^2 - r_1^2}{2} + \dfrac{r_3^2 - r_2^2}{2}\right) & -\left(\dfrac{r_3^2 - r_2^2}{2}\right) \\
0 & -\left(\dfrac{r_3^2 - r_2^2}{2}\right) & \left(\dfrac{r_3^2 - r_2^2}{2}\right)
\end{bmatrix}
$$

$$
\begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix}
- \frac{\rho C_p}{l^2}
\begin{bmatrix}
F_{11} & -F_{21} & 0 \\
-F_{12} & F_{22} & -F_{32} \\
0 & -F_{23} & F_{33}
\end{bmatrix}
\begin{bmatrix} \dfrac{\partial T_1}{\partial t} \\ \dfrac{\partial T_2}{\partial t} \\ \dfrac{\partial T_3}{\partial t} \end{bmatrix}
= 0 \qquad (3.2\text{-}26)
$$

or, in simplified finite element method notation

$$
[RR] - [R] * [TR] - [S] * [\dot{T}] = 0 \qquad (3.2\text{-}27)
$$

where

$[RR]$ = column vector of radial Neumann boundary conditions

$[R]$ = assemblage matrix of radial element conductive properties

$[S]$ = assemblage matrix of radial element capacitance properties

$[TR]$ = column vector of unknown radial nodal temperatures

$[\dot{T}]$ = column vector of time derivatives of unknown radial nodal temperatures

Eq. (3.2-27) may be rearranged to

$$[R] * [TR] + [S] * [\dot{T}] = [RR] \qquad (3.2\text{-}28)$$

The Crank-Nicholson method for evaluating $[\dot{T}]$ with finite difference approximations may be applied as in the axial dimension. The resulting assembled finite element equations

$$\left([R] + \frac{2}{\Delta t}[S]\right)[TR]_{t+\Delta t} = 2[RR]_{t+\Delta t} - \left([R] - \frac{2}{\Delta t}[S]\right)[TR]_{t} \qquad (3.2\text{-}29)$$

may be solved for nodal temperatures at time $t + \Delta t$ since nodal temperatures at the previous time interval and the boundary conditions are known.

3.3 Initial and Boundary Conditions

In the unsteady-state heat conduction problem under examination, initial conditions are established by specifying the temperature at all axial and radial nodes at time $t \leq 0$. Therefore for a finite cylinder of radius

R and length L at an initial uniform temperature T^0 the initial conditions are

$$T\ (x,t) \;=\; T^0 \qquad 0 \le x \le L \qquad t \le 0 \qquad (3.3\text{-}1)$$

$$T\ (r,t) \;=\; T^0 \qquad 0 \le r \le R \qquad t \le 0 \qquad (3.3\text{-}2)$$

The unknown nodal temperature column vectors of Eqs. (3.1-38) and (3.2-29) for two finite elements in each the axial and radial dimensions become

$$\left[TX\right]_{t=0} = \begin{bmatrix} T^0 \\ T^0 \\ T^0 \end{bmatrix} \qquad (3.3\text{-}3)$$

$$\left[TR\right]_{t=0} = \begin{bmatrix} T^0 \\ T^0 \\ T^0 \end{bmatrix} \qquad (3.3\text{-}4)$$

For a finite cylinder having its entire surface instantaneously changed to temperature T^1 and maintained at this temperature, the Dirichlet boundary conditions may be expressed as

$$T\ (0,t) \;=\; T\ (L,t) \;=\; T^1 \qquad t > 0 \qquad (3.3\text{-}5)$$

$$T\ (R,t) \;=\; T^1 \qquad t > 0 \qquad (3.3\text{-}6)$$

For two finite elements in the axial dimension, Eq. (3.1-38) becomes at time t

$$
\begin{bmatrix}
\left(\frac{K}{l} + \frac{2\rho Cpl}{3\Delta t}\right) & \left(\frac{-K}{l} + \frac{\rho Cpl}{3\Delta t}\right) & 0 \\
\left(\frac{-K}{l} + \frac{\rho Cpl}{3\Delta t}\right) & \left(\frac{2K}{l} + \frac{4\rho Cpl}{3\Delta t}\right) & \left(\frac{-K}{l} + \frac{\rho Cpl}{3\Delta t}\right) \\
0 & \left(\frac{-K}{l} + \frac{\rho Cpl}{3\Delta t}\right) & \left(\frac{K}{l} + \frac{2\rho Cpl}{3\Delta t}\right)
\end{bmatrix}
\begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix}
=
\begin{bmatrix}
-2K \left.\frac{\partial T}{\partial x}\right|_{x_1} \\
0 \\
2K \left.\frac{\partial T}{\partial x}\right|_{x_3}
\end{bmatrix}
-
\begin{bmatrix}
\left(\frac{K}{l} - \frac{2\rho Cpl}{3\Delta t}\right) & \left(\frac{-K}{l} - \frac{\rho Cpl}{3\Delta t}\right) & 0 \\
\left(\frac{-K}{l} - \frac{\rho Cpl}{3\Delta t}\right) & \left(\frac{2K}{l} - \frac{4\rho Cpl}{3\Delta t}\right) & \left(\frac{-K}{l} - \frac{\rho Cpl}{3\Delta t}\right) \\
0 & \left(\frac{-K}{l} - \frac{\rho Cpl}{3\Delta t}\right) & \left(\frac{K}{l} - \frac{2\rho Cpl}{3\Delta t}\right)
\end{bmatrix}
\begin{bmatrix} T^0 \\ T^0 \\ T^0 \end{bmatrix}
\tag{3.3-7}
$$

Dirichlet boundary conditions introduced into the finite element equations by the following modifications of Eq. (3.3-7) :

$$\begin{bmatrix} 1 & 0 & 0 \\ \left(\frac{-K}{l} + \frac{\rho\, Cpl}{3\Delta t}\right) & \left(\frac{2K}{l} + \frac{4\rho\, Cpl}{3\Delta t}\right) & \left(\frac{-K}{l} + \frac{\rho\, Cpl}{3\Delta t}\right) \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} =$$

$$\begin{bmatrix} 2T^1 \\ 0 \\ 2T^1 \end{bmatrix} \quad -$$

$$
\begin{bmatrix}
1 & 0 & 0 \\
\left(\frac{-K}{1} - \frac{\rho\, Cpl}{3\Delta t}\right) & \left(\frac{2K}{1} - \frac{4\rho\, Cpl}{3\Delta t}\right) & \left(\frac{-K}{1} - \frac{\rho\, Cpl}{3\Delta t}\right) \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix} T^1 \\ T^0 \\ T^1 \end{bmatrix}
\qquad (3.3\text{-}8)
$$

The nodal temperature solutions at times $t > 0$ will now yield temperatures at nodes 1 and 3 equal to T^1. This can be seen by solving Eq. (3.3-8) for T_1 and T_3 :

$$T_1 = 2T^1 - T^1 = T^1 \qquad (3.3\text{-}9)$$

$$T_3 = 2T^1 - T^1 = T^1 \qquad (3.3\text{-}10)$$

The finite element equations in the radial dimension are modified analogously for Dirichlet boundary conditions.

3.4 Method of Solution

The solution of the assembled finite element equations for one-dimensional unsteady-state heat conduction, such as Eqs. (3.1-38) and (3.2-29), consists of the temperatures at the nodes at a particular time, t. The assembled finite element equations are appropriately modified for the boundary conditions. The time t is broken down into n time intervals, Δt, such that $t = n\,\Delta t$. Beginning at time

0, solutions are found at each successive time interval until the solution at time $t = n \Delta t$ is obtained. This is done by solving the assembled finite element equations for $[T]_{t+\Delta t}$ where $[T]_t$ is the solution at the previous time interval.

The solution for the three-dimensional unsteady-state heat transfer problem involving a cylinder of finite length is easily obtained by application of Newman's method. In Newman's method, the dimensionaless solution at any point within the three dimensional object is the product of the dimensionaless solutions in each dimension. Assuming no variation of temperature with angular position, the temperature at any point within the cylinder is simply

$$\theta_{x,r} = \theta_x \theta_r \tag{3.4-1}$$

where

$$\theta = \frac{T^1 - T}{T^1 - T^0} \tag{3.4-2}$$

and

T^0 = initial condition, uniform temperature

T^1 = temperature boundary condition

T = nodal temperature at time t

Dimensionaless nodal temperature vectors are directly substituted for $[TX]$ and $[TR]$ in the assembled finite element equations. Thus the temperature at a node with coordinates x and r at time t is obtained by solving Eqs. (3.1-38) and (3.2-29), individually for θ_x and θ_r as defined by Eq. (3.4-2) at time t. For the node of interest, Eq. (3.4-1) is applied to calculate θ_{xr}.

In the solution of both Eqs. (3.1-38) and (3.2-29), a tridiagonal system of linear equations in terms of the unknown nodal temperatures results. The Crout Reduction method, which is an efficient, direct method of solution for tridiagonal linear systmes of equations, may be used to solve for the vector of nodal unknown temperatures.

3.5 Computer Program

A FORTRAN computer program written to calculate the temperature distribution throughout the cylinder is shown in Appendix A. In the program, the thermal conductivity, heat capacity, and density of the material are specified. The cylinder is assumed homogeneous with respect to these properties. The length and radius of the cylinder, and the time at which a solution is desired are specified. The number of finite elements in both the axial and radial dimensions must be set and the length of the time interval for successive solutions in time must be indicated. All matrices and vectors are dimensioned to permit a maximum of 50 finite elements in both the axial and radial dimensions; any lesser number of elements may also be used.

The program constructs the assembled finite element equations, Eqs. (3.1-38) and (3.2-29), for the desired number of nodes. The matrices and vectors for problems with more than two elements are constructed based on the symmetry and form evident in Eqs. (3.1-33) and (3.2-26) which were derived for two elements.

Solutions, in terms of dimensionaless nodal temperatures, are obtained independently in both the axial and radial dimension at successive time intervals by solving the assembled finite element equations using the Crout Reduction method. At each successive time interval, Newman's method is applied using the independent axial and radial solutions to calculate dimensionless temperatures throughout the cylinder.

4.0 RESULTS

4.1 Temperature Profiles

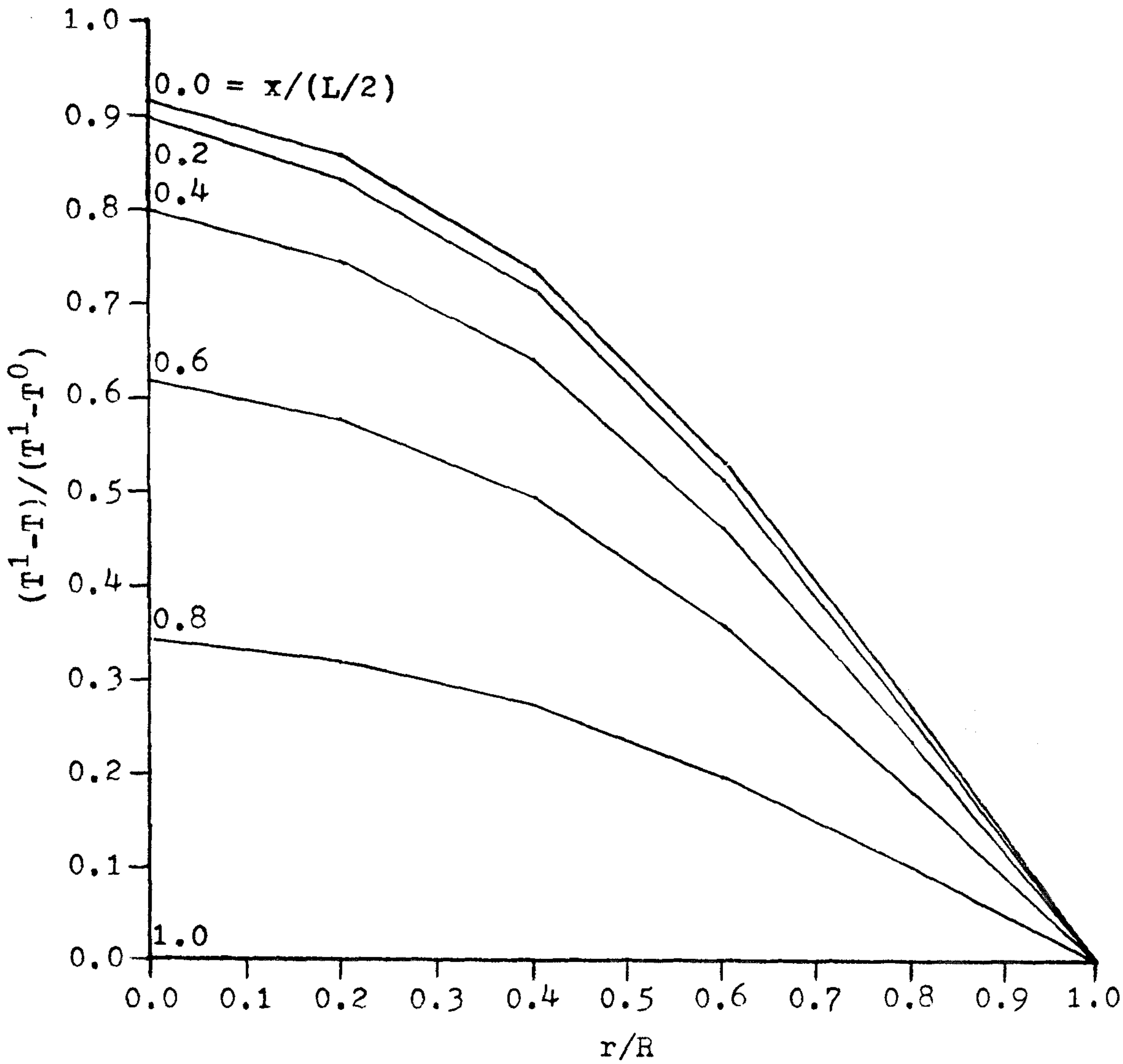
For purposes of illustration, the solution was obtained for an unsteady-state conduction problem for a cylinder with the following parameters: L = 10.0 inches, 10 axial finite elements, R = 5.0 inches, 5 radial finite elements, K = 20.0 Btu-ft/hr-ft^2-^{o}F, ρ = 490.0 lb/ft^3, Cp = 0.12 Btu/lb-^{o}F. The time increment, Δt, at which successive solutions in time was selected as 36.75 seconds. The physical and thermal properties chosen approximately correspond to those of a mild steel.

