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BY

MOSES NEE BUERNOR AYIKU

A DISSERTATION  
PRESENTED IN PARTIAL FULFILLMENT OF  
THE REQUIREMENTS FOR THE DEGREE  
OF  
DOCTOR OF ENGINEERING SCIENCE  
AT  
NEW JERSEY INSTITUTE OF TECHNOLOGY

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THEORY TO SOLVE CERTAIN  
RELIABILITY PROBLEMS

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FOR

DEPARTMENT OF ELECTRICAL ENGINEERING  
NEW JERSEY INSTITUTE OF TECHNOLOGY

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ABSTRACT

It is well known that in actual control systems there are uncontrolled parameter changes caused by aging of elements, temperature, pressure and effects of external medium, among others, which may impair the performance of the system. Hence, there is the need to develop systems that weakly react to these parameter fluctuations. Sensitivity theory has been developed to study some of these problems. Most of the investigations in this area, however, deal with problems in continuous systems modeled by ordinary differential equations. Even the recent publications on sensitivity problems in distributed parameter control systems are largely concerned with systems modeled by first order partial differential equations.

In this dissertation, the study of parameter sensitivity is extended to higher order distributed parameter control systems. The dynamic system of interest is represented by a non-linear higher order vector partial differential equation and its associated matrix sensitivity equation. The problem posed is that of minimizing a cost functional consisting of both the performance and trajectory sensitivity indices subject to the state and sensitivity equations of the system. By means of variational techniques, the necessary conditions for optimality are obtained. The sufficient conditions are also derived using the theory of convexity.



The theory developed is applied to two classes of reliability models of wide-applicability. These are represented by the standby redundant system and the semi-infinite parabolic PDE. The resulting co-state equations, together with the system's equations, are discretized in both space and time. Algorithms are then developed to integrate these equations. The results are presented in the form of state and sensitivity profiles for a given set of conditions. The variation of the functional performance index with respect to changes in system parameter is also presented.

The study concludes with a series of numerical examples to illustrate the theory and technique in modeling various sensitivity problems in distributed parameter control systems. In particular, the problem of achieving a compromise among a set of design objectives is emphasized.

This dissertation is essentially an extension of low sensitivity design theory to distributed parameter systems.

TO MY FAMILY - IN THE WIDEST  
POSSIBLE SENSE

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CHAPTER I  
INTRODUCTION

Dynamic systems have been studied by classical mechanics and solved satisfactorily. Like any other branch of science or mathematics, studies of dynamic systems, which are physical processes mathematically modeled by differential equations, have to surmount certain problems. One of these problems is stability of motion due to the fact that real dynamic systems are exposed to disturbances which may affect the response of the system.

These disturbances, partly due to parameter changes, provide an important link between physical dynamic systems and their mathematical models. Hence, the most significant investigation in this connection is the study of stability as a result of variations in the system's parameters. Such investigations have led to new concepts such as parameter sensitivity, and trajectory sensitivity among others.

Earlier investigations into problems of parameter sensitivity of dynamic systems proved mathematically cumbersome. The advent of high speed electronic computers, however, facilitated computations and thereby eased the problem of extending the study of parameter sensitivity in dynamic systems to areas such as multi-variable systems (91). In addition, more sophisticated types of investigations other than straight forward stability analysis or study of behavior of sensitivity coefficients had been encouraged and

continue to be encouraged as better and faster computers became available. For example, the study of the effects of disturbances in dynamic systems has been extended to include the study of trajectory, initial condition, eigenvalue and eigenvector sensitivities. (92).

In spite of the advent of high speed computers, sensitivity studies have not been vigorously pursued outside the area of dynamic systems modeled by ordinary differential equations. It has not been easy to handle the different types of partial differential equations. For, while there is a fairly general and unified theory for ordinary differential equations, there is no such theory for partial differential equations.

In this investigation, the study of sensitivity in distributed parameter systems represented by some classes of reliability problems is undertaken. Prior published works in the area of sensitivity analysis in control systems, with particular reference to multiple-index optimization in distributed systems, will be considered in this chapter. The specific objectives of this research will also be stated.

#### A. Brief Survey of Control System Sensitivity

Recent investigators have developed techniques for the study of sensitivity problems in conjunction with system optimization. (50). This technique of optimizing the system's performance index while at the same time reducing

the sensitivity of the system due to parameter variations appears particularly attractive. Unfortunately, this approach is not without its problems. For, it raises some computational issues in addition to questions concerning the efficiency of such an approach and also practical implementation of the control system. (25, 50). But the study of sensitivity, especially techniques of design, have not reached the point where definitive answers can be given to these questions posed by Kreindler. (50). Although sensitivity studies were first initiated by Bode (10) as far back as 1945, it was not until 1963 when the usefulness of sensitivity analysis in the design of optimum control systems was realized. (28). Since then problems in control system sensitivity have captured the attention of researchers and textbook writers. (25, 50, 92). Considerable attention also continues to be devoted to sensitivity in optimal control systems at national and international conferences. (69).

Since 1963, three major surveys on the subject of sensitivity and sensitivity problems in optimal control systems have been published. The first paper published in 1964 gave a systematic survey of researches on the sensitivity of automatic control systems and of the application of the results of these researches to the synthesis of control systems including adaptive systems. (47). This first survey dealt mostly with Soviet literature. The second survey was published by Sobral in 1968. (87). It summarized the

then current developments on the subject, and essentially updated the earlier survey. The third survey, apparently the most recent, was published in 1971 by Ngo. (59). This survey reviewed the publications on sensitivity in discrete control systems.

A recent important monograph (92), written in the style of a survey primarily considered problems related to trajectory sensitivity in continuous systems.

The above monograph, papers, books and surveys account for most of the literature on the subject. In later sections, various aspects of sensitivity in optimal control systems will be considered. Before doing so, the general problem of control of distributed parameter systems will be briefly outlined.

#### B. Optimum Control of Distributed Parameter Systems

The distributed parameter system of interest in this study has some spatial domain  $D$  which is contained in the Euclidean space  $R^n$  ( $n \geq 1$ ). In many such systems this domain is fixed, closed and bounded. It may even be infinite or semi-infinite in extent. In more difficult problems the boundaries may be movable. There is also a time domain  $T$  which is usually finite or semi-infinite.

These distributed parameter systems are mathematically characterized by either partial differential equations or

integral equations. In the case of partial differential equations, there are also boundary conditions to be considered. There is then some functional dependent on the state of the system, its boundary conditions, its controls, or some combination of these. The usual optimal control problem is either to find the control which minimizes the given functional (open-loop control) or to find the functional relationship between the control and the system state in order that the functional is minimized (closed-loop control).

