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INPLANE AND OUT OF PLANE BUCKLING OF THICK RINGS
SUBJECTED TO HYDROSTATIC PRESSURE

by

Thomas Michael Juliano

Dissertation submitted to the Faculty of the Graduate School
of the New Jersey Institute of Technology in partial
fulfillment of the requirements for the degree of
Doctor of Engineering Science
1979

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ABSTRACT

Title of Thesis: Inplane and Out of Plane Buckling of Thick
Rings Subjected to Hydrostatic Pressure

Thomas Michael Juliano, Doctor of Engineering Science, 1979

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The problem of inplane and out of plane buckling of a thick circular ring subjected to a hydrostatic pressure is analyzed from first principals. The equations of equilibrium from nonlinear elasticity theory are used to describe two adjacent equilibrium positions, i.e., an initial and final state. The initial position problem is assumed to be governed by linear theory. The resulting incremental value equations obtained by subtracting the two equilibrium states, are also linear in nature. These problems are reduced to a one dimensional ring theory problem by the method of power series expansion of the displacements in the radial and axial directions and Fourier series expansions in the circumferential direction, and then by integrating through the thickness and depth of the ring. The coefficients of the terms in the power series displacement expansions are treated as unknown variables and the resulting eigenvalue problem is solved for the lowest root in each of the two perpendicular directions. The number of terms in the power series are reduced and a thin

ring theory is defined, also in terms of unknown coefficients. These coefficients are determined and compared with published equations for thin rings.

Several rings with rectangular cross sections were analyzed by these three methods, i.e., (a) thick ring theory with unknown coefficients, (b) thin ring theory with unknown coefficients, (c) thin ring theory with known coefficients. Both thin ring theories were found to agree with the thick ring theory to within ten percent for rings with diameter to thickness ratios of twenty or more. However, the thick ring theory does not become inaccurate until this ratio is less than five. In all of the analyzed cases of diameter to thickness ratios, the square cross sections had the largest critical buckling loads in both the inplane and out of plane directions. In each case the out of plane critical buckling load was the smaller. When the ratio of the radial thickness to the axial depth, called aspect ratio, was approximately equal to one half, the rings had equal critical buckling loads in both the inplane and out of plane directions. If the aspect ratio was greater than 0.5, the out of plane direction had a smaller critical load. If it was less than this value, the inplane direction would control.

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INTRODUCTION

The problem of the stability of a thin ring has been studied quite extensively during the last century. A. E. Love¹ derives the equation of motion for inplane and out of plane vibration of a thin circular ring. These equations are analagous to the stability equations in these directions. He first develops the relationships for the forces and moment resultants in terms of the curvatures, strains and twists for curved rods,² and then obtains the equilibrium equations. He also presents the solution to the inplane buckling of a thin ring loaded by a normal pressure as given by M. Lévy in 1884.³

S. Timoshenko⁴ develops the stability equations for thin curved bars. He neglects the extension of the center line and considers inplane buckling due to hydrostatic pressure.⁵ He obtained a value of three for the critical buckling load parameter, which is plotted in section IX.D of this work. This result agrees with Lévy's. He also solves the problem of the lateral buckling of a curved bar with circular axis,⁶ i.e., a ring segment. These results are extended to include a complete ring which buckles into four half waves. The

¹A.E.H. Love, A Treatise on the Mathematical Theory of Elasticity, 4th ed. (New York, 1944), pp. 426-431, 451.

²IBID., pp. 381-398.

³IBID., pp. 424-425.

⁴S.P. Timoshenko and J.M. Gere, Theory of Elastic Stability, 2nd ed. (New York: 1961), pp. 278-318.

⁵IBID., p. 291.

⁶IBID., pp. 313-318.

loadings treated are radial line loads which either undergo a parallel translation or rotate and remained centrally directed.

A. P. Boresi⁷ solves the problem of inplane buckling of thin rings subjected to two type of loadings, i.e., normal and centrally directed pressure. He uses a variational approach which states that a stationary conservative mechanical system is in stable equilibrium only if its total potential energy is at a relative minimum. He attributes the origination of this method to E. Trefftz in 1933. The smallest diameter to thickness ratio which he considers is twenty. His results agree with Lévy's for an infinitely thin ring with Poission's ratio equal to zero.

J. E. Goldberg and J. L. Bogdanoff⁸ investigated the out of plane buckling of a thin ring subjected to an inwardly directed radial line load, which undergoes a parallel translation during deformation. An example problem was solved for a ring with an I-shaped cross section with a diameter to thickness ratio of 115 to 1. They also considered the case of having intermediate supports by solving for the higher order critical buckling loads.

T. Wah⁹ deduced the inplane and out of plane buckling

⁷A.P. Boresi, "A Refinement of the Theory of Buckling of Rings Under Unifrom Pressure," Journal of Applied Mechanics, Vol. 22 (March, 1955), pp. 95-102.

⁸J.E. Goldberg and J.L. Bogdanoff, "Out of Plane Buckling of I-Section Rings," International Association of Bridge and Structural Engineers, Vol. 22 (1962), pp. 73-92.

⁹T. Wah, "Buckling of Thin Circular Rings Under Uniform Pressure," International Journal of Solids and Structures, Vol. 3 (1967), pp. 967-974.

equations of thin rings from the equations of motion of a vibrating circular ring given, as previously mentioned, by Love. He replaced the inertia terms by fictitious loads whose intensity is the compressive hoop force times the appropriate curvature term. He attributes this method to Timoshenko¹⁰ in his investigation of the torsional buckling of open section columns. He retains the extension of the center line only in the terms in the circumferential direction and assumes the load to be a uniform inward radial pressure. He obtains quadratic equations for the critical buckling loads for both directions. The inplane equation reduces to Lévy's result for thin rings under hydrostatic pressure. The out of plane results agree with Goldberg and Bagdanoff's values if their boundary condition of parallel translation of the load is removed. A more detailed description of Wah's analysis is given in Chapter V.

H.R. Mech¹¹ demonstrated that the inplane and out of plane deformations of a circular ring of arbitrary cross section were coupled, unless at least one of the principal axes of the section is parallel to the axis of revolution of the ring. This indicates that the ring cross section rotates under the initial deformation due to the loading and prior to any condition of buckling. He neglects the extension of the

¹⁰Timoshenko and Gere, p. 225.

¹¹H.R. Mech, "Three-Dimensional Deformation and Buckling of a Circular Ring of Arbitrary Section," Journal of Engineering for Industry, (February, 1969), pp. 266-272.

ring center line and considers rings which are loaded by various combinations of concentrated radial forces and twisting moments. He also treats as a special case a ring with its principal axis of inertia parallel to its axis of revolution and for a uniform radial pressure obtains the same results as Timoshenko.

S. C. Batterman¹² and A. I. Soler use the rate equation method to account for the disturbance that incipient buckling has on the stress resultants and curvatures given in the equilibrium equations. They consider only inplane buckling of thin rings with the effects of the extension of the center line retained. They analyze hydrostatic pressure loading, centrally directed pressure loading and dead pressure, i.e., where the direction of the loading remains unchanged.

S. C. Anand and A. R. Griffith¹³ consider the effects which residual stresses have upon the buckling of thin rings. Residual stresses may be introduced during manufacturing or heat treatment of a ring. There may also be plastic deformation during these processes. They analyze the problem by dividing the cross section into an elastic and a plastic zone, and consider the strain energy of each. They use the Donnell kinematic relationships for the ring and assume that the load remains unchanged by the deformation. Their results are ex-

¹²S.C. Batterman and A.I. Soler, "Buckling of Rings," Journal of the Engineering Mechanics Division, ASCE, Vol. 96, No. EM6 (1970), pp. 1291-1297.

¹³S.C. Anand and A.R. Griffith, "Inelastic Buckling of Rings with Residual Stresses," Journal of the Engineering Mechanics Division, ASCE, Vol. 99, No. EM5 (1973), pp. 927-942.

pressed in terms of the moment of inertia of the elastic portion of the cross section.

The thermally induced buckling of a constrained elastic ring is analyzed by L. El-Bayoumy.¹⁴ He considers a thin ring which is initially loaded by a uniformly contracted circular boundary. The ring is subjected to various temperature distributions which cause it to buckle by inwardly detaching itself from the boundary. He uses the energy method to find the buckling loads. His results indicate that the larger the initial contraction the smaller the minimum buckling temperature.

The foregoing has been a summary of some of the works which were of interest in gaining some familiarity with the buckling of thin rings. The thickest ring which was analyzed had a diameter to thickness ratio of twenty to one. The only paper which considered any thick rings was Wah. However, his equations were developed using thin ring theory and not all of his results are accurate.

In this work the problem of the inplane and out of plane buckling of a thick ring is analyzed beginning with the nonlinear theory of elasticity stress equilibrium equations in curvilinear coordinates. The technique of adjacent equilibrium positions is used to separate the problem into two parts:¹⁵ the initial position and final position problems. The equi-

¹⁴L. El-Bayoumy, "Thermal Buckling of a Constrained Elastic Ring," International Journal of Non-Linear Mechanics, Vol. 9 (1974), pp. 529-549.

¹⁵V.V. Novozhilov, Foundations of the Nonlinear Theory of Elasticity, (Rochester, 1953), p. 156.

librium equations which describe these two positions are non-linear in form. However, it is assumed that the initial position problem can be solved using linear theory, since the elongations, shears and angles of rotations are small compared to unity. The incremental problem is defined by subtracting the initial and final position equations. The resulting equations are linear in terms of the incremental values, and are also simplified under the assumption of a linear initial position problem.

The ring is assumed to be loaded by a uniform, inwardly directed radial pressure. The load is hydrostatic and remains normal to the ring outer surface. An additional assumption is made of no rotation of the cross section at the initial position of equilibrium. This requires that the cross section of the ring be symmetrical about the radial axis, and allows the initial problem to be solved with the Lamé formulae.¹⁶

The procedure used to solve the incremental problem was developed from the method of displacement expansions. The displacement of a point is described in terms of the centroid of the cross section plus a power series expansion through the thickness and depth of the ring. This method has also been used to solve problems in the theory of plates and shells.¹⁷ The coefficients of the individual terms of the

¹⁶S. Timoshenko and J.N. Goodier, Theory of Elasticity, 2nd ed. (New York, 1951), p. 58.

¹⁷Novozhilov, p. 198-216.

series are usually determined by making certain assumptions about the behavior of the points within the cross section. For example, the Kirchoff-Love¹⁸ Hypothesis assumes that normals to the center line remain straight and normal. The relationships between the displacements at a point away from the center line and that of a point on the center line can then be determined using geometry.

The technique used in this work considers the coefficients to be unknowns leading to a large set of simultaneous equations with homogeneous right hand sides. This resulting eigenvalue problem is then solved for the roots in each of the two buckling mode directions, i.e., inplane and out of plane buckling. Only the lowest root in each direction is of interest, since the ring is assumed to have failed after it has buckled.

The above procedure was first used to solve the problem of the thin ring. The equations obtained from the unknown coefficient series expansions reduced to those used by Wah¹⁹ when the proper substitutions were made.

The same procedure was extended to the case of a thick ring. Additional terms were added to the series expansions leading to more simultaneous equations. Rings of various aspect ratios with rectangular cross sections were analyzed, and the variation of the critical buckling load with the ring diameter, thickness, and shape were plotted.

¹⁸IBID., p. 197.

¹⁹Wah, p. 969.

Comparisons were made between the three methods, i.e., (a) thick ring theory with unknown coefficients, (b) thin ring theory with unknown coefficients, (c) thin ring theory with known coefficients, and also with the classical thin ring theory results.

I. EQUILIBRIUM EQUATIONS

A. Nonlinear Equilibrium Equations for an Orthogonal Curvilinear Coordinate System

A differential element of volume may be constructed in terms of a general orthogonal curvilinear coordinate system as shown in figure 1.

The edges are defined as,¹

(a) unstrained state

$$\bar{K}_1 H_1 d\alpha_1$$

$$\bar{K}_2 H_2 d\alpha_2$$

$$K_3 H_3 d\alpha_3$$

(b) strained state

$$\bar{K}'_1 H_1 (1 + E_{\alpha_1}) d\alpha_1$$

$$\bar{K}'_2 H_2 (1 + E_{\alpha_2}) d\alpha_2$$

$$\bar{K}'_3 H_3 (1 + E_{\alpha_3}) d\alpha_3$$

where

α_i are orthogonal curvilinear coordinates

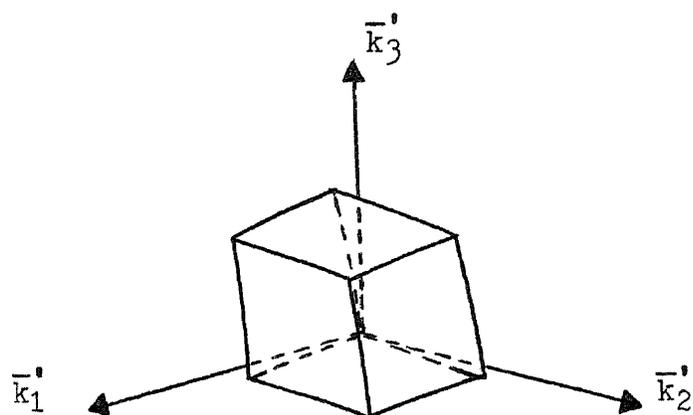
\bar{K}_i are unit vectors in unstrained state

\bar{K}'_i are unit vectors in strained state

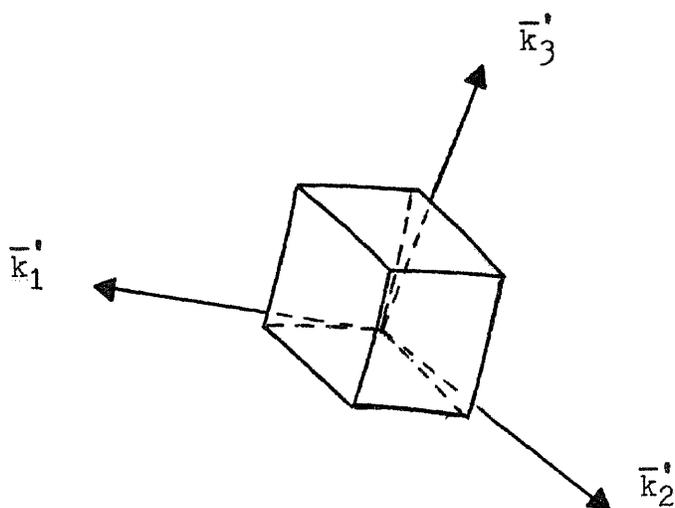
H_i are Lamé coefficients

E_{α_i} are elongation of line element which was originally in the α_i direction.

¹Novozhilov, p. 88.



UNSTRAINED STATE



STRAINED STATE

Figure 1 Differential Volume Elements

The nonlinear equilibrium equations for the differential volume element shown in figure 1 may be written as the following, with zero body forces,²

$$\begin{aligned} \frac{1}{H_1 H_2 H_3} \left\{ \frac{\partial}{\partial \alpha_1} (H_2 H_3 S_{11}) + \frac{\partial}{\partial \alpha_2} (H_3 H_1 S_{21}) + \frac{\partial}{\partial \alpha_3} (H_1 H_2 S_{31}) \right\} + \frac{1}{H_1 H_2} \frac{\partial H_1}{\partial \alpha_2} S_{12} \\ + \frac{1}{H_1 H_3} \frac{\partial H_1}{\partial \alpha_3} S_{13} - \frac{1}{H_1 H_2} \frac{\partial H_2}{\partial \alpha_1} S_{22} - \frac{1}{H_1 H_3} \frac{\partial H_3}{\partial \alpha_1} S_{33} = 0 \end{aligned} \quad (1)$$

$$\begin{aligned} \frac{1}{H_1 H_2 H_3} \left\{ \frac{\partial}{\partial \alpha_1} (H_2 H_3 S_{12}) + \frac{\partial}{\partial \alpha_2} (H_3 H_1 S_{22}) + \frac{\partial}{\partial \alpha_3} (H_1 H_2 S_{32}) \right\} + \frac{1}{H_2 H_3} \frac{\partial H_2}{\partial \alpha_3} S_{23} \\ + \frac{1}{H_2 H_1} \frac{\partial H_2}{\partial \alpha_1} S_{21} - \frac{1}{H_2 H_3} \frac{\partial H_3}{\partial \alpha_2} S_{33} - \frac{1}{H_2 H_1} \frac{\partial H_1}{\partial \alpha_2} S_{11} = 0 \end{aligned} \quad (2)$$

$$\begin{aligned} \frac{1}{H_1 H_2 H_3} \left\{ \frac{\partial}{\partial \alpha_1} (H_2 H_3 S_{13}) + \frac{\partial}{\partial \alpha_2} (H_3 H_1 S_{23}) + \frac{\partial}{\partial \alpha_3} (H_1 H_2 S_{33}) \right\} + \frac{1}{H_3 H_1} \frac{\partial H_3}{\partial \alpha_1} S_{31} \\ + \frac{1}{H_3 H_2} \frac{\partial H_3}{\partial \alpha_2} S_{32} - \frac{1}{H_3 H_1} \frac{\partial H_1}{\partial \alpha_3} S_{11} - \frac{1}{H_3 H_2} \frac{\partial H_2}{\partial \alpha_3} S_{22} = 0 \end{aligned} \quad (3)$$

where,

H_1, H_2, H_3 are the Lamé coefficients

$\sigma_{\alpha_1}, \sigma_{\alpha_2}, \sigma_{\alpha_3}$ are the stresses on the areas perpendicular to the dihedrals in the strained state

$$[\bar{K}'_2, \bar{K}'_3], [\bar{K}'_3, \bar{K}'_1], [\bar{K}'_1, \bar{K}'_2]$$

S_{ij} are the components of the stress σ_{α_i} along the directions $\bar{K}_1, \bar{K}_2, \bar{K}_3$ in the unstrained state

The S_{ij} are components of the stresses on the deformed differential volume element acting along the undeformed co-

²IBID., p. 89.

ordinate system directions. For the case of small elongations and shears, the S_{ij} may be expressed in terms of the undeformed volume element as the following relationships,³

$$\begin{aligned}
 S_{11} &= (1 + e_{11})\sigma_{11} + \left(\frac{e_{12}}{2} - \omega_3\right)\sigma_{12} + \left(\frac{e_{13}}{2} + \omega_2\right)\sigma_{13} \\
 S_{12} &= \left(\frac{e_{12}}{2} + \omega_3\right)\sigma_{11} + (1 + e_{22})\sigma_{12} + \left(\frac{e_{23}}{2} - \omega_1\right)\sigma_{13} \\
 S_{13} &= \left(\frac{e_{13}}{2} - \omega_2\right)\sigma_{11} + \left(\frac{e_{23}}{2} + \omega_1\right)\sigma_{12} + (1 + e_{33})\sigma_{13} \\
 S_{21} &= (1 + e_{11})\sigma_{21} + \left(\frac{e_{12}}{2} - \omega_3\right)\sigma_{22} + \left(\frac{e_{13}}{2} + \omega_2\right)\sigma_{23} \\
 S_{22} &= \left(\frac{e_{12}}{2} + \omega_3\right)\sigma_{21} + (1 + e_{22})\sigma_{22} + \left(\frac{e_{23}}{2} - \omega_1\right)\sigma_{23} \\
 S_{23} &= \left(\frac{e_{13}}{2} - \omega_2\right)\sigma_{21} + \left(\frac{e_{23}}{2} + \omega_1\right)\sigma_{22} + (1 + e_{33})\sigma_{23} \\
 S_{31} &= (1 + e_{11})\sigma_{31} + \left(\frac{e_{12}}{2} - \omega_3\right)\sigma_{32} + \left(\frac{e_{13}}{2} + \omega_2\right)\sigma_{33} \\
 S_{32} &= \left(\frac{e_{12}}{2} + \omega_3\right)\sigma_{31} + (1 + e_{22})\sigma_{32} + \left(\frac{e_{23}}{2} - \omega_1\right)\sigma_{33} \\
 S_{33} &= \left(\frac{e_{13}}{2} - \omega_2\right)\sigma_{31} + \left(\frac{e_{23}}{2} + \omega_1\right)\sigma_{32} + (1 + e_{33})\sigma_{33} \quad (4)
 \end{aligned}$$

where

σ_{ij} are components of the stresses on the undeformed differential volume element acting along the undeformed coordinate directions

e_{ij} are the strains

ω_j are the rotations

Therefore, σ_{ij} , e_{ij} and ω_j correspond to the linear theory

³ IBID., pp. 76-80, 90-93.

stresses, strains and rotations.

For a cylindrical coordinate system where

$$\begin{aligned}\alpha_1 &= r \\ \alpha_2 &= \phi \\ r_3 &= x\end{aligned}\tag{5}$$

and

$$\begin{aligned}\bar{X} &= r \cos \phi \\ \bar{Y} &= r \sin \phi \\ \bar{Z} &= X\end{aligned}\tag{6}$$

the Lamé coefficients are given as⁴

$$H_i = \sqrt{\left(\frac{\partial \bar{X}}{\partial \alpha_i}\right)^2 + \left(\frac{\partial \bar{Y}}{\partial \alpha_i}\right)^2 + \left(\frac{\partial \bar{Z}}{\partial \alpha_i}\right)^2}\tag{7}$$

or

$$\begin{aligned}H_1 &= 1 \\ H_2 &= r \\ H_3 &= 1\end{aligned}\tag{8}$$

Substitute equations (4), (5), (8) into equations (1), (2), (3) to obtain the nonlinear equilibrium equations with zero body forces.

⁴IBID., p. 58

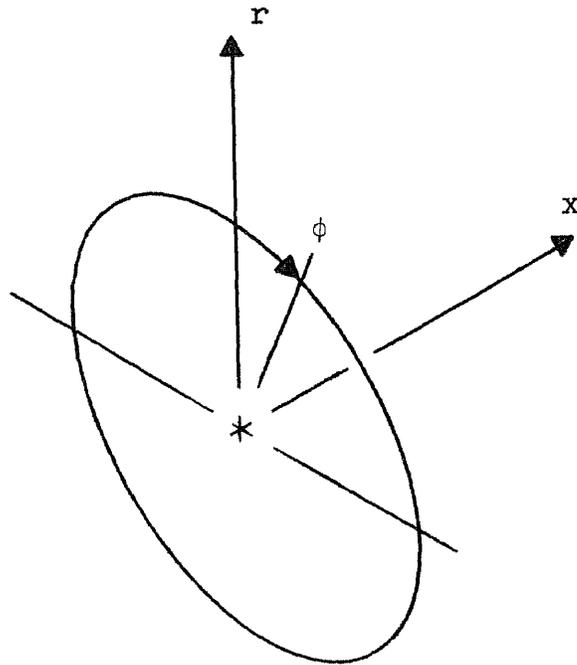


Figure 2 Coordinate Axes

$$\begin{aligned}
& \frac{1}{r} \left[\frac{\partial}{\partial r} \{ r [(1 + e_{11})\sigma_{11} + (\frac{e_{12}}{2} - \omega_3)\sigma_{12} + (\frac{e_{13}}{2} + \omega_2)\sigma_{13}] \} \right. \\
& \quad + \frac{\partial}{\partial \phi} [(1 + e_{11})\sigma_{21} + (\frac{e_{12}}{2} - \omega_3)\sigma_{22} + (\frac{e_{13}}{2} + \omega_2)\sigma_{23}] \\
& \quad + \frac{\partial}{\partial z} \{ r [(1 + e_{11})\sigma_{31} + (\frac{e_{12}}{2} - \omega_3)\sigma_{32} + (\frac{e_{13}}{2} + \omega_2)\sigma_{33}] \} \\
& \quad \left. - [(\frac{e_{12}}{2} + \omega_3)\sigma_{21} + (1 + e_{22})\sigma_{22} + (\frac{e_{23}}{2} - \omega_1)\sigma_{23}] \right] = 0 \quad (9)
\end{aligned}$$

$$\begin{aligned}
& \frac{1}{r} \left[\frac{\partial}{\partial r} \{ r [(\frac{e_{12}}{2} + \omega_3)\sigma_{11} + (1 + e_{22})\sigma_{12} + (\frac{e_{23}}{2} - \omega_1)\sigma_{13}] \} \right. \\
& \quad + \frac{\partial}{\partial \phi} [(\frac{e_{12}}{2} + \omega_3)\sigma_{21} + (1 + e_{22})\sigma_{22} + (\frac{e_{23}}{2} - \omega_1)\sigma_{23}] \\
& \quad + \frac{\partial}{\partial x} \{ r [(\frac{e_{12}}{2} + \omega_3)\sigma_{31} + (1 + e_{22})\sigma_{32} + (\frac{e_{23}}{2} - \omega_1)\sigma_{33}] \} \\
& \quad \left. + [(1 + e_{11})\sigma_{21} + (\frac{e_{12}}{2} - \omega_3)\sigma_{22} + (\frac{e_{13}}{2} + \omega_2)\sigma_{23}] \right] = 0 \quad (10)
\end{aligned}$$

$$\begin{aligned}
& \frac{1}{r} \left[\frac{\partial}{\partial r} r [(\frac{e_{13}}{2} - \omega_2)\sigma_{11} + (\frac{e_{23}}{2} + \omega_1)\sigma_{12} + (1 + e_{33})\sigma_{13}] \right. \\
& \quad + \frac{\partial}{\partial \phi} [(\frac{e_{13}}{2} - \omega_2)\sigma_{21} + (\frac{e_{23}}{2} + \omega_1)\sigma_{22} + (1 + e_{33})\sigma_{23}] \\
& \quad \left. + \frac{\partial}{\partial x} \{ r [(\frac{e_{13}}{2} - \omega_2)\sigma_{31} + (\frac{e_{23}}{2} + \omega_1)\sigma_{23} + (1 + e_{33})\sigma_{33}] \} \right] = 0 \quad (11)
\end{aligned}$$

B. Adjacent Equilibrium Equations

For impending buckling, i.e., when the loading reaches the critical value, there exists the possibility of two infinitely close equilibrium positions. Let u^0, v^0, w^0 be the displacements to the initial equilibrium position, and u', v', w' be the incremental displacements from that position to the final equilibrium position. Thus the total displacements are

given as⁵

$$u = u^0 + \alpha u'; \quad v = v^0 + \alpha v'; \quad w = w^0 + \alpha w' \quad (12)$$

where u' , v' , w' are finite and α is an infinitely small value which is independent of the r , ϕ , x coordinates. From this assumption and the linear theory strains and rotations and Hooke's law the following expressions can be obtained,

$$\begin{aligned} \sigma_{ij} &= \sigma_{ij}^0 + \alpha \sigma'_{ij} \\ e_{ij} &= e_{ij}^0 + \alpha e'_{ij} \\ \omega_j &= \omega_j^0 + \alpha \omega'_j \end{aligned} \quad (13)$$

noting that the initial and final equilibrium positions coincide when α approaches zero.

Substituting equations (12) and (13) into equations (9), (10), (11) gives the final position equilibrium equations. The α^2 terms are dropped as higher order contributions and the equations are multiplied by r .

The final position equilibrium equations are,

⁵IBID., p. 156.

$$\begin{aligned}
& \frac{\partial}{\partial r} \{ r [(1 + e_{11}^0 + \alpha e_{11}') \sigma_{11}^0 + (1 + e_{11}^0) \alpha \sigma_{11}' + (\frac{e_{12}^0}{2} - \omega_3^0) \alpha \sigma_{12}' \\
& + (\frac{e_{12}^0 + \alpha e_{12}'}{2} - \omega_3^0 - \alpha \omega_3') \sigma_{12}^0 + (\frac{e_{13}^0 + \alpha e_{13}'}{2} + \omega_2^0 + \alpha \omega_2') \sigma_{13}^0 \\
& + (\frac{e_{13}^0}{2} + \omega_2^0) \alpha \sigma_{13}'] \} + \frac{\partial}{\partial \phi} [(1 + e_{11}^0 + \alpha e_{11}') \sigma_{21}^0 + (1 + e_{11}^0) \alpha \sigma_{21}' \\
& + (\frac{e_{12}^0 + \alpha e_{12}'}{2} - \omega_3^0 - \alpha \omega_3') \sigma_{22}^0 + (\frac{e_{12}^0}{2} - \omega_3^0) \alpha \sigma_{22}' \\
& + (\frac{e_{13}^0 + \alpha e_{13}'}{2} + \omega_2^0 + \alpha \omega_2') \sigma_{23}^0 + (\frac{e_{13}^0}{2} + \omega_2^0) \alpha \sigma_{23}'] \\
& + \frac{\partial}{\partial x} \{ r [(1 + e_{11}^0 + \alpha e_{11}') \sigma_{31}^0 + (1 + e_{11}^0) \alpha \sigma_{31}' \\
& + (\frac{e_{12}^0 + \alpha e_{12}'}{2} - \omega_3^0 - \alpha \omega_3') \sigma_{32}^0 + (\frac{e_{12}^0}{2} - \omega_3^0) \alpha \sigma_{32}' \\
& + (\frac{e_{13}^0 + \alpha e_{13}'}{2} + \omega_2^0 + \alpha \omega_2') \sigma_{33}^0 + (\frac{e_{13}^0}{2} + \omega_2^0) \alpha \sigma_{33}'] \} \\
& - [(\frac{e_{12}^0 + \alpha e_{12}'}{2} + \omega_3^0 + \alpha \omega_3') \sigma_{21}^0 + (\frac{e_{12}^0}{2} + \omega_3^0) \alpha \sigma_{21}' + (1 + e_{22}^0 + \alpha e_{22}') \sigma_{22}^0 \\
& + (1 + e_{22}^0) \alpha \sigma_{22}' + (\frac{e_{23}^0 + \alpha e_{23}'}{2} - \omega_1^0 - \alpha \omega_1') \sigma_{23}^0 \\
& + (\frac{e_{23}^0}{2} - \omega_1^0) \alpha \sigma_{23}'] = 0 \tag{14}
\end{aligned}$$

$$\begin{aligned}
& \frac{\partial}{\partial r} \left\{ r \left[\left(\frac{e_{12}^0 + \alpha e_{12}'}{2} + \omega_3^0 + \alpha \omega_3' \right) \sigma_{11}^0 + \left(\frac{e_{12}^0}{2} + \omega_3^0 \right) \alpha \sigma_{11}' \right. \right. \\
& + (1 + e_{22}^0 + \alpha e_{22}') \sigma_{12}^0 + (1 + e_{22}^0) \alpha \sigma_{12}' + \left(\frac{e_{23}^0 + \alpha e_{23}'}{2} - \omega_1^0 - \alpha \omega_1' \right) \sigma_{13}^0 \\
& \left. \left. + \left(\frac{e_{23}^0}{2} - \omega_1^0 \right) \alpha \sigma_{13}' \right] \right\} + \frac{\partial}{\partial \phi} \left[\left(\frac{e_{12}^0 + \alpha e_{12}'}{2} + \omega_3^0 + \alpha \omega_3' \right) \sigma_{21}^0 \right. \\
& + \left(\frac{e_{12}^0}{2} + \omega_3^0 \right) \alpha \sigma_{21}' + (1 + e_{22}^0 + \alpha e_{22}') \sigma_{22}^0 + (1 + e_{22}^0) \alpha \sigma_{22}' \\
& \left. + \left(\frac{e_{23}^0 + \alpha e_{23}'}{2} - \omega_1^0 - \alpha \omega_1' \right) \sigma_{23}^0 + \left(\frac{e_{23}^0}{2} - \omega_1^0 \right) \alpha \sigma_{23}' \right] \\
& + \frac{\partial}{\partial x} \left\{ r \left[\left(\frac{e_{12}^0 + \alpha e_{12}'}{2} + \omega_3^0 + \alpha \omega_3' \right) \sigma_{31}^0 + \left(\frac{e_{12}^0}{2} + \omega_3^0 \right) \alpha \sigma_{31}' + (1 + e_{22}^0 + \alpha e_{22}') \sigma_{32}^0 \right. \right. \\
& + (1 + e_{22}^0) \alpha \sigma_{32}' + \left(\frac{e_{23}^0 + \alpha e_{23}'}{2} - \omega_1^0 - \alpha \omega_1' \right) \sigma_{33}^0 + \left(\frac{e_{23}^0}{2} - \omega_1^0 \right) \alpha \sigma_{33}' \left. \right] \left. \right\} \\
& + \left[(1 + e_{11}^0 + \alpha e_{11}') \sigma_{21}^0 + (1 + e_{11}^0) \alpha \sigma_{21}' + \left(\frac{e_{12}^0 + \alpha e_{12}'}{2} - \omega_3^0 - \alpha \omega_3' \right) \sigma_{22}^0 \right. \\
& \left. + \left(\frac{e_{12}^0}{2} - \omega_3^0 \right) \alpha \sigma_{22}' + \left(\frac{e_{13}^0 + \alpha e_{13}'}{2} + \omega_2^0 + \alpha \omega_2' \right) \sigma_{23}^0 + \left(\frac{e_{13}^0}{2} + \omega_2^0 \right) \alpha \sigma_{23}' \right] = 0 \quad (15)
\end{aligned}$$

and

$$\begin{aligned}
& \frac{\partial}{\partial r} \left\{ r \left[\left(\frac{e_{13}^0 + \alpha e_{13}'}{2} - \omega_2^0 - \alpha \omega_2' \right) \sigma_{11}^0 + \left(\frac{e_{13}^0}{2} - \omega_2^0 \right) \alpha \sigma_{11}' \right. \right. \\
& + \left. \left(\frac{e_{23}^0 + \alpha e_{23}'}{2} + \omega_1^0 + \alpha \omega_1' \right) \sigma_{12}^0 + \left(\frac{e_{23}^0}{2} + \omega_1^0 \right) \alpha \sigma_{12}' + (1 + e_{33}^0 + \alpha e_{33}') \sigma_{13}^0 \right. \\
& + \left. (1 + e_{33}^0) \alpha \sigma_{13}' \right\} + \frac{\partial}{\partial \phi} \left[\left(\frac{e_{13}^0 + \alpha e_{13}'}{2} - \omega_2^0 - \alpha \omega_2' \right) \sigma_{21}^0 + \left(\frac{e_{13}^0}{2} - \omega_2^0 \right) \alpha \sigma_{21}' \right. \\
& + \left. \left(\frac{e_{23}^0 + \alpha e_{23}'}{2} + \omega_1^0 + \alpha \omega_1' \right) \sigma_{22}^0 + \left(\frac{e_{23}^0}{2} + \omega_1^0 \right) \alpha \sigma_{22}' + (1 + e_{33}^0 + \alpha e_{33}') \sigma_{23}^0 \right. \\
& + \left. (1 + e_{33}^0) \alpha \sigma_{23}' \right] + \frac{\partial}{\partial x} \left\{ r \left[\left(\frac{e_{13}^0 + \alpha e_{13}'}{2} - \omega_2^0 - \alpha \omega_2' \right) \sigma_{31}^0 \right. \right. \\
& + \left. \left(\frac{e_{13}^0}{2} - \omega_2^0 \right) \alpha \sigma_{31}' + \left(\frac{e_{23}^0 + \alpha e_{23}'}{2} + \omega_1^0 + \alpha \omega_1' \right) \sigma_{32}^0 + \left(\frac{e_{23}^0}{2} + \omega_1^0 \right) \alpha \sigma_{32}' \right. \\
& + \left. (1 + e_{33}^0 + \alpha e_{33}') \sigma_{33}^0 + (1 + e_{33}^0) \alpha \sigma_{33}' \right\} = 0 \quad (16)
\end{aligned}$$

These equations are nonlinear in the initial position values. The equilibrium equations at the initial position can be similarly obtained by substituting only the initial parts of equation (13) into equations (9), (10) and (11), and dividing them by r .

$$\begin{aligned}
& \frac{\partial}{\partial r} \{ r [(1 + e_{11}^0) \sigma_{11}^0 + (\frac{e_{12}^0}{2} - \omega_3^0) \sigma_{12}^0 + (\frac{e_{13}^0}{2} + \omega_2^0) \sigma_{13}^0] + \frac{\partial}{\partial \phi} [(1 + e_{11}^0) \sigma_{21}^0 \\
& + (\frac{e_{12}^0}{2} - \omega_3^0) \sigma_{22}^0 + (\frac{e_{13}^0}{2} + \omega_2^0) \sigma_{23}^0] + \frac{\partial}{\partial x} \{ r [(1 + e_{11}^0) \sigma_{31}^0 + (\frac{e_{12}^0}{2} - \omega_3^0) \sigma_{32}^0 \\
& + (\frac{e_{13}^0}{2} + \omega_2^0) \sigma_{33}^0] \} - [(\frac{e_{12}^0}{2} + \omega_3^0) \sigma_{21}^0 + (1 + e_{22}^0) \sigma_{22}^0 \\
& + (\frac{e_{23}^0}{2} - \omega_1^0) \sigma_{23}^0] = 0
\end{aligned} \tag{17}$$

and

$$\begin{aligned}
& \frac{\partial}{\partial r} \{ r [(\frac{e_{12}^0}{2} + \omega_3^0) \sigma_{11}^0 + (1 + e_{22}^0) \sigma_{12}^0 + (\frac{e_{23}^0}{2} - \omega_1^0) \sigma_{13}^0] \} \\
& + \frac{\partial}{\partial \phi} [(\frac{e_{12}^0}{2} + \omega_3^0) \sigma_{21}^0 + (1 + e_{22}^0) \sigma_{22}^0 + (\frac{e_{23}^0}{2} - \omega_1^0) \sigma_{23}^0] + \frac{\partial}{\partial x} \{ r [(\frac{e_{12}^0}{2} + \omega_3^0) \sigma_{31}^0 \\
& + (1 + e_{22}^0) \sigma_{32}^0 + (\frac{e_{23}^0}{2} - \omega_1^0) \sigma_{33}^0] \} + [(1 + e_{11}^0) \sigma_{21}^0 + (\frac{e_{12}^0}{2} - \omega_3^0) \sigma_{22}^0 \\
& + (\frac{e_{13}^0}{2} + \omega_2^0) \sigma_{23}^0] = 0
\end{aligned} \tag{18}$$

$$\begin{aligned}
& \frac{\partial}{\partial r} \{ r [(\frac{e_{13}^0}{2} - \omega_2^0) \sigma_{11}^0 + (\frac{e_{23}^0}{2} + \omega_1^0) \sigma_{12}^0 + (1 + e_{33}^0) \sigma_{13}^0] + \frac{\partial}{\partial \phi} [(\frac{e_{13}^0}{2} - \omega_2^0) \sigma_{21}^0 \\
& + (\frac{e_{23}^0}{2} + \omega_1^0) \sigma_{22}^0 + (1 + e_{22}^0) \sigma_{23}^0] + \frac{\partial}{\partial x} \{ r [(\frac{e_{13}^0}{2} - \omega_2^0) \sigma_{31}^0 + (\frac{e_{23}^0}{2} + \omega_1^0) \sigma_{32}^0 \\
& + (1 + e_{33}^0) \sigma_{33}^0] \} = 0
\end{aligned} \tag{19}$$

These equations also are nonlinear in the initial position values. To obtain a linear set of equations, subtract the

previous two sets of equations and divide through by α . The resulting equations are linear in terms of the incremental values.