For both the axial and radial dimension, the value of the ratio $K \Delta t/\rho Cpl^2$, where l is the size of the finite element, is 0.50. The effect of the value of this ratio on the stability and accuracy of solutions obtained by the finite element method developed here is explored in Sections 4.2 and 4.3.

The solutions at successive time increments obtained by the computer program developed are presented in Appendix B. The dimensionaless nodal temperatures are defined as $(T^1 - T)/(T^1 - T^o)$ where T^o is the initial uniform temperature throughout the cylinder and T^1 is the temperature applied to the surface of the cylinder at time t = 0. As the Tables show, the temperatures are symmetric about the axial midpoint of the cylinder, L/2.

Figure 4.1-1 shows the temperature profiles in the cylinder at time t = 147.0 seconds. The parameter of the curves, x/(L/2), is the dimensionaless distance from the axial midpoint of the cylinder at L/2. Figures 4.1-2 through 4.1-6 show the temperature profiles at x/(L/2) = 0.0, 0.2, 0.4, 0.6, and 0.8, respectively, at different times. The curves of each Figure are at times 147.0, 294.0, 441.0, 588.0, and 735.0 seconds corresponding to Fourier numbers in the axial and radial dimensions, $Kt/\rho Cp(L/2)^2$ and $Kt/\rho CpR^2$, of 0.08, 0.16, 0.24, 0.32, and 0.40, respectively.

The breakpoints on these curves are the nodal temperature solutions in the Tables of Appendix B. The straight lines between the nodal temperature solutions are the profiles imposed by the assumption of a linear approximation model for temperature. For better estimates of temperature solutions between nodes, a smooth curve could be drawn through the nodal solutions with a zero slope at r/R = 0.0, the center of the cylinder.



