77-22,451

JHAVERI, Madhusudan Hiralal, 1936-
NONLINEAR BUCKLING OF DEEP CYLINDRICAL SHELL PANELS WITH IMPERFECTIONS.

New Jersey Institute of Technology,
D.Eng.Sc., 1977
Engineering, civil

Xerox University Microfilms, Ann Arbor, Michigan 48106

NONLINEAR BUCKLING OF DEEP CYLINDRICAL

SHELL PANELS WITH IMPERFECTIONS

BY

MADHUSUDAN H. JHAVERI

A DISSERTATION

PRESENTED IN PARTIAL FULFILLMENT OF

THE REQUIREMENTS FOR THE DEGREE

OF

DOCTOR OF ENGINEERING SCIENCE

IN CIVIL ENGINEERING

AT

NEW JERSEY INSTITUTE OF TECHNOLOGY

Newark, New Jersey

1977

APPROVAL OF DISSERTATION

NONLINEAR BUCKLING OF DEEP CYLINDRICAL

SHELL PANELS WITH IMPERFECTIONS

BY

MADHUSUDAN H. JHAVERI

FOR

DEPARTMENT OF CIVIL ENGINEERING

NEW JERSEY INSTITUTE OF TECHNOLOGY

BY

FACULTY COMMITTEE

APPROVED ______________________ CHAIRMAN

____

____

____

____

NEWARK, NEW JERSEY

1977

ABSTRACT

The work presented in this dissertation is devoted to the analysis of nonlinear buckling of thin cylindrical shells with imperfections. The nonlinearity of the problem treated in this dissertation is that associated with large displacements in the linear elastic range. The method used is not restricted by the magnitude of the displacements provided that the strains do not exceed the limit of proportionality.

Thus, based upon the large deflection theory, cylindrical shell panel is investigated for buckling under the action of uniform external pressure. Certain higher order infinitesimal terms which are usually neglected in the shallow shell theories, have been retained in the present paper to study their effect on buckling, and also to test the validity of shallow shell assumptions. The shell is clamped at the two longitudinal edges while it is simply supported at the transverse edges.

The general nonlinear theory with respect to strains is applied to deep shells to formulate the set of equations, which are nonlinear partial differential equations. These nonlinear partial differential equations are reduced to a set of nonlinear ordinary differential equations by applying the Kantorovitch method.

Further these differential equations are reduced to a set of nonlinear finite difference equations by conventional methods in terms of central differences. These nonlinear finite difference equations are linearized by incremental method, and transformed into a suitable matrix form.

An iterative procedure to solve this system of incremental equations in the matrix form is developed. A computer program is written in Fortran IV to solve these equations, following the interative procedure.

The following different cases of buckling have been investigated in this paper.

a) Symmetric buckling of deep shells.

b) Symmetric buckling of shallow shells.

c) The effect of initial imperfections on the buckling loads.

d) Asymmetric buckling as a bifurcational buckling of symmetric case.

The findings of this study may be stated as follows:

From buckling point of view, deep shells are stronger than shallow shells.

When only shallow shell parameters are employed, neglecting deep shell parameters, the corresponding deflection caused by an increment in load tends to be smaller. And therefore the buckling load when shallow shell parameters are employed tends to be higher.

In case of shallower shells, though, no appreciable difference in the unit load is found by employing deep shell parameters.

Initial imperfections if present even to a minute degree can affect the buckling load significantly. Also, this paper establishes the validity of certain assumptions of shallow shell theory for the first time.

PREFACE

Much recent research has been devoted to the problem of elastic buckling of thin plates and shells. This is understandable in light of the fact that thin walled shell constructions incorporate material economy with possible high strength. But it also renders the structure prone to buckling. The thin shells have wide applications in aeronautical, naval and civil engineering. Although, the practical potential for the use of thin shells has by no means exhausted.

The work presented in this dissertation is devoted to the analysis of nonlinear buckling of thin cylindrical shells with imperfections. The nonlinearity of the problem treated in this dissertation is that associated with large displacements in the linear elastic range. The method used is not restricted by the magnitude of the displacements provided that the strains do not exceed the limit of proportionality.

ACKNOWLEDGEMENTS

The author wishes to express his sincere appreciation to Dr. William J. Stack-Staikidis for his extreme patience, unselfish guidance and assistance throughout the course of this work. In addition, the author would like to thank Dr. F.G. Lehman for granting him the teaching fellowship during 1970-71, and 1971-72 while he was a graduate student at Newark College of Engineering. He also thanks Dr. Golub, Dr. Khera, Dr. Craig, Dr. Peyser and Dr. Salek for serving on his doctoral committee.

Appreication is also expressed to the N.J.I.T. Computing Center and S.M.U. Computing Center for the use of the computing facilities.

He specially thanks his good friend Mrs. Beverly Zexter for her help with the typing.

Finally, he wishes to thank his wife Pallavi, for her help and understanding.

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

LIST OF NOTATIONS

A	coefficient matrix determined by geometry and material properties of the shell.
A1,...A5	matrices at the intermediate points for the symmetric case.
$A_1 A_2 \ldots A_{13}$	shell parameters determined by geometry of the shell and material properties.
a, b	sides of shell
B1, B2...B4	matrices at the boundaries for symmetric case.
$B_1, B_2 \ldots B_{11}$	shell parameters determined by geometry of the shell and material properties.
C1, C2	matrices at the intermediate points for the asymmetric case.
$C_1, C_2, \ldots C_{21}$	shell parameters determined by geometry of the shell and material properties.
C	shell parameter
D1, D2	matrices at the boundaries for asymmetric case.
$(DD1)_i^j \ldots (DD12)_i^j$	summation terms introduced in (5-3).
Df, Dh, Dp_z	small increments.
$(DS1)_i^j \ldots (DS19)_i^j$	summation terms of Dh Df introducted in (5-2).
$(DST)_i^j$	summation term introduced in (5-5).
E	modulus of elasticity.
e_x, e_y, e_{xy}	strains on middle surface.
e_x^z, e_y^z, e_{xy}^z	strains on z surface.

FE1,...FE5	submatrices introduced in (5-10).
FF1,...FF5	submatrices introduced in (6-15).
f_n, g_n, h_n	functions of x introduced in (3-2).
$\underline{f}_n$, $\underline{g}_n$, $\underline{h}_n$	functions of x introduced in (6-5).
HH1,...HH5	submatrices introduced in (6-15).
h	mesh size.
i	integer denoting the nodal position.
j, m, n, p	integral numbers.
k_x^o, K_y^o, K_{xy}^o	curvatures of unstrained surface.
K_x, K_y, K_{xy}	curvatures of strained surface.
k	imperfection parameter introduced in (7-3).
M_x, M_y, M_{xy}	bending and twisting moments per unit distance in the shell.
N_x, N_y, N_{xy}	membrane forces per unit distance in the shell.
p_x, p_y, p_z	components of external pressure.
$P_1 \ldots P_{14}$	parameters defined in (4-5).
p	external pressure.
Q_x, Q_y	shearing forces per unit distance in the shell.
$Q_1 \ldots Q_{22}$	definite integrals introduced in (3-5) defined in Appendix A and B.
$R_1 \ldots R_{19}$	parameters introduced in (4-6)
$(S1)_i^j \ldots (S66)_i^j$	summation introduced in (5-1)
$(SS1)_i^j \ldots (SS33)_i^j$	summation terms introduced in (5-4)
$(T1)_i^n \ldots (T19)_i^n$	summation terms introduced in (6-9)

$(U0)_i^n \ldots (U27)_i^n$	summation terms introduced in (6-10).
$\underline{u}, \underline{v}, \underline{w}$	displacements due to small asymmetric deformation.
$\bar{w}$	average deflection divided by thickness.
$(V1)_i^n \ldots (V16)_i^n$	summation terms introduced in (6-11)
w	average deflection
w_L	average deflection corresponding to limit load.
X_i, Y_i	column matrices introduced in (5-9).
$\bar{X}, \bar{Y}, \bar{Z}$	orthogonal system of axes as shown in Fig. 3.
x, y, z	parametric coordinates on middle surface.
Z_i	column matrix introduced in (6-13).
$\sigma_x, \sigma_y, \sigma_{xy}$	unit normal and shear stresses.
ν	Poisson's ratio.
ρ	load parameter equal to $(p \times 10^7/E)$
$\Delta\delta_N$	column matrix representing Nth increments in the functions f_j and h_j.
Δp_n	column matrix determined by Nth increment in the external load.
ΔA_{N-1}	corrections to the matrix A, as a result of updating the displacements.
$f(\Delta\delta_{N-1})$	additional corrections due to "equivalent load terms."

1. INTRODUCTION

1.1 General

Minimum weight, optimum structural design has forced thinner and thinner structural shapes of various kinds into engineering.

The characteristic property of such shapes is their flexibility and their relatively small resistance to bending and torsion. Therefore, when deformed under load such structures have large displacements compared to their thickness. In this respect their behavior is geometrically nonlinear.

The object of this paper is to investigate such geometrically nonlinear buckling of a thin deep cylindrical shell under the action of uniform external pressure.

There has been a considerable research devoted to the problem of elastic buckling of closed shells. Such research has applications in aircraft and naval industry. Little work, however, is done on the stability investigations of open shells. The historical review of the previous work done in this field is presented in the next article.

1.2 Historical Review

The theory of elastic stability of thin shells has been investigated by many researchers in last sixty years. In the first twenty five years of the above period though most research workers dealt only with linear theory of elastic stability, Thielemann [26].

The buckling loads of thin cylindrical shells have been in serious disagreement with those predicted by the linear theory, e.g. In the case of the axially loaded cylinder the test results shown in Figure [1], have been about 20 to 30% of the classical buckling load, whereas in the case of torsion and external pressure the tests give about 70 to 80% of the buckling loads expected from the classical theory. In addition the test results show unusually large scatter in the buckling loads, Thielemann [26] Stein [25].

Thus, classical linear theory of shells failed to explain the discrepancy that existed between the theoretical and experimental results.

Donnell[4] achieved a major simplification of the linear theory of buckling of circular cylindrical shells in 1934, which played an important role in the development of an adequate nonlinear theory.

In 1940, the tests of simple struts with a central nonlinear elastic support, in England and in United States, showed that small imperfections in these nonlinear structures reduced the critical load drastically. Guided by these tests, Von Karman and Tsien examined the post buckling behavior of cylindrical shells under axial compression, Karman and Tsien [13].

These investigators made use of nonlinear large-deflection theory proposed by Donnell. Their method of attack on the problem was based on the pair of nonlinear differential equations for the normal deflection w and the Airy stress function of the theory of shallow shells.

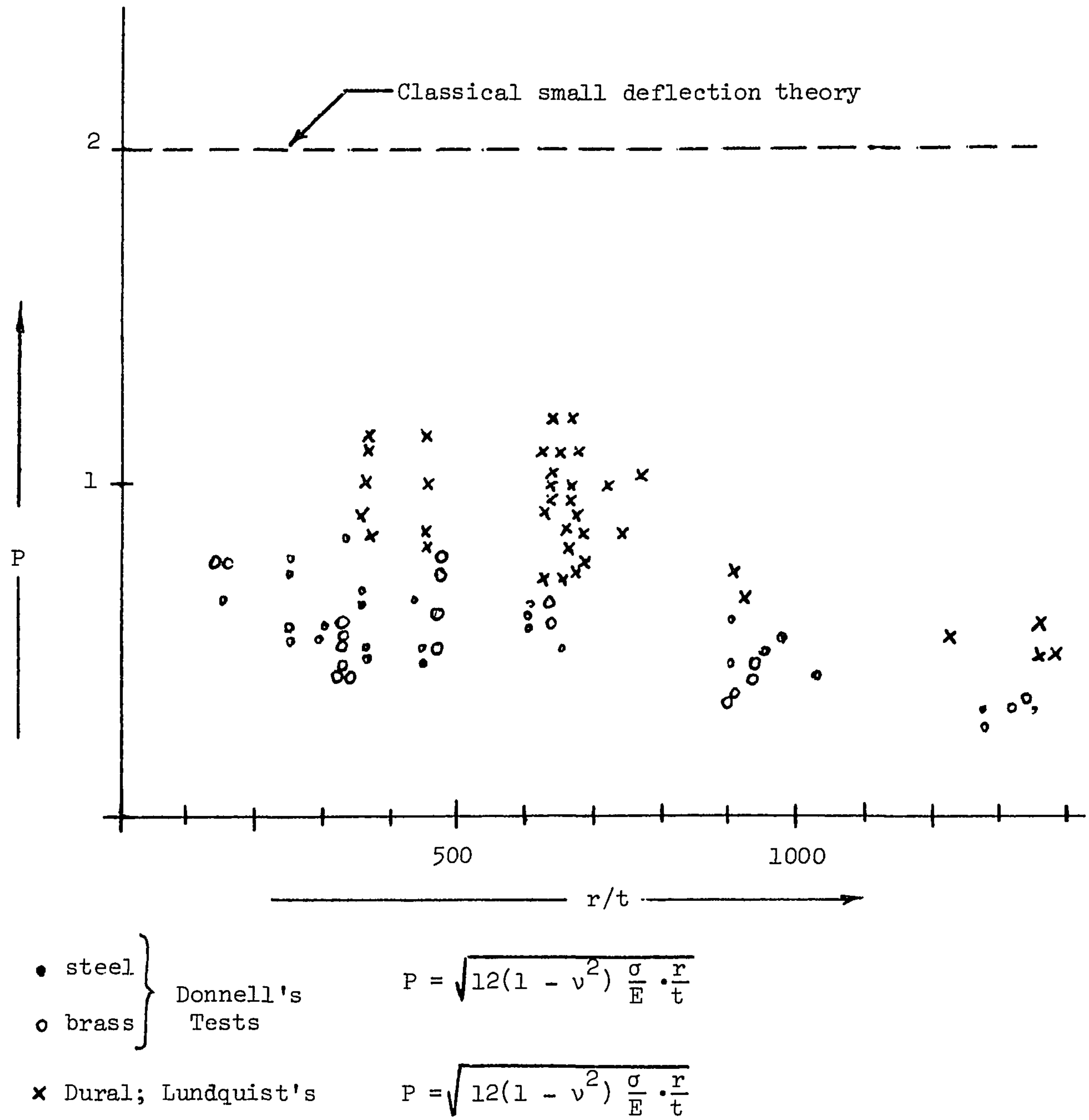


Figure 1 Comparison of experimental strengths of thin cylindrical shells under axial compression with classical theory.

In a single statement the result of investigations of Karman and Tsien [13] may be stated as follows: The axially compressed cylinder after having reached the critical buckling load must snap through into another state of equilibrium which is connected with a considerably smaller axial load.

The analysis of Von Karman and Tsien was refined and enlarged by Kempner [14], Leggett and Jones [19], Michielsen [22].

Donnell and Wan [6] made an important contribution toward the explanation of the discrepancies between experimental and theoretical buckling loads of axially loaded cylindrical shells. They introduced the assumption of initial imperfections into the analysis of Karman and Tsien [13]. These initial imperfections they showed, present even in the order of a few tenths of the thickness of the wall, considerably reduce the buckling load of the axially compressed cylinder.

A different kind of approach to the nonlinear buckling problem was developed by Koiter [15], [16]. In this approach, the attention was focussed on the initial stage of post-buckling behavior. A number of additional examples have been investigated by Budiansky [1], Hutchinson [10], [11] and Thompson [27].

Most of the above mentioned research investigations have been devoted to the problem of elastic buckling of closed shells, such as circular cylinders. The research findings have applications primarily in the design of aircraft fuselages, missile casings etc. It is only in the last few years that the problem of the stability of open shells, such as cylindrical panels has drawn attention of some researchers.

The stability investigations of open cylindrical shells have been carried out by Karakas and Scalz [12], Yang and Guralnick [31], W.J. Stack-staikidis [24], Chu and Turula [30] and Mak and Wen [20]. The first two of these, are experimental investigations whereas the last three are the theoretical investigations.

A fiberglass-reinforced plastic shell was tested at Case Institute of Technology by Karakas and Scalz [12]. The shell was simply supported by resting the end diaphragms on rollers. The usual practice of incremental loading was followed, and deflections and strains were recorded at each increment of load. The compressive stress at the center of the shell was found to be about 32 percent of the critical buckling stress, calculated according to the suggestions of the ASCE Manual No. 31.

Shell models fabricated from sheets of 5052 H-32 aluminum were tested for buckling by T.H. Yang and S.A. Guralnick [31] at Illinois Institute of Technology. The shells were simply supported on rollers at the ends. A uniformly distributed live load on the horizontal projection of the shell, was simulated by means of a series of closely spaced concentrated loads. The results of this study compare well with those calculated by shallow shell theory. The models tested in this study were all relatively shallow.

A method for the study of the nonlinear buckling of shallow cylindrical shells is presented by W.J. Stack-Staikidis [24]. In this method the governing differential equations are expressed in terms of stress function and the normal displacement. The Kantorovitch method and a finite difference scheme are applied to solve the resulting equations. The general nonlinear theory with respect to strains and shallow shell assumptions are used for the mathematical formulation of the problem.

A technique for obtaining the critical load of open cylindrical shells is presented by Kuang-Han Chu and Peter Turula [30]. The shell considered is simply supported at the ends and is free at the longitudinal edges. General equations based on the large deflection theory are developed and a set of nonlinear finite difference equations are solved. In developing this technique also, the shell is assumed to be shallow.

Cary Mak and Robert Wen [20] have investigated a cylindrical shell panel supported by flexible longitudinal beams on edges and rollers on the curved edge of the shell. The problem is solved by means of Rayleigh-Ritz type approach. In this investigation also the shell was assumed to be shallow and loaded by its own weight or radial pressure.

1.3 The Method of Present Investigation

All of the aforementioned theoretical investigations were based on shallow shell theory assumptions. The present investigation distinguishes itself from all the others by taking into consideration certain higher order infinitesimal terms, which in other words is equivalent to assuming the shell to be deep. This makes the method more general.

This investigation also takes into account, the effect of initial imperfections of various amplitudes of the order of the thickness of the shell. As a result of this general approach, the mathematical formulation has involved many terms making the calculation very laborious.

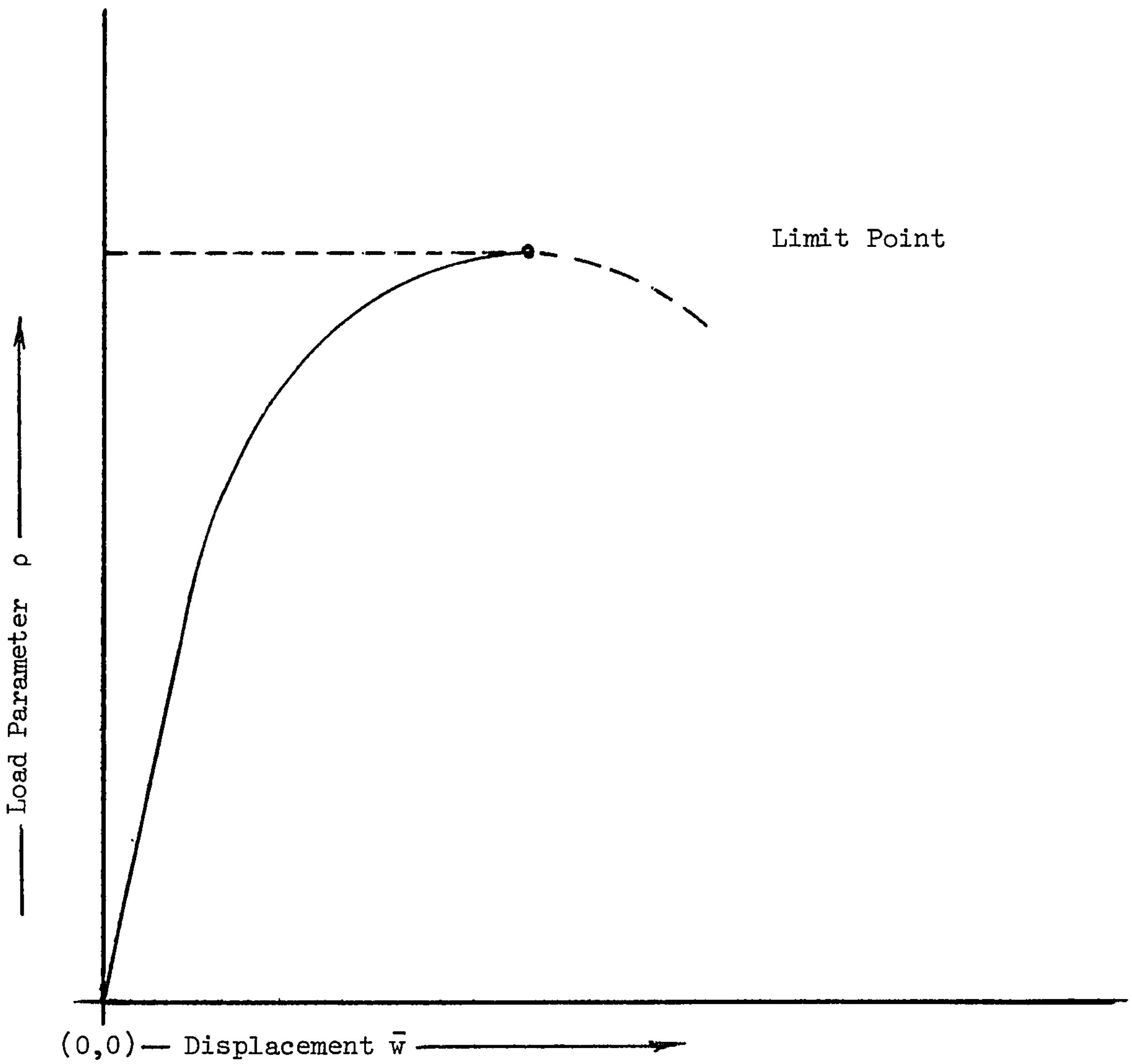
In this investigation the governing equations for deep shells are obtained by applying the general nonlinear theory with respect to strains. The nonlinearity treated is that associated with large displacements in the linear elastic range. The method used is not restricted by the mangitude of the displacements provided that the strains do not exceed the limit of proportionality. The shell is assumed to be clamped at the two longitudinal edges and simply supported at the transverse edges.

The governing equations mentioned above are nonlinear and partial differential equations. These nonlinear partial differential equations are reduced to a set of nonlinear ordinary differential equations by the application of the Kantorovitch method. Further, these nonlinear ordinary differential equations are reduced to a set of nonlinear finite difference equations by conventional methods in terms of central differences. These nonlinear finite difference equations are linearized by incremental technique, and transformed into a suitable matrix form.

An iterative procedure to solve the above mentioned matrix in the incremental form is developed and a computer program is written in Fortran IV to solve these equations using the iterative procedure. Symmetric buckling of deep as well as shallow shells, the effects of imperfections on buckling, and asymmetric buckling are investigated.

1.4 Stability Criteria

1.4.1 Classical Stability. The object of theory of elastic stability is to investigate the states of equilibrium under varying



Heavy lines denote stable equilibrium, dotted lines denote unstable equilibrium

Figure 2

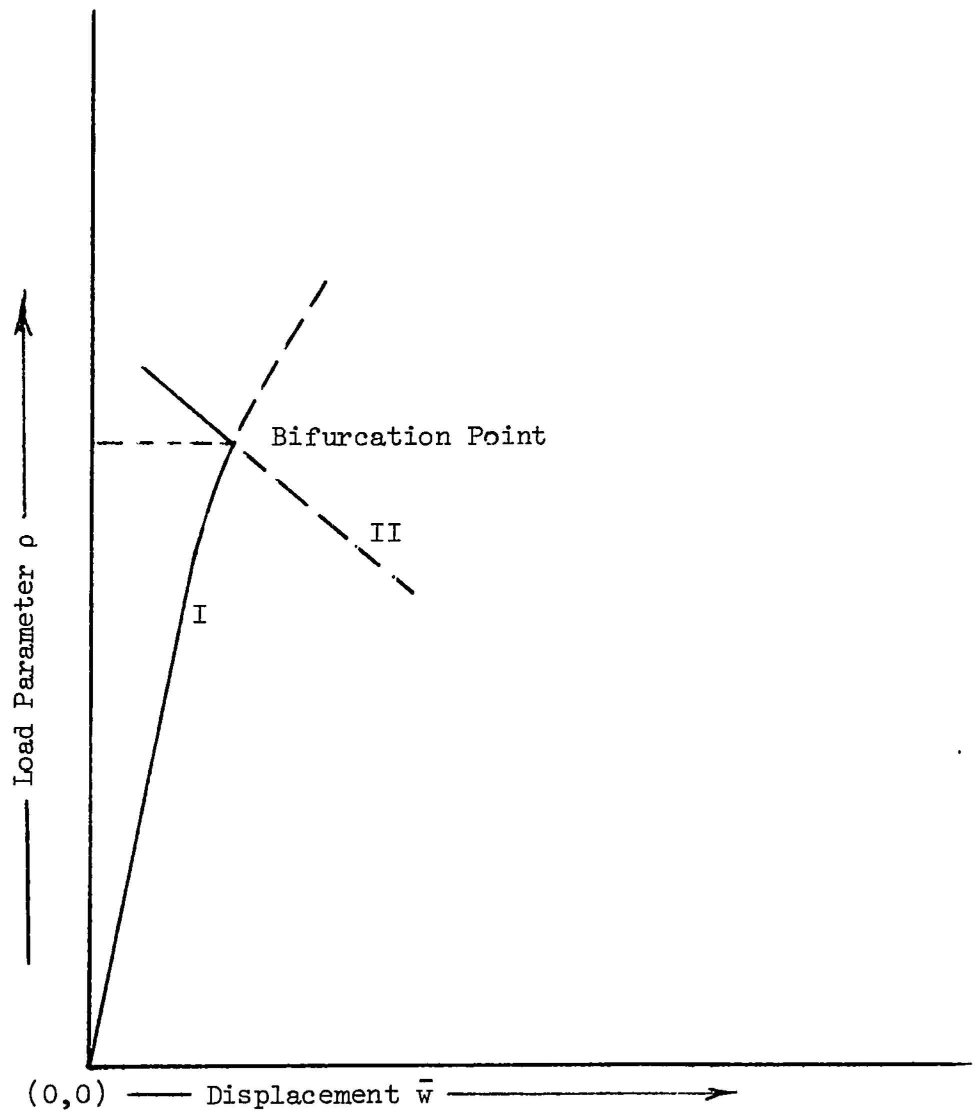
loads. The classical stability analysis in particular investigates the stable equilibrium configurations that develop through successive deformations as shown in Figure [2] from unstrained state, when loads are gradually increased.

Critical state of equilibrium is defined as that equilibrium configuration at which the loss of stability occurs. This critical state of equilibrium can be identified with a point at which further increase of the load may result in a discontinuous change of the configuration.

1.4.2 Bifurcation Point Criterion. The second type of loss of stability that may occur is represented by the concept of "bifurcation point."

According to this concept, it is possible under certain conditions, that the fundamental load deflection curve is intersected by another equilibrium path as shown in Figure [2]. The point at which this occurs is called bifurcation or branching point. The adjacent states for loads in excess of the critical load are stable and those corresponding to loads below the critical value are unstable. In other words, an exchange of stability takes place between the fundamental state and the adjacent state. The state of equilibrium at a bifurcation point in general is unstable.

1.4.3 The Effect of Imperfections on Buckling. No shell can be manufactured to a perfect geometric shape. Therefore, there are bound to be certain deviations from the desired shape which are termed as initial



Heavy lines denote stable equilibrium, dotted lines denote unstable equilibrium.

Figure 3 Bifurcation Point

imperfections. Again these initial imperfections may be different even for the same geometric specifications.

E.H. Dill [7] in order to simplify the problem states that:

> "In theory, if the actual shape is known one could write the equation of the middle surface and from the general equations one could write the differential equations governing the behavior of the shell. However, it will be more simple to derive the equation by supposing that an initially perfect shell is deformed into the imperfect form by deflections $\underline{w}$ of its middle surface but that no stress is caused by this deformation. In this case the total deflection will be $w + \underline{w}$ where w is the deflection due to load. The strain due to load will be found by taking the total strain less the initial strain."