The analytical study of the design of optimum distributed parameter systems was first initiated by Butkovskii (14) who dealt with the integral representation and the maximum principle for distributed parameter systems. The dynamic programming approach was studied by Wang and Tung (95), among others. Kim and Gajwani (43) derived the canonical equations, i.e., the necessary conditions for optimality, by variational techniques and thereby improved upon the earlier work by Lurie. (53). The most comprehensive survey on optimal control of distributed parameter systems was published by Robinson (74) in 1969. Earlier surveys include those by Wang (94) and Butkovskii et al. (17). The latter dealt with the Soviet literature on the subject. Thus, the theory of optimal control of distributed parameter systems is apparently well established. Unfortunately, the inherent difficulty in handling the mathematics of distributed systems inhibit extensive applied research in this area. The problem

of searching for an efficient mathematical approach to distributed parameter problems continues to engage the attention of researchers (79, 94). Indeed, a few dissertations have already been written on this subject. (83, 48).

The classical theory of distributed control followed closely that of single parameter control modeled by ordinary differential equations. The objective, for example, is still the minimization of some cost functional subject to some constraints. On the other hand, the solutions of the resulting co-state equations are more difficult to solve than those for continuous systems. Techniques have been developed for tackling the distributed two-point boundary value problems. These follow closely some of the techniques developed for solving two-point boundary value problems in lumped parameter systems. (83).

### C. Sensitivity in Continuous Systems

We have already noted that the study of control system sensitivity started with the origins of feedback system theory. In fact, the basic concepts in this area first appeared in the fundamental work of Bode (10) which constituted the beginnings of modern theory of feedback systems. The usefulness of sensitivity in system design was soon apparent (28, 40, 90). Cruz and Perkins (22), Rohrer and Sobral (75) and others have also investigated sensitivity problems in optimal control. An excellent overview of sensitivity theory is provided by Kreindler. (49). Most of the

earlier works on sensitivity in continuous systems were concerned with deterministic parameter sensitivity. Thus, Gonzales' (35) extension into multiparameter sensitivity measures which are applicable to cases where the parameters are stochastic is significant. White (97) reviewed several types of sensitivity measures and defined two new measures, integral and peak sensitivity, in terms of sensitivity functions.

Horowitz (40) showed that sensitivity analysis need not be restricted to small parameter deviations. He, therefore, introduced the concept of "sensitivity in the large" and demonstrated that systems which suitably **compensated** for large parameter deviations exhibited qualities generally considered attainable only by means of adaptive control.

Dorato's (28) earlier formulation of the performance index sensitivity problem for both open-loop and closed-loop control systems was extended by Pagurek (60) to include a broader class of systems. But it was Kreindler (49) who in 1967 considered the general problem of reducing performance index sensitivity in a system subject to a single parameter variation, by means of adding a measure of performance index sensitivity to the original performance index.

Other measures of sensitivity of importance in optimal control are trajectory sensitivity (50), relative sensitivity (75) and terminal condition sensitivity. (39, 36).

In spite of the multiplicity of measures of sensitivity, the effects of parameter changes on the behavior of a system is manifested in two main ways. (87). First, the motion of the system - the state trajectory - may deviate from the nominal trajectory. This is called the trajectory sensitivity. Second, the performance index may also differ from that value associated with the nominal trajectory. This will be called performance index sensitivity. In the theory of synthesis of low sensitivity control systems these two types of sensitivity undoubtedly receive most of the attention of researchers. Accordingly, it is proposed here to define and examine each of them.

1. Performance index sensitivity

Sensitivity, in the classical sense, is defined as the ratio of a relative change in a desired quantity to the relative change in a parameter of the system (10, 91). It is assumed, in this definition, that changes in the parameters are uniform and small. (This may not be true in practical systems.) By a limiting process, the above definition leads to the sensitivity coefficient of a system. (91). This sensitivity coefficient is defined as the partial derivative of a variable relative to the system parameter evaluated at the nominal values of the parameter. Thus, the sensitivity coefficients only convey information about a small neighborhood of a point in parametric space. (45).



A more general approach to sensitivity studies is obtained by using the performance index as a measure of the behavior of the system. Performance index sensitivity is thus defined as the derivative of the performance index with respect to system parameters.

In extending Dorato's (28) method of computing performance index sensitivity functions, Pagurek (60) obtained the rather strange result that the first variation of the performance index caused by a variation of the system parameter is the same whether an open-loop or a closed-loop implementation is used. It turned out, however, that Pagurek's result is a special case of a more general result. (62). Also, Pagurek's result applies if one is concerned only with variations of the performance index. But if variations in the state are the main concern, then in a certain sense a closed-loop system is better than an open-loop system. (87).

## 2. Trajectory sensitivity

The trajectory sensitivity matrix,  $\bar{v}(t)$ , is defined as the variation of the state variable  $\underline{x}(t)$ , due to small variations of the system parameter,  $\underline{a}$ . This is given by the relation,

$$\bar{v} = \frac{\partial \underline{x}}{\partial \underline{a}} \quad (1.1)$$

The elements of the matrix  $\bar{v}$  constitute what we have already defined as sensitivity coefficients.

Many authors devote considerable attention to the development of techniques for generating and solving for these coefficients, notably by means of sensitivity equations. (9, 91).

In this connection we note Butkovskii's (1958), structural approach which essentially eliminated the need to solve the sensitivity equations.

Since trajectory sensitivity coefficients are independent of state vectors, they are normally treated as another type of state vector. This is important since it means that a trajectory sensitivity coefficient,  $v(t)$ , may be adjoined to a state to form an augmented State  $X = [x, v]$ . In the same fashion, the original performance index may be augmented by addition of some positive definite function of the sensitivity vector. This, in fact, provides the basis and legitimacy of multiple-index optimization. (49). This approach has been successfully applied to continuous systems. (54).

#### D. Parameter Sensitivity in Discrete Systems

An early investigator of sensitivity in discrete systems was Lindorff. (52). Others include Radanovic (69) who in 1966 presented the theory of sensitivity analysis for sampled-data (discrete and discrete-continuous) systems. King (44) extended the techniques of sensitivity analysis to the class of systems whose mathematical description leads to a set of linear differential-difference equations.

Methods for systematic adjustment of sampling rates based on sensitivity considerations were developed by Tomovic and Bekey (89) and Bennet and Sage (7), among others.

We note the fact that the fundamental work in sensitivity was parameter sensitivity. In discrete systems this was studied either by expressing the change in state in terms of a perturbation matrix or by means of a sensitivity-vector function. (79). Generally, discrete sensitivity problems are studied via appropriate adaptations of continuous sensitivity definitions and sensitivity theory.