<u>Figure 4.1-1</u>

Temperature Profile at t = 147.0 seconds

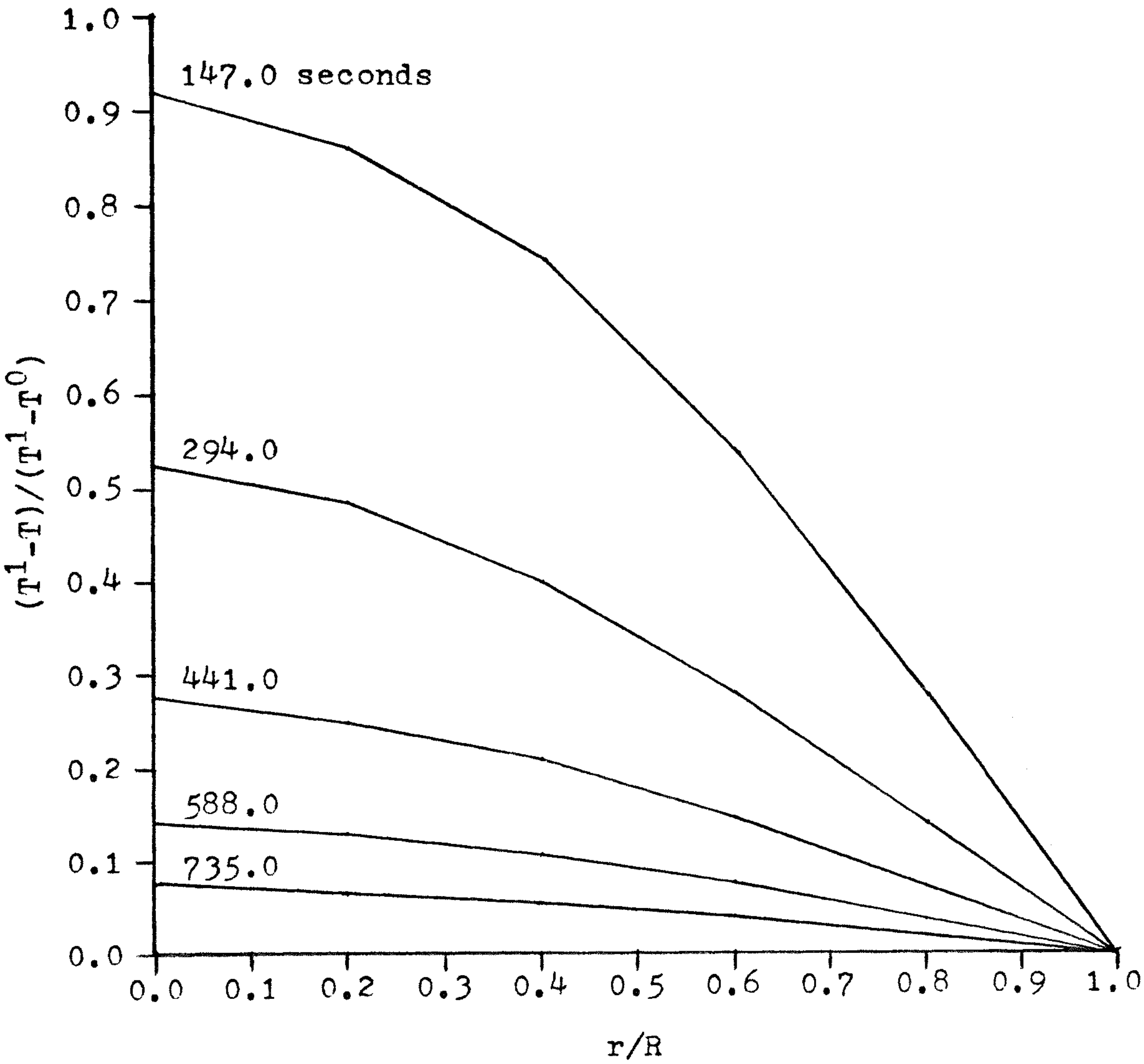


Figure 4.1-2

Temperature Profile at x/(L/2) = 0.0

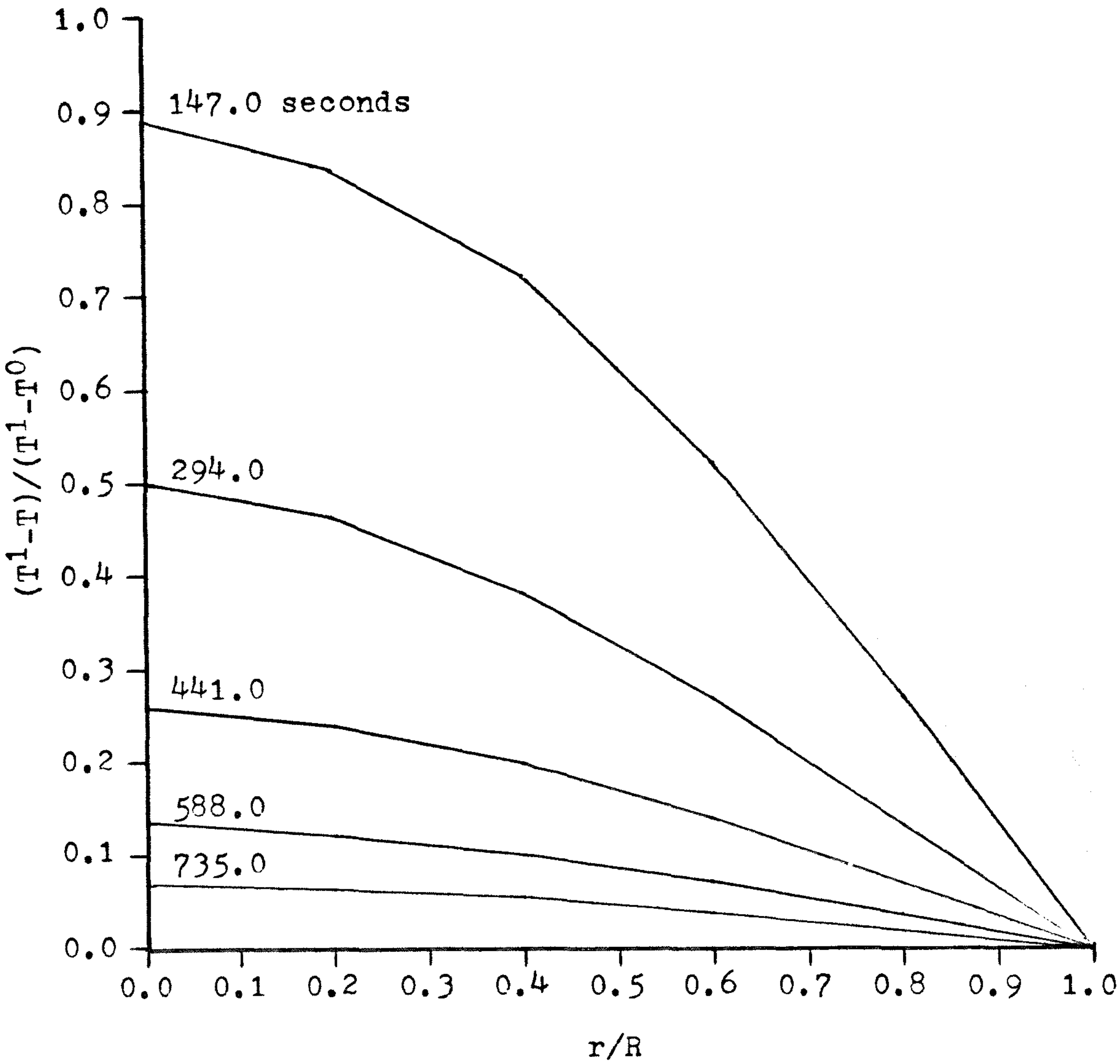


Figure 4.1-3

Temperature Profile at x/(L/2) = 0.2

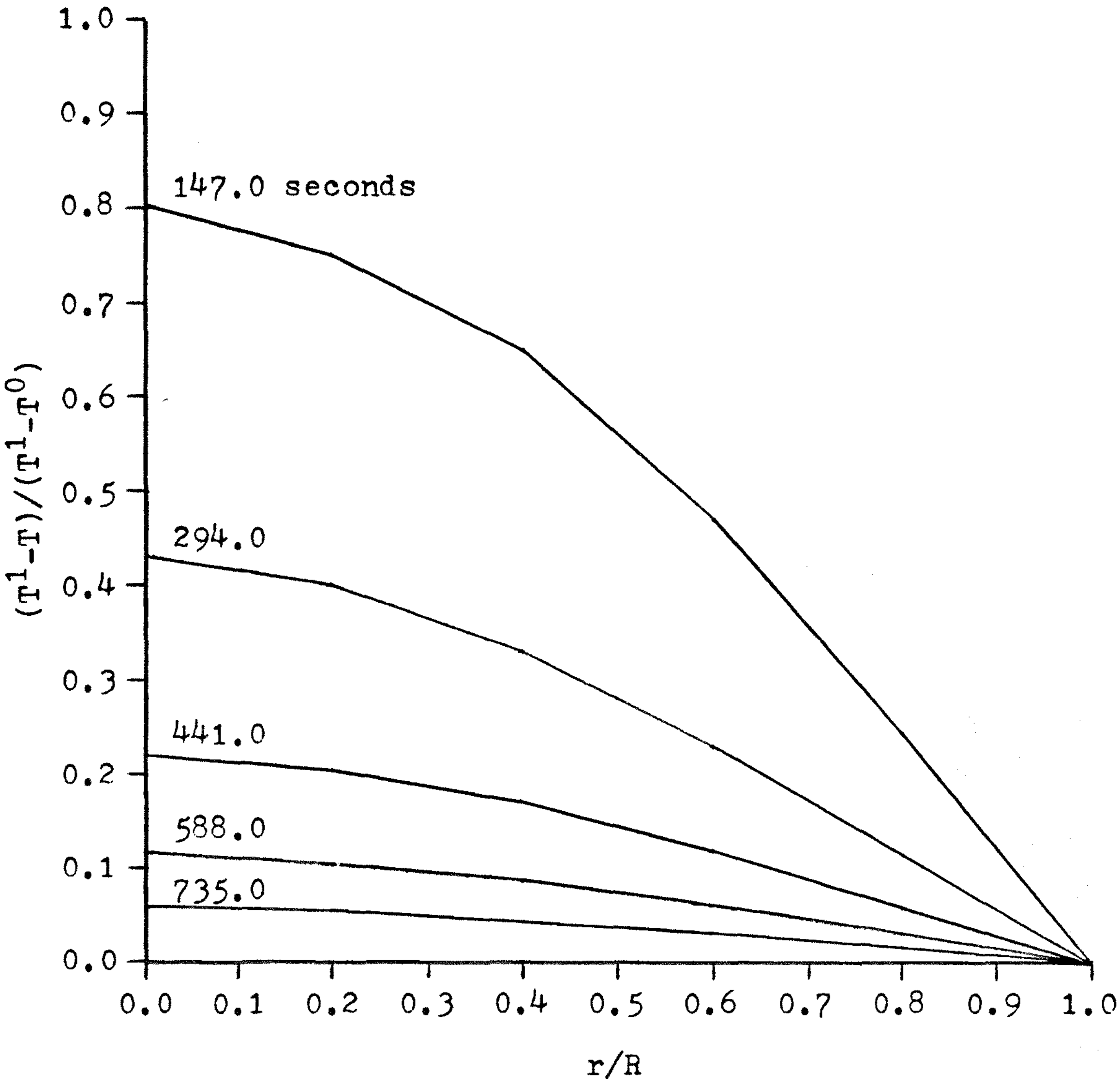
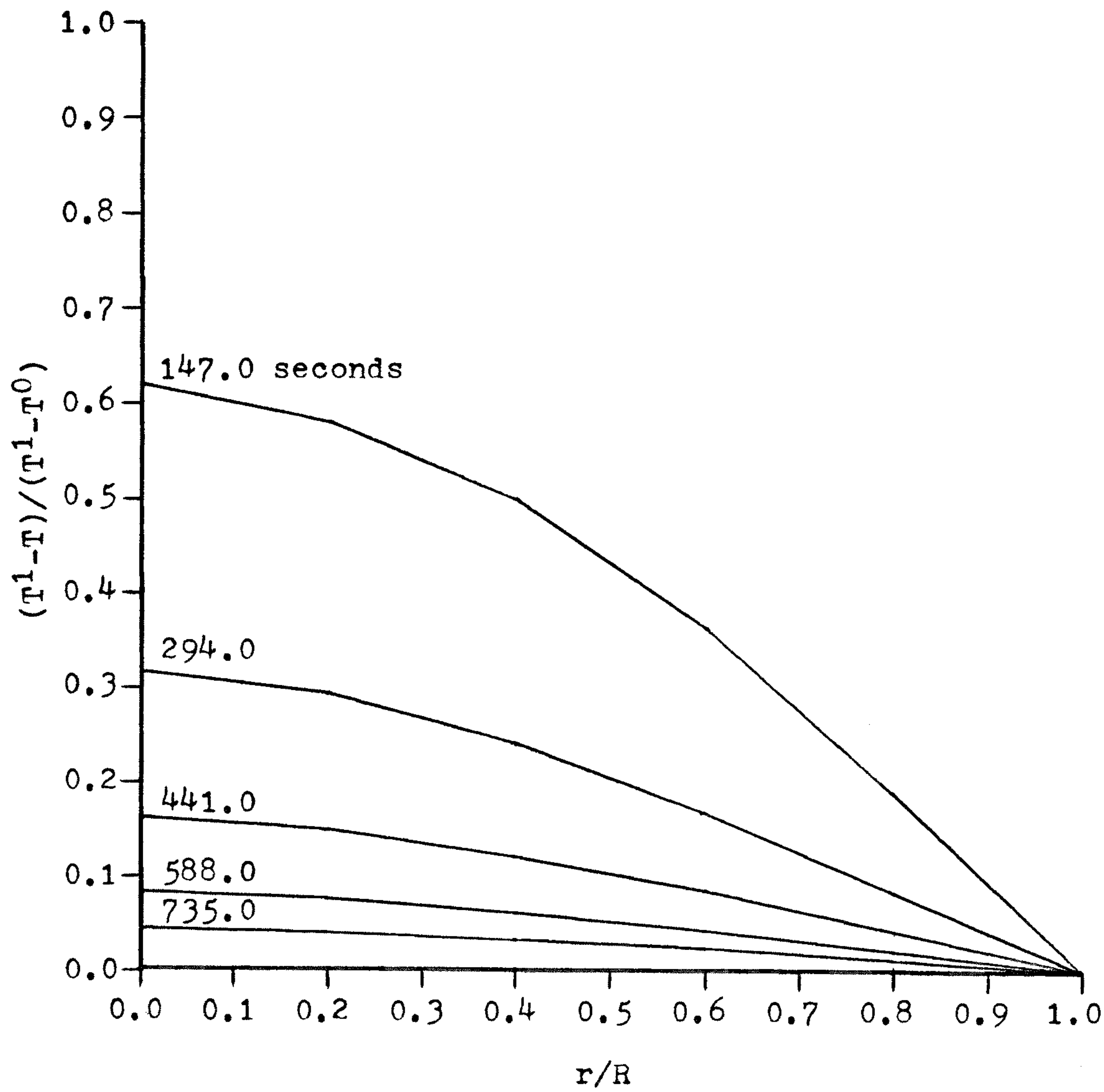


Figure 4.1-4

Temperature Profile at $x/(L/2) = 0.4$



<u>Figure 4.1-5</u>

Temperature Profile at x/(L/2) = 0.6

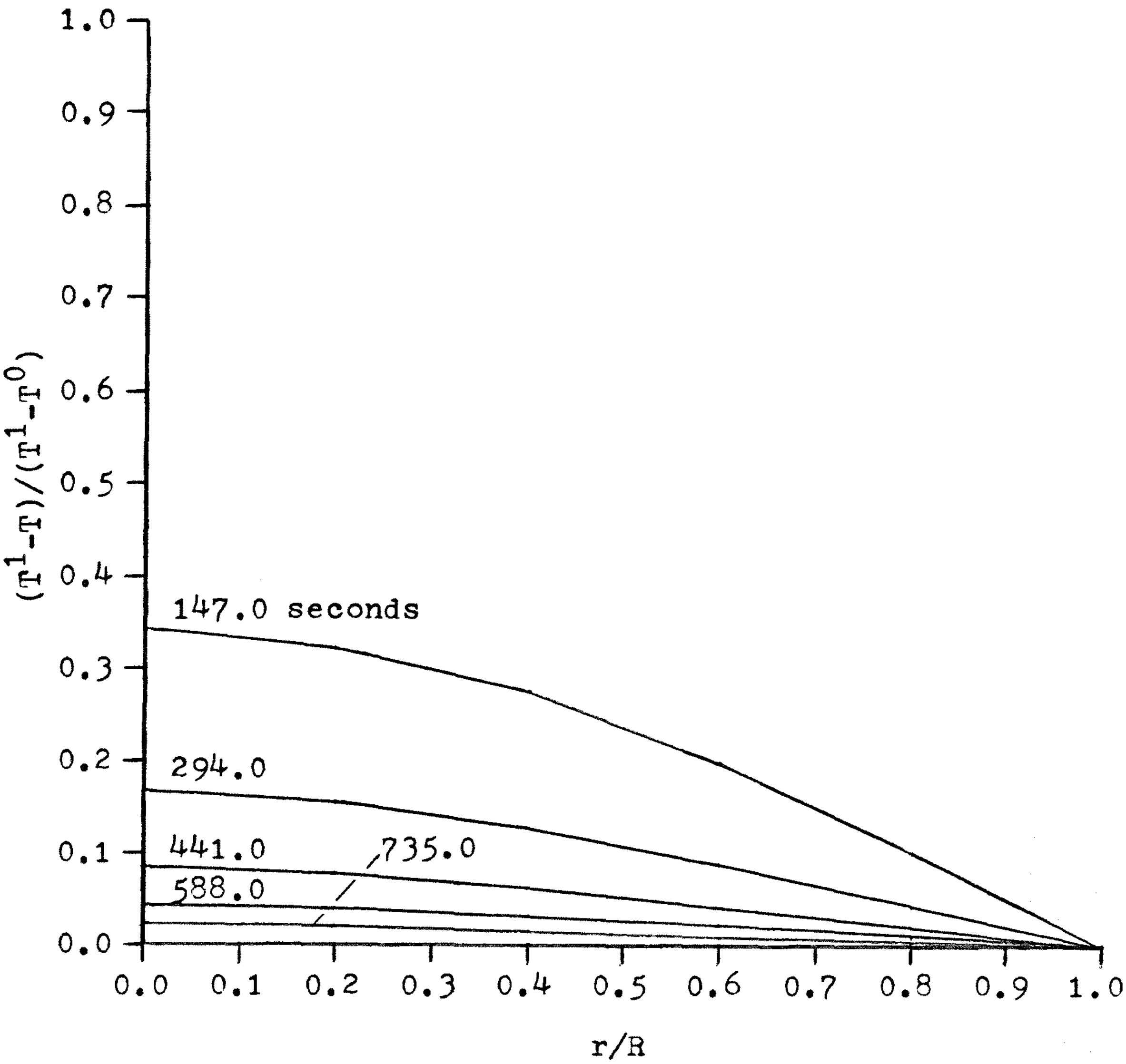


Figure 4.1-6

Temperature Profile at x/(L/2) = 0.8

4.2 Stability of Finite Element Solutions

In the development of computer programs for solution of the finite element equations in both the axial and radial dimensions, physically impossible solutions or solutions with very large errors were frequently obtained. For a cylinder of a given size, physical and thermal properties, the accuracy of the solution was found to be dependent on the length of the finite element, l, and the size of the time increment, Δt, at which successive solutions in time were obtained. As in certain finite difference methods for the solution of partial differential equations, the solutions from the finite element method of analysis are conditionally stable and the stability of the solutions are dependent on the magnitude of the dimensionaless ratio $K\Delta t/\rho Cpl^2$. A well-defined lower bound for $K\Delta t/\rho Cpl^2$ was found to exist. However, a well-defined upper bound for the ratio was not discerned.

Dimensionaless nodal temperature vectors at successive time increments are shown in Table 4.2-1 for a problem in the axial dimension with the following parameters: L = 10.0 inches, 10 finite elements, K = 20.0 Btu-ft/ft^2-hr-^{o}F, ρ = 490.0 lb/ft^3, Cp = 0.12 Btu/lb-^{o}F, t = 20.0 seconds. The corresponding value of $K\Delta t/\rho Cpl^2$ is 0.272. The solutions shown in Table 4.2-1 are for initial conditions and boundary conditions with $T^1>T^0$.