Subtracting equation (17) from equation (14) yields,

$$\begin{aligned}
& \frac{\partial}{\partial r} \{ r [e'_{11} \sigma_{11}^0 + (1 + e_{11}^0) \sigma'_{11} + (\frac{e'_{12}}{2} - \omega_3^0) \sigma'_{12} + (\frac{e'_{12}}{2} - \omega_3^0) \sigma_{12}^0 \\
& + (\frac{e'_{13}}{2} + \omega_2^0) \sigma'_{13} + (\frac{e'_{13}}{2} + \omega_2^0) \sigma_{13}^0] \} + \frac{\partial}{\partial \phi} [e'_{11} \sigma_{21}^0 + (1 + e_{11}^0) \sigma'_{21} \\
& + (\frac{e'_{12}}{2} - \omega_3^0) \sigma_{22}^0 + (\frac{e'_{12}}{2} - \omega_3^0) \sigma'_{22} + (\frac{e'_{13}}{2} + \omega_2^0) \sigma_{23}^0 + (\frac{e'_{13}}{2} + \omega_2^0) \sigma'_{23}] \\
& + \frac{\partial}{\partial x} \{ r [e'_{11} \sigma_{31}^0 + (1 + e_{11}^0) \sigma'_{31} + (\frac{e'_{12}}{2} - \omega_3^0) \sigma'_{32} + (\frac{e'_{12}}{2} - \omega_3^0) \sigma_{32}^0 \\
& + (\frac{e'_{13}}{2} + \omega_2^0) \sigma_{33}^0 + (\frac{e'_{13}}{2} + \omega_2^0) \sigma'_{33}] \} - [(\frac{e'_{12}}{2} + \omega_3^0) \sigma_{21}^0 + (\frac{e'_{12}}{2} + \omega_3^0) \sigma'_{21} \\
& + e'_{22} \sigma_{22}^0 + (1 + e_{22}^0) \sigma'_{22} + (\frac{e'_{23}}{2} - \omega_1^0) \sigma_{23}^0 + (\frac{e'_{23}}{2} - \omega_1^0) \sigma'_{23}] = 0 \quad (20)
\end{aligned}$$

Subtracting equation (18) from equation (15) yields,

$$\begin{aligned}
& \frac{\partial}{\partial r} \{ r [(\frac{e'_{12}}{2} + \omega_3^0) \sigma_{11}^0 + (\frac{e'_{12}}{2} + \omega_3^0) \sigma'_{11} + e'_{22} \sigma_{12}^0 + (1 + e_{22}^0) \sigma'_{12} + (\frac{e'_{23}}{2} - \omega_1^0) \sigma_{13}^0 \\
& + (\frac{e'_{23}}{2} - \omega_1^0) \sigma'_{13}] \} + \frac{\partial}{\partial \phi} [(\frac{e'_{12}}{2} + \omega_3^0) \sigma_{21}^0 + (\frac{e'_{12}}{2} + \omega_3^0) \sigma'_{21} + e'_{22} \sigma_{22}^0 \\
& + (1 + e_{22}^0) \sigma'_{22} + (\frac{e'_{23}}{2} - \omega_1^0) \sigma_{23}^0 + (\frac{e'_{23}}{2} - \omega_1^0) \sigma'_{23}] + \frac{\partial}{\partial x} \{ r [(\frac{e'_{12}}{2} + \omega_3^0) \sigma_{31}^0 \\
& + (\frac{e'_{12}}{2} + \omega_3^0) \sigma'_{31} + e'_{22} \sigma_{32}^0 + (1 + e_{22}^0) \sigma'_{32} + (\frac{e'_{23}}{2} - \omega_1^0) \sigma_{33}^0 + (\frac{e'_{23}}{2} - \omega_1^0) \sigma'_{33}] \} \\
& + [e'_{11} \sigma_{21}^0 + (1 + e_{11}^0) \sigma'_{21} + (\frac{e'_{12}}{2} - \omega_3^0) \sigma_{22}^0 + (\frac{e'_{12}}{2} - \omega_3^0) \sigma'_{22} + (\frac{e'_{13}}{2} + \omega_2^0) \sigma_{23}^0 + \\
& (\frac{e'_{13}}{2} + \omega_2^0) \sigma'_{23}] = 0 \quad (21)
\end{aligned}$$

Subtracting equation (19) from equation (16) yields,

$$\begin{aligned}
& \frac{\partial}{\partial r} \left\{ r \left[\left(\frac{e_{13}'}{2} - \omega_2' \right) \sigma_{11}^0 + \left(\frac{e_{13}^0}{2} - \omega_2^0 \right) \sigma_{11}' + \left(\frac{e_{23}'}{2} + \omega_1' \right) \sigma_{12}^0 + \left(\frac{e_{23}^0}{2} + \omega_1^0 \right) \sigma_{12}' \right. \right. \\
& + e_{33}' \sigma_{13}^0 + (1 + e_{33}^0) \sigma_{13}' \left. \right\} + \frac{\partial}{\partial \phi} \left[\left(\frac{e_{13}'}{2} - \omega_2' \right) \sigma_{21}^0 + \left(\frac{e_{13}^0}{2} - \omega_2^0 \right) \sigma_{21}' \right. \\
& + \left. \left(\frac{e_{23}'}{2} + \omega_1' \right) \sigma_{22}^0 + \left(\frac{e_{23}^0}{2} + \omega_1^0 \right) \sigma_{22}' + e_{33}' \sigma_{23}^0 + (1 + e_{33}^0) \sigma_{23}' \right] \\
& + \frac{\partial}{\partial x} \left\{ r \left[\left(\frac{e_{13}'}{2} - \omega_2' \right) \sigma_{31}^0 + \left(\frac{e_{13}^0}{2} - \omega_2^0 \right) \sigma_{31}' + \left(\frac{e_{23}'}{2} + \omega_1' \right) \sigma_{32}^0 + \left(\frac{e_{23}^0}{2} + \omega_1^0 \right) \sigma_{32}' \right. \right. \\
& \left. \left. + e_{33}' \sigma_{33}^0 + (1 + e_{33}^0) \sigma_{33}' \right] \right\} = 0 \tag{22}
\end{aligned}$$

these equations are linear.

C. Linearization of the Initial Position Equilibrium Equations

In a large number of practical problems, the angles of rotation corresponding to the initial position of equilibrium are either zero or of the same order of magnitude as the shears and elongations. In section I.A it was assumed that the shears and elongations were small compared to unity, thus reducing the stress, strain and rotation expressions to the linear expressions. Since it is assumed that these angles of rotation are small, then their products with the stresses may be neglected compared to the stresses themselves.

Implementing these assumptions in equations (17), (18), (19) the initial position equilibrium equations reduce to the following forms,

$$\frac{\partial \sigma_{11}^0}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{21}^0}{\partial \phi} + \frac{\partial \sigma_{31}^0}{\partial x} + \frac{\sigma_{11}^0 - \sigma_{22}^0}{r} = 0$$

$$\frac{\partial \sigma_{12}^0}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{22}^0}{\partial \phi} + \frac{\partial \sigma_{32}^0}{\partial x} + \frac{\sigma_{12}^0 + \sigma_{21}^0}{r} = 0$$

$$\frac{\partial \sigma_{13}^0}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{23}^0}{\partial \phi} + \frac{\partial \sigma_{33}^0}{\partial x} + \frac{\sigma_{13}^0}{r} = 0 \quad (23)$$

II. BOUNDARY CONDITIONS

A. Initial Position Boundary Conditions

Since linear theory applies to the initial position problem, equilibrium on the surface yields,¹

$$\begin{aligned}
 \sigma_{11}^0 \cos (n,r) + \sigma_{21}^0 \cos (n,\phi) + \sigma_{31}^0 \cos (n,x) &= f_1(w^0,v^0,u^0) \\
 \sigma_{12}^0 \cos (n,r) + \sigma_{22}^0 \cos (n,\phi) + \sigma_{32}^0 \cos (n,x) &= f_2(w^0,v^0,u^0) \\
 \sigma_{13}^0 \cos (n,r) + \sigma_{23}^0 \cos (n,\phi) + \sigma_{33}^0 \cos (n,x) &= f_3(w^0,v^0,u^0) \quad (24)
 \end{aligned}$$

where

n = surface normal before deformation

f_1, f_2, f_3 = projections of the surface forces

for cylindrical coordinates as shown in figure 2.

$$\begin{aligned}
 f_1^0 &= -p \cos (n^0,r) \\
 f_2^0 &= -p \cos (n^0,\phi) \\
 f_3^0 &= -p \cos (n^0,x) \quad (25)
 \end{aligned}$$

Substituting these values into equation (24) after noting that the direction cosines are given as (1,0,0) yields,

$$\begin{aligned}
 \sigma_{11}^0 &= -p \\
 \sigma_{12}^0 &= 0 \\
 \sigma_{13}^0 &= 0 \quad (26)
 \end{aligned}$$

¹Novozhilov, p. 166.

These are the boundary conditions for the initial position problem.

B. Final Position Boundary Conditions

At the final position, equilibrium on the surface yields by a similar method,

$$\begin{aligned}
 S_{11} &= f_1(w^0 + \alpha w', v^0 + \alpha v', u^0 + \alpha u') \\
 S_{12} &= f_2(w^0 + \alpha w', v^0 + \alpha v', u^0 + \alpha u') \\
 S_{13} &= f_3(w^0 + \alpha w', v^0 + \alpha v', u^0 + \alpha u') \quad (27)
 \end{aligned}$$

where the S's are given by equation (4).

From equations (27), (4) and (13) and after dropping the α^2 terms, the final position boundary conditions are obtained in the following form,

$$\begin{aligned}
 f_1(w^0 + \alpha w', v^0 + \alpha v', u^0 + \alpha u') &= \sigma_{11}^0 + \alpha[\sigma_{11}' + e_{11}^0 \sigma_{11}' + e_{11}' \sigma_{11}^0 \\
 &+ \sigma_{12}' \left(\frac{e_{12}^0}{2} - \omega_3^0\right) + \sigma_{12}^0 \left(\frac{e_{12}'}{2} - \omega_3'\right) + \sigma_{13}' \left(\frac{e_{13}^0}{2} + \omega_2^0\right) + \sigma_{13}^0 \left(\frac{e_{13}'}{2} + \omega_2'\right)] \\
 f_2(w^0 + \alpha w', v^0 + \alpha v', u^0 + \alpha u') &= \sigma_{12}^0 + \alpha[\sigma_{12}' + e_{22}^0 \sigma_{12}' + e_{22}' \sigma_{12}^0 \\
 &+ \left(\frac{e_{12}^0}{2} + \omega_3^0\right) \sigma_{11}' + \left(\frac{e_{23}'}{2} - \omega_1'\right) \sigma_{13}^0 + \left(\frac{e_{12}'}{2} + \omega_3'\right) \sigma_{11}^0 + \left(\frac{e_{23}^0}{2} - \omega_1^0\right) \sigma_{13}'] \\
 f_3(w^0 + \alpha w', v^0 + \alpha v', u^0 + \alpha u') &= \sigma_{13}^0 + \alpha[\sigma_{13}' + e_{33}^0 \sigma_{13}' + e_{33}' \sigma_{13}^0 \\
 &+ \left(\frac{e_{13}^0}{2} - \omega_2^0\right) \sigma_{11}' + \left(\frac{e_{13}'}{2} - \omega_2'\right) \sigma_{11}^0 + \left(\frac{e_{23}^0}{2} + \omega_1^0\right) \sigma_{12}' + \left(\frac{e_{23}'}{2} + \omega_1'\right) \sigma_{12}^0] \quad (28)
 \end{aligned}$$

C. Incremental Problem Boundary Conditions

The incremental value boundary conditions can be obtained by subtracting equation (26) from equation (28), dividing through by α , and taking the limit as α approaches zero.

$$[\sigma_{11}' + e_{11}^0 \sigma_{11}' + e_{11}' \sigma_{11}^0 + \sigma_{12}' (\frac{e_{12}^0}{2} - \omega_3^0) + \sigma_{12}^0 (\frac{e_{12}'}{2} - \omega_3') + \sigma_{13}' (\frac{e_{13}^0}{2} + \omega_2^0) + \sigma_{13}^0 (\frac{e_{13}'}{2} + \omega_2')] = \lim_{\alpha \rightarrow 0} \{ \frac{1}{\alpha} [f_1(w^0 + \alpha w', v^0 + \alpha v', u^0 + \alpha u') - f_1(w^0, v^0, u^0)] \}$$

$$[\sigma_{12}' + e_{22}^0 \sigma_{12}' + e_{22}' \sigma_{12}^0 + (\frac{e_{12}^0}{2} + \omega_3^0) \sigma_{11}' + (\frac{e_{12}'}{2} + \omega_3') \sigma_{11}^0 + (\frac{e_{23}^0}{2} - \omega_1^0) \sigma_{13}' + (\frac{e_{23}'}{2} - \omega_1') \sigma_{13}^0] = \lim_{\alpha \rightarrow 0} \{ \frac{1}{\alpha} [f_2(w^0 + \alpha w', v^0 + \alpha v', u^0 + \alpha u') - f_2(w^0, v^0, u^0)] \}$$

$$[\sigma_{13}' + e_{33}^0 \sigma_{13}' + e_{33}' \sigma_{13}^0 + (\frac{e_{13}^0}{2} - \omega_2^0) \sigma_{11}' + (\frac{e_{13}'}{2} - \omega_2') \sigma_{11}^0 + (\frac{e_{23}^0}{2} + \omega_1^0) \sigma_{12}' + (\frac{e_{23}'}{2} + \omega_1') \sigma_{12}^0] = \lim_{\alpha \rightarrow 0} \{ \frac{1}{\alpha} [f_3(w^0 + \alpha w', v^0 + \alpha v', u^0 + \alpha u') - f_3(w^0, v^0, u^0)] \} \quad (29)$$

If the surface forces f_1, f_2, f_3 are independent of displacement, then the right hand sides of equations (29) become zero in the limit as α approaches zero. This is not always the case. For example, under a hydrostatic pressure the surface forces remain normal to the surface after deformation.

In the remainder of this work, only hydrostatic pressure loadings are considered. The other types of loadings do not effect the theory being developed to solve the ring problem. The line of action of the loading is assumed to remain normal to ring during bifurcation. The terms on the right hand sides of equation (29) are the difference between the initial position values given in equation (25) and the final position values given by the following,

$$\begin{aligned} f'_1 &= -p \cos (n', r) \\ f'_2 &= -p \cos (n', \phi) \\ f'_3 &= -p \cos (n', d) \end{aligned} \quad (30)$$

After substituting into equation (30) for the cosines in terms of the strains, rotations, and elongations,² the right hand sides of equation (29) become,

$$\begin{aligned} \lim_{\alpha \rightarrow 0} \left[\frac{1}{\alpha} (f'_1 - f_1^0) \right] &= -p \lim_{\alpha \rightarrow 0} \left\{ \frac{1}{\alpha} \left[\frac{1 + \alpha e'_{11}}{1 + \alpha E'_1} - 1 \right] \right\} = -p e'_{11} \\ \lim_{\alpha \rightarrow 0} \left[\frac{1}{\alpha} (f'_2 - f_2^0) \right] &= -p \lim_{\alpha \rightarrow 0} \left\{ \frac{1}{\alpha} \left[\frac{\alpha \left(\frac{e'_{12}}{2} + \omega'_3 \right)}{1 + \alpha E'_1} \right] \right\} = -p \left(\frac{e'_{12}}{2} + \omega'_3 \right) \\ \lim_{\alpha \rightarrow 0} \left[\frac{1}{\alpha} (f'_3 - f_3^0) \right] &= -p \lim_{\alpha \rightarrow 0} \left\{ \frac{1}{\alpha} \left[\frac{\alpha \left(\frac{e'_{13}}{2} - \omega'_2 \right)}{1 + \alpha E'_1} \right] \right\} = -p \left(\frac{e'_{13}}{2} - \omega'_2 \right) \end{aligned} \quad (31)$$

Substituting equation (31) into equation (29) yields,

²IBID., p. 7.

$$\begin{aligned}
& \sigma_{11}' + e_{11}^0 \sigma_{11}' + e_{11}' \sigma_{11}^0 + \sigma_{12}' \left(\frac{e_{12}^0}{2} - \omega_3^0 \right) + \sigma_{12}^0 \left(\frac{e_{12}'}{2} - \omega_3' \right) \\
& \quad \sigma_{13}' \left(\frac{e_{13}^0}{2} + \omega_2^0 \right) + \sigma_{13}^0 \left(\frac{e_{13}'}{2} + \omega_2' \right) = -p e_{11}' \\
& \sigma_{12}' + e_{22}^0 \sigma_{12}' + e_{22}' \sigma_{12}^0 + \sigma_{11}' \left(\frac{e_{12}^0}{2} + \omega_3^0 \right) + \sigma_{11}^0 \left(\frac{e_{12}'}{2} + \omega_3' \right) \\
& \quad + \sigma_{13}' \left(\frac{e_{23}^0}{2} - \omega_1^0 \right) + \sigma_{13}^0 \left(\frac{e_{23}'}{2} - \omega_1' \right) = -p \left(\frac{e_{12}'}{2} - \omega_2' \right) \\
& \sigma_{13}' + e_{33}^0 \sigma_{13}' + e_{33}' \sigma_{13}^0 + \sigma_{11}' \left(\frac{e_{13}^0}{2} - \omega_2^0 \right) + \sigma_{11}^0 \left(\frac{e_{13}'}{2} - \omega_2' \right) \\
& \quad + \sigma_{12}' \left(\frac{e_{23}^0}{2} + \omega_1^0 \right) + \sigma_{12}^0 \left(\frac{e_{23}'}{2} + \omega_1' \right) = -p \left(\frac{e_{13}'}{2} - \omega_2' \right) \quad (32)
\end{aligned}$$

III. SOLUTION TO THE INITIAL POSITION PROBLEM

A. Lamé Type Solution to the Thick Ring Initial Problem

The equilibrium equations given in equation (23) may be written in the familiar shell type coordinate system as shown in figure 3,

$$r = a + z$$

and ϕ and x are unchanged.

The above substitution yields,

$$\begin{aligned} \frac{\partial \sigma_{11}^0}{\partial z} + \frac{1}{a+z} \frac{\partial \sigma_{21}^0}{\partial \phi} + \frac{\partial \sigma_{31}^0}{\partial x} + \frac{\sigma_{11}^0 - \sigma_{22}^0}{a+z} &= 0 \\ \frac{\partial \sigma_{12}^0}{\partial z} + \frac{1}{a+z} \frac{\partial \sigma_{22}^0}{\partial \phi} + \frac{\partial \sigma_{32}^0}{\partial x} + \frac{\sigma_{12}^0 + \sigma_{21}^0}{a+z} &= 0 \\ \frac{\partial \sigma_{13}^0}{\partial z} + \frac{1}{a+z} \frac{\partial \sigma_{23}^0}{\partial \phi} + \frac{\partial \sigma_{33}^0}{\partial x} + \frac{\sigma_{13}^0}{a+z} &= 0 \end{aligned} \quad (33)$$

The load is assumed to be an inwardly directed uniform radial pressure, p , in pounds per square inch. It is further assumed that the magnitude of this pressure does not vary with the deformations.

The boundary conditions on the stress are given by equation (26) on the top surface of the ring.

$$\begin{aligned} \sigma_{11}^0 &= -p \\ \sigma_{12}^0 &= \sigma_{13}^0 = 0 \end{aligned} \quad (34)$$

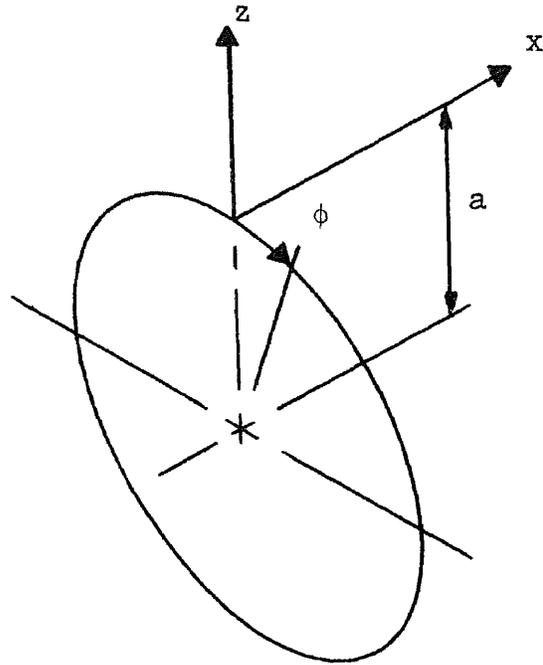


Figure 3 Shell-type Coordinate System

The displacement boundary conditions are such that the cross section of the ring does not rotate at the initial equilibrium position. For this condition to be satisfied, the cross section must be symmetrical about the radial axis.

To find the solution to the initial problem of a thick ring a particular cross section must be chosen. A rectangular cross section was selected since its initial position solution was given by Lamé¹ in the form of a long cylinder subjected to a uniform pressure.

$$w^0 = A(a + z) + \frac{B}{(a + z)}$$

$$v^0 = u^0 = 0 \quad (35)$$

where A and B are constants.

The displacement w^0 may be expressed in the following form by expanding the second term in a geometric series and retaining only the linear term.

$$w^0 = C + Dz \quad (36)$$

where

$$C = aA + \frac{B}{a} \quad (37)$$

and

$$D = A - \frac{B}{a^2} \quad (38)$$

The expressions for A and B are given as the following:²

¹Timoshenko and Goodier, p. 58.

²S. Timoshenko, Strength of Materials, 3rd ed. (Princeton, 1956), p. 210.

$$A = - \frac{pR_o^2(1 - \mu)}{(R_o^2 - R_i^2)E} \quad (39)$$

and

$$B = - \frac{pR_o^2R_i^2(1 + \mu)}{(R_o^2 - R_i^2)E} \quad (40)$$

where

p = external pressure

R_o = outer radius

R_i = inner radius

E = Young's modulus of the material

μ = Poisson's ratio of the material

Substituting equations (39) and (40) into equations (37) and (38), with K defined as the following expression,

$$K = - \frac{pR_o^2}{E(R_o^2 - R_i^2)} \quad (41)$$

yields

$$C = a(1 - \mu)K + \frac{(1 + \mu)KR_i^2}{a} \quad (42)$$

and

$$D = (1 - \mu)K - \frac{(1 + \mu)KR_i^2}{a} \quad (43)$$

The initial position stresses, strains and rotations can be evaluated using the following relationships:

Hooke's Law³

$$\begin{aligned}\sigma_{11} &= \frac{E}{1 + \mu} \left[e_{11} + \frac{\mu}{1 - 2\mu} (e_{11} + e_{22} + e_{33}) \right] \\ \sigma_{22} &= \frac{E}{1 + \mu} \left[e_{22} + \frac{\mu}{1 - 2\mu} (e_{11} + e_{22} + e_{33}) \right] \\ \sigma_{33} &= \frac{E}{1 + \mu} \left[e_{33} + \frac{\mu}{1 - 2\mu} (e_{11} + e_{22} + e_{33}) \right] \\ \sigma_{12} &= G e_{12} \\ \sigma_{23} &= G e_{23} \\ \sigma_{13} &= G e_{13}\end{aligned}\tag{44}$$

strain-displacement relationships⁴

$$\begin{aligned}e_{11} &= w, z & e_{12} &= v, x + \frac{w, \phi - v}{a + z} \\ e_{22} &= \frac{v, \phi + w}{a + z} & e_{13} &= u, z + w, x \\ e_{33} &= u, x & e_{23} &= v, x + \frac{u, \phi}{a + z}\end{aligned}\tag{45}$$

and rotation-displacement⁵

$$\begin{aligned}\omega_1 &= \frac{1}{2} \left(\frac{u, \phi}{a + z} - v, x \right) \\ \omega_2 &= \frac{1}{2} (w, x - u, z) \\ \omega_3 &= \frac{1}{2} \left(\frac{v - w, \phi}{a + z} + v, z \right)\end{aligned}\tag{46}$$

³Novozhilov, p. 119.

⁴I.S. Sokolnikoff, Mathematical Theory of Elasticity, 2nd ed. (New York, 1956), p. 183.

⁵Novozhilov, pp. 57-58.

Hooke's Law in terms of displacements

$$\begin{aligned}
 \sigma_{11} &= \frac{E}{1 + \mu} \left[w_{,z} + \frac{\mu}{1 - 2\mu} \left(w_{,z} + \frac{v_{,\phi} + w}{a + z} + u_{,x} \right) \right] \\
 \sigma_{22} &= \frac{E}{1 + \mu} \left[\frac{v_{,\phi} + w}{a + z} + \frac{\mu}{1 - 2\mu} \left(w_{,z} + \frac{v_{,\phi} + w}{a + z} + u_{,x} \right) \right] \\
 \sigma_{33} &= \frac{E}{1 + \mu} \left[u_{,x} + \frac{\mu}{1 - 2\mu} \left(w_{,z} + \frac{v_{,\phi} + w}{a + z} + u_{,x} \right) \right] \\
 \sigma_{12} &= G \left[v_{,z} + \frac{w_{,\phi} - v}{a + z} \right] \\
 \sigma_{13} &= G [u_{,z} + w_{,x}] \\
 \sigma_{23} &= G \left[v_{,x} + \frac{u_{,\phi}}{a + z} \right] \tag{47}
 \end{aligned}$$

Substituting the following values for the critical position displacements

$$\begin{aligned}
 w^0 &= C + Dz \\
 u^0 &= v^0 = 0 \tag{48}
 \end{aligned}$$

into the preceding relationships yields for the initial position strains,

$$\begin{aligned}
 e_{11}^0 &= D \\
 e_{22}^0 &= \frac{C + Dz}{a + z} \tag{49}
 \end{aligned}$$

All the initial rotations are zero,

$$\omega_j^0 = 0 \tag{50}$$

the initial position stresses are given by the following

$$\begin{aligned}\sigma_{11}^0 &= \frac{E}{1 + \mu} \left[MD + N \left(\frac{C + Dz}{a + z} \right) \right] \\ \sigma_{22}^0 &= \frac{E}{1 + \mu} \left[M \left(\frac{C + Dz}{a + z} \right) + ND \right] \\ \sigma_{33}^0 &= \frac{E}{1 + \mu} \left[N \left(Dz + \frac{C + Dz}{a + z} \right) \right]\end{aligned}\quad (51)$$

where

$$M = \frac{1 - \mu}{1 - 2\mu} \quad (52)$$

and

$$N = \frac{\mu}{1 - 2\mu} \quad (53)$$

B. Constant Radial Displacement Solution to the Thin Ring Initial Problem

For the case of the thin ring, the dimensionless parameter K given in equation (41) reduces to

$$K = - \frac{Pa}{2EA} \quad (54)$$

where P is a radial line in pounds per inch and A is the cross sectional area. The intermediate steps to this result are given later in equations (188), (189), (190) and (191).

The initial position displacement of a thin ring is assumed to be a constant through the thickness. From equation (36) with $D = 0$, and equation (42) with $R_i = a$, and $\mu = 0$

the following well-known expression⁶ is obtained

$$w^0 = - \frac{Pa^2}{AE} \quad (55)$$

Substituting equation (51) and also $v^0 = u^0 = 0$ into equation (45) and (46) yields for the thin ring initial position strains and rotations,

$$e_{22}^0 = \frac{w^0}{a+z}$$

$$e_{11}^0 = e_{33}^0 = e_{12}^0 = e_{13}^0 = e_{23}^0 = 0 \quad (56)$$

and

$$\omega_1^0 = \omega_2^0 = \omega_3^0 = 0 \quad (57)$$

Similarly, equation (47) yields for the thin ring initial position stresses,

$$\sigma_{11}^0 = \sigma_{33}^0 = \frac{EN}{1+\mu} \frac{w^0}{(a+z)}$$

$$\sigma_{22}^0 = \frac{EM}{1+\mu} \frac{w^0}{(a+z)}$$

$$\sigma_{12}^0 = \sigma_{13}^0 = \sigma_{23}^0 = 0 \quad (58)$$

⁶Wah, p. 969.

IV. DEFINITION OF THE INCREMENTAL PROBLEM FOR THE
PARTICULAR INITIAL PROBLEM SOLUTION

A. Reduced Form of the Incremental Problem Equilibrium Equations

By using equations (49), (50) and (51) to define the initial position strains, rotations and stresses respectively, the zero valued terms may be eliminated from the incremental equilibrium equations in equations (20), (21) and (22). Also, by incorporating the assumptions used to linearize the initial problem given in section I.C the three equilibrium equations may be reduced to the following forms,

$$\begin{aligned} & \frac{\partial}{\partial z} [(a + z)(e_{11}'\sigma_{11}^0 + \sigma_{11}')] + \frac{\partial}{\partial \phi} [\sigma_{21}' + (\frac{e_{12}'}{2} - \omega_3')\sigma_{22}^0] \\ & + \frac{\partial}{\partial x} \{ (a + z) [\sigma_{31}' + (\frac{e_{13}'}{2} + \omega_2')\sigma_{33}^0] \} - (e_{22}'\sigma_{22}^0 + \sigma_{22}') = 0 \quad (59) \end{aligned}$$

$$\begin{aligned} & \frac{\partial}{\partial z} (a + z) [(\frac{e_{12}'}{2} + \omega_3')\sigma_{11}^0 + \sigma_{12}'] + \frac{\partial}{\partial \phi} (e_{22}'\sigma_{22}^0 + \sigma_{22}') \\ & + \frac{\partial}{\partial x} \{ (a + z) [\sigma_{32}' + (\frac{e_{23}'}{2} - \omega_1')\sigma_{33}^0] \} + [\sigma_{21}' + (\frac{e_{12}'}{2} - \omega_3')\sigma_{22}^0] = 0 \quad (60) \end{aligned}$$

$$\begin{aligned} & \frac{\partial}{\partial z} \{ (a + z) [(\frac{e_{13}'}{2} - \omega_2')\sigma_{11}^0 + \sigma_{13}'] \} + \frac{\partial}{\partial \phi} [(\frac{e_{23}'}{2} + \omega_1')\sigma_{22}^0 + \sigma_{23}'] \\ & + \frac{\partial}{\partial x} [(a + z)(e_{33}'\sigma_{33}^0 + \sigma_{33}')] = 0 \quad (61) \end{aligned}$$

B. Reduced Form of Incremental Boundary Conditions

By using equations (49), (50) and (51) for the initial position values and the same linearizing assumptions used above in equations (32) and (34) yields

$$\sigma'_{11} = \sigma'_{12} = \sigma'_{13} = 0 \quad (62)$$

V. REDUCTION OF THE INCREMENTAL PROBLEM TO A
THIN RING THEORY PROBLEM

A. Displacement Expansions in Terms of Unknown Functions

The linear equilibrium equations for the incremental problem as expressed by equations (59) and (60) represents a three dimensional theory of elasticity problem. These equations may be reduced to a one dimensional ring theory problem by using several methods. In this analysis the displacements are expressed in the form of a power series. In the theory of the deformation of rods as given by Novozhilov¹, the displacement of an arbitrary point on a rod can be expressed in terms of the following power series,

$$\begin{aligned}w' &= w_0 + x w_1 + z w_2 + x z w_3 + \dots \\u' &= u_0 + x u_1 + z u_2 + x z u_3 + \dots \\v' &= v_0 + x v_1 + z v_2 + x z v_3 + \dots\end{aligned}\tag{63}$$

where w_0 , u_0 , v_0 are the displacements of the origin of the x-z axes, which corresponds to the centroid of the cross section of the ring. The number of terms retained in each series is based on the desired accuracy. The warping function, v_3 , must be retained for non-circular cross sections. From the exact series which is used for each direction one equation must be generated for each variable retained in the series. The procedure for obtaining these equations is the following.

¹Novozhilov, p. 198-216

The displacement series are substituted into the strain-displacement and rotation-displacement relationships given in equations (45) and (46) respectively, and also into Hooke's Law in terms of displacements given in equation (47). The results of these substitutions are then substituted into the incremental equilibrium equations i.e. equations (59), (60) and (61).

The equilibrium equations are written in each of the three directions w' , v' , u' , respectively. A system of equations are generated by taking the product of each retained displacement function coefficient with the equilibrium equations in the same directions. This set of equations is then integrated over the cross section of the ring.

The resulting system of differential equations may be reduced to a system of linear algebraic equations by expanding each of the displacement functions in a Fourier series and eliminating the angle ϕ from the equations. These equations may be manipulated into the form of an eigenvalue problem which may be solved directly.

The procedures stated above are used in the succeeding sections in both the thin ring and thick ring analysis.

1. Equilibrium in terms of displacement functions. In the buckling of thin rings, it can be assumed that the normal strains in the radial and out of plane directions are negligible compared to the others. The displacement expansions may be assumed to be the following:

$$\begin{aligned}
w' &= w_0 + x w_1 \\
v' &= v_0 + z v_1 + x v_2 + x z v_3 \\
u' &= u_0 + z u_1
\end{aligned} \tag{64}$$

Using these expansions, the strain-displacement and rotation-displacement relationships given by equations (45) and (46) respectively, Hooke's Law in terms of displacement given by equation (47), and the boundary conditions for the incremental problem given by equation (62), one can generate eight differential equations. By integrating the product of each displacement coefficient with the equilibrium equation in the same direction as that displacement, one equation will be obtained for each unknown displacement function retained in the series.

For example, equation (59) may be integrated, term by term. The first term yields after integrating on z ,

$$\begin{aligned}
&\iint \frac{\partial}{\partial z} [(a + z)(e_{11}' \sigma_{11}^0 + \sigma_{11}')] dz dx \\
&= \int [(a + z)(e_{11}' \sigma_{11}^0 + \sigma_{11}')] dx \\
&\hspace{15em} (Z\text{-BDNY})
\end{aligned} \tag{65}$$

From the strain-displacement relationships given in equations (45) and the displacement expansions given in equation (64),

$$e_{11}' = w, \quad z' = 0 \tag{66}$$

and from the boundary conditions given in equation (65) $\sigma_{11}' = 0$ on the boundary. Therefore, equation (66) equals zero.

The integral of the second term of equation (59) is given as

$$\iint \frac{\partial}{\partial \phi} [\sigma_{21}' + (\frac{e_{12}'}{2} - \omega_3') \sigma_{22}^0] dz dx \quad (67)$$

From the stress-displacement relationships given in equations (47),

$$\sigma_{21}' = G[v_{,x}' + \frac{w_{,\phi}' - v_{,z}'}{a+z}] \quad (68)$$

then, after substituting into equation (68) for the displacements from equations (64), the first term of equation (67) may be written as,

$$G \iiint \left[(v_{2,\phi}' + zv_{3,\phi}') + \frac{w_{0,\phi\phi}' + x w_{1,\phi\phi}' - v_{0,\phi}' - zv_{1,\phi}' - xv_{2,\phi}' - xzv_{3,\phi}'}{(a+z)} \right] dz dx \quad (69)$$

After integrating over the cross section and utilizing the symmetry in the x-direction as given in section III.A, and the integral forms given in Appendix III equation (69) becomes,

$$G[v_{1,\phi}' A + (w_{0,\phi\phi}' - v_{0,\phi}') I_1 - v_{1,\phi}' I_2] \quad (70)$$

The initial position stress term of equation (67) is given in terms of the initial position displacements in equation (58). The strain-displacement relationships given in equation (45), and the rotation-displacement relationships given in equation (46) when substituted into the second term of equation (67) yield,

$$\begin{aligned}
& \iint \frac{\partial}{\partial \phi} \left[\sigma_{22}^0 \left(\frac{e_{12}'}{2} - \omega_3' \right) \right] dz dx \\
= & \iint \left[\frac{EM}{(1+\mu)} \left(\frac{w^0}{a+z} \right) \left(\frac{w, \phi \phi^{-v, \phi}}{a+z} \right) \right] dz dx \quad (71)
\end{aligned}$$

M is given in equation (52). After substituting into equation (71) for the displacements given by equation (64) the following integral is obtained,

$$\left(\frac{EM}{1+\mu} \right) w^0 \iint \frac{(w_{0,\phi\phi} + xw_{1,\phi\phi} - v_{0,\phi} - zv_{1,\phi} - xv_{2,\phi} - xzv_{3,\phi}) dz dx}{(a+z)^2} \quad (72)$$

After integrating in a similar manner as in equation (69) the following form is obtained,

$$\left(\frac{EM}{1+\mu} \right) w^0 [(w_{0,\phi\phi} - v_{0,\phi}) I_3 - v_{1,\phi} I_4] \quad (73)$$

This term contains the effect of the change of inplane slope on the load.