#### E. Sensitivity in Distributed Parameter Systems

Extending Cruz and Perkins' (22) generalization of the Bode (10) sensitivity criterion to multivariable stationary systems, Porter (65) demonstrated that an effective design procedure for the multivariate case can be developed from the generalized criterion. In a second paper, Porter (68) showed that certain results in system sensitivity analysis when properly formulated are valid in the domain of distributed systems.

Recently, Davis and Perkins (26) reviewed the comparison sensitivity criterion for distributed systems and derived sufficient conditions which insure satisfaction of the comparison sensitivity criterion for distributed parameter systems described by non-separable partial differential equations.

Both Seinfeld (83) and Gembicki et al (34) have considered the problem of multiple-index sensitivity optimization. Seinfeld studied sensitivity of open- and closed-loop

distributed parameter systems and solved the augmented problem for double-pipe heat exchangers with distributed and boundary controls. Gembicki's work involved only static control problems in power systems.

A highly theoretical study of eigenvalue and eigenfunction sensitivity of distributed parameter systems was recently published. (25). We may also mention the growing body of distributed parameter sensitivity studies with application to water pollution problems. (63). These water pollution problems are modeled by first order partial differential equations.

#### F. Sensitivity in Reliability Systems

Sensitivity problems encountered in reliability studies have mostly been documented as studies of parameter variations, component tolerances and application of the concept of drift failures to electrical circuits. (85). These studies are obviously useful in design of circuits or devices, since they allow the designer to gain a useful insight into the effect of parameter changes on the system. Belove (6) developed important sensitivity theorems for two- and three-element networks to facilitate the design of low sensitivity networks.

Apart from the above studies, there does not appear to be much interest in the literature on problems of sensitivity in reliability systems such as repairable structures. In

the context of reliability theory, therefore, this investigation is probably the first attempt at developing the theory and extending the multiple-index optimization techniques to the solution of distributed systems with application to reliability.

#### G. Multiple-index Optimization Involving Sensitivity Functions.

The important work of Kreindler (50) in which he proposed and discussed the optimization of an augmented performance index subject to augmented state equations has already been noted. Although this technique has been successfully applied to the linear regulator problem (54) and chemical control processes (83), it has not gained sufficient popularity among researchers. In the area of distributed parameter sensitivity, therefore, very little work has been done to date.

It is fair to note the fact that Kreindler himself raised some theoretical problems in connection with the multiple-index optimization approach. These include the problem of comparing the ordinary optimization results to those obtained after augmenting both the state and performance index. There is also the problem of implementing such an augmented control system. Kreindler, however, agreed that further experimentation was necessary.

It is this experimentation preceded by the development of the appropriate multiple-index optimization theory for

distributed parameter systems that constitute the core of this investigation.

#### H. Objectives

Recent contributions to the theory of sensitivity in optimal control have been concerned primarily with continuous systems. However, a majority of physical systems are essentially of a distributed nature. These require partial differential equations for their formulation. (83). Among these are classes of reliability problems represented by heat conduction phenomenon and standby repairable systems. Although some work has been done on the control of distributed parameter systems (48), they have not been applied specifically to reliability problems. The literature on sensitivity in distributed parameter systems is even poorer.

Therefore, this investigation will be primarily concerned with the development of sensitivity theory for distributed parameter systems, and its application to classes of reliability problems.

Specifically, the main objectives of this work are three-fold:

1. To develop by means of the calculus of variations, necessary and sufficient conditions for the optimality of the optimal distributed parameter sensitivity problem modeled by vector-

matrix partial differential equations.

2. To extend the technique of sythesis of low sensitivity optimal control to distributed parameter systems.

3. To develop some numerical algorithms for solving sensitivity problems with mixed boundary conditions.

## Chapter II

### PROBLEM FORMULATION AND STATEMENT

It is proposed in this chapter to formulate and state the general dynamic model for the classes of problems under investigation. The appropriate functional performance index, boundary conditions and classes of admissible controls will also be considered.

#### A. Formulation of the Model

The following notations are pertinent:

$\underline{a} \in R^p$ , parameter vector.

$\underline{x}(t,y) = [x_1(t,y) \dots x_n(t,y)] \in R^n$ , state vector.

$(\cdot)_t = \frac{\partial(\cdot)}{\partial t}$

$(\cdot)_{y^r} = \frac{\partial^r(\cdot)}{\partial y^r}$ , where  $r$  is a positive integer.

$\underline{u}(t,y) = [u_1(t,y) \dots u_m(t,y)] \in R^m$ , control vector.

$\underline{v}(t,y) = \frac{\partial \underline{x}}{\partial \underline{a}} \in R^n \times R^p$ , sensitivity matrix.

$\underline{w}(t,y) = \frac{\partial \underline{u}}{\partial \underline{a}} \in R^m \times R^p$ , sensitivity control matrix.

$T = [t_0, t_f]$

$\Omega = [y_0, y_f]$

$t, y, \underline{a}$  are independent variables.

$\partial \Omega$  denotes the boundary of  $\Omega$

and  $R^n$  is  $n$ -dimensional Euclidean space.



## 1. The Vector-Matrix PDE Dynamic Model

The distributed parameter system of interest may be represented by the vector partial differential equation of the form,

$$\frac{\partial^k \underline{x}(t,y)}{\partial t^k} = \underline{f}_1(\underline{a}, t, y, \underline{x}, \underline{x}_y^r, \underline{u}) \quad (2.1)$$

where  $k$  and  $r$  are positive integers

and  $\underline{f}_1(\cdot)$  represents an  $n \times 1$  continuous and differentiable vector function of the variables inside  $(\cdot)$ . For convenience the argument of  $\underline{x}$  and  $\underline{u}$  are omitted. It is clear that a multi-spatial version of equation (2.1) can easily be obtained by making the state variable,  $\underline{x}$ , a function of a vector spatial variable,  $\underline{y}$ . (78) and employing Sage's (79) notation and definition for the differential  $\frac{\partial^r \underline{x}(t, \underline{y})}{\partial \underline{y}^r}$ .

The multi-spatial case will not be considered in this investigation. For, it only complicates the mathematics without necessarily providing any additional insight into the problem. Also, only space-distributed controls, belonging to the so-called Class II systems (48), will be considered.

The important assumption is now made that the dynamic model is subject to uncontrolled parameter changes. These changes may be due to:

- (a) 'aging' of elements,
  - (b) effects of external medium,
  - (c) interaction with other systems,
  - (d) inaccuracy of calculated data,
  - (e) impossibility of precise realization of the control device,
- and (f) environmental and other effects.