Therefore all physically realistic dimensionaless temperatures, defined by Eq. 3.4-2, should lie between 0 and 1. However, nodal solutions greater than 1, as denoted by an asterisk, were obtained. Dimensionaless solutions greater than 1 are, of course, a physical impossibility corresponding to temperatures less than T^o, the initial temperature. As shown in Table 4.2-1, the dimensionaless solutions greater than 1 move closer to the nodal midpoint with increasing time until they disappear altogether.

The phenomenon noted in Table 4.2-1 was found to occur in all problems where $K\Delta t/\rho Cpl^2$ is less than 1/3. This lower limit was established by varying one parameter in $K\Delta t/\rho Cpl^2$ with all others constant; dimensionaless solutions greater than 1 occurred when the ratio dropped below 1/3. In both the axial and radial dimensions, all parameters in the ratio $K\Delta t/\rho Cpl^2$ exhibited this effect.

Solutions at larger times for the same problem are shown in Table 4.2-2 with time increments $t = 20$ seconds and $t = 30$ seconds and the corresponding ratios $K\Delta t/\rho Cpl^2 = 0.272$ and $K\Delta t/\rho Cpl^2 = 0.408$, respectively. The solutions, at earlier times, with $K\Delta t/\rho Cpl^2 = 0.408$ did not exceed 1. At $t = 300$ seconds, dimensionaless solutions greater than 1 for $K\Delta t/\rho Cpl^2 = 0.272$ have disappeared. After a sufficiently long time, $t = 600$

seconds, the solutions for $K\Delta t/\rho Cpl^2 = 0.272$ approach and become identical to those for $K\Delta t/\rho Cpl^2 = 0.408$.

For solutions at early times, therefore, the dimensionaless ratio $K\Delta t/\rho Cpl^2$ must be greater than 1/3. It is not sufficient to use a small time increment Δt. For early times, the length of the finite element, l, must be adjusted together with the time increment Δt to maintain $K\Delta t/\rho Cpl^2 \geq 1/3$.

Nodal temperature solutions in the axial and radial dimensions at time t = 735.0 seconds for a cylinder with the properties L = 10.0 inches, R = 5.0 inches, K = 20.0 Btu-ft/hr-ft^3-^{o}F, ρ = 490.0 lb/ft^3, and Cp = 0.12 Btu/lb-^{o}F are shown in Tables 4.2-3 and 4.2-4, respectively. In each case, a time increment of 73.5 seconds was used. Solutions were obtained using 10, 20, and 30 finite elements.

In the axial dimension, Table 4.2-3, the dimensionaless coordinates are from the midpoint of the cylinder as the solution is symmetric about the midpoint. For solutions with 10, 20, and 30 finite elements, the ratio $K\Delta t/\rho Cpl^2$ has the values 1.0, 4.0, and 9.0 respectively. It might be expected that increasing the number of finite elements in a problem would increase the accuracy of the solution. However, as shown in Table 4.2-3, the average absolute deviation of the finite element solutions from the analytical solutions, $\left(\sum|T-\bar{T}|\right)/n$, increases from 0.0059 to 0.0127. For large values of $K \Delta t/\rho Cpl^2$, extreme errors occur at the nodes near the surface of the cylinder rendering the solution useless. For the case with $K \Delta t/\rho Cpl^2 = 1.0$, the nodal solution deviations are acceptable.

The same phenomenon is noted in the radial dimension in Table 4.2-4. The average absolute deviations increase from 0.0093 to 0.0173 as the ratio $K \Delta t/\rho Cpl^2$ increases from 4.0 to 36.0. In each case, extreme errors occur at nodes near the surface of the cylinder. For the cases with $K \Delta t/\rho Cpl^2$ values of 16.0 and 36.0, some nodal solutions are negative. Negative dimensionaless temperatures indicate temperatures greater than T^1, the temperature imposed at the surface of the cylinder. Clearly, this is a physical impossibility. Also, the solution for the case with $K \Delta t/\rho Cpl^2 = 4.0$ has unacceptable errors.

The value of the dimensionaless ratio $K \Delta t/\rho Cpl^2$ in the solution of unsteady-state conduction in cylinders is of singular importance in the stability and accuracy of the solutions. The ratio must exceed 1/3 for reasonable and accurate solutions at small times. Extreme errors can occur with large values of $K \Delta t/\rho Cpl^2$. No well-defined upper limit for $K \Delta t/\rho Cpl^2$ was found; however, values of 2.0 and less were found to give useful solutions.

Table 4.2-1

Unstable Solutions - Dimensionaless Temperature Vectors

Axial Dimension

$K\Delta t/\rho\, Cpl^2 = 0.272$

Time, seconds

Node	20	40	60	80	100
1	0.0	0.0	0.0	0.0	0.0
2	0.7098	0.5937	0.5194	0.4680	0.4293
3	1.0095*	0.9234	0.8548	0.7971	0.7492
4	0.9997	1.0054*	0.9822	0.9531	0.9224
5	1.0000	0.9997	1.0022*	0.9966	0.9865
6	1.0000	1.0000	0.9997	1.0015*	0.9993
7	1.0000	0.9997	1.0022*	0.9966	0.9865
8	0.9997	1.0054*	0.9822	0.9531	0.9924
9	1.0095*	0.9234	0.8548	0.7971	0.7492
10	0.7098	0.5937	0.5194	0.4680	0.4293
11	0.0	0.0	0.0	0.0	0.0

Table 4.2-2

Stability of Finite Element Solutions

Axial Dimension

Node	60 seconds $K\Delta t/\rho Cpl^2$ = 0.272	60 seconds $K\Delta t/\rho Cpl^2$ = 0.408	300 seconds $K\Delta t/\rho Cpl^2$ = 0.272	300 seconds $K\Delta t/\rho Cpl^2$ = 0.408	600 seconds $K\Delta t/\rho Cpl^2$ = 0.272	600 seconds $K\Delta t/\rho Cpl^2$ = 0.408
1	0.0	0.0	0.0	0.0	0.0	0.0
2	0.5194	0.5190	0.2662	0.2660	0.1733	0.1733
3	0.8548	0.8484	0.5018	0.5016	0.3295	0.3295
4	0.9822	0.9900	0.6830	0.6829	0.4534	0.4534
5	1.0022*	0.9995	0.7955	0.7956	0.5329	0.5329
6	0.9997	1.0000	0.8335	0.8336	0.5602	0.5602
7	1.0022*	0.9995	0.7955	0.7956	0.5329	0.5329
8	0.9822	0.9900	0.6830	0.6829	0.4534	0.4534
9	0.8548	0.8484	0.5018	0.5016	0.3295	0.3295
10	0.5194	0.5190	0.2662	0.2660	0.1733	0.1733
11	0.0	0.0	0.0	0.0	0.0	0.0

Table 4.2-3

Stability of Finite Element Solutions

Axial Dimension
Solutions at time t = 735 seconds

T = Analytical solution, dimensionaless
$\bar{T}$ = Finite element solution, dimensionaless
NL = Number of finite elements

x/(L/2)	T	NL = 10, $K\Delta t/\rho C p l^2 = 1.0$: $\bar{T}$	NL = 10: $T-\bar{T}$	NL = 20, $K\Delta t/\rho C p l^2 = 4.0$: $\bar{T}$	NL = 20: $T-\bar{T}$	NL = 30, $K\Delta t/\rho C p l^2 = 9.0$: $\bar{T}$	NL = 30: $T-\bar{T}$
0.0	0.4745	0.4670	0.0075	0.4721	0.0024	0.4732	0.0013
0.067	0.4719					0.4706	0.0013
0.100	0.4687			0.4664	0.0023		
0.133	0.4641					0.4629	0.0012
0.200	0.4513	0.4429	0.0084	0.4493	0.0020	0.4502	0.0011
0.267	0.4335					0.4326	0.0009
0.300	0.4228			0.4210	0.0018		
0.333	0.4110					0.4100	0.0010
0.400	0.3839	0.3781	0.0058	0.3815	0.0024	0.3827	0.0012
0.467	0.3527					0.3508	0.0019
0.500	0.3356			0.3323	0.0033		
0.533	0.3176					0.3155	0.0021
0.600	0.2790	0.2735	0.0055	0.2794	-0.0004	0.2786	0.0004
0.667	0.2373					0.2419	-0.0046
0.700	0.2155			0.2249	-0.0094		
0.733	0.1931					0.2008	-0.0077
0.800	0.1467	0.1446	0.0021	0.1162	0.0305	0.1367	0.0100
0.867	0.0987					0.0401	0.0586
0.900	0.0743			0.1034	-0.0291		
0.933	0.0496					0.1473	-0.0977
1.000	0.0	0.0	0.0	0.0	0.0	0.0	0.0
$\left(\sum\lvert T-\bar{T}\rvert\right)/n =$			0.0059		0.0084		0.0127

Table 4.2-4

Stability of Finite Element Solutions

Radial Dimension
Solutions at time t = 735 seconds

$\underline{T}$ = Analytical solution, dimensionaless
$\bar{T}$ = Finite element solution, dimensionaless
NR = Number of finite elements

r/R	$\underline{T}$	NR = 10, $K\Delta t/\rho C p l = 4.0$ $\bar{T}$	$\underline{T}-\bar{T}$	NR = 20, $K\Delta t/\rho C p l = 16.0$ $\bar{T}$	$\underline{T}-\bar{T}$	NR = 30, $K\Delta t/\rho C p l = 36.0$ $\bar{T}$	$\underline{T}-\bar{T}$
0.0	0.1585	0.1563	0.0022	0.1566	0.0019	0.1567	0.0018
0.033	0.1583					0.1564	0.0019
0.050	0.1579			0.1559	0.0020		
0.067	0.1575					0.1556	0.0019
0.100	0.1562	0.1535	0.0027	0.1543	0.0019	0.1544	0.0018
0.133	0.1545					0.1527	0.0018
0.150	0.1534			0.1516	0.0018		
0.167	0.1522					0.1506	0.0016
0.200	0.1495	0.1471	0.0024	0.1478	0.0017	0.1480	0.0015
0.233	0.1463					0.1449	0.0014
0.250	0.1445			0.1430	0.0015		
0.267	0.1426					0.1413	0.0013
0.300	0.1386	0.1362	0.0024	0.1371	0.0015	0.1373	0.0013
0.333	0.1341					0.1327	0.0014
0.350	0.1317			0.1301	0.0016		
0.367	0.1292					0.1277	0.0015
0.400	0.1237	0.1206	0.0031	0.1220	0.0017	0.1222	0.0015
0.433	0.1183					0.1163	0.0020
0.450	0.1152			0.1129	0.0023		
0.467	0.1124					0.1101	0.0013

Table 4.2-4 (contd.)

r/R	T	NR = 10 $\bar{T}$	NR = 10 $T-\bar{T}$	NR = 20 $\bar{T}$	NR = 20 $T-\bar{T}$	NR = 30 $\bar{T}$	NR = 30 $T-\bar{T}$
0.500	0.1059	0.1017	0.0042	0.1033	0.0026	0.1036	0.0023
0.533	0.0997					0.0971	0.0026
0.550	0.0964			0.0938	0.0026		
0.567	0.0930					0.0909	0.0021
0.600	0.0861	0.0867	-0.0006	0.0852	0.0009	0.0852	0.0009
0.633	0.0791					0.0802	-0.0011
0.650	0.0755			0.0783	-0.0028		
0.667	0.0719					0.0758	-0.0039
0.700	0.0646	0.0758	-0.0112	0.0719	-0.0073	0.0713	-0.0067
0.733	0.0573					0.0652	-0.0079
0.750	0.0536			0.0609	-0.0073		
0.767	0.0499					0.0551	-0.0052
0.800	0.0425	0.0083	0.0342	0.0353	0.0072	0.0376	0.0049
0.833	0.0351					0.0106	0.0245
0.850	0.0315			-0.0112	0.0427		
0.867	0.0278					-0.0224	0.0502
0.900	0.0207	0.0504	-0.0297	-0.0383	0.0590	-0.0391	0.0598
0.933	0.0136					0.0223	-0.0087
0.950	0.0101			0.1901	-0.1800		
0.967	0.0067					0.3211	-0.3144
1.000	0.0	0.0	0.0	0.0	0.0	0.0	0.0
	$\left(\sum \lvert T-\bar{T}\rvert\right)/n =$		0.0093		0.0165		0.0173

4.3 Accuracy and Convergence of Finite Element Solutions

Due to the assumption of the linear approximation model for temperature in the development of the finite element equations, the temperature distributions obtained by finite element analysis are not expected to be exact. The accuracy of the finite element solution may be seen by comparing the nodal temperatures in the axial and radial dimensions with the temperatures predicted by the analytical solutions.