The integral of the third term of equation (59) is given as

$$\iint \frac{\partial}{\partial x} \{ (a+z) [\sigma_{31}' + \left(\frac{e_{13}'}{2} + \omega_z' \right) \sigma_{33}^0] \} dz dx = 0 \quad (74)$$

since after integrating on x, the result of zero is given by the boundary conditions of both the incremental and initial problems given in equations (62) and (34) respectively.

The integral of the fourth term of equation (59) is given as

$$- \iint (e_{22}' \sigma_{22}^0 + \sigma_{22}') dz dx \quad (75)$$

Taking the incremental circumferential stress σ_{22}' , and performing similar substitutions from equations (47) the resulting integral is given by,

$$\frac{-E}{(1+\mu)} \iint \left[\frac{v', \phi + w'}{a+z} + \frac{\mu}{1-2\mu} (w', 'z + \frac{v, ' \phi + w'}{a+z} + u, 'x) \right] dz dx \quad (76)$$

The substitution of the displacement expansions given in equation (64) yields,

$$\frac{-EM}{1+\mu} \iint \left[\frac{v_0, \phi + z v_{1, \phi} + x v_{2, \phi} + w_0 + x w_1}{a+z} \right] dz dx \quad (77)$$

After integrating, equation (77) becomes,

$$\frac{-EM}{1+\mu} [(v_0, \phi + w_0) I_1 + v_{1, \phi} I_2] \quad (78)$$

After substituting equations (45) and (58) into the initial position circumferential stress term of equation (75), the following form results.

$$\frac{-EMw_0}{1+\mu} \left[\frac{v, ' \phi + w'}{(a+z)^2} \right] \quad (79)$$

This term gives the effect of the incremental circumferential strain on the initial circumferential stress, or load term. These effects along with those produced by the incremental rotations as given by the last term of equation (66) are neglected compared to those effects produced by the incremental curvatures and the change in the incremental circumferential strain.

The integral of equation (59) is equal to the sum of equations (70), (73) and (78) and is given by equation (80). The next six equations, i.e. equations (81) to (87) were obtained

by a similar procedure.

The inplane equations are:

$$\begin{aligned}
 \iint \text{Eq. (59)} dz dx &= G[v_{1,\phi} A + (w_{0,\phi\phi} - v_{0,\phi}) I_1 - v_{1,\phi} I_2] \\
 &\quad - \frac{EM}{1+\mu} [(v_{0,\phi} + w_{0,\phi}) I_1 + v_{1,\phi} I_2] \\
 &\quad + \frac{EM}{1+\mu} w^0 [(w_{0,\phi\phi} - v_{0,\phi}) I_3 - v_{1,\phi} I_4] = 0
 \end{aligned} \tag{80}$$

$$\begin{aligned}
 \iint \text{Eq. (60)} dz dx &= \\
 \frac{EM}{1+\mu} [(v_{0,\phi\phi} + w_{0,\phi}) I_1 + v_{1,\phi\phi} I_2] &+ G[v_{1,\phi} A + (w_{0,\phi} - v_{0,\phi}) I_1 - v_{1,\phi} I_2] \\
 &\quad + \frac{EM}{1+\mu} w^0 [(v_{0,\phi\phi} + w_{0,\phi}) I_3 + v_{1,\phi\phi} I_4] = 0
 \end{aligned} \tag{81}$$

$$\begin{aligned}
 \iint z \text{ Eq. (60)} dz dx &= \frac{EM}{1+\mu} [(v_{0,\phi\phi} + w_{0,\phi}) I_2 + v_{1,\phi\phi} I_8] \\
 &\quad - aG[v_{1,\phi} A + (w_{0,\phi} - v_{0,\phi}) I_1 - v_{1,\phi} I_2] \\
 &\quad + \frac{EM}{1+\mu} w^0 [(v_{0,\phi\phi} + w_{0,\phi}) I_4 + v_{1,\phi\phi} I_9] = 0
 \end{aligned} \tag{82}$$

The out of plane equations are:

$$\begin{aligned}
\iint \text{Eq. (61)} dz dx &= G[v_{2,\phi}^A + v_{3,\phi}^{I_6} + u_{0,\phi\phi}^{I_1} + u_{1,\phi\phi}^{I_2}] \\
&+ \frac{EM}{1+\mu} w^0 [u_{0,\phi\phi}^{I_3} + u_{1,\phi\phi}^{I_4}] = 0
\end{aligned} \tag{83}$$

$$\begin{aligned}
\iint_x \text{Eq. (59)} dz dx &= G[v_{3,\phi}^{I_{13}} + (w_{1,\phi\phi}^{-v_{2,\phi}})^{I_5} - v_{3,\phi}^{-I_{10}}] \\
&- G(u_1 + w_1) (aA + I_6) - \frac{EM}{1+\mu} [(v_{2,\phi}^{+w_1})^{I_5} + v_{3,\phi}^{I_{10}}] \\
&+ \frac{EM}{1+\mu} w^0 [(w_{1,\phi\phi}^{-v_{2,\phi}})^{I_7} - v_{3,\phi}^{I_{11}}] = 0
\end{aligned} \tag{84}$$

$$\begin{aligned}
\iint_z \text{Eq. (61)} dz dx &= G[v_{2,\phi}^{I_6} + v_{3,\phi}^{I_{12}} + u_{0,\phi\phi}^{I_2} + u_{1,\phi\phi}^{I_8}] \\
&- G(u_1 + w_1) (aA + I_6) + \frac{EM}{1+\mu} w^0 [u_{0,\phi\phi}^{I_4} + u_{1,\phi\phi}^{I_9}] = 0
\end{aligned} \tag{85}$$

$$\begin{aligned}
\iint x \text{ Eq. (60)} dz dx &= \frac{EM}{1+\mu} [(v_{2,\phi\phi} + w_{1,\phi}) I_5 + v_{3,\phi\phi} I_{10}] \\
&+ G[v_3 I_{13} + (w_{1,\phi} - v_2) I_5 - v_3 I_{10}] \\
- G[v_2 (aA + I_6) + v_3 (aI_6 + I_{12}) + u_{0,\phi} (aI_1 + I_2) + u_{1,\phi} (aI_2 + I_8)] \\
&+ \frac{EM}{1+\mu} w^0 [(v_{2,\phi\phi} + w_{1,\phi}) I_7 + v_{3,\phi\phi} I_{11}] = 0
\end{aligned} \tag{86}$$

$$\begin{aligned}
\iint x \text{ Eq. (61)} &= -G[v_3 (aI_{13} + I_{14}) + (w_{1,\phi} - v_2) (aI_5 + I_{10}) - v_3 (aI_{10} + I_{15})] \\
&+ \frac{EM}{1+\mu} [(v_{2,\phi\phi} + w_{1,\phi}) I_{10} + v_{3,\phi\phi} I_{15}] + G[v_3 I_{14} + (w_{1,\phi} - v_2) I_{10} - v_3 I_{15}] \\
- G[v_2 (aI_{11} + I_{12}) + v_3 (aI_{13} + I_{17}) + u_{0,\phi} (aI_2 + I_8) + u_{1,\phi} (aI_8 + I_{18})] \\
&+ \frac{EM}{1+\mu} w^0 [(v_{2,\phi\phi} + w_{1,\phi}) I_{11} + v_{3,\phi\phi} I_{16}] = 0
\end{aligned} \tag{87}$$

The integrals $I_1, I_2, I_3, \dots, I_n$ given in the above equations are evaluated in Appendix I.

2 Reduction to a system of linear algebraic equations using Fourier series. The unknown displacement functions given in equation (64) may be expanded in a Fourier series separating out the circumferential variable ϕ . When substituted into the eight developed equations, the following Fourier series are used:²

$$\begin{aligned}
 w_0 &= W_0 \sin (n\phi + \phi_0) \\
 w_1 &= W_1 \sin (n\phi + \phi_0) \\
 v_0 &= V_0 \cos (n\phi + \phi_0) \\
 v_1 &= V_1 \cos (n\phi + \phi_0) \\
 v_2 &= V_2 \cos (n\phi + \phi_0) \\
 v_3 &= V_3 \cos (n\phi + \phi_0) \\
 u_0 &= U_0 \sin (n\phi + \phi_0) \\
 u_1 &= U_1 \sin (n\phi + \phi_0)
 \end{aligned} \tag{88}$$

These expansions are then substituted into equations (80) through (87) and coefficients of like terms can be collected after dividing through by the common sine or cosine term. The resulting equation are a system of eight linear algebraic equations.

²Wah, p. 970-971.

The inplane equations yield,

$$\begin{aligned}
W_0 \left[-I_1 \left(\frac{EM}{1+\mu} + n^2 G \right) - \frac{n^2 EM}{1+\mu} w^0 I_3 \right] + V_0 \left[n I_1 \left(\frac{EM}{1+\mu} + G \right) + \frac{n EM}{1+\mu} w^0 I_4 \right] \\
+ V_1 \left[n G (I_2 - A) + n \frac{EM}{1+\mu} I_2 + \frac{n EM}{(1+\mu)} w^0 I_4 \right] = 0
\end{aligned} \tag{89}$$

$$\begin{aligned}
W_0 \left[n I_1 \left(\frac{EM}{1+\mu} + G \right) + \frac{n EM}{1+\mu} w^0 I_3 \right] + V_0 \left[-I_1 \left(\frac{n^2 EM}{1+\mu} + G \right) - \frac{n^2 EM}{1+\mu} w^0 I_3 \right] \\
+ V_1 \left[-\frac{n^2 EM}{1+\mu} I_2 + G (A - I_2) - \frac{n^2 EM}{1+\mu} w^0 I_4 \right] = 0
\end{aligned} \tag{90}$$

$$\begin{aligned}
W_0 \left[n \left(\frac{EM}{1+\mu} I_2 - a G I_1 \right) + \frac{n EM}{1+\mu} w^0 I_4 \right] + V_0 \left[a G I_1 - n^2 \frac{EM}{1+\mu} I_2 - \frac{n^2 EM}{1+\mu} w^0 I_4 \right] \\
+ V_1 \left[-\frac{n^2 EM}{1+\mu} I_8 - a G (A - I_2) - \frac{n^2 EM}{1+\mu} w^0 I_9 \right] = 0
\end{aligned} \tag{91}$$

and the out of plane equations yield:

$$U_0 \left[-n^2 G I_1 - \frac{n^2 EM}{1+\mu} w^0 I_3 \right] + W_1 [0] + U_1 \left[-n^2 G I_2 - \frac{n^2 EM}{1+\mu} w^0 I_4 \right] + V_2 [-nGA] + V_3 [-nGI_6] = 0 \tag{92}$$

The out of plane equations (continued)

$$\begin{aligned}
 & U_0 [0] + W_1 [-I_5 (n^2 G + \frac{EM}{1+\mu}) - G(aA + I_6) - \frac{n^2 EM}{1+\mu} w^0 I_7] + U_1 [-G(aA + I_6)] \\
 & + V_2 [nI_5 (G + \frac{EM}{1+\mu}) + \frac{nEM}{1+\mu} w^0 I_7] + V_3 [nI_{10} (G + \frac{EM}{1+\mu}) - nGI_{13} + \frac{nEM}{1+\mu} w^0 I_{11}] = 0
 \end{aligned} \tag{93}$$

$$\begin{aligned}
 & U_0 [-n^2 GI_2 - \frac{n^2 EM}{1+\mu} w^0 I_4] + W_1 [-G(aA + I_6)] + U_1 [-G(n^2 I_8 + aA + I_6) - \frac{n^2 EM}{1+\mu} w^0 I_9] \\
 & + V_2 [-nGI_6] + V_3 [-nGI_{12}] = 0
 \end{aligned} \tag{94}$$

$$\begin{aligned}
 & U_0 [-nG(aI_1 + I_2)] + W_1 [nI_5 (\frac{EM}{1+\mu} + G) + \frac{nEM}{1+\mu} w^0 I_7] + U_1 [-nG(aI_2 + I_8)] \\
 & + V_2 [-I_5 (\frac{n^2 EM}{1+\mu} + G) - G(aA + I_6) - \frac{n^2 EM}{1+\mu} w^0 I_7] + V_3 [-\frac{n^2 EM}{1+\mu} I_{10} + G(I_{13} - I_{10}) \\
 & - G(aI_6 + I_{12}) - \frac{n^2 EM}{1+\mu} w^0 I_{11}] = 0
 \end{aligned} \tag{95}$$

The out-of plane equations (continued)

$$\begin{aligned}
 & U_0 [-nG(aI_2 + I_8)] + W_1 [n(\frac{EM}{1+\mu}I_{10} - aGI_5) + \frac{nEM}{1+\mu}w^0 I_{11}] \\
 & + U_1 [-nG(aI_8 + I_{18})] + V_2 [G(aI_5 - aI_6 - I_{12}) - \frac{n^2 EM}{1+\mu}I_{10} - \frac{n^2 EM}{1+\mu}w^0 I_{11}] \\
 & + V_3 [G(aI_{10} - aI_{13} - aI_{12} - I_{17}) - \frac{n^2 EM}{1+\mu}I_{15} - \frac{n^2 EM}{1+\mu}w^0 I_{16}] = 0 \quad (96)
 \end{aligned}$$

The numerical order of the equations and the location of the variables within each equation were chosen to give symmetry in the matrices formed by the coefficient of the non-load and load terms, and also to eliminate any zero diagonal terms in the matrix formed by the coefficients of the load terms. The above arrangement facilitates more efficient matrix operations used in the later sections.

B. Displacement Expansions in Terms of Known Functions

The unknown displacement functions $w_1, v_1, v_2, v_3,$ and u_1 given in equations (64) may be replaced with combinations of the three median surface displacements w_0, v_0, u_0 and their derivatives. These displacement functions must be determined by actual geometric computations or by using a method in which a set of displacement functions is assumed and made to conform to a known result, for example, the stress resultant-curvature relationships for thin rods given by Love³ and Lamb⁴. The solution proceeds from that point as in the previous section.

The definitions of the resultant forces and moments in terms of the stresses are given by the following integrals,⁵

$$\begin{aligned}
 V &= \iint \sigma_{21} dz dx \\
 N &= \iint \sigma_{23} dz dx \\
 T &= \iint \sigma_{22} dz dx \\
 M_z &= -\iint x \sigma_{22} dz dx \\
 M_x &= \iint z \sigma_{22} dz dx \\
 M_\phi &= \iint (x \sigma_{21} - z \sigma_{23}) dz dx
 \end{aligned} \tag{97}$$

³Love, p. 388.

⁴Lamb, Dynamical Theory of Sound, 2nd ed. (New York, 1960), p. 136.

⁵Love, p. 386.

where the forces V, N, T and moments M_z, M_x, M_ϕ are shown in figure 4.

A system of loading may be applied at the center of the element, with its components along the coordinate directions. These forces and moments are P_z, P_x, P_ϕ and Q_z, Q, Q_ϕ .

The six equilibrium equations for this element may be obtained by direct summation of forces and moments, or by integrating the three stress equilibrium equations given as equations (59), (60) and (61). This second method will be used since this integration was already performed in the previous section on thin rings with unknown displacement functions.

Integrating these equations and substituting for the forces and moments as given in equation (97) yields,

$$\iint \text{Eq. (59)} dz dx = V_{,\phi} - T + P_z = 0 \quad (98)$$

$$\iint \text{Eq. (60)} dz dx = T_{,\phi} + V + P_\phi = 0 \quad (99)$$

$$\iint \text{Eq. (61)} dz dx = N_{,\phi} + P_x = 0 \quad (100)$$

$$\iint x \text{Eq. (60)} dz dx = M_{z,\phi} - M_\phi + aN + Qz = 0 \quad (101)$$

$$\iint z \text{Eq. (60)} dz dx = M_{x,\phi} - aV + Q_x = 0 \quad (102)$$

$$\iint [x \text{ Eq. (59)} - z \text{ Eq. (61)}] dz dx = M_{\phi,\phi} + M_z + Q_\phi = 0 \quad (103)$$

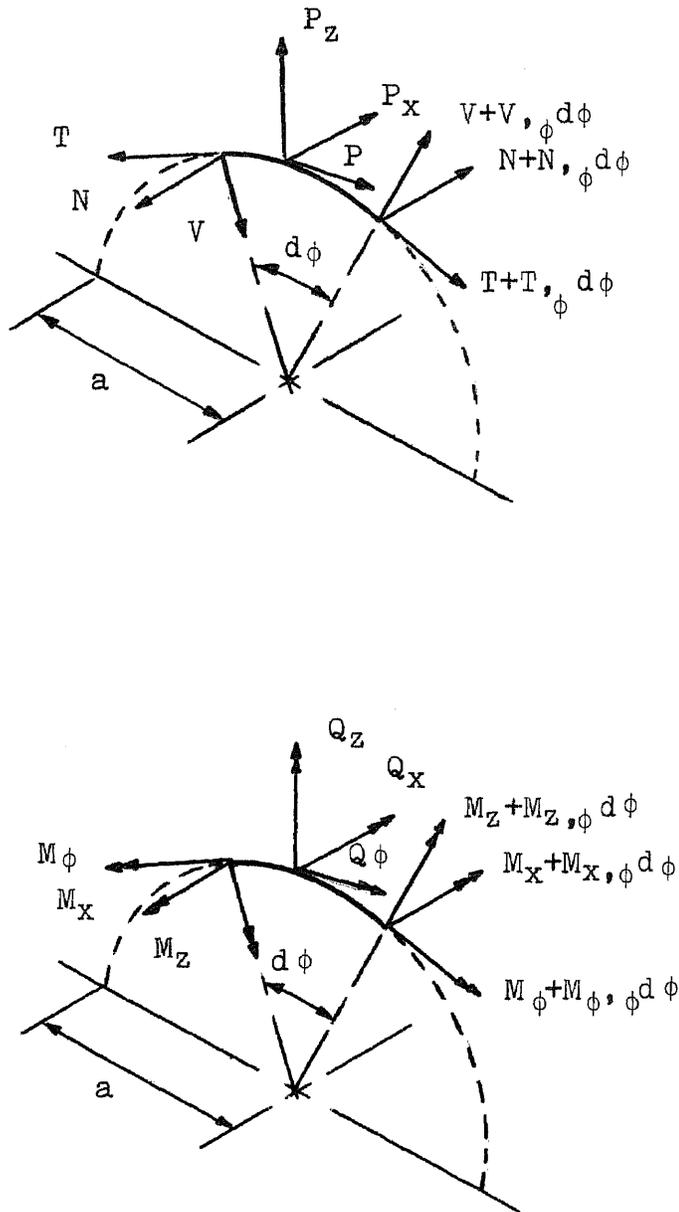


Figure 4 Force and Moment Resultants on a Differential Ring Element

This set of equations is analogous to the equations of motion for the vibration of a circular ring as given by Love.⁶ The load terms replace the inertia terms. Love also gives the moment resultant-curvature and twist relationships in terms of centerline displacement values as,

$$M_z = \frac{EI}{a^2} Z(u,_{\phi\phi} - a\theta) \quad (104)$$

$$M_x = \frac{EI}{a^2} X(v,_{\phi} - w,_{\phi\phi}) \quad (105)$$

$$M_{\phi} = \frac{GJ}{a} Z(a\theta,_{\phi} + u,_{\phi}) \quad (106)$$

where θ is the rotation of the cross section, and J is its torsional constant.

The circumferential force is given as,

$$T = 0 \quad (107)$$

since Love assumes that the circumferential strain of the centerline is zero i.e.

$$v,_{\phi} = -w \quad (108)$$

If the extension of the centerline is not assumed to be zero, then the following forms are given by Lamb⁷ for the circumferential force,

⁶Love, p. 451

⁷Lamb, p. 136

$$T = \frac{EA}{a}(w + v, \phi) \quad (109)$$

and for the inplane moment,

$$M_x = - \frac{EI}{a^2}(w, \phi\phi + w) \quad (110)$$

In his solution to the buckling of thin rings Wah⁸ begins from this point and uses Love's curvature and twist relationships i.e. equations (104), (105), (106) and Lamb's expression for the circumferential force i.e. equation (109). Wah substitutes these four relationships into the equilibrium equations and eliminates V and N to give four equations with four unknowns. He then replaces the inertia terms with fictitious load terms, which are obtained by multiplying the buckling load by the appropriate curvature, twist or change in circumferential strain. This fictitious load technique has been used by Timoshenko in analyzing the torsional buckling of open section columns.⁹

The thin ring displacement expansions in terms of unknown functions were given in equation (64). The expressions for the coefficients of the linear terms may be geometrically determined, since they result from the rotation of a straight line about the three coordinate directions. The functions w_1 and u_1 are the positive and negative values of the rotation of the ring cross section. The function v_1 is the inplane slope, and v_2

⁸Wah, p. 969

⁹Timoshenko and Gere, p. 225

is the negative of the out of plane slope. The function v_3 is the warping function, which gives the distortion of the plane of the cross section due to twist.

This form is given by Love¹⁰, where c is a constant dependent upon the shape of a cross section. After substituting the above for the unknown functions equation (64) becomes

$$\begin{aligned}
 w' &= w + x\theta \\
 v' &= v + z\left(\frac{v-w}{a}, \phi\right) - x\frac{u}{a}, \phi + xz\left[\frac{c}{a^2}(a\theta, \phi + u, \phi)\right] \\
 u &= u - z\theta
 \end{aligned} \tag{111}$$

The value of the constant c is determined by substituting equation (101) into Hooke's Law in terms of displacements given in equation (47) and substituting these results into the expression for the torsional moment in equation (97).

After integrating, the torsional moment is given by the following,

$$\begin{aligned}
 M_\phi &= G\{(u, \phi + a\theta, \phi)\left[\frac{I_5}{a} + \frac{c}{a^2}(I_{13} - I_{10} - I_{12})\right] \\
 &\quad + u, \phi\left(\frac{I_6}{a} - I_2\right) + \theta, \phi I_8\}
 \end{aligned} \tag{112}$$

¹⁰Love, p. 311-324

Then, substituting into equation (112) the expressions for the I 's given for the thin ring in Appendix I yields,

$$M_{\phi} = \frac{G}{a^2} (u,_{\phi} + a\theta,_{\phi}) [I_p + c(I_z - I_x)] \quad (113)$$

The constant c may be obtained by equating the two expressions for M_{ϕ} given by equations (106) and (113). Solving for c yields

$$c = \frac{J - I_p}{I_z - I_x} \quad (114)$$

A similar procedure is followed in evaluating the remainder of the required forces and moments given in equation (97). The circumferential force was found to be

$$T = \frac{EA}{a} (v,_{\phi} + w) \quad (115)$$

The inplane moment was found to be

$$M_x = \frac{EI_x}{a^2} [(v,_{\phi} - w,_{\phi\phi}) - (v,_{\phi} + w)] \quad (116)$$

If the circumferential strain is considered to be zero, equation (116) yields Love's¹¹ form

$$M_x = \frac{EI_x}{a^2} (v,_{\phi} - w,_{\phi\phi}) \quad (117)$$

and when $v,_{\phi} \neq -w$, Lamb's¹² form is obtained,

$$M_x = -\frac{EI_x}{a^2} (w + w,_{\phi\phi}) \quad (118)$$

¹¹Love, p. 451

¹²Lamb, p. 136

The out of plane moment was found to be

$$M_z = -\frac{EI}{a^2}(a\theta - u, \phi\phi) \quad (119)$$

1. Equilibrium in terms of displacement functions. Substituting equations (106), (115), (117), (119) into the force and moment equilibrium equations (98) through (103) yield

$$V, \phi - \frac{EA}{a}(v, \phi + w) + P_z = 0 \quad (120)$$

$$\frac{EA}{a}(v, \phi\phi + w, \phi) + V + P_\phi = 0 \quad (121)$$

$$N, \phi + P_z = 0 \quad (122)$$

$$-\frac{EI}{a^2}(a\theta, \phi - u, \phi\phi\phi) - \frac{GJ}{a^2}(a\theta, \phi + u, \phi) + aN + Q_z = 0 \quad (123)$$

$$\frac{EI}{a^2}(v, \phi\phi - w, \phi\phi\phi) - aV + Q_x = 0 \quad (124)$$

$$\frac{GJ}{a^2}(a\theta, \phi\phi + u, \phi\phi) - \frac{EI}{a^2}(a\theta - u, \phi\phi) + Q_\phi = 0 \quad (125)$$

The equations for the inplane and out of plane buckling may be handled separately. For the inplane case, equation (124) can be solved for V and substituted into equations (120) and (121),

$$\frac{EI}{a^3}(w, \phi\phi\phi\phi - v, \phi\phi\phi) + \frac{EA}{a}(v, \phi + w) - (P_z + \frac{Q_x}{a}) = 0 \quad (126)$$

$$\frac{EI}{a^3}(w, \phi\phi\phi - v, \phi\phi) - \frac{EA}{a}(v, \phi\phi + w, \phi) - (P_\phi + \frac{Q_x}{a}) = 0 \quad (127)$$

and for the out of plane case, equation (123) can be solved for N and substituted into equation (122), while equation (125) stands unchanged.

$$\frac{EI}{a^3} \frac{z}{\phi\phi\phi\phi} (u, -a\theta, \phi) - \frac{GJ}{a^3} (u, \phi\phi + a\theta, \phi\phi) - (P_x - \frac{Q_z}{a}) = 0 \quad (128)$$

$$\frac{EI}{a^2} \frac{z}{\phi\phi} (a\theta - u, \phi\phi) - \frac{GJ}{a^2} (u, \phi\phi + a\theta, \phi) - Q_\phi = 0 \quad (129)$$

The non-load terms given above are the same as those given by Wah¹³, the load terms have to still be evaluated. The applied load is a radially inward uniform line load P per length of circumference. The only load terms which are used are those due to the curvature, the twist and the change in circumferential strain, thus neglecting the effects of the circumferential strain and also any rigid body rotations.

The load terms may be obtained by substituting the displacement expansions, equations (111) into the appropriate load terms derived in section V.A using the unknown displacement functions. For example, P_z appears in the radial force equilibrium equation (98) and also in equation (80) as

$$P_z = \frac{EM}{1+\mu} w^0 [(w_{0,\phi\phi} - v_{0,\phi}) I_3 - v_{1,\phi} I_4] \quad (130)$$

After substituting in for the displacement functions and integrals and also for the initial position displacement and M from equations (53) and (54), P_z can be written as,

$$P_z = - P(w, \phi\phi - v, \phi) \quad (131)$$

¹³Wah, p. 968

In a similar manner, P_ϕ may be obtained by comparing equation (99) to equation (81). Using the same substitutions,

$$P_\phi = - P(w, \phi + v, \phi\phi) \quad (132)$$

The comparison of equation (100) to equation (82) yields,

$$P_x = - P(u, \phi\phi) \quad (133)$$

And the comparison of equation (101) to the difference of equation (83) and equation (84) yields,

$$Q_\phi = - \frac{P}{A} \frac{I}{a} (a\theta, \phi\phi) \quad (134)$$

The substitution of equations (131) through (134) into equation (126) through (129) yields for the inplane case,

$$\frac{EI}{a^4} x (w, \phi\phi\phi\phi - v, \phi\phi\phi) + \frac{EA}{a^2} (v, \phi + w) + \frac{P}{a} (w, \phi\phi - v, \phi) = 0 \quad (135)$$

$$\frac{EI}{a^4} x (w, \phi\phi\phi - v, \phi\phi) - \frac{EA}{a^2} (v, \phi\phi + w, \phi) + \frac{P}{a} (w, \phi + v, \phi\phi) = 0 \quad (136)$$

and for the out of plane case,

$$\frac{EI}{a^4} z (u, \phi\phi\phi\phi - a\theta, \phi) - \frac{GJ}{a^4} (u, \phi\phi + a\theta, \phi\phi) + \frac{P}{a} u, \phi\phi = 0 \quad (137)$$

$$\frac{EI}{a^2} z (a\theta - u, \phi\phi) - \frac{GJ}{a^2} (u, \phi\phi + a\theta, \phi\phi) + \frac{P}{A} \frac{I}{a} (a\theta, \phi\phi) = 0 \quad (138)$$

2. Critical buckling loads. These four equations are the starting point of Wah's¹⁴ solution. He expresses the displacements in a Fourier series and sets the determinant of the coefficients of the resulting two sets of equations equal to zero. The two resulting quadratic equations are given as:

Inplane case:

$$\gamma^2 - \gamma(\kappa + n^2) + \kappa(n^2 - 1) = 0 \quad n \neq 0,1 \quad (139)$$

Out of plane case:

$$\zeta^2 - \zeta(n^2 + \rho + \frac{\rho}{\beta} + \frac{1}{\beta n^2}) + \frac{\rho(n^2 - 1)}{\beta n^2} = 0 \quad (140)$$

where

$$\kappa = \frac{Aa^2}{I_x} ; \gamma = \frac{Pa^3}{EI_x} \quad (141)$$

and

$$\rho = \frac{GJ}{EI_z} ; \zeta = \frac{Pa^3}{EI_z} ; \beta = \frac{I_p}{a^2 A} \quad (142)$$

For thin rings κ becomes a large quantity and for the inplane case the root is given as

$$\gamma_{cr} \approx (n^2 - 1) \quad (143)$$

Therefore, from equation (141) the classical solution for the inplane buckling load is given by

$$P_{cr} = \frac{(n^2 - 1)EI}{a^3} \quad n = 2,3,\dots \quad (144)$$

For the out of plane case, β becomes small for thin rings and the roots to equation (14) are given by

¹⁴Wah, p. 969-973.

$$\zeta_{cr} \approx \frac{\alpha(n^2-1)^2}{(\alpha n^2+1)} \quad n = 2, 3, \dots \quad (145)$$

Substituting equation (142) into equation (145) with n equal to 2 yields

$$P_{cr} = \frac{9\rho}{1+4\rho} \frac{EI_z}{a^3} \quad (146)$$

This is the classical solution for the out of plane buckling load. The two cases $n=0,1$ represent rigid body movement of the ring and are therefore not of interest.

Plots of the solution to equations (139) and (140) are given in section IX.D for various size rings. The roots were obtained by a short FORTRAN program using the quadratic formula. Also shown on these plots are the classical thin ring theory results as given in equations (144) and (146).

VI. THICK RING THEORY SOLUTION TO THE
INCREMENTAL PROBLEM

A. Thick Ring Displacement Expansions

The following expansions were chosen to represent the thick ring displacement variation through the thickness and depth of the ring.

$$\begin{aligned}
 w' &= w_0 + xw_1 + (zw_2 + xzw_3 + x^2w_4) \\
 u' &= u_0 + zu_1 + (xu_2 + zu_3 + z^2u_4) \\
 v' &= v_0 + zv_1 + xv_2 + xzv_3 + (z^2v_4 + x^2v_5 \\
 &\quad + z^2xv_6 + x^2zv_7)
 \end{aligned}
 \tag{147}$$

The terms inside the parenthesis represent the unknown displacement functions which were not included in the thin ring theory. One additional power is added to x, z and xz in each direction as seen in equation (64). These additional terms increase the accuracy of the expansions and permit the analysis of thicker rings. The extra first degree terms give the variation of the radial displacement w, in the radial direction, and also the variation of the out of plane displacement u, in the out of plane direction. The xz product terms provide a coupling between the inplane and out of plane effects. The second degree terms in x and z allow the normals to the median surface to deviate from straight lines. The z^2x and x^2z terms were added to the v expansion to increase the accuracy in v_3 , the warping term, which was included in the thin ring expansions given in

equation (64).

B. Equilibrium in Terms of Displacement Functions

There are eighteen variables on the right hand side of equations (147). To obtain a solution eighteen equations are needed. The techniques given for the thin ring problem in section V are used to generate the equations. The procedure is exactly the same.

As described for the initial problem in section III,A, the cross section of the ring is symmetrical about a radial line, and as given in the boundary conditions in section II.C, the load is a hydrostatic pressure, which also is uniform around the circumference. Due to these two conditions, the eighteen equations are not coupled in the inplane and out of plane directions.

The following equations are obtained by integrating over the ring cross section the product of the coefficient of each unknown displacement function in equation (147) with the equilibrium equation in the same direction, as given by equations (59) through (61).