It is further assumed that changes in the system parameter vector (which has a nominal value  $\underline{a}_0$ ) are small. Hence, the sensitivity equation can be obtained by partial differentiation of equation (2.1) with respect to the parameter  $\underline{a}$  of the system. (49). The result is:

$$\frac{\partial \bar{v}}{\partial t^k} = \frac{\partial f_1}{\partial \underline{x}} \bar{v} + \frac{\partial f_1}{\partial \underline{x}_y^r} \bar{v}_y^r + \frac{\partial f_1}{\partial \underline{u}} \bar{w} + \frac{\partial f_1}{\partial \underline{a}}$$

$$\underline{\Delta} \bar{f}_2(\underline{a}, t, y, \underline{x}, \underline{x}_y^r, \bar{v}, \bar{v}_y^r, \underline{u}, \bar{w}) \quad (2.2)$$

where  $\bar{f}_2 \in \mathbb{R}^n \times \mathbb{R}^p$

Without any loss of generality, let  $k = 1$ . Therefore, from (2.1) and (2.2), the general dynamic model of interest may be represented by:

$$\frac{\partial \underline{x}}{\partial t} = \underline{f}_1(\underline{a}, t, y, \underline{x}, \underline{x}_y^r, \underline{u}) \quad (2.3)$$

and 
$$\frac{\partial \bar{v}}{\partial t} = \bar{f}_2(\underline{a}, t, y, \underline{x}, \underline{x}_y^r, \bar{v}, \bar{v}_y^r, \underline{u}, \bar{w}) \quad (2.4)$$

## 2. The Functional Performance Index

Now, given

$$J_1(\underline{a}, \underline{x}, \underline{u}) = \int_{\Omega} \theta_1(\underline{x}_f, \underline{a}, t_f) d\Omega + \int_{t_0}^{t_f} \int_{\Omega} \phi_1(\underline{a}, t, y, \underline{x}, \underline{x}_y^r, \underline{u}) d\Omega dt \quad (2.5)$$

Let the scalars  $c_1$  and  $c_3$ , and px1 vector  $\underline{c}_2$  be weighting factors.

Consider the performance index,

$$J(\underline{a}, \underline{x}, \bar{v}, \underline{u}, \bar{w}) = c_1 J_1(\underline{a}, \underline{x}, \underline{u}) + c_2 \frac{\partial J_1}{\partial \underline{a}} + c_3 J_2(\underline{a}, \bar{v}, \bar{w}) \quad (2.6)$$

where,

$$J_2(\underline{a}, \bar{v}, \bar{w}) = J_1(\underline{a}, \underline{x}, \underline{u}) \left| \begin{array}{l} \underline{x} \quad \bar{v} \quad \bar{v} \\ \underline{u} \quad \bar{v} \quad \bar{w} \end{array} \right.$$

$$= \int_{\Omega} \theta_2(\bar{v}_f, \underline{a}, t_f) d\Omega + \int_{t_0}^{t_f} \int_{\Omega} \phi_2(\underline{a}, t, y, \bar{v}, \bar{v}_y r, \bar{w}) d\Omega dt \quad (2.7)$$

$$\theta_2(\bar{v}_f, \underline{a}, t_f) = \theta_1(\underline{x}_f, \underline{a}, t_f) \left| \begin{array}{l} \underline{x} \quad \bar{v} \quad \bar{v} \end{array} \right. \quad (2.8)$$

$$\phi_2(\underline{a}, t, y, \bar{v}, \bar{v}_y r, \bar{w}) = \phi_1(\underline{a}, t, y, \underline{x}, \underline{x}_y r, \underline{u}) \left| \begin{array}{l} \underline{x} \quad \bar{v} \quad \bar{v} \\ \underline{u} \quad \bar{v} \quad \bar{w} \end{array} \right. \quad (2.9)$$

$(\cdot)^{\wedge}$  denotes the transpose of  $(\cdot)$ .

and  $(\cdot)_f$  denotes  $(\cdot)$  evaluated at  $t = t_f$ .

Remark: In equation (2.7),  $\underline{x} \quad \bar{v} \quad \bar{v}$  means  $\underline{x}$  is replaced by  $\bar{v}$  in the argument of  $J_1$ . Similar notation applies to (2.8) and (2.9).

The first term in equation (2.6) is the performance index defined by (2.5). The second term is the sensitivity function of the performance index and the third term is the trajectory sensitivity. This

last term reduces the variations of trajectory and control caused by the variations of the system's parameter,  $\underline{a}$ . Hence, the performance index,  $J$ , in equation (2.6) seeks to approximately reach a compromise among the system optimality, the sensitivity of the performance index and the sensitivity of the trajectory. In addition, by suitable choices of  $c_1$ ,  $c_2$  and  $c_3$ , the appropriate type of index may be emphasized or de-emphasized to meet a predetermined set of design criteria.

For convenience, denote  $\theta_{1f} = \theta_1(\underline{x}_f, \underline{a}, t_f)$ . Similar notations apply to  $\theta_{2f}, \bar{v}_f$ , etc.

Now, by defining,

$$\theta(\underline{x}_f, \bar{v}_f, \underline{a}, t_f) \triangleq c_1 \theta_{1f} + c_3 \theta_{2f} + c_2 \bar{v}_f \frac{\partial \theta_{1f}}{\partial \underline{x}_f} + c_2 \frac{\partial \theta_{1f}}{\partial \underline{a}} \quad (2.10)$$

$$\begin{aligned} \text{and } \phi(\underline{a}, t, y, \underline{x}, \underline{x}_y r, \bar{v}, \bar{v}_y r, \underline{u}, \bar{w}) \triangleq & c_1 \phi_1 + c_3 \phi_2 + c_2 \bar{v} \frac{\partial \phi_1}{\partial \underline{x}} + c_2 \bar{v}_y r \frac{\partial \phi_1}{\partial \underline{x}_y r} \\ & + c_2 \frac{\partial \phi_1}{\partial \underline{a}} + c_2 \bar{w} \frac{\partial \phi_1}{\partial \underline{u}} \end{aligned} \quad (2.11)$$

then, from (2.5) and (2.7) through (2.11), equation (2.6) reduces to

$$J = \int_{\Omega} \theta_f d\Omega + \int_{t_0}^{t_f} \int_{\Omega} \phi d\Omega dt \quad (2.12)$$

Remark: The functional performance index (2.12) is very general. It includes the various types of performance indices suggested for multiple-index optimization. (86)

### 3. Boundary Conditions

The specified initial and boundary conditions for (2.3) are:

$$\underline{x}(t_0, y) = b_1, \quad \underline{x}(t, y) \Big|_{\partial\Omega} = b_2, \quad \frac{\partial \underline{x}}{\partial y} \Big|_{\partial\Omega} = b_3, \quad \dots \quad \frac{\partial^{r-2} \underline{x}}{\partial y^{r-2}} \Big|_{\partial\Omega} = b_r \quad (2.13)$$

where  $b_i$  ( $i = 1, 2, \dots, r$ ) are constants.