Donnell [6] employs a very ingeneous device to simplify this problem further. Again in the E.H. Dill's [7] language:

> "Precisely the manner of deviation from a perfect shape is not generally known; however, control over the maximum amplitude of the initial imperfection can generally be maintained. It should be assumed then for the purpose of design that any shape may be present initially whose magnitude does not exceed a certain amount. For example, the initial deviations might be expanded in a Fourier series. Then the coefficients of this series should be determined in such a way that the buckling load is minimized subject to the restraint that the amplitude of initial imperfection cannot exceed a given amount. The determination of these coefficients would be a complicated task. Donnell assumes, instead, that the worst shape that could be present would be the shape into which the shell will eventually buckle."

The above technique is used later for the derivations to account for imperfections.

2. DERIVATIONS OF THE DIFFERENTIAL EQUATIONS FOR NONSHALLOW CYLINDRICAL SHELL

2.1 Equilibrium Equations of a Shell Element

Consideration of the equilibrium of a shell element with respect to x and y reference axes as shown in Fig. 4, result into the following equations.

$$
\begin{aligned}
&N_{x,x} + N_{xy,y} - K_x Q_x - K_{xy} Q_y + p_x = 0 \\
&N_{xy,x} + N_{y,y} - K_y Q_y - K_{xy} Q_x + p_y = 0 \\
&M_{x,x} + M_{xy,y} - Q_x = 0 \\
&M_{xy,x} + M_{y,y} - Q_y = 0 \\
&Q_{x,x} + Q_{y,y} + K_x N_x + 2K_{xy} N_{xy} + K_y N_y + p_z = 0
\end{aligned}
\tag{2-1}
$$

where N_x, N_y, N_{xy}, M_x, M_y, M_{xy}, Q_x, Q_y denote forces and moments acting on the element as shown in Fig. 4 and Fig. 5 and K_x, K_y, K_{xy} are curvatures of the deformed middle surface.

Substituting for Q_x and Q_y from third and fourth of equations (2-1) into the remaining three equations, the following equations (2-2) are obtained.

$$
\begin{aligned}
&N_{x,x} + N_{xy,y} - K_x(M_{x,x} + M_{xy,y}) - K_{xy}(M_{xy,x} + M_{y,y}) + p_x = 0 \\
&N_{xy,x} + N_{y,y} - K_y(M_{xy,x} + M_{y,y}) - K_{xy}(M_{x,x} + M_{xy,y}) + p_y = 0 \\
&M_{x,xx} + 2M_{xy,xy} + M_{y,yy} + K_x N_x + 2K_{xy} N_{xy} + K_y N_y + p_z = 0
\end{aligned}
\tag{2-2}
$$

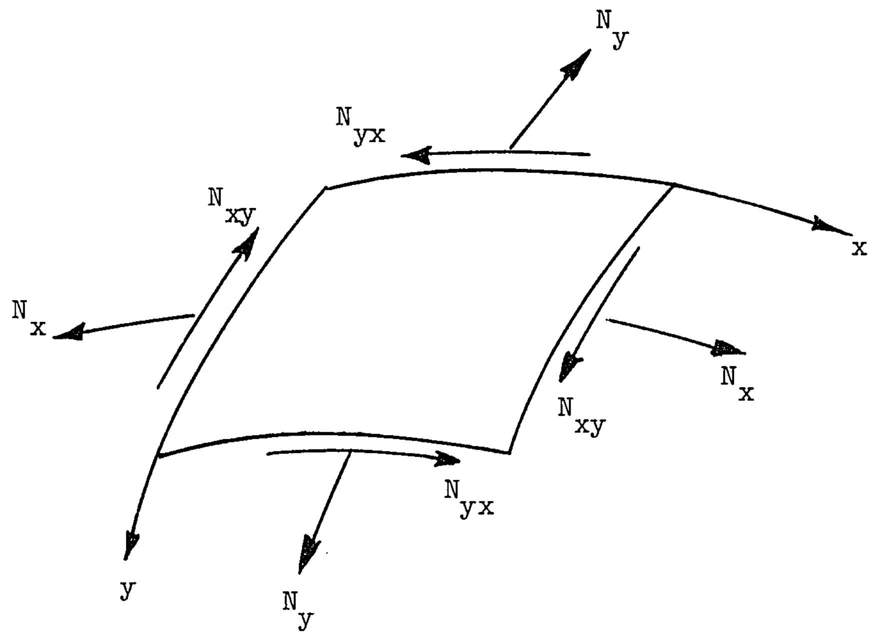


Figure 4 EXTENSIONAL ACTIONS

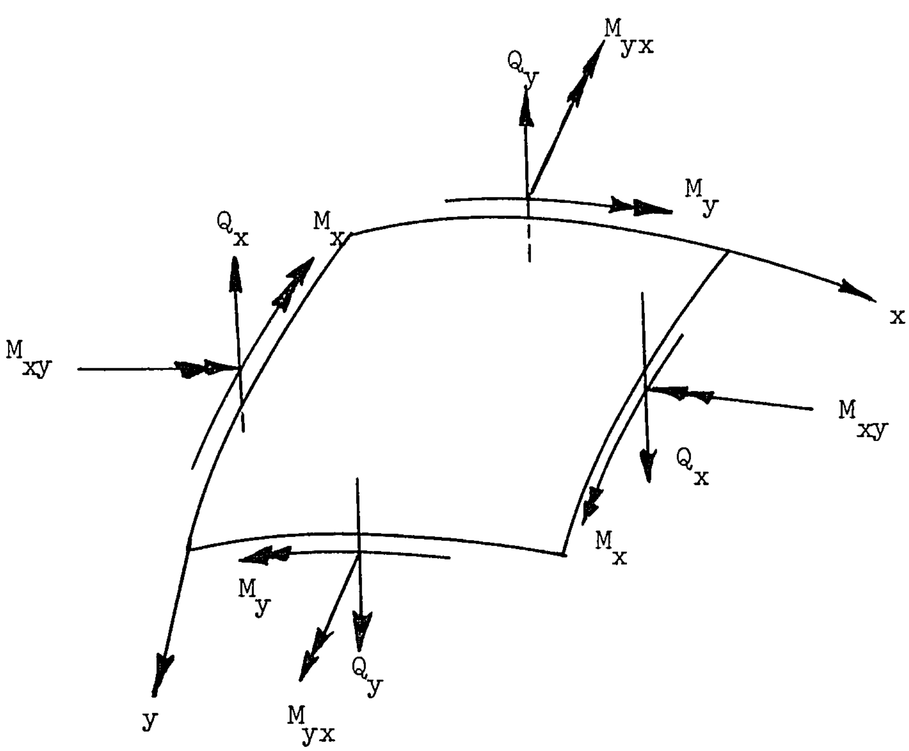


Figure 5 FLEXURAL ACTIONS

2.2 Stress-Strain Relations

If the strains at the middle surface of the shell are denoted by e_x, e_y and e_{xy} and those at a distance z from the middle surface are denoted by e_x^z, e_y^z, e_{xy}^z then the stresses σ_x, σ_y, σ_{xy} in terms of strains are given by the equations,

$$\sigma_x = \frac{E}{1 - \nu^2} (e_x + \nu e_y)$$

$$\sigma_y = \frac{E}{1 - \nu^2} (e_y + \nu e_x) \tag{2-3}$$

$$\sigma_{xy} = \frac{E}{2(1 + \nu)} (e_{xy}).$$

and the strains in terms of stresses are given by the equations,

$$e_x = \frac{1}{E} (\sigma_x - \nu\sigma_y)$$

$$e_y = \frac{1}{E} (\sigma_y - \nu\sigma_x) \tag{2-4}$$

$$e_{xy} = \frac{2(1 + \nu)}{E} (\sigma_{xy})$$

The strains on z surface in terms of the strains on the middle surface are obtained as follows

$$e_x^z = e_x - z\dot{w}_{,xx}$$

$$e_y^z = e_y - z\dot{w}_{,yy} \tag{2-5}$$

$$e_{xy}^z = e_{xy} - 2z\dot{w}_{,xy}$$

Equations (2-5) hold with the assumption of Love's first approximation.

2.3 Strain-Displacement Relations

For the general case of large deflections of shells, including the second degree infinitesimal terms the nonlinear expressions for strains in terms of displacements are given as follows:

$$e_x = u_{,x} - K_x^o w + \frac{1}{2} w_{,x}^2$$

$$e_y = v_{,y} - K_y^o w + \frac{1}{2} w_{,y}^2 \qquad (2\text{-}6)$$

$$e_{xy} = u_{,y} + v_{,x} - 2K_{xy}^o w + w_{,x} w_{,y}$$

where K_x^o, K_y^o, and K_{xy}^o are the corresponding curvatures of the original unstrained surface.

For the shells with initial imperfections, it is assumed that the midsurface of the unloaded shell is displaced radially from the perfect cylinder by w_1. The shape of this initial deflection may be assumed to be proportional to the final deflection w. That is

$$w_1 = kw.$$

With this assumption, the following strain displacement relations may be established.

$$e_x = u_{,x} + \left(\frac{1 + 2k}{2} \right) w_{,x}^2 - K_o w$$

$$e_y = v_{,y} + \left(\frac{1 + 2k}{2} \right) w_{,y}^2 \qquad (2\text{-}7)$$

$$e_{xy} = u_{,y} + v_{,x} + (1 + 2k) w_{,x} w_{,y}$$

2.4 Actions in terms of Strains and Displacements

The expressions of the membrane forces in terms of strains and of the moments in terms of displacements are as follows:

$$N_x = \frac{Et}{1 - \nu^2} \left[e_x + \nu e_y \right]$$

$$N_y = \frac{Et}{1 - \nu^2} \left[e_y + \nu e_x \right]$$

$$N_{xy} = \frac{Et}{2(1 + \nu)} \left[e_{xy} \right] \qquad (2\text{-}8)$$

$$M_x = \frac{- Et^3}{12(1 - \nu^2)} \left[w_{,xx} + \nu w_{,yy} \right]$$

$$M_y = \frac{- Et^3}{12(1 - \nu^2)} \left[w_{,yy} + \nu w_{,xx} \right]$$

$$M_{xy} = \frac{- Et^3}{12(1 + \nu)} \left[w_{,xy} \right]$$

2.5 Differential Equations for Shells

Differentiating each of the equations (2-7) once with respect to x and y, the following equations are obtained:

$$e_{x,x} = u_{,xx} + (1 + 2k)w_{,x}w_{,xx} - K_o w_{,x}$$

$$e_{x,y} = u_{,xy} - K_o w_{,y} + (1 + 2k)w_{,x}w_{,xy}$$

$$e_{y,x} = v_{,yx} + (1 + 2k)w_{,y}w_{,xy} \qquad (2\text{-}9)$$

$$e_{y,y} = v_{,yy} + (1 + 2k)w_{,y}w_{,yy}$$

$$e_{xy,x} = u_{,xy} + v_{,xx} + (1 + 2k)w_{,x}w_{,xy} + (1 + 2k)w_{,y}w_{,xx}$$

$$e_{xy,y} = u_{,yy} + v_{,xy} + (1 + 2k)w_{,x}w_{,yy} + (1 + 2k)w_{,y}w_{,xy}$$

In shallow shells the products like $Q_x K_x$, $Q_y K_y$, $Q_x K_{xy}$, $Q_y K_{xy}$ in the equilibrium equations can be considered of higher order in comparison to the rest of the terms and hence can be neglected. In nonshallow shells however the above terms can not be neglected, which being the case under investigation they shall be retained.

The parametric equations of the shell under consideration are as follows:

$$\overline{X} = x$$

$$\overline{Y} = y \qquad (2\text{-}10)$$

$$\overline{Z} = \frac{1}{2} cx(x - a)$$

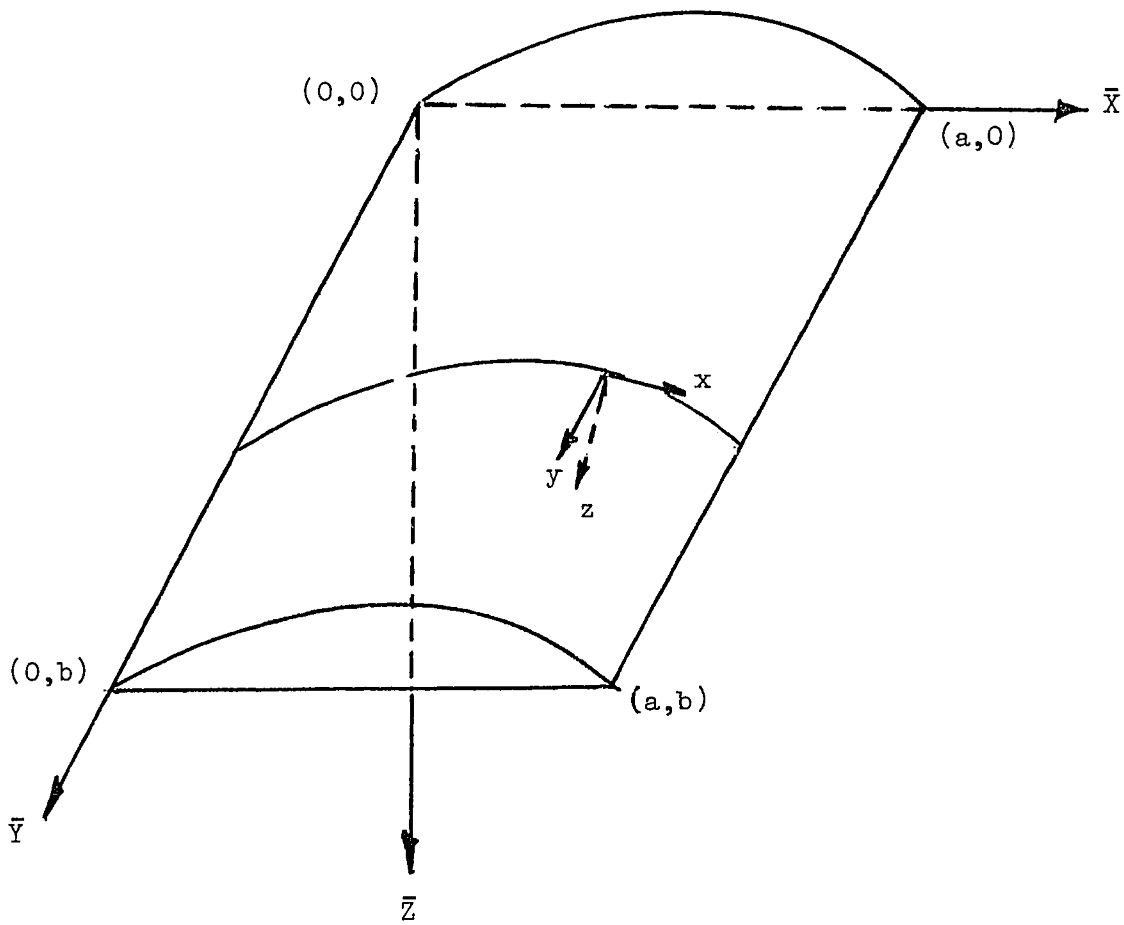


Figure 6 CYLINDRICAL SHELL EXPRESSED BY EQUATION (2-10)

where the parametric lines, as chosen, are also the lines of principal curvature of the surface. Therefore c is the principal curvature of the middle surface.

From these equations (2-9) of the undeformed middle surface the values of curvatures are obtained as follows:

$$K_x^o = Z_{,xx} = c$$

$$K_y^o = Z_{,yy} = 0 \qquad (2\text{-}11)$$

$$K_{xy}^o = Z_{,xy} = 0$$

From the geometry of the deformed middle surface the values of the curvatures are given by

$$K_x = w_{,xx} + c$$

$$K_y = w_{,yy} \qquad (2\text{-}12)$$

$$K_{xy} = w_{,xy}$$

Using the above values (2-11)and (2-12) into the equations (2-9), substituting which into equations (2-2) the following set of partial differential equations in terms of displacements are obtained.

$$\frac{Et}{(1-\nu^2)}\left[\{u_{,xx} - cw_{,x} + (1+2k)w_{,x}w_{,xx}\} + \nu\{v_{,xy} + (1+2k)w_{,y}w_{,xy}\}\right]$$

$$+\frac{Et}{2(1+\nu)}\left[u_{,yy} + v_{,xy} + (1+2k)w_{,x}w_{,yy} + (1+2k)w_{,y}w_{,xy}\right]$$

$$+\{(1+k)w_{,xx} + c\}\left[\frac{Et^3}{12(1-\nu^2)}(w_{,xxx} + w_{,xyy}) + \frac{Et^3}{12(1+\nu)}(w_{,xyy})\right]$$

$$+(1+k)w_{,xy}\left[\frac{Et^3}{12(1+\nu)}w_{,xxy} + \frac{Et^3}{12(1-\nu^2)}\{w_{,yyy}+\nu w_{,xxy}\}\right] + p_x = 0$$

$$\frac{Et}{2(1+\nu)}\left[u_{,xy} + v_{,xx} + (1+2k)w_{,x}w_{,xy} + (1+2k)w_{,y}w_{,xx}\right]$$

$$+\frac{Et}{(1-\nu^2)}\left[v_{,yy} + (1+2k)w_{,y}w_{,yy}\right] \qquad (2\text{-}13)$$

$$+\frac{\nu Et}{(1-\nu^2)}\left[u_{,xy} - cw_{,y} + (1+2k)w_{,x}w_{,xy}\right]$$

$$+(1+k)w_{,yy}\left[\frac{Et^3}{12(1+\nu)}w_{,xxy} + \frac{Et^3}{12(1-\nu^2)}\{w_{,yyy}+\nu w_{,xxy}\}\right]$$

$$+(1+k)w_{,xy}\left[\frac{Et^3}{12(1-\nu^2)}\{w_{,xxx}+w_{,xyy}\} + \frac{Et^3}{12(1+\nu)}w_{,xyy}\right] + p_y = 0$$

$$\frac{-Et^3}{12(1-\nu^2)}\left[w_{,xxxx} + w_{,xxyy}\right] - \frac{+2Et^3}{12(1+\nu)}(w_{,xxyy})$$

$$\frac{-Et^3}{12(1-\nu^2)}\left[w_{,yyyy} + \nu w_{,xxyy}\right]$$

$$+ \frac{Et}{(1-\nu^2)}\left[\{(1+k)w_{,xx} + c\}\{u_{,x} + (\frac{1+2k}{2})w^2_{,x} - cw\} + \nu\{v_{,y} + (\frac{1+2k}{2})w^2_{,y}\}\right]$$

$$+ \frac{(1+k)Et}{(1+\nu)}\{w_{,xy}\}\left[u_{,y} + v_{,x} + (1+2k)w_{,x}w_{,y}\right]$$

$$+ \frac{(1+k)Et}{(1-\nu^2)}\{w_{,yy}\}\left[\{v_{,y} + (\frac{1+2k}{2})w^2_{,y}\} + \nu\{u_{,x} + (\frac{1+2k}{2})w^2_{,x} - cw\}\right]$$

$$+ p_z = 0$$

3. REDUCTION OF GOVERNING EQUATIONS BY THE KANTOROVITCH METHOD

3.1 Simplification of the Equations

The differential equations as derived in article 2-5 are nonlinear fifth order partial differential equations, and represent the mathematical formulation of the problem. Since these equations as such can not be solved by direct methods Kantorovitch method will be applied to these equations in order to reduce them to a set of nonlinear ordinary differential equations.

Dividing all the terms of the equations by E and t, simplifying and rearranging the terms, following set of equations (3-1) is obtained:

$$A_1 u_{,xx} - A_2 w_{,x} + A_3 w_{,x} w_{,xx} + A_4 w_{,y} w_{,xy} + A_5 u_{,yy} + A_6 v_{,xy} + A_7 w_{,x} w_{,yy}$$
$$+ A_8 w_{,xx} w_{,xxx} + A_9 w_{,xx} w_{,xyy} + A_{10} w_{,xyy} + A_{11} w_{,xxx} + A_{12} w_{,xy} w_{,xxy}$$
$$+ A_{13} w_{,xy} w_{,yyy} + \frac{p_x}{Et} = 0 \qquad (3\text{-}1)$$

$$B_1 u_{,xy} + B_2 v_{,xx} + B_3 w_{,x} w_{,xy} + B_4 w_{,y} w_{,xx} + B_5 v_{,yy} + B_6 w_{,y} w_{,yy} - B_7 w_{,y}$$
$$+ B_8 w_{,yy} w_{,xxy} + B_9 w_{,yy} w_{,yyy} + B_{10} w_{,xy} w_{,xxx} + B_{11} w_{,xy} w_{xyy} + \frac{p_y}{Et} = 0$$

$$C_1 w_{,xxyy} + C_2 w_{,xxxx} + C_3 w_{,yyyy} + C_4 u_{,x} w_{,xx} + C_5 w_{,xx} w^2_{,x} + C_6 w_{,xx} w$$
$$+ C_7 w_{,x} + C_8 w^2_{,x} + C_9 w + C_{10} w_{,xx} v_{,y} + C_{11} w_{,xx} w^2_{,y} + C_{12} v_{,y} + C_{13} w^2_{,y}$$
$$+ C_{14} w_{,xy} u_{,y} + C_{15} w_{,xy} v_{,x} + C_{16} w_{,x} w_{,y} w_{,xy} + C_{17} w_{,yy} v_{,y} + C_{18} w_{,yy} w^2_{,y}$$
$$+ C_{19} w_{,yy} u_{,x} + C_{20} w_{,yy} w^2_{,x} + C_{21} w_{,yy} w + \frac{p_z}{Et} = 0 \qquad (3\text{-}1)$$

Where following parameters which depend on the material properties and the geometry of the shell are defined and substituted.

$$A_1 = \frac{1}{1 - \nu^2} \qquad A_2 = \frac{C}{1 - \nu^2}$$

$$A_3 = \frac{(1 + 2k)}{1 - \nu^2} \qquad A_4 = \frac{\nu(1 + 2k)}{1 - \nu^2} + \frac{(1 + 2k)}{2(1 + \nu)}$$

$$A_5 = \frac{1}{2(1 + \nu)} \qquad A_6 = \frac{\nu}{1 - \nu^2} + \frac{1}{2(1 + \nu)}$$

$$A_7 = \frac{(1 + 2k)}{2(1 + \nu)} \qquad A_8 = \frac{(1 + k)t^2}{12(1 - \nu^2)}$$

$$A_9 = \frac{\nu(1 + k)t^2}{12(1 - \nu^2)} + \frac{(1 + k)t^2}{12(1 + \nu)}$$

$$A_{10} = \frac{C\nu t^2}{12(1 - \nu^2)} + \frac{Ct^2}{12(1 + \nu)}$$

$$A_{11} = \frac{Ct^2}{12(1 - \nu^2)}$$

$$A_{12} = \frac{(1 + k)t^2}{12(1 + \nu)} + \frac{\nu(1 + k)t^2}{12(1 - \nu^2)}$$

$$A_{13} = \frac{(1 + k)t^2}{12(1 - \nu^2)} ;$$

$$B_1 = \frac{1}{1 + \nu} + \frac{\nu}{(1 - \nu^2)} \qquad B_2 = \frac{1}{1 + \nu}$$

$$B_3 = \frac{(1 + 2k)}{(1 + \nu)} + \frac{\nu(1 + 2k)}{(1 - \nu^2)} \qquad B_4 = \frac{(1 + 2k)}{1 + \nu}$$

$$B_5 = \frac{1}{(1 - \nu^2)} \qquad B_6 = \frac{(1 + 2k)}{(1 - \nu^2)}$$

$$B_7 = \frac{- C\nu}{(1 - \nu^2)} \qquad B_8 = \frac{(1 + k)t^2}{12(1 + \nu)} + \frac{\nu(1 + k)t^2}{12(1 - \nu^2)}$$

$$B_9 = \frac{(1 + k)t^2}{12(1 - \nu^2)} \qquad B_{10} = \frac{(1 + k)t^2}{12(1 - \nu^2)}$$

$$B_{11} = \frac{\nu(1 + k)t^2}{12(1 - \nu^2)} + \frac{(1 + k)t^2}{12(1 + \nu)}$$

and

$$C_1 = \frac{-\nu t^2}{6(1 - \nu^2)} - \frac{t^2}{6(1 + \nu)} \qquad C_2 = \frac{- t^2}{12(1 - \nu^2)}$$

$$C_3 = \frac{- t^2}{12(1 - \nu^2)} \qquad C_4 = \frac{1 + k}{(1 - \nu^2)}$$

$$C_5 = \frac{(1 + k)(1 + 2k)}{2(1 - \nu^2)} \qquad C_6 = \frac{- C(1 + k)}{(1 - \nu^2)}$$

$$C_7 = \frac{C}{(1 - \nu^2)} \qquad C_8 = \frac{C(1 + 2k)}{2(1 - \nu^2)}$$

$$C_9 = \frac{- C^2}{(1 - \nu^2)} \qquad C_{10} = \frac{\nu(1 + k)}{(1 - \nu^2)}$$

$$C_{11} = \frac{(1 + k)(1 + 2k)\nu}{2(1 - \nu^2)} \qquad C_{12} = \frac{\nu C}{(1 - \nu^2)}$$

$$C_{13} = \frac{C\nu(1 + 2k)}{2(1 - \nu^2)} \qquad C_{14} = \frac{2(1 + k)}{(1 + \nu)}$$

$$C_{15} = \frac{2(1 + k)}{(1 + \nu)} \qquad C_{16} = \frac{2(1 + k)(1 + 2k)}{(1 + \nu)}$$

$$C_{17} = \frac{(1 + k)}{(1 - \nu^2)} \qquad C_{18} = \frac{(1 + k)(1 + 2k)}{2(1 - \nu^2)}$$

$$C_{19} = \frac{\nu(1 + k)}{(1 - \nu^2)} \qquad C_{20} = \frac{\nu(1 + k)(1 + 2k)}{2(1 - \nu^2)}$$

$$C_{21} = \frac{- C\nu(1 + k)}{(1 - \nu^2)} ,$$

It may be observed in the above equations that, parameters A_1 to A_7, B_1 to B_7 and C_1 to C_{21} represent the shallow shell parameters.

3.2 Approximating Functions

Let the solutions of the equations (3-1) be represented by the approximate set of finite series as follows:

$$u = \sum_n f_n \sin(\frac{n\pi y}{b})$$

$$v = \sum_n g_n \sin(\frac{n\pi y}{b}) \qquad (3\text{-}2)$$

$$w = \sum_n h_n \sin(\frac{n\pi y}{b})$$

where the functions f_n, g_n and h_n are functions of x only and $n = 1, 2, 3, \ldots N$. The unknown functions f_n, g_n and h_n are to be determined so that they satisfy the governing equations (3-1) along with the boundary conditions to be set up.