In the axial dimension, the analytical solution to this unsteady-state conduction problem is

$$\frac{T^1-T}{T^1-T^o} = 2\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+\frac{1}{2})\pi}\, e^{-(n+\frac{1}{2})^2\pi^2\alpha t/b^2} \cos\left[(n+\tfrac{1}{2})\pi x/b\right] \quad (4.3\text{-}1)$$

where

t = time

$\alpha = K/\rho Cp$

$b = L/2$

x = coordinate in axial dimension measured from midpoint of cylinder, L/2.

Similarly, in the radial dimension, the analytical solution is

$$\frac{T^1-T}{T^1-T^o} = \frac{2}{R}\sum_{i=1}^{\infty}\frac{e^{-a_i^2 Kt/\rho Cp}}{a_i J(a_i R)}\, J_o(a_i r)$$

where

R = radius of cylinder

t = time

J_n = Bessel function of first kind of order n,
$n = 0, 1.$

a_i = i'th root of $J_o(a_i R) = 0$

Finite element solutions in both the axial and radial dimensions were obtained for a cylinder with the following parameters:

$L = 10.0$ inches

$R = 5.0$ inches

$K = 20.0$ Btu-ft/ft^2-hr-^oF

$\rho = 490.0$ lb/ft^3

$Cp = 0.12$ Btu/lb-^oF

Solutions were obtained at 183.75 seconds and 735.0 seconds corresponding to Fourier numbers of 0.1 and 0.4, respectively, with the cylinder discretized into 10, 20, and 30 finite elements in each dimension. In each case, the time increment, Δt, was chosen to maintain $K \Delta t/\rho Cpl^2$ within the identified permissible bounds. For the axial dimension cases, $K \Delta t/\rho Cpl^2 = 0.5$, and for the radial dimension cases, $K \Delta t/\rho Cpl^2 = 2.0$. The results are shown in Tables 4.3-1 through 4.3-4 along with the analytical solutions. In the axial dimension, the dimensionaless coordinates are from the midpoint of the cylinder; nodal temperatures are only shown for one half of the cylinder because of the symmetry about L/2.

By examining Tables 4.3-1 through 4.3-4, the finite element solutions can be seen to be reasonably good approximations of the analytical solutions even for the axial and radial cases with only 10 finite elements. For each case, the nodal deviations, $T-\overline{T}$, and the average nodal deviation, $\left(\sum |T-\overline{T}|\right)/n$, are shown. By comparing the nodal deviations with the analytical temperature profiles, it can be seen that the largest nodal deviations occur in the areas of largest rate of change of slope,

$$\frac{\partial^2 T}{\partial x^2} \quad \text{or} \quad \frac{\partial^2 T}{\partial r^2}.$$

Furthermore, in the areas of the largest rate of change of slope, the nodal deviations decrease with an increasing number of elements. In view of the linear approximation model used in the finite element analysis, this is a reasonable result.

The finite element solutions converge to the analytical solution with an increasing number of finite elements. Since the dimensionaless temperatures are defined as $(T^1-T)/(T^1-T^o)$, the generally positive nodal deviations indicate the finite element method of analysis converges to an upper bound solution in this problem.

For most engineering applications, solutions obtained from 20 finite elements provide adequate accuracy.

Table 4.3-1

Convergence of Finite Element Solutions

Axial Dimension
$Kt/\rho Cp(L/2)^2 = 0.1$

T = Analytical solution, dimensionaless
$\bar{T}$ = Finite element solution, dimensionaless
NL = Number of finite elements

x/(L/2)	T	NL = 10 $\bar{T}$	NL = 10 $T-\bar{T}$	NL = 20 $\bar{T}$	NL = 20 $T-\bar{T}$	NL = 30 $\bar{T}$	NL = 30 $T-\bar{T}$
0.0	0.9493	0.9496	-0.0003	0.9494	-0.0001	0.9493	0.0
0.067	0.9460					0.9460	0.0
0.100	0.9419			0.9418	0.0001		
0.133	0.9361					0.9359	0.0002
0.200	0.9191	0.9169	0.0022	0.9183	0.0008	0.9187	0.0004
0.267	0.8943					0.8937	0.0006
0.300	0.8788			0.8771	0.0017		
0.333	0.8611					0.8602	0.0009
0.400	0.8185	0.8065	0.0120	0.8158	0.0027	0.8173	0.0012
0.467	0.7659					0.7644	0.0015
0.500	0.7357			0.7320	0.0037		
0.533	0.7027					0.7010	0.0017
0.600	0.6286	0.6119	0.0167	0.6244	0.0042	0.6267	0.0019
0.667	0.5438					0.5419	0.0019
0.700	0.4975			0.4935	0.0040		
0.733	0.4489					0.4472	0.0017
0.800	0.3452	0.3317	0.0135	0.3420	0.0032	0.3438	0.0014
0.867	0.2344					0.2333	0.0011
0.900	0.1769			0.1751	0.0018		
0.933	0.1185					0.1179	0.0006
1.000	0.0	0.0	0.0	0.0	0.0	0.0	0.0
	$\left(\sum \lvert T-\bar{T} \rvert\right) /n =$		0.0089		0.0022		0.0010

Table 4.3-2

Convergence of Finite Element Solutions

Axial Dimension
$Kt/\rho\, Cp(L/2)^2 = 0.4$

T = Analytical solution, dimensionaless
$\bar{T}$ = Finite element solution, dimensionaless
NL = Number of finite elements

x/(L/2)	T	NL = 10 $\bar{T}$	NL = 10 $T-\bar{T}$	NL = 20 $\bar{T}$	NL = 20 $T-\bar{T}$	NL = 30 $\bar{T}$	NL = 30 $T-\bar{T}$
0.0	0.4745	0.4667	0.0078	0.4725	0.0020	0.4736	0.0009
0.067	0.4719					0.4710	0.0009
0.100	0.4687			0.4667	0.0020		
0.133	0.4641					0.4633	0.0008
0.200	0.4513	0.4439	0.0074	0.4494	0.0019	0.4504	0.0009
0.267	0.4335					0.4327	0.0008
0.300	0.4228			0.4211	0.0017		
0.333	0.4110					0.4102	0.0008
0.400	0.3839	0.3776	0.0063	0.3823	0.0016	0.3832	0.0007
0.467	0.3527					0.3520	0.0007
0.500	0.3356			0.3342	0.0014		
0.533	0.3176					0.3170	0.0006
0.600	0.2790	0.2743	0.0047	0.2778	0.0012	0.2785	0.0005
0.667	0.2373					0.2369	0.0004
0.700	0.2155			0.2146	0.0009		
0.733	0.1931					0.1927	0.0004
0.800	0.1467	0.1442	0.0025	0.1461	0.0006	0.1464	0.0003
0.867	0.0987					0.0985	0.0002
0.900	0.0743			0.0740	0.0003		
0.933	0.0496					0.0495	0.0001
1.000	0.0	0.0	0.0	0.0	0.0	0.0	0.0
$\left(\sum\lvert T-\bar{T}\rvert\right)/n =$			0.0057		0.0014		0.0006

Table 4.3-3

Convergence of Finite Element Solutions

Radial Dimension
$Kt/\rho CpR^2 = 0.1$

$\underline{T}$ = Analytical solution, dimensionaless
$\bar{T}$ = Finite element solution, dimensionaless
NR = Number of finite elements

r/R	T	NR = 10		NR = 20		NR = 30	
		$\bar{T}$	$T-\bar{T}$	$\bar{T}$	$T-\bar{T}$	$\bar{T}$	$T-\bar{T}$
0.0	0.8485	0.8550	-0.0065	0.8494	-0.0009	0.8489	-0.0004
0.033	0.8474					0.8475	-0.0001
0.050	0.8462			0.8463	-0.0001		
0.067	0.8444					0.8443	0.0001
0.100	0.8392	0.8416	-0.0024	0.8391	0.0001	0.8391	0.0001
0.133	0.8320					0.8318	0.0002
0.150	0.8276			0.8272	0.0004		
0.167	0.8227					0.8224	0.0003
0.200	0.8112	0.8107	0.0005	0.8106	0.0006	0.8109	0.0003
0.233	0.7976					0.7973	0.0003
0.250	0.7900			0.7893	0.0007		
0.267	0.7819					0.7815	0.0004
0.300	0.7640	0.7608	0.0032	0.7630	0.0010	0.7635	0.0005
0.333	0.7438					0.7433	0.0005
0.350	0.7331			0.7317	0.0014		
0.367	0.7214					0.7209	0.0005
0.400	0.6955	0.6929	0.0026	0.6956	-0.0001	0.6963	-0.0008
0.433	0.6671					0.6695	-0.0024
0.450	0.6555			0.6545	0.0011		
0.467	0.6413					0.6406	0.0007

Table 4.3-3 (contd.)

r/R	T	NR = 10 $\bar{T}$	NR = 10 $T-\bar{T}$	NR = 20 $\bar{T}$	NR = 20 $T-\bar{T}$	NR = 30 $\bar{T}$	NR = 30 $T-\bar{T}$
0.500	0.6088	0.6062	0.0026	0.6088	0.0	0.6096	-0.0008
0.533	0.5774					0.5766	0.0008
0.550	0.5601			0.5585	0.0016		
0.567	0.5425					0.5418	0.0007
0.600	0.5058	0.4959	0.0099	0.5042	0.0016	0.5051	0.0007
0.633	0.4675					0.4668	0.0007
0.650	0.4478			0.4463	0.0015		
0.667	0.4277					0.4271	0.0006
0.700	0.3867	0.3664	0.0203	0.3851	0.0016	0.3860	0.0007
0.733	0.3446					0.3440	0.0006
0.750	0.3232			0.3222	0.0010		
0.767	0.3016					0.3011	0.0005
0.800	0.2581	0.2857	-0.0276	0.2563	0.0018	0.2576	0.0005
0.833	0.2142					0.2138	0.0004
0.850	0.1923			0.1922	0.0001		
0.867	0.1703					0.1700	0.0003
0.900	0.1267	0.0934	0.0333	0.1253	0.0014	0.1264	0.0003
0.933	0.0836					0.0834	0.0002
0.950	0.0623			0.0625	-0.0002		
0.967	0.0412					0.0411	0.0001
1.000	0.0	0.0	0.0	0.0	0.0	0.0	0.0
	$\left(\sum\lvert T-\bar{T}\rvert\right)$ /n =		0.0109		0.0009		0.0005

Table 4.3-4

Convergence of Finite Element Solutions

Radial Dimension
$Kt/\rho CpR^2 = 0.4$

T = Analytical solution, dimensionaless
$\bar{T}$ = Finite element solution, dimensionaless
NR = Number of finite elements

r/R	T	NR = 10 $\bar{T}$	NR = 10 $T-\bar{T}$	NR = 20 $\bar{T}$	NR = 20 $T-\bar{T}$	NR = 30 $\bar{T}$	NR = 30 $T-\bar{T}$
0.0	0.1585	0.1578	0.0007	0.1584	0.0001	0.1585	0.0
0.033	0.1583					0.1582	0.0001
0.050	0.1579			0.1577	0.0002		
0.067	0.1575					0.1574	0.0001
0.100	0.1562	0.1547	0.0015	0.1559	0.0003	0.1561	0.0001
0.133	0.1545					0.1543	0.0002
0.150	0.1534			0.1530	0.0004		
0.167	0.1522					0.1520	0.0002
0.200	0.1495	0.1478	0.0017	0.1491	0.0004	0.1493	0.0002
0.233	0.1463					0.1461	0.0002
0.250	0.1445			0.1441	0.0004		
0.267	0.1426					0.1425	0.0001
0.300	0.1386	0.1369	0.0017	0.1382	0.0004	0.1384	0.0002
0.333	0.1341					0.1339	0.0002
0.350	0.1317			0.1313	0.0004		
0.367	0.1292					0.1290	0.0002
0.400	0.1237	0.1222	0.0015	0.1236	0.0001	0.1238	-0.0001
0.433	0.1183					0.1182	0.0001
0.450	0.1152			0.1151	0.0001		
0.467	0.1124					0.1123	0.0001

Table 4.3-4 (contd.)