The corresponding conditions for equation (2.4) are:

$$\bar{v}(t_0, y) = 0, \quad \bar{v}(t, y) \Big|_{\partial\Omega} = 0, \quad \frac{\partial \bar{v}}{\partial y} \Big|_{\partial\Omega} = 0, \quad \dots \quad \frac{\partial^{r-2} \bar{v}}{\partial y^{r-2}} \Big|_{\partial\Omega} = 0. \quad (2.14)$$

It is assumed that the non-linear partial differential equations (2.3) and (2.4) are so posed that a unique solution can be obtained in terms of the other variables by applying the initial and boundary conditions (2.13) and (2.14).

### 4. Classes of Admissible Controls

We assume that control vector  $\underline{u}$  and control matrix  $\bar{w}$  belong to a prescribed set  $U \in R^m$  and  $W \in R^m \times R^p$ , respectively.

We say that  $\underline{u}(t, y)$  or  $\bar{w}(t, y)$  is admissible if,

(i)  $\underline{u}(t, y)$  or  $\bar{w}(t, y)$  is defined and piecewise continuous

on  $[t_0, t_f] \times [y_0, y_f]$

and (ii)  $\underline{u}(t, y) \in U$  or  $\bar{w}(t, y) \in W$  for all  $[t_0, t_f] \times [y_0, y_f]$

## B. Statement of the Problem

Problem P is stated as follows:

Problem P: Find  $(\underline{u}^*(t,y), \bar{w}^*(t,y))$  which minimize

$$J = \int_{\Omega} \theta_f d\Omega + \int_{t_0}^{t_f} \int_{\Omega} \phi d\Omega dt \quad (2.12)$$

subject to the constraints

$$\underline{x}_t = \underline{f}_1(\underline{a}, t, y, \underline{x}, \underline{x}_y, r, \underline{u}) \quad (2.3)$$

and

$$\bar{v}_t = \bar{f}_2(\underline{a}, t, y, \underline{x}, \underline{x}_y, r, \bar{v}, \bar{v}_y, r, \underline{u}, \bar{w}) \quad (2.4)$$

with specified initial and boundary conditions,

$$\underline{x}(t_0, y) = b_1, \quad \underline{x}(t, y) \Big|_{\partial\Omega} = b_2, \quad \frac{\partial \underline{x}}{\partial y} \Big|_{\partial\Omega} = b_3 \dots \frac{\partial^{r-2} \underline{x}}{\partial y^{r-2}} \Big|_{\partial\Omega} = b_r \quad (2.13)$$

and

$$\bar{v}(t_0, y) = 0, \quad \bar{v}(t, y) \Big|_{\partial\Omega} = 0, \quad \frac{\partial \bar{v}}{\partial y} \Big|_{\partial\Omega} = 0 \dots \frac{\partial^{r-2} \bar{v}}{\partial y^{r-2}} \Big|_{\partial\Omega} = 0. \quad (2.14)$$

## C. Conclusion

The formulation and statement of problem P have been presented. The primary objective is to obtain conditions for optimality of the functional performance index (2.12) subject to the vector-matrix partial differential equations (2.3) and (2.4) and the boundary

conditions (2.13) and (2.14). The solution, to be presented in the next chapter, is essentially a compromise among the system optimality, the sensitivity of the performance index and the trajectory sensitivity.

CHAPTER III

THEORY OF SYNTHESIS OF LOW SENSITIVITY DISTRIBUTED PARAMETER CONTROL

The necessary and sufficient conditions for problem P of chapter II will be derived. The derivation of the necessary conditions will be done by means of the calculus of variations. The sufficient conditions will be obtained by applying the conditions for convexity. (77)

A. Necessary Conditions for Problem P

An optimal control problem is essentially a minimizing problem. Such problems are usually solved in two stages. First, necessary conditions are obtained. These conditions limit the number of possible candidates for a solution to the problem. The required solution is then obtained by imposing some other conditions which insure a minimum. These latter conditions are the sufficient conditions. It is important to note that a necessary condition does not necessarily guarantee existence of a solution. It does, however, narrow the area of search for a solution.

The necessary conditions for optimality of  $(\underline{u}^*, \bar{w}^*)$  for problem P are summarized in Theorem 3.1.

(a) Theorem 3.1: Let  $\underline{u}^*(t, y) \in R^m$  and  $\bar{w}^*(t, y) \in R^m \times R^p$  for  $(t, y) \in T \times \Omega$  be the extremal controls which transfer  $(\underline{x}(t_0, y_0), \bar{v}(t_0, y_0))$  to  $(\underline{x}(t_f, y_f), \bar{v}(t_f, y_f))$ . Also let  $\underline{x}^*(t, y)$  and  $\bar{v}^*(t, y)$  be the trajectories of (3.1) and (3.2) generated by  $\underline{u}^*(t, y)$  and  $\bar{w}^*(t, y)$  respectively. In order that  $\underline{u}^*$  and  $\bar{w}^*$  be optimal for problem P, it is necessary that there exists non-zero  $p \times 1$  vector



$\underline{q}_1(t, y)$  and a non-zero  $n \times p$  matrix  $\bar{q}_2(t, y)$  such that  $\underline{q}_1^*, \bar{q}_2^*, \underline{x}^*, \bar{v}^*$  are a solution to the following system:

$$\left( \frac{\partial \theta}{\partial \underline{x}} \right)^* = \underline{q}_1^* \quad , \quad t = t_f \quad (3.1)$$

$$\left( \frac{\partial \theta}{\partial \bar{v}} \right)^* = \bar{q}_2^* \quad , \quad t = t_f \quad (3.2)$$

$$\left( \frac{\partial H}{\partial \underline{x}_Y^r} \right)^* = 0 \quad , \quad \Omega = \partial \Omega \quad (3.3)$$

$$\left( \frac{\partial H}{\partial \bar{v}_Y^r} \right)^* = 0 \quad , \quad \Omega = \partial \Omega \quad (3.4)$$

$$\left( \frac{\partial H}{\partial \underline{x}} \right)^* + (-1)^r \frac{\partial^r}{\partial Y^r} \left( \frac{\partial H}{\partial \underline{x}_Y^r} \right)^* + \frac{\partial \underline{q}_1^*}{\partial t} = 0 \quad , \quad (t, y) \in T \times \Omega \quad (3.5)$$