Differentiating the approximating functions given by (3-2) , with respect to x and y as required by the equations (3-1) and substituting in the same the following equations (3-3) are obtained.

$$A_1 \sum_n f''_n \sin(\frac{n\pi y}{b})$$

$$- A_2 \sum_n h'_n \sin(\frac{n\pi y}{b})$$

$$+ A_3 \sum_m \sum_n h'_m h'_n \sin(\frac{m\pi y}{b}) \sin(\frac{n\pi y}{b}) \qquad (3\text{-}3)$$

$$+ A_4 \sum_m \sum_n (\frac{mn\pi^2}{b^2}) h'_m h_n \cos(\frac{m\pi y}{b}) \cos(\frac{n\pi y}{b})$$

$$+ A_5 \sum_m (\frac{n\pi}{b})^2 f_n \sin(\frac{n\pi y}{b})$$

$$+ A_6 \sum_n (\frac{n\pi}{b}) g'_n \cos(\frac{n\pi y}{b})$$

$$- A_7 \sum_m \sum_n h'_m h_n \left(\frac{n\pi}{b}\right)^2 \sin\left(\frac{m\pi y}{b}\right) \sin\left(\frac{n\pi y}{b}\right)$$

$$+ A_8 \sum_m \sum_n h''_m h'''_n \sin\left(\frac{m\pi y}{b}\right) \sin\left(\frac{n\pi y}{b}\right)$$

$$- A_9 \sum_m \sum_n h''_m h'_n \left(\frac{n\pi}{b}\right)^2 \sin\left(\frac{m\pi y}{b}\right) \sin\left(\frac{n\pi y}{b}\right)$$

$$- A_{10} \sum_n h'_n \left(\frac{n\pi}{b}\right)^2 \sin\left(\frac{n\pi y}{b}\right)$$

$$+ A_{11} \sum_n h'''_n \sin\left(\frac{n\pi y}{b}\right)$$

$$+ A_{12} \sum_m \sum_n \left(\frac{mn\pi^2}{b^2}\right) h'_m h''_n \cos\left(\frac{m\pi y}{b}\right) \cos\left(\frac{n\pi y}{b}\right) \tag{3-3}$$

$$- A_{13} \sum_m \sum_n \left(\frac{mn^3\pi^4}{b^4}\right) h'_m h_n \cos\left(\frac{m\pi y}{b}\right) \cos\left(\frac{n\pi y}{b}\right) + \frac{p_x}{Et} = 0$$

$$B_1 \sum_n \left(\frac{n\pi}{b}\right) f'_n \cos\left(\frac{n\pi y}{b}\right)$$

$$+ B_2 \sum_n g''_n \sin\left(\frac{n\pi y}{b}\right)$$

$$+ B_3 \sum_m \sum_n \left(\frac{n\pi}{b}\right) h'_m h'_n \sin\left(\frac{m\pi y}{b}\right) \cos\left(\frac{n\pi y}{b}\right)$$

$$+ B_4 \sum_m \sum_n \left(\frac{n\pi}{b}\right) h''_m h_n \sin\left(\frac{m\pi y}{b}\right) \cos\left(\frac{n\pi y}{b}\right)$$

$$- B_5 \sum_n g_n \left(\frac{n\pi}{b}\right)^2 \sin\left(\frac{n\pi y}{b}\right)$$

$$- B_6 \sum_m \sum_n \left(\frac{mn^2\pi^3}{b^3}\right) h_m h_n \cos\left(\frac{m\pi y}{b}\right) \sin\left(\frac{n\pi y}{b}\right)$$

$$- B_7 \sum_n \left(\frac{n\pi}{b}\right) h_n \cos\left(\frac{n\pi y}{b}\right)$$

$$- B_8 \sum_m \sum_n \left(\frac{mn^2\pi^3}{b^3}\right) h''_m h_n \cos\left(\frac{m\pi y}{b}\right) \sin\left(\frac{n\pi y}{b}\right)$$

$$+ B_9 \sum_m \sum_n \left(\frac{m^2 n^3 \pi^5}{b^5}\right) h_m h_n \sin\left(\frac{m\pi y}{b}\right) \cos\left(\frac{n\pi y}{b}\right)$$

$$+ B_{10} \sum_m \sum_n \left(\frac{m\pi}{b}\right) n'_m h'''_n \cos\left(\frac{m\pi y}{b}\right) \sin\left(\frac{n\pi y}{b}\right)$$

$$- B_{11} \sum_m \sum_n \left(\frac{mn^2\pi^3}{b^3}\right) h'_m h'_n \cos\left(\frac{m\pi y}{b}\right) \sin\left(\frac{n\pi y}{b}\right) + \frac{p_y}{Et} = 0 \tag{3-3}$$

$$- C_1 \sum_n h''_n \left(\frac{n\pi}{b}\right)^2 \sin\left(\frac{n\pi y}{b}\right)$$

$$+ C_2 \sum_n h_n^{IV} \sin\left(\frac{n\pi y}{b}\right)$$

$$+ C_3 \sum_n \left(\frac{n\pi}{b}\right)^4 h_n \sin\left(\frac{n\pi y}{b}\right)$$

$$+ C_4 \sum_m \sum_n f'_m h''_n \sin\left(\frac{m\pi y}{b}\right) \sin\left(\frac{n\pi y}{b}\right)$$

$$+ C_5 \sum_m \sum_n \sum_p h'_m h'_n h''_p \sin\left(\frac{m\pi y}{b}\right) \sin\left(\frac{n\pi y}{b}\right) \sin\left(\frac{p\pi y}{b}\right)$$

$$+ C_6 \sum_m \sum_n h''_m h_n \sin\left(\frac{m\pi y}{b}\right) \sin\left(\frac{n\pi y}{b}\right)$$

$$+ C_7 \sum_n f'_n \sin\left(\frac{n\pi y}{b}\right)$$

$$+ C_8 \sum_m \sum_n h'_m h'_n \sin(\frac{m\pi y}{b}) \sin(\frac{n\pi y}{b})$$

$$+ C_9 \sum_n h_n \sin(\frac{n\pi y}{b})$$

$$+ C_{10} \sum_m \sum_n (\frac{n\pi}{b}) h''_m g_n \sin(\frac{m\pi y}{b}) \cos(\frac{n\pi y}{b})$$

$$+ C_{11} \sum_m \sum_n \sum_p (\frac{mn\pi^2}{b^2}) h_m h_n h''_p \cos(\frac{m\pi y}{b}) \cos(\frac{n\pi y}{b}) \sin(\frac{p\pi y}{b})$$

$$+ C_{12} \sum_n (\frac{n\pi}{b}) g_n \cos(\frac{n\pi y}{b})$$

$$+ C_{13} \sum_m \sum_n (\frac{mn\pi^2}{b^2}) h_m h_n \cos(\frac{m\pi y}{b}) \cos(\frac{n\pi y}{b})$$

$$+ C_{14} \sum_m \sum_n (\frac{mn\pi^2}{b^2}) h'_m f_n \cos(\frac{m\pi y}{b}) \cos(\frac{n\pi y}{b}) \qquad (3\text{-}3)$$

$$+ C_{15} \sum_m \sum_n (\frac{m\pi}{b}) h'_m g'_n \cos(\frac{m\pi y}{b}) \sin(\frac{n\pi y}{b})$$

$$+ C_{16} \sum_m \sum_n \sum_p (\frac{mn\pi^2}{b^2}) h'_m h_n h'_p \cos(\frac{m\pi y}{b}) \cos(\frac{n\pi y}{b}) \sin(\frac{p\pi y}{b})$$

$$- C_{17} \sum_m \sum_n (\frac{mn^2\pi^3}{b^3}) g_m h_n \cos(\frac{m\pi y}{b}) \sin(\frac{n\pi y}{b})$$

$$- C_{18} \sum_m \sum_n \sum_p (\frac{mnp^2\pi^4}{b^4}) h_m h_n h_p \cos(\frac{m\pi y}{b}) \cos(\frac{n\pi y}{b}) \sin(\frac{n\pi y}{b})$$

$$- C_{19} \sum_m \sum_n f'_m h_n (\frac{n\pi}{b})^2 \sin(\frac{m\pi y}{b}) \sin(\frac{n\pi y}{b})$$

$$- C_{20} \sum_m \sum_n \sum_p h'_m h'_n h_p (\frac{p\pi}{b})^2 \sin(\frac{m\pi y}{b}) \sin(\frac{n\pi y}{b}) \sin(\frac{p\pi y}{b})$$

$$- C_{21} \sum_m \sum_n h_m h_n (\frac{n\pi}{b})^2 \sin(\frac{m\pi y}{b}) \sin(\frac{n\pi y}{b}) + \frac{p_z}{Et} = 0 \qquad (3\text{-}3)$$

where m = 1, 2, 3,.....N.

3.3 Transformation to Ordinary Differential Equations

The partial differential equations (3-3) are transformed into a system of ordinary simultaneous equations by the Kantorovitch method. Each of the governing equations after the substitution by the approximating functions, is multiplied successively by each of the j functions sin $(\frac{j\pi y}{b})$, for j = 1, 2, 3,,.....N, and integrated with respect to y from 0 to b. And the governing equations resulted into a system of simultaneous ordinary differential equations. These equations are obtained as follows:

As originally stated the object of this work was the investigation of a thin cylindrical shell subject to uniform external normal pressure. And therefore, for the normal pressure p,

$$p_x = 0$$

$$p_y = 0 \qquad (3\text{-}4)$$

$$
\begin{aligned}
& A_1 \sum_n f''_n Q_1 - A_2 \sum_n h'_n Q_1 + A_2 \sum_m \sum_n h'_m h'_n Q_{11} + A_4 \sum_m \sum_n \left(\frac{mn\pi^2}{b^2}\right) h'_m h_n Q_{22} \\
& - A_5 \sum_n \left(\frac{n\pi}{b}\right)^2 f_n Q_1 - A_7 \sum_m \sum_n h'_m h_n \left(\frac{n\pi}{b}\right)^2 Q_{11} + A_8 \sum_m \sum_n h''_m h'''_n Q_{11} \\
& - A_9 \sum_m \sum_n h''_m h'_n \left(\frac{n\pi}{b}\right)^2 Q_{11} - A_{10} \sum_n h'_n \left(\frac{n\pi}{b}\right)^2 Q_1 + A_{11} \sum_n h'''_n Q_1 \\
& - A_{12} \sum_m \sum_n \left(\frac{mn\pi^2}{b^2}\right) h'_m h''_n Q_{22} - A_{13} \sum_m \sum_n \left(\frac{mn^3\pi^4}{b^4}\right) h'_m h_n Q_{22} = 0
\end{aligned}
\tag{3-5}
$$

$$
\begin{aligned}
& B_1 \sum_n \left(\frac{n\pi}{b}\right) f'_n Q_2 + B_2 \sum_n g''_n Q_1 + B_3 \sum_m \sum_n \left(\frac{n\pi}{b}\right) h'_m h'_n Q_{12} + B_4 \sum_m \sum_n \left(\frac{n\pi}{b}\right) h''_m h_n Q_{12} \\
& - B_5 \sum_n g_n \left(\frac{n\pi}{b}\right)^2 Q_1 - B_6 \sum_m \sum_n \left(\frac{mn^2\pi^3}{b^3}\right) h_m h_n Q_{21} - B_7 \sum_n \left(\frac{n\pi}{b}\right) h_n Q_2 \\
& - B_8 \sum_m \sum_n \left(\frac{mn^2\pi^3}{b^3}\right) h''_m h_n Q_{21} + B_9 \sum_m \sum_n \left(\frac{m^2n^3\pi^5}{b^5}\right) h_m h_n Q_{12} \\
& + B_{10} \sum_m \sum_n \left(\frac{m\pi}{b}\right) h'_m h'''_n Q_{21} - B_{11} \sum_m \sum_n \left(\frac{mn^2\pi^3}{b^3}\right) h'_m h'_n Q_{21} = 0
\end{aligned}
\tag{3-5}
$$

$$
\begin{aligned}
& - C_1 \sum_n h''_n \left(\frac{n\pi}{b}\right)^2 Q_1 + C_2 \sum_n h^{IV}_n Q_1 + C_3 \sum_n \left(\frac{n\pi}{b}\right)^4 h_n Q_1 + C_4 \sum_m \sum_n f'_m h''_n Q_{11} \\
& + C_5 \sum_m \sum_n \sum_p h'_m h'_n h''_p Q_{111} + C_6 \sum_m \sum_n h''_m h_n Q_{11} + C_7 \sum_n f'_n Q_1 + C_8 \sum_m \sum_n h'_m h'_n Q_{11} \\
& + C_9 \sum_n h_n Q_1 + C_{11} \sum_m \sum_n \sum_p \left(\frac{mn\pi^2}{b^2}\right) h_m h_n h''_p Q_{221} + C_{13} \sum_m \sum_n \left(\frac{mn\pi^2}{b^2}\right) h_m h_n Q_{22}
\end{aligned}
$$

$$+ C_{14} \sum_m \sum_n \left(\frac{mn\pi^2}{b^2}\right) h'_m f_n Q_{22} + C_{16} \sum_m \sum_n \sum_p \left(\frac{mn\pi^2}{b^2}\right) h'_m h_n h'_p Q_{221}$$

$$- C_{18} \sum_m \sum_n \sum_p \left(\frac{mnp^2\pi^4}{b^4}\right) h_m h_n h_p Q_{221} - C_{19} \sum_m \sum_n f'_m h_n \left(\frac{n\pi}{b}\right)^2 Q_{11} \qquad (3\text{-}5)$$

$$- C_{20} \sum_m \sum_n \sum_p h'_m h'_n h_p \left(\frac{p\pi}{b}\right)^2 Q_{111} - C_{21} \sum_m \sum_n h_m h_n \left(\frac{n\pi}{b}\right)^2 Q_{11} + \frac{p_z}{Et} Q = 0$$

where Q, Q_1, Q_2, Q_{11}, Q_{22}, Q_{12}, Q_{21}, Q_{111} , Q_{221} etc. are the definite integrals of the products of trigonometric functions derived in the Appendix A at the end.

4. NUMERICAL FORMULATION FOR SYMMETRIC BUCKLING

4.1 Symmetric Buckling Considerations

It may be observed that the definite integral Q is equal to zero when j is an even number, whereas it exists when j is an odd number. As a result, therefore, the last term of the last of equations (3-5) exists only if j is an odd number, while if j is an even number this term vanishes. Therefore, the only values that j, and hence m, n and p can take for symmetric buckling are 1, 3, 5,....N, where N is an odd integer.

Due to this restriction on m, n, p and j of taking only the odd number values for symmetric buckling the definite integrals Q_2^{jn}, Q_{12}^{jmn} and Q_{21}^{jmn} disappear. Also the definite integral

$$Q_1^{jn} = 0 \quad \text{when} \quad n^2 \neq j^2,$$

$$= \frac{b}{2} \quad \text{when } n = j.$$

This results into simplification of equations (3-4). Second of the equations (3-4) is considerably simplified and takes the following form:

$$B_2 \sum_n g_n'' Q_1^{jn} - B_5 \sum_n g_n \left(\frac{n\pi}{b}\right)^2 Q_1^{jn} = 0 \qquad (4\text{-}1)$$

where the function g_n has uncoupled itself from the rest of the functions.

As the object of this investigation is the study of the variation of the radial deflection w by varying the external pressure p, the equations of concern are the remaining ones. These equations are further reduced to finite difference equations as expalined in the next article.

4.2 Finite-Difference Equations

The equations noted in the previous article constitute a system of ordinary, nonlinear, differential equations containing six unknowns. The governing differential equations are replaced by corresponding finite difference equations. This then reduces the problem to a set of nonlinear simultaneous algebraic equations confining the range of the independent variables to a network of meshpoints in the direction of x.

Let h be the mesh size, chosen in such a way that x is represented as follows:

$x = ih$, where $i = 0, 1, 2, 3, \ldots M$.

Let $f_{ni} = f_n(ih)$ and $h_{ni} = h_n(ih)$

To transform the differential equations to difference equations the following formulae in terms of central differences are employed.

$$f' = \frac{(-f_{i-1} + f_{i+1})}{2h}$$

$$f'' = \frac{(f_{i-1} - 2f_i + f_{i+1})}{h^2} \qquad (4\text{-}2)$$

$$f''' = \frac{(-f_{i-2} + 2f_{i-1} - 2f_{i+1} + f_{i+2})}{2h3}$$

$$f^{IV} = \frac{(f_{i-2} - 4f_{i-1} + 6f_i - 4f_{i+1} + f_{i+2})}{h^4}$$

Similarly the difference equations are also obtained for the function h_n. Substituting the above difference equations into equation (3-5), and simplifying along with as noted in Article 4.1. The following equations (4-3) and (4-4) are obtained.

$$P_1 h_{ni-2} + P_2^j h_{ni-1} + P_3^j h_{ni+1} + P_4 h_{ni+2} + P_5 f_{ni-1} + P_6^j f_{ni} + P_7 f_{ni+1}$$

$$+ P_8 \sum_m \sum_n (-h_{mi-1} + h_{mi+1})(h_{ni-1} - 2h_{ni} + h_{ni+1}) Q_{11}^{jmn}$$

$$+ P_9 \sum_m \sum_n mn(-h_{mi-1} + h_{mi+1})(h_{ni}) Q_{22}^{jmn} \qquad (4\text{-}3)$$

$$- P_{10} \sum_m \sum_n n^2(-h_{mi-1} + h_{mi+1})(h_{ni}) Q_{11}^{jmn}$$

$$+ P_{11} \sum_m \sum_n (h_{mi-1} - 2h_{mi} + h_{mi+1})(-h_{ni-2} + 2h_{ni-1} - 2h_{ni+1} + h_{ni+2}) Q_{11}^{jmn}$$

$$- P_{12} \sum_m \sum_n n^2(h_{mi-1} - 2h_{mi} + h_{mi+1})(-h_{ni-1} + h_{ni+1}) Q_{11}^{jmn}$$

$$+ P_{13} \sum_m \sum_n mn(-h_{mi-1} + h_{mi+1})(h_{ni-1} - 2h_{ni} + h_{ni+1})Q_{22}^{jmn}$$

$$- P_{14} \sum_{mn}\sum mn^3(-h_{mi-1} + h_{mi+1})(h_{ni})Q_{22}^{jmn} = 0 \qquad (4\text{-}3)$$

$$R_1 h_{ni-2} + R_2^j h_{ni-1} + R_3^j h_{ni} + R_4^j h_{ni+1} + R_5 h_{ni+2} + R_{18}(-f_{ni-1} + f_{ni+1})$$

$$+ R_6 \sum_m \sum_n (-f_{mi-1} + f_{mi+1})(h_{ni-1} - 2h_{ni} + h_{ni+1})Q_{11}^{jmn}$$

$$+ R_7 \sum_m \sum_n (h_{mi-1} - 2h_{mi} + h_{mi+1})(h_{ni})Q_{11}^{jmn}$$

$$+ R_8 \sum_m \sum_n (-h_{mi-1} + h_{mi+1})(-h_{ni-1} + h_{ni+1})Q_{11}^{jmn} \qquad (4\text{-}4)$$

$$+ R_9 \sum_m \sum_n (mnh_{mi}h_{ni}Q_{22}^{jmn} + R_{10} \sum_m \sum_n mn(-h_{mi-1} + h_{mi+1})f_{ni}Q_{22}^{jmn}$$

$$+ R_{11} \sum_m \sum_n n^2(-f_{mi-1} + f_{mi+1})h_{ni}Q_{11}^{jmn} + R_{12} \sum_m \sum_n n^2 h_{mi}h_{ni}Q_{11}^{jmn}$$

$$+ R_{13} \sum_m \sum_n \sum_p (-h_{mi-1} + h_{mi+1})(-h_{ni-1} + h_{ni+1})(h_{pi-1} - 2h_{pi} + h_{pi+1})Q_{111}^{jmn}$$

$$+ R_{14} \sum_m \sum_n \sum_p mnh_{mi}h_{ni}(h_{pi-1} - 2h_{pi} + h_{pi+1})Q_{221}^{jmnp}$$

$$+ R_{15} \sum_m \sum_n \sum_p mn(-h_{mi-1} + h_{mi+1})h_{ni}(-h_{pi-1} + h_{pi+1})Q_{221}^{jmnp}$$

$$- R_{16} \sum_m \sum_n \sum_p mnp^2 h_{mi}h_{ni}h_{pi}Q_{221}^{jmnp}$$

$$+ R_{17} \sum_m \sum_n \sum_p p^2(-h_{mi-1} + h_{mi+1})(-h_{ni-1} + h_{ni+1})h_{pi}Q_{111}^{jmnp} = - R_{19}^j \frac{p_z}{Et}$$

Each of the above equations represents three equations for j = 1,3,5, a total of six equations. And each of the double summation terms represents a total of nine terms where as each of the triple summation terms represents a total of twenty-seven terms. The summation terms are the combinations of m, n, p = 1, 3, 5

These equations have been arranged setting up linear terms at the beginning, followed by nonlinear terms successively in increasing order, with the following parameters defined and substituted.

$$P_1 = \frac{-A_{11}b}{4h^3}, \qquad P_2^j = \frac{A_2 b}{4h} + \frac{A_{10} j^2 \pi^2}{4bh} + \frac{A_{11}b}{2h^3}$$

$$P_3^j = \frac{-A_2 b}{4h} - \frac{A_{10} j^2 \pi^2}{4bh} - \frac{A_{11}b}{2h^3}, \qquad P_4 = \frac{A_{11}b}{4h^3}$$

$$P_5 = \frac{A_1 b}{2h^2}, \qquad P_6^j = \frac{-A_1 b}{h^2} - \frac{A_5 j^2 \pi^2}{2b}, \qquad P_7 = \frac{A_1 b}{2b^2} \tag{4-5}$$

$$P_8 = \frac{A_3}{4h^2}, \qquad P_9 = \frac{A_4 \pi^2}{2hb^2}, \qquad P_{10} = \frac{-A_7 \pi^2}{2b^2 h}$$

$$P_{11} = \frac{A_8}{2h^5}, \qquad P_{12} = \frac{A_9 \pi^2}{2h^3 b^2}, \qquad P_{13} = \frac{A_{12} \pi^2}{2b^2 h^3}, \qquad P_{14} = \frac{A_{13} \pi^4}{2b^4 h}$$

and

$$R_1 = \frac{C_2 b}{2h^4}, \qquad R_2^j = \frac{-C_1 j^2 \pi^2}{2bh^2} - \frac{4C_2 b}{2h^4}$$

$$R_3^j = \frac{C_1 j^2 \pi^2}{bh^2} + \frac{6C_2 b}{2h^4} + \frac{C_3 \pi^4 j^4}{2b^3} + \frac{C_9 b}{2} \tag{4-6}$$

$$R_4^j = \frac{-C_1 j^2 \pi^2}{2bh^2} - \frac{4C_2 b}{2h^4}, \quad R_5 = \frac{C_2 b}{2h^4}, \quad R_6 = \frac{C_4}{2h^3}, \quad R_7 = \frac{C_6}{h^2}$$

$$R_8 = \frac{C_8}{4h^2}, \quad R_9 = \frac{C_{13}\pi^2}{b^2}, \quad R_{10} = \frac{C_{14}\pi^2}{2b^2 h}, \quad R_{11} = \frac{C_{19}\pi^2}{2b^2 h}$$

$$R_{12} = \frac{C_{21}\pi^2}{b^2}, \quad R_{13} = \frac{C_5}{4h^4}, \quad R_{14} = \frac{C_{11}\pi^2}{b^2 h^2}, \quad R_{15} = \frac{C_{16}\pi^2}{4b^2 h^2} \qquad (4\text{-}6)$$

$$R_{16} = \frac{C_{18}\pi^4}{b^4}, \quad R_{17} = \frac{C_{20}\pi^2}{4b^2 h^2}, \quad R_{18} = \frac{C_7 b}{4h}, \quad R_{19} = \frac{-2}{j}$$

4.3 Linearization of Algebraic Equations

Equations (4-3) and (4-4) represent a system of nonlinear, simultaneous, algebraic equations with the six unknown functions f_{1i}, f_{3i}, f_{5i} and h_{1i}, h_{3i}, h_{5i}. The solution of this system determines the above unknown functions at each point i.

Linearization of the above system is achieved by incremental method. According to this method an increment is given to the external pressure and the corresponding increments of the functions are determined in terms of the values of the functions at the previous point. Thus by incrementing the pressure successively and determining the corresponding increments of the functions the curve between the variables can be found. This method requires that the values of the functions be at least known at a certain point. Such a point for the case under investigation is zero pressure - zero deflection point, which is taken as the origin of the reference axes for the pressure-deflection curve.

In order to set up the above equations (4-3) and (4-4) in the incremental form, let Df and Dh be the increments in the functions f and h caused by an increment in the external pressure Dp_z. After the increment the corresponding values of f, h and p_z are f + DF, h + Dh, and $p_z + Dp_z$. Substituting these values into equations (4-3) and (4-4), and subtracting from which the original equations (4-3) and (4-4), the following set of equations is obtained.

$$P_1 Dh_{ji-2} + P_2^j Dh_{ji-1} + P_3^j Dh_{ji+1} + P_4 Dh_{ji+2} + P_5 Df_{ji-1} + P_6^j Df_{ji} + P_7 Df_{ji+1}$$

$$+ P_8 \sum_m \sum_n (-Dh_{mi-1} + Dh_{mi+1})(h_{ni-1} - 2h_{ni} + h_{ni+1}) Q_{11}^{jmn}$$

$$+ P_8 \sum_m \sum_n (-h_{mi-1} + h_{mi+1})(Dh_{ni-1} - 2Dh_{ni} + Dh_{ni+1}) Q_{11}^{jmn}$$

$$+ P_8 \sum_m \sum_n (-Dh_{mi-1} + Dh_{mi+1})(Dh_{ni-1} - 2Dh_{ni} + Dh_{ni+1}) Q_{11}^{jmn} \qquad (4\text{-}7)$$

$$+ P_9 \sum_m \sum_n m(-Dh_{mi-1} + Dh_{mi+1}) n(h_{ni}) Q_{22}^{jmn}$$

$$+ P_9 \sum_m \sum_n m(-h_{mi-1} + h_{mi+1}) n(Dh_{ni}) Q_{22}^{jmn}$$

$$+ P_9 \sum_m \sum_n m(-Dh_{mi-1} + Dh_{mi+1}) n(Dh_{ni}) Q_{22}^{jmn}$$

$$- P_{10} \sum_m \sum_n (-Dh_{mi-1} + Dh_{mi+1}) n^2 (h_{ni}) Q_{11}^{jmn}$$

$$- P_{10} \sum_m \sum_n (-h_{mi-1} + h_{mi+1}) n^2 (Dh_{ni}) Q_{11}^{jmn}$$

$$- P_{10} \sum_m \sum_n (-Dh_{mi-1} + Dh_{mi+1}) n^2 (Dh_{ni}) Q_{11}^{jmn}$$

$$+ P_{11} \sum_m \sum_n (Dh_{mi-1} - 2Dh_{mi} + Dh_{mi+1})(-h_{ni-2} + 2h_{ni+1} + h_{ni+2} - 2h_{ni+1}) Q_{11}^{jmn}$$

$$+ P_{11} \sum_m \sum_n (h_{mi-1} - 2h_{mi} + h_{mi+1})(-Dh_{ni-2} + 2Dh_{ni-1} - 2Dh_{ni+1} + Dh_{ni+2}) Q_{11}^{jmn}$$

$$+ P_{11} \sum_m \sum_n (Dh_{mi-1} - 2Dh_{mi} + Dh_{mi+1})(Dh_{ni-2} + 2Dh_{ni-1} - 2D_{ni+1} + Dh_{ni+2}) Q_{11}^{jmn}$$

$$- P_{12} \sum_m \sum_n (Dh_{mi-1} - 2Dh_{mi} + Dh_{mi+1}) n^2 (-h_{ni-1} + h_{ni+1}) Q_{11}^{jmn}$$

$$- P_{12} \sum_m \sum_n (h_{mi-1} - 2h_{mi} + h_{mi+1}) n^2 (-Dh_{ni-1} + Dh_{ni+1}) Q_{11}^{jmn} \qquad (4\text{-}7)$$

$$- P_{12} \sum_m \sum_n (Dh_{mi-1} - 2Dh_{mi} + Dh_{mi+1}) n^2 (-Dh_{ni-1} + Dh_{ni+1}) Q_{11}^{jmn}$$

$$+ P_{13} \sum_m \sum_n m(-Dh_{mi-1} + Dh_{mi+1}) n (h_{ni-1} - 2h_{ni} + h_{ni+1}) Q_{22}^{jmn}$$

$$+ P_{13} \sum_m \sum_n m(-h_{mi-1} + h_{mi+1}) n (Dh_{ni-1} - 2Dh_{ni} + Dh_{ni+1}) Q_{22}^{jmn}$$

$$+ P_{13} \sum_m \sum_n m(-Dh_{mi-1} + Dh_{mi+1}) n (Dh_{ni-1} - 2Dh_{ni} + Dh_{ni+1}) Q_{22}^{jmn}$$

$$- P_{14} \sum_m \sum_n m(-Dh_{mi-1} + Dh_{mi+1})n^3(Dh_{ni})Q_{22}^{jmn}$$

$$- P_{14} \sum_m \sum_n m(-Dh_{mi-1} + Dh_{mi+1})n^3(h_{ni})Q_{22}^{jmn} \qquad (4\text{-}7)$$

$$- P_{14} \sum_m \sum_n m(-h_{mi-1} + h_{mi+1})n^3(Dh_{ni})Q_{22}^{jmn} = 0$$

$$R_1 Dh_{ji-2} + R_2^j Dh_{ji-1} + R_3^j Dh_{ji} + R_4^j Dh_{ji+1} + R_5 Dh_{ji+2} + R_{18}(-Df_{ji-1} + Df_{ji+1})$$