r/R	T	NR = 10		NR = 20		NR = 30	
		$\bar{T}$	$T-\bar{T}$	$\bar{T}$	$T-\bar{T}$	$\bar{T}$	$T-\bar{T}$
0.500	0.1059	0.1053	0.0006	0.1059	0.0	0.1061	-0.0002
0.533	0.0997					0.0996	0.0001
0.550	0.0964			0.0961	0.0003		
0.567	0.0930					0.0929	0.0001
0.600	0.0861	0.0843	0.0018	0.0859	0.0002	0.0860	0.0001
0.633	0.0791					0.0790	0.0001
0.650	0.0755			0.0753	0.0002		
0.667	0.0719					0.0718	0.0001
0.700	0.0646	0.0647	-0.0001	0.0644	0.0002	0.0645	0.0001
0.733	0.0573					0.0572	0.0001
0.750	0.0536			0.0534	0.0002		
0.767	0.0499					0.0498	0.0001
0.800	0.0425	0.0410	0.0015	0.0423	0.0002	0.0424	0.0001
0.833	0.0351					0.0351	0.0
0.850	0.0315			0.0314	0.0001		
0.867	0.0278					0.0278	0.0
0.900	0.0207	0.0209	-0.0002	0.0206	0.0001	0.0206	0.0001
0.933	0.0136					0.0136	0.0
0.950	0.0101			0.0101	0.0		
0.967	0.0067					0.0067	0.0
1.000	0.0	0.0	0.0	0.0	0.0	0.0	0.0
$\left(\sum \lvert T-\bar{T} \rvert\right)/n$			0.0011		0.0002		0.0001

5.0 CONCLUSIONS

The finite element method of analysis may be easily applied to three-dimensional unsteady-state heat conduction in a cylinder of constant thermal and physical properties in which the cylinder, initially at some temperature T^0 throughout, has its entire surface temperature instantaneously changed to a temperature T^1. The symmetry of the cylinder about an axis through the center and parallel with the curved surface of the cylinder reduces the analysis to a two-dimensional problem. The use of Newman's method reduces the problem still further to two independent one-dimensional problems. A Crank-Nicholson scheme is used for the approximation of time derivatives of temperature.

Solutions for this problem obtained by the finite element method of analysis are conditionally stable. The stability of the solutions in both the axial and radial dimensions is dependent on the value of the ratio $K\Delta t/\rho Cpl^2$ where t is the time increment used in the approximation of time derivatives and l is the length of the finite element. For stability, $K\Delta t/\rho Cpl^2$ must be greater than 1/3 but less than approximately 2.0.

An absolute upper limit for $K\Delta t/\rho Cpl^2$ was not found, but for values greater than 2.0 very large errors result. The magnitude of the errors increase with increasing values of $K\Delta t/\rho Cpl^2$. The large errors occur at nodes near the surface of the cylinder in both the axial and radial dimensions. Physically impossible solutions may result where nodal temperatures are greater than the surface temperature in the case where the cylinder is being heated.

If the value of the ratio $K\Delta t/\rho Cpl^2$ is less than 1/3, errors in solutions at small times result with temperatures in the interior of the cylinder less than the initial uniform temperature of the cylinder in the case where the cylinder is being heated. Again, this is a physically impossible solution. After a sufficiently long time, these deviations disappear and no longer affect the accuracy of the solution.

Solutions for this problem obtained by the finite element method of analysis converge to the analytical solution forming an upper bound limit. Convergence may be obtained by increasing the number of finite elements in the axial and radial dimensions in the formulation of the problem provided the value of the ratio $K\Delta t/\rho Cpl^2$ is maintained within the identified acceptable limits. The greatest deviations of the finite element solutions from the analytical solutions occur where the rate of change of the slope of the temperature profile is large. Solutions with sufficient accuracy for most engineering applications may be obtained with 20 finite elements.

6.0 RECOMMENDATIONS

For a given problem, care must be exercised in the selection of the size of finite elements used and the time increment on which time derivatives are based. Improper selection can result in instabilities and rather large errors.

A useful extension of this work would be the application of the methods of numerical analysis to the formulation and solution of the finite element equations to determine the origin of, and better define, the conditional stability of the method, and to predict bounds for the maximum errors in the solutions.

REFERENCES

Bennet, C. O., and Meyers, J. E., *Momentum, Heat, and Mass Transfer*. New York: McGraw-Hill Book Company, 1962, pp. 279-281.

Bird, R. Byron; Stewart, Warren E.; and Lightfoot, Edwin N., *Transport Phenomena*. New York: John Wiley & Sons, 1960, pp.319, 354-357.

Burden, Richard L.; Faires, J. Douglas; and Reynolds, Albert C., *Numerical Methods*. Boston, Massachusetts: Prindle, Weber, & Schmidt, 1978, pp. 355-359, 508-518.

Desai, Chandrakant S., *Elementary Finite Element Method*. Englewood Cliffs, New Jersey: Prentice Hall, 1979, pp. 35-36, 114-120, 206.

Jahnke, Eugene and Emde, Fritz, *Tables of Functions*. New York: Dover Publications, 1945, pp. 156-163.

Mickley, H. S.; Sherwood, T. K.; and Reed, C. E., *Applied Mathematics in Chemical Engineering*. New York: McGraw-Hill Book Company, 1957, pp. 170-173.

APPENDIX A

Program Listing

```
      DIMENSION A(51,51), AR(51,51), B(51,51), BR(51,51), C(51),
     . D(51), RR(51), RX(51), TX(51), TR(51), R(51,51), S(51,51),
     . Z(51)
      REAL LENGTH, K
      INTEGER T
C    PHYSICAL DATA HAVE THE FOLLOWING UNITS:
C         LENGTH     INCHES
C         RADIUS     INCHES
C         TIME       SECONDS
C         K          BTU*FT/FT2*HR*DEGF
C         RHO        LB/FT3
C         CP         BTU/LB*DEGF
C    THE VARIABLE TIME IS THE TIME INTERVAL OVER WHICH A SOLUTION
C    IS OBTAINED.
      LENGTH = 10.0
      RADIUS = 5.0
      TIME = 735.0
      K = 20.0
      RHO = 490.0
      CP = 0.12
C    NL IS THE NUMBER OF FINITE ELEMENTS IN THE AXIAL DIMENSION.
C    DELTAL IS THE LENGTH OF THE AXIAL FINITE ELEMENTS
      NL = 10
      DELTAL = LENGTH/NL
C    NR IS THE NUMBER OF FINITE ELEMENTS IN THE RADIAL DIMENSION.
C    DELTAR IS THE LENGTH OF THE RADIAL FINITE ELEMENTS.
      NR = 5
      DELTAR = RADIUS/NR
C    THE VARIABLE NT IS THE DESIRED NUMBER OF TIME INCREMENTS IN
C    THE TIME INTERVAL.
C    THE VARIABLE DELTAT IS THE LENGTH OF THE TIME INCREMENTS.
C    A SOLUTION IS OBTAINED AT EACH SUCCESSIVE TIME INCREMENT.
      NT = 20
      DELTAT = TIME/NT
```

```
C   ALPHA AND BETA ARE THE CONSTANTS ASSOCIATED WITH THE ASSEMBLAGE
C   PROPERTY MATRICES IN THE AXIAL DIMENSION.
      ALPHA = K*12.0/DELTAL
      BETA = RHO*CP*DELTAL*100.0/DELTAT
C   GAMMA AND EPSILO ARE THE CONSTANTS ASSOCIATED WITH THE ASSEMBLAGE
C   PROPERTY MATRICES IN THE RADIAL DIMENSION.
      GAMMA = K*12.0/(DELTAR*DELTAR)
      EPSILO = RHO*CP*600.0/(DELTAR*DELTAR*DELTAT)
C   TO = 1.0 IS THE INITIAL DIMENSIONALESS TEMPERATURE THROUGHOUT
C   THE CYLINDER.
C   T1 = 0.0 IS THE DIMENSIONALESS TEMPERATURE APPLIED AT THE
C   SURFACE OF THE CYLINDER AT TIME ZERO.
      TO = 1.0
      T1 = 0.0
C   NM IS THE NUMBER OF NODES IN THE AXIAL DIMENSION.
C   NN IS THE NUMBER OF NODES IN THE RADIAL DIMENSION.
      NM = NL + 1
      NN = NR + 1
      DO 5 I = 1, NM
      DO 5 J = 1, NM
      A(I,J) = 0.0
      B(I,J) = 0.0
    5 CONTINUE
      DO 10 I = 1, NN
      DO 10 J = 1, NN
      BR(I,J) = 0.0
      R(I,J) = 0.0
      S(I,J) = 0.0
   10 CONTINUE
```

```
C   THE MODIFIED ASSEMBLAGE MATRICES FOR CRANK-NICHOLSON SOLUTION
C   IN THE AXIAL DIMENSION ARE COMPUTED.
       DO 15 I = 2, NM
       A(I,I) = 2.0*ALPHA + 4.0*BETA
       B(I,I) = 2.0*ALPHA - 4.0*BETA
       A(I,I-1) = -ALPHA + BETA
       B(I,I-1) = -ALPHA - BETA
       A(I-1,I) = -ALPHA + BETA
       B(I-1,I) = -ALPHA - BETA
    15 CONTINUE
C   THE ASSEMBLAGE CONDUCTIVITY MATRIX R AND THE ASSEMBLAGE
C   CAPACITANCE MATRIX S ARE COMPUTED FOR THE RADIAL DIMENSION.
       R1 = 0.0
       R2 = DELTAR
       R(1,1) = R2*R2/2.0
       R(1,2) = -R(1,1)
       S(1,1) = FUN(R2,R2,R2,R1)
       S(1,2) =-FUN(R2,R1,R2,R1)
       DO 20 I = 2, NR
       R1 = (I-2)*DELTAR
       R2 = (I-1)*DELTAR
       R3 = I*DELTAR
       R(I,I-1) = R(I-1,I)
       S(I,I-1) = S(I-1,I)
       R(I,I+1) = -(R3*R3 - R2*R2)/2.0
       S(I,I+1) = -FUN(R3,R2,R3,R2)
       R(I,I) = -R(I,I-1) - R(I,I+1)
       S(I,I) = FUN(R1,R1,R2,R1) + FUN(R3,R3,R3,R2)
    20 CONTINUE
C   THE MODIFIED ASSEMBLAGE MATRICES FOR CRANK-NICHOLSON SOLUTION
C   IN THE RADIAL DIMENSION ARE COMPUTED.
       DO 25 I = 1, NN
       DO 25 J = 1, NN
       AR(I,J) = GAMMA*R(I,J) + EPSILO*S(I,J)
       BR(I,J) = GAMMA*R(I,J) - EPSILO*S(I,J)
    25 CONTINUE
```

```
C   BOUNDARY CONDITIONS ARE IMPOSED ON THE MODIFIED ASSEMBLAGE
C   MATRICES.
      A(1,1) = 1.0
      A(1,2) = 0.0
      A(NL+1,NL) = 0.0
      A(NL+1,NL+1) = 1.0
      B(1,1) = 1.0
      B(1,2) = 0.0
      B(NL+1,NL) = 0.0
      B(NL+1,NL+1) = 1.0
      AR(NR+1,NR) = 0.0
      AR(NR+1,NR+1) = 1.0
      BR(NR+1,NR) = 0.0
      BR(NR+1,NR+1) = 1.0
C   THE AXIAL AND RADIAL TEMPERATURE VECTORS, TX AND TR, ARE
C   DIMENSIONALESS.
C   INITIAL CONDITIONS AND BOUNDARY CONDITIONS ARE ESTABLISHED IN
C   THE VECTOR RX AND THE TEMPERATURE VECTOR TX.
      RX(1) = T1
      TX(1) = T1
      DO 30 I = 2, NL
      RX(I) = 0.0
      TX(I) = T0
   30 CONTINUE
      RX(NL+1) = T1
      TX(NL+1) = T1
C   INITIAL CONDITIONS AND BOUNDARY CONDITIONS ARE ESTABLISHED IN
C   THE VECTOR RR AND THE TEMPERATURE VECTOR TR.
      DO 35 I = 1, NR
      RR(I) = 0.0
      TR(I) = T0
   35 CONTINUE
      RR(NR+1) = T1
      TR(NR+1) = T1
```

```
C   THE MATRICES A AND AR ARE RECALCULATED AS REQUIRED BY THE
C   CROUT REDUCTION METHOD FOR TRIDIAGONAL SYSTEMS OF EQUATIONS.
      A(1,2) = A(1,2)/A(1,1)
      DO 40 I = 2, NL
      A(I,I) = A(I,I) - A(I,I-1)*A(I-1,I)
      A(I,I+1) = A(I,I+1)/A(I,I)
   40 CONTINUE
      A(NL+1,NL+1) = A(NL+1,NL+1) - A(NL+1,NL)*A(NL,NL+1)
      AR(1,2) = AR(1,2)/AR(1,1)
      DO 45 I = 2, NR
      AR(I,I) = AR(I,I) - AR(I,I-1)*AR(I-1,I)
      AR(I,I+1) = AR(I,I+1)/AR(I,I)
   45 CONTINUE
      AR(NR+1,NR+1) = AR(NR+1,NR+1) - AR(NR+1,NR)*AR(NR,NR+1)
C   SOLUTIONS IN THE AXIAL AND RADIAL DIMENSIONS ARE COMPUTED AT
C   EACH SUCCESSIVE TIME INCREMENT.
      DO 100 T = 1, NT
C   THE VECTOR C IS COMPUTED WHERE (C) = 2(RX) - (B)*(TX).
      DO 55 I = 1, NM
      C(I) = 0.0
      DO 50 J = 1, NM
      C(I) = C(I) + B(I,J)*TX(J)
   50 CONTINUE
      C(I) = 2.0*RX(I) - C(I)
   55 CONTINUE
C   THE VECTOR TX IS COMPUTED BY THE CROUT REDUCTION METHOD
C   WHERE (A)*(TX) = (C).
      Z(1) = C(1)/A(1,1)
      DO 60 I = 2, NM
      Z(I) = ( C(I) - A(I,I-1)*Z(I-1) )/A(I,I)
   60 CONTINUE
      TX(NL+1) = Z(NL+1)
      DO 65 I = 1, NL
      J = NL + 1 - I
      TX(J) = Z(J) - A(J,J+1)*TX(J+1)
   65 CONTINUE
```

```
C   THE VECTOR D IS COMPUTED WHERE (D) = 2(RR) - (BR)*(TR).
      DO 75 I = 1, NN
      D(I) = 0.0
      DO 70 J = 1, NN
      D(I) = D(I) + BR(I,J)*TR(J)
   70 CONTINUE
      D(I) = 2.0*RR(I)-D(I)
   75 CONTINUE
C   THE VECTOR TR IS COMPUTED BY THE CROUT REDUCTION METHOD
C   WHERE (AR)*(TR) = (D).
      Z(1) = D(1)/AR(1,1)
      DO 80 I = 2, NN
      Z(I) = ( D(I) - AR(I,I-1)*Z(I-1) )/AR(I,I)
   80 CONTINUE
      TR(NR+1) = Z(NR+1)
      DO 85 I = 1, NR
      J = NR + 1 - I
      TR(J) = Z(J) - AR(J,J+1)*TR(J+1)
   85 CONTINUE
      TIME = T*DELTAT
      WRITE (6,110) TIME
      DO 95 I = 1, NM
      DO 90 J = 1, NN
C   NEWMAN'S METHOD IS USED TO COMPUTE THE DIMENSIONALESS TEMPERATURES
C   AT NODES THROUGHOUT THE CYLINDER.
      Z(J) = TX(I)*TR(J)
   90 CONTINUE
C     DIMENSIONALESS TEMPERATURES ARE OUTPUTTED AT EACH AXIAL NODE
C     FROM R = 0 TO R = R.
      WRITE (6,120) ( Z(J), J = 1, NN )
   95 CONTINUE
  100 CONTINUE
  110 FORMAT('1',5X,'TIME = ',F8.2,' SECONDS')
  120 FORMAT('0',6(11F8.4,/))
      STOP
      END
```

```
      FUNCTION FUN(F1,F2,F3,F4)
C     THIS FUNCTION COMPUTES THE ELEMENTS OF THE ASSEMBLAGE CAPACITANCE
C     PROPERTY MATRIX IN THE RADIAL DIMENSION.
      F5 = F1*F2*(F3*F3 - F4*F4)/2.0
      F6 = (F1 + F2)*(F3**3.0 - F4**3.0)/3.0
      F7 = (F3**4.0 - F4**4.0)/4.0
      FUN = F5 - F6 + F7
      RETURN
      END
```