$$\left( \frac{\partial H}{\partial \bar{v}} \right)^* + (-1)^r \frac{\partial^r}{\partial Y^r} \left( \frac{\partial H}{\partial \bar{v}_Y^r} \right)^* + \frac{\partial \bar{q}_2^*}{\partial t} = 0 \quad , \quad (t, y) \in T \times \Omega \quad (3.6)$$

$$\left( \frac{\partial H}{\partial \underline{u}} \right)^* = 0 \quad (t, y) \in T \times \Omega \quad (3.7)$$

$$\left( \frac{\partial H}{\partial \bar{w}} \right)^* = 0 \quad (t, y) \in T \times \Omega \quad (3.8)$$

where  $(\cdot)^*$  represents  $(\cdot)$  evaluated at  $(\underline{x}^*, \underline{x}_Y^*, \bar{v}^*, \bar{v}_Y^*, \underline{u}^*, \bar{w}^*)$ ; the Hamiltonian,  $H$ , is defined by,

$$H(\underline{a}, t, y, \underline{x}, \underline{x}_Y^r, \bar{v}, \bar{v}_Y^r, \underline{u}, \bar{w}, \underline{q}_1, \bar{q}_2) = \phi + \underline{q}_1^T \underline{f}_1 + \text{tr} (\bar{q}_2 \bar{f}_2) \quad (3.9)$$

and  $\text{tr} (\cdot)$  denotes the trace operation on  $(\cdot)$ .

(b) Proof of Theorem 3.1

In order to simplify the derivation, we omit the dependence on the variables. Also, define the first variation  $\delta(\cdot) = (\cdot) - (\cdot)^*$  where  $(\cdot)^*$  represents  $\underline{x}^*$ ,  $\underline{y}^*$ ,  $\bar{v}^*$ ,  $\bar{w}^*$ ,  $\underline{u}^*$  or  $\bar{w}^*$ . From (2.3), (2.4), (2.12) and  $\underline{q}_1$ , and  $\bar{q}_2$ , we obtain the adjoined performance index,

$$\begin{aligned}
 J_a &= \int_{\Omega} \theta_f d\Omega + \int_{t_0}^{t_f} \int_{\Omega} \left[ \phi + \underline{q}_1' \left( \underline{f}_1 - \frac{\partial \underline{x}}{\partial t} \right) + \text{tr} \left( \bar{q}_2 \left( \bar{f}_2 - \frac{\partial \bar{v}}{\partial t} \right)' \right) \right] d\Omega dt \\
 &= \int_{\Omega} \theta_f d\Omega + \int_{t_0}^{t_f} \int_{\Omega} \left[ H - \underline{q}_1' \frac{\partial \underline{x}}{\partial t} - \text{tr} \left( \bar{q}_2 \frac{\partial \bar{v}}{\partial t} \right)' \right] d\Omega dt \quad (3.10)
 \end{aligned}$$

We note that the first variation  $\delta g$  of a scalar function  $g(\bar{B})$ , where  $\bar{B}$  is an  $M \times N$  matrix, is given by (12)

$$\delta g = \sum_j \sum_i \left[ \delta B_{j,i} \left( \frac{\partial g}{\partial B_{i,j}} \right)' \right] = \text{tr} \left[ \delta \bar{B} \left( \frac{\partial g}{\partial \bar{B}} \right)' \right] \quad (3.11)$$

where  $\frac{\partial g}{\partial \bar{B}}$  is an  $M \times N$  gradient matrix,

Taking the first variation of (3.10) and applying (3.11) it is seen that;

$$\begin{aligned}
 \delta J_a &= \int_{\Omega} [ \delta \underline{x}_f' \left( \frac{\partial \theta}{\partial \underline{x}} \right)^* + \text{tr} \left( \delta \bar{v}_f \left( \frac{\partial \theta}{\partial \bar{v}} \right)^* \right) ] d\Omega \\
 &+ \int_{t_0}^{t_f} \int_{\Omega} [ \delta \underline{x}' \left( \frac{\partial H}{\partial \underline{x}} \right)^* + \delta \underline{x}'_y \left( \frac{\partial H}{\partial \underline{x}_y} \right)^* \\
 &+ \text{tr} \left( \delta \bar{v} \left( \frac{\partial H}{\partial \bar{v}} \right)^* \right) + \text{tr} \left( \delta \bar{v}'_y \left( \frac{\partial H}{\partial \bar{v}'_y} \right)^* \right) + \delta \underline{u}' \left( \frac{\partial H}{\partial \underline{u}} \right)^* \\
 &+ \text{tr} \left( \delta \bar{w} \left( \frac{\partial H}{\partial \bar{w}} \right)^* \right) - \text{tr} \left( q_2^* \frac{\partial (\delta \bar{v})}{\partial t} \right) - q_1^* \left( \frac{\partial \underline{x}}{\partial t} \right) ] d\Omega dt
 \end{aligned} \tag{3.12}$$

Integrating the fourth term on the right hand side of (3.12) and applying the boundary conditions in (2.13), we have,

$$\begin{aligned}
 \int_{t_0}^{t_f} \int_{\Omega} \delta \underline{x}'_y \left( \frac{\partial H}{\partial \underline{x}_y} \right)^* d\Omega dt &= \int_{t_0}^{t_f} [ \delta \underline{x}'_y^{r-1} \left( \frac{\partial H}{\partial \underline{x}_y^r} \right)^* - \delta \underline{x}'_y^{r-2} \frac{\partial}{\partial y} \left( \frac{\partial H}{\partial \underline{x}_y^r} \right)^* \\
 &\dots + (-1)^{r-1} \delta \underline{x}'_y \frac{\partial^{r-1}}{\partial y^{r-1}} \left( \frac{\partial H}{\partial \underline{x}_y^r} \right)^* ] dt + (-1)^r \int_{t_0}^{t_f} \int_{\Omega} \delta \underline{x}'_y^r \left( \frac{\partial H}{\partial \underline{x}_y^r} \right)^* d\Omega dt \\
 &= \int_{t_0}^{t_f} \delta \underline{x}'_y^{r-1} \left( \frac{\partial H}{\partial \underline{x}_y^r} \right)^* dt + (-1)^r \int_{t_0}^{t_f} \int_{\Omega} \delta \underline{x}'_y^r \left( \frac{\partial H}{\partial \underline{x}_y^r} \right)^* d\Omega dt
 \end{aligned} \tag{3.13}$$

Similarly, for the sixth term,

$$\begin{aligned}
 \int_{t_0}^{t_f} \int_{\Omega} \operatorname{tr} \left[ \delta \bar{v}_{y^r} \left( \frac{\partial H}{\partial \bar{v}_{y^r}} \right)^* \right] d\Omega dt &= \int_{t_0}^{t_f} \operatorname{tr} \left[ \delta \bar{v}_{y^{r-1}} \left( \frac{\partial H}{\partial \bar{v}_{y^r}} \right)^* \right] dt \\
 &\quad + (-1)^r \int_{t_0}^{t_f} \int_{\Omega} \operatorname{tr} \left[ \delta \bar{v} \frac{\partial F}{\partial y^r} \left( \frac{\partial H}{\partial \bar{v}_{y^r}} \right)^* \right] d\Omega dt
 \end{aligned} \tag{3.14}$$

in view of (2.14).