$$+ R_6 \sum_m \sum_n (-Df_{mi-1} + Df_{mi+1})(h_{ni-1} - 2h_{ni} + h_{ni+1})Q_{11}^{jmn}$$

$$+ R_6 \sum_m \sum_n (-f_{mi-1} + f_{mi+1})(Dh_{ni-1} - 2Dh_{ni} + Dh_{ni+1})Q_{11}^{jmn}$$

$$+ R_6 \sum_m \sum_n (-Df_{mi-1} + Df_{mi+1})(Dh_{ni-1} - 2Dh_{ni} + Dh_{ni+1})Q_{11}^{jmn} \qquad (4\text{-}8)$$

$$+ R_7 \sum_m \sum_n (Dh_{mi-1} - 2Dh_{mi} + Dh_{mi+1})(h_{ni})Q_{11}^{jmn}$$

$$+ R_7 \sum_m \sum_n (h_{mi-1} - 2h_{mi} + h_{mi+1})(Dh_{ni})Q_{11}^{jmn}$$

$$+ R_7 \sum_m \sum_n (Dh_{mi-1} - 2Dh_{mi} + Dh_{mi+1})(Dh_{ni})Q_{11}^{jmn}$$

$$+ R_8 \sum_m \sum_n (-Dh_{mi-1} + Dh_{mi+1})(-h_{ni-1} + h_{ni+1})Q_{11}^{jmn}$$

$$+ R_8 \sum_m \sum_n (-h_{mi-1} + h_{mi+1})(-Dh_{ni-1} + Dh_{ni+1})Q_{11}^{jmn}$$

$$+ R_8 \sum_m \sum_n (-Dh_{mi-1} + Dh_{mi+1})(-Dh_{ni-1} + Dh_{ni+1})Q_{11}^{jmn}$$

$$+ R_9 \sum_m \sum_n mn(Dh_{mi})(h_{ni})Q_{22}^{jmn}$$

$$+ R_9 \sum_m \sum_n mn(h_{mi})(Dh_{ni})Q_{22}^{jmn}$$

$$+ R_9 \sum_m \sum_n mn(Dh_{mi})(Dh_{ni})Q_{22}^{jmn}$$

$$+ R_{10} \sum_m \sum_n mn(-Dh_{mi-1} + Dh_{mi+1})(f_{ni})Q_{22}^{jmn}$$

$$+ R_{10} \sum_m \sum_n mn(-h_{mi-1} + h_{mi+1})Df_{ni}\ Q_{22}^{jmn}$$

$$+ R_{10} \sum_m \sum_n mn(-Dh_{mi-1} + Dh_{mi+1})(Df_{ni})Q_{22}^{jmn}$$

$$- R_{11} \sum_m \sum_n n^2(-Df_{mi-1} + Df_{mi+1})h_{ni}Q_{11}^{jmn}$$

$$- R_{11} \sum_m \sum_n n^2(-f_{mi-1} + f_{mi+1})(Dh_{ni})Q_{11}^{jmn} \qquad (4\text{-}8)$$

$$- R_{11} \sum_m \sum_n n^2(-Df_{mi-1} + Df_{mi+1})(Dh_{ni})Q_{11}^{jmn}$$

$$- R_{12} \sum_m \sum_n n^2(Dh_{mi})(h_{ni})Q_{11}^{jmn}$$

$$- R_{12} \sum_m \sum_n n^2(h_{mi})(Dh_{ni})Q_{11}^{jmn}$$

$$- R_{12} \sum_m \sum_n n^2(Dh_{mi})(Dh_{ni})Q_{11}^{jmn}$$

$$+ R_{13} \sum_m \sum_n \sum_p (-Dh_{mi-1} + Dh_{mi+1})(-h_{ni-1} + h_{ni+1})(h_{pi-1} - 2h_{pi} + h_{pi+1})Q_{111}^{jmnp}$$

$$+ R_{13} \sum_m \sum_n \sum_p (-h_{mi-1} + h_{mi+1})(-Dh_{ni-1} + Dh_{ni+1})(h_{pi-1} - 2h_{pi} + h_{pi+1})Q_{111}^{jmnp}$$

$$+ R_{13} \sum_m \sum_n \sum_p (-h_{mi-1} + h_{mi+1})(-h_{ni-1} + h_{ni+1})(Dh_{pi-1} - 2Dh_{pi} + Dh_{pi+1})Q_{111}^{jmnp}$$

$$+ R_{13} \sum_m \sum_n \sum_p (-Dh_{mi-1} + Dh_{mi+1})(-Dh_{ni-1} + Dh_{ni+1})(h_{pi-1} - 2h_{pi} + h_{pi+1})Q_{111}^{jmnp}$$

$$+ R_{13} \sum_m \sum_n \sum_p (-h_{mi-1} + h_{mi+1})(-Dh_{ni-1} + Dh_{ni+1})(Dh_{pi-1} - @Dh_{pi} + Dh_{pi+1})Q_{111}^{jmnp}$$

$$+ R_{13} \sum_m \sum_n \sum_p (-Dh_{mi-1} + Dh_{mi+1})(-h_{ni-1} + h_{ni+1})(Dh_{pi-1} - 2Dh_{pi} + Dh_{pi+1})Q_{111}^{jmnp}$$

$$+ R_{13} \sum_m \sum_n \sum_p (-Dh_{mi-1} + Dh_{mi+1})(-Dh_{ni-1} + Dh_{ni+1})(Dh_{pi-1} - 2Dh_{pi} + Dh_{pi+1})Q_{111}^{jmnp}$$

$$+ R_{14} \sum_m \sum_n \sum_p mn(Dh_{mi})(h_{ni})(h_{pi-1} - 2h_{pi} + h_{pi+1})Q_{221}^{jmnp} \qquad (4\text{-}8)$$

$$+ R_{14} \sum_m \sum_n \sum_p mn(h_{mi})(Dh_{ni})(h_{pi-1} - 2h_{pi} + h_{pi+1})Q_{221}^{jmnp}$$

$$+ R_{14} \sum_m \sum_n \sum_p mn(h_{mi})(h_{ni})(Dh_{pi-1} - 2Dh_{pi} + Dh_{pi+1})Q_{221}^{jmnp}$$

$$+ R_{14} \sum_m \sum_n \sum_p mn(Dh_{mi})(Dh_{ni})(h_{pi-1} - 2h_{pi} + h_{pi+1})Q_{221}^{jmnp}$$

$$+ R_{14} \sum_m \sum_n \sum_p mn(h_{mi})(Dh_{ni})(Dh_{pi-1} - 2Dh_{pi} + Dh_{pi+1})Q_{221}^{jmnp}$$

$$+ R_{14} \sum_m \sum_n \sum_p mn(Dh_{mi})h_{ni} (Dh_{pi-1} - 2Dh_{pi} + Dh_{pi+1})Q_{221}^{jmnp}$$

$$+ R_{14} \sum_m \sum_n \sum_p mn(Dh_{mi})(Dh_{ni})(Dh_{pi-1} - 2Dh_{pi} + Dh_{pi+1})Q_{221}^{jmnp}$$

$$+ R_{15} \sum_m \sum_n \sum_p mn(-Dh_{mi-1} + Dh_{mi+1})(h_{ni})(-h_{pi-1} + h_{pi+1})Q_{221}^{jmnp}$$

$$+ R_{15} \sum_m \sum_n \sum_p mn(-h_{mi-1} + h_{mi+1})(Dh_{ni})(-h_{pi-1} + h_{pi+1})Q_{221}^{jmnp}$$

$$+ R_{15} \sum_m \sum_n \sum_p mn(-h_{mi-1} + h_{mi+1})(h_{ni})(-Dh_{pi-1} + Dh_{pi+1})Q_{221}^{jmnp}$$

$$+ R_{15} \sum_m \sum_n \sum_p mn(-Dh_{mi-1} + Dh_{mi+1})(dh_{ni})(-h_{pi-1} + h_{pi+1})Q_{221}^{jmnp}$$

$$+ R_{15} \sum_m \sum_n \sum_p mn(-Dh_{mi-1} + Dh_{mi+1})(h_{ni})(-Dh_{pi-1} + Dh_{pi+1})Q_{221}^{jmnp} \quad (4\text{-}8)$$

$$+ R_{15} \sum_m \sum_n \sum_p mn(-h_{mi-1} + h_{mi+1})(Dh_{ni})(-Dh_{pi-1} + Dh_{pi+1})Q_{221}^{jmnp}$$

$$+ R_{15} \sum_m \sum_n \sum_p mn(-Dh_{mi-1} + Dh_{mi+1})(Dh_{ni})(-Dh_{pi-1} + Dh_{pi+1})Q_{221}^{jmnp}$$

$$- R_{16} \sum_m \sum_n \sum_p mnp^2(Dh_{mi})(h_{ni})(h_{pi})Q_{221}^{jmnp}$$

$$- R_{16} \sum_m \sum_n \sum_p mnp^2(h_{mi})(Dh_{ni})(h_{pi})Q_{221}^{jmnp}$$

$$- R_{16} \sum_m \sum_n \sum_p mnp^2(h_{mi})(h_{ni})(Dh_{pi})Q_{221}^{jmnp}$$

$$- R_{16} \sum_m \sum_n \sum_p mnp^2(Dh_{mi})(Dh_{ni})(h_{pi})Q_{221}^{jmnp}$$

$$- R_{16} \sum_m \sum_n \sum_p mnp^2(h_{mi})(Dh_{ni})(Dh_{pi})Q_{221}^{jmnp}$$

$$- R_{16} \sum_m \sum_n \sum_p mnp^2(Dh_{mi})(h_{ni})(Dh_{pi})Q_{221}^{jmnp}$$

$$- R_{16} \sum_m \sum_n \sum_p mnp^2(Dh_{mi})(Dh_{ni})(Dh_{pi})Q_{221}^{jmnp}$$

$$- R_{17} \sum_m \sum_n \sum_p p^2(-Dh_{mi-1} + Dh_{mi+1})(-h_{ni-1} + h_{ni+1})(h_{pi})Q_{111}^{jmnp}$$

$$- R_{17} \sum_m \sum_n \sum_p p^2(-h_{mi-1} + h_{mi+1})(-Dh_{ni-1} + Dh_{ni+1})(h_{pi})Q_{111}^{jmnp}$$

$$- R_{17} \sum_m \sum_n \sum_p p^2(-h_{mi-1} + h_{mi+1})(-h_{ni-1} + h_{ni+1})(Dh_{pi})Q_{111}^{jmnp}$$

$$- R_{17} \sum_m \sum_n \sum_p p^2(-Dh_{mi-1} + Dh_{mi+1})(-Dh_{ni-1} + Dh_{ni+1})(h_{pi})Q_{111}^{jmnp}$$

$$- R_{17} \sum_m \sum_n \sum_p p^2(-h_{mi-1} + h_{mi+1})(-Dh_{mi-1} + Dh_{ni+1})(Dh_{pi})Q_{111}^{jmnp}$$

$$- R_{17} \sum_m \sum_n \sum_p p^2(-Dh_{mi-1} + Dh_{mi+1})(-h_{ni-1} + h_{ni+1})(Dh_{pi})Q_{111}^{jmnp}$$

$$- R_{17} \sum_m \sum_n \sum_p p^2(-Dh_{mi-1} + Dh_{mi+1})(-Dh_{ni-1} + Dh_{ni+1})(Dh_{pi})Q_{111}^{jmnp} = - R_{19}^{j} \frac{Dp_z}{Et}$$

In the above system of equations (4-7) and (4-8) nonlinear terms exist with respect to the increments Df and Dh. These terms are of higher order in comparison to the other terms and hence their contribution is very small. These terms are treated as equivalent loads.

These equations if solved for a known increment of pressure, determine the values of the unknown increments of functions f and h at any point i.

4.4 Boundary Conditions

As already mentioned the shell under investigation is clamped at the two boundaries parallel to the y axis and is simply supported on the other two sides. Hence the boundary conditions are expressed as follows:

at $x = 0$, and $x = a$,

$$u = 0, \; v = 0, \; w = 0, \quad \text{and } w_{,x} = 0 \tag{4-9}$$

at $y = 0$, and $y = b$,

$$u = 0, \; v = 0, \; w = 0, \quad \text{and } w_{,yy} = 0 \tag{4-10}$$

It may be observed that the conditions expressed by equation (4-10) are identically satisfied as the functions chosen to represent the displacements are as given by equation (3-2).

As the mathematical formulation of the problem is in terms of displacements, conditions(4-9) yield the following equation.

at $x = 0$, and $x = a$

$$f_j = 0, \quad g_j = 0, \quad h_j = 0 \tag{4-11}$$

and $w_{,x} = \sum_j h_j' \sin\left(\frac{j\pi y}{b} \right) = 0$

This equation may be satisfied by letting

$$h'_j = 0 \qquad (4\text{-}12)$$

In terms of central differences this means

$$h'_j = \frac{-h_{i-1} + h_{i+1}}{2h} = 0 \qquad (4\text{-}13)$$

and in incremental form this means

$$-Dh_{i-1} + Dh_{i+1} = 0 \qquad (4\text{-}13)$$

5. MATRIX FORMULATION

5.1 Summation Terms

In order to set the system of simultaneous algebraic equations (4-7) and (4-8) in the matrix form the unknown increments Df and Dh are expanded for m, n, p = 1, 3, 5 in the summation terms. The values of the functions f and h are considered to be known at the point immediately before the increments and hence are treated as known coefficients.

Let

$$\sum_n (-h_{ni-1} + h_{ni+1})Q_{11}^{j1n} = (S1)_i^j$$

$$\sum_n (-h_{ni-1} + h_{ni+1})Q_{11}^{j3n} = (S2)_i^j$$

$$\sum_n (-h_{ni-1} + h_{ni+1})Q_{11}^{j5n} = (S3)_i^j$$

$$\sum_n n(h_{ni})Q_{22}^{j1n} = (S4)_i^j \qquad (5\text{-}1)$$

$$\sum_n n(h_{ni})Q_{22}^{j3n} = (S5)_i^j$$

$$\sum_n n(h_{ni})Q_{22}^{j5n} = (S6)_i^j$$

$$\sum_m m(-h_{mi-1} + h_{mi+1})Q_{22}^{jm1} = (S7)_i^j$$

$$\sum_m m(-h_{mi-1} + h_{mi+1})Q_{22}^{jm3} = (S8)_i^j$$

$$\sum_m m(-h_{mi-1} + h_{mi+1})Q_{22}^{jm5} = (S9)_i^j$$

$$\sum_m n^2(h_{ni})Q_{11}^{j1n} = (S10)_i^j$$

$$\sum_n n^2(h_{ni})Q_{11}^{j3n} = (S11)_i^j$$

$$\sum_n n^2(h_{ni})Q_{11}^{j5n} = (S12)_i^j$$

$$\sum_n (-h_{ni-2} + 2h_{ni-1} - 2h_{ni+1} + h_{ni+2})Q_{11}^{j1n} = (S13)_i^j$$

$$\sum_n (-h_{ni-2} + 2h_{ni-1} - 2h_{ni+1} + h_{ni+2})Q_{11}^{j3n} = (S14)_i^j$$

$$\sum_n (-h_{ni-2} + 2h_{ni-1} - 2h_{ni+1} + h_{ni+2})Q_{11}^{j5n} = (S15)_i^j \qquad (5\text{-}1)$$

$$\sum_m (h_{mi-1} - 2h_{mi} + h_{mi+1})Q_{11}^{jm1} = (S16)_i^j$$

$$\sum_m (h_{mi-1} - 2h_{mi} + h_{mi+1})Q_{11}^{jm3} = (S17)_i^j$$

$$\sum_m (h_{mi-1} - 2h_{mi} + h_{mi+1})Q_{11}^{jm5} = (S18)_i^j$$

$$\sum_n n^2(-h_{ni-1} + h_{ni+1})Q_{11}^{j1n} = (S19)_i^j$$

$$\sum_n n^2(-h_{ni-1} + h_{ni+1})Q_{11}^{j3n} = (S20)_i^j$$

$$\sum_n n^2(-h_{ni-1} + h_{ni+1})Q_{11}^{j5n} = (S21)_i^j$$

$$\sum_n n(h_{ni-1} - 2h_{ni} + h_{ni+1})Q_{22}^{j1n} = (S22)_i^j$$

$$\sum_n n(h_{ni-1} - 2h_{ni} + h_{ni+1})Q_{22}^{j3n} = (S23)_i^j$$

$$\sum_n n(h_{ni-1} - 2h_{ni} + h_{ni+1})Q_{22}^{j5n} = (S24)_i^j$$

$$\sum_n n^3(h_{ni})Q_{22}^{j1n} = (S25)_i^j$$

$$\sum_n n^3(h_{ni})Q_{22}^{j3n} = (S26)_i^j$$

$$\sum_n n^3(h_{ni})Q_{22}^{j5n} = (S27)_i^j$$

(5-1)

$$\sum_m (-f_{mi-1} + f_{mi+1})Q_{11}^{jm1} = (S28)_i^j$$

$$\sum_m (-f_{mi-1} + f_{mi+1})Q_{11}^{jm3} = (S29)_i^j$$

$$\sum_m (-f_{mi-1} + f_{mi+1})Q_{11}^{jm5} = (S30)_i^j$$

$$\sum_n (h_{ni})Q_{11}^{j1n} = (S31)_i^j$$

$$\sum_n (h_{ni})Q_{11}^{j3n} = (S32)_i^j$$

$$\sum_n (h_{ni})Q_{11}^{j5n} = (S33)_i^j$$

$$\sum_n n(h_{ni})Q_{22}^{j1n} = (S34)_i^j$$

$$\sum_n n(h_{ni})Q_{22}^{j3n} = (S35)_i^j$$

$$\sum_n n(h_{ni})Q_{22}^{j5n} = (S36)_i^j$$

$$\sum_n n(f_{ni})Q_{22}^{j1n} = (S37)_i^j$$

$$\sum_n n(f_{ni})Q_{22}^{j3n} = (S38)_i^j$$

$$\sum_n n(f_{ni})Q_{22}^{j5n} = (S39)_i^j \qquad (5\text{-}1)$$

$$\sum_n \sum_p (-h_{ni-1} + h_{ni+1})(h_{pi-1} - 2h_{pi} + h_{pi+1})Q_{111}^{j1np} = (S40)_i^j$$

$$\sum_n \sum_p (-h_{ni-1} + h_{ni+1})(h_{pi-1} - 2h_{pi} + h_{pi+1})Q_{111}^{j3np} = (S41)_i^j$$

$$\sum_n \sum_p (-h_{ni-1} + h_{ni+1})(h_{pi-1} - 2h_{pi} + h_{pi+1})Q_{111}^{j5np} = (S42)_i^j$$

$$\sum_m \sum_n (-h_{mi-1} + h_{mi+1})(-h_{ni-1} + h_{ni+1})Q_{111}^{jmn1} = (S43)_i^j$$

$$\sum_m \sum_n (-h_{mi-1} + h_{mi+1})(-h_{ni-1} + h_{ni+1})Q_{111}^{jmn3} = (S44)_i^j$$

$$\sum_m \sum_n (-h_{mi-1} + h_{mi+1})(-h_{ni-1} + h_{ni+1})Q_{111}^{jmn5} = (S45)_i^j$$

$$\sum_n \sum_p n(h_{ni})(h_{pi-1} - 2h_{pi} + h_{pi+1})Q_{221}^{j1np} = (S46)_i^j$$

$$\sum_n \sum_p n(h_{ni})(h_{pi-1} - 2h_{pi} + h_{pi+1})Q_{221}^{j3np} = (S47)_i^j$$

$$\sum_n \sum_p n(h_{ni})(h_{pi-1} - 2h_{pi} + h_{pi+1})Q_{221}^{j5np} = (S48)_i^j$$

$$\sum_m \sum_n mn(h_{mi})(h_{ni})Q_{221}^{jmn1} = (S49)_i^j$$

$$\sum_m \sum_n mn(h_{mi})h_{ni}\ Q_{221}^{jmn3} = (S50)_i^j$$

$$\sum_m \sum_n mn(h_{mi})(h_{ni})Q_{221}^{jmn5} = (S51)_i^j \qquad (5\text{-}1)$$

$$\sum_n \sum_p n(h_{ni})(-h_{pi-1} + h_{pi+1})Q_{221}^{j1np} = (S52)_i^j$$

$$\sum_n \sum_p n(h_{ni})(-h_{pi-1} + h_{pi+1})Q_{221}^{j3np} = (S53)_i^j$$

$$\sum_n \sum_p n(h_{ni})(-h_{pi-1} + h_{pi+1})Q_{221}^{j5np} = (S54)_i^j$$

$$\sum_m \sum_p m(-h_{mi-1} + h_{mi+1})(-h_{pi-1} + h_{pi+1})Q_{221}^{jm1p} = (S55)_i^j$$

$$\sum_m \sum_p m(-h_{mi-1} + h_{mi+1})(-h_{pi-1} + h_{pi+1})Q_{221}^{jm3p} = (S56)_i^j$$

$$\sum_m \sum_p m(-h_{mi-1} + h_{mi+1})(-h_{pi-1} + h_{pi+1})Q_{221}^{jm5p} = (S57)_i^j$$

$$\sum_m \sum_n mn(-h_{mi-1} + h_{mi+1})(h_{ni})Q_{221}^{jmn1} = (S58)_i^j$$

$$\sum_m \sum_n mn(-h_{mi-1} + h_{mi+1})(h_{ni})Q_{221}^{jmn3} = (S59)_i^j$$

$$\sum_m \sum_n mn(-h_{mi-1} + h_{mi+1})(h_{ni})Q_{221}^{jmn5} = (S60)_i^j$$

$$\sum_n \sum_p np^2(h_{ni})(h_{pi})Q_{221}^{j1np} = (S61)_i^j \qquad (5\text{-}1)$$

$$\sum_n \sum_p np^2(h_{ni})(h_{pi})Q_{221}^{j3np} = (S62)_i^j$$

$$\sum_n \sum_p np^2(h_{ni})(h_{pi})Q_{221}^{j5np} = (S63)_i^j$$

$$\sum_n \sum_p p^2(-h_{ni-1} + h_{ni+1})(h_{pi})Q_{111}^{j1np} = (S64)_i^j$$

$$\sum_n \sum_p p^2(-h_{ni-1} + h_{ni+1})(h_{pi})Q_{111}^{j3np} = (S65)_i^j$$

$$\sum_n \sum_p p^2(-h_{ni-1} + h_{ni+1})(h_{pi})Q_{111}^{j5np} = (S66)_i^j$$

The above summations have to be evaluated for $j = 1, 3, 5$, at each point i.

The nonlinear terms of higher order are designated as follows:

Let:

$$+ P_8 \sum_m \sum_n (-Dh_{mi-1} + Dh_{mi+1})(Dh_{ni-1} - 2Dh_{ni} + Dh_{ni+1})Q_{111}^{jmn} = (DS1)_i^j$$

$$P_9 \sum_m \sum_n m(-Dh_{mi-1} + Dh_{mi+1})n(Dh_{ni})Q_{22}^{jmn} = (DS2)_i^j$$

$$P_{10} \sum_m \sum_n (-Dh_{mi-1} + Dh_{mi+1})n^2(Dh_{ni})Q_{11}^{jmn} = (DS3)_i^j$$

$$P_{11} \sum_m \sum_n (Dh_{mi-1} - 2Dh_{mi} + Dh_{mi+1})(Dh_{ni-2} + 2Dh_{ni-1} - 2Dh_{ni+1} + Dh_{ni+2})Q_{11}^{jmn} = (DS4)_i^j$$

$$P_{12} \sum_m \sum_n (Dh_{mi-1} - 2Dh_{mi} + Dh_{mi+1})n^2(-Dh_{ni-1} + Dh_{ni+1})Q_{11}^{jmn} = (DS5)_i^j$$

$$P_{13} \sum_m \sum_n m(-Dh_{mi-1} + Dh_{mi+1})n(Dh_{ni-1} - 2Dh_{ni} + Dh_{ni+1})Q_{22}^{jmn} = (DS6)_i^j$$

$$P_{14} \sum_m \sum_n m(-Dh_{mi-1} + Dh_{mi+1})n^3(Dh_{ni})Q_{22}^{jmn} = (DS7)_i^j \qquad (5\text{-}2)$$

$$R_6 \sum_m \sum_n (-Df_{mi-1} + Df_{mi+1})(Dh_{ni-1} - 2Dh_{ni} + Dh_{ni+1})Q_{11}^{jmn} = (DS8)_i^j$$

$$R_7 \sum_m \sum_n (Dh_{mi-1} - 2Dh_{mi} + Dh_{mi+1})(Dh_{ni})Q_{11}^{jmn} = (DS9)_i^j$$

$$R_8 \sum_m \sum_n (-Dh_{mi-1} + Dh_{mi+1})(-Dh_{ni-1} + Dh_{ni+1})Q_{11}^{jmn} = (DS10)_i^j$$

$$R_9 \sum_m \sum_n (Dh_{mi})(Dh_{ni})Q_{22}^{jmn} = (DS11)_i^j$$

$$R_{10} \sum_m \sum_n mn(-Dh_{mi-1} + Dh_{mi+1})(Df_{ni})Q_{22}^{jmn} = (DS12)_i^j$$

$$R_{11} \sum_m \sum_n n^2(-Df_{mi-1} + Df_{mi+1})(Dh_{ni})Q_{11}^{jmn} = (DS13)_i^j$$

$$R_{12} \sum_m \sum_n n^2(Dh_{mi})(Dh_{ni})Q_{11}^{jmn} = (DS14)_i^j$$

$$R_{13} \sum_m \sum_n \sum_p (-Dh_{mi-1} + Dh_{mi+1})(-Dh_{ni-1} + Dh_{ni+1})(Dh_{pi-1} - 2Dh_{pi} + Dh_{pi+1}) Q_{111}^{jmnp} = (DS15)_i^j$$

$$R_{14} \sum_m \sum_n \sum_p mn(Dh_{mi})(Dh_{ni})(Dh_{pi-1} - 2Dh_{pi} + Dh_{pi+1})Q_{221}^{jmnp} = (DS16)_i^j$$

$$R_{15} \sum_m \sum_n \sum_p mn(-Dh_{mi-1} + Dh_{mi+1})(Dh_{ni})(-Dh_{pi-1} + Dh_{pi+1})Q_{221}^{jmnp} = (DS17)_i^j$$

$$R_{16} \sum_m \sum_n \sum_p mnp^2(Dh_{mi})(Dh_{ni})(Dh_{pi})Q_{221}^{jmnp} = (DS18)_i^j \qquad (5\text{-}3)$$

$$R_{17} \sum_m \sum_n \sum_p p^2(-Dh_{mi-1} + Dh_{mi+1})(-Dh_{ni-1} + Dh_{ni+1})(Dh_{pi})Q_{111}^{jmnp} = (DS19)_i^j$$

$$R_{13} \sum_m \sum_n \sum_p (-Dh_{mi-1} + Dh_{mi+1})(-Dh_{ni-1} + Dh_{ni+1})(h_{pi-1} - 2h_{pi} + h_{pi+1}) Q_{111}^{jmnp} = (DD1)_i^j$$

$$R_{13} \sum_m \sum_n \sum_p (-h_{mi-1} + h_{mi+1})(-Dh_{ni-1} + Dh_{ni+1})(Dh_{pi-1} - 2Dh_{pi} + Dh_{pi+1}) Q_{111}^{jmnp} = (DD2)_i^j$$

$$R_{13} \sum_m \sum_n \sum_p (-Dh_{mi-1} + Dh_{mi+1})(-h_{ni-1} + h_{ni+1})(Dh_{pi-1} - 2Dh_{pi} + Dh_{pi+1}) Q_{111}^{jmnp} = (DD3)_i^j$$

$$R_{14} \sum_m \sum_n \sum_p mn(Dh_{mi})(Dh_{ni})(h_{pi-1} - 2h_{pi} + h_{pi+1})Q_{221}^{jmnp} = (DD4)_i^j$$

$$R_{14} \sum_m \sum_n \sum_p mn(h_{mi})(Dh_{ni})(Dh_{pi-1} - 2Dh_{pi} + Dh_{pi+1})Q_{221}^{jmnp} = (DD5)_i^j$$

$$R_{14} \sum_m \sum_n \sum_p mn(Dh_{mi})(h_{ni})(Dh_{pi-1} - 2Dh_{pi} + Dh_{pi+1})Q_{221}^{jmnp} = (DD6)_i^j$$

$$R_{15} \sum_m \sum_n \sum_p mn(-Dh_{mi-1} + Dh_{mi+1})(Dh_{ni})(-h_{pi-1} + h_{pi+1})Q_{221}^{jmnp} = (DD7)_i^j$$

$$R_{15} \sum_m \sum_n \sum_p mn(-Dh_{mi-1} + Dh_{mi+1})(h_{ni})(-Dh_{pi-1} + Dh_{pi+1})Q_{221}^{jmnp} = (DD8)_i^j$$

$$R_{15} \sum_m \sum_n \sum_p mn(-h_{mi-1} + h_{mi+1})(Dh_{ni})(-Dh_{pi-1} + Dh_{pi+1})Q_{221}^{jmnp} = (DD9)_i^j$$

$$R_{16} \sum_m \sum_n \sum_p mnp^2(Dh_{mi})(Dh_{ni})(h_{pi})Q_{221}^{jmnp} = (DD10)_i^j \qquad (5\text{-}3)$$

$$R_{16} \sum_m \sum_n \sum_p mnp^2(h_{mi})(Dh_{ni})(Dh_{pi})Q_{221}^{jmnp} = (DD11)_i^j$$

$$R_{16} \sum_m \sum_n \sum_p mnp^2(Dh_{mi})(h_{ni})(Dh_{pi})Q_{221}^{jmnp} = (DD12)_i^j$$