Inplane Equations

$$\begin{aligned}
 \iint \text{Equation (59)} \, dzdx = & G[v_{1,\phi}A + 2v_{4,\phi}I_6 + (w_{0,\phi\phi} - v_{0,\phi})I_1 + (w_{2,\phi\phi} - v_{1,\phi})I_2 + (w_{4,\phi\phi} - v_{5,\phi})I_5 \\
 & - v_{4,\phi}I_8 + v_{7,\phi}(I_{13} - I_{10})] - \frac{E}{1+\mu}\{M[(v_{0,\phi} + w_0)I_1 + (v_{1,\phi} + w_2)I_2 + v_{4,\phi}I_8 + (v_{5,\phi} + w_4)I_5 \\
 & + v_{7,\phi}I_{10}] + N[(w_2 + u_2)A + u_3I_6]\} + \frac{E}{1+\mu}(w_{0,\phi\phi} - v_{0,\phi})(NDI_1 + MCI_3 + MDI_4) \quad (148) \\
 & + (w_{2,\phi\phi} - v_{1,\phi})(NDI_2 + MCI_4 + MDI_9) + (w_{4,\phi\phi} - v_{5,\phi})(NDI_5 + MCI_7 + MDI_{11}) \\
 & - v_{7,\phi}(NDI_{10} + MCI_{11} + MDI_{16}) - v_{4,\phi}(NDI_8 + MCI_9 + MDI_{19})] = 0
 \end{aligned}$$

$$\begin{aligned}
 \iint \text{Equation (60)} \, dzdx = & \frac{E}{1+\mu}\{M[(v_{0,\phi\phi} + w_0,\phi\phi)I_1 + (v_{1,\phi\phi} + w_2,\phi\phi)I_2 + v_{4,\phi\phi}I_8 + (v_{5,\phi\phi} + w_4,\phi\phi)I_5 + v_{7,\phi\phi}I_{10}] \\
 & + N[(w_{2,\phi} + u_{2,\phi})A + u_{3,\phi}I_6]\} + G[v_{1,\phi}A + 2v_{4,\phi}I_6 + (w_{0,\phi} - v_{0,\phi})I_1 + (w_{2,\phi} - v_{1,\phi})I_2 \\
 & + (w_{4,\phi} - v_{5,\phi})I_5 - v_{4,\phi}I_8 + v_{7,\phi}(I_{13} - I_{10})] + \frac{E}{1+\mu}\{v_{0,\phi\phi} + w_0,\phi\phi\}(MCI_3 + MDI_4 + NDI_1) \quad (149) \\
 & + (v_{1,\phi\phi} + w_2,\phi\phi)(MCI_4 + MDI_9 + NDI_2) + v_{4,\phi\phi}(MCI_9 + MDI_{19} + NDI_8)
 \end{aligned}$$

$$+ (v_5, \phi\phi + w_4, \phi) (MCI_7 + MNI_{11} + NDI_5) + v_7, \phi\phi (MCI_{11} + MDI_{16} + NDI_{10})] = 0$$

$$\iint z \text{ Equation (61) } dzdx = - G[v_1 aA + 2v_4 aI_6 + (w_0, \phi - v_0) aI_1 + (w_2, \phi - v_1) aI_2 + (w_4, \phi - v_5) aI_5 \\ - v_4 aI_8 - v_7 a(I_{10} - I_{13})] + \frac{E}{1+\mu} \{ M[(v_0, \phi\phi + w_0, \phi) I_2 + (v_1, \phi\phi + w_2, \phi) I_8 + v_7, \phi\phi I_{15} \\ + v_4, \phi\phi I_{18} + v_5, \phi\phi + w_4, \phi) I_{10}] + N[(w_2, \phi) I_6 + u_3, \phi I_{12}] \} \quad (150)$$

$$+ \frac{E}{1+\mu} \{ (v_0, \phi\phi + w_0, \phi) (MCI_4 + MDI_9 + NDI_2) + v_4, \phi\phi (MCI_{19} + MDI_{20} + NDI_{18})$$

$$+ (v_5, \phi\phi + w_4, \phi) (MCI_{11} + MDI_{16} + NDI_{10}) + v_7, \phi\phi (MCI_{16} + MDI_{22} + NDI_{15}) \} = 0$$

$$\iint z \text{ Equation (59) } dzdx = - \frac{E}{1+\mu} \{ M[w_2 (aA + I_6)] + N[(v_0, \phi + w_0) (aI_1 + I_2) + (v_1, \phi + w_2) (aI_2 + I_8) \\ + v_4, \phi (aI_8 + I_{18}) + (v_5, \phi + w_4) (aI_5 + I_{10}) + v_7, \phi (aI_{10} + I_{15}) + u_2 (aA + I_{16}) + u_3 (aI_6 + I_{12})] \} \quad (151)$$

$$+ G[v_1, \phi I_6 + 2v_4, \phi I_{12} + v_7, \phi I (I_{14} - I_{15}) + (w_0, \phi\phi - v_0, \phi) I_2 + (w_2, \phi\phi - v_1, \phi) I_8$$

$$+ (w_4, \phi\phi - v_5, \phi) I_{10} - v_4, \phi I_{18}] - \frac{E}{1+\mu} \{ M[(v_0, \phi + w_0) I_2 + (v_1, \phi + w_2) I_8 + v_4, \phi I_{18}$$

$$\begin{aligned}
& + (v_5, \phi + w_4)I_{10} + v_7, \phi I_{15}] + N[(w_2 + u_2)I_6 + u_3 I_{12}] \} \\
& + \frac{E}{1+\mu} [(w_0, \phi\phi - v_0, \phi)(MCI_4 + MDI_9 + NDI_2) + (w_2, \phi\phi - v_1, \phi)(MCI_9 + MDI_{19} + NDI_8) \\
& + (w_4, \phi\phi - v_5, \phi)(MCI_{11} + MDI_{16} + NDI_{10}) - v_4, \phi(MCI_{19} + MDI_{20} + NDI_{18}) \quad (151) \\
& - v_7, \phi(MCI_{16} + MDI_{22} + NDI_{15})] = 0
\end{aligned}$$

$$\begin{aligned}
\iint z^2 \text{ Equation (60) } dzdx = & - G[v_1(2aI_6 + I_{12}) + 2v_4(2aI_{12} + I_{17}) + v_7(2aI_{14} + I_{25} - 2aI_{15} - I_{31}) \\
& + (w_0, \phi - v_0)(2aI_2 + I_8) + (w_2, \phi - v_1)(2aI_8 + I_{18}) + (w_4, \phi - v_5)(2aI_{10} + I_{15}) - v_4(2aI_{18} + I_{21}) \\
& + \frac{E}{1+\mu} \{M[(v_0, \phi\phi + w_0, \phi)I_8 + (v_1, \phi\phi + w_2, \phi)I_{18} + v_4, \phi\phi I_{21} + (v_5, \phi\phi + w_4, \phi)I_{15} \\
& + v_7, \phi\phi I_{30}] \quad (152)
\end{aligned}$$

$$\begin{aligned}
& + N[(w_2, \phi + u_2, \phi)I_{12} + u_3, \phi I_{17}] \} + \frac{E}{1+\mu} [(v_0, \phi\phi + w_0, \phi)(MCI_9 + MDI_{19} + NDI_8) \\
& + (v_1, \phi\phi + w_2, \phi)(MCI_{19} + MDI_{20} + NDI_{18}) + v_4, \phi\phi(MCI_{20} + MDI_{23} + NDI_{21})
\end{aligned}$$

$$+ (v_5, \phi\phi + w_4, \phi)(MCI_{16} + MDI_{22} + NDI_{16}) + v_7, \phi\phi MCI_{22} + MDI_{34} + MCI_{30}] = 0$$

$$\begin{aligned} \iint x \text{ Equation (61)} dzdx &= G[2v_5, \phi I_{13} + 2v_7, \phi I_{14} + u_2, \phi\phi I_5 + u_3, \phi\phi I_{10}] - \frac{E}{1+\mu} \{M[u_2(aA + I_6) \\ &+ u_3(aI_6 + I_{12})] + N[w_2(aA + I_6) + (v_0, \phi + w_0)(aI_1 + I_2) + (v_1, \phi + w_2)(aI_2 + I_8) \\ &+ v_4\phi(aI_8 + I_{18}) + (v_5, \phi + w_4)(aI_5 + I_{10}) + v_7, \phi(aI_{10} + I_{15})]\} \end{aligned} \quad (153)$$

$$+ \frac{E}{1+\mu} [u_2, \phi\phi(MCI_7 + MDI_{11} + NDI_5) + u_3, \phi\phi(MCI_{11} + MDI_{16} + NDI_{10})] = 0$$

$$\begin{aligned} \iint xz \text{ Equation (61)} dzdx &= G[(u_3 + 2w_4)(aI_{13} + I_{14})] + G[2v_5, \phi I_{14} + 2v_7, \phi\phi I_{25} + u_2, \phi\phi I_{10} \\ &+ u_3, \phi\phi I_{15}] \end{aligned}$$

$$- \frac{E}{1+\mu} \{M[u_2(aI_6 + I_{12}) + u_3(aI_{12} + I_{17})] + N[w_2(aI_6 + I_{12}) + (v_0, \phi + w_0)(aI_2 + I_8)$$

$$+ (v_1, \phi + w_2)(aI_8 + I_{18}) + v_4, \phi(aI_{18} + I_{21}) + (v_5, \phi + w_4)(aI_{10} + I_{15}) + v_7, \phi(aI_{15} + I_{30})]\}$$

$$+ \frac{E}{1+\mu} [u_2, \phi\phi(MCI_{11} + MDI_{16} + NDI_{10}) + u_3, \phi\phi(MCI_{16} + MDI_{22} + NDI_{15})] = 0 \quad (154)$$

$$\iint x^2 \text{ Equation (59) } dzdx = G[v_1, \phi I_{13} + 2v_4, \phi I_{14} + (w_0, \phi\phi - v_0, \phi) I_5 + (w_2, \phi\phi - v_1, \phi) I_{22} \\ + (w_4, \phi\phi - v_5, \phi) I_{26} - v_4, \phi I_{15} + v_7, \phi (I_{36} - I_{38})] - 2G[u_3 + 2w_4] (aI_{13} + I_{14})] \quad (155)$$

$$- \frac{E}{1+\mu} \{ M[(v_0, \phi + w_0) I_5 + (v_1, \phi + w_2) I_{10} + v_4, \phi I_{15} + (v_5, \phi + w_4) I_{26} + v_7, \phi I_{38}]$$

$$+ N[(w_2 + u_2) I_{13} + u_3 I_{14}] \} + \frac{E}{1+\mu} [(w_0, \phi\phi - v_0, \phi) (MCI_7 + MDI_{11} + NDI_5)$$

$$+ (w_2, \phi\phi - v_1, \phi) (MCI_{11} + MDI_{16} + NDI_{10}) + (w_4, \phi\phi - v_5, \phi) (MCI_{27} + MDI_{28} + NDI_{26})$$

$$- v_4, \phi (MCI_{16} + MDI_{22} + NDI_{15}) - v_7, \phi (MCI_{28} + MDI_{40} + NDI_{38})] = 0$$

$$\iint x^2 \text{ Equation (60) } dzdx = \frac{E}{1+\mu} \{ M[(v_0, \phi\phi + w_0, \phi) I_5 + (v_1, \phi\phi + w_2, \phi) I_{10} + v_4, \phi\phi I_{15}$$

$$+ (v_5, \phi\phi + w_4, \phi) I_{26} + v_7, \phi I_{38}] + N[w_2, \phi + u_2, \phi) I_{13} + u_3, \phi I_{14}] \} \quad (156)$$

$$- 2G[2v_5(aI_{13} + I_{14}) + u_2, \phi(aI_5 + I_{10}) + 2v_7(aI_{14} + I_{25}) + u_3, \phi(aI_{10} + I_{16})]$$

$$+ G[v_1 I_{13} + 2v_4 I_{14} + (w_0, \phi - v_0) I_5 + (w_2, \phi - v_1) I_{10} + (w_4, \phi - v_5) I_{26} - v_4 I_{15}$$

$$\begin{aligned}
& +v_7(I_{36} - I_{38})] + \frac{E}{1+\mu}[(v_0, \phi\phi + w_0, \phi)(MCI_7 + MDI_{11} + NDI_5) \\
& + (v_1, \phi\phi + w_2, \phi)(MCI_{11} + MDI_{16} + NDI_{10}) + v_4, \phi\phi(MCI_{16} + MDI_{22} + NDI_{15}) \\
& + (v_5, \phi\phi + w_4, \phi)(MCI_{27} + MDI_{38} + NDI_{26}) + v_7, \phi\phi(MCI_{28} + MDI_{40} + NDI_{38})] = 0 \\
\iint x^2 z \text{ Equation (60) } dz dx = & -G[v_1(aI_{13} + I_{14}) + 2v_4(aI_{14} + I_{25}) + v_7(aI_{36} + I_{37}) \\
& + (w_0, \phi - v_0)(aI_5 + I_{10}) \tag{157} \\
& + (w_2, \phi - v_1)(aI_{10} + I_{15}) + (w_4, \phi - v_5)(aI_{26} + I_{38}) - v_4(aI_{15} + I_{30}) - v_7(aI_{38} + I_{39})] \\
& + \frac{E}{1+\mu}\{M[(w_0, \phi\phi + w_0, \phi)I_{10} + v_4, \phi\phi I_{30} + (v_1, \phi\phi + w_2, \phi)I_{15} + (v_5, \phi\phi + w_4, \phi)I_{38} + v_7, \phi\phi I_{39}] \\
& + N[w_2, \phi + u_2, \phi)I_{14} + u_3, \phi I_{25}] - 2G[2v_5(aI_{14} + I_{25}) + 2v_7(aI_{25} + I_{29}) \\
& + u_2, \phi(aI_{10} + I_{15}) + u_3, \phi(aI_{15} + I_{30})] + G[v_1 I_{14} + 2v_4 I_{25} + v_7 I_{37} + (w_0, \phi - v_0)I_{10}
\end{aligned}$$

$$\begin{aligned}
& + (w_2, \phi - v_1)I_{15} + (w_4, \phi - v_5)I_{38} - v_4 I_{30} - v_7 I_{39}] + \frac{E}{1+\mu} [v_0, \phi\phi + w_0, \phi) (MCI_{11} + MDI_{16} + NDI_{10}) \\
& \quad + (v_1, \phi\phi + w_2, \phi) (MCI_{16} + MDI_{22} + NDI_{15}) + v_4, \phi\phi (MCI_{22} + MDI_{34} + NDI_{30}) \\
& \quad + (v_5, \phi\phi + w_4, \phi) (MCI_{28} + MDI_{40} + NDI_{38}) + v_7, \phi\phi (MCI_{40} + MDI_{41} + NDI_{39})] = 0 \quad (158)
\end{aligned}$$

Out of Plane Equations

$$\begin{aligned}
\iint \text{Equation (61)} \, dzdx &= G[(v_2, \phi A + v_3, \phi I_6 + v_6, \phi I_{12}) + u_0, \phi\phi I_1 + u_1, \phi\phi I_2 + u_4, \phi\phi I_8] \\
& + \frac{E}{1+\mu} [u_0, \phi\phi (MCI_3 + MDI_4 + NDI_1) + u_1, \phi\phi (MCI_4 + MDI_9 + NDI_2) \\
& \quad + u_4, \phi\phi (MCI_9 + MDI_{19} + NDI_8)] = 0
\end{aligned}$$

$$\begin{aligned}
\iint x \text{ Equation (59)} \, dzdx &= G[v_3, \phi I_{13} + v_6, \phi (2I_{14} - I_{15}) + (w_1, \phi\phi - v_2, \phi) I_5 + (w_3, \phi\phi - v_3, \phi) I_{10}] \\
& - G[(u_1 + w_1)(aA + I_6) + (2u_4 + w_3)(aI_6 + I_{12})] - \frac{E}{1+\mu} \{M[(v_2, \phi + w_1) I_5 + (v_3, \phi + w_3) I_{10} \\
& \quad + v_6, \phi I_{15}] + N(w_3 I_{13})\} + \frac{E}{1+\mu} [(w_1, \phi\phi - v_2, \phi) (MCI_7 + MDI_{11} + NDI_5)
\end{aligned} \quad (159)$$

$$+ (w_3, \phi\phi - v_3, \phi)(MCI_{11} + MDI_{16} + NDI_{10}) - v_6, \phi(MCI_{16} + MDI_{22} + NDI_{15})] = 0$$

$$\begin{aligned} \iint z \text{ Equation (61) } dzdx = & - G[(u_1 + w_1)(aA + I_6) + (2u_4 + w_3)(aI_6 + I_{12})] + G[v_2, \phi I_6 + v_3, \phi I_{12} \\ & + u_0, \phi\phi I_2 + u_1, \phi\phi I_8 + u_4, \phi\phi I_{18}] + \frac{E}{1+\phi}[u_0, \phi\phi(MCI_4 + MDI_9 + NDI_2) \\ & + u_1, \phi\phi(MCI_9 + MDI_{19} + NDI_8) + u_4, \phi\phi(MCI_{19} + MDI_{20} + NDI_{18})] = 0 \end{aligned} \quad (160)$$

$$\begin{aligned} \iint x \text{ Equation (60) } dzdx = & \frac{E}{1+\mu}\{M[(v_2, \phi\phi + w_1, \phi)I_5 + (v_3, \phi\phi + w_3, \phi)I_{10} + v_6, \phi\phi I_{15}] \\ & + N(w_3, \phi I_{13})\} - G[v_2(aA + I_6) + v_3(aI_6 + I_{12}) + v_6(aI_{12} + I_{17}) + u_0, \phi(aI_1 + I_2) + u_1, \phi[aI_2 + I_8) \\ & + u_4, \phi(aI_8 + I_{18})] + G[v_3 I_{13} + (w_1, \phi - v_2)I_5 + (w_3, \phi - v_3)I_{10} + v_6(2I_{14} - I_{15})] \\ & + \frac{E}{1+\mu}[(v_2, \phi\phi + w_1, \phi)(MCI_7 + MDI_{11} + NDI_5) + (v_3, \phi\phi + w_3, \phi)(MCI_{11} + MDI_{16} + NDI_{10}) \\ & + v_6, \phi\phi(MCI_{16} + MDI_{22} + NDI_{15})] = 0 \end{aligned} \quad (161)$$

$$\begin{aligned}
\iint \text{xz Equation (60) dzdx} = & -G[v_3(aI_{13} + I_{14}) + v_6(2aI_{14} + 2I_{25}aI_{15} - I_{31}) + (w_{1,\phi} - v_2)(aI_5 + I_{10}) \\
& + (w_{3,\phi} - v_3)(aI_{10} + I_{15})] + \frac{E}{1+\mu}\{M[(v_2\phi\phi + w_{1,\phi})I_{10} + (v_3\phi\phi + w_{3,\phi})I_{15} + v_6\phi\phi I_{30}] \\
& + N(w_{3,\phi}I_{14})\} - G[v_2(aI_6 + I_{12}) + v_3(aI_{12} + I_{17}) + v_6(aI_{17} + I_{24}) + u_{0,\phi}(aI_2 + I_8) \\
& + u_{1,\phi}(aI_8 + I_{18}) + u_{4,\phi}(aI_{18} + I_{21})] + (G[v_3I_{14} + v_6(2I_{25} - I_{30}) + (w_{1,\phi} - v_2)I_{10} \\
& + (w_{3,\phi} - v_3)I_{15}] + \frac{E}{1+\mu}[v_2\phi\phi + w_{1,\phi})(MCI_{12} + MDI_{16} + NDI_{10}) \\
& + (v_3\phi\phi + w_{3,\phi})(MCI_{16} + MDI_{22} + NDI_{15}) + v_6\phi\phi(MCI_{22} + MDI_{34} + NDI_{30})] = 0 \quad (162)
\end{aligned}$$

$$\begin{aligned}
\iint \text{xz Equation (59) dzdx} = & -\frac{E}{1+\mu}\{M[w_3(aI_{13} + I_{14}) + N[(v_2\phi + w_1)(aI_5 + I_{10}) + (v_3\phi + w_3)(aI_{10} + I_{15}) \\
& + v_6\phi(aI_{15} + I_{30})]\} + G[v_3\phi I_{14} + (v_6\phi(2I_{25} - I_{30}) + (w_{1,\phi} - v_2)\phi I_{10} + (w_{3,\phi} - v_3)\phi I_{15}] \\
& - G[(u_1 + w_1)(aI_6 + I_{12}) + (2u_4 + w_3)(aI_{12} + I_{17})] - \frac{E}{1+\mu}\{M[(v_2\phi + w_1)I_{10} + (v_3\phi + w_3)I_{15} \\
& + v_6\phi I_{30}]
\end{aligned}$$

$$\begin{aligned}
& + N[w_3 I_{14}] \} + \frac{E}{1+\mu} [(w_1, \phi\phi - v_2, \phi) (MCI_{11} + MDI_{16} + NDI_{10}) + (w_3, \phi\phi - v_3, \phi) (MCI_{16} + MDI_{22} \\
& \qquad \qquad \qquad + NDI_{15}) \\
& \qquad \qquad \qquad - v_6, \phi (MCI_{22} + MDI_{34} + NDI_{30})] = 0 \qquad (163)
\end{aligned}$$

$$\begin{aligned}
\iint x^2 \text{ Equation (61) } dzdx &= -2G[(u_1 + w_1)(aI_6 + I_{12}) + (2u_4 + w_3)(aI_{12} + I_{17})] + G[v_2, \phi I_{12} + v_3, \phi I_{17} \\
& + v_6, \phi (aI_{17} + I_{24}) + u_0, \phi\phi I_8 + u_1, \phi\phi I_{18} + u_4, \phi\phi I_{21}] + \frac{E}{1+\mu} [u_0, \phi\phi (MCI_9 + MDI_{19} + NDI_8) \\
& \qquad \qquad \qquad + u_1, \phi\phi (MCI_{19} + MDI_{20} + NDI_{18}) + u_4, \phi\phi (MCI_{20} + MDI_{23} + NDI_{21})] = 0 \qquad (164)
\end{aligned}$$

$$\begin{aligned}
\iint z^2 x \text{ Equation (60) } dzdx &= -2G[v_3 (aI_{14} + I_{25}) + 2v_6 (aI_{25} + I_{29}) + (w, \phi - v_2) (aI_{10} + I_{15}) \\
& + (w_3, \phi - v_3) (aI_{15} + I_{30}) - v_6 (aI_{30} + I_{31})] + \frac{E}{1+\mu} \{ M[(v_2, \phi\phi + w_1, \phi) I_{15} + (v_3, \phi\phi + w_3, \phi) I_{30} \\
& \qquad \qquad \qquad + v_6, \phi\phi I_{31}] + N[w_3, \phi I_{25}] \} - G[v_2 (aI_{12} + I_{17}) + v_3 (aI_{17} + I_{24}) + v_6 (aI_{24} + I_{32}) \\
& \qquad \qquad \qquad + u_0, \phi (aI_8 + I_{18}) + u_1, \phi (aI_{18} + I_{21}) + u_4, \phi (aI_{21} + I_{33})] + G[v_3 I_{25} - 2v_6 I_{29} + (w_1, \phi - v_2) I_{27}
\end{aligned}$$

$$\begin{aligned}
& + (w_3, \phi - v_3) I_{30} - v_6 I_{31}] + \frac{E}{1+\mu} [(v_2, \phi\phi + w_1, \phi) (MCI_{16} + MDI_{22} + NDI_{15}) \\
& + (v_3, \phi\phi + w_3, \phi) (MCI_{22} + MDI_{34} + NDI_{30}) + v_6, \phi\phi (MCI_{34} + MDI_{35} + NDI_{31})] = 0 \quad (165)
\end{aligned}$$

The integrals I_1, I_2, \dots given in the above equations are evaluated in Appendix I.

C. Reduction To a System of Linear Algebraic Equations
Equations Using Fourier Series

The unknown coefficients in the displacements expansions given in equations (147) may be expressed as Fourier series. The circumferential variable ϕ may then be isolated and the differential equations (148) through (165) lead to algebraic equations. The Fourier series expansions are given by the following:

In the radial direction,

$$w_i = W_i \sin(n\phi + \phi_0)$$

where $i = 0$ to 4 (166)

and in the circumferential direction

$$v_i = V_i \cos(n\phi + \phi_0)$$

where $i = 0$ to 7 (167)

and in the out of plane direction

$$u_i = U_i \sin(n\phi + \phi_0)$$

where $i = 0$ to 4 (168)

These expansions can be substituted into equations (148) through (165) to give eighteen algebraic equations in the variables,

$$W_1, W_2, W_3, W_4, V_1, V_2, V_3, V_4, V_5, V_6, V_7$$

$$W_0, V_0, U_0, U_1, U_2, U_3, U_4$$

After collecting coefficients of like terms, equations (148) through (157) give the inplane equations as the following,

$$\begin{aligned}
& W_0[-I_1(n^2G + \frac{EM}{1+\mu}) - \frac{n^2E}{1+\mu}(MCI_3 + MDI_4 + NDI_1)] + V_0[nI_1(G + \frac{EM}{1+\mu}) + \frac{nE}{1+\mu}(MCI_3 + MDI_4 + NDI_1)] \\
& + V_1\{-n[G(A - I_2) - \frac{EM}{1+\mu}I_2] + \frac{nE}{1+\mu}(MCI_4 + MDI_9 + NDI_2)\} + W_2[-(n^2G - \frac{EM}{1+\mu})I_2 - \frac{NE}{1+\mu}A \\
& - n^2\frac{E}{1+\mu}(MCI_4 + MDI_9 + NDI_2)] + V_4[nG(I_8 - 2I_6) + \frac{nEM}{1+\mu}I_8 + \frac{nE}{1+\mu}(MCI_9 + MDI_{19} + NDI_8)] \\
& + U_2[-\frac{EN}{1+\mu}A] + U_3[-\frac{EN}{1+\mu}I_6] + W_4[-(n^2G + \frac{EM}{1+\mu})I_5 - \frac{n^2E}{1+\mu}(MCI_7 + MDI_{11} + NDI_5)] \\
& + V_5[nI_5(G + \frac{EM}{1+\mu}) + \frac{nE}{1+\mu}(MCI_7 + MDI_{11} + NDI_5)] + V_7[-nG(I_{13} - I_{10}) + \frac{nEM}{1+\mu}I_{10} \\
& + \frac{nE}{1+\mu}(MCI_{11} + MDI_{16} + NDI_{10})] = 0 \tag{169}
\end{aligned}$$

$$\begin{aligned}
& W_0[nI_1(G + \frac{EM}{1+\mu}) + \frac{nE}{1+\mu}(MCI_3 + MDI_4 + MDI_{11})] + V_0[-I_1(G + \frac{n^2EM}{1+\mu}) - \frac{n^2E}{1+\mu}(MCI_3 + MDI_4 + NDI_1)] \\
& + V_1[G(A - I_2) - \frac{n^2EM}{1+\mu}I_2 - \frac{n^2E}{1+\mu}(MCI_4 + MDI_9 + NDI_2)] + W_2[\frac{nE}{1+\mu}(MI_2 + NA) + nGI_2 + \frac{nE}{1+\mu}(MCI_4 + MDI_9 + NDI_2)] \\
& + V_4[G(2I_6 - I_8) - \frac{n^2EM}{1+\mu}I_8 - \frac{n^2E}{1+\mu}(MCI_9 + MDI_{19} + NDI_8)] + U_2[\frac{nEN}{1+\mu}A] + U_3[\frac{nEN}{1+\mu}I_6] \\
& + W_4[nI_5(\frac{EM}{1+\mu} + G) + \frac{nE}{1+\mu}(MCI_7 + MDI_{11} + NDI_5)] + V_5[-I_9(\frac{n^2EM}{1+\mu} + G) - \frac{n^2E}{1+\mu}(MCI_7 + MDI_{11} + NDI_5)] \\
& + V_7[G(I_{13} - I_{10}) - \frac{n^2EM}{1+\mu}I_{10} - \frac{n^2E}{1+\mu}(MCI_{11} + MDI_{16} + NDI_{10})] = 0 \tag{170}
\end{aligned}$$

Inplane Equations (cont'd.)

$$\begin{aligned}
& W_0 \left[n \left(\frac{EM}{1+\mu} I_2 - aGI_1 \right) + \frac{nE}{1+\mu} (MCI_4 + MDI_9 + NDI_2) \right] + V_0 \left[aGI_1 - \frac{n^2 EM}{1+\mu} I_2 - \frac{n^2 E}{1+\mu} (MCI_4 + MDI_9 + NDI_2) \right] \\
& + V_1 \left[G(aI_2 - I_{11}) - aAG - \frac{n^2 EM}{1+\mu} I_8 - \frac{n^2 E}{1+\mu} (MCI_9 + MDI_{19} + NDI_8) \right] + W_2 \left[\frac{nE}{1+\mu} (MI_8 + NI_6) - naGI_2 \right. \\
& + \left. \frac{nE}{1+\mu} (MCI_9 + MDI_{19} + NDI_8) \right] + V_4 \left[aG(I_8 - 2I_6) - \frac{n^2 EM}{1+\mu} I_{18} - \frac{n^2 E}{1+\mu} (MCI_{19} + MDI_{20} + NDI_{18}) \right] \\
& + U_2 \left[\frac{nEN}{1+\mu} I_6 \right] + U_3 \left[\frac{nEN}{1+\mu} I_{12} \right] + W_4 \left[n \left(\frac{EM}{1+\mu} I_{10} - aGI_5 \right) + \frac{nE}{1+\mu} (MCI_{11} + MDI_{16} + NDI_{10}) \right] \\
& + V_5 \left[GaI_5 - \frac{n^2 EM}{1+\mu} I_{10} - \frac{n^2 E}{1+\mu} (MCI_{11} + MDI_{16} + NDI_{10}) \right] + V_7 \left[aG(I_{10} - I_{13}) - \frac{n^2 EM}{1+\mu} I_{15} \right. \\
& \quad \left. - \frac{n^2 E}{1+\mu} (MCI_{16} + MDI_{22} + NDI_{15}) \right] = 0
\end{aligned} \tag{171}$$

Inplane Equations (cont'd.)

$$\begin{aligned}
& W_0 \left[-\frac{EN}{1+\mu} (aI_1 + I_2) - n^2 I_2 \left(G + \frac{EM}{1+\mu} \right) - \frac{n^2 E}{1+\mu} (MCI_4 + MDI_9 + NDI_2) \right] + V_0 \left[\frac{nEN}{1+\mu} (aI_1 + I_2) + nI_2 \left(G + \frac{EM}{1+\mu} \right) \right. \\
& + \left. \frac{nE}{1+\mu} (MCI_4 + MDI_9 + NDI_2) \right] + V_1 \left[\frac{nEN}{1+\mu} (aI_2 + I_8) + nG(I_8 - I_6) + \frac{nEM}{1+\mu} I_8 + \frac{nE}{1+\mu} (MCI_9 + MDI_{19} + NDI_8) \right] \\
& + W_2 \left\{ -\frac{E}{1+\mu} [M(aA + I_6) + N(aI_2 + I_8)] - n^2 GI_8 - \frac{E}{1+\mu} (MI_8 + NI_6) - \frac{n^2 E}{1+\mu} (MCI_9 + MDI_{19} + NDI_8) \right\} \\
& + V_4 \left[\frac{nEN}{1+\mu} (aI_8 + I_{18}) + nG(I_{18} - 2I_{12}) + \frac{nEM}{1+\mu} I_{18} + \frac{nE}{1+\mu} (MCI_{19} + MDI_{20} + NDI_{18}) \right] + U_2 \left[-\frac{EN}{1+\mu} (aA + 2I_6) \right] \\
& + U_3 \left[-\frac{EN}{1+\mu} (aI_6 + 2I_{12}) \right] + W_4 \left[-\frac{EN}{1+\mu} (aI_5 + I_{10}) - I_{22} \left(n^2 G + \frac{EM}{1+\mu} \right) - \frac{n^2 E}{1+\mu} (MCI_{11} + MDI_{16} + NDI_{10}) \right] \\
& + V_5 \left[\frac{nEN}{1+\mu} (aI_5 + I_{10}) + nI_{10} \left(G + \frac{EM}{1+\mu} \right) + \frac{EM}{1+\mu} (MCI_{11} + MDI_{16} + NDI_{10}) \right] \\
& + V_7 \left[\frac{nEN}{1+\mu} (aI_{10} + I_{15}) - nG(I_{14} - I_{15}) + \frac{nEM}{1+\mu} I_{15} + \frac{nE}{1+\mu} (MCI_{16} + MDI_{22} + NDI_{15}) \right] = 0 \quad (172)
\end{aligned}$$

Inplane Equations (cont'd.)

$$\begin{aligned}
& W_0 [-nG(2aI_2 + I_8) + \frac{nEM}{1+\mu}I_8 + \frac{nE}{1+\mu}(MCI_9 + MDI_{19} + NDI_8)] + V_0 [G(2aI_2 + I_8) - \frac{n^2ME}{1+\mu}I_8 \\
& - \frac{n^2E}{1+\mu}(MCI_9 + MDI_{19} + NDI_8)] + V_1 \{G[2a(I_8 - I_6) + I_{18} - I_{12}] - \frac{n^2EM}{1+\mu}I_{17} - \frac{n^2E}{1+\mu}(MCI_{19} + MDI_{20} + NDI_{18})\} \\
& + W_2 [-nG(2aI_8 + I_{18}) + \frac{nE}{1+\mu}(MI_{18} + NI_{12}) + \frac{nE}{1+\mu}(MCI_{19} + MDI_{20} + NDI_{18})] \\
& + V_4 \{G[2a(I_{18} - 2I_{12}) + I_{21} - 2I_{17}] - \frac{n^2EM}{1+\mu}I_{21} - \frac{n^2E}{1+\mu}(MCI_{20} + MDI_{23} + NDI_{21})\} \\
& + U_2 [\frac{nEN}{1+\mu}I_{12}] + U_3 [\frac{nEN}{1+\mu}I_{17}] + W_4 [\frac{nEM}{1+\mu}I_{15} - nG(2aI_{10} + I_{15}) + \frac{nE}{1+\mu}(MCI_{16} + MDI_{22} + NDI_{15})] \\
& + V_5 [G(2aI_{10} + I_{15}) - \frac{n^2EM}{1+\mu}I_{15} - \frac{n^2E}{1+\mu}(MCI_{16} + MDI_{22} + NDI_{15})] \\
& + V_7 \{-G[2a(I_{14} + I_{15}) + I_{25} + I_{30}] - \frac{n^2EM}{1+\mu}I_{30} - \frac{n^2E}{1+\mu}(MCI_{22} + MDI_{34} + NDI_{30})\} = 0 \tag{173}
\end{aligned}$$

$$\begin{aligned}
 & W_0 \left[-\frac{I+n}{EN} (a_{I1} + I_2) \right] + V_0 \left[\frac{I+n}{nEN} (a_{I1} + I_2) \right] + V_1 \left[\frac{I+n}{nEN} (a_{I2} + I_8) \right] + V_2 \left[-\frac{I+n}{EN} (a_{I2} + I_8) \right] + V_3 \left[-\frac{I+n}{EN} (a_{I6} + I_6) \right] + V_4 \left[\frac{I+n}{nEN} (a_{I8} + I_{18}) \right] \\
 & - \frac{I+n}{n^2 E} (MCI_{11} + MDI_{16} + NDI_{10}) + W_4 \left[-\frac{I+n}{EN} (a_{I5} + I_{10}) \right] + V_5 \left[-2nGI_{13} + \frac{I+n}{nEN} (a_{I5} + I_{10}) \right] \\
 & + U_2 \left[-n^2 GI_5 - \frac{I+n}{EM} (a_{I5} + I_6) - \frac{I+n}{n^2 E} (MCI_7 + MDI_{11} + NDI_5) \right] + U_3 \left[-n^2 GI_{10} - \frac{I+n}{EM} (a_{I6} + I_{12}) \right] \\
 & + V_7 \left[-2nGI_{14} + \frac{I+n}{nEN} (a_{I10} + I_{15}) \right] = 0 \quad (174)
 \end{aligned}$$

$$\begin{aligned}
 & W_0 \left[-\frac{I+n}{EN} (a_{I2} + I_8) \right] + V_0 \left[\frac{I+n}{nEN} (a_{I2} + I_8) \right] + V_1 \left[\frac{I+n}{nEN} (a_{I8} + I_{18}) \right] + W_2 \left[-\frac{I+n}{EN} (a_{I8} + I_6) + I_{12} + I_{18} \right] \\
 & + V_4 \left[\frac{I+n}{nEN} (a_{I8} + I_{21}) \right] + U_2 \left[-n^2 GI_{10} - \frac{I+n}{EM} (a_{I6} + I_{12}) - \frac{I+n}{n^2 E} (MCI_{11} + MDI_{16} + NDI_{10}) \right] \\
 & + U_3 \left[-G(a_{I3} + I_{14}) - n^2 GI_{15} - \frac{I+n}{EM} (a_{I12} + I_{17}) - \frac{I+n}{n^2 E} (MCI_{16} + MDI_{22} + NDI_{15}) \right] \\
 & + W_4 \left[-2G(a_{I13} + I_{14}) - \frac{I+n}{nEN} (a_{I10} + I_{15}) \right] + V_5 \left[-2nGI_{14} + \frac{I+n}{nEN} (a_{I10} + I_{15}) \right] \\
 & + V_7 \left[-2nGI_{25} + \frac{I+n}{nEN} (a_{I15} + I_{21}) \right] = 0 \quad (175)
 \end{aligned}$$

Inplane Equations (cont'd.)

$$\begin{aligned}
& W_0 \left[-n^2 GI_5 - \frac{EM}{1+\mu} I_5 - \frac{n^2 E}{1+\mu} (MCI_7 + MDI_{11} + NDI_5) \right] + V_0 \left[nGI_5 + \frac{nEM}{1+\mu} I_5 + \frac{nE}{1+\mu} (MCI_7 + MDI_{11} + NDI_5) \right] \\
& + V_1 \left[-nGI_{13} + nGI_{10} + \frac{nEM}{1+\mu} I_{10} + \frac{nE}{1+\mu} (MCI_{11} + MDI_{16} + NDI_{10}) \right] + W_2 \left[-n^2 GI_{10} - \frac{EM}{1+\mu} I_{10} - \frac{EN}{1+\mu} I_{13} \right. \\
& \quad \left. - \frac{n^2 E}{1+\mu} (MCI_{11} + MDI_{16} + NDI_{10}) \right] + V_4 \left[nG(I_{15} - 2I_{14}) + \frac{nEM}{1+\mu} I_{15} + \frac{nE}{1+\mu} (MCI_{16} + MDI_{22} + NDI_{15}) \right] \\
& + U_2 \left[-\frac{EN}{1+\mu} I_{13} \right] + U_3 \left[-2G(aI_{13} + I_{14}) - \frac{EN}{1+\mu} I_{14} \right] + W_4 \left[-n^2 GI_{26} - 4G(aI_{13} + I_{14}) - \frac{EM}{1+\mu} I_{26} \right. \\
& \quad \left. - \frac{En^2}{1+\mu} (MCI_{27} + MDI_{28} + NDI_{26}) \right] + V_5 \left[nI_{26} \left(G + \frac{EM}{1+\mu} \right) + \frac{nE}{1+\mu} (MCI_{27} + MDI_{28} + NDI_{26}) \right] \\
& \quad + V_7 \left[-nG(I_{36} - I_{38}) + \frac{nEM}{1+\mu} I_{38} + \frac{nE}{1+\mu} (MCI_{28} + MDI_{40} + NDI_{38}) \right] = 0 \quad (176)
\end{aligned}$$

Inplane Equations (cont'd.)

$$\begin{aligned}
& W_0 \left[nI_5 \left(\frac{EM}{1+\mu} + G \right) + \frac{nE}{1+\mu} (MCI_7 + MDI_{11} + NDI_5) \right] + V_0 \left[-I_5 \left(\frac{n^2 EM}{1+\mu} + G \right) - \frac{n^2 E}{1+\mu} (MCI_7 + MDI_{11} + NDI_5) \right] \\
& + V_1 \left[-\frac{n^2 EM}{1+\mu} I_{10} + G(I_{13} - I_{10}) - \frac{n^2 E}{1+\mu} (MCI_{11} + MDI_{16} + NDI_{10}) \right] + V_4 \left[-\frac{n^2 EMI_{15}}{1+\mu} + G(2I_{14} - I_{15}) \right] \\
& - \frac{n^2 E}{1+\mu} (MCI_{16} + MDI_{22} + NDI_{15}) \left. \right] + W_2 \left[\frac{nEM}{1+\mu} (MI_{10} + NI_{13}) + nGI_{10} + \frac{nE}{1+\mu} (MCI_{11} + MDI_{16} + NDI_{10}) \right] \\
& + U_2 \left[\frac{nEN}{1+\mu} I_{13} - 2nG(aI_5 + I_{10}) \right] + U_3 \left[\frac{nEN}{1+\mu} I_{14} - 2nG(aI_{10} + I_{15}) \right] + W_4 \left[nI_{26} \left(\frac{EM}{1+\mu} + G \right) + \frac{E}{1+\mu} (MCI_{27} \right. \\
& \left. + MDI_{28} + NDI_{26}) \right] + V_5 \left[-\frac{n^2 EM}{1+\mu} I_{26} - 4G(aI_{13} + I_{14}) - GI_{26} - \frac{En^2}{1+\mu} (MCI_{27} + MDI_{28} + NDI_{26}) \right] \\
& + V_7 \left[-\frac{n^2 EM}{1+\mu} I_{38} - 4G(aI_{14} + I_{25}) + G(I_{36} - I_{38}) - \frac{n^2 E}{1+\mu} (MCI_{28} + MDI_{40} + NDI_{38}) \right] = 0 \quad (177)
\end{aligned}$$

Inplane Equations (cont'd.)