Integrating the last two terms in (3.12), we have,

$$\begin{aligned}
 &\int_{t_0}^{t_f} \int_{\Omega} \left[ \bar{q}_1^* \frac{\partial (\delta \underline{x})}{\partial t} + \operatorname{tr} \left( \bar{q}_2^* \frac{\partial (\delta \bar{v})'}{\partial t} \right) \right] d\Omega dt \\
 &= \int_{\Omega} \left[ \delta \underline{x}'_{-f} \bar{q}_{1f}^* + \operatorname{tr} \left( \delta v_f \bar{q}_{2f}^* \right) \right] d\Omega \\
 &\quad - \int_{t_0}^{t_f} \int_{\Omega} \left[ \delta \underline{x}' \frac{\partial \bar{q}_1^*}{\partial t} + \operatorname{tr} \left( \delta \bar{v} \frac{\partial \bar{q}_2^*}{\partial t} \right) \right] d\Omega dt
 \end{aligned} \tag{3.15}$$

in view of (2.13) and (2.14), i.e.  $\underline{x}(t_0, y) = \bar{v}(t_0, y) = 0$

Substituting (3.13) through (3.15) into (3.12) and simplifying, we have,

$$\begin{aligned}
 \delta J_a = & \int_{\Omega} \left[ \delta \underline{x}'_f \left( \frac{\partial \theta}{\partial \underline{x}} - \underline{q}_1 \right)^* \Big|_{t_f} + \text{tr} \left( \delta \underline{v}'_f \left( \frac{\partial \theta}{\partial \underline{v}} - \underline{q}_2 \right)^* \right) \Big|_{t_f} \right] d\Omega \\
 & + \int_{t_0}^{t_f} \left[ \delta \underline{x}'_{y^{r-1}} \left( \frac{\partial H}{\partial \underline{x}_{y^r}} \right)^* \Big|_{\partial \Omega} + \text{tr} \left( \delta \underline{v}'_{y^{r-1}} \left( \frac{\partial H}{\partial \underline{v}_{y^r}} \right)^* \right) \Big|_{\partial \Omega} \right] dt \\
 & + \int_{t_0}^{t_f} \int_{\Omega} \left[ \delta \underline{x}' \left( \frac{\partial H}{\partial \underline{x}} + (-1)^r \frac{\partial^r}{\partial y^r} \left( \frac{\partial H}{\partial \underline{x}_{y^r}} \right) + \frac{\partial \underline{q}_1}{\partial t} \right)^* \right. \\
 & + \left. \delta \underline{u}' \left( \frac{\partial H}{\partial \underline{u}} \right)^* + \text{tr} \left( \delta \underline{w}' \left( \frac{\partial H}{\partial \underline{w}} \right)^* \right) \right. \\
 & + \left. \text{tr} \left( \delta \underline{v}' \left( \frac{\partial H}{\partial \underline{v}} + (-1)^r \frac{\partial^r}{\partial y^r} \left( \frac{\partial H}{\partial \underline{v}_{y^r}} \right) + \frac{\partial \underline{q}_2}{\partial t} \right) \right)^* \right] d\Omega dt
 \end{aligned} \tag{3.16}$$

Applying the fundamental theorem of the calculus of variations (20), the necessary conditions for problem P of Chapter II are obtained by setting  $\delta J_a = 0$ . Theorem 3.1 is immediately established.

LEMMA : If the specified initial and boundary conditions given in (2.13) and (2.14) are modified as :

$$\underline{x}(t_0, y) = b_1, \quad \left. \frac{\partial \underline{x}}{\partial y} \right|_{\partial \Omega} = b_3, \quad \dots \dots \quad \left. \frac{\partial^{r-1} \underline{x}}{\partial y^{r-1}} \right|_{\partial \Omega} = b_{r+1} \quad (3.17)$$

and

$$\bar{v}(t_0, y) = 0, \quad \left. \frac{\partial \bar{v}}{\partial y} \right|_{\partial \Omega} = 0, \quad \dots \dots \quad \left. \frac{\partial^{r-1} \bar{v}}{\partial y^{r-1}} \right|_{\partial \Omega} = 0 \quad (3.18)$$

then, the necessary conditions are the same as those stated in Theorem 3.1 of Chapter III, except that equations (3.3) and (3.4) are replaced by,

$$\frac{\partial^{r-1}}{\partial y^{r-1}} \left( \frac{\partial H}{\partial \underline{x}_{y^r}} \right)^* = 0, \quad \Omega = \partial \Omega \quad (3.19)$$

and

$$\frac{\partial^{r-1}}{\partial y^{r-1}} \left( \frac{\partial H}{\partial \bar{v}_{y^r}} \right)^* = 0, \quad \Omega = \partial \Omega \quad (3.20)$$

Remark : The above Lemma is similar to that considered by Sage and Chaudhuri (78) . The results derived here, however, are of wider applicability since they apply to both vector and matrix partial differential equations.

To establish the lemma, note that it follows from (3.17) and (3.18) that;

$$\int_{t_0}^{t_f} \int_{\Omega} \delta \underline{x}_{y^r} \left( \frac{\partial H}{\partial \underline{x}_{y^r}} \right)^* d\Omega dt = (-1)^{r-1} \int_{t_0}^{t_f} \delta \underline{x}^r \frac{\partial^{r-1}}{\partial y^{r-1}} \left( \frac{\partial H}{\partial \underline{x}_{y^r}} \right)^* \Big|_{\partial \Omega} dt$$

$$+ (-1)^r \int_{t_0}^{t_f} \int_{\Omega} \delta \underline{x}^r \frac{\partial^r}{\partial y^r} \left( \frac{\partial H}{\partial \underline{x}_{y^r}} \right)^* d\Omega dt$$

(3.21)

and

$$\int_{t_0}^{t_f} \int_{\Omega} \text{tr} \left[ \delta \underline{v}_{y^r} \left( \frac{\partial H}{\partial \underline{v}_{y^r}} \right)^* \right] d\Omega dt = (-1)^{r-1} \int_{t_0}^{t_f} \text{tr} \left[ \delta \underline{v}^r \frac{\partial^{r-1}}{\partial y^{r-1}} \left( \frac{\partial H}{\partial \underline{v}_{y^r}} \right)^* \right] \Big|_{\partial \Omega} dt$$

$$+ (-1)^r \int_{t_0}^{t_f} \int_{\Omega} \text{tr} \left[ \delta \underline{v}^r \frac{\partial^r}{\partial y^r} \left( \frac{\partial H}{\partial \underline{v}_{y^r}} \right)^* \right] d\Omega dt$$

(3.22)

This leads to (3.19) and (3.20), respectively. The lemma is thus immediately established.