If the above defined summation terms (5-1) and (5-2) are substituted into the equations (4-7) and (4-8), simplified and rearranged, the coefficients of the unknown increments are found to be the following combinations of the summation terms.

$$P_2^j d_{j1} + P_8(S1)_i^j - P_9(S4)_i^j + P_{10}(S10)_i^j - P_8(S16)_i^j + P_{11}(S13)_i^j$$

$$+ 2P_{11}(S16)_i^j = P_{12}(S19)_i^j + P_{12}(S16)_i^j - P_{13}(S22)_i^j + P_{13}(S7)_i^j$$

$$+ P_{14}(S25)_i^j = (SS1)_i^j$$

$$P_2^j d_{j3} + P_8(S2)_i^j - (3)(P_9)(S5)_i^j + P_{10}(S11)_i^j = P_8(S17)_i^j + P_{11}(S14)_i^j$$

$$+ 2P_{11}(S17)_i^j - P_{12}(S20)_i^j + P_{12}(9)(S17)_i^j - P_{13}(3)(S23)_i^j$$

$$+3P_{13}(S8)_i^j + (3)(P_{14})(S26)_i^j = (SS2)_i^j \tag{5-4}$$

$$P_2^j d_{5j} + P_8(S3)_i^j - 5P_9(S6)_i^j + P_{10}(S12)_i^j + P_{11}(S15)_i^j - P_8(S18)_i^j$$

$$+ 2P_{11}(S18)_i^j - P_{12}(S21)_i^j + P_{12}(25)(S18)_i^j - (5)P_{13}(S24)_i^j$$

$$+ 5P_{13}(S9)_i^j + 5P_{14}(S27)_i^j = (SS3)_i^j$$

$$P_3^j d_{1j} + P_8(S1)_i^j + P_9(S4)_i^j - P_{10}(S10)_i^j + P_{11}(S13)_i^j - 2P_{11}(S16)_i^j$$

$$- P_{12}(S19)_i^j - P_{12}(S16)_i^j + P_{13}(S22)_i^j + P_{13}(S7)_i^j - P_{14}(S25)_i^j$$

$$+ P_8(S16)_i^j = (SS4)_i^j$$

$$P_3^j d_{3j} + P_8(S2)_i^j + 3P_9(S5)_i^j - P_{10}(S11)_i^j + P_{11}(S14)_i^j - 2P_{11}(S17)_i^j$$

$$- P_{12}(S20)_i^j - 9P_{12}(S17)_i^j + 3P_{13}(S23)_i^j + 3P_{13}(S8)_i^j - 3P_{14}(S26)_i^j$$

$$+ P_8(S17)_i^j = (SS5)_i^j$$

$$P_3^j d_{5j} + P_8(S3)_i^j + 5P_9(S6)_i^j - P_{10}(S12)_i^j + P_{11}(S15)_i^j - 2P_{11}(S18)_i^j$$

$$- P_{12}(S21)_i^j - 25P_{12}(S18)_i^j + 5P_{13}(S24)_i^j + 5P_{13}(S9)_i^j - 5P_{14}(S27)_i^j$$

$$+ P_8(S18)_i^j = (SS6)_i^j$$

$$P_1 d_{j1} - P_{11}(S16)_i^j = (SS7)_i^j \qquad (5\text{-}4)$$

$$P_1 d_{j3} - P_{11}(S17)_i^j = (SS8)_i^j$$

$$P_1 d_{j5} - P_{11}(S18)_i^j = (SS9)_i^j$$

$$P_4 d_{j1} + P_{11}(S16)_i^j = (SS10)_i^j$$

$$P_4 d_{j3} + P_{11}(S17)_i^j = (SS11)_i^j$$

$$P_4 d_{j5} + P_{11}(S18)_i^j = (SS12)_i^j$$

$$P_9(S7)_i^j - P_{10}(S1)_i^j - 2P_{11}(S13)_i^j + 2P_{12}(S19)_i^j - 2P_8(S1)_i^j - 2P_{13}(S7)_i^j$$

$$- P_{14}(S7)_i^j = (SS13)_i^j$$

$$+ 3P_9(S8)_i^j - 9P_{10}(S2)_i^j - 2P_{11}(S14)_i^j + 2P_{12}(S20)_i^j - 2P_8(S2)_i^j - 6P_{13}(S8)_i^j$$

$$-27P_{14}(S8)_i^j = (SS14)_i^j$$

$$5P_9(S9)_i^j - 25P_{10}(S3)_i^j - 2P_{11}(S15)_i^j + 2P_{12}(S21)_i^j - 2P_8(S3)_i^j - 10P_{13}(S9)_i^j$$

$$- 125P_{14}(S9)_i^j = (SS15)_i^j \quad (5\text{-}4)$$

$$R_2^j d_{j1} + R_6(S28)_i^j + R_7(S31)_i^j - 2R_8(S1)_i^j - R_{10}(S37)_i^j - 2R_{13}(S40)_i^j$$

$$+ R_{13}(S43)_i^j + R_{14}(S49)_i^j - R_{15}(S52)_i^j - R_{15}(S58)_i^j + 2R_{17}(S64)_i^j = (SS16)_i^j$$

$$R_2^j d_{j3} + R_6(S29)_i^j + R_7(S32)_i^j - 2R_8(S2)_i^j - 3R_{10}(S38)_i^j - 2R_{13}(S41)_i^j$$

$$+ R_{13}(S44)_i^j + R_{14}(S50)_i^j - 3R_{15}(S53)_i^j - R_{15}(S59)_i^j + 2R_{17}(S65)_i^j = (SS17)_i^j$$

$$R_2^j d_{j5} + R_6(S30)_i^j + R_7(S33)_i^j - 2R_8(S3)_i^j - 5R_{10}(S39)_i^j - 2R_{13}(S42)_i^j$$

$$+ R_{13}(S45)_i^j + R_{14}(S51)_i^j - 5R_{15}(S54)_i^j - R_{15}(S60)_i^j + 2R_{17}(S66)_i^j = (SS18)_i^j$$

$$R_4^j d_{j1} + R_6(S28)_i^j + R_7(S31)_i^j + 2R_8(S1)_i^j + R_{10}(S37)_i^j + 2R_{13}(S40)_i^j$$

$$+ R_{13}(S43)_i^j + R_{14}(S49)_i^j + R_{15}(S52)_i^j + R_{15}(S58)_i^j - 2R_{17}(S64)_i^j = (SS19)_i^j$$

$$R_4^j d_{j3} + R_6(S29)_i^j + R_7(S32)_i^j + 2R_8(S2)_i^j + 3R_{10}(S38)_i^j + 2R_{13}(S41)_i^j$$

$$+ R_{13}(S44)_i^j + R_{14}(S50)_i^j + 3R_{15}(S53)_i^j + R_{15}(S59)_i^j - 2R_{17}(S65)_i^j = (SS20)_i^j$$

$$R_4^j d_{j5} + R_6(S30)_i^j + R_7(S33)_i^j + 2R_8(S3)_i^j + 5R_{10}(S39)_i^j + 2R_{13}(S42)_i^j$$

$$+ R_{13}(S45)_i^j + R_{14}(S51)_i^j + 5R_{15}(S54)_i^j + R_{15}(S60)_i^j - 2R_{17}(S66)_i^j = (SS21)_i^j$$

$$R_3^j d_{1j} - 2R_6(S28)_i^j - 2R_7(S31)_i^j + R_7(S16)_i^j - R_{12}(S10)_i^j + 2R_9(S34)_i^j \qquad (5\text{-}4)$$

$$- R_{11}(S28)_i^j - R_{12}(S31)_i^j - 2R_{13}(S43)_i^j + 2R_{14}(S46)_i^j - 2R_{14}(S49)_i^j$$

$$+ R_{15}(S55)_i^j - 2R_{16}(S61)_i^j - R_{16}(S49)_i^j - R_{17}(S43)_i^j = (SS22)_i^j$$

$$R_3^j d_{3j} - 2R_6(S29)_i^j - 2R_7(S32)_i^j + R_7(S17)_i^j + 6R_9(S35)_i^j - 9R_{11}(S29)_i^j$$

$$- R_{12}(S11)_i^j - 9R_{12}(S32)_i^j - 2R_{13}(S44)_i^j + 6R_{14}(S47)_i^j - 2R_{14}(S50)_i^j$$

$$+ 3R_{15}(S56)_i^j - 6R_{16}(S62)_i^j - 9R_{16}(S50)_i^j - 9R_{17}(S44)_i^j = (SS23)_i^j$$

$$R_3^j d_{5j} - 2R_6(S30)_i^j - 2R_7(S33)_i^j + R_7(S18)_i^j + 10R_9(S36)_i^j - 25R_{11}(S30)_i^j$$

$$- R_{12}(S12)_i^j - 25R_{12}(S33)_i^j - 2R_{13}(S45)_i^j + 10R_{14}(S48)_i^j - 2R_{14}(S51)_i^j$$

$$+ 5R_{15}(S57)_i^j - 10R_{16}(S63)_i^j - 25R_{16}(S51)_i^j - 25R_{17}(S45)_i^j = (SS24)_i^j$$

$$R_{18}d_{1j} + R_6(S16)_i^j - R_{11}(S10)_i^j = (SS25)_i^j$$

$$R_{18}d_{3j} + R_6(S17)_i^j - R_{11}(S11)_i^j = (SS26)_i^j$$

$$R_{18}d_{5j} + R_6(S18)_i^j - R_{11}(S12)_i^j = (SS27)_i^j \qquad (5\text{-}4)$$

$$- R_{18}d_{1j} - R_6(S16)_i^j + R_{11}(S10)_i^j = (SS28)_i^j$$

$$- R_{18}d_{3j} - R_6(S17)_i^j + R_{11}(S11)_i^j = (SS29)_i^j$$

$$- R_{18}d_{5j} - R_6(S18)_i^j + R_{11}(S12)_i^j = (SS30)_i^j$$

$$R_{10}(S7)_i^j = (SS31)_i^j$$

$$3R_{10}(S8)_i^j = (SS32)_i^j$$

$$5R_{10}(S9)_i^j = (SS33)_i^j$$

where $d_{ab} = 1$ when $a = b$

$= 0$ when $a \neq b$

To simplify the right hand side of the equations call the following equivalent load terms as:

$$-(DS1)_i^j - (DS2)_i^j + (DS3)_i^j - (DS4)_i^j$$

$$+(DS5)_i^j - (DS6)_i^j + (DS7)_i^j = (DST)_i^j$$

$$-R_{19}^j \frac{Dp_z}{Et} - (DS8)_i^j - (DS9)_i^j - (DS10)_i^j$$

$$-(DS11)_i^j - (DS12)_i^j + (DS13)_i^j + (DS14)_i^j \qquad (5\text{-}5)$$

$$-(DS15)_i^j - (DS16)_i^j - (DS17)_i^j + (DS18)_i^j + (DS19)_i^j - (DD1)_i^j$$

$$-(DD2)_i^j - (DD3)_i^j$$

$$-(DD4)_i^j - (DD5)_i^j - (DD6)_i^j - (DD7)_i^j$$

$$-(DD8)_i^j - (DD9)_i^j + (DD10)_i^j + (DD11)_i^j$$

$$+(DD12)_i^j + (DD13)_i^j + (DD14)_i^j + (DD15)_i^j = (DDD)_i^j$$

Using the above summations (5-1), (5-2), (5-3), (5-4) and (5-5) the system of equations (4-7) and (4-8) takes the following form

$$(SS7)_i^j \, dh_{1i-2} + (SS1)_i^j \, Dh_{1i-1} + (SS13)_i^j \, Dh_{1i} + (SS4)_i^j \, Dh_{1i+1}$$

$$+ (SS10)_i^j \, Dh_{1i+2}$$

$$(SS8)_i^j \, Dh_{3i-2} + (SS2)_i^j \, Dh_{3i-1} + (SS14)_i^j \, Dh_{3i} + (SS5)_i^j \, Dh_{3i+1}$$

$$+ (SS11)_i^j \, Dh_{3i+2}$$

$$(SS9)_i^j \, Dh_{5i-2} + (SS3)_i^j \, Dh_{5i-1} + (SS15)_i^j \, Dh_{5i} + (SS6)_i^j \, Dh_{5i+1}$$

$$+ (SS12)_i^j \, Dh_{5i+2}$$

$$+ P_5 d_{1j} Df_{1i-1} + P_6^j d_{1j} Df_{1i} + P_7 d_{1j} Df_{1i+1} \qquad (5\text{-}6)$$

$$+ P_5 d_{3j} Df_{3i-1} + P_6^j d_{3j} Df_{3i} + P_7 d_{3j} Df_{3i+1}$$

$$+ P_5 d_{5j} Df_{5i-1} + P_6^j d_{5j} Df_{5i} + P_7 d_{5j} Df_{5i+1} = (DST)_i^j$$

$$(R_1 d_{1j}) Dh_{1i-2} + (SS16)_i^j \, Dh_{1i-1} + (SS22)_i^j \, Dh_{1i} + (SS19)_i^j Dh_{1i+1}$$

$$+ (R_5 d_{ij}) Dh_{1i+2}$$

$$(R_1 d_{3j}) Dh_{3i-2} + (SS17)_i^j \, Dh_{3i-1} + (SS23)_i^j \, Dh_{3i} + (SS20)_i^j \, Dh_{3i+1}$$

$$+ (R_5 d_{3j}) DH_{3i+2}$$

$$+ (R_1 d_{5j}) D_{h5i-2} + (SS18)_i^j \, Dh_{5i-1} + (SS24)_i^j \, Dh_{5i} + (SS21)_i^j \, Dh_{5i+1}$$

$$+ (R_5 d_{5j}) Dh_{5i+2}$$

$$(SS28)_i^j \, Df_{1i-1} + (SS31)_i^j \, Df_{1i} + (SS25)_i^j \, Df_{1i+1} \qquad (5\text{-}7)$$

$$(SS29)_i^j \, Df_{3i-1} + (SS32)_i^j \, Df_{3i} + (SS26)_i^j \, Df_{3i+1}$$

$$(SS30)_i^j \, Df_{5i-1} + (SS33)_i^j \, Df_{5i} + (SS27)_i^j \, Df_{5i+1} = (DDD)_i^j$$

If the six unknown increments Dh_1, Dh_3, Dh_5, Df_1, Df_3, Df_5 at a mesh point i are considered as a six by one column matrix and called X and the right hand side of the above system of equations as Y, then the above system of equations can be set in the following matrix form at each mesh point i

$$A_1 X_{i-2} + A_2 X_{i-1} + A_3 X_i + A_4 X_{i+1} + A_5 X_{i+2} = Y_i \qquad (5\text{-}8)$$

where X and Y are the following column matrices.

$$X_i = \begin{bmatrix} \left[Dh_j^i \right] \\ \hline \left[Df_j^i \right] \end{bmatrix}$$

$$Y_i = \begin{bmatrix} \left[DST_j^i \right] \\ \hline \left[DDD_i^j \right] \end{bmatrix} \qquad (5\text{-}9)$$

The A matrices are defined later in terms of the following FE and HE submatrices.

$$FE1 = \left[\begin{array}{ccc} 0 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right]$$

$$FE2 = \left[\begin{array}{ccc} P_5 d_{j1} & P_5 d_{j3} & P_5 d_{j5} \\ \hline (SS28)_i^j & (SS29)_i^j & (SS30)_i^j \end{array}\right]$$

$$FE3 = \left[\begin{array}{ccc} P_6^j d_{ij} & P_6^j d_{3j} & P_6^j d_{5j} \\ \hline (SS31)_i^j & (SS32)_i^j & (SS33)_i^j \end{array}\right]$$

$$FE4 = \left[\begin{array}{ccc} P_7 d_{1j} & P_7 d_{3j} & P_7 d_{5j} \\ \hline (SS25)_i^j & (SS26)_i^j & (SS27)_i^j \end{array}\right] \qquad (5\text{-}10)$$

$$FE5 = \left[\begin{array}{ccc} 0 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right]$$

$$HE1 = \left[\begin{array}{ccc} (SS7)_i^j & (SS8)_i^j & (SS9)_i^j \\ \hline R_1 d_{1j} & R_1 d_{3j} & R_1 d_{5j} \end{array}\right] \qquad (5\text{-}11)$$

$$HE2 = \left[\begin{array}{ccc} (SS1)_i^j & (SS2)_i^j & (SS3)_i^j \\ \hline (SS16)_i^j & (SS17)_i^j & (SS18)_i^j \end{array}\right]$$

$$HE3 = \left[\begin{array}{ccc} (SS13)_i^j & (SS14)_i^j & (SS15)_i^j \\ \hline (SS22)_i^j & (SS23)_i^j & (SS24)_i^j \end{array}\right]$$

$$HE4 = \begin{bmatrix} \begin{bmatrix} (SS4)_i^j & (SS5)_i^j & (SS6)_i^j \end{bmatrix} \\ \hline \begin{bmatrix} (SS19)_i^j & (SS20)_i^j & (SS21)_i^j \end{bmatrix} \end{bmatrix}$$

$$HE5 = \begin{bmatrix} \begin{bmatrix} (SS10)_i^j & (SS11)_i^j & (SS12)_i^j \end{bmatrix} \\ \hline \begin{bmatrix} R_5 d_{1j} & R_5 d_{3j} & R_5 d_{5j} \end{bmatrix} \end{bmatrix} \quad (5\text{-}11)$$

Each of the above submatrices is a three by three matrix with each row consisting of the values of the elements for $j = 1, 3, 5$.

The A matrices then are defined as follows.

$$A_1 = \left[HE1 \mid FE1 \right]$$

$$A2 = \left[HE2 \mid FE2 \right]$$

$$A3 = \left[HE3 \mid FE3 \right] \quad (5\text{-}12)$$

$$A4 = \left[HE4 \mid FE4 \right]$$

$$A5 = \left[HE5 \mid FE5 \right]$$

These matrices are of the order of six by six and have to be determined at each mesh point $i = 1, 2, 3, \ldots M$.

If the matrix equation (5-8) is applied at mesh point $i = 1$, it becomes.

$$A1X_{-1} + A2X_o + A3X_1 + A4X_2 + A5X_3 = Y \quad (5\text{-}13)$$

The X matrices if substituted in terms of submatrices Df_j and Dh_j given by equation (5-9) and **A** matrices substituted in terms of HE and FE submatrices yield the following equation.

$$HE1Dh_{j\,-1} + FE1Df_{j\,-1} + HE2Dh_{j\,0} + FE2Df_{j\,0} + HE3Dh_{j\,1}$$

$$+ FE3Df_{j\,1} + HE4Dh_{j\,2} + FE4Df_{j\,2}\ HE5Dh_{j\,3} + FE5Df_{j\,3} = Y_1 \qquad (5\text{-}14)$$

5.2 Boundary Conditions

The boundary conditions at x = 0 and x = a expressed by equations (4-11) and (4-13) in the incremental form require, that

$$Df_j = 0, \quad Dh_i = 0$$

and

$$-Dh_{ji-1} + DH_{ji+1} = 0 \qquad (5\text{-}15)$$

substituting the unknown increments at mesh points -1, in terms of increments at +1 and satisfying the above conditions, the equation (5-8) is reduced to

$$B1x_1 + A4X_2 + A5X_3 = Y_1 \qquad (5\text{-}16)$$

where

$$B1 = \left[HE1 + HE2 \;\middle|\; FE1 + FE3 \right] \qquad (5\text{-}17)$$

Because the shell and the loading are symmetrical, the numerical solution would be more efficient if half of the shell is considered. Therefore, it is also necessary to determine the boundary conditions on the crown of the shell at $x = \frac{a}{2}$.

Let i = M, be the mesh point at the crown of the shell, then due to symmetry

$$Dh_{M-k} = Dh_{M+k}, \quad Df_{M-k} = Df_{M+k} \tag{5-18}$$

where k = 0, 1, 2,...M

Therefore substituting the unknown increments at mesh points M + 1, and M + 2 in terms of increments at points M - 1 and M - 2; equation (5-9), if applied to mesh points M - 1, and M reduces to

$$\begin{aligned} A1X_{M-3} + A2X_{M-2} + B2X_{M-1} + A4X_M &= Y_{M-1} \\ B3X_{M-2} + B4X_{M-1} + A3X_M &= Y_M \end{aligned} \tag{5-19}$$

where

$$\begin{aligned} B2 &= \left[HE2 + HE4 \;\middle|\; FE2 + FE4 \right] \\ B3 &= \left[HE1 + HE5 \;\middle|\; FE1 + FE5 \right] \\ B4 &= \left[HE2 + HE4 \;\middle|\; FE2 + FE4 \right] \end{aligned} \tag{5-20}$$

5.3 Matrix Formulation

The matrix equation (5-8) when applied at all the mesh points from i = 1 to i = M, generates a system of 6M simultaneous equations with 6M unknowns. The resulting matrix is bandtype matrix with non-zero submatrices appearing only along the principal five diagonals.

Each submatrix is of the order of six by six and each row of the band may contain up to thirty non-zero elements. The solution of this system determines the values of the unknown increments Df_j and Dh_i, j = 1, 3, 5 at all the mesh points i = 1, 2, 3,...M. The total matrix in terms of submatrices takes the following form.

$$\begin{bmatrix} B1 & A4 & A5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ A2 & A3 & A4 & A5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ A1 & A2 & A3 & A4 & A5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & A1 & A2 & A3 & A4 & A5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & A1 & A2 & A3 & A4 & A5 & 0 & 0 & 0 & 0 \\ - & - & - & - & - & - & - & - & - & - & - \\ - & - & - & - & - & - & - & - & - & - & - \\ 0 & 0 & 0 & 0 & 0 & A1 & A2 & A3 & A4 & A5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & A1 & A2 & A3 & A4 & A5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & A1 & A2 & B2 & A4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & B3 & B4 & A3 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ \\ \\ \\ \\ \\ X_{M-2} \\ X_{M-1} \\ X_M \end{bmatrix} = \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \\ \\ \\ \\ \\ \\ Y_{M-2} \\ Y_{M-1} \\ Y_M \end{bmatrix} \tag{5-21}$$

6. ANALYTICAL AND NUMERICAL FORMULATION FOR ASYMMETRIC BUCKLING

6.1 General

As discussed in the first chapter asymmetric buckling is one of the possible mode of buckling. In this chapter this asymmetric buckling is investigated. It is assumed that the shell deforms symmetrically until the bifurcation of the solution occurs.

By mathematically adding a small asymmetric deformation to already accumulated symmetric deformations, it is tested if an unstable state equilibrium exists in the vicinity of the above state of equilibrium. The lowest value of the load for which the bifurcation occurs is considered as the stability limit.

Physically the above procedure is equivalent to forcing a feeble asymmetric deformation on the original symmetric deformation, without any increase in the load; and checking if the resulting deformed shape is in equilibrium.

The mathematical formulation of the problem leads to an eigenvalue problem.

6.2 Mathematical Formulation

As discussed above to determine the states of small asymmetric deformation in the vicinity of a finite symmetric deformation, it may be assumed that the total deflection at a point consists of two parts, first representing the displacements due to symmetric deformation and the second representing the displacements due to small asymmetric deformation.

Thus

$$
\begin{aligned}
\text{total } u &= u + \underline{u} \\
\text{total } v &= v + \underline{v} \qquad (6\text{-}1) \\
\text{total } w &= w + \underline{w}
\end{aligned}
$$

where $\underline{u}$, $\underline{v}$ and $\underline{w}$ are the displacements corresponding to small asymmetric deformation.

Substituting the above equations (6-1) into the first of equation (3-1), and letting $p_x = 0$, the latter yields the following equation (6-2)

$$
\begin{aligned}
& A_1(u + \underline{u})_{,xx} - A_2(w + \underline{w})_{,x} + A_3(w + \underline{w})_{,x}(w + \underline{w})_{,xx} \\
& + A_4(w + \underline{w})_{,y}(w + \underline{w})_{,xy} + A_5(u + \underline{u})_{,yy} + A_6(v + \underline{v})_{,xy} \\
& + A_7(w + \underline{w})_{,x}(w + \underline{w})_{,yy} + A_8(w + \underline{w})_{,xx}(w + \underline{w})_{,xxx} \qquad (6\text{-}2) \\
& + A_9(w + \underline{w})_{,xx}(w + \underline{w})_{,xyy} + A_{10}(w + \underline{w})_{,xyy} + A_{11}(w + \underline{w})_{,xxx} \\
& A_{12}(w + \underline{w})_{,xy}(w + \underline{w})_{,xxy} + A_{13}(w + \underline{w})_{,xy}(w + \underline{w})_{,yyy} = 0
\end{aligned}
$$

Expanding the above equation, and subtracting the first of equation (3-1) from the expanded equation; and as $\underline{u}$, $\underline{v}$, $\underline{w}$ are higher order infinitesimals compared to u, v, w, neglecting the nonlinear terms with respect to $\underline{u}$, $\underline{v}$, $\underline{w}$ the following equation (6-3) is obtained.

$$
\begin{aligned}
& A_1\underline{u}_{,xx} - A_2\underline{w}_{,x} + A_3\underline{w}_{,x}w_{,xx} + A_3 w_{,x}\underline{w}_{,xx} + A_4\underline{w}_{,y}w_{,xy} + A_4 w_{,y}\underline{w}_{,xy} \\
& + A_5\underline{u}_{,yy} + A_6\underline{v}_{,xy} + A_7\underline{w}_{,x}w_{,yy} + A_7 w_{,x}\underline{w}_{,yy} + A_8\underline{w}_{,xx}w_{,xxx} \\
& + A_8 w_{,xx}\underline{w}_{,xxx} + A_9\underline{w}_{,xx}w_{,xyy} + A_9 w_{,xx}\underline{w}_{,xyy} + A_{10}\underline{w}_{,xyy} + A_{11}\underline{w}_{,xxx} \\
& + A_{12}\underline{w}_{,xy}w_{,xxy} + A_{12}w_{,xy}\underline{w}_{,xxy} + A_{13}\underline{w}_{,xy}w_{,yyy} + A_{13}w_{,xy}\underline{w}_{,yyy} = 0
\end{aligned}
\tag{6-3}
$$

Similarly working in the same manner with the last of equations (3-1) and equations (6-1) and neglecting the nonlinear terms with respect to $\underline{u}$, $\underline{v}$, $\underline{w}$ the following equation (6-4) is obtained.

$$
\begin{aligned}
& C_1\underline{w}_{,xxyy} + C_2\underline{w}_{,xxxx} + C_3\underline{w}_{,yyyy} + C_4 u_{,x}\underline{w}_{,xx} + C_4\underline{u}_{,x}w_{,xx} \\
& + 2C_5 w_{,xx}w_{,x}\underline{w}_{,x} + C_5\underline{w}_{,xx}w_{,x}^2 + C_6 w_{,xx}\underline{w} + C_6\underline{w}_{,xx}w \\
& + C_7\underline{u}_{,x} + 2C_8 w_{,x}\underline{w}_{,x} + C_9\underline{w} + C_{10}w_{,xx}\underline{v}_{,y} + C_{10}\underline{w}_{,xx}v_{,y} \\
& + 2C_{11}w_{,xx}w_{,y}\underline{w}_{,y} + C_{11}\underline{w}_{,xx}w_{,y}^2 + C_{12}\underline{v}_{,y} + 2C_{13}w_{,y}\underline{w}_{,y}
\end{aligned}
\tag{6-4}
$$

$$
\begin{aligned}
&+ C_{14} w_{,xy}\underline{u}_{,y} + C_{14}\underline{w}_{,xy} u_{,y} + C_{15} w_{,xy}\underline{v}_{,x} + C_{15}\underline{w}_{,xy} v_{,x} \\
&+ C_{16} w_{,x} w_{,y}\underline{w}_{,xy} + C_{16} w_{,x}\underline{w}_{,y} w_{,xy} + C_{16}\underline{w}_{,x} w_{,y} w_{,xy} \\
&+ C_{17} w_{,yy}\underline{v}_{,y} + C_{17}\underline{w}_{,yy} v_{,y} + 2C_{18} w_{,yy} w_{,y}\underline{w}_{,y} + C_{18}\underline{w}_{,yy} w^{2}_{,y} \\
&+ C_{19} w_{,yy}\underline{u}_{,x} + C_{19}\underline{w}_{,yy} u_{,x} + 2C_{20} w_{,yy} w_{,x}\underline{w}_{,x} + C_{20}\underline{w}_{,yy} w^{2}_{,x} \\
&+ C_{21} w_{,yy}\underline{w} + C_{21}\underline{w}_{,yy} w = 0
\end{aligned}
\qquad (6\text{-}4)
$$

It may be observed that the above equations (6-3) and (6-4) constitute a set of two homogeneous, partial differential equations and hence the problem has been reduced to an eigenvalue problem. The value of the load p_z is present in the above equation in an implicit way, through the displacement functions u and w.