$$\begin{aligned}
& W_0 \left[\frac{nEM}{1+\mu} I_{10} - nGaI_9 + \frac{nE}{1+\mu} (MCI_{11} + MDI_{16} + NDI_{10}) \right] + V_0 \left[GaI_5 - \frac{n^2 EM}{1+\mu} I_{10} - \frac{n^2 E}{1+\mu} (MCI_{11} \right. \\
& \quad \left. + MDI_{16} + NDI_{10}) \right] + V_1 \left[Ga(I_{10} - I_{13}) - \frac{n^2 EM}{1+\mu} I_{15} - \frac{n^2 E}{1+\mu} (MCI_{16} + MDI_{22} + NDI_{15}) \right] \\
& \quad + W_2 \left[\frac{nE}{1+\mu} (MI_{15} + NI_{14}) - nGaI_{10} + \frac{nE}{1+\mu} (MCI_{16} + MDI_{22} + NDI_{15}) \right] \\
& \quad + V_4 \left[Ga(I_{15} - 2I_{14}) - \frac{n^2 EM}{1+\mu} I_{30} - \frac{n^2 E}{1+\mu} (MCI_{22} + MDI_{34} + NDI_{30}) \right] \\
& \quad + U_2 \left[\frac{nEN}{1+\mu} I_{14} - 2nG(aI_{10} + I_{15}) \right] + U_3 \left[\frac{nEN}{1+\mu} I_{25} - 2nG(aI_{15} + I_{30}) \right] \\
& \quad + W_4 \left[\frac{nEM}{1+\mu} I_{38} - nGaI_{26} + \frac{nE}{1+\mu} (MCI_{28} + MDI_{40} + NDI_{38}) \right] \\
& \quad + V_5 \left[GaI_{26} - \frac{n^2 EM}{1+\mu} I_{38} - 4G(aI_{14} + I_{25}) - \frac{n^2 E}{1+\mu} (MCI_{28} + MDI_{40} + NDI_{38}) \right] \\
& \quad + V_7 \left[Ga(I_{38} - I_{36}) - 4G(aI_{25} + I_{29}) - \frac{n^2 EM}{1+\mu} I_{39} - \frac{n^2 E}{1+\mu} (MCI_{40} + MDI_{41} + NDI_{39}) \right] = 0 \quad (178)
\end{aligned}$$

After collecting coefficients of like terms, equations (158) through (165) give the out of plane equations as the following,

$$U_0[-n^2GI_1 - \frac{n^2E}{1+\mu}(MCI_3 + MDI_4 + NDI_1)] + W_0[0] + U_1[-n^2GI_2 - \frac{n^2E}{1+\mu}(MCI_4 + MDI_9 + NDI_2)] \\ + V_2[-nGA] + V_3[-nGI_6] + W_3[0] + U_4[-n^2GI_8 - \frac{n^2E}{1+\mu}(MCI_9 + MDI_{19} + NDI_8)] + V_6[-nGI_{12}] = 0 \quad (179)$$

$$U_0[0] + W_1[-(n^2G + \frac{EM}{1+\mu})I_5 - G(aA + I_6) - \frac{n^2E}{1+\mu}(MCI_7 + MDI_{11} + NDI_5)] + U_1[-G(aA + I_6)] \\ + V_2[nI_9(G + \frac{EM}{1+\mu}) + \frac{nE}{1+\mu}(MCI_7 + MDI_{11} + NDI_5)] + V_3[nG(I_{10} - I_{13}) + \frac{nEM}{1+\mu}I_{10} + \frac{nE}{1+\mu}(MCI_{11} + MDI_{16} \\ + NDI_{10})] + W_3[-G(n^2I_{10} + aI_6 + I_{12}) - \frac{E}{1+\mu}(MI_{10} + NI_{13} - \frac{n^2E}{1+\mu}(MCI_{11} + MDI_{16} + NDI_{10}))] \\ + U_4[-2G(aI_6 + I_{12})] + V_6[-nG(2I_{14} - I_{15}) + \frac{nEM}{1+\mu}I_{15} + \frac{nE}{1+\mu}(MCI_{16} + MDI_{22} + NDI_{15})] = 0 \quad (180)$$

$$U_0[-n^2GI_2 - \frac{n^2E}{1+\mu}(MCI_4 + MDI_9 + NDI_2)] + W_1[-G(aA + I_6)] + U_1[-G(aA + I_6) - n^2GI_8 - \frac{n^2E}{1+\mu}(MCI_9 + MDI_{19} \\ + NDI_8)] + V_2[-nGI_6] + V_3[-nGI_{12}] + W_3[-G(aI_6 + I_{12})] + U_4[-2G(aI_6 + I_{12}) - n^2GI_{18} \\ - \frac{n^2E}{1+\mu}(MCI_{19} + MDI_{20} + NDI_{18})] + V_6[0] = 0 \quad (181)$$

Out of Plane Equations (cont'd.)

$$\begin{aligned}
& U_0[-nG(aI_1 + I_2)] + W_1[nI_5(\frac{EM}{1+\mu} + G) + \frac{nE}{1+\mu}(MCI_7 + MDI_{11} + NDI_5)] + U_1[-nG(aI_2 + I_8)] \\
& + V_2[-(\frac{n^2EM}{1+\mu} + G)I_5 - G(aA + I_6) - \frac{n^2E}{1+\mu}(MCI_7 + MDI_{11} + NDI_5)] + V_3[GI_{13} - (\frac{n^2EM}{1+\mu} + G)I_{10} - G(aI_6 + I_{12}) \\
& - \frac{n^2E}{1+\mu}(MCI_{11} + MDI_{16} + NDI_{10})] + W_3[\frac{nE}{1+\mu}(MI_{10} + NI_{13}) + nGI_{10} + \frac{nE}{1+\mu}(MCI_{11} + MDI_{16} + NDI_{10})] \\
& + U_4[-nG(aI_8 + I_{18})] + V_6[G(2I_{14} - I_{15}) - \frac{n^2EM}{1+\mu}I_{15} - G(aI_{12} + I_{17}) - \frac{n^2E}{1+\mu}(MCI_{16} + MDI_{22} + NDI_{15})] = 0
\end{aligned} \tag{182}$$

$$\begin{aligned}
& U_0[-nG(aI_2 + I_8)] + W_1[nI_{22}(\frac{EM}{1+\mu} + G) - nG(aI_5 + I_{10}) + \frac{nE}{1+\mu}(MCI_{11} + MDI_{16} + NDI_{10})] \\
& + U_1[-nG(aI_8 + I_{18})] + V_2[G(aI_5 - aI_6 - I_{12}) - \frac{n^2EM}{1+\mu}I_{10} - \frac{n^2E}{1+\mu}(MCI_{11} + MDI_{16} + NDI_{10})] \\
& + V_3[-G(aI_{13} + aI_{12} - aI_{10} + I_{17}) - \frac{n^2E}{1+\mu}MI_{15} - \frac{n^2E}{1+\mu}(MCI_{16} + MDI_{22} + NDI_{15})] \\
& + W_3[\frac{nE}{1+\mu}(MI_{15} + NI_{14}) - nGaI_{10} + \frac{nE}{1+\mu}(MCI_{16} + MDI_{22} + NDI_{15})] + U_4[-nG(aI_{18} + I_{21})] \\
& + V_6[-G(2aI_{14} - aI_{15} + aI_{17} + I_{24}) - \frac{n^2E}{1+\mu}MI_{30} \\
& - \frac{n^2E}{1+\mu}(MCI_{22} + MDI_{34} + NDI_{30})] = 0
\end{aligned} \tag{183}$$

Out of Plane Equations (cont'd.)

$$\begin{aligned}
& U_0[0] + W_1 \left\{ \frac{E}{1+\mu} [N(aI_5 + I_{10}) + MI_{10}] - G[n^2 I_{10} + aI_6 + I_{12}] - \frac{n^2 E}{1+\mu} (MCI_{11} + MDI_{16} + NDI_{10}) \right\} \\
& + U_1 [-G(aI_6 + I_{12})] + V_2 \left\{ \frac{nE}{1+\mu} [N(aI_5 + I_{10}) + NI_{10}] + nGI_{10} + \frac{nE}{1+\mu} (MCI_{11} + MDI_{16} + NDI_{10}) \right\} \\
& + V_3 \left\{ \frac{nE}{1+\mu} [MI_{15} + N(aI_{10} + I_{15})] + nG(I_{15} - I_{14}) + \frac{nE}{1+\mu} (MCI_{16} + MDI_{22} + NDI_{15}) \right\} \\
& + W_3 \left\{ -\frac{E}{1+\mu} [M(aI_{13} + I_{14} + I_{15}) + N(aI_{10} + I_{15} + I_{14})] - G(n^2 I_{15} + aI_{12} + I_{17}) \right. \\
& \quad \left. - \frac{n^2 E}{1+\mu} (MCI_{16} + MDI_{22} + NDI_{15}) \right\} + U_4 [-2G(aI_{12} + I_{17})] \\
& + V_6 \left\{ \frac{nE}{1+\mu} [MI_{30} + N(aI_{15} + I_{30}) - nG(2I_{25} - I_{30}) \right. \\
& \quad \left. + \frac{nE}{1+\mu} (MCI_{22} + MDI_{34} + NDI_{30}) \right\} = 0
\end{aligned} \tag{184}$$

Out of Plane Equations (cont'd.)

$$\begin{aligned}
& U_0 [-n^2GI_8 - \frac{n^2E}{1+\mu}(MCI_9 + MDI_{19} + NDI_8)] + W_1 [-2G(aI_6 + I_{12})] + U_1 [-2G(aI_6 + I_{12}) - n^2GI_{18} \\
& - \frac{n^2E}{1+\mu}(MCI_{11} + MDI_{16} + NDI_{10})] + U_1 [-G(aI_6 + I_{12})] + V_2 [\frac{nEN}{1+\mu}(aI_5 + I_{10}) + nI_{10}(\frac{EM}{1+\mu} + G) \\
& + \frac{nE}{1+\mu}(MCI_{11} + MDI_{16} + NDI_{10})] + V_3 [\frac{nEN}{1+\mu}(aI_{10} + I_{16}) + nG(I_{15} - I_{14}) + \frac{nEM}{1+\mu}I_{15} + \frac{nE}{1+\mu}(MCI_{16} + MDI_{22} \\
& + ND_{15})] + W_3 [-2G(aI_{12} + I_{17})] + U_4 [-4G(aI_{12} + I_{17}) - n^2GI_{21} - \frac{n^2E}{1+\mu}(MCI_{20} + MDI_{23} + NDI_{21})] \\
& + V_6 [-nG(aI_{17} + I_{24})] = 0 \tag{185}
\end{aligned}$$

$$\begin{aligned}
& U_0 [-nG(aI_8 + I_{18})] + W_1 [-nG(a2I_{10} + I_{15}) + nI_{15} \frac{EM}{1+\mu} + \frac{nE}{1+\mu}(MCI_{16} + MDI_{22} + NDI_{15})] \\
& + U_1 [-nG(aI_{18} + I_{21})] + V_2 [G(2aI_{10} + I_{15} - aI_{12} - I_{17}) - \frac{n^2EM}{1+\mu}I_{15} - \frac{n^2E}{1+\mu}(MCI_{16} + MDI_{22} + NDI_{15})] \\
& + V_3 [-G(2aI_{14} + I_{25} - aI_{15} - I_{30} + aI_{17} + I_{24}) - \frac{n^2EM}{1+\mu}I_{30} - \frac{n^2E}{1+\mu}(MCI_{22} + MDI_{34} + NDI_{30})] \\
& + W_3 [-nG(2aI_{15} + I_{30}) + \frac{nE}{1+\mu}(MI_{30} + NI_{25}) + \frac{nE}{1+\mu}(MCI_{22} + MDI_{34} + NDI_{30})] + U_4 [-nG(aI_{21} + I_{33})] \\
& + V_6 [G(2aI_{30} + I_{31} - 4aI_{25} - 2I_{29} - aI_{24} - I_{32}) - \frac{n^2E}{1+\mu}MI_{31} - \frac{n^2E}{1+\mu}(MCI_{34} + MDI_{35} + NDI_{31})] = 0 \tag{186}
\end{aligned}$$

The integrals over the ring cross section are given in Appendix I. Thus far, in this analysis only symmetry in the x-direction (out of plane) has been assumed, so that the initial position problem satisfies the conditions of the Lamé problem.

In the example problems which were solved a rectangular cross section was chosen. This selection eliminated some of the integrals since such rings have double symmetry. However, the technique of solution applies to a one directional symmetry case as well.

A description of the cross section of the ring is given in figure 5.

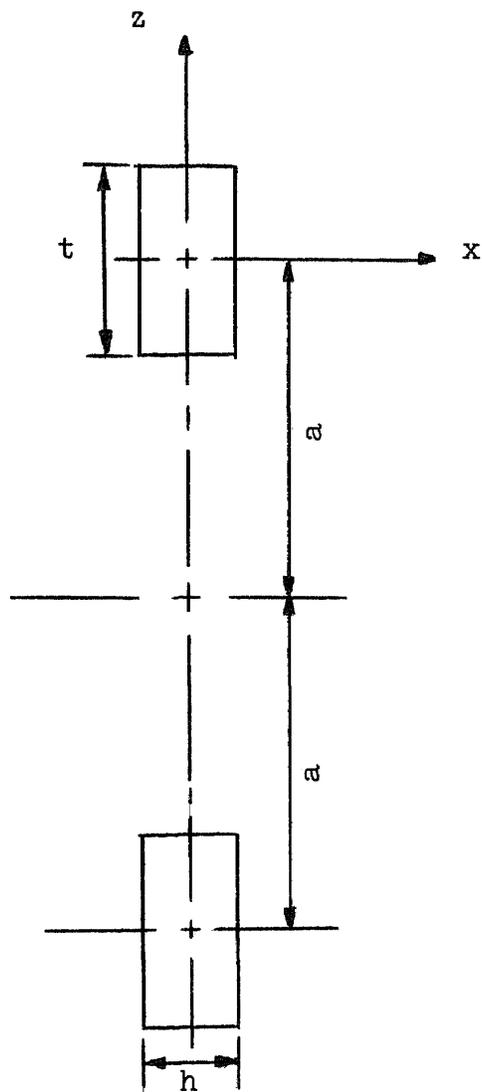


Figure 5 Cross Section of Rectangular Ring

VII. SOLUTION TO A SYSTEM OF HOMOGENEOUS
LINEAR ALGEBRAIC EQUATIONS

A. Definition of The Eigenvalues

In the foregoing sections, three systems of homogeneous linear algebraic equations have been developed in terms of displacement expansions with unknown coefficients. The in-plane and out of plane buckling of a thin ring is given by equations (89) through (96). The inplane buckling of a thick ring is given by equations (169) through (178) and the out of plane buckling of a thick ring is given by equations (179) through (186).

Each of these sets of equations can be written in indicial notation in the following form, where a repeated subscript indicates summation,

$$(A_{ij} + \alpha_{ij}\lambda) X_j = 0 \quad (187)$$

The X_j are the displacement functions and the A_{ij} are their coefficients. The λ is the load term or eigenvalue and the α_{ij} are the coefficients of λ . For equation (187) to have a solution the determinant of the coefficients of X_j must be equal to zero. The values of λ for which this condition is satisfied are called eigenvalues, and equation (187) is said to be an eigenvalue problem.

In each set of equations, the expressions for the terms in equation (187) are different. For the inplane case of the

thick ring, i.e. equations (169) through (178) each variable X_j for ($j=1, 10$) is given by ($W_0, V_0, V_1, W_2, V_4, U_2, U_3, W_4, V_5, V_7$) respectively. For the out of plane case of the thick ring i.e. equations (179) through (186) each X_j for ($j=1, 8$) is given by ($U_0, W_0, U_1, V_2, V_3, W_3, U_4, V_6$) respectively. In both cases, the eigenvalue λ , was chosen to be equal to κ , the dimensionless common factor of the coefficients C and D, which were given in the initial problem solution in section III.A by equations (54), (42) and (43) respectively. It is given by the following,

$$\lambda = \kappa = \frac{-p R_o^2}{E(R_o^2 - R_i^2)} \quad (188)$$

In the case of a ring with a rectangular cross section, the inner and outer radii may be expressed as

$$\begin{aligned} R_o &= a + t/2 \\ R_i &= a - t/2 \end{aligned} \quad (189)$$

where t is the radial thickness of the ring. After substituting equations (189) into equation (188) and multiplying by h/h , the eigenvalue λ can be expressed as

$$\lambda = \frac{-Pa \left(1 + \frac{t}{2a}\right)^2}{2EA} \quad (190)$$

the radial line load P is given by

$$P = ph \quad (191)$$

in pounds per inch of circumference.

For the thin ring problem, the inplane and out of plane buckling conditions are described by one set of combined equations, i.e. equations (89) through (96). In this case each variable X_j for $(j=1,8)$ is given by $(W_0, V_0, V_1, U_0, W_1, U_1, V_2, V_3)$ respectively. The eigenvalue λ is chosen to be the common factor in all of the load terms in equations (89) through (96) i.e.

$$\lambda = Mw^0 = \frac{-Pa^2 (1-\mu)}{AE (1-2\mu)} \quad (192)$$

where M and w^0 are given in equations (52) and (55) respectively.

B. Standard Eigenvalue Problem Format

The eigenvalue problem equation, i.e. equation (187) may be written as,

$$[A_{ij} - (-\alpha_{ij})\lambda]x_j = 0 \quad (193)$$

Then pre-multiplying by the negative inverse of the α matrix yields,

$$[(-\alpha_{ik})^{-1}A_{kj} - (-\alpha_{ik})^{-1}(\alpha_{kj})\lambda]x_j = 0 \quad (194)$$

and using the definition of the identity matrix E_n to substitute for the coefficient of λ in equation (194) yields

$$(B_{ij} - E_n \lambda)x_j = 0 \quad (195)$$

with

$$B_{ij} = (-\alpha_{ik})^{-1}A_{kj} \quad (196)$$

Equation (195) is in the standard form of an eigenvalue problem. This equation is the general form for the three previously mentioned sets of equations which are to be solved:

- (a) thick ring inplane buckling, (10x10) matrices
- (b) thick ring out of plane buckling, (8x8) matrices
- (c) thin ring with unknown displacements and combined inplane and out of plane buckling, (8x8) matrices.

VIII. COMPUTER PROGRAMS TO SOLVE THE EIGENVALUE PROBLEMS

The computer programs used to solve the three eigenvalue problems consists of four parts:

- (a) Main program
- (b) Subprogram to evaluate the integrals (thick ring only)
- (c) Matrix inversion subprogram
- (d) Eigenvalue package of subprograms

An abbreviated flow chart is given in figure 6, and a complete listing of the programs is given in Appendix II. A summary of each program is given in the next sections.

A. Main Program

The main program allows for the input of the data values, such as the order, and size of the root and matrices respectively. It also describes the ring parameters such as material properties E , G and μ and geometrical properties such as r , t , h for the radius, thickness and depth of the ring. The main program also evaluates the coefficients of all of the variables as specified in section VII. It also performs a matrix multiplication to obtain the final form of the standard eigenvalue problem. The critical buckling load is extracted from the eigenvalue. And the results are printed. The length of the output may be varied by selecting the number of matrices not to be printed and assigning that value to the variable called SKIP. It will skip in reverse order, starting with the B_{ij} matrix for SKIP equal to unity and proceeding to the A_{ij} matrix for SKIP equal to four.

B. Integral Subroutine

The subroutine subprogram AINTEG evaluates the cross sectional property integrals as given in Appendix I. Each integral which involved an infinite series was evaluated to three term accuracy. This would indicate that the first term which was dropped was of the order of

$$\left(\frac{t}{2a}\right)^6 \ll 1 \quad (196)$$

This subroutine was used to evaluate forty-one integrals for the thick ring cases. It was not used for the thin ring and the eighteen required integrals were evaluated in the main program. However, the same forms were used as in the AINTEG subroutine. This modification was made since the thin ring main program was smaller in size and ran at a higher system priority than the thick ring program. It was therefore desirable to keep the number of subprograms at a minimum, to retain this priority level.

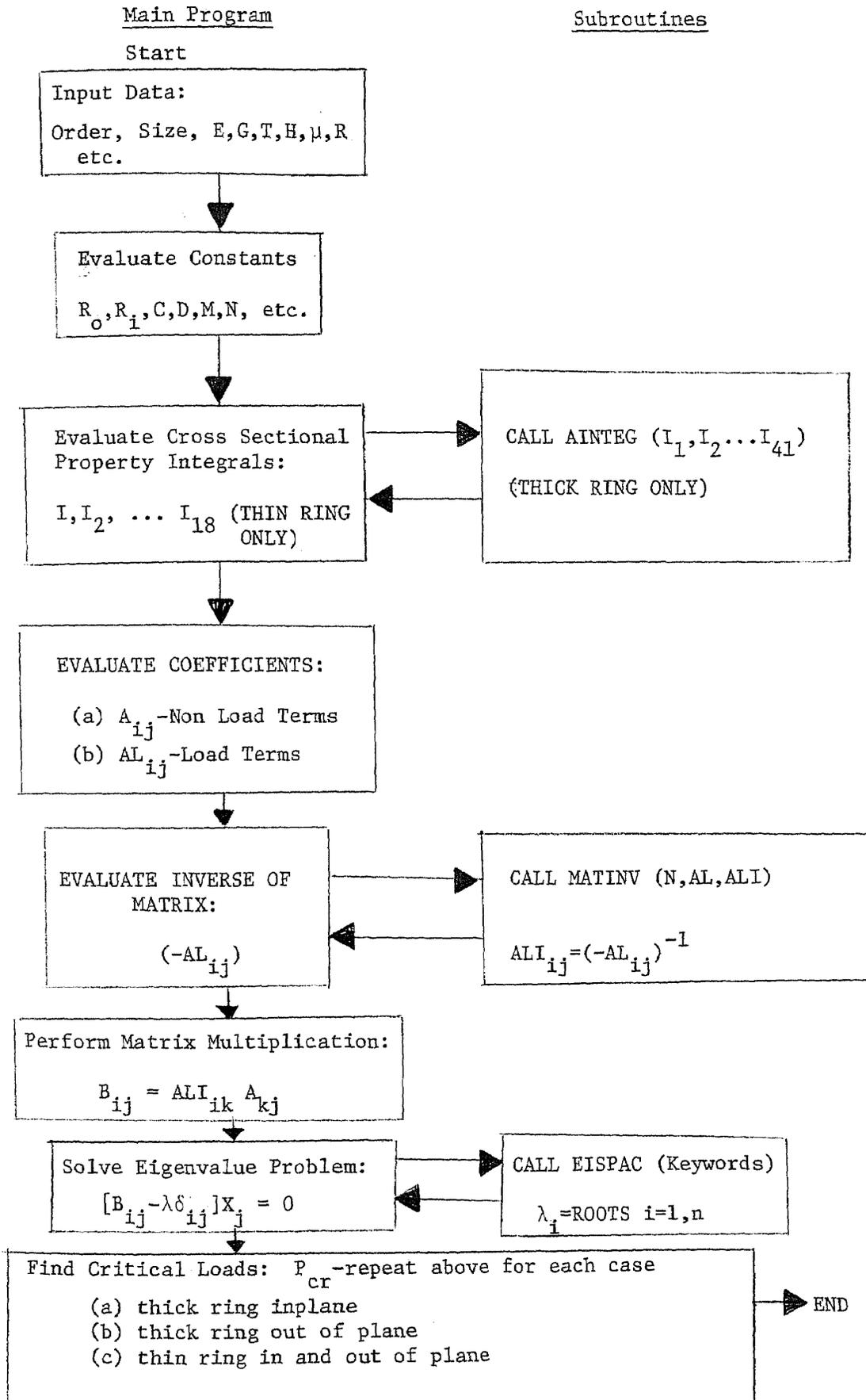
C. Matrix Inversion Subroutine

The matrix inversion subroutine subprogram utilized the Gauss-Jordan method with pivoting along the main diagonal.^{1,2} Since the pivot terms are used to normalize each row, there cannot be any zero terms along the main diagonal. The algorithm which is the basis for this method is the following theorem,

¹B. Carnahan, et.al., Applied Numerical Methods (New York, 1969), pp. 272-276.

²P.C. Wang, Numerical and Matrix Methods in Structural Mechanics (New York, 1966), pp. 341-342.

Figure 6 Abbreviated Flow Chart



If

$$[0] A \rightarrow E \quad (198)$$

then

$$[[0]] E \rightarrow A^{-1} \quad (199)$$

Where $[0]A \rightarrow E$, means a series of matrix operations which transforms matrix A into the identity matrix E , and $[[0]]E \rightarrow A^{-1}$, means that these same series of operations will transform the identity matrix E into A^{-1} .³

The operations to be performed upon the A matrix to produce the identity matrix E are the following:

- (a) Normalize the first row by dividing by its diagonal element
- (b) Take the product of this normalized first row with the first element of the second row and subtract that result from the second row
- (c) Repeat step (b) for each row in the matrix to make the first column have a one in the first row and zeros for the rest.
- (d) the procedure is repeated for the second diagonal term and so on, along the main diagonal.

If the same sequence of operations are performed upon the identity matrix E , the result will be the inverse of the A matrix.

³S. F. Borg, Matrix-Tensor Methods in Continuum Mechanics (Princeton, 1963), p. 11.

D. Eigenvalue Subroutine Package

The evaluation of the eigenvalues and eigenvectors was accomplished through the use of the "Eigensystem Subroutine Package" (EISPACK).⁴ It was developed at Argonne National Laboratory under the auspices of the United States Atomic Energy Commission and the National Science Foundation.

EISPACK consists of a subroutine EISPAC which is the calling program for a large number of other specialized subroutines contained in the eigensystem package. Through the use of selected keywords in the CALL statement for EISPAC, a path is selected to be followed in calling these various subroutines. The CALL statement used for the ring buckling problems included the following keywords,⁵

```
CALL EISPAC (NM, N, MATRIX('REAL', B), VALUES(WR,
WI), VECTOR(ZP), ERROR(IERROR))
```

Where

NM	gives the size of in input matrix
N	gives the order of the eigenvalue problem
MATRIX('REAL',B)	indicates that the input matrix, B, is a real general matrix
VALUES(WR,WI)	indicates that all real and imaginary eigenvalues are to be found

⁴B. T. Smith, et.al., "Matrix Eigensystem Routines-EISPACK Guide," Lecture Notes in Computer Science (New York, 1974), pp. 7.2-1 to 7.2-22.

⁵"Eigensystem Subroutine Package (EISPACK) Documentation," CCIS Technical Documentation, Rutgers, University Document No. A01-0040, (1974), p. 6

VECTOR(ZP)	indicates that all real and imaginary eigenvectors are to be found.
ERROR(IERROR)	provides for the listing of any EISPACK error codes which might have been generated during the execution of the subroutines.

A description of each of the subroutines can be obtained by requesting a computer listing. There is documentation given as to the testing of the accuracy of the results for each subroutine performed by the various universities and governmental agencies which participated in the projects.

IX. SUMMARY OF RESULTS

A. Type of Rings Which Were Analyzed

Six different rectangular cross sections were analyzed, and for each cross section several values of radius were used. The cross sections which were used had aspect ratios, i.e. the ratio of the radial thickness (t) to the axial depth (h), of

$$\frac{t}{h} = 3, 2, 1, 0.5, 0.333, 2.71 \quad (200)$$

The ring diameters were selected to give a full range of values which are plotted in figure 6 through 15. The diameter to thickness ratios of these rings had a range of

$$3.33 \leq \frac{d}{t} \leq 24.0 \quad (201)$$

This range is consistent with the accuracy maintained in the evaluation of the cross sectional property integrals as given in equation (197) i.e.,

$$\left(\frac{t}{d}\right)^6 = 0.000733 \ll 1.0 \quad (202)$$

which is less than 0.1% for the thick ring integrals. In the evaluation of the thin ring cross sectional property integrals, which were used only in the known displacement function analysis, terms of the second power were neglected compared to unity. This would indicate a maximum variation of 9% for the lowest ratio ring, and slightly less than

0.2% for the highest ratio (thinnest) ring. These integrals were used in section V.B to arrive at the same set of equations which are given by Wah¹ for the buckling of thin rings.

B. Methods Used in the Analysis

Each of the rings were analyzed by four methods. The summary of these results are given in tables 1 through 6. The following is a discription of each of the methods used.

1. Classical thin ring theory. The critical buckling loads for inplane and out of plane buckling were given in equations (144) and (146) respectively. The ring is assumed to be thin, and the extension of the center line is neglected. Any Poisson effects are also neglected.

2. Wah's thin ring theory. The critical buckling loads for inplane and out of plane buckling are given in equations (139) and (140) respectively. The rings are assumed to be thin since only the first degree terms were retained in the cross sectional property integrals. Wah retains the extension of the center line only in the circumferential force and load expressions and assumes it to be zero elsewhere. No Poisson effects are included. The roots were obtained by using the quadratic formula.

3. Thin ring theory with unknown displacement func-

¹Wah, pp. 967-974.

tions. In the thin ring theory, a power series expansion was utilized for the displacements as given in equation (64). The expansions consisted of eight unknown terms i.e., three center line displacements and five unknown functions of these displacements. Eight equations were generated by integrating the incremental equilibrium equations, i.e. (59), (60) and (61) and the products of these equations with the coefficients of the unknown displacement functions in their corresponding directions. The resulting eight equations are given in section V.A.2 as equations (89) through (96). The accuracy of the integrals in these equations is given in equation (202). The roots were obtained by solving the (8 x 8) eigenvalue problem as given in section VII. B. This format contained both the inplane and out of plane buckling problems in one system of equations. They were uncoupled, however, and the direction of the roots could easily be determined by observing their corresponding eigenvectors. The inplane buckling loads had zeros for the out of plane eigenvectors and, similarly, the out of plane eigenvalues had zeros for the inplane eigenvectors.

4. Thick ring theory. The thick ring theory also utilized a power series expansion for the displacements. However, ten additional unknown displacement functions were added to the thin ring case. The expansions are given in equations (147) in section VI.A. By using the same procedure used for the thin ring solution with unknown dis-

placement functions, eighteen equations were generated. The ten inplane equations are given by equations (169) through (178), and the eight out of plane equations are given by equations (179) through (186). These equations are also uncoupled and could have been solved as in the previous section. However, due to the size of the combined matrix which would have to be inverted and then later substituted into the eigenvalue package, the decision was made to treat each direction separately, to conserve computer time and storage space.

C. Tabulation of Results

The critical buckling loads for the various rings which were analyzed are given in table 9 in Appendix III. These results are given in the form of dimensionless parameters in tables 1 through 6. These parameters were used by Wah² and were defined in section V.B.2 in equations (141) and (142).

In the tables mentioned above, some values of the critical buckling load parameter were not evaluated. This was done because the result was needed for only one direction and exceeded the maximum plotted values for the other direction. For example, in table 4 for rings with radii of 12 and 9 inches, the inplane buckling load parameters were not needed since the thickness parameter for the 7.5 inch radius ring

²Wah, pp. 970-971.

was very large. However, in the out of plane direction, the values for the thickness parameters were in the range of interest for plotting purposes.

Table 7 compares the three methods of solution for rings with aspects ratios of 3.0 and 2.71. Comparisons were made between the two thin ring theories and also between both thin ring theories and the thick ring theory.

SUMMARY OF RESULTS FOR GIVEN RING GEOMETRY

No.	Ring Radius a (Inch)	Ratio $\frac{2a}{t}$	Thickness Parameters		Critical Buckling			Load Parameters		
			Inplane	Out of Plane	Inplane			Out of Plane		
			$\kappa = \frac{Aa^2}{I_x}$	$\frac{1}{\beta} = \frac{Aa^2}{I_p}$	$\gamma_{cr} = \frac{P_{cr} a^3}{E I_x}$			$\zeta_{cr} = \frac{P_{cr} a^3}{E I_z}$		
					γ_{cr} (Classical Thin Ring Theory) = 3.00			ζ_{cr} (Classical Thin Ring Theory) = 1.87		
Wah's Thin Ring	Thin Ring Theory (8 x 8)	Thick Ring Theory (18 x 18)	Wah's Thin Ring	Thin Ring Theory (8 x 8)	Thick Ring Theory (18 x 18)					
1	15	10.0	300	270	2.990	2.847	2.407	1.849	1.816	1.590
2	12	8.0	192	172.8	2.984	2.769	2.289	1.840	1.820	1.556
3	10	6.67	133.3	120	2.977	2.678	2.170	1.829	1.825	1.525
4	8	5.33	85.33	76.80	2.964	2.528	1.990	1.810	1.836	1.482
5	5	3.33	33.3	30.0	2.905	2.043	1.497	1.727	1.892	1.374
6	4	2.67	21.3	19.2	2.846	1.740	1.234	1.655	1.964	1.316
7	15 ($\mu=0.0$)	10.0	300	270	2.990	2.879	2.623	1.849	1.914	1.747

Table 1. Summary of Results For Rings With Aspect Ratio of 3 to 1.

SUMMARY OF RESULTS FOR GIVEN RING GEOMETRY

No.	Ring Radius a (Inch)	Ratio $\frac{2a}{t}$	Thickness Parameters		Critical Buckling			Load Parameters		
			Inplane	Out of Plane	Inplane			Out of Plane		
			$\kappa = \frac{Aa^2}{I_x}$	$\frac{1}{\beta} = \frac{Aa^2}{I_p}$	$\gamma_{cr} = \frac{P_{cr} a^3}{E I_x}$			$\zeta_{cr} = \frac{P_{cr} a^3}{E I_z}$		
					γ_{cr} (Classical Thin Ring Theory) = 3.00			ζ_{cr} (Classical Thin Ring Theory) = 1.82		
				Wah's Thin Ring	Thin Ring Theory (8 x 8)	Thick Ring Theory (18 x 18)	Wah's Thin Ring	Thin Ring Theory (8 x 8)	Thick Ring Theory (18 x 18)	
1	15	10	300.	240.	2.99	2.847	2.407	1.800	1.782	1.568
2	12	8	192.	154.	2.984	2.768	2.290	1.789	1.781	1.531
3	9	6	108.	86.4	2.972	2.613	2.090	1.767	1.778	1.475
4	6	4	48.04	38.4	2.935	2.258	1.704	1.704	1.777	1.373
5	5	3.33	33.33	26.67	2.905	2.043	1.498	1.657	1.782	1.307
6										
7										

Table 2. Summary of Results For Rings With Aspect Ratio of 2 to 1.

SUMMARY OF RESULTS FOR GIVEN RING GEOMETRY

No.	Ring Radius a (Inch)	Ratio $\frac{2a}{t}$	Thickness Parameters		Critical Buckling			Load Parameters		
			Inplane	Out of Plane	Inplane			Out of Plane		
			$\kappa = \frac{Aa^2}{I_x}$	$\frac{1}{\beta} = \frac{Aa^2}{I_p}$	$\gamma_{cr} = \frac{P_{cr} a^3}{E I_x}$			$\zeta_{cr} = \frac{P_{cr} a^3}{E I_z}$		
					γ_{cr} (Classical Thin Ring Theory) = 3.00			ζ_{cr} (Classical Thin Ring Theory) = 1.62		
				Wah's Thin Ring	Thin Ring Theory (8 x 8)	Thick Ring Theory (18 x 18)	Wah's Thin Ring	Thin Ring Theory (8 x 8)	Thick Ring Theory (18 x 18)	
1	15	10	300.	150	2.990	2.847	2.409	1.588	1.525	1.373
2	12	8	192.	96	2.984	2.768	2.292	1.568	1.504	1.326
3	9	6	108.	54	2.972	2.613	2.093	1.526	1.463	1.250
4	6	4	48.	24	2.935	2.258	1.710	1.412	1.366	1.099
5	5	3.33	33.33	16.67	2.905	2.043	1.504	1.329	1.305	1.013
6										
7										

Table 3. Summary of Results For Rings With Aspect Ratio of 1 to 1.

SUMMARY OF RESULTS FOR GIVEN RING GEOMETRY

No.	Ring Radius (Inch)	Ratio $\frac{2a}{t}$	Thickness Parameters		Critical Buckling			Load Parameter		
			Inplane	Out of Plane	Inplane			Out of Plane		
			$\kappa = \frac{Aa^2}{I_x}$	$\frac{1}{\beta} = \frac{Aa^2}{I_p}$	$\gamma_{cr} = \frac{P_{cr}}{E} \frac{a^3}{I_x}$			$\zeta_{cr} = \frac{P_{cr}}{E} \frac{a^3}{I_z}$		
					γ_{cr} (Classical Thin Ring Theory) = 3.00			ζ_{cr} (Classical Thin Ring Theory) = 1.15		
					Wah's Thin Ring	Thin Ring Theory (8 x 8)	Thick Ring Theory (18 x 18)	Wah's Thin Ring	Thin Ring Theory (8 x 8)	Thick Ring Theory (18 x 18)
1	12	16	768.	154.	2.996	2.938	-	1.110	1.030	1.005
2	9	12	432.	86.4	2.993	2.892	-	1.092	0.9990	0.9598
3	7.5	10	300.	60.	2.990	2.847	2.425	1.060	0.9709	0.9218
4	6	8	192.	38.4	2.984	2.768	2.315	1.005	0.9239	0.8629
5	4.5	6	108.	21.6	2.972	2.613	2.127	0.8991	0.8400	0.7647
6	3	4	48.	9.60	2.935	2.258	1.757	0.6800	0.6798	-
7										

Table 4. Summary of Results for Rings of Aspect Ratio of 1 to 2

SUMMARY OF RESULTS FOR GIVEN RING GEOMETRY

No.	Ring Radius (Inch)	Ratio $\frac{2a}{t}$	Thickness Parameters		Critical Buckling			Load Parameters		
			Inplane	Out of Plane	Inplane			Out of Plane		
			$\kappa = \frac{Aa^2}{I_x}$	$\frac{1}{\beta} = \frac{Aa^2}{I_p}$	$\gamma_{cr} = \frac{P_{cr}}{E} \frac{a^3}{I_x}$			$\zeta_{cr} = \frac{P_{cr}}{E} \frac{a^3}{I_z}$		
					γ_{cr} (Classical Thin Ring Theory) = 3.00			ζ_{cr} (Classical Thin Ring Theory) = 0.787		
Wah's Thin Ring	Thin Theory (8 x 8)	Thick Ring Theory (18 x 18)	Wah's Thin Ring	Thin Ring Theory (8 x 8)	Thick Ring Theory (18 x 18)					
1	12	24	1728.	172.8	2.998	2.972	-	0.7486	0.6765	0.6963
2	8	16	768.	76.8	2.996	2.938	-	0.7049	0.6457	0.6538
3	5	10	300.	30.	2.990	2.847	2.468	0.6023	0.5758	0.5683
4	4	8	192.	19.2	2.984	2.768	2.370	0.5286	0.5267	0.5113
5	3	6	108.	10.8	2.972	2.613	2.195	0.4150	0.4506	0.4250
6	2	4	48.	4.8	2.935	2.258	1.823	0.2535	0.3334	0.2953
7										

Table 5. Summary of Results for Rings With Aspect Ratio of 1 to 3.