APPENDIX B

Program Output - Dimensionaless Nodal Temperatures

Table B-1

Program Output - Dimensionaless Nodal Temperatures

t = 36.75 seconds

x \ r	0.0 R	0.2 R	0.4 R	0.6 R	0.8 R	1.0 R
0.0 L	0.0	0.0	0.0	0.0	0.0	0.0
0.1 L	0.5692	0.5690	0.5671	0.5461	0.2933	0.0
0.2 L	0.9690	0.9687	0.9655	0.9297	0.4994	0.0
0.3 L	0.9977	0.9974	0.9941	0.9573	0.5142	0.0
0.4 L	0.9998	0.9994	0.9962	0.9592	0.5152	0.0
0.5 L	0.9999	0.9996	0.9963	0.9594	0.5153	0.0
0.6 L	0.9998	0.9994	0.9962	0.9592	0.5152	0.0
0.7 L	0.9977	0.9974	0.9941	0.9573	0.5142	0.0
0.8 L	0.9690	0.9687	0.9655	0.9297	0.4994	0.0
0.9 L	0.5692	0.5690	0.5671	0.5461	0.2933	0.0
1.0 L	0.0	0.0	0.0	0.0	0.0	0.0

t = 73.50 seconds

0.0 L	0.0	0.0	0.0	0.0	0.0	0.0
0.1 L	0.4839	0.4815	0.4642	0.3588	0.2034	0.0
0.2 L	0.8009	0.7970	0.7684	0.5939	0.3367	0.0
0.3 L	0.9726	0.9678	0.9331	0.7212	0.4089	0.0
0.4 L	0.9956	0.9907	0.9551	0.7383	0.4186	0.0
0.5 L	0.9978	0.9929	0.9572	0.7399	0.4195	0.0
0.6 L	0.9956	0.9907	0.9551	0.7383	0.4186	0.0
0.7 L	0.9726	0.9678	0.9331	0.7212	0.4089	0.0
0.8 L	0.8009	0.7970	0.7684	0.5939	0.3367	0.0
0.9 L	0.4839	0.4815	0.4642	0.3588	0.2034	0.0
1.0 L	0.0	0.0	0.0	0.0	0.0	0.0

Table B-1 (contd.)

t = 110.25 seconds

x \ r	0.0 R	0.2 R	0.4 R	0.6 R	0.8 R	1.0 R
0.0 L	0.0	0.0	0.0	0.0	0.0	0.0
0.1 L	0.4019	0.3910	0.3435	0.2624	0.1352	0.0
0.2 L	0.7188	0.6992	0.6143	0.4693	0.2418	0.0
0.3 L	0.8911	0.8668	0.7615	0.5818	0.2998	0.0
0.4 L	0.9688	0.9424	0.8279	0.6325	0.3259	0.0
0.5 L	0.9811	0.9544	0.8385	0.6406	0.3300	0.0
0.6 L	0.9688	0.9424	0.8279	0.6325	0.3259	0.0
0.7 L	0.8911	0.8668	0.7615	0.5818	0.2998	0.0
0.8 L	0.7188	0.6992	0.6143	0.4693	0.2418	0.0
0.9 L	0.4019	0.3910	0.3435	0.2624	0.1352	0.0
1.0 L	0.0	0.0	0.0	0.0	0.0	0.0

t = 147.00 seconds

0.0 L	0.0	0.0	0.0	0.0	0.0	0.0
0.1 L	0.3430	0.3217	0.2783	0.2011	0.1035	0.0
0.2 L	0.6187	0.5802	0.5020	0.3627	0.1866	0.0
0.3 L	0.8022	0.7523	0.6508	0.4703	0.2420	0.0
0.4 L	0.8906	0.8352	0.7225	0.5221	0.2686	0.0
0.5 L	0.9188	0.8616	0.7454	0.5386	0.2772	0.0
0.6 L	0.8906	0.8352	0.7225	0.5221	0.2686	0.0
0.7 L	0.8022	0.7523	0.6508	0.4703	0.2420	0.0
0.8 L	0.6187	0.5802	0.5020	0.3627	0.1866	0.0
0.9 L	0.3430	0.3217	0.2783	0.2011	0.1035	0.0
1.0 L	0.0	0.0	0.0	0.0	0.0	0.0

Table B-1 (contd.)

t = 183.75 seconds

x \ r	0.0 R	0.2 R	0.4 R	0.6 R	0.8 R	1.0 R
0.0 L	0.0	0.0	0.0	0.0	0.0	0.0
0.1 L	0.2827	0.2664	0.2241	0.1602	0.0804	0.0
0.2 L	0.5215	0.4915	0.4134	0.2956	0.1483	0.0
0.3 L	0.6873	0.6478	0.5449	0.3896	0.1954	0.0
0.4 L	0.7814	0.7364	0.6194	0.4429	0.2222	0.0
0.5 L	0.8093	0.7627	0.6416	0.4587	0.2301	0.0
0.6 L	0.7814	0.7364	0.6194	0.4429	0.2222	0.0
0.7 L	0.6873	0.6478	0.5449	0.3896	0.1954	0.0
0.8 L	0.5215	0.4915	0.4134	0.2956	0.1483	0.0
0.9 L	0.2827	0.2664	0.2241	0.1602	0.0804	0.0
1.0 L	0.0	0.0	0.0	0.0	0.0	0.0

t = 220.50 seconds

0.0 L	0.0	0.0	0.0	0.0	0.0	0.0
0.1 L	0.2383	0.2212	0.1852	0.1302	0.0651	0.0
0.2 L	0.4432	0.4115	0.3445	0.2422	0.1210	0.0
0.3 L	0.5942	0.5516	0.4618	0.3247	0.1622	0.0
0.4 L	0.6824	0.6335	0.5303	0.3729	0.1863	0.0
0.5 L	0.7118	0.6607	0.5531	0.3890	0.1943	0.0
0.6 L	0.6824	0.6335	0.5303	0.3729	0.1863	0.0
0.7 L	0.5942	0.5516	0.4618	0.3247	0.1622	0.0
0.8 L	0.4432	0.4115	0.3445	0.2422	0.1210	0.0
0.9 L	0.2383	0.2212	0.1852	0.1302	0.0651	0.0
1.0 L	0.0	0.0	0.0	0.0	0.0	0.0

Table B-1 (contd.)

t = 257.25 seconds

x \ r	0.0 R	0.2 R	0.4 R	0.6 R	0.8 R	1.0 R
0.0 L	0.0	0.0	0.0	0.0	0.0	0.0
0.1 L	0.1988	0.1851	0.1536	0.1076	0.0533	0.0
0.2 L	0.3730	0.3474	0.2882	0.2018	0.1000	0.0
0.3 L	0.5041	0.4695	0.3895	0.2728	0.1352	0.0
0.4 L	0.5843	0.5442	0.4515	0.3162	0.1567	0.0
0.5 L	0.6107	0.5688	0.4719	0.3305	0.1638	0.0
0.6 L	0.5843	0.5442	0.4515	0.3162	0.1567	0.0
0.7 L	0.5041	0.4695	0.3895	0.2728	0.1352	0.0
0.8 L	0.3730	0.3474	0.2882	0.2018	0.1000	0.0
0.9 L	0.1988	0.1851	0.1536	0.1076	0.0533	0.0
1.0 L	0.0	0.0	0.0	0.0	0.0	0.0

t = 294.00 seconds

0.0 L	0.0	0.0	0.0	0.0	0.0	0.0
0.1 L	0.1676	0.1553	0.1286	0.0896	0.0443	0.0
0.2 L	0.3159	0.2926	0.2424	0.1688	0.0835	0.0
0.3 L	0.4299	0.3981	0.3298	0.2298	0.1137	0.0
0.4 L	0.5005	0.4636	0.3841	0.2675	0.1324	0.0
0.5 L	0.5245	0.4858	0.4024	0.2803	0.1387	0.0
0.6 L	0.5005	0.4636	0.3841	0.2675	0.1324	0.0
0.7 L	0.4299	0.3981	0.3298	0.2298	0.1137	0.0
0.8 L	0.3159	0.2926	0.2424	0.1688	0.0835	0.0
0.9 L	0.1676	0.1553	0.1286	0.0896	0.0443	0.0
1.0 L	0.0	0.0	0.0	0.0	0.0	0.0