6.3 Approximating Functions

Let $\underline{u}$ and $\underline{w}$ be approximated by the series

$$
\begin{aligned}
\underline{u} &= \sum_{n} \underline{f}_n \sin\left(\frac{n\pi y}{b}\right) \\
\underline{w} &= \sum_{n} \underline{h}_n \sin\left(\frac{n\pi y}{b}\right)
\end{aligned}
\qquad (6\text{-}5)
$$

then each mode n can be represented by the expressions:

$$
\begin{aligned}
\underline{u}_n &= \underline{f}_n \sin\left(\frac{n\pi y}{b}\right) \\
\underline{w}_n &= \underline{h}_n \sin\left(\frac{n\pi y}{b}\right)
\end{aligned}
\qquad (6\text{-}6)
$$

substituting in the equations (6-3) and (6-4), the values of u and w given in article 3.2, and the values of $\underline{u}$ and $\underline{w}$ given by equations (6-6), and using Kantorovitch's method the following equations (6-7) and (6-8) are obtained.

$$
\begin{aligned}
& A_1 \underline{f}''_n Q_1^n - A_2 \underline{h}'_n Q_1^n + A_3 \underline{h}'_n \sum_m h''_m Q_{11}^{mn} + A_3 \underline{h}''_n \sum_m h'_m Q_{11}^{mn} \\
& + A_4 \left(\frac{\pi}{b}\right)^2 \underline{h}_n \sum_m mn h'_m Q_{22}^{mn} + A_4 \left(\frac{\pi}{b}\right)^2 \underline{h}'_n \sum_m mn h_m Q_{22}^{mn} - A_5 \underline{f}_n \left(\frac{n\pi}{b}\right)^2 Q_1^n \\
& - A_7 \left(\frac{\pi}{b}\right)^2 \underline{h}'_n \sum_m m^2 h_m Q_{11}^{mn} - A_7 \left(\frac{\pi}{b}\right)^2 \underline{h}_n \sum_m n^2 h'_m Q_{11}^{mn} + A_8 \underline{h}''_n \sum_m h'''_m Q_{11}^{mn} \\
& + A_8 \underline{h}'''_n \sum_m h''_m Q_{11}^{mn} - A_9 \left(\frac{\pi}{b}\right)^2 \underline{h}''_n \sum_m m^2 h'_m Q_{22}^{mn} - A_9 \left(\frac{\pi}{b}\right)^2 \underline{h}'_n \sum_m n^2 h''_m Q_{22}^{mn} \\
& - A_{10} \left(\frac{\pi}{b}\right)^2 \underline{h}'_n n^2 Q_1^n + A_{11} \underline{h}'''_n Q_1^n + A_{12} \underline{h}'_n \left(\frac{\pi}{b}\right)^2 \sum_m mn h''_m Q_{22}^{mn} \\
& + A_{12} \underline{h}''_n \left(\frac{\pi}{b}\right)^2 \sum_m mn h'_m Q_{22}^{mn} - A_{13} \underline{h}'_n \left(\frac{\pi}{b}\right)^4 \sum_m m^3 n h_m Q_{22}^{mn} \\
& - A_{13} \underline{h}_n \left(\frac{\pi}{b}\right)^4 \sum_m mn^3 h'_m Q_{22}^{mn} = 0
\end{aligned}
\tag{6-7}
$$

$$
\begin{aligned}
& - C_1 \underline{h}''_n \left(\frac{n\pi}{b}\right)^2 Q_1^n + C_2 \underline{h}^{IV}_n Q_1^n + C_3 \underline{h}_n \left(\frac{n\pi}{b}\right)^2 Q_1^n + C_4 \underline{h}''_n \sum_m f'_m Q_{11}^{mn} \\
& + C_4 \underline{f}'_n \sum_m h''_m Q_{11}^{mn} + 2C_5 \underline{h}'_n \sum_m \sum_p h'_m h'_p Q_{111}^{mpn} + C_5 \underline{h}''_n \sum_m \sum_p h'_m h'_p Q_{111}^{mpn} \\
& + C_6 \underline{h}_n \sum_m h''_m Q_{11}^{mn} + C_6 h''_n \sum_m h_m Q_{11}^{mn} + C_7 \underline{f}'_n Q_1^n + 2C_8 \underline{h}'_n \sum_m h'_m Q_{11}^{mn} + C_9 \underline{h}_n Q_1^n
\end{aligned}
\tag{6-8}
$$

$$+ 2C_{11}\underline{h}_n(\frac{\pi}{b})^2 \sum_m \sum_p mnh_m h'_p Q_{212}^{mpn} + C_{11}\underline{h}''_n(\frac{\pi}{b})^2 \sum_m \sum_p mph_m h_p Q_{221}^{mpn}$$

$$+ 2C_{13}\underline{h}_n(\frac{\pi}{b})^2 \sum_m mnh_m Q_{22}^{mn} + C_{14}\underline{f}_n(\frac{\pi}{b})^2 \sum_m mnh'_m Q_{22}^{mn}$$

$$+ C_{14}\underline{h}'_n(\frac{\pi}{b})^2 \sum_m mnf_m Q_{22}^{mn} + C_{16}\underline{h}'_n(\frac{\pi}{b})^2 \sum_m \sum_p nph'_m h_p Q_{122}^{mpn}$$

$$+ C_{16}\underline{h}_n(\frac{\pi}{b})^2 \sum_m \sum_p nph'_m h'_p Q_{122}^{mpn} + C_{16}\underline{h}'_n(\frac{\pi}{b})^2 \sum_m \sum_p mph_m h'_p Q_{221}^{mpn}$$

$$- 2C_{18}\underline{h}_n(\frac{\pi}{b})^4 \sum_m \sum_p nm^2 ph_m h_p Q_{122}^{mpn} - C_{18}\underline{h}_n(\frac{\pi}{b})^4 \sum_m \sum_p n^2 mph_m h_p Q_{221}^{mpn}$$

$$- C_{19}\underline{f}'_n(\frac{\pi}{b})^2 \sum_m m^2 h_m Q_{11}^{mn} - C_{19}\underline{h}_n(\frac{\pi}{b})^2 \sum_m n^2 f'_m Q_{11}^{mn} \qquad (6\text{-}8)$$

$$- 2C_{20}\underline{h}'_n(\frac{\pi}{b})^2 \sum_m \sum_p m^2 h_m h'_p Q_{11}^{mpn} - C_{20}\underline{h}_n(\frac{\pi}{b})^2 \sum_m \sum_p n^2 h'_m h'_p Q_{111}^{mpn}$$

$$- C_{21}\underline{h}_n(\frac{\pi}{b})^2 \sum_m m^2 h_m Q_{11}^{mn} - C_{21}\underline{h}_n(\frac{\pi}{b})^2 \sum_m n^2 h_n Q_{11}^{mn} = 0$$

where m, p = 1, 3, 5,....N and Q_1^n, Q_{11}^{mn}, Q_{22}^{mn}, Q_{111}^{mpn}, Q_{122}^{mpn}, Q_{212}^{mpn}, Q_{221}^{mpn} are the definite integrals of the trigonometric functions explained in the appendix B at the end.

6.4 Finite Difference Formulation

Considering the same location of mesh points as in the symmetrical buckling case, the differential equations (6-7) and (6-8) are transformed into difference equations. These difference equations may be simplified by defining the following summation terms.

$$(T1)_i^n = \frac{A_1 Q_1^n}{h^2}$$

$$(T2)_i^n = \frac{- A_5 h^2 \pi^2 Q_1^n}{b^2}$$

$$(T3)_i^n = \frac{- A_2 Q_1^n}{2h}$$

$$(T4)_i^n = \frac{A_3}{2h^3} \sum_m (h_{mi-1} - 2h_{mi} + h_{mi+1}) Q_{11}^{mn}$$

$$(T5)_i^n = \frac{A_3}{2h^3} \sum_m (- h_{mi-1} + h_{mi+1}) Q_{11}^{mn}$$

$$(T6)_i^n = \frac{A_4 \pi^2}{2b^2 h} \sum_m mn(- h_{mi-1} + h_{mi+1}) Q_{22}^{mn}$$

$$(T7)_i^n = \frac{A_4 \pi^2}{2b^2 h} \sum_m mn(h_{mi}) Q_{22}^{mn}$$

$$(T8)_i^n = \frac{- A_7 \pi^2}{2b^2 h} \sum_m m^2 h_{mi} Q_{11}^{mn}$$

$$(T9)_i^n = \frac{- A_7 \pi^2}{2b^2 h} \sum_m n^2(- h_{mi-1} + h_{mi+1}) Q_{11}^{mn} \qquad (6\text{-}9)$$

$$(T10)_i^n = \frac{A_8}{2h^5} \sum_m (- h_{mi-2} + 2h_{mi-1} - 2h_{mi+1} + h_{mi+2}) Q_{11}^{mn}$$

$$(T11)_i^n = \frac{A_8}{2h^5} \sum_m (h_{mi-1} - 2h_{mi} + h_{mi+1}) Q_{11}^{mn}$$

$$(T12)_i^n = \frac{- A_9 \pi^2}{2b^2 h^3} \sum_m m^2(- h_{mi-1} + h_{mi+1}) Q_{11}^{mn}$$

$$(T13)_i^n = \frac{- A_9 \pi^2}{2b^2 h^3} \sum_m n^2(h_{mi-1} - 2h_{mi} + h_{mi+1}) Q_{11}^{mn}$$

$$(T14)_i^n = \frac{-A_{10}\pi^2 h^2 Q_1^n}{2b^2 h}$$

$$(T15)_i^n = \frac{A_{11} Q_1^n}{2h^3}$$

$$(T16)_i^n = \frac{A_{12}\pi^2}{2b^2 h^3} \sum_m mn(h_{mi-1} - 2h_{mi} + h_{mi+1}) Q_{22}^{mn}$$

$$(T17)_i^n = \frac{A_{12}\pi^2}{2b^2 h^3} \sum_m mn(-h_{mi-1} + h_{mi+1}) Q_{22}^{mn} \qquad (6\text{-}9)$$

$$(T18)_i^n = \frac{-A_{13}\pi^4}{2b^4 h} \sum_m m^3 h(h_{mi}) Q_{22}^{mn}$$

$$(T19)_i^n = \frac{-A_{13}\pi^4}{2b^4 h} \sum_m mn^3(-h_{mi-1} + h_{mi+1}) Q_{22}^{mn}$$

$$(U0)_i^n = \frac{-C_{19}\pi^2}{2b^2 h} \sum_m m^2 h_{mi} Q_{11}^{mn}$$

$$(U1)_i^n = \frac{C_4}{2h^3} \sum_m (h_{mi-1} - 2h_{mi} + h_{mi+1}) Q_{11}^{mn}$$

$$(U2)_i^n = \frac{C_7 Q_1^n}{2h}$$

$$(6\text{-}10)$$

$$(U3)_i^n = \frac{C_{14}\pi^2}{2b^2 h} \sum_m mn(-h_{mi-1} + h_{mi+1}) Q_{22}^{mn}$$

$$(U4)_i^n = \frac{-C_1 n^2 \pi^2 Q_1^n}{b^2 h^2}$$

$$(U5)_i^n = \frac{C_2 Q_1^n}{h^4}$$

$$(U6)_i^n = \frac{C_3 n^4 \pi^4 Q_1^n}{b^4}$$

$$(U7)_i^n = \frac{C_4}{2h^3} \sum_m (- f_{mi-1} + f_{mi+1}) Q_{11}^{mn}$$

$$(U8)_i^n = \frac{C_5}{4h^3} \sum_m \sum_p (- h_{mi-1} + h_{mi+1})(- h_{pi-1} + h_{pi+1}) Q_{111}^{mpn}$$

$$(U9)_i^n = \frac{C_5}{4h^4} \sum_m \sum_p (- h_{mi-1} + h_{mi+1})(- h_{pi-1} + h_{pi+1}) Q_{111}^{mpn}$$

$$(U10)_i^n = \frac{C_6}{h^2} \sum_m (h_{mi-1} - 2h_{mi} + h_{mi+1}) Q_{11}^{mn}$$

$$(U11)_i^n = \frac{C_6}{h^2} \sum_m h_{mi} Q_{11}^{mn}$$

$$(U12)_i^n = \frac{C_8}{2h^2} \sum_m (- h_{mi-1} + h_{mi+1}) Q_{11}^{mn}$$

(6-10)

$$(U13)_i^n = C_9 Q_1^n$$

$$(U14)_i^n = \frac{C_{11}\pi^2}{b^2 h} \sum_m \sum_p mn(h_{mi})(- h_{pi-1} + h_{pi+1}) Q_{212}^{mpn}$$

$$(U15)_i^n = \frac{C_{11}\pi^2}{b^2 h^2} \sum_m \sum_p mp(h_{mi})(h_{pi}) Q_{221}^{mpn}$$

$$(U16)_i^n = \frac{2C_{13}\pi^2}{b^2} \sum_m mn(h_{mi}) Q_{22}^{mn}$$

$$(U17)_i^n = \frac{C_{14}\pi^2}{2b^2 h} \sum_m mn(f_{mi}) Q_{22}^{mn}$$

$$(U18)_i^n = \frac{C_{16}\pi^2}{4b^2 h^2} \sum_m \sum_p np(- h_{mi-1} + h_{mi+1})(h_{pi}) Q_{122}^{mpn}$$

$$(U19)_i^n = \frac{C_{16}\pi^2}{4b^2h^2} \sum_m \sum_p np(-h_{mi-1} + h_{mi+1})(-h_{pi-1} + h_{pi+1})Q_{122}^{mpn}$$

$$(U20)_i^n = \frac{C_{16}\pi^2}{4b^2h^2} \sum_m \sum_p mp(h_{mi})(-h_{pi-1} + h_{pi+1})Q_{221}^{mpn}$$

$$(U21)_i^n = \frac{-2C_{18}\pi^4}{b^4} \sum_m \sum_p nm^2p(h_{mi})(h_{pi})Q_{122}^{mpn}$$

$$(U22)_i^n = \frac{-C_{18}\pi^4}{b^4} \sum_m \sum_p n^2mp(h_{mi})(h_{pi})Q_{221}^{mpn}$$

$$(U23)_i^n = \frac{-C_{19}\pi^2}{2b^2h} \sum_m n^2(-f_{mi-1} + f_{mi+1})Q_{11}^{mn}$$

$$(6\text{-}10)$$

$$(U24)_i^n = \frac{-C_{20}\pi^2}{2b^2h^2} \sum_m \sum_p m^2(h_{mi})(-h_{pi-1} + h_{pi+1})Q_{11}^{mn}$$

$$(U25)_i^n = \frac{-C_{20}\pi^2}{4b^2h^2} \sum_m \sum_p n^2(-h_{mi-1} + h_{mi+1})(-h_{pi-1} + h_{pi+1})Q_{111}^{mpn}$$

$$(U26)_i^n = \frac{-C_{21}\pi^2}{b^2} \sum_m m^2(h_{mi})Q_{11}^{mn}$$

$$(U27)_i^n = \frac{-C_{21}\pi^2}{b^2} \sum_m n^2(h_{mi})Q_{11}^{mn}$$

When the above defined summation terms are substituted into the difference equations,simplified and rearranged; the coefficients of the functions $\underline{f}_n$ and $\underline{h}_n$ are found to be the following combinations of summation terms.

$$(V1)_i^n = (V3)_i^n = (T1)_i^n$$

$$(V2)_i^n = -2(T1)_i^n + (T2)_i^n$$

$$(V4)_i^n = -(T11)_i^n - (T15)_i^n$$

$$(V5)_i^n = -(T3)_i^n - (T4)_i^n + (T5)_i^n - (T7)_i^n - (T8)_i^n + (T10)_i^n$$

$$+ 2(T11)_i^n + (T12)_i^n - (T13)_i^n - (T14)_i^n + 2(T15)_i^n - (T16)_i^n$$

$$+ (T17)_i^n - (T18)_i^n$$

$$(V6)_i^n = -2(T5)_i^n + (T6)_i^n + (T9)_i^n - 2(T10)_i^n - 2(T12)_i^n$$

$$-2(T17)_i^n + (T19)_i^n \qquad (6\text{-}11)$$

$$(V7)_i^n = (T3)_i^n + (T4)_i^n + (T5)_i^n + (T7)_i^n + (T8)_i^n + (T10)_i^n - 2(T11)_i^n$$

$$+ (T12)_i^n + (T13)_i^n + (T14)_i^n - 2(T15)_i^n + (T16)_i^n + (T17)_i^n + (T18)_i^n$$

$$(V8)_i^n = (T11)_i^n + (T15)_i^n$$

$$(V9)_i^n = -(U0)_i^n - (U1)_i^n - (U2)_i^n$$

$$(V10)_i^n = (U3)_i^n$$

$$(V11)_i^n = (U1)_i^n + (U2)_i^n + (U0)_i^n$$

$$(V12)_i^n = (U5)_i^n$$

$$(V13)_i^n = (U4)_i^n - 4(U5)_i^n + (U7)_i^n - (U8)_i^n + (U9)_i^n + (U11)_i^n - (U12)_i^n$$

$$+ (U15)_i^n - (U17)_i^n - (U18)_i^n - (U20)_i^n - (U24)_i^n$$

$$(V14)_i^n = - 2(U4)_i^n + 6(U5)_i^n + (U6)_i^n - 2(U7)_i^n - 2(U9)_i^n + (U10)_i^n$$

$$- 2(U11)_i^n + (U13)_i^n + (U14)_i^n - 2(U15)_i^n + (U16)_i^n + (U19)_i^n + (U21)_i^n$$

$$+ (U23)_i^n + (U25)_i^n + (U26)_i^n + (U27)_i^n + (U22)_i^n \qquad (6\text{-}11)$$

$$(V15)_i^n = (U4)_i^n - 4(U5)_i^n + (U7)_i^n + (U8)_i^n + (U9)_i^n + (U11)_i^n + (U12)_i^n$$

$$+ (U15)_i^n + (U17)_i^n + (U18)_i^n + (U20)_i^n + (U24)_i^n$$

$$(V16)_i^n = (U5)_i^n$$

Using the above summations (6-9), (6-10) and (6-11), the difference equations take the following form.

$$(V1)_i^n \underline{f}_{n,i-1} + (V2)_i^n \underline{f}_{n,i} + (V3)_i^n \underline{f}_{n,i+1} + (V4)_i^n \underline{h}_{n,i-2}$$

$$+ (V5)_i^n \underline{h}_{n,i-1} + (V6)_i^n \underline{h}_{n,i} + (V7)_i^n \underline{h}_{n,i+1}) \; (V8)_i^n \underline{h}_{n,i+2}) = 0 \qquad (6\text{-}12)$$

$$(V9)_i^n \underline{f}_{n,i-1} + (V10)_i^n \underline{f}_{n,i} + (V11)_i^n \underline{f}_{n,i+1} \quad (V12)_i^n \underline{h}_{n,i-2}) +$$

$$(V13)_i^n \underline{h}_{n,i-1} + (V14)_i^n \underline{h}_{n,i} + (V15)_i^n \underline{h}_{n,i+1} + (V16)_i^n \underline{h}_{n,i+2} = 0 \qquad (6\text{-}12)$$

The equations (6-12) applied to all mesh points constitute a system of simultaneous equations. In this case, the network of the mesh points covers the whole middle surface of the shell and therefore the total number of mesh points is 2M.

6.5 Matrix Formulation

If the unknown functions $\underline{f}_n$ and $\underline{h}_n$ at a mesh point i are considered as a two by one column matrix and called Z, that is

$$Z_i = \begin{bmatrix} \underline{f}_n \\ \underline{h}_n \end{bmatrix} \qquad (6\text{-}13)$$

then the equations (6-12) can be set in the matrix form as follows:

$$C1Z_{i-2} + C2Z_{i-1} + C3Z_i + C4Z_{i+1} + C5Z_{i+2} = 0 \qquad (6\text{-}14)$$

where the following substitutions have been made.

$$FF1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad HH1 = \begin{bmatrix} (V4)_i \\ (V12)_i \end{bmatrix}$$

$$FF2 = \begin{bmatrix} (V1)_i \\ (V9)_i \end{bmatrix}, \quad HH2 = \begin{bmatrix} (V5)_i \\ (V13)_i \end{bmatrix}$$

$$FF3 = \begin{bmatrix} (V2)_i \\ (V10)_i \end{bmatrix}, \quad HH3 = \begin{bmatrix} (V6)_i \\ (V14)_i \end{bmatrix} \qquad (6\text{-}15)$$

$$FF4 = \begin{bmatrix} (V3)_i \\ (V11)_i \end{bmatrix}, \quad HH4 = \begin{bmatrix} (V7)_i \\ (V15)_i \end{bmatrix}$$

$$FF5 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad HH5 = \begin{bmatrix} (V8)_i \\ (V16)_i \end{bmatrix}$$

and

$$C1 = \left[FF1 \mid HH1 \right]$$

$$C2 = \left[FF2 \mid HH2 \right]$$

$$C3 = \left[FF3 \mid HH3 \right] \qquad (6\text{-}16)$$

$$C4 = \left[FF4 \mid HH4 \right]$$

$$C5 = \left[FF5 \mid HH5 \right]$$

If the equation (6-14) is applied to mesh points $i = 1$ and $i = 2M - 1$, it becomes respectively.

$$C1Z_{-1} + C1Z_{o} + C3Z_{1} + C4Z_{2} + C5Z_{3} = 0 \qquad (6\text{-}17)$$

$$C1Z_{2M-3} + C2Z_{2M-2} + C3Z_{2M-1} + C4Z_{2M} + C5Z_{2M+1} = 0$$

6.6 Matrix Formulation for Boundary Conditions

The shell under investigation is clamped at the two boundaries parallel to the Y axis. And since the additional asymmetric deformations must also satisfy the boundary conditions expressed by equations (4-11) and (4-12), the boundary conditions expressed by the above equation (4-11) and (4-12) are transformed to those in terms of $\underline{f}_n$ and $\underline{h}_n$.

Thus at x = 0 and x = a i.e. at i = 0 and i = 2M.

$$\underline{f}_n = 0, \ \underline{h}_n = 0 \qquad (6\text{-}18)$$

and

$$- \underline{h}_{n-1} + \underline{h}_{n+1} = 0; \ - \underline{h}_{2M-1} + \underline{h}_{2M+1} = 0 \qquad (6\text{-}19)$$

Substituting the unknown functions f_n, h_n at mesh point - 1 in terms of functions at +1, and at the mesh point M + 1 in terms of those at 2M - 1 and satisfying the above equation, the equation (6-17) becomes

$$D1Z_{1} + C4Z_{2} + C5Z_{3} = 0 \qquad (6\text{-}20)$$

and

$$C1Z_{2M-3} + C2Z_{2M-2} + D2Z_{2M-1} = 0 \qquad (6\text{-}21)$$

where

$$D1 = \left[FF1 + FF3 \;\middle|\; HH1 + HH3 \right] \qquad (6\text{-}22)$$

$$D2 = \left[FF3 + FF5 \;\middle|\; HH3 + HH5 \right]$$

It may be noted that the displacement functions at i = 0 and i = 2M each being equal to zero the submatrix C2 at i = 1 and C4 at i = 2M - 1 may be neglected.

6.7 Characteristic Equation

The matrix equation (6-14) when applied to all the mesh points from i = 1 to i = 2M - 1 generates a system of 2(2M - 1) simultaneous equations with equal number of unknowns. The resulting matrix is also a band type matrix with non-zero submatrices along the principal five diagonals.

$$\begin{bmatrix} D1 & C4 & C5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ C2 & C3 & C4 & C5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ C1 & C2 & C3 & C4 & C5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & C1 & C2 & C3 & C4 & C5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & C1 & C2 & C3 & C4 & C5 & 0 & 0 & 0 & 0 \\ - & - & - & - & - & - & - & - & - & - & - \\ - & - & - & - & - & - & - & - & - & - & - \\ 0 & 0 & 0 & 0 & 0 & C1 & C2 & C3 & C4 & C5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & C1 & C2 & C3 & C4 & C5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & C1 & C2 & C3 & C4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & C1 & C2 & D2 \end{bmatrix} \begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \\ - \\ - \\ \\ \\ \\ Z_{2M-3} \\ Z_{2M-2} \\ Z_{2M-1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \\ \\ \\ \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad (6\text{-}23)$$

In order for the system of equations represented by the above matrix to have non-trivial solutions the determinant of the above coefficient matrix has to be equal to zero. If the determinant of the above coefficient matrix is called D, then the condition for the non-trivial solution is

$$\text{Determinant } D = 0 \tag{6-24}$$

This equation determines explicitly the critical pressure for unsymmetrical buckling, the method of solution of which is discussed in the next chapter.

7. METHOD OF SOLUTION

7.1 General

Having derived the matrix for symmetrical buckling in Chapter Five, and the characteristic equation for asymmetric buckling in Chapter Six, this chapter is devoted to the method of solution of the above.

In order to explain the method of solution, let the matrix equation (5-21) be represented as follows.

$$\left[[A] + [\Delta A_{N-1}]\right]\left[\Delta\delta_N\right] = \left[[\Delta p_N] + f(\Delta\delta_{N-1})\right] \qquad (7\text{-}1)$$

where A = the coefficient matrix consisting of the elements depending upon the geometry of the shell, material properties and the mesh size h.

$[\Delta\delta_N]$ = column matrix representing the Nth increments in the functions f_j and h_j which determine the displacements.

Δp_N = column matrix determined by the Nth increment in the external load.

ΔA_{N-1} = additional corrections to the coefficient matrix A, as a result of updating the displacements, in the previous step.

$[f(\Delta\delta_{N-1})]$ = additional corrections due to "equivalent load terms."

The characteristic equation (6-24) is

$$|D| = 0 \tag{7-2}$$

where D = Determinant of the coefficient matrix represented by equation (6-23).

7.2 Linear Incremental Step Method

In order to solve the above system of algebraic equations, represented in the incremental form by equation (7-1) the linear incremental technique is employed. According to this technique the load is divided into a number of equal steps, whose size is chosen to yield displacement increments sufficiently small so that the linear theory applies. The solution technique consists of the following steps in sequence.