SUMMARY OF RESULTS FOR GIVEN RING GEOMETRY

No.	Ring Radius (Inch)	Ratio $\frac{2a}{t}$	Thickness Parameters		Critical Buckling			Load Parameters		
			Inplane	Out of Plane	Inplane			Out of Plane		
			$\kappa = \frac{Aa^2}{I_x}$	$\frac{1}{\beta} = \frac{Aa^2}{I_p}$	$\gamma_{cr} = \frac{P_{cr} a^3}{E I_x}$			$\zeta_{cr} = \frac{P_{cr} a^3}{E I_z}$		
			γ_{cr} (Classical Thin Ring Theory) = 3.00			ζ_{cr} (Classical Thin Ring Theory) = 1.911				
					Wah's Thin Ring	Thin Ring Theory (8 x 8)	Thick Ring Theory (18 x 18)	Wah's Thin Ring	Thin Ring Theory (8 x 8)	Thick Ring Theory (18 x 18)
1	727.7	115.3	39,903.	35 124.	3.000	2.999	2.782	1.911	1.802	1.735
2	72.77	11.53	399.03	351.24	2.996	2.883	2.469	1.900	1.820	1.608
3	36.39	5.77	99.78	87.83	2.969	2.587	2.059	1.866	1.819	1.490
4	727.7	115.3	39,903.	35,124.	3.000	2.999	2.973	1.911	1.898	1.882
5	($\mu=0.0$)									
6										
7										

Table 6. Summary of Results For Rings With Aspect Ratio of 2.71 to 1

$\frac{t}{h}$	$\frac{D}{t}$	Percent Variation Between Methods					
		Inplane			Out of Plane		
		$\frac{Wah}{Thin}$	$\frac{Wah}{Thick}$	$\frac{Thin}{Thick}$	$\frac{Wah}{Thin}$	$\frac{Wah}{Thick}$	$\frac{Thin}{Thick}$
3	10	5.0	24.2	18.3	1.8	16.3	14.2
3	8	7.8	30.4	21.0	1.1	18.3	17.0
3	6.7	11.2	37.2	23.4	0.2	19.9	19.7
3	5.3	17.2	48.9	27.0	1.4	22.1	23.9
3	3.3	42	94.1	36.5	9.6	25.7	37.7
3	2.7	63.6	131.	41.0	18.7	25.8	49.2
3	10 ($\mu=0$)	3.9	14.0	9.8	3.5	5.8	9.6
2.71	115 ($\mu=0$)	0.03	0.91	0.88	0.69	1.5	0.85
2.71	11.5	3.9	21.4	16.8	4.4	18.2	13.2
2.71	115.	0.03	7.8	7.8	6.1	10.1	3.9

Table 7. Comparison of Results Using Three Methods of Analysis

D. Plots of Results

The values of dimensionless parameters given in table 1 through 5 are plotted for the various rings in figures 7 through 16. There are two figures for each aspect ratio ring. One depicts the variation of the inplane critical buckling load parameter with the inplane thickness parameter. The other graph shows this variation for the corresponding out of plane parameters. The expressions defining these parameters were given in equations (141) and (142) respectively.

Figures 17 and 18 show the comparisons of the three ring theories for the inplane and out of plane directions, respectively. These results were given in table 7. Only rings with aspect ratios of 3.0 and 2.71 were considered since they have the widest separation of results as shown in the plots in figures 7 through 16.

The direction of the lowest critical buckling load is shown by the curves in figure 19. One curve is plotted for each of the five major aspect rings which were analyzed.