Table B-1 (contd.)

t = 330.75 seconds

x \ r	0.0 R	0.2 R	0.4 R	0.6 R	0.8 R	1.0 R
0.0 L	0.0	0.0	0.0	0.0	0.0	0.0
0.1 L	0.1410	0.1308	0.1080	0.0751	0.0371	0.0
0.2 L	0.2666	0.2472	0.2042	0.1420	0.0701	0.0
0.3 L	0.3641	0.3378	0.2790	0.1940	0.0958	0.0
0.4 L	0.4255	0.3947	0.3260	0.2268	0.1119	0.0
0.5 L	0.4464	0.4140	0.3419	0.2379	0.1174	0.0
0.6 L	0.4255	0.3947	0.3260	0.2268	0.1119	0.0
0.7 L	0.3641	0.3378	0.2790	0.1940	0.0958	0.0
0.8 L	0.2666	0.2472	0.2042	0.1420	0.0701	0.0
0.9 L	0.1410	0.1308	0.1080	0.0751	0.0371	0.0
1.0 L	0.0	0.0	0.0	0.0	0.0	0.0

t = 367.50 seconds

0.0 L	0.0	0.0	0.0	0.0	0.0	0.0
0.1 L	0.1191	0.1102	0.0910	0.0632	0.0312	0.0
0.2 L	0.2256	0.2089	0.1725	0.1197	0.0591	0.0
0.3 L	0.3091	0.2861	0.2362	0.1640	0.0809	0.0
0.4 L	0.3619	0.3350	0.2766	0.1921	0.0947	0.0
0.5 L	0.3800	0.3518	0.2904	0.2017	0.0995	0.0
0.6 L	0.3619	0.3350	0.2766	0.1921	0.0947	0.0
0.7 L	0.3091	0.2861	0.2362	0.1640	0.0809	0.0
0.8 L	0.2256	0.2089	0.1725	0.1197	0.0591	0.0
0.9 L	0.1191	0.1102	0.0910	0.0632	0.0312	0.0
1.0 L	0.0	0.0	0.0	0.0	0.0	0.0

Table B-1 (contd.)

t = 404.25 seconds

x \ r	0.0 R	0.2 R	0.4 R	0.6 R	0.8 R	1.0 R
0.0 L	0.0	0.0	0.0	0.0	0.0	0.0
0.1 L	0.1006	0.0931	0.0768	0.0533	0.0263	0.0
0.2 L	0.1908	0.1767	0.1458	0.1012	0.0499	0.0
0.3 L	0.2618	0.2425	0.2000	0.1388	0.0684	0.0
0.4 L	0.3070	0.2843	0.2345	0.1628	0.0802	0.0
0.5 L	0.3225	0.2987	0.2463	0.1710	0.0843	0.0
0.6 L	0.3070	0.2843	0.2345	0.1628	0.0802	0.0
0.7 L	0.2618	0.2425	0.2000	0.1388	0.0684	0.0
0.8 L	0.1908	0.1767	0.1458	0.1012	0.0499	0.0
0.9 L	0.1006	0.0931	0.0768	0.0533	0.0263	0.0
1.0 L	0.0	0.0	0.0	0.0	0.0	0.0

t = 441.00 seconds

0.0 L	0.0	0.0	0.0	0.0	0.0	0.0
0.1 L	0.0851	0.0787	0.0649	0.0450	0.0222	0.0
0.2 L	0.1615	0.1495	0.1233	0.0855	0.0421	0.0
0.3 L	0.2219	0.2054	0.1694	0.1175	0.0579	0.0
0.4 L	0.2604	0.2411	0.1988	0.1379	0.0679	0.0
0.5 L	0.2737	0.2533	0.2089	0.1449	0.0714	0.0
0.6 L	0.2604	0.2411	0.1988	0.1379	0.0679	0.0
0.7 L	0.2219	0.2054	0.1694	0.1175	0.0579	0.0
0.8 L	0.1615	0.1495	0.1233	0.0855	0.0421	0.0
0.9 L	0.0851	0.0787	0.0649	0.0450	0.0222	0.0
1.0 L	0.0	0.0	0.0	0.0	0.0	0.0

Table B-1 (contd.)

t = 477.75 seconds

x \ r	0.0 R	0.2 R	0.4 R	0.6 R	0.8 R	1.0 R
0.0 L	0.0	0.0	0.0	0.0	0.0	0.0
0.1 L	0.0720	0.0666	0.0549	0.0381	0.0188	0.0
0.2 L	0.1367	0.1266	0.1044	0.0724	0.0357	0.0
0.3 L	0.1880	0.1740	0.1434	0.0995	0.0490	0.0
0.4 L	0.2207	0.2043	0.1685	0.1168	0.0576	0.0
0.5 L	0.2320	0.2148	0.1771	0.1228	0.0605	0.0
0.6 L	0.2207	0.2043	0.1685	0.1168	0.0576	0.0
0.7 L	0.1880	0.1740	0.1434	0.0995	0.0490	0.0
0.8 L	0.1367	0.1266	0.1044	0.0724	0.0357	0.0
0.9 L	0.0720	0.0666	0.0549	0.0381	0.0188	0.0
1.0 L	0.0	0.0	0.0	0.0	0.0	0.0

t = 514.50 seconds

0.0 L	0.0	0.0	0.0	0.0	0.0	0.0
0.1 L	0.0609	0.0564	0.0465	0.0322	0.0159	0.0
0.2 L	0.1158	0.1072	0.0884	0.0613	0.0302	0.0
0.3 L	0.1592	0.1474	0.1215	0.0843	0.0415	0.0
0.4 L	0.1871	0.1731	0.1427	0.0990	0.0488	0.0
0.5 L	0.1967	0.1820	0.1500	0.1040	0.0513	0.0
0.6 L	0.1871	0.1731	0.1427	0.0990	0.0488	0.0
0.7 L	0.1592	0.1474	0.1215	0.0843	0.0415	0.0
0.8 L	0.1158	0.1072	0.0884	0.0613	0.0302	0.0
0.9 L	0.0609	0.0564	0.0465	0.0322	0.0159	0.0
1.0 L	0.0	0.0	0.0	0.0	0.0	0.0

Table B-1 (cOntd.)

t = 551.25 seconds

x \ r	0.0 R	0.2 R	0.4 R	0.6 R	0.8 R	1.0 R
0.0 L	0.0	0.0	0.0	0.0	0.0	0.0
0.1 L	0.0516	0.0477	0.0393	0.0273	0.0134	0.0
0.2 L	0.0981	0.0908	0.0748	0.0519	0.0256	0.0
0.3 L	0.1349	0.1249	0.1029	0.0714	0.0351	0.0
0.4 L	0.1585	0.1467	0.1209	0.0839	0.0413	0.0
0.5 L	0.1666	0.1542	0.1271	0.0882	0.0434	0.0
0.6 L	0.1585	0.1467	0.1209	0.0839	0.0413	0.0
0.7 L	0.1349	0.1249	0.0129	0.0714	0.0351	0.0
0.8 L	0.0981	0.0908	0.0748	0.0519	0.0256	0.0
0.9 L	0.0516	0.0477	0.0393	0.0273	0.0134	0.0
1.0 L	0.0	0.0	0.0	0.0	0.0	0.0

t = 588.00 seconds

0.0 L	0.0	0.0	0.0	0.0	0.0	0.0
0.1 L	0.0437	0.0404	0.0333	0.0231	0.0114	0.0
0.2 L	0.0831	0.0769	0.0634	0.0439	0.0216	0.0
0.3 L	0.1143	0.1058	0.0872	0.0604	0.0298	0.0
0.4 L	0.1343	0.1243	0.1025	0.0710	0.0350	0.0
0.5 L	0.1412	0.1307	0.1077	0.0747	0.0368	0.0
0.6 L	0.1343	0.1243	0.1025	0.0710	0.0350	0.0
0.7 L	0.1143	0.1058	0.0872	0.0604	0.0298	0.0
0.8 L	0.0831	0.0769	0.0634	0.0439	0.0216	0.0
0.9 L	0.0437	0.0404	0.0333	0.0231	0.0114	0.0
1.0 L	0.0	0.0	0.0	0.0	0.0	0.0

Table B-1 (contd.)

t = 624.75 seconds

x \ r	0.0 R	0.2 R	0.4 R	0.6 R	0.8 R	1.0 R
0.0 L	0.0	0.0	0.0	0.0	0.0	0.0
0.1 L	0.0370	0.0342	0.0282	0.0196	0.0096	0.0
0.2 L	0.0703	0.0651	0.0537	0.0372	0.0183	0.0
0.3 L	0.0968	0.0896	0.0738	0.0512	0.0252	0.0
0.4 L	0.1138	0.1053	0.0868	0.0602	0.0296	0.0
0.5 L	0.1196	0.1107	0.0913	0.0633	0.0312	0.0
0.6 L	0.1138	0.1053	0.0868	0.0622	0.0296	0.0
0.7 L	0.0968	0.0896	0.0738	0.0512	0.0252	0.0
0.8 L	0.0703	0.0651	0.0537	0.0372	0.0183	0.0
0.9 L	0.0370	0.0342	0.0282	0.0196	0.0096	0.0
1.0 L	0.0	0.0	0.0	0.0	0.0	0.0

t = 661.50 seconds

0.0 L	0.0	0.0	0.0	0.0	0.0	0.0
0.1 L	0.0313	0.0290	0.0239	0.0166	0.0082	0.0
0.2 L	0.0596	0.0551	0.0455	0.0315	0.0155	0.0
0.3 L	0.0820	0.0759	0.0626	0.0434	0.0214	0.0
0.4 L	0.0964	0.0892	0.0735	0.0510	0.0251	0.0
0.5 L	0.1013	0.0938	0.0773	0.0536	0.0264	0.0
0.6 L	0.0964	0.0892	0.0735	0.0510	0.0251	0.0
0.7 L	0.0820	0.0759	0.0626	0.0434	0.0214	0.0
0.8 L	0.0596	0.0551	0.0455	0.0315	0.0155	0.0
0.9 L	0.0313	0.0290	0.0239	0.0166	0.0082	0.0
1.0 L	0.0	0.0	0.0	0.0	0.0	0.0

Table B-1 (contd.)

t = 698.25 seconds

x \ r	0.0 R	0.2 R	0.4 R	0.6 R	0.8 R	1.0 R
0.0 L	0.0	0.0	0.0	0.0	0.0	0.0
0.1 L	0.0265	0.0246	0.0202	0.0140	0.0069	0.0
0.2 L	0.0505	0.0467	0.0385	0.0267	0.0131	0.0
0.3 L	0.0695	0.0643	0.0530	0.0367	0.0181	0.0
0.4 L	0.0816	0.0756	0.0623	0.0432	0.0213	0.0
0.5 L	0.0858	0.0795	0.0655	0.0454	0.0224	0.0
0.6 L	0.0816	0.0756	0.0623	0.0432	0.0213	0.0
0.7 L	0.0695	0.0643	0.0530	0.0376	0.0181	0.0
0.8 L	0.0505	0.0467	0.0385	0.0267	0.0131	0.0
0.9 L	0.0265	0.0246	0.0202	0.0140	0.0069	0.0
1.0 L	0.0	0.0	0.0	0.0	0.0	0.0

t = 735.00 seconds

x \ r	0.0 R	0.2 R	0.4 R	0.6 R	0.8 R	1.0 R
0.0 L	0.0	0.0	0.0	0.0	0.0	0.0
0.1 L	0.0225	0.0208	0.0171	0.0119	0.0059	0.0
0.2 L	0.0428	0.0396	0.0326	0.0226	0.0111	0.0
0.3 L	0.0588	0.0545	0.0449	0.0311	0.0153	0.0
0.4 L	0.0692	0.0640	0.0528	0.0366	0.0180	0.0
0.5 L	0.0727	0.0673	0.0555	0.0385	0.0189	0.0
0.6 L	0.0692	0.0640	0.0528	0.0366	0.0180	0.0
0.7 L	0.0588	0.0545	0.0449	0.0311	0.0153	0.0
0.8 L	0.0428	0.0396	0.0326	0.0226	0.0111	0.0
0.9 L	0.0225	0.0208	0.0171	0.0119	0.0059	0.0
1.0 L	0.0	0.0	0.0	0.0	0.0	0.0