1. The entire shell which is in equilibrium in the beginning has no load on it with no resulting deflection. Therefore, the starting point on the load deflection curve is determined.

2. With the initial geometry of the shell being now known, the coefficient matrix A_N is generated with $N = 1$ (the 1st step). In this initial cycle the higher order non-linear terms dependent upon the displacement functions are taken as zero, i.e.

   $$[f(\delta_{N-1})] = 0$$

3. Also the "equivalent load" terms in the load column matrix $f(\Delta\delta_{N-1})$ are also taken equal to zero.

4. An increment in the load Δp_N is considered and the system is solved.

5. The first cycle of the solution of the system determines $\Delta\delta_N$, corresponding to the initial increment Δp_1 at all the mesh points.

6. The resulting increments of functions f and h are used to compute corresponding increment of the average deflection.

7. The increments of the function Df and Dh are added to the values of the functions f and h of the preceeding cycle, with these values the correction to the coefficient matrix is determined.

8. Also with the values of Df and Dh (i.e. $\Delta\delta_N$) determined in the previous cycle the additional corrections to "equivalent load" terms consisting of non-linear terms are determined.

9. A new increment of load is applied and whole process is repeated.

As described above, the curve of load against the corresponding deflection can be determined till the deflection becomes excessively high for a particular increment in load indicating stability limit.

7.3 Asymmetric Buckling

1. For asymmetric buckling, to determine the bifurcation point on the load deflection curve, everytime the displacements are

determined for an increment in the load. The value of the determinant D of the characteristic equation is calculated.

2. By interpolation on the load deflection curve the point corresponding to zero value of the determinant is found.

3. The corresponding value of the load indicates the asymmetric buckling load.

7.4 Buckling with Initial Imperfections

1. To determine the buckling load with initial imperfections, the deflection W_L corresponding to the limit load is determined from the load deflection curve.

2. Then the imperfection parameter k is determined from the following equation

$$k = \frac{t}{W_L} \tag{7-3}$$

3. Using different values of k like k, -k, 2k, -2k, etc. and following the same steps as described in article 7.2 the load deflection curve is determined.

4. The limiting load on this curve corresponds to the buckling load for the shell with initial imperfections.

8. NUMERICAL RESULTS AND CONCLUSION

8.1 Computer Programming

The iterative method of solution using linear incremental approach described in Chapter seven, requires repeated formulation and solution of a large number of simultaneous equations, expressed by matrix equation (5-21). To accomplish this, a computer program is written for and executed on the I.B.M. 360-40 Model at Southeastern Massachusetts University. Program being excessively long, some effort is also devoted in optimizing the numerical computations [3]. Subroutines are written to evaluate integrals in appendices A and B, and the summation terms defined in Chapters five and six.

8.2 Computational Effort

The principal short coming of the linear incremental approach to large displacement problems is the computational effort involved. Every incremental step requires the setting up of the large matrix and its inversion. The large number of subroutines also consume substantial computation time. Moreover, each incremental (or iterative) step requires the complete solution of a small deflection problem. For these reasons, even though some effort is made to optimize computations, machine running time is understandably high.

The total solution time depends of course on the number of mesh points employed and even more significantly on the number of incremental steps required to reach the limiting point on the load-deflection curve.

The solution time for the six cases treated ranged from 65 to 75 minutes on the I.B.M. Model 360-40 at Southeastern Massachusetts University Computing Center.

8.3 Mathematical Models

Six mathematical models of shells are investigated with the following values of parameters. All models have Poisson's ratio $\nu = 0.33$, $t = .077''$, and $a/b = 1$. In the first three cases the models are relatively shallow ($C = 0.02$) as compared to the last three models ($C = 0.37$)

Dimensionless load parameter ρ and dimensionless deflection parameter $\bar{w}$ are defined as follows:

$$\rho = \frac{P}{E} \times 10^7$$

$$\bar{w} = \frac{w}{t}$$

where w is the average deflection.

In addition to the parameters stated above, some additional parameters with different values are employed in the various cases as noted below in Table 1.

TABLE 1 - PARAMETERS OF SHELL MODELS

Case No. 1	c	k	Shell Parameters
1	0.02	0.0	deep, and shallow
2.	0.02	+0.5	deep
3.	0.02	-0.5	deep
4.	0.037	0	deep
5.	0.037	0	shallow
6.	0.037	-0.02	deep

An additional case with all the parameters equal to that of Case No. 1, but with shallow shell parameters, instead of deep shell parameters, has also been investigated independently.

The load-deflection curves of the above models are shown in Figures 7, 8, 9, 10, 11, 12, and 13.

8.4 Results and Comparison of the Results with Experimental Investigations

Because of different ways the shells are supported, difference in the manner of loading and different geometries of the shell used by the experimental investigators, explicit comparison of the results is neither appropriate nor logical. Although an attempt is made to state the results of the two available experimental investigations, along with relevant data. In presenting the data some additional parameters are calculated which are believed to affect the results; though they are not presented as such by the original investigators.

The Results of Other Experimental Investigations

	Shell Thickness (t), in	Width (a) in	Span in	Shell Radius (R) in	$\frac{R}{t}$	ρ	Rise/Span	Remarks
1	0.05	23.6"	27	18	360	1.36	0.16	Yang and Guralnick
2	0.05	23.6"	27	18	360	1.46	0.16	
3	0.063	23.6"	54	18	285	1.65	0.08	
4	0.063	23.6"	54	18	285	1.55	0.08	
5	0.063	23.6"	81	18	285	1.41	0.054	
6	0.063	23.6"	81	18	285	1.26	0.054	
7	.156"	32.11"	125	25	160	1.337	.182	Karakas & Scalz

Present Investigation

	a/b	k	c	Rise/Span	R/t	ρ	Shell Parameters
1	1	0	0.02	0.1	650	2.1	deep
2	1	+0.5	0.02	0.1	650	1.875	deep
3	1	-0.5	0.02	0.1	650	2.42	deep
4	1	0	0.037	0.185	350	3.22	deep
5	1	0	0.037	0.185	350	3.87	shallow
6	1	-0.02	0.037	0.185	350	3.48	deep

The shell tested by Yang and Guralnick (31), was simply supported at the transverse edges and was free at the longitudinal edges. Also a uniformly distributed live load on the horizontal projection of the shell was simulated, by means of closely spaced concentrated loads.

Karakas and Scalz (12) also tested the shell with simulated uniformly distributed load on the horizontal projection of the shell. This shell was simply supported on rollers at the ends.

It can be readily seen that the buckling loads of the present investigation are relatively high compared with those of Karakas and Scalz, and Yang and Guralnick. But this is understandable as the boundary conditions of the shell considered make it more rigid, and the load considered was applied normal to the shell surface.

8.5 Conclusions

By comparing the results of the load-deflection curves of the six models, the following observations may be made. It may be observed from Table 1, that first three cases deal with shallower shells with ($c = 0.02$, Rise/Span = 0.1) whereas the last three cases deal with deeper shells with ($c = .037$, Rise/Span = .185). Relatively, a shallower shell ($c = .02$, Rise/Span = 0.1) is investigated in case No. 1 by employing deep as well as shallow shell parameters. The load-deflection curves for both (Figure No.7) coincide resulting into a common value ($\rho = 2.1$) for buckling load. Hence, it is concluded that there is no improvement in the value of the buckling load for this shell ($c = 0.02$, Rise/Span = 0.1) by employing deep shell parameters, instead of shallow shell parameters.

This also proves that, the following assumptions used in the shallow shell theory are correct, and the buckling loads calculated by using equations based on them are not in error for the shells with $c \leqslant 0.02$, and Rise/Span $\leqslant 0.1$, and hence establishes their validity.

Assumptions of Shallow Shell Theory

1. The slope of the shell is small compared with some reference plane (usually the horizontal plane for roofs)

2. The curvature of the surface is small.

3. The changes in curvature of the surface is small.

On the other hand, in case No. 4 and case No. 5, relatively a deeper shell (with c = 0.037, and Rise/Span = 0.185) has been investigated by employing deep (case No. 4) as well as shallow (cas No. 5) shell parameters. The load-deflection curves for these two cases (Figure 11, and Figure 12) coincide in the initial range (up to $\rho = 2.5$). However they do not do so at higher values of loads and hence result into different values of buckling loads ($\rho = 3.22$ with deep shell paramters) for case No. 4 and ($\rho = 3.87$ with shallow shell parameters) for case No. 5. Hence, it is concluded that there is some improvement in the value of the buckling load for this shell (with c = 0.037, and Rise/Span = 0.185) by employing deep shell parameters instead of shallow shell parameters.

In the initial range, though the load deflection curves for the two cases coincide which proves the validity of shallow shell theory, for small deflection problems or ordinary statical problems where no buckling or excessive deflections are involved, for the shells with $c \leqslant 0.037$, and Rise/Span $\leqslant 0.185$.

Next, case No. 1, case No. 2, and case No. 3 are compared for the effect of imperfections on buckling load where a shell with $c = 0.02$, and Rise/Span = 0.1 is investigated for $k = 0$ (case No. 1), $k = +0.5$ (case No. 2) and $k = -0.5$ (case No. 3). The load deflection curves for these three cases are shown in Figure Nos. 7, 8 and 9 respectively. The buckling loads in these three cases respectively are $\rho = 2.1$ for $k = 0$, $\rho = 1.875$ for $k = +0.5$ and $\rho = 2.42$ for $k = -0.5$. From these observations, therefore, it is concluded that, the initial imperfections in the shell if present, can affect the buckling load significantly. The effect of initial imperfections on buckling load is not always of weakening kind. For imperfections in the direction of deflection (k positive) the buckling load is reduced, whereas for the initial imperfections in the direction opposite to the direction of deflection (k negative) the buckling load is increased.

Figure 10, where load deflection curves for case No.2 ($k = +0.5$) and case No. 3 ($k = -0.5$) are plotted together, indicates the upper and lower range of load-deflection curves due to imperfections for k between -0.5 to +0.5. For all other amplitudes of imperfections, with k between -0.5 to +0.5. the plot of load-deflection curve may lie within the shaded portion. This indicates that the experimental values of the buckling loads of shell models constructed to same specifications may vary within a certain range depending upon the degree of initial imperfections present.

If comparison is made between case No. 4 ($k = 0$, $c = 0.037$, Rise/Span = 0.185) with $\rho = 3.22$; and case No. 6 ($k = -0.02$, $c = 0.037$, Rise/Span = 0.185) and $\rho = 3.48$, the following conclusions may be drawn. The effect of imperfections on buckling loads is not only limited to relatively shallower shells ($c = 0.02$, Rise/Span = 0.1) but is also seen in deeper shells ($c = 0.037$, Rise/Span = 0.185). Also, the introduction of even feeble imperfections ($k = -0.02$) as in Case No. 6 affects the solution. Hence, generalizing, it may be stated that the initial imperfections, if present can affect the buckling load significantly for any shell.

Next, comparing the first three cases where the shells are relatively shallow ($c = 0.02$, Rise/Span = 0.1) and $\rho = 2.1, 1.875, 2.42$ respectively with the last three cases where shells are relatively deep ($c = 0.037$, Rise/Span = 0.185) and $\rho = 3.22, 3.87, 3.48$ respectively, it may be stated in general that the deep shells are stronger than the shallow shells.

8.6 Suggestions for Future Research

Further investigations of deep shells or shells in general with different boundary conditions and different geometries would be of considerable interest.

The present practice of treating a shell as shallow or deep is more or less arbitrary, i.e., it has not been evolved through theoretical investigations. Hence, one of the focus of future research in this area can be to determine theoretically the critical Rise/Span ratio and curvature at which the shell changes from shallow to deep. This investigator believes that the boundary conditions, apart from Rise/Span ratio and curvature would affect the treatment of a shell as shallow or deep and hence should also be taken into account in order to establish the critical ratios mentioned above.

It is also suggested that more investigations, experimental as well as theoretical should be carried out to refine and confirm the results of this investigation.

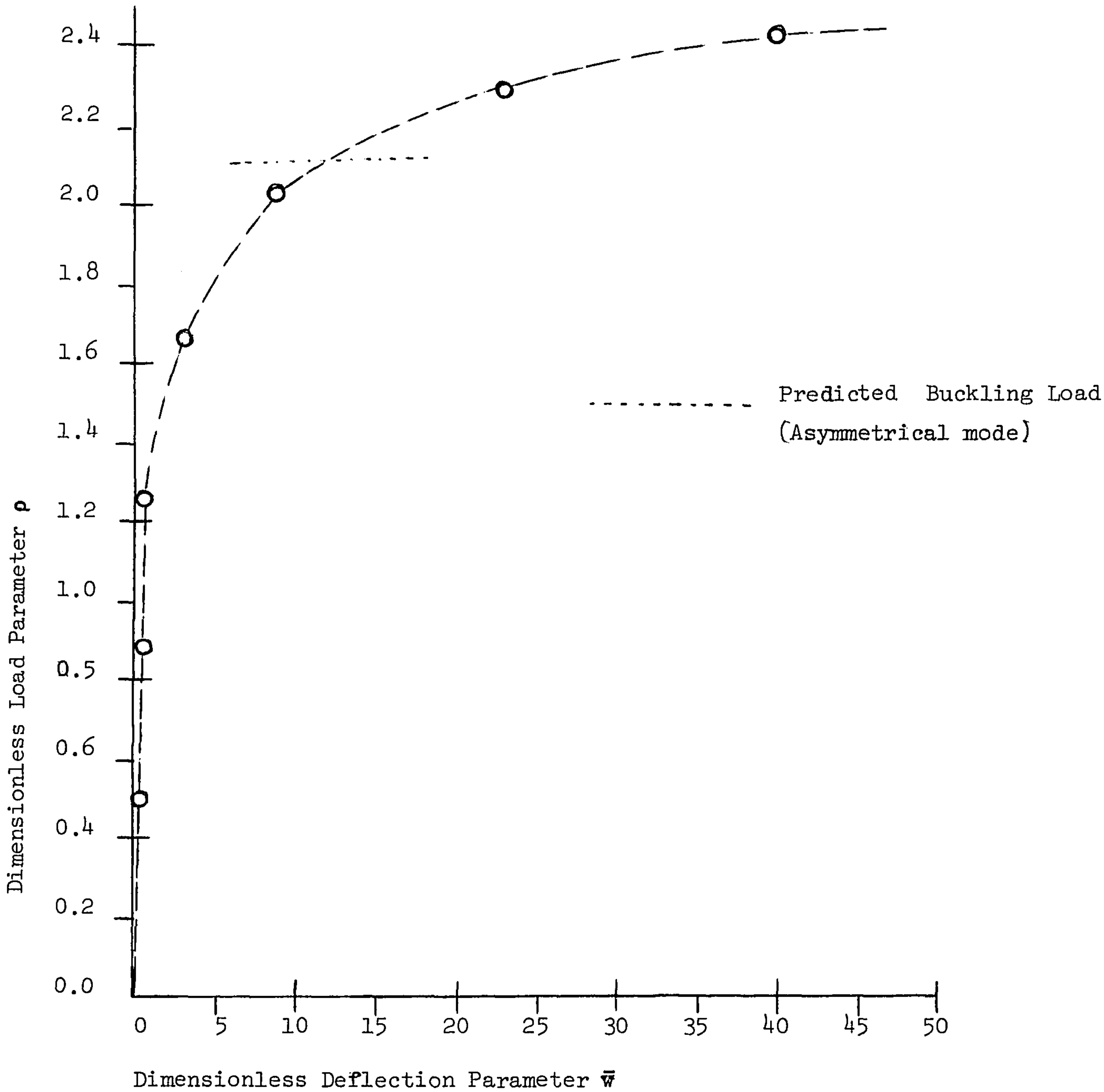


Figure 7 - Load vs Average Deflection Curve (Case No. 1)

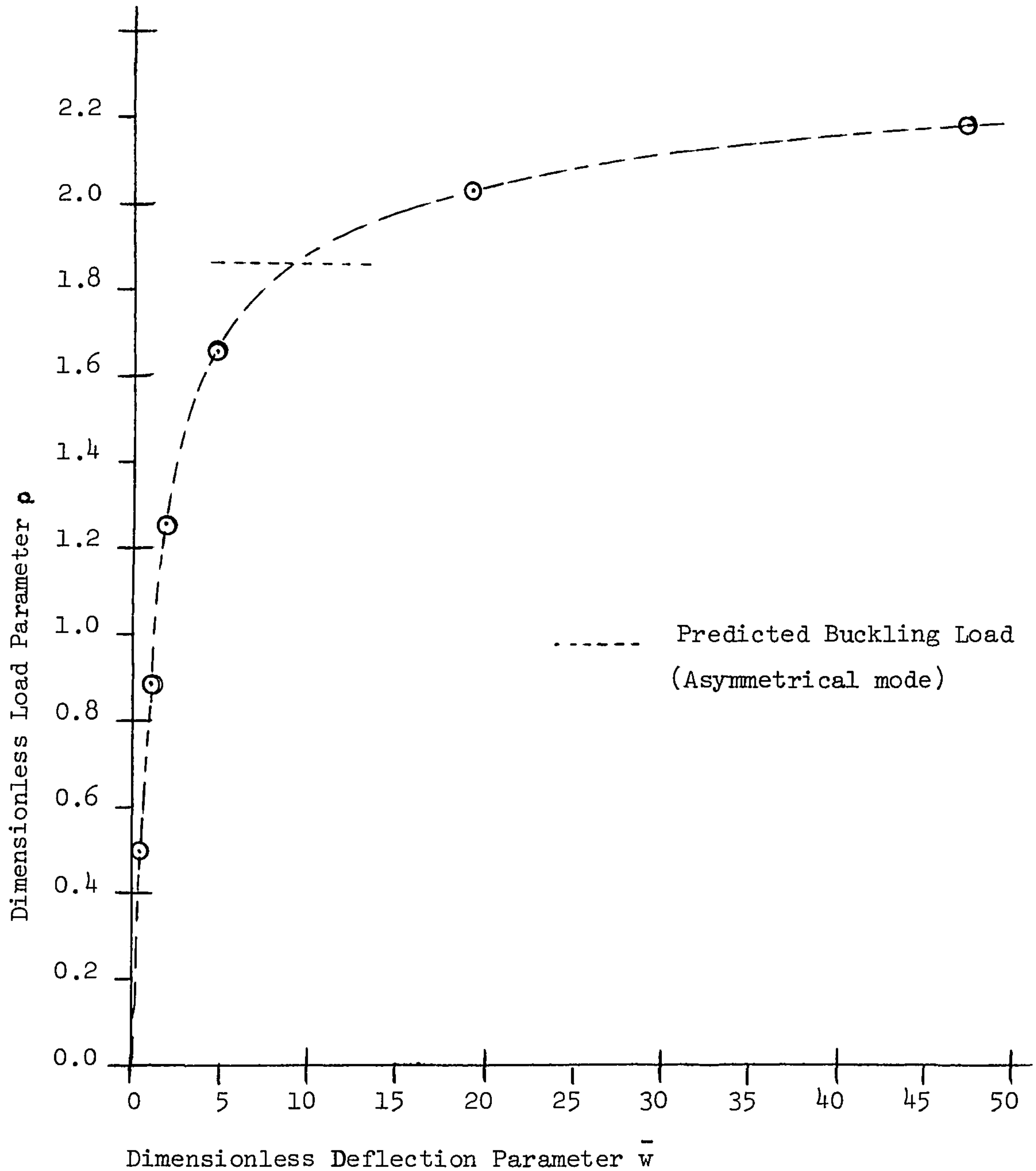


Figure 8 Load vs Average Deflection Curve (Case No. 2)

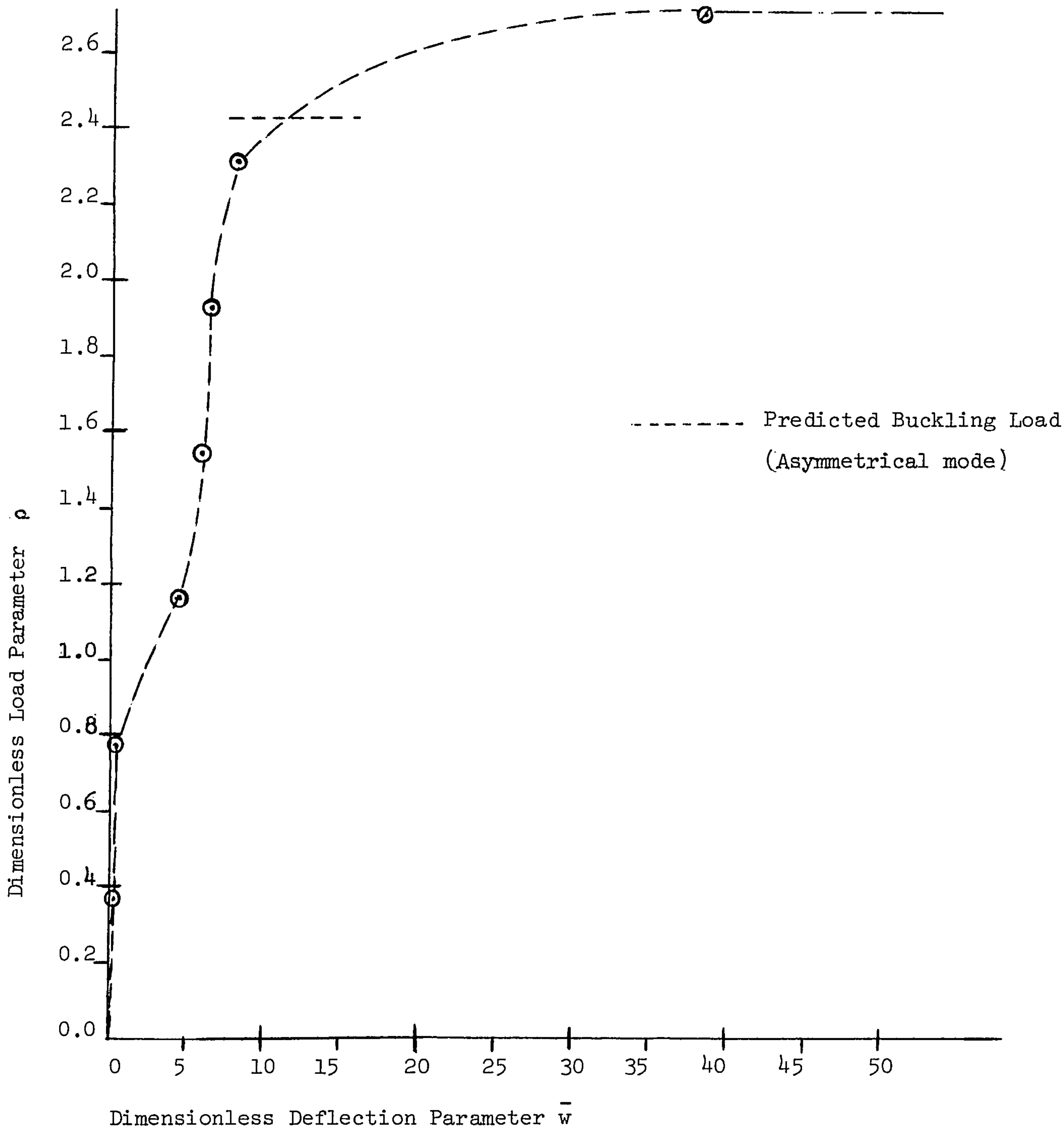


Figure 9 Load vs Average Deflection Curve (Case No. 3)

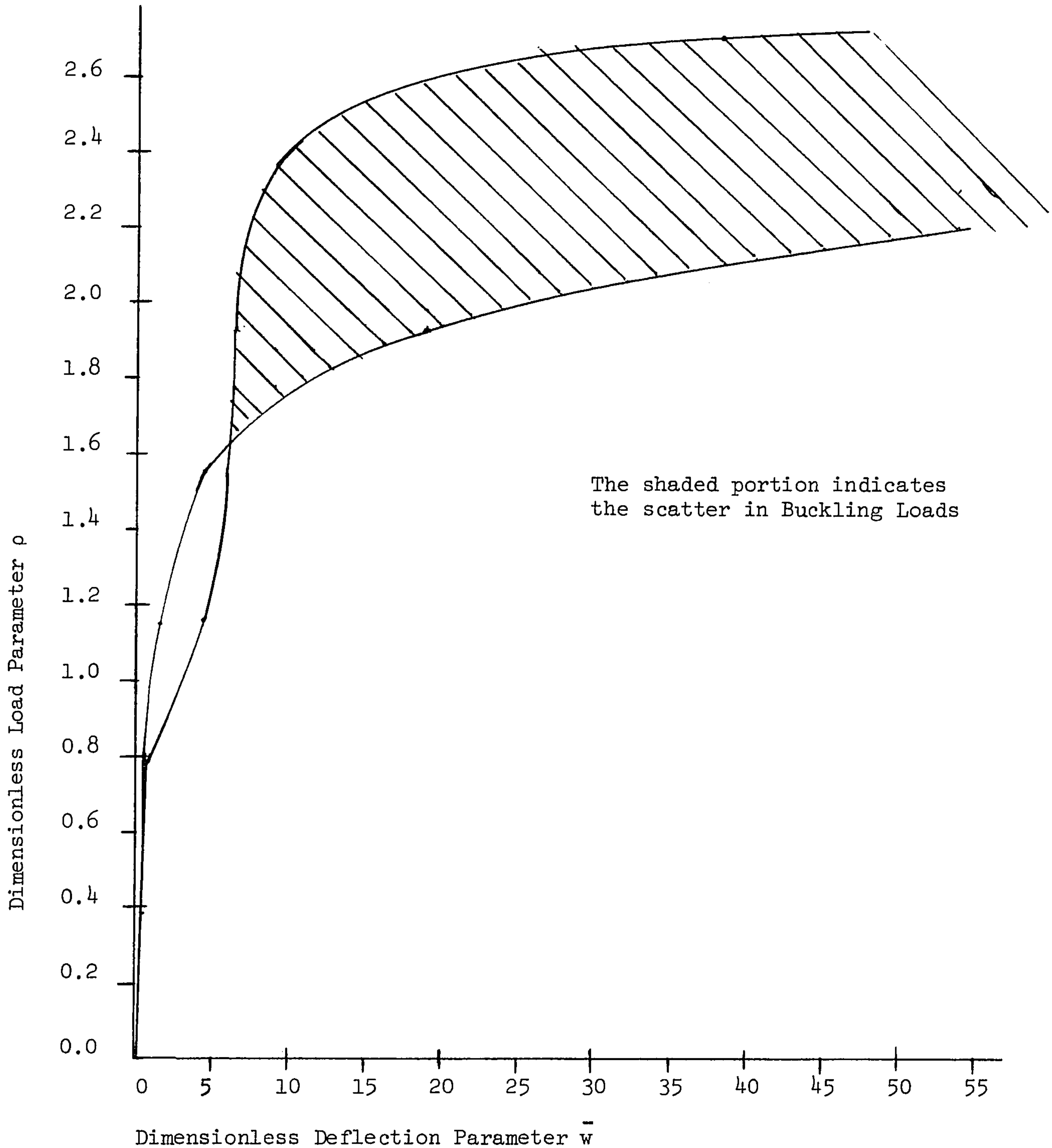


Figure 10 Load vs Average Deflection Curve Illustrating Scatter

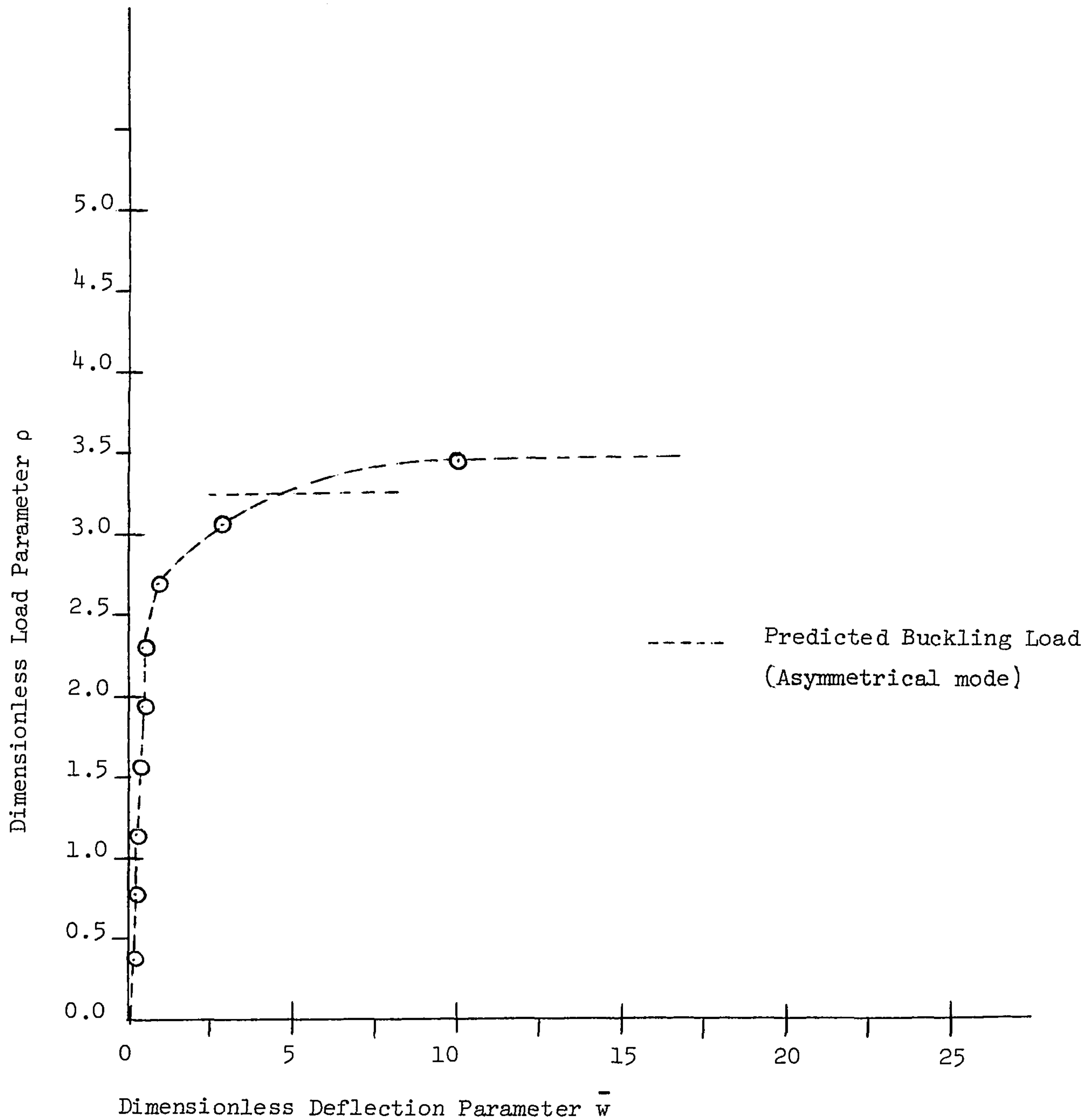


Figure 11 Load vs Average Deflection Curve (Case No. 4)