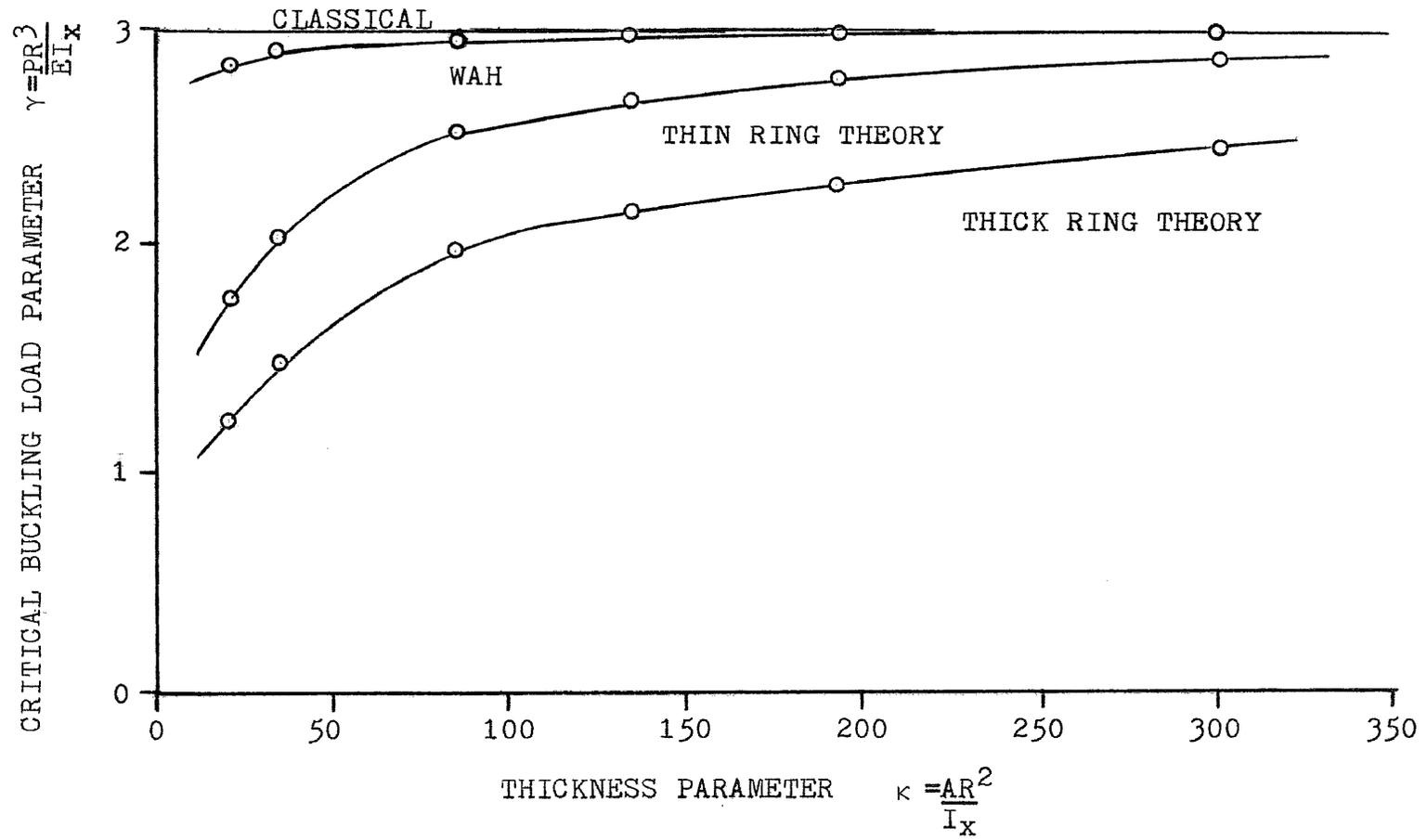


Figure 7 Inplane Buckling of Rings with Aspect Ratio of 3 to 1

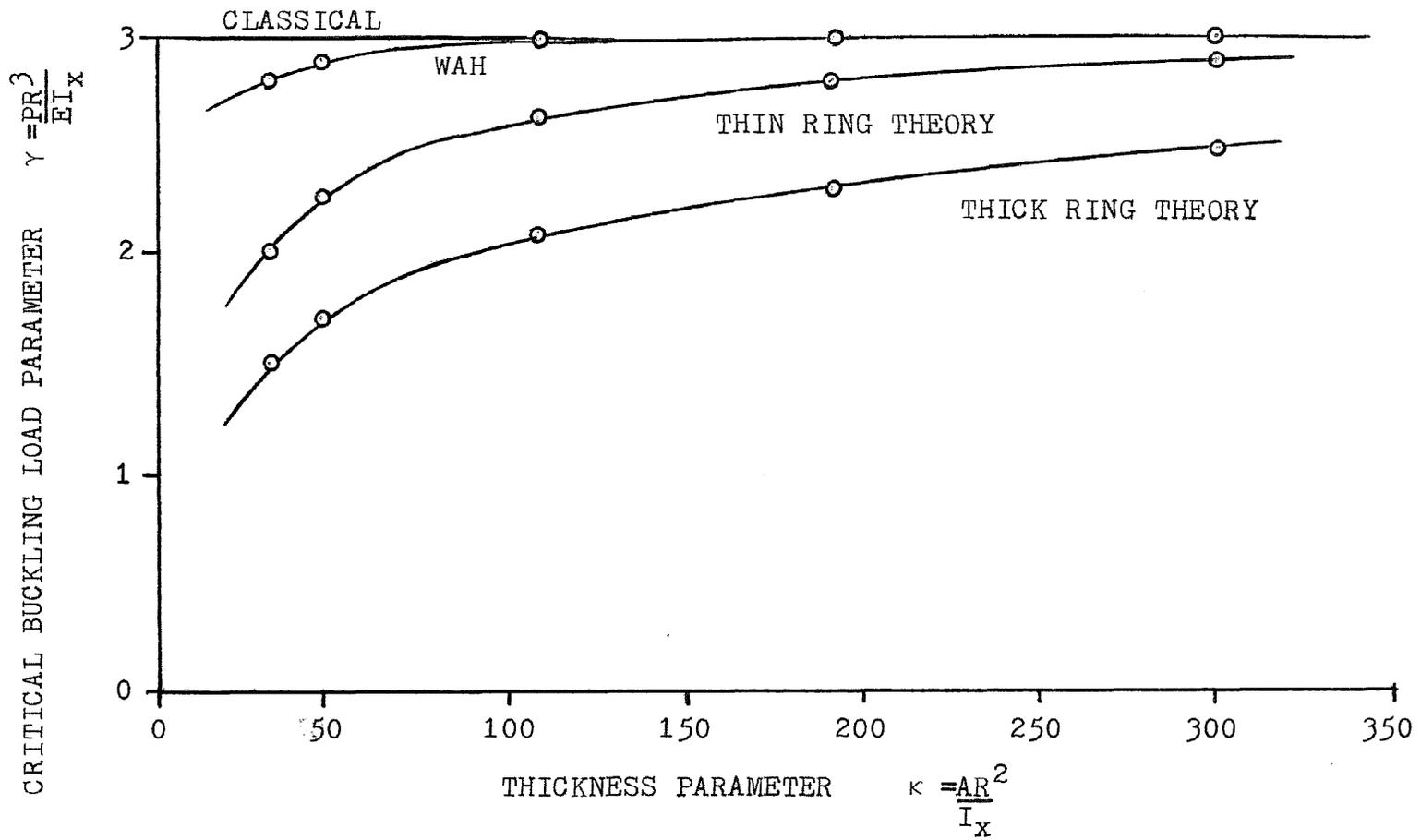


Figure 8 Inplane Buckling of Rings with Aspect Ratio of 2 to 1

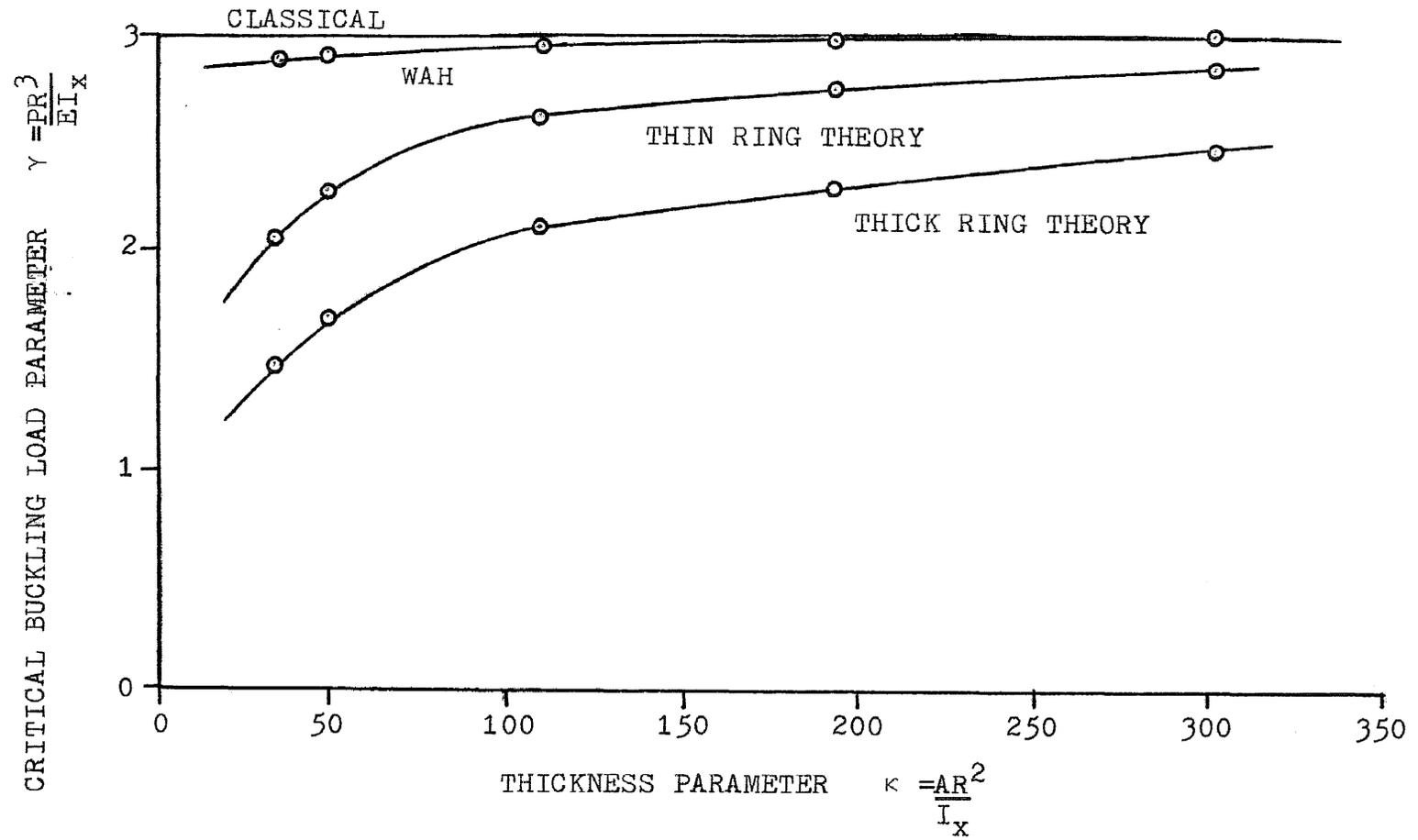


Figure 9 Inplane Buckling of Rings with Aspect Ratio of 1 to 1

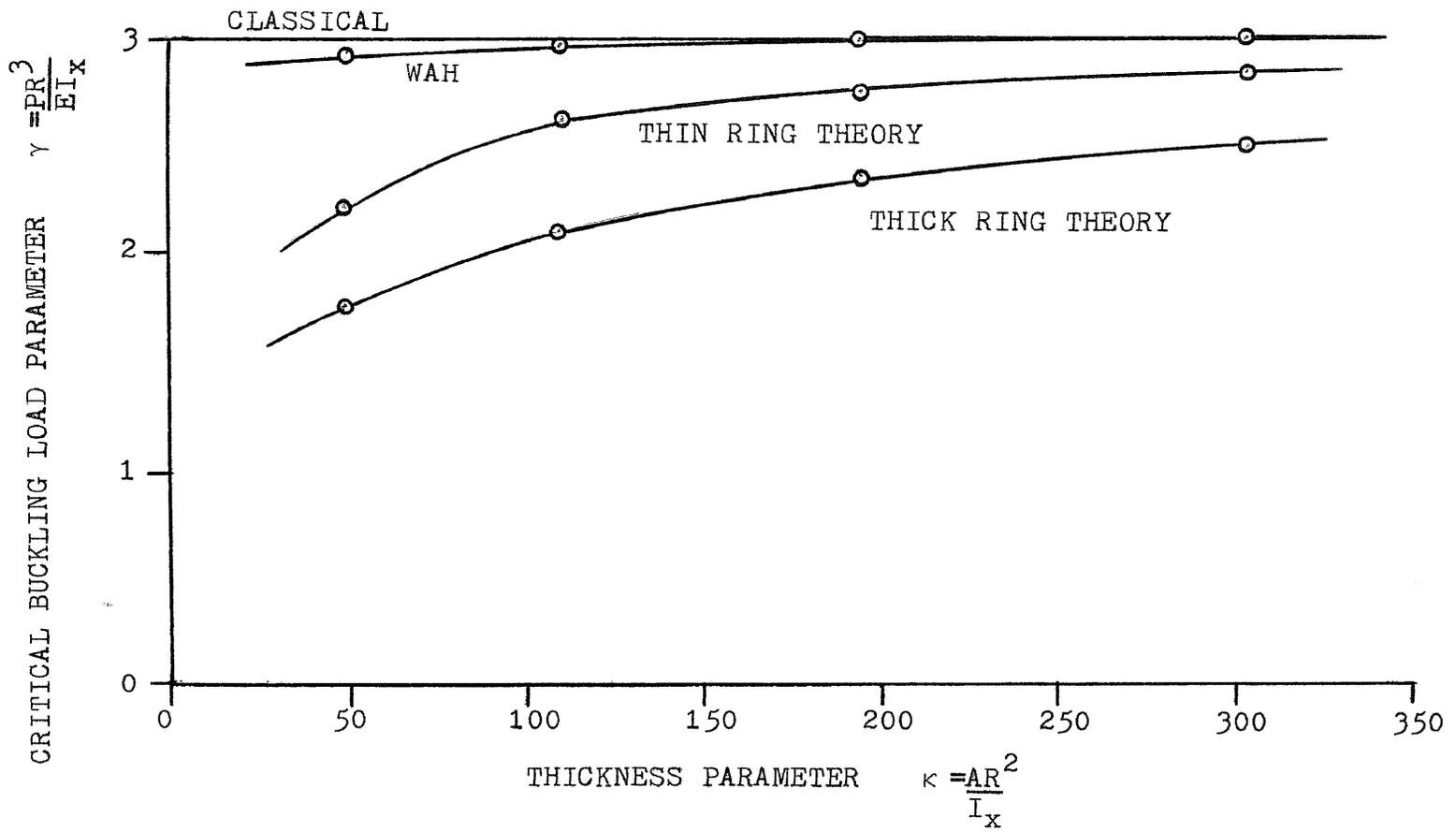


Figure 10 Inplane Buckling of Rings with Aspect Ratio of 1 to 2

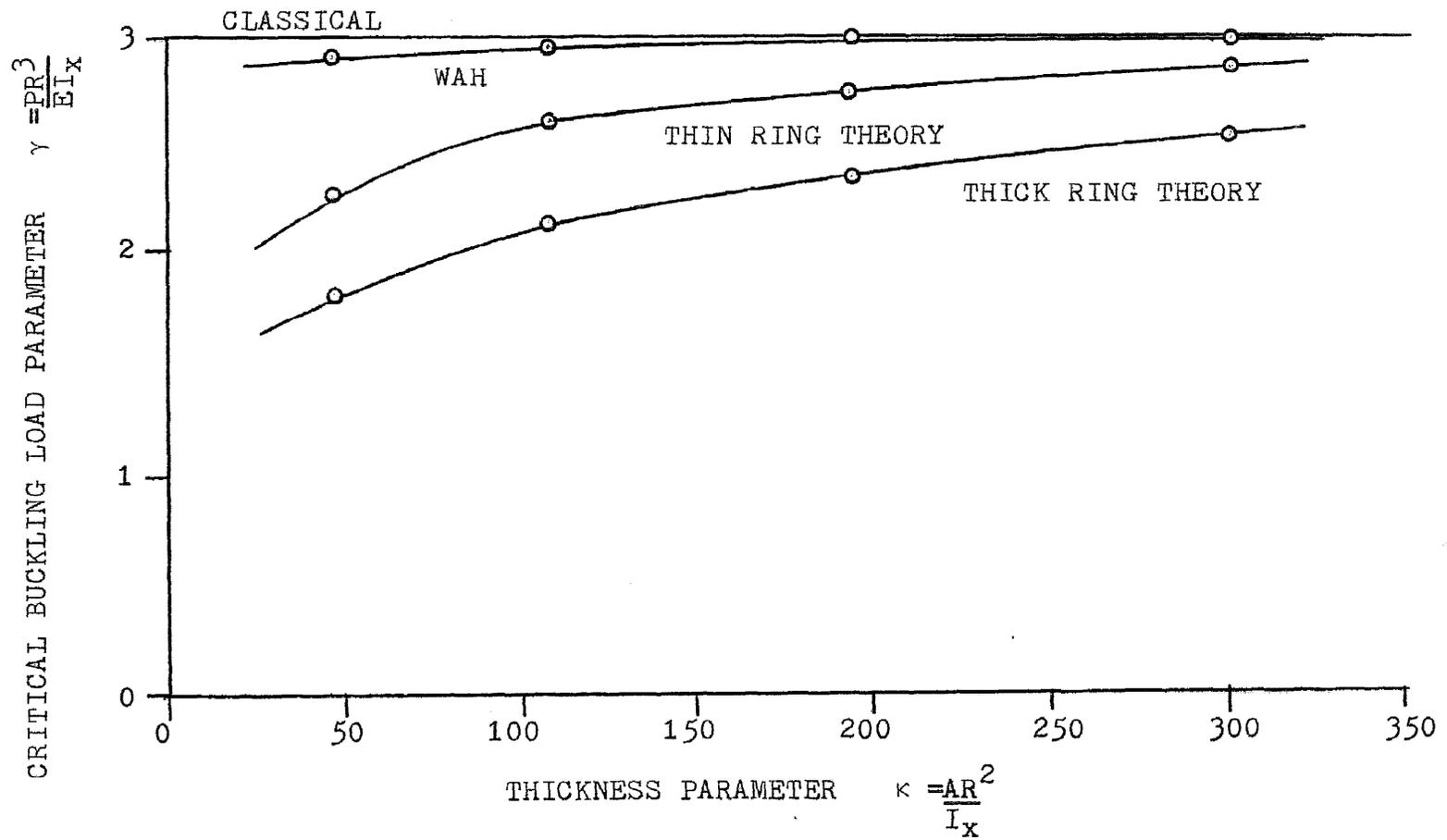


Figure 11 Inplane Buckling of Rings with Aspect Ratio of 1 to 3

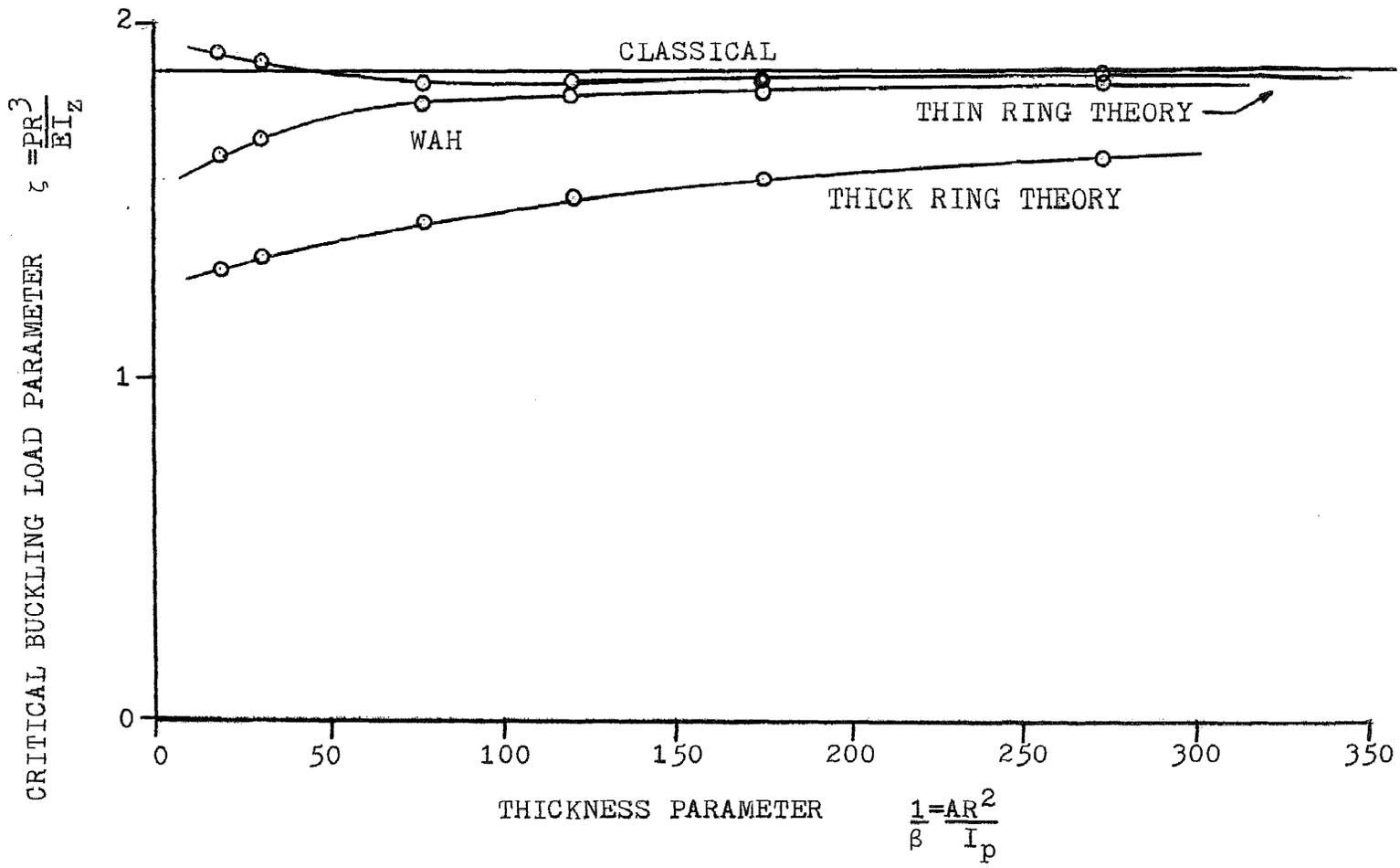


Figure 12 Out of Plane Buckling of Rings with Aspect Ratio of 3 to 1

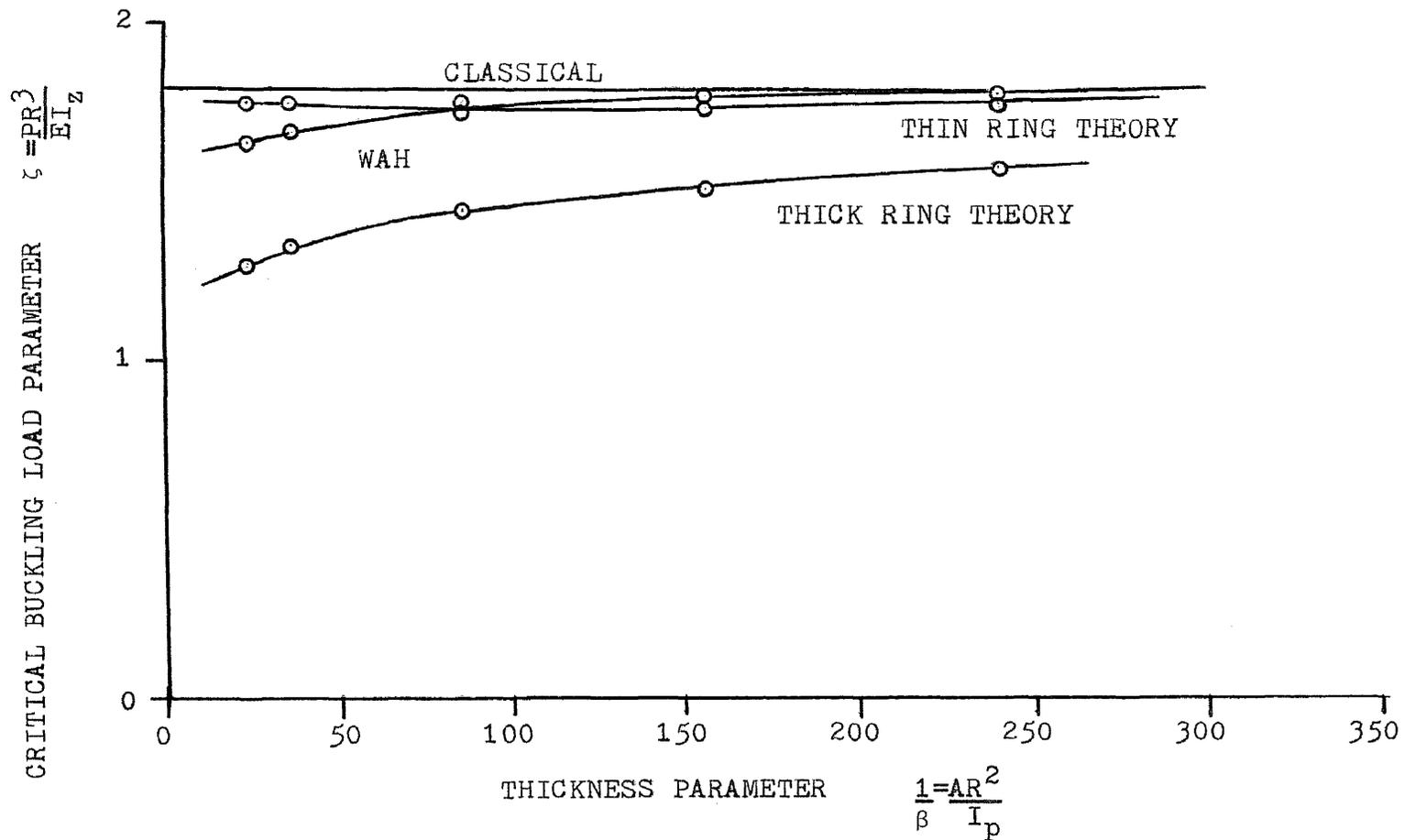


Figure 13 Out of Plane Buckling of Rings with Aspect Ratio of 2 to 1

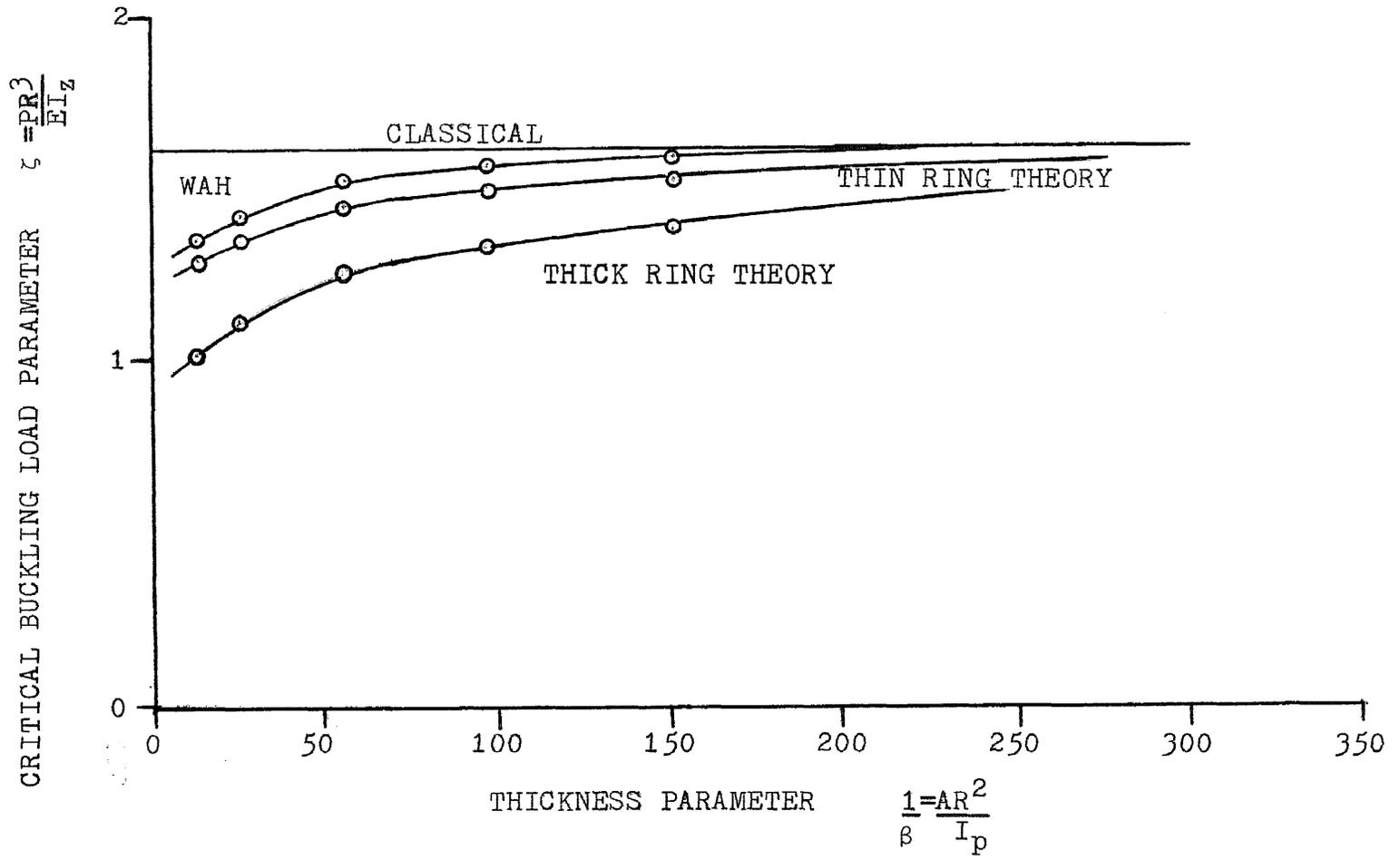


Figure 14 Out of Plane Buckling of Rings with Aspect Ratio of 1 to 1

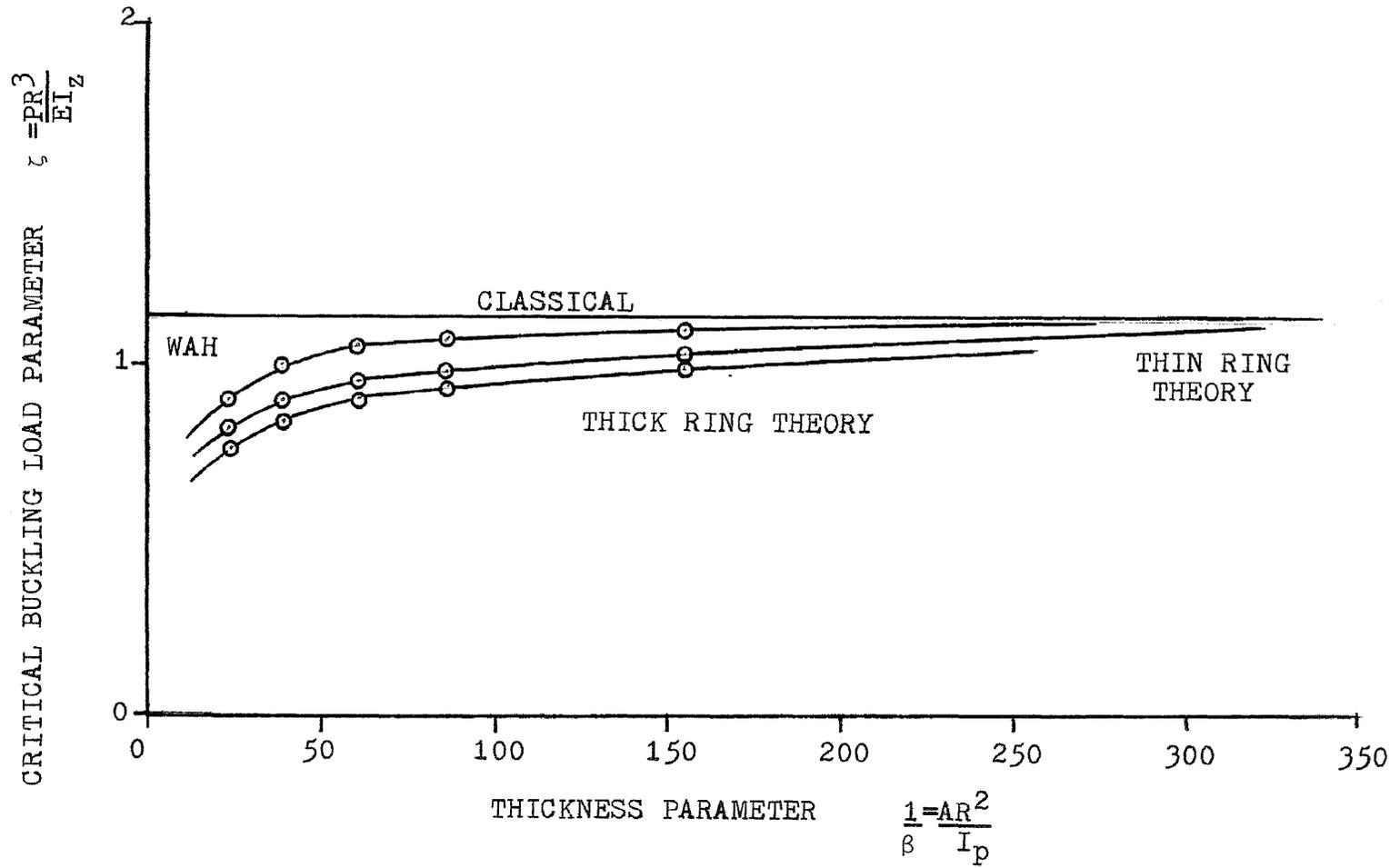


Figure 15 Out of Plane Buckling of Rings with Aspect Ratio of 1 to 2

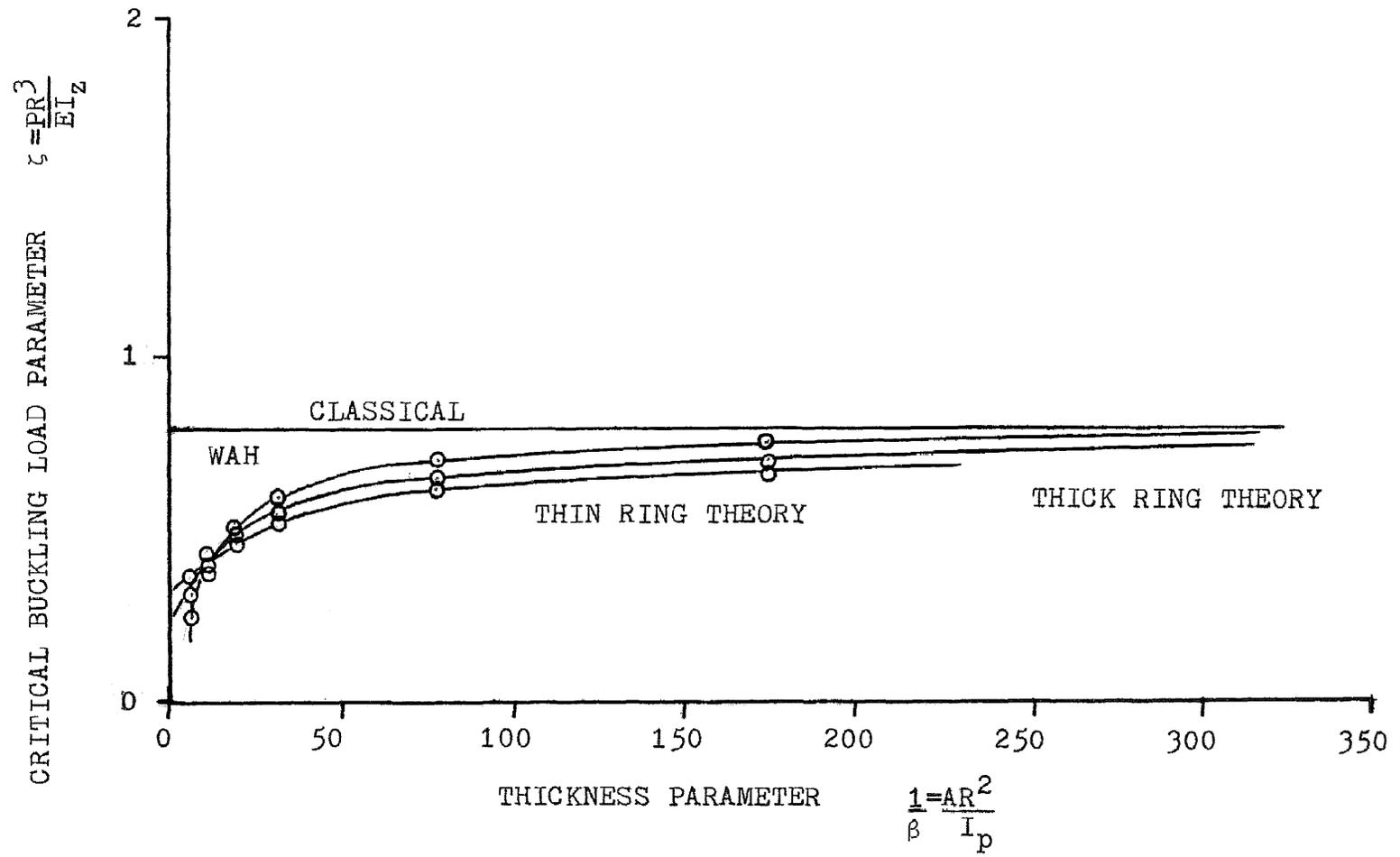


Figure 16 Out of Plane Buckling of Rings with Aspect Ratio of 1 to 3

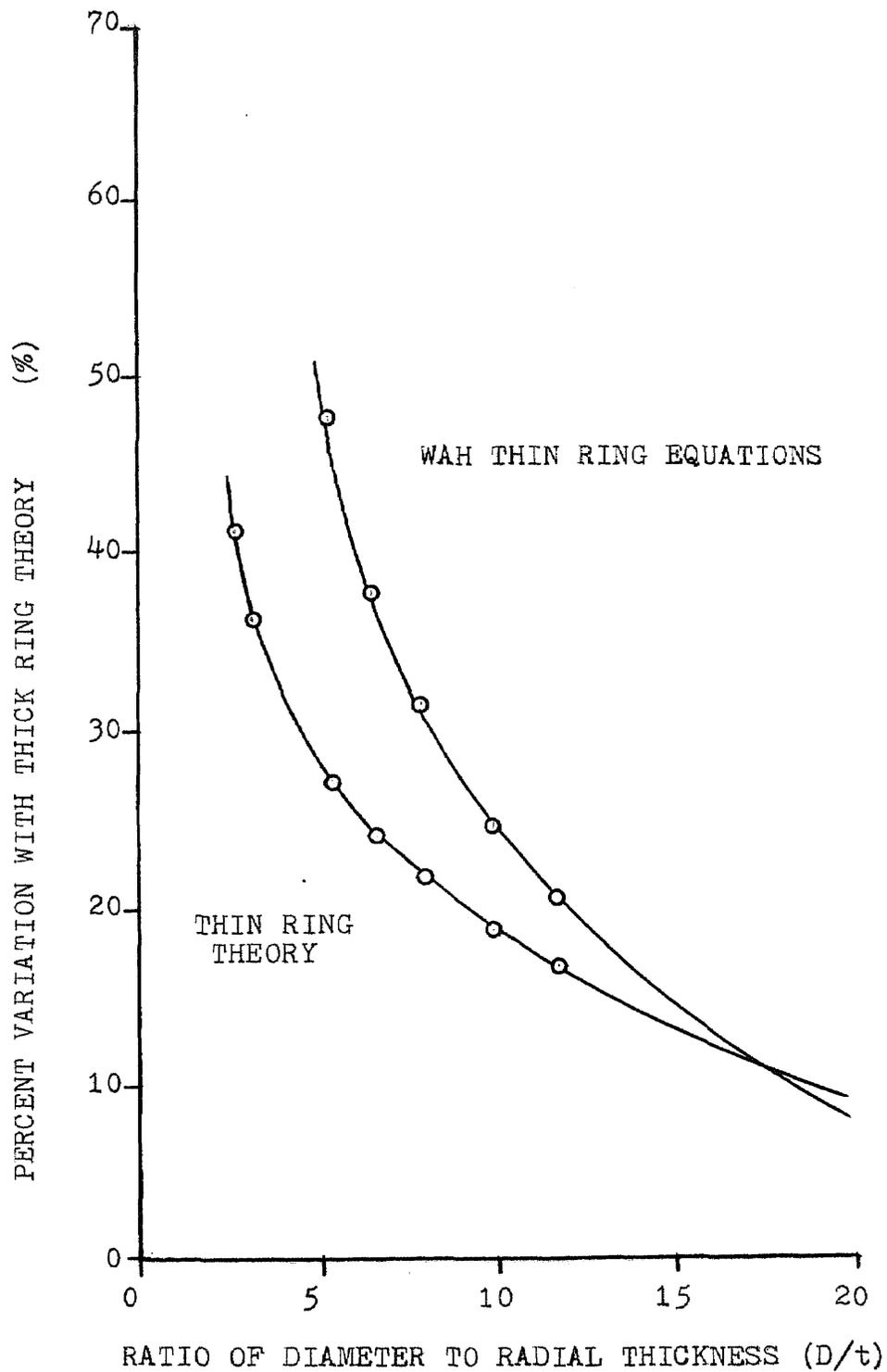


Figure 17 Comparison of Two Thin Ring Theories with Thick Ring Theory for the Inplane Buckling of a Ring

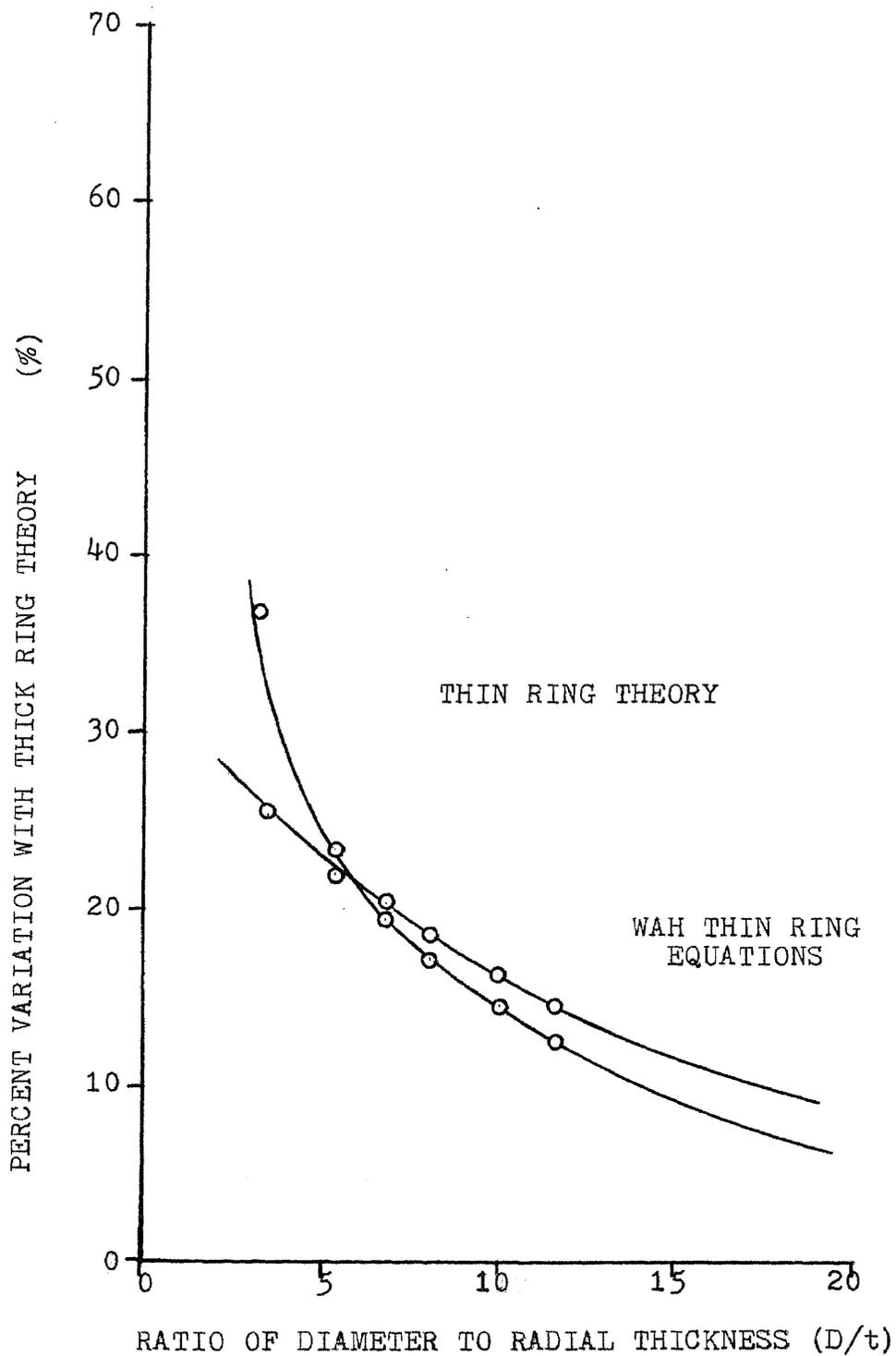


Figure 18 Comparison of Two Thin Ring Theories with Thick Ring Theory for the Out of Plane Buckling of a Ring

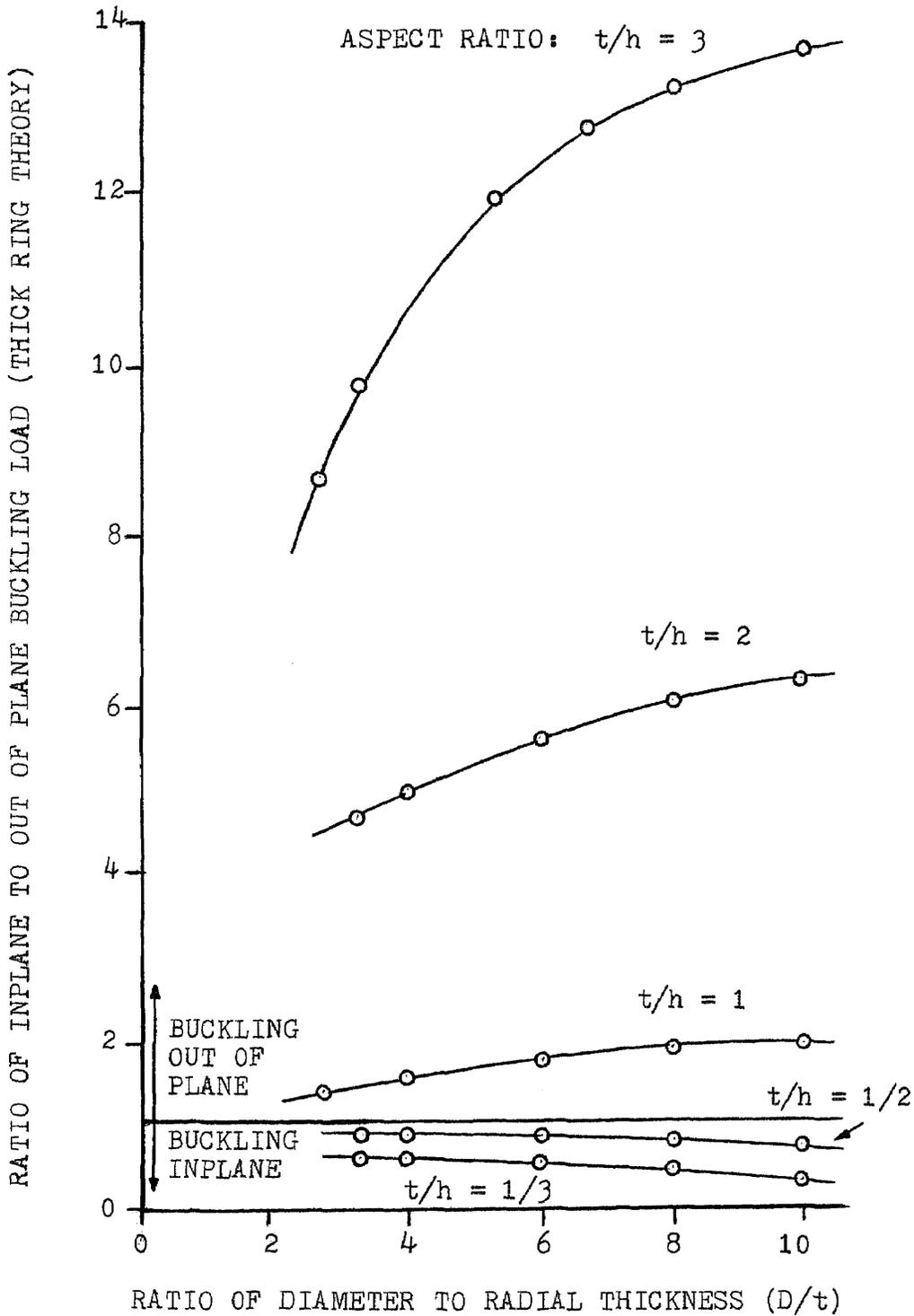


Figure 19 Direction of Buckling for Circular Rings with Rectangular Cross Section

E. Discussion of Results

1. Inplane buckling. The results of the inplane buckling problem are plotted in figures 7 through 11. The dimensionless parameters used in these graphs were defined in equation (141). The four methods of analysis follow the same pattern in each of the figures. The classical value of the buckling parameter is equal to three in all cases. The values obtained by Wah's thin ring theory do not fall below 2.85. This is the case even for the thickest ring, which has a diameter to thickness ratio of only 2.67.

The plots of the critical buckling parameters obtained by the thin ring theory, i.e., (8 x 8) matrices and the thick ring theory i.e., (18 x 18) matrices show a rapid decrease in value for thickness parameters which are less than eighty. This value corresponds to a diameter to thickness ratio of approximately five, which indicates a thick ring. A ratio of twenty or more is considered to indicate a thin ring.

Table 7 gives a comparison of the results obtained by using Wah's equations, the thin ring theory equations, and the thick ring theory equations for two aspect ratio ring cross sections. These two ratios, i.e., 3.0 and 2.71 were chosen because they showed the widest separation of values in the plots in figures 7 through 16. The ratios of the two thin ring methods compared to the thick ring theory is plotted for the inplane direction in figure 17. It can be seen that for a D/t ratio of 10, the difference between Wah's result and the thick ring theory is 24 percent. The thin ring theory varies by 18.3 percent from the thick ring

results. Additional computer runs were made with the Poisson effects eliminated and these differences were reduced to 14 percent and 9.8 percent respectively for this same ring. Figure 7 indicates that both of these methods would differ from the thick ring theory by less than 10 percent for D/t ratios which are greater than 20.

Another thin ring was also analyzed, but not included in the plots because the D/t ratio of 115 would compress the scale until the variation in the area of interest, i.e., $D/t < 20$ would be distorted. The results obtained using Wah's thin ring solution agreed with the unknown coefficient thin ring theory to 0.03 percent. This indicates the effects of the extension of the center line are negligible in thin rings. However, both of these methods vary 7.8 percent from the thick ring (18 x18) theory. This would indicate the effects of the distortion of the rectangular cross section, neglected in the simpler theories due to the lack of the higher order terms in the displacement expansions are still of some significance in thin rings. When the Poisson effects are eliminated, the variation between all three methods is less than 1 percent.

Some of the variation between the methods can be attributed to the difference in the initial problem solution. The radial displacement to the initial position

for the thin ring theory was given by equation (55),

$$w^0 = - \frac{Pa^2}{AE} \quad (203)$$

The initial problem displacement from the thick ring theory is given by substituting equation (190) into equations (36), (42) and (43) yielding,

$$w^0 = - \frac{Pa^2}{2AE} \left[1 + \left(\frac{t}{a}\right) + \left(\frac{t}{2a}\right)^2 \right] \left\{ [(1 - \mu) + (1 + \mu) \left(\frac{R_i}{a}\right)^2] \right. \\ \left. + \frac{z}{a} [(1 - \mu) - (1 + \mu) \left(\frac{R_i}{a}\right)^2] \right\} \quad (204)$$

The variation in these two values has been found for several D/t ratios, and the results are given in table 8. The displacements are evaluated at the center line i.e., $z = 0$. The variation is more than four percent for values of diameter to thickness ratio of less than 15 and is as high as 25 percent for a ratio of 2.67. This variation influences the value of the buckling load, since the critical buckling load represents impending buckling based on the values at the initial equilibrium position.

The calculated values of the buckling loads for all of

D/t	$\% = \frac{ w^{\circ}_{\text{Thin}} - w^{\circ}_{\text{Thick}} }{w^{\circ}_{\text{Thick}}} \times 100$
2.67	25.6
15.	4.3
20.	2.8
115.	0.6

Table. 8. Percent Variation in Initial
Position Displacement

the rings which were analyzed are given in table 9 in Appendix III. They show that the inplane buckling load reaches a maximum value for each diameter ring when the aspect ratio is equal to unity. The critical buckling load decreases from this value when the aspect ratio is either increased or decreased. The square cross section ring had the largest values of moment of inertia about each major axis. It also had the largest cross sectional area and torsional constant of the first five cross section rings, i.e., those rings with aspect ratios of 3, 2, 1, 1/2, 1/3. These properties indicate the ability to withstand larger critical buckling loads.

2. Out of plane buckling. The results of the out of plane buckling problem are plotted in figures 12 through 16. The dimensionless parameters used in these graphs were defined in equation (121). Wah's thin ring equation and the thin ring equations with unknown coefficients agree within two percent for the first two cases i.e., aspect ratios of 3 and 2, if the D/t ratio is greater than five. These ratios are given in table 7. This ratio corresponds to a thickness parameter $1/\beta$ of approximately sixty. Both thin ring theories vary from the thick ring results by over twenty percent for these rings. As was the case for the inplane results, the thin ring theory varies by less than ten percent from the thick ring theory for rings with a diameter to thickness ratio greater than 20. This is demon-

strated in the plot given in figure 18. The difference in the buckling load obtained from the thin and thick ring theories becomes greater than 20 percent for D/t ratios of 6.5 or less for rings with aspect ratios of 3 and 2.7.

Figures 14 through 16 show the out of plane results for rings with aspect ratios of 1.0, 0.5, 0.333, respectively. The three methods give increasingly similar results for the rings with smaller aspect ratios. In particular, in figure 16, the thin ring theory varies by approximately 6 percent variation for a D/t ratio of 6. The smaller aspect ratio indicates a larger resistance to out of plane bending due to a larger moment of inertia about the radial axis. The out of plane slope term as seen in equation (111) for the thin ring theory with known coefficients is inversely proportional to the radius, whereas the twist terms are not. Therefore, rings with smaller aspect ratios are less sensitive to changes in their radius. However, this does not mean that they have higher buckling loads. The calculated critical buckling loads given in table 9 in Appendix III show that for a constant diameter ring, the out of plane buckling load peaks for an aspect ratio of unity, and decreases for higher or lower values. This behavior is similar to that of the inplane buckling load.

3. Direction of buckling. Figure 19 shows which direction, i.e., inplane or out of plane, has the smaller critical buckling load. The ratio of the two buckling loads i.e., inplane divided by out of plane is plotted as a func-

tion of diameter to thickness ratio for each aspect ratio. The buckling load ratio becomes larger for increasing aspect ratios. This indicates that these rings will buckle in the out of plane direction. An increase in aspect ratio indicates an increase in the moment of inertia of the ring cross section about the $x - x$ axis indicating a greater resistance to inplane bending. On the other hand, a decrease in the aspect ratio increases the moment of inertia of the ring cross section about the radial axis. This increases the out of plane stiffness and raises the magnitude of the out of plane critical buckling load.

The tabulation of critical buckling loads given in table 9 in Appendix III shows that the square cross section, i.e., aspect ratio equal to unity, gives the maximum critical buckling loads in both directions. However, the inplane value is greater than the out of plane for each of the square rings which were analyzed. Figure 19 indicates an aspect ratio of slightly more than 0.5 is needed to give a ring with the same critical buckling load in both directions. This figure also shows that the ratio of the critical loads in the two directions remains relatively constant for large variations in the diameter to thickness ratio for aspect ratios which are less than unity. The load ratio drops off rapidly when the aspect ratio is equal to three, indicating that the out of plane critical buckling load increases at a faster rate than the inplane critical buckling load for decreases in ring radius.

F. Conclusions

The following conclusions may be drawn from the rectangular cross section rings analyzed in the previous sections.

- (a) Both thin ring theories agree with the thick ring theory within a 10% variation for rings with a D/t ratios of greater than twenty.
- (b) The thick ring theory becomes inaccurate for rings with D/t ratios which are less than five.
- (c) Both the inplane and out of plane directions have their maximum critical buckling loads occur when the rings have square cross sections.
- (d) Rings with aspect ratios which are greater than approximately 0.5 have lower out of plane critical buckling loads.
- (e) Rings with aspect ratios which are less than approximately 0.5 have lower inplane critical buckling loads.
- (f) Rings with aspect ratios of approximately 0.5 have equal critical buckling loads in both the inplane and out of plane directions.

X. APPENDICES

A. Appendix I, Integrals Over Cross Section

The integrals over the cross section of the ring are given by the following forms,

$$I_1 = \iint \frac{dzdx}{(a+z)}$$

$$I_2 = \iint \frac{zdzdx}{(a+z)}$$

$$I_3 = \iint \frac{dzdx}{(a+z)^2}$$

$$I_4 = \iint \frac{zdzdx}{(a+z)^2}$$

$$I_5 = \iint \frac{x^2 dzdx}{(a+z)}$$

$$I_6 = \iint z dzdx$$

$$I_7 = \iint \frac{x^2 dzdx}{(a+z)^2}$$

$$I_8 = \iint \frac{z^2 dzdx}{(a+z)}$$

$$I_9 = \iint \frac{z^2 dzdx}{(a+z)^2}$$

$$I_{10} = \iint \frac{x^2 z dzdx}{(a+z)}$$

$$I_{11} = \iint \frac{x^2 z dzdx}{(a+z)^2}$$

$$I_{12} = \iint z^2 dzdx$$

$$\begin{aligned}
I_{13} &= \iint x^2 dz dx \\
I_{14} &= \iint x^2 z dz dx \\
I_{15} &= \iint \frac{x^2 z^2 dz dx}{(a+z)} \\
I_{16} &= \iint \frac{x^2 z^2 dz dx}{(a+z)^2} \\
I_{17} &= \iint z^3 dz dx \\
I_{18} &= \iint \frac{z^3 dz dx}{(a+z)} \\
I_{19} &= \iint \frac{z^3 dz dx}{(a+z)^2} \\
I_{20} &= \iint \frac{z^4 dz dx}{(a+z)^2} \\
I_{21} &= \iint \frac{z^4 dz dx}{(a+z)} \\
I_{22} &= \iint \frac{z^3 x^2 dz dx}{(a+z)^2} \\
I_{23} &= \iint \frac{z^5 dz dx}{(a+z)^2} \\
I_{24} &= \iint z^4 dz dx \\
I_{25} &= \iint x^2 z^2 dz dx \\
I_{26} &= \iint \frac{x^4 dz dx}{(a+z)} \\
I_{27} &= \iint \frac{x^4 dz dx}{(a+z)^2}
\end{aligned}$$

$$I_{28} = \iint \frac{x^4 z dz dx}{(a+z)^2}$$

$$I_{29} = \iint x^2 z^3 dz dx$$

$$I_{30} = \iint \frac{x^2 z^3 dz dx}{(a+z)}$$

$$I_{31} = \iint \frac{x^2 z^4 dz dx}{(a+z)}$$

$$I_{32} = \iint z^5 dz dx$$

$$I_{33} = \iint \frac{z^5 dz dx}{(a+z)}$$

$$I_{34} = \iint \frac{x^2 z^4 dz dx}{(a+z)^2}$$

$$I_{35} = \iint \frac{x^2 z^5 dz dx}{(a+z)^2}$$

$$I_{36} = \iint x^4 dz dx$$

$$I_{37} = \iint x^4 z dz dx$$

$$I_{38} = \iint \frac{x^4 z dz dx}{(a+z)}$$

$$I_{39} = \iint \frac{x^4 z^2 dz dx}{(a+z)}$$

$$I_{40} = \iint \frac{x^4 z^2 dz dx}{(a+z)^2}$$

$$I_{41} = \iint \frac{x^4 z^3 dz dx}{(a+z)^2}$$

The previously defined integrals, I_1 through I_{41} were evaluated using the following indefinite integral forms,¹

$$\int \frac{dx}{x+1} = \ln(x+1)$$

$$\int \frac{xdx}{x+1} = x - \ln(x+1)$$

$$\int \frac{x^2 dx}{x+1} = \frac{x^2}{2} - x + \ln(x+1)$$

and the general form for the higher power is given by,

$$\int \frac{x^m}{x+1} dx = \frac{x^m}{m} - \int \frac{x^{m-1}}{x+1} dx$$

The load term integrals have the denominator to the second power, and were evaluated using the following forms,²

$$\int \frac{dx}{(x+1)^2} = -\frac{1}{x+1}$$

$$\int \frac{xdx}{(x+1)^2} = \frac{1}{x+1} + \ln(x+1)$$

$$\int \frac{x^2 dx}{(x+1)^2} = \frac{x(x+2)}{x+1} - 2 \ln(x+1)$$

and the general form for the higher power is given by,

¹G. P. Bois, Tables of Indefinite Integrals, (New York, 1961)
pp. 1, 8, 77.

²IBID., pp. 2, 4, 77.

$$\int \frac{x^m dx}{(x+1)^2} = \frac{x^m}{(m-1)(x+1)} - \frac{m}{(m-1)} \int \frac{x^{m-1} dx}{(x+1)^2}$$

These cross sectional parameters may be used to evaluate the integrals and give the following forms,

$$I_1 = \frac{ht}{a} \left(1 + \frac{t^2}{12a^2} + \frac{t^4}{80a^4} \right)$$

$$I_2 = - \frac{ht^3}{a^2} \left(\frac{1}{12} + \frac{t^2}{80a^2} + \frac{t^4}{448a^4} \right)$$

$$I_3 = \frac{1}{a} I_1$$

$$I_4 = - \frac{ht^3}{a^3} \left(\frac{1}{6} + \frac{t^2}{20a^2} + \frac{3t^4}{224a^4} \right)$$

$$I_5 = \frac{h^2}{12} I_1$$

$$I_6 = 0$$

$$I_7 = \frac{h^2}{12} I_3$$

$$I_8 = - a I_2$$

$$I_9 = \frac{ht^3}{a^2} \left(\frac{1}{12} + \frac{3t^2}{80a^2} + \frac{5t^4}{448a^4} \right)$$

$$I_{10} = \frac{h^2}{12} I_2$$

$$I_{11} = \frac{h^2}{12} I_4$$

$$I_{12} = I_x$$

$$I_{13} = I_z$$

$$I_{14} = 0$$

$$I_{15} = \frac{h^2}{12} I_8$$

$$I_{16} = \frac{h^2}{12} I_9$$

$$I_{17} = 0$$

$$I_{18} = -\frac{ht^5}{a^2} \left(\frac{1}{80} + \frac{t^2}{448a^2} + \frac{t^4}{2304a^4} \right)$$

$$I_{19} = -\frac{ht^5}{a^3} \left(\frac{1}{40} + \frac{t^2}{112a^2} + \frac{t^4}{384a^4} \right)$$

$$I_{20} = \frac{ht^5}{a^2} \left(\frac{1}{80} + \frac{3t^2}{448a^2} + \frac{5t^4}{2304a^4} \right)$$

$$I_{21} = -a I_{18}$$

$$I_{22} = \frac{h^2}{12} I_{19}$$

$$I_{23} = -\frac{ht^4}{a^3} \left(\frac{1}{224} + \frac{t^2}{576a^2} + \frac{t^4}{1756.945a^4} \right)$$

$$I_{24} = \frac{ht^5}{80}$$

$$I_{25} = \frac{h^3 t^3}{144}$$

$$I_{26} = \frac{h^4}{80} I_1$$

$$I_{27} = \frac{h^4}{80} I_3$$

$$I_{28} = \frac{h^4}{80} I_4$$

$$I_{29} = 0$$

$$I_{30} = \frac{h^2}{12} I_{18}$$

$$I_{31} = \frac{h^2}{12} I_{21}$$

$$I_{32} = 0$$

$$I_{33} = \frac{ht^7}{a^2} \left(\frac{1}{448} + \frac{t^2}{2304a^2} + \frac{t^4}{11,264a^4} \right)$$

$$I_{34} = \frac{h^2}{12} I_{20}$$

$$I_{35} = \frac{h^2}{12} I_{23}$$

$$I_{36} = \frac{h^5 t}{80}$$

$$I_{37} = 0$$

$$I_{38} = \frac{h^4}{80} I_2$$

$$I_{39} = \frac{h^4}{80} I_8$$

$$I_{40} = \frac{h^4}{80} I_9$$

$$I_{41} = \frac{h^4}{80} I_{19}$$

In the case of the thin ring, the integrals can be reduced by retaining only up to the first power of (z/a) . The following forms were used only in the known displacement function formulation. They were not used in the un-

known function eigenvalue problem.

$$I_1 = \frac{A}{a}$$

$$I_2 = -\frac{I_x}{a^2}$$

$$I_3 = \frac{A}{a^2}$$

$$I_4 = 0$$

$$I_5 = \frac{I_z}{a}$$

$$I_6 = 0$$

$$I_7 = \frac{I_z}{a^2}$$

$$I_8 = \frac{I_x}{a}$$

$$I_9 = \frac{I_x}{a^2}$$

$$I_{10} = -\frac{h^2}{12} \frac{I_x}{a^2}$$

$$I_{11} = 0$$

$$I_{12} = I_x$$

$$I_{13} = I_z$$

$$I_{14} = 0$$

$$I_{15} = -\frac{h^2}{12} \frac{I_x}{a}$$

$$I_{16} = \frac{h^2}{12} \frac{I_x}{a^2}$$

B. Appendix II, Listing of Computer Programs

This section contains listings of the computer programs which were used in the foregoing analyses.

```

C          DATA SET TMJ          AT LEVEL 001 AS OF 04/12/79
C          T. M. JULIANO DOCTORAL DISSERTATION
C          MECHANICAL ENGINEERING DEPARTMENT
C          NEW JERSEY INSTITUTE OF TECHNOLOGY
C          INPLANE BUCKLING OF A THICK RING (10X10) MATRICES
IMPLICIT REAL*8(A-H,O-Z),INTEGER*4(I-N)
DIMENSION AA(10,10),WR(10),WI(10),ZP(10,10),B(10,10),BI(10)
DIMENSION A(10,10),AL(10,10),ALI(10,10)
DATA XN/2.0D0/,E/.3D08/,G/.115D08/,R/36.38D0/
DATA T/12.62D0/,H/4.656D0/,U/0.3D0/
DATA A,AL,B/300*0.0D0/
DATA N/10/,SKIP/2.0/
WRITE(6,99) XN
99  FORMAT(5X,'THICK RING, IN PLANE EQS. 10X10 ',' N= ',F7.1//
1  5X,'REVISION 16 06-17-78, SYMMETRY IN A + AL'//
2  5X,'SUBROUTINE AINTEG HAS MIN OF 3 TERMS'//)
XA= H*T
AIX= H*T**3/1.2D1
AIZ= T*H**3/1.2D1
PRINT 310,T,H,R,XA,AIX,AIZ
310 FORMAT(5X,'T= ',D12.4,' H= ',D12.4,' R= ',D12.4/
1  5X,'A= ',D12.4,' IX= ',D12.4,' IZ= ',D12.4//)
ONE= 1.0D0
TWO= 2.0D0
N2=N*2
XN2=XN*XN
RO= R+T/TWO
RI= R-T/TWO
U1= ONE+U
U2= ONE-U
D1= R*U2+U1*RI*RI/R
D2= U2-U1*RI*RI/(R*R)
C1= U2/(ONE-TWO*U)
C2= C1/U2*U
C11= C1*D1
C12= C1*D2
C22= C2*D2
EU1= E/U1
TR=T/R
CALL AINTEG (H,T,R,TR,ONE,TWO,AI1,AI2,AI4, AI6,AI9,AI13,AI14,
1  AI15,AI22,AI23,AI28,AI29,AI31,AI32,AI34,AI37,AI39,AI40,AI47,
2  AI49,AI52,AI54,AI55,AI59,AI61, AI66,AI67,AI70,AI72,AI74,AI75,
3  AI77,AI78)
A(1,1)= -AI1*(XN2*G+EU1*C1)
A(1,2)= XN*AI1*(G+EU1*C1)
A(1,3)= -XN*(G*(XA+(-AI2))+EU1*C1*(-AI2))
A(1,4)= (XN2*G+EU1*C1)*(-AI2)-EU1*C2*XA
A(1,5)= XN*(G+EU1*C1)*AI14
A(1,6)= -EU1*C2*XA
A(1,8)= -AI9*(XN2*G+EU1*C1)