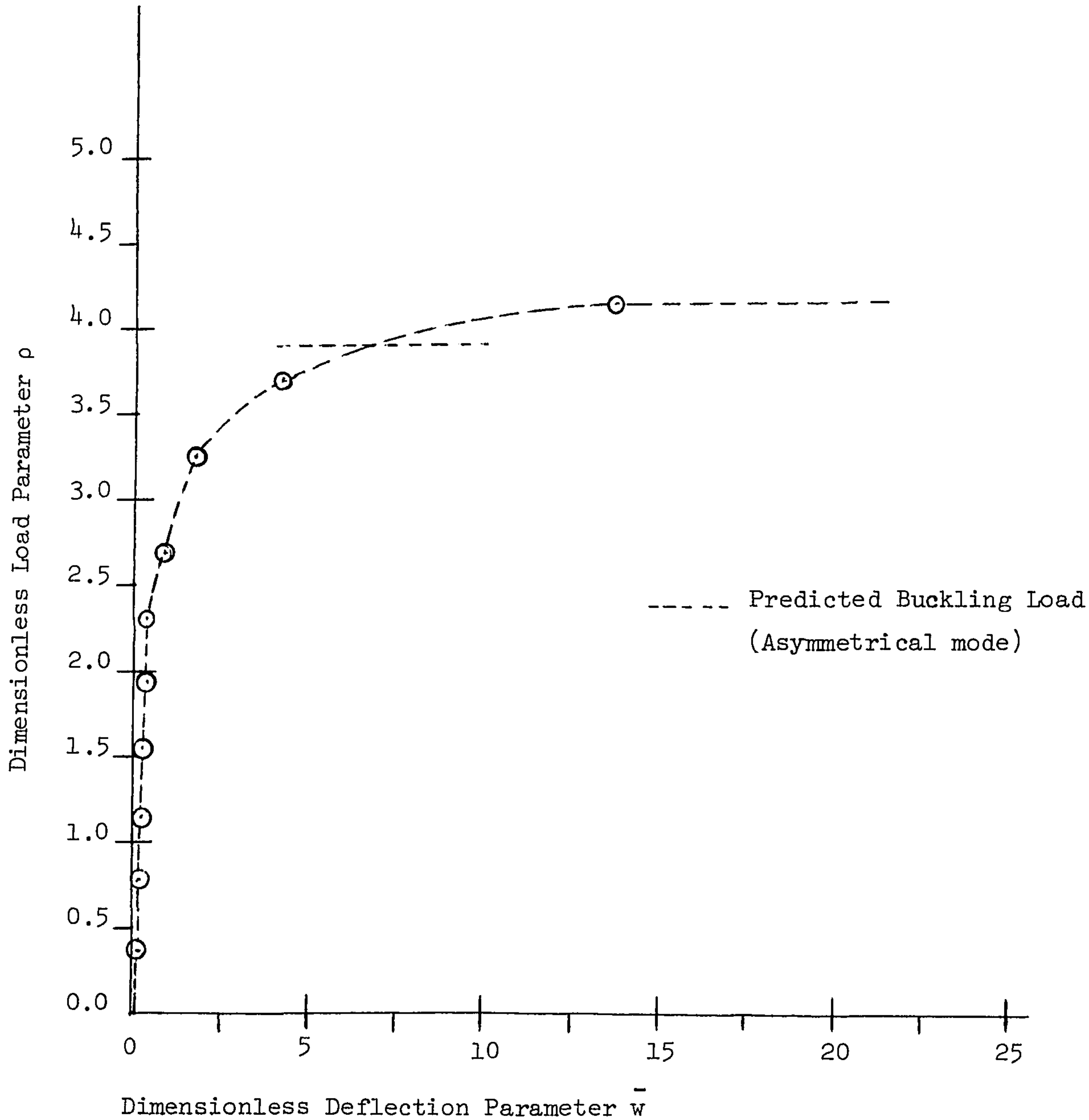


Figure 12 Load vs Average Deflection Curve (Case No. 5)

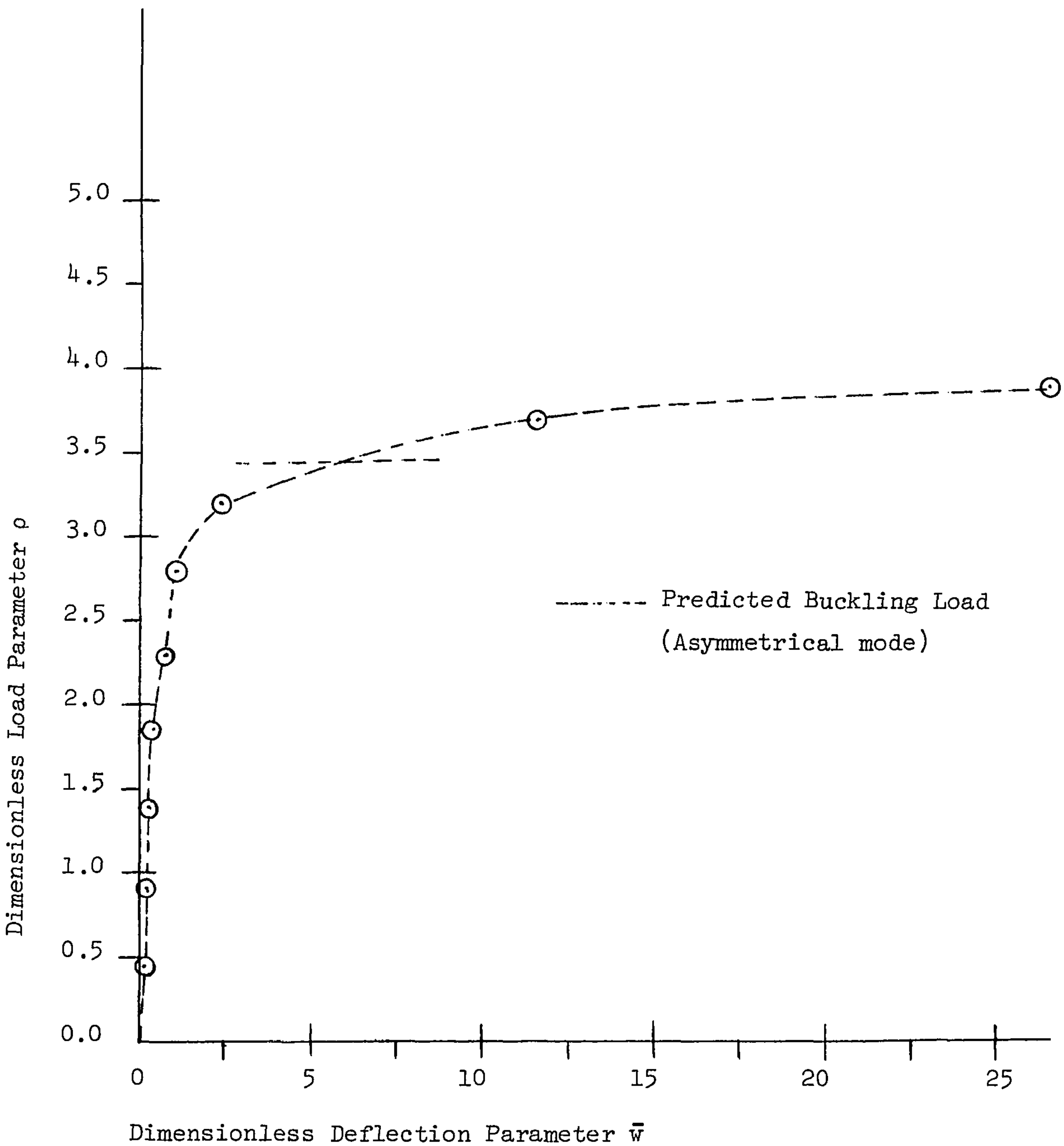


Figure 13 Load vs Average Deflection Curve (Case No. 6)

APPENDIX A

$$Q^{j} = \int_{o}^{b} \sin\left(\frac{j\pi y}{b}\right) dy = \frac{2b}{j\pi}$$

$$Q_{1}^{jn} = \int_{o}^{b} \sin\left(\frac{n\pi y}{b}\right)\sin\left(\frac{j\pi y}{b}\right) dy = \begin{Bmatrix} 0 & \text{if} & n \neq j \\ \frac{b}{2} & \text{if} & n = j \end{Bmatrix}$$

$$Q_{2}^{jn} = \int_{o}^{b} \cos\left(\frac{m\pi y}{b}\right)\sin\left(\frac{j\pi y}{b}\right) dy = 0 \quad \text{if } (j - n) \text{ is even}$$

$$Q_{12}^{jmn} = Q_{21}^{jmn} = \int_{o}^{b} \sin\left(\frac{m\pi y}{b}\right)\cos\left(\frac{n\pi y}{b}\right)\sin\left(\frac{j\pi y}{b}\right) dy = 0$$

$$Q_{11}^{jmn} = \int_{o}^{b} \sin\left(\frac{m\pi y}{b}\right)\sin\left(\frac{n\pi y}{b}\right)\sin\left(\frac{j\pi y}{b}\right) dy$$

$$= \frac{b}{\pi}\left\{\frac{j}{j^{2} - (m - n)^{2}}\right\} - \frac{b}{\pi}\left\{\frac{j}{j^{2} - (m + n)^{2}}\right\}$$

$$Q_{22} = \int_{o}^{b} \cos\left(\frac{m\pi y}{b}\right)\cos\left(\frac{n\pi y}{b}\right)\sin\left(\frac{j\pi y}{b}\right) dy$$

$$= \frac{b}{\pi}\left\{\frac{j}{j^{2} - (m - n)^{2}}\right\} + \frac{b}{\pi}\left\{\frac{j}{j^{2} - (m + n)^{2}}\right\}$$

$$Q_{221}^{jmnp} = \int_{o}^{b} \cos\left(\frac{m\pi y}{b}\right)\cos\left(\frac{n\pi y}{b}\right)\sin\left(\frac{p\pi y}{b}\right)\sin\left(\frac{j\pi y}{b}\right) dy$$

$$Q_{221}^{jmnp} = \begin{Bmatrix} 0 & \text{if } (m - n)^{2} \neq (p - j)^{2} \\ b/8 & \text{if } (m - n)^{2} = (p - j)^{2} \end{Bmatrix} - \begin{Bmatrix} 0 & \text{if } (m - n)^{2} \neq (p + j)^{2} \\ b/8 & \text{if } (m - n)^{2} = (p + j)^{2} \end{Bmatrix}$$

$$+ \begin{Bmatrix} 0 & \text{if } (m + n)^{2} \neq (p - j)^{2} \\ b/8 & \text{if } (m + n)^{2} = (p - j)^{2} \end{Bmatrix} - \begin{Bmatrix} 0 & \text{if } (m + n)^{2} \neq (p + j)^{2} \\ b/8 & \text{if } (m + n)^{2} = (p + j)^{2} \end{Bmatrix}$$

$$Q_{111}^{jmnp} = \int_0^b \sin\left(\frac{m\pi y}{b}\right)\sin\left(\frac{n\pi y}{b}\right)\sin\left(\frac{p\pi y}{b}\right)\sin\left(\frac{j\pi y}{b}\right)dy$$

$$Q_{111}^{jmnp} = \begin{Bmatrix} 0 & \text{if } (m-n)^2 \neq (p-j)^2 \\ b/8 & \text{if } (m-n)^2 = (p-j)^2 \end{Bmatrix} - \begin{Bmatrix} 0 & \text{if } (m-n)^2 \neq (p+j)^2 \\ b/8 & \text{if } (m-n)^2 = (p+j)^2 \end{Bmatrix}$$

$$- \begin{Bmatrix} 0 & \text{if } (m+n)^2 \neq (p-j)^2 \\ b/8 & \text{if } (m+n)^2 = (p-j)^2 \end{Bmatrix} - \begin{Bmatrix} 0 & \text{if } (m+n)^2 \neq (p+j)^2 \\ b/8 & \text{if } (m+n)^2 = (p+j)^2 \end{Bmatrix}$$

APPENDIX B

$$Q_1^n = \int_0^b \sin^2(\frac{n\pi y}{b})\,dy = \frac{b}{2}$$

$$Q_{11}^{mn} = \int_0^b \sin(\frac{m\pi y}{b})\sin^2(\frac{n\pi y}{b})\,dy$$

$$= \begin{bmatrix} \frac{b}{\pi m} & \text{if m is odd} \\ 0 & \text{if m is even} \end{bmatrix} - \begin{bmatrix} \frac{b}{\pi}(\frac{m}{m^2 - 4n^2}) & \text{if (m - 2n) is odd} \\ 0 & \text{if (m - 2n) is even} \end{bmatrix}$$

$$Q_{22}^{mn} = \int_0^b \cos(\frac{m\pi y}{b})\cos(\frac{n\pi y}{b})\sin(\frac{n\pi y}{b})\,dy$$

$$= \begin{bmatrix} \frac{b}{\pi}\left(\frac{n}{n^2 - (m+n)^2}\right) & \text{if } [n - (m+n)] \text{ is odd} \\ 0 & \text{if } [n - (m+n)] \text{ is even} \end{bmatrix}$$

$$+ \begin{bmatrix} \frac{b}{\pi}\left(\frac{n}{n^2 - (m-n)^2}\right) & \text{if } [n - (m-n)] \text{ is odd} \\ 0 & \text{if } [n - (m-n)] \text{ is even} \end{bmatrix}$$

$$Q_{111}^{mpn} = \int_0^b \sin(\frac{m\pi y}{b})\sin(\frac{p\pi y}{b})\sin^2(\frac{n\pi y}{b})\,dy$$

$$= \begin{Bmatrix} 0 & \text{if } m - p \neq 2n \\ -b/8 & \text{if } m - p = 2n \end{Bmatrix} + \begin{Bmatrix} 0 & \text{if } m + p \neq 2n \\ b/8 & \text{if } m + p = 2n \end{Bmatrix}$$

$$Q_{122}^{mpn} \equiv Q_{212}^{mpn} = \int_0^b \cos(\frac{m\pi y}{b})\sin(\frac{p\pi y}{b})\cos(\frac{n\pi y}{b})\sin(\frac{n\pi y}{b})\,dy$$

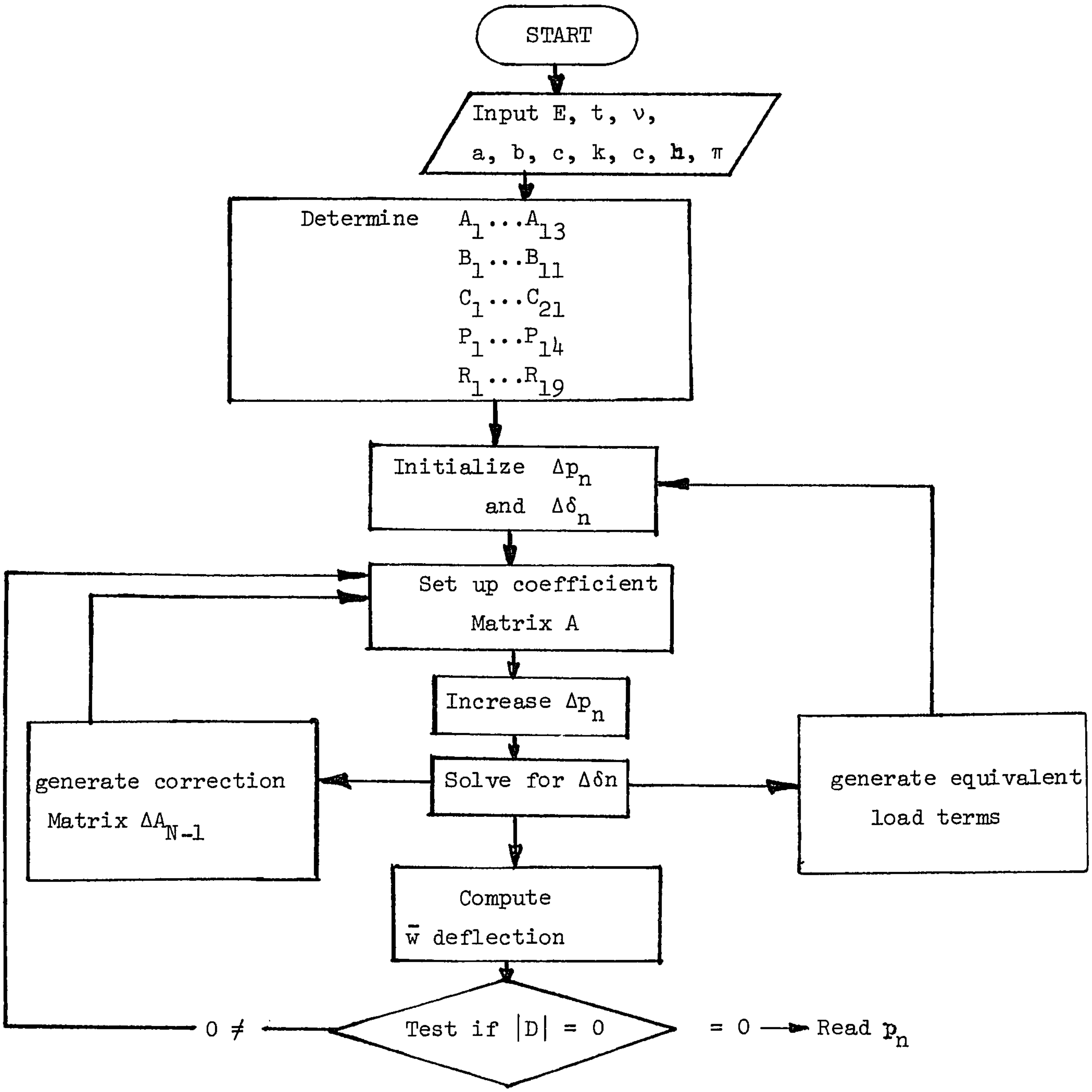
$$= \begin{Bmatrix} 0 & \text{if } m + p \neq 2n \\ b/8 & \text{if } m + p = 2n \end{Bmatrix} - \begin{Bmatrix} 0 & \text{if } m - p \neq 2n \\ b/8 & \text{if } m - p = 2n \end{Bmatrix}$$

$$Q_{221}^{mpn} = \int_{0}^{b} \cos\left(\frac{m\pi y}{b}\right)\cos\left(\frac{p\pi y}{b}\right)\sin^2\left(\frac{n\pi y}{b}\right)dy$$

$$= \begin{Bmatrix} 0 & \text{if } m + p \neq 2n \\ -b/8 & \text{if } m + p = 2n \end{Bmatrix} - \begin{Bmatrix} 0 & \text{if } m - p \neq 2n \\ b/8 & \text{if } m - p = 2n \end{Bmatrix}$$

APPENDIX C

FLOW CHART FOR THE COMPUTER PROGRAM



REFERENCES

(1) Budiansky,B. and Hutchinson J.W., A Survey of Some Buckling Problems. AIAA Journal, Vol. 4, No. 9, September 1966, pp. 1505-1510.

(2) Chu,K.H. and Turula,P., Buckling of Open Cylindrical Shells Under Lateral Load. Journal of Engineering Mechanics Division, December 1970, A.S.C.E.

(3) Cohn,C.E., Efficient Programming in Fortran. Software Age, Vol. 2, No. 5, June 1968, pp. 22-31.

(4) Donnell,L.H., A New Theory for the Buckling of Thin Cylinders Under Axial Compression and Bending. Trans. A.S.M.E. Vol. 56, 1934, p. 795.

(5) Donnell,L.H., Effect of Imperfections on Buckling of Thin Cylinders with Fixed Edges Under External Pressure. Proceedings of Third U.S. National Congress of Applied Mechanics (June 1958, Providence, R.I.), ASME, 1958, pp. 305-311.

(6) Donnell,L.H. and Wan,C.C., Effect of Imperfections on Buckling of Thin Cylinders and Columns Under Axial Compression. Journal of Applied Mechanics, March 1950, pp. 73-82.

(7) Dill,E.H., General Theory of large Deflections of Thin Shells. National Aeronautics and Space Administration, Technical Note D - 826,1959, pp. 16-18.

(8) Feodos'ev,V.I., On a Method of Solution of the Nonlinear Problems of Stability of Deformable Systems. Journal of Applied Mechanics, Vol. 27, 1963, pp. 392-404.

(9) Hiroyuki,Abe, A Nonlinear Shell Theory and its Application to Circular Cylinders. Nonlinear Mechanics, Vol. 4, pp. 107-121, Pergamon Press 1969.

(10) Hutchinson, J.W., Imperfection-Sensitivity of Externally Pressurized Spherical Shells. Report SM-5, Harvard University (1965).

(11) Hutchinson, J.W., Initial Post-Buckling Behavior of Toroidal Shell Segments. Report SM-6, Harvard University (1965).

(12) Karakas, J. and Scalz, M., Test Load of Shell Fails at Design Load. Civil Engineering, March 1961.

(13) Karman, Th. Von, and Tsien, H.S., The Buckling of Thin Cylindrical Shells under Axial Compression. Journal Aero. Sci. 8, (1941) 303.

(14) Kempner, J., Postbuckling Behavior of Axially Compressed Circular Cylindrical Shells. Journal Aero Sci. 21 (1954) 329.

(15) Koiter, W.T., On the Stability of Elastic Equilibrium.Thesis Delft (in Dutch with English summary) H.J. Paris, Amsterdam (1945).

(16) Koiter, W.T., Elastic Stability and Post Buckling Behavior. Proceedings of the Symposium on Nonlinear Problems, Edited by R.E. Langer, University of Wisconsin Press 1963, p. 257.

(17) Koiter, W.T., On the Stability of Elastic Stability. Translation from the Dutch, Technical Report No. AFFDL - TR 20-25, Air Force Flight Dynamics Laboratory, Air Force Systems Command, WPAFB, Ohio.

(18) Kraus, H., Thin Elastic Shells. John Wiley and Sons, Inc., 1967.

(19) Leggett, D.M.A. and Jones, R.P.N., The Behavior of a Cylindrical Shell under Axial Compression when the Buckling Load Has Been Exceeded. Aero. Res. Council, London, Reports and Memoranda 2190 (1942).

(20) Mak, C.K.K. and Wen, R.K., Large Deflection Behavior of Cylindrical Shell Panels. Developments in Mechanics pp. 187-205. Proceedings of the 10th Midwestern Mechanics Conference, 1968.

(21) Mushtari, Kh.M. and Galimov, K.Z., Nonlinear Theory of Thin Elastic Shells. (English Translation from Russian) Published for the National Science Foundation and the National Aeronautics and Space Administration by the Israel Program for Scientific Translation, 1961.

(22) Michielsen, H.F., The Behavior of Thin Cylindrical Shells after Buckling under Axial Compression. Journal Aero. Sci. 15 (1948) 738.

(23) Soare, M., Application of Finite Difference Equations to Shell Analysis. Pergamon Press, 1967.

(24) Stack Staikidis, W.J., Nonlinear Buckling of Cylindrical Shells. Computer and Structures, Vol. 2, pp. 615-624, Pergamon Press, 1972.

(25) Stein, M., Some Recent Advances in the Investigation of Shell Buckling. pp. 2339-2345, AIAA Jouranl, Vol. 6, No. 12, December 1968.

(26) Thielemann, W.F., New Developments in the Nonlinear Theories of the Buckling of Thin Cylindrical Shells. International Series on Aeronautical Sciences and Space Flight, AFOSR TR 1960, pp. 76-117.

(27) Thompson, J.M.T., The Rotationally-Symmetric Branching Behavior of a Complete Spherical Shell. Proc. Kon.Ned.Ak.Wet, Amsterdam B67 (1964) 295.

(28) Timoshenko, S.P. and Gere, J.M., Theory of Elastic Stability. McGraw-Hill Book Co., 1961.

(29) Timoshenko, S. and Woinowsky, Krieger S., Theory of Plates and Shells. McGraw-Hill Book Co., 1959.

(30) Turula, P. and Chu, K.H., Buckling of Open Cylindrical Shells with Imperfections. Journal of the Engineering Mechanics Division, December 1970, A.S.C.E.

(31) Yang, T.H. and Guralnick, S.A., An Experimental Study of the Buckling of Open Cylindrical Shells. pp. 121-127, Experimental Mechanics, April 1975.

VITA

Madhusudan H. Jhaveri, the son of Mr. and Mrs. Hiralal Jhaveri was born on   in

Received B.E. (C.E.) with First class honors from S.V.V. India in 1963.

1963	Presented a Technical Paper titled "Plastic Methods of Design as Applied to Steel Structures" in a Graduating Seniors competition and received the First Prize from Institute of Engineers (India).
1963 - 1966	Served as Associate Lecturer in Civil Engineering at Sardar Patel College of Engineering, Bombay.
1967	Received M.S. (C.E.) from North Dakota State University, Fargo, North Dakota.
1968 - 1970	Served as Structural Engineer in Port of New York Authority and New York City Transit Authority.
1970 - 1972	Teaching Fellow at Newark College of Engineering
1972 - 1973	Served as Senior Design Engineer - in P.W.D. Franklin Township, Franklin, New Jersey.
1973 - 1974	Served as Instructor in C.E. Department of C.C.N.Y.
1974 to date	Serving as Assistant Professor in Civil Engineering at Southeastern Massachusetts University, North Dartmouth, Massachusetts