```

$A(1,9) = XN * AI9 * (G + EU1 * C1)$
 $A(1,10) = XN * (EU1 * C1 * AI22 - G * (AI2 - AI22))$
 $A(2,1) = A(1,2)$
 $A(2,2) = -AI1 * (XN2 * EU1 * C1 + G)$
 $A(2,3) = +(-AI2) * (G + XN2 * EU1 * C1) + G * XA$
 $A(2,4) = -XN * (-AI2) * (EU1 * C1 + G) + XN * EU1 * C2 * XA$
 $A(2,5) = -AI14 * (XN2 * EU1 * C1 + G)$
 $A(2,6) = XN * EU1 * C2 * XA$
 $A(2,8) = XN * AI9 * (EU1 * C1 + G)$
 $A(2,9) = -AI9 * (XN2 * EU1 * C1 + G)$
 $A(2,10) = -AI22 * (XN2 * EU1 * C1 + G) + G * AI2$
 $A(3,1) = XN * (-R * G * AI1 - EU1 * C1 * (-AI2))$
 $A(3,2) = G * R * AI1 + XN2 * EU1 * C1 * (-AI2)$
 $A(3,3) = -R * G * (XA + (-AI2)) - XN2 * EU1 * AI14 * C1$
 $A(3,4) = XN * (R * G * (-AI2) + EU1 * C1 * AI14)$
 $A(3,5) = R * G * AI14 - XN2 * EU1 * C1 * AI31$
 $A(3,7) = XN * EU1 * C2 * AIX$
 $A(3,8) = XN * (EU1 * C1 * AI22 - R * G * AI9)$
 $A(3,9) = R * G * AI9 - XN2 * EU1 * C1 * AI22$
 $A(3,10) = R * G * AI22 - XN2 * EU1 * C1 * AI28 - R * G * AI2$
 $A(4,1) = EU1 * ((-AI2) * (C1 + C2) - C2 * R * AI1) + XN2 * G * (-AI2)$
 $A(4,2) = XN * EU1 * (AI2 * (C2 + C1) + C2 * R * AI1) + XN * G * AI2$
 $A(4,3) = XN * EU1 * (AI14 * (C2 + C1) - C2 * (-AI2 * R)) + XN * G * AI14$
 $A(4,4) = -EU1 * (C1 * (R * XA + AI14) + C2 * (AI14 + AI2 * R)) - XN2 * G * AI14$
 $A(4,5) = XN * (EU1 * (C2 * (R * AI14 + AI31) + C1 * AI31) + G * (AI31 - TWO * AIX))$
 $A(4,6) = -EU1 * C2 * R * XA$
 $A(4,7) = -EU1 * C2 * TWO * AIX$
 $A(4,8) = -EU1 * (AI22 * (C2 + C1) + C2 * AI9 * R) - XN2 * G * AI22$
 $A(4,9) = XN * EU1 * (AI22 * (C2 + C1) + C2 * R * AI9) + XN * G * AI22$
 $A(4,10) = XN * (EU1 * (C2 * (R * AI22 + AI28) + C1 * AI28) + G * AI28)$
 $A(5,1) = +XN * G * (2.0D0 * (-AI2 * R) - AI14) + XN * EU1 * C1 * AI14$
 $A(5,2) = G * (-2.0D0 * (-AI2 * R) + AI14) - XN2 * EU1 * C1 * AI14$
 $A(5,3) = G * (TWO * R * AI14 + AI31) - XN2 * EU1 * C1 * AI31 - G * AIX$
 $A(5,4) = -G * XN * (2.0D0 * R * AI14 + AI31) + XN * EU1 * (C1 * AI31 + C2 * AIX)$
 $A(5,5) = -G * (4.0D0 * R * AIX - 2.0D0 * R * AI31 - AI37) - XN2 * EU1 * C1 * AI37$
 $A(5,6) = XN * EU1 * C2 * AIX$
 $A(5,8) = -XN * G * (2.0D0 * R * AI22 + AI28) + XN * EU1 * C1 * AI28$
 $A(5,9) = G * (2.0D0 * R * AI22 + AI28) - EU1 * C1 * XN2 * AI28$
 $A(5,10) = G * (TWO * R * AI28 - AI49 + AI59) - XN2 * EU1 * C1 * AI59$
 $A(6,1) = -EU1 * C2 * (R * AI1 + AI2)$
 $A(6,2) = -XN * A(6,1)$
 $A(6,3) = XN * EU1 * C2 * (AI14 + AI2 * R)$
 $A(6,4) = -EU1 * C2 * R * XA$
 $A(6,5) = XN * EU1 * C2 * (R * AI14 + AI31)$
 $A(6,6) = -XN2 * G * AI9 - EU1 * C1 * R * XA$
 $A(6,7) = -XN2 * G * AI22 - EU1 * C1 * AIX$
 $A(6,8) = -EU1 * C2 * (R * AI9 + AI22)$
 $A(6,9) = -TWO * XN * G * AI2 + XN * EU1 * C2 * (R * AI9 + AI22)$
 $A(6,10) = XN * EU1 * C2 * (R * AI22 + AI28)$
 $A(7,1) = -EU1 * C2 * (AI14 + AI2 * R)$

```

A(7,2)= XN*EU1*C2*(AI14+AI2*R)
A(7,3)= XN*EU1*C2*(R*AI14+AI31)
A(7,4)= -EU1*C2*(AIX+R*AI14+AI31)
A(7,5)= XN*EU1*C2*(R*AI31+AI37)
A(7,6)= -XN2*G*AI22-EU1*C1*AIX
A(7,7)= -G*(R*AI2+XN2*AI28)-EU1*C1*R*AIX
A(7,8)= -TWO*G*R*AI2-EU1*C2*(R*AI22+AI28)
A(7,9)= XN*EU1*C2*(R*AI22+AI28)
A(7,10)= XN*(-TWO*G*AI49+EU1*C2*(AI28*R+AI59))
A(8,1)= -AI9*(XN2*G+EU1*C1)
A(8,2)= XN*AI9*(G+EU1*C1)
A(8,3)= -XN*(G*(AI2-AI22)-EU1*C1*AI22)
A(8,4)= -AI22*(XN2*G+EU1*C1)-EU1*C2*AI2
A(8,5)= XN*AI28*(G+EU1*C1)
A(8,6)= -EU1*C2*AI2
A(8,7)= -TWO*G*R*AI2
A(8,8)= -AI52*(XN2*G+EU1*C1)-4.ODO*G*R*AI2
A(8,9)= XN*AI52*(G+EU1*C1)
A(8,10)= XN*(G*(AI74-AI72)+EU1*C1*AI74)
A(9,1)= XN*AI9*(G+EU1*C1)
A(9,2)= -AI9*(G+XN2*EU1*C1)
A(9,3)= G*(AI2-AI22)-XN2*EU1*C1*AI22
A(9,4)= XN*(EU1*(C1*AI22+C2*AI2)+G*AI22)
A(9,5)= -AI28*(XN2*EU1*C1+G)
A(9,6)= XN*(EU1*C2*AI2-TWO*G*(R*AI9+AI22))
A(9,7)= -TWO*XN*G*(R*AI22+AI28)
A(9,8)= XN*(EU1*C1+G)*AI52
A(9,9)= -XN2*EU1*C1*AI52-G*(4.ODO*R*AI2+AI52)
A(9,10)= -XN2*EU1*C1*AI74-G*(4.ODO*AI49-AI72+AI74)
A(10,1)= XN*(EU1*C1*AI22-G*R*AI9)
A(10,2)= G*R*AI9-XN2*EU1*C1*AI22
A(10,3)= G*R*(AI22-AI2)-XN2*EU1*C1*AI28
A(10,4)= XN*(EU1*C1*AI28-G*(R*AI22+AI28)+G*AI28)
A(10,5)= G*R*AI28-XN2*EU1*C1*AI59
A(10,6)= -TWO*XN*G*(R*AI22+AI28)
A(10,7)= XN*(EU1*C2*AI49-TWO*G*(R*AI28+AI59))
A(10,8)= XN*(EU1*C1*AI74-G*R*AI52)
A(10,9)= G*(R*AI52-4.ODO*AI49)-XN2*EU1*C1*AI74
A(10,10)= G*R*(AI74-AI72-4.ODO*AI49)-XN2*EU1*C1*AI75
DO 300 I=1,N
DO 300 J=1,N
300 A(I,J)= A(I,J)/(G)
IF(SKIP.GE.4.0) GO TO 215
WRITE(6,100)
100 FORMAT(1X,'MATRIX A COEFFICIENTS'/)
DO 200 I=1,N
200 WRITE(6,210) (A(I,J),J=1,N)
210 FORMAT(1X,5(1PD11.3,3X)/5X,5(1PD11.3,3X))
215 AL(1,1)= -XN2*EU1*(C22*AI1+C11*AI4+C12*AI6)
AL(1,2)= -AL(1,1)/XN

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AL(1,3)= XN*EU1*(C22*AI2+C12*AI15+C11*AI6)
AL(1,4)= -XN*AL(1,3)
AL(1,5)= XN*EU1*(C22*AI14+C11*AI15+C12*AI32)
AL(1,8)= -XN2*EU1*(C22*AI9+C11*AI13+C12*AI23)
AL(1,9)= -AL(1,8)/XN
AL(1,10)= XN*EU1*(C12*AI29+C22*AI22+C11*AI23)
AL(2,2)= -AL(1,2)*XN
AL(2,3)= -XN2*EU1*(C12*AI15-C22*(-AI2)+C11*AI6)
AL(2,4)= -AL(2,3)/XN
AL(2,5)= -XN2*EU1*(C11*AI15+C12*AI32+C22*AI14)
AL(2,8)= XN*EU1*(C11*AI13+C12*AI23+C22*AI9)
AL(2,9)= -AL(2,8)*XN
AL(2,10)= -XN2*EU1*(C12*AI29+C22*AI22+C11*AI23)
AL(3,3)= -XN2*EU1*(C11*AI15+C12*AI32+C22*AI14)
AL(3,4)= -AL(3,3)/XN
AL(3,5)= -XN2*EU1*(C11*AI32+C12*AI34+C22*AI31)
AL(3,8)= XN*EU1*(C12*AI29+C22*AI22+C11*AI23)
AL(3,9)= -XN*AL(3,8)
AL(3,10)= -XN2*EU1*(C11*AI29+C12*AI39+C22*AI28)
AL(4,4)= -XN2*EU1*(C11*AI15+C12*AI32+C22*AI14)
AL(4,5)= XN*EU1*(C11*AI32+C12*AI34+C22*AI31)
AL(4,8)= -XN2*EU1*(C12*AI29+C22*AI22+C11*AI23)
AL(4,9)= -AL(4,8)/XN
AL(4,10)= -AL(3,10)/XN
AL(5,5)= -XN2*EU1*(C11*AI34+C12*AI40+C22*AI37)
AL(5,8)= AL(4,10)
AL(5,9)= -XN*AL(4,10)
AL(5,10)= -XN2*EU1*(C11*AI39+C12*AI67+C22*AI59)
AL(6,6)= -XN2*EU1*(C11*AI13+C12*AI23+C22*AI9)
AL(6,7)= -XN*AL(4,9)
AL(7,7)= AL(5,9)
AL(8,8)= -XN2*EU1*(C11*AI54+C22*AI52+C12*AI55)
AL(8,9)= -AL(8,8)/XN
AL(8,10)= XN*EU1*(C12*AI77+C22*AI74+C11*AI55)
AL(9,9)= -XN*AL(8,9)
AL(9,10)= -XN*AL(8,10)
AL(10,10)= -XN2*EU1*(C11*AI77+C12*AI78+C22*AI75)
DO 3 I=1,N
DO 3 J=I,N
3 AL(J,I)=AL(I,J)
DO 4 I=1,N
DO 4 J=1,N
4 AL(I,J)= -AL(I,J)/(G)
IF(SKIP.GE.3.0) GO TO 221
WRITE(6,101)
101 FORMAT('//1X,'MATRIX MINUS ALPHA'/)
DO 220 I=1,N
220 WRITE(6,210) (AL(I,J),J=1,N)
221 NM=N
MM=N

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CALL MATINV (N,AL,ALI)
IF(SKIP.GE.2.0) GO TO 231
WRITE(6,102)
102  FORMAT(//1X,'MATRIX (MINUS ALPHA)INVERSE'//)
      DO 230 I=1,N
230  WRITE(6,210) (ALI(I,J),J=1,N)
231  DO 11 I1=1,N
      DO 11 I2=1,N
      DO 10 I3=1,N
10  B(I1,I2)=ALI(I1,I3)*A(I3,I2)+B(I1,I2)
11  AA(I1,I2)=B(I1,I2)
      IF(SKIP.GE.1.0) GO TO 241
      WRITE(6,105)
105  FORMAT(//1X,'MATRIX AA --- COEFFICIENTS'//)
      DO 240 I=1,N
240  WRITE(6,210) (AA(I,J),J=1,N)
241  CALL EISPAC (NM,N,MATRIX('REAL',AA),VALUES
1(WR,WI),VECTOR(ZP),ERROR(IERROR))
      DO 50 I=1,N
50  WRITE(6,120) I,WR(I),WI(I),(ZP(J,I),J=1,N)
120  FORMAT(//5X,I6,10X,'EIGENVALUE ',1PD16.7//
1 10X,'CORRESPONDING EIGENVECTORS'//
2 1X,5(1PD11.3,3X)/5X,5(1PD11.3,3X))
      WRITE(6,130) IERROR,ESP1
130  FORMAT(//,' ERROR= ',I10,5X,'ESP1= ',D16.8)
      DO 60 I=1,N
      P= -WR(I)*TWO*E*XA/(ONE+T/R+(T/(TWO*R))**2)/R
      PR=P*R
60  WRITE(6,140) I,P,PR
140  FORMAT(//10X,I5,5X,'P= ',1PD16.7,' PR= ',1PD16.7)
      CONTINUE
      END FILE 6
      STOP
      END

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C      T. M. JULIANO DOCTORAL DISSERTATION
C      MECHANICAL ENGINEERING DEPARTMENT
C      NEW JERSEY INSTITUTE OF TECHNOLOGY
C      OUT OF PLANE BUCKLING OF A THICK RING (8X8) MATRICES
      IMPLICIT REAL*8(A-H,O-Z),INTEGER*4(I-N)
      DIMENSION AA(8,8),WR(8),WI(8),ZP(8,8),B(8,8),BI(8)
      DIMENSION A(8,8),AL(8,8),ALI(8,8)
      DATA XN/2.0D0/,E/.3D08/,G/.115D08/,R/36.385D0/
      DATA T/1.262D1/,H/4.656D0/,U/0.3D0/
      DATA A,AL,B/192*0.0D0/
      DATA N/8/,SKIP/2.0/
      PRINT 99, XN
99     FORMAT(5X,'THICK RING, OUT OF PLANE EQS. 8X8 ', ' N= ',F7.1//
1      5X,'REV NO 15 DATE 06-17-78, SYMMETRY IN A + AL'//
2      5X,'SUBROUTINE AINTEG HAS MIN OF 3 TERMS'//)
      XA= H*T
      AIX= H*T**3/1.2D1
      AIZ= T*H**3/1.2D1
      PRINT 310,T,H,R,XA,AIX,AIZ
310    FORMAT(5X,'T= ',F10.3,' H= ',F10.3,' R= ',F10.3/
1      5X,'A= ',F10.3,' IX= ',F10.3,' IZ= ',F10.3//)
      ONE= 1.0D0
      TWO= 2.0D0
      N2=N*2
      XN2=XN*XN
      RO= R+T/TWO
      RI= R-T/TWO
      U1= ONE+U
      U2= ONE-U
      D1= R*U2+U1*RI*RI/R
      D2= U2-U1*RI*RI/(R*R)
      C1= U2/(ONE-TWO*U)
      C2= C1/U2*U
      C11= C1*D1
      C12= C1*D2
      C22= C2*D2
      EU1= E/U1
      TR= T/R
      CALL AINTEG (H,T,R,TR,ONE,TWO,AI1,AI2,AI4,AI6,AI9,AI13,
1      AI14,AI15,AI22,AI23,AI28,AI29,AI31,AI32,AI34,AI37,AI39,
2      AI40,AI47,AI49,AI52,AI54,AI55,AI59,AI61,AI66,AI67,AI70,
3      AI72,AI74,AI75,AI77,AI78)
      A(1,1)= -XN2*G*AI1
      A(1,3)= -XN2*G*AI2
      A(1,4)= -XN*G*XA
      A(1,7)= -XN2*G*AI14
      A(1,8)= -XN*G*AIX
      A(2,2)= -AI9*(EU1*C1+XN2*G)-G*R*XA
      A(2,3)= -G*R*XA
      A(2,4)= XN*AI9*(G+EU1*C1)

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A(2,5)= XN*(G*(-AIZ+AI22)+EU1*C1*AI22)
A(2,6)= -G*(XN2*AI22+AI2)+EU1*(C1*AI22+C2*AI2)
A(2,7)= -TWO*G*AI2
A(2,8)= XN*AI28*(G+EU1*C1)
A(3,1)= A(1,3)
A(3,2)= A(2,3)
A(3,3)= -G*(R*XA+XN2*AI14)
A(3,5)= -XN*G*AI2
A(3,6)= -G*AI2
A(3,7)= -G*(TWO*AI2+XN2*AI31)
A(4,1)= -XN*G*(R*AI1+AI2)
A(4,2)= A(2,4)
A(4,3)= -XN*G*(AI14+AI2*R)
A(4,4)= -AI9*(G+XN2*EU1*C1)-G*R*XA
A(4,5)= G*(AI2-AI2)-(XN2*EU1*C1+G)*AI2
A(4,6)= XN*EU1*(C1*AI22+C2*AI2)+XN*G*AI2
A(4,7)= -XN*G*(R*AI14+AI31)
A(4,8)= -G*(R*AI2+AI28)-XN2*EU1*C1*AI28
A(5,1)= -XN*G*(R*AI2+AI14)
A(5,2)= XN*(EU1*C1*AI22-G*R*AI9)
A(5,3)= -XN*G*(R*AI14+AI31)
A(5,4)= G*(R*AI9-AI2)-XN2*EU1*C1*AI2
A(5,5)= -G*(R*(AI2-AI22+AI2)-AI28)-XN2*EU1*C1*AI28
A(5,6)= XN*(EU1*C1*AI28-G*AI22*R)
A(5,7)= -XN*G*(R*AI31+AI37)
A(5,8)= -G*(AI47-R*AI28)-XN2*EU1*C1*AI59
A(6,2)= -EU1*(C2*(R*AI9+AI22)+C1*AI22)-G*(XN2*AI22+AI2)
A(6,3)= -G*AI2
A(6,4)= XN*EU1*(C2*(R*AI9+AI22)+C1*AI22)+XN*G*AI2
A(6,5)= XN*EU1*(C2*(R*AI22+AI28)+C1*AI28)+XN*G*AI28
A(6,6)= -EU1*((C1*R*AI2+AI28)+C2*(R*AI22+AI28))-G*R*AI2
1 -XN*G*AI28
A(6,7)= -TWO*G*R*AI2
A(6,8)= XN*EU1*(C2*R*AI28+AI59*(C2+C1))-XN*G*(TWO*AI49-AI59)
A(7,1)= A(1,7)
A(7,2)= A(2,7)
A(7,3)= A(3,7)
A(7,4)= -XN*G*AI2
A(7,6)= A(6,7)
A(7,7)= -G*(4.0DO*R*AI2+XN2*AI37)
A(7,8)= -XN*G*AI47
A(8,1)= -XN*G*(R*AI14+AI31)
A(8,2)= XN*(AI28*(EU1*C1-G)-TWO*G*R*AI22)
A(8,3)= -XN*G*(R*AI31+AI37)
A(8,4)= G*(TWO*R*AI22+AI28-R*AI2)-XN2*EU1*C1*AI28
A(8,5)= G*(TWO*R*AI28-AI49+AI59-AI47)-XN2*EU1*C1*AI59
A(8,6)= XN*EU1*(C1*AI59+C2*AI49)-XN*G*(TWO*R*AI28+AI59)
A(8,7)= -XN*G*(R*AI37+AI66)
A(8,8)= G*(R*(AI59*TWO-AI49*4.0DO-AI47)+AI61)-XN2*EU1*C1*AI61
DO 300 I=1,N

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DO 300 J=1,N
300 A(I,J)= A(I,J)/(G)
   IF(SKIP.GE.4.0) GO TO 215
   WRITE(6,100)
100 FORMAT(1X,'MATRIX A COEFFICIENTS'/)
   DO 200 I=1,N
200 WRITE(6,210) (A(I,J),J=1,N)
210 FORMAT(1X,5(1PD11.3,3X)/30X,3(1PD11.3,3X))
215 AL(1,1)= -XN2*EU1*(C11*AI4+C22*AI1+C12*AI6)
   AL(1,3)= -XN2*EU1*(C11*AI6+C12*AI15+C22*AI2)
   AL(1,7)= -XN2*EU1*(C11*AI15+C12*AI32+C22*AI14)
   AL(2,2)= -XN2*EU1*(C11*AI13+C12*AI23+C22*AI9)
   AL(2,4)= -AL(2,2)/XN
   AL(2,5)= XN*EU1*(C12*AI29+C22*AI22+C11*AI23)
   AL(2,6)= -XN2*EU1*(C11*AI23+C12*AI29+C22*AI22)
   AL(2,8)= XN*EU1*(C11*AI29+C12*AI39+C22*AI28)
   AL(3,3)= AL(1,7)
   AL(3,7)= -XN2*EU1*(C11*AI32+C12*AI34+C22*AI31)
   AL(4,4)= AL(2,2)
   AL(4,5)= -XN*AL(2,5)
   AL(4,6)= AL(2,5)
   AL(4,8)= -XN*AL(2,8)
   AL(5,5)= AL(4,8)
   AL(5,6)= AL(2,8)
   AL(5,8)= -XN2*EU1*(C11*AI39+C12*AI67+C22*AI59)
   AL(6,6)= AL(4,8)
   AL(6,8)= -XN*AL(5,8)
   AL(7,7)= -XN2*EU1*(C11*AI34+C12*AI40+C22*AI37)
   AL(8,8)= -XN2*EU1*(C11*AI67+C12*AI70+C22*AI61)
   DO 3 I=1,N
   DO 3 J=I,N
3   AL(J,I)=AL(I,J)
   DO 4 I=1,N
   DO 4 J=1,N
4   AL(I,J)= -AL(I,J)/(G)
   IF(SKIP.GE.3.0) GO TO 221
   WRITE(6,101)
101 FORMAT(//1X,'MATRIX MINUS ALPHA'/)
   DO 220 I=1,N
220 WRITE(6,210) (AL(I,J),J=1,N)
221 NM= N
   MM=N
   CALL MATINV (N,AL,ALI)
   IF(SKIP.GE.2.0) GO TO 231
   WRITE(6,102)
102 FORMAT(//1X,'MATRIX (MINUS ALPHA)INVERSE'/)
   DO 230 I=1,N
230 WRITE(6,210) (ALI(I,J),J=1,N)
231 DO 11 I1=1,N
   DO 11 I2=1,N

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DO 10 I3=1,N
10 B(I1,I2)=ALI(I1,I3)*A(I3,I2)+B(I1,I2)
11 AA(I1,I2)=B(I1,I2)
   IF(SKIP.GE.1.0) GO TO 241
   WRITE(6,105)
105 FORMAT(/1X,'MATRIX AA --- COEFFICIENTS'/)
   DO 240 I=1,N
240 WRITE(6,210) (AA(I,J),J=1,N)
   241 CALL EISPAC (NM,N,MATRIX('REAL',AA),VALUES
   1(WR,WI),VECTOR(ZP),ERROR(IERROR))
   DO 50 I=1,N
50 WRITE(6,120) I,WR(I),WI(I),(ZP(J,I),J=1,N)
120 FORMAT(/5X,I6,10X,'EIGENVALUE ',1P2D16.7//
1 10X,'CORRESPONDING EIGENVECTORS'//
2 1X,5(1PD11.3,3X)/5X,3(1PD11.3,3X))
   WRITE(6,130) IERROR,ESP1
130 FORMAT(/,' ERROR= ',I10,5X,'ESP1= ',D16.8)
   DO 60 I=1,N
   P= -WR(I)*TWO*E*XA/(ONE+T/R+(T/(TWO*R))**2)/R
   PR= P*R
60 WRITE(6,140) I,P,PR
140 FORMAT(/10X,I5,5X,'P= ',1PD16.7,' PR=',1PD16.7)
CONTINUE
END FILE 6
STOP
END

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C      T. M. JULIANO DOCTORAL DISSERTATION
C      MECHANICAL ENGINEERING DEPARTMENT
C      NEW JERSEY INSTITUTE OF TECHNOLOGY
C      INPLANE AND OUT OF PLANE BUCKLING OF A THIN RING
C      (8X8) MATRICES
      IMPLICIT REAL*8(A-H,O-Z),INTEGER*4(I-N)
      DIMENSION AA(8,8),WR(8),WI(8),ZP(8,8),B(8,8),BI(8)
      DIMENSION A(8,8),AL(8,8),ALI(8,8)
      DATA XN/2.0D0/,E/.3D08/,G/.115D08/,R/727.7D0/
      DATA T/1.262D1/,H/4.656D0/,U/0.3D0/
      DATA A,AL,B/192*0.0D0/
      DATA N/8/,SKIP/2.0/
      PRINT 99, XN
99     FORMAT(5X,'THIN RING, IN + OUT OF PLANE 8X8 ', ' N= ',F7.1//
1      5X,'REV NO 7 DATE 06-19-78, SYMMETRY IN A + AL'//
2      5X,'MIN THREE TERMS IN ALL AI'//)
      XA= H*T
      AIX= H*T**3/1.2D1
      AIZ= T*H**3/1.2D1
      PRINT 310,T,H,R,XA,AIX,AIZ
310    FORMAT(5X,'T= ',F10.3,' H= ',F10.3,' R= ',F10.3/
1      5X,'A= ',F10.3,' IX= ',F10.3,' IZ= ',F10.3//)
      ONE= 1.0D0
      TWO= 2.0D0
      N2=N*2
      XN2=XN*XN
      U1= ONE+U
      C1= (ONE-U)/((ONE-U*TWO)*U1)
      EU1= E/U1
      TR= T/R
C      EVALUATES INTEGRALS OVER RING CROSS SECTION
      AI1= H*TR*(ONE+TR**2/12.0D0+TR**4/8.0D1)
      AI2= -H*TR**3*R*(ONE/1.2D1+TR**2/8.0D1+TR**4/4.48D2)
      AI4= AI1/R
      AI6= -H*TR**3*(ONE/6.0D0+TR**2/2.0D1+3.0D0*TR**4/2.24D2)
      AI9= H*H/1.2D1*AI1
      AI13= H*H/1.2D1*AI4
      AI14= -R*AI2
      AI15= H*TR**3*R*(ONE/1.2D1+3.0D0*TR**2/8.0D1+5.0D0*TR**4/
1      4.48D2)
      AI22= H*H*AI2/1.2D1
      AI23= H*H*AI6/1.2D1
      AI28= -R*AI22
      AI29= H*H*AI15/1.2D1
      AI31= -H*T**5/R**2*(ONE/8.0D1+TR**2/4.48D2+TR**4/
1      2.304D3)
      A(1,1)= -(XN2*G+E*C1)*AI1
      A(1,2)= XN*AI1*(G+E*C1)
      A(1,3)= -XN*(G*(XA-AI2)-E*C1*AI2)
      A(2,1)= A(1,2)

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```

A(2,2)= -AI1*(XN2*E*C1+G)
A(2,3)= -XN2*E*C1*AI2+G*(XA-AI2)
A(3,1)=-XN*(R*G*AI1-E*C1*AI2)
A(3,2)= R*G*AI1-XN2*E*C1*AI2
A(3,3)= -XN2*E*C1*AI14-R*G*(XA-AI2)
A(4,4)= -XN2*G*AI1
A(4,6)= -XN2*G*AI2
A(4,7)= -XN*G*XA
A(5,5)= -AI9*(XN2*G+E*C1)-R*G*XA
A(5,6)= -R*G*XA
A(5,7)= XN*AI9*(G+E*C1)
A(5,8)= -XN*(G*(AI2-AI22)-E*C1*AI22)
A(6,4)= A(4,6)
A(6,5)= A(5,6)
A(6,6)= -G*(XN2*AI14+R*XA)
A(6,8)= -XN*G*AI2
A(7,4)= -XN*G*(R*AI1+AI2)
A(7,5)= A(5,7)
A(7,6)= -XN*G*(R*AI2+AI14)
A(7,7)= -AI9*(XN2*E*C1+G)-R*G*XA
A(7,8)= G*(AI2-AIX-AI22)-XN2*E*C1*AI22
A(8,4)= -XN*G*(R*AI2+AI14)
A(8,5)= -XN*(R*G*AI9-E*C1*AI22)
A(8,6)= -XN*G*(R*AI14+AI31)
A(8,7)= G*(R*AI9-AIX)-XN2*E*C1*AI22
A(8,8)= -G*R*(AI2+AIX-AI22)-XN2*E*C1*AI28
DO 300 I=1,N
DO 300 J=1,N
300  A(I,J)= A(I,J)/(G)
      IF(SKIP.GE.4.0) GO TO 215
      WRITE(6,100)
100  FORMAT(1X,'MATRIX A COEFFICIENTS'/)
DO 200 I=1,N
200  WRITE(6,210) (A(I,J),J=1,N)
210  FORMAT(1X,5(1PD11.3,3X)/30X,3(1PD11.3,3X))
215  AL(1,1)= -XN2*EU1*AI4
      AL(1,2)= -AL(1,1)/XN
      AL(1,3)= XN*EU1*AI6
      AL(2,2)= AL(1,1)
      AL(2,3)= -AL(1,3)*XN
      AL(3,3)= -XN2*EU1*AI15
      AL(4,4)= AL(1,1)
      AL(4,6)= AL(2,3)
      AL(5,5)= -XN2*EU1*AI13
      AL(5,7)= -AL(5,5)/XN
      AL(5,8)= XN*EU1*AI23
      AL(6,6)= AL(3,3)
      AL(7,7)= AL(5,5)
      AL(7,8)= -AL(5,8)*XN
      AL(8,8)= -XN2*EU1*AI29

```

```

DO 3 I=1,N
DO 3 J=I,N
3 AL(J,I)=AL(I,J)
DO 4 I=1,N
DO 4 J=1,N
4 AL(I,J)= -AL(I,J)/(G)
IF(SKIP.GE.3.0) GO TO 221
WRITE(6,101)
101 FORMAT(/1X,'MATRIX MINUS ALPHA'/)
DO 220 I=1,N
220 WRITE(6,210) (AL(I,J),J=1,N)
221 NM= N
MM=N
CALL MATINV (N,AL,ALI)
IF(SKIP.GE.2.0) GO TO 231
WRITE(6,102)
102 FORMAT(/1X,'MATRIX (MINUS ALPHA)INVERSE'/)
DO 230 I=1,N
230 WRITE(6,210) (ALI(I,J),J=1,N)
231 DO 11 I1=1,N
DO 11 I2=1,N
DO 10 I3=1,N
10 B(I1,I2)=ALI(I1,I3)*A(I3,I2)+B(I1,I2)
11 AA(I1,I2)=B(I1,I2)
IF(SKIP.GE.1.0) GO TO 241
WRITE(6,105)
105 FORMAT(/1X,'MATRIX AA --- COEFFICIENTS'/)
DO 240 I=1,N
240 WRITE(6,210) (AA(I,J),J=1,N)
241 CALL EISPAC (NM,N,MATRIX('REAL',AA),VALUES
1(WR,WI),VECTOR(ZP),ERROR(IERROR))
DO 50 I=1,N
50 WRITE(6,120) I,WR(I),WI(I),(ZP(J,I),J=1,N)
120 FORMAT(/5X,I6,10X,'EIGENVALUE ',1PD16.7//
1 10X,'CORRESPONDING EIGENVECTORS'//
2 1X,5(1PD11.3,3X)/5X,3(1PD11.3,3X))
WRITE(6,130) IERROR,ESP1
130 FORMAT(/,' ERROR= ',I10,5X,'ESP1= ',D16.8)
DO 60 I=1,N
P= -XA*E*WR(I)/(R*R*C1*U1)
PR= P*R
60 WRITE(6,140) I,P,PR
140 FORMAT(/10X,I5,5X,'P= ',1PD16.7,' PR= ',1PD16.7)
CONTINUE
END FILE 6
STOP
END

```

```

SUBROUTINE AINTEG (H,T,R,TR,ONE,TWO,AI1,AI2,AI4,AI6,AI9,AI13,
1 AI14,AI15,AI22,AI23,AI28,AI29,AI31,AI32,AI34,AI37,AI39,AI40,
2 AI47,AI49,AI52,AI54,AI55,AI59,AI61,AI66,AI67,AI70,AI72,
3 AI74,AI75,AI77,AI78)
C EVALUATES INTEGRALS OVER RING CROSS SECTION
  IMPLICIT REAL*8(A-H,O-Z),INTEGER*4(I-N)
  AI1= H*TR*(ONE+TR**2/12.0D0+TR**4/8.0D1)
  AI2= -H*TR**3*R*(ONE/1.2D1+TR**2/8.0D1+TR**4/4.48D2)
  AI4= AI1/R
  AI6= -H*TR**3*(ONE/6.0D0+TR**2/2.0D1+3.0D0*TR**4/2.24D2)
  AI9= H*H/1.2D1*AI1
  AI13= H*H/1.2D1*AI4
  AI14= -R*AI2
  AI15= H*TR**3*R*(ONE/1.2D1+3.0D0*TR**2/8.0D1+5.0D0*TR**4/
1 4.48D2)
  AI22= H*H*AI2/1.2D1
  AI23= H*H*AI6/1.2D1
  AI28= -R*AI22
  AI29= H*H*AI15/1.2D1
  AI31= -H*T**5/R**2*(ONE/8.0D1+TR**2/4.48D2+TR**4/
1 2.304D3)
  AI32= -H*T**5/R**3*(ONE/4.0D1+TR**2/1.12D2+TR**4/
1 3.84D2)
  AI34= H*T**5/R**2*(ONE/8.0D1+3.0D0*TR**2/4.48D2+
1 5.0D0*TR**4/2.304D3)
  AI37= -AI31*R
  AI39= H*H*AI32/1.2D1
  AI40= -H*TR**4*R*(ONE/2.24D2+TR**2/5.76D2+TR**4/
1 1.756945D3)
  AI47= H*T**5/8.0D1
  AI49= (H*T)**3/1.44D2
  AI52= H**4*AI1/8.0D1
  AI54= AI52/AI1*AI4
  AI55= AI52/AI1*AI6
  AI59= H*H*AI31/1.2D1
  AI61= H*H*AI37/1.2D1
  AI66= -H*T**7/(R*R)*(ONE/4.48D2+TR**2/2.304D3+TR**4/
1 1.1264D4)
  AI67= H*H*AI34/1.2D1
  AI70= H*H*AI40/1.2D1
  AI72= H**5*T/8.0D1
  AI74= H**4*AI2/8.0D1
  AI75= AI74/AI2*AI14
  AI77= AI74/AI2*AI15
  AI78= AI74/AI2*AI32
  RETURN
  END

```

```
      SUBROUTINE MATINV(N,A,B)
C      INVERTS MATRIX WITH NON-ZERO DIAGONAL TERMS
      IMPLICIT REAL*8(A-H,O-Z),INTEGER*4(I-N)
      DIMENSION A(10,10),B(10,10),C(10,10),D(10,10)
      DO 30 I=1,N
      DO 30 J=1,N
      C(I,J)=A(I,J)
      IF(I-J) 10,20,10
10     B(I,J)=0.0DO
      GO TO 30
20     B(I,J)=1.0DO
30     CONTINUE
      DO 70 NT=1,N
      DO 40 I=1,N
40     D(I,NT)=A(I,NT)
      DO 50 J=1,N
      A(NT,J)=A(NT,J)/D(NT,NT)
50     B(NT,J)=B(NT,J)/D(NT,NT)
      DO 70 I=1,N
      DO 70 J=1,N
      IF(NT-I)60,70,60
60     A(I,J)=A(I,J)-A(NT,J)*D(I,NT)
      B(I,J)=B(I,J)-B(NT,J)*D(I,NT)
70     CONTINUE
      DO 80 I=1,N
      DO 80 J=1,N
      A(I,J)=C(I,J)
80     CONTINUE
      RETURN
      END
```

C. Appendix III, Computer Program Results

The actual results obtained by the computer analyses are given on the following pages in table 9. This table also contains the physical dimensions of the rings, i.e. radial thickness (T), axial depth (H), and mean radius (R).

Cross sectional Dimensions		Ring Mean Radius	Critical Buckling Force (Kips) ($P_{cr} \times R$)			
T	H	R	Thin Ring Theory		Thick Ring Theory	
In	In	In	Inplane	Out of Plane	Inplane	Out of Plane
3.0	1.0	15	854.0	60.53	722.1	52.98
3.0	1.0	12	1298.	94.79	1073.	81.03
3.0	1.0	10	1808.	136.9	1464.	114.4
3.0	1.0	8	2666.	215.1	2099.	173.7
3.0	1.0	5	5515.	567.5	4043.	412.3
3.0	1.0	4	7342.	920.5	5205.	616.9
3.0	1.0	15($\mu=0.0$)	863.8	63.80	786.9	58.22
3.0	1.5	15	1281.	198.3	1083.	174.5
3.0	1.5	12	1946.	309.6	1610.	266.3
3.0	1.5	9	3266.	549.8	2612.	456.0
3.0	1.5	6	6352.	1237.	4793.	955.2
3.0	1.5	5	8272.	1785.	6066.	1320.
3.0	3.0	15	2562.	1372.	2168.	1235.
3.0	3.0	12	3893.	2115.	3223.	1865.
3.0	3.0	9	6533.	3658.	5232.	3124.
3.0	3.0	6	12,700.	7683.	9617.	6181.
3.0	3.0	5	16,550.	10,570.	12,180.	8204.
1.5	3.0	12	516.5	724.0	-	706.6
1.5	3.0	9	903.7	1249.	-	1200.
1.5	3.0	7.5	1281.	1748.	1091.	1659.

Table 9. Computer Program Results.

Cross sectional Dimensions		Ring Mean Radius	Critical Buckling Force (Kips)			
			(P _{cr} x R)			
T	H	R	Thin Ring Theory		Thick Ring Theory	
In	In	In	Inplane	Out of Plane	Inplane	Out of Plane
1.5	3.0	6	1946.	2599.	1628.	2427.
1.5	3.0	4.5	3266.	4200.	2659.	3823.
1.5	3.0	3	6352.	7648.	4941.	6616.
1.0	3.0	12	154.8	317.1	-	326.4
1.0	3.0	8	344.3	681.0	-	690.0
1.0	3.0	5	854.0	1555.	740.4	1534.
1.0	3.0	4	1298.	2222.	1111.	2157.
1.0	3.0	2	4234.	5626.	3419.	4984.
12.62	4.66	728.	132.5	10.83	122.9	10.43
12.62	4.66	72.68	12,740.	1094.	10,910.	966.6
12.62	4.66	36.4	45,700.	4372.	36,380.	3582.
12.62	4.66	728 ($\mu=0.0$)	132.5	11.42	131.4	11.32
1.0	3.0	3	2178.	3380.	1829.	3188.

Table 9. Computer Program Results (Continued)

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