### B. Sufficient Conditions for Problem P

In optimal control problems sufficient conditions determine whether the extremal obtained by the necessary conditions is indeed the minimizing ( or maximizing ) solution. A common method for determining this is to take the second variation of the augmented performance index and determine its sign. If the second variation is positive it implies the solutions yield a minimum, a negative sign, of course, implies a maximum. This technique unfortunately leads to only local sufficient conditions. Global solutions are generally obtained by the use of the theory of convex functions ( 77 ).

By the theory of convex functions, the global solution is obtained by comparing it to all possible values, not only those due to neighboring solutions. This theory, which will be used here, has been successfully applied to optimal control problems including those involving differential games (12).

Sufficient conditions for problem P of Chapter II are stated in Theorem 3.2. The proof follows immediately thereafter.



(a) Theorem 3.2 : Let  $(\underline{x}^*, \bar{v}^*, \underline{u}^*, \bar{w}^*, \underline{q}_1^*, \bar{q}_2^*)$  be continuous, differentiable and a solution of equations (3.1) through (3.8) and (2.3), (2.13) and (2.14). If the following conditions hold;

(i)  $\theta_f$ , defined in (2.10), is convex in  $\underline{x}_f$  and  $\bar{v}_f$ , at  $t=t_f$

(ii)  $H$ , defined in (3.9), is convex in  $(\underline{x}, \underline{x}_{y^r}, \bar{v}, \bar{v}_{y^r}, \underline{u}, \bar{w})$

then,  $\underline{u}^*$ , and  $\bar{w}^*$ , are minima for problem P.

(b) Proof of Theorem 3.2

For convenience let

$$J = J(\underline{a}, \underline{x}, \bar{v}, \underline{u}, \bar{w}, \underline{q}_1^*, \bar{q}_2^*)$$

and  $J^* = J(\underline{a}, \underline{x}^*, \bar{v}^*, \underline{u}^*, \bar{w}^*, \underline{q}_1^*, \bar{q}_2^*)$

Similar notation applies to  $\theta(\underline{x}_f^*, \bar{v}_f^*, \underline{a}, t_f)$

and  $H(\underline{a}, t, y, \underline{x}, \underline{x}_{y^r}, \bar{v}, \bar{v}_{y^r}, \underline{u}, \bar{w}, \underline{q}_1^*, \bar{q}_2^*)$

Consider,

$$\begin{aligned}
 J - J^* &= \int_{\Omega} (\theta_f - \theta_f^*) \, d\Omega \\
 &+ \int_{t_0}^{t_f} \int_{\Omega} [ H - H^* - \underline{q}_1^* \left( \frac{\partial x}{\partial t} - \frac{\partial x^*}{\partial t} \right) - \text{tr}(\bar{q}_2^* (\frac{\partial \bar{v}}{\partial t} - \frac{\partial \bar{v}^*}{\partial t})) ] \, d\Omega dt
 \end{aligned} \tag{3.23}$$

If  $\theta_f$  is convex in  $\underline{x}_f$  and  $\bar{v}_f$ , then from the condition for convexity (77) and also equations (3.1) and (3.2), it is seen that,

$$(\theta_f - \theta_f^*) \geq [\delta \underline{x}_f \underline{q}_1^* + \text{tr}(\delta \bar{v}_f \bar{q}_2^*)] \tag{3.24}$$

where  $(\cdot)_f$  denotes  $(\cdot)$  evaluated at  $t = t_f$

Similarly, we have,

$$\begin{aligned}
 (H - H^*) &\geq \delta \underline{x}' \left( \frac{\partial H}{\partial \underline{x}} \right)^* + \delta \underline{x}'_{yr} \left( \frac{\partial H}{\partial \underline{x}_{yr}} \right)^* \\
 &+ \text{tr} \left[ \delta \bar{v} \left( \frac{\partial H}{\partial \bar{v}} \right)^* \right] + \text{tr} \left[ \delta \bar{v}_{yr} \left( \frac{\partial H}{\partial \bar{v}_{yr}} \right)^* \right] \\
 &+ \delta \underline{u}' \left( \frac{\partial H}{\partial \underline{u}} \right)^* + \text{tr} \left[ \delta \bar{w} \left( \frac{\partial H}{\partial \bar{w}} \right)^* \right]
 \end{aligned} \tag{3.25}$$

Now,

$$\int_{t_0}^{t_f} \int_{\Omega} \delta \underline{x}'_{y^r} \frac{(\partial H)}{\partial \underline{x}_{y^r}}^* d\Omega dt = (-1)^r \int_{t_0}^{t_f} \int_{\Omega} \delta \underline{x}'_{y^r} \frac{\partial^r}{\partial y^r} \frac{(\partial H)}{\partial \underline{x}_{y^r}}^* d\Omega dt \quad (3.26)$$

in view of (3.3) and (3.13).

$$\text{and } \int_{t_0}^{t_f} \int_{\Omega} \text{tr} \left[ \delta \bar{v}'_{y^r} \frac{(\partial H)}{\partial \bar{v}_{y^r}}^* \right] d\Omega dt = (-1)^r \int_{t_0}^{t_f} \int_{\Omega} \text{tr} \left[ \delta \bar{v}'_{y^r} \frac{\partial^r}{\partial y^r} \frac{(\partial H)}{\partial \bar{v}_{y^r}}^* \right] d\Omega dt \quad (3.27)$$

in view of (3.4) and (3.14).

Also,  $\underline{x}(t_0, y) = \bar{v}(t_0, y) = 0$  (in view of (3.3) and (3.4)), hence,

$$\int_{t_0}^{t_f} \int_{\Omega} q_1^* \left( \frac{\partial \underline{x}}{\partial t} - \frac{\partial \underline{x}^*}{\partial t} \right) d\Omega dt = \int_{\Omega} \delta \underline{x}'_{1f} q_{1f}^* d\Omega - \int_{t_0}^{t_f} \int_{\Omega} \delta \underline{x}'_{1f} \frac{\partial q_1^*}{\partial t} d\Omega dt \quad (3.28)$$

$$\int_{t_0}^{t_f} \int_{\Omega} \text{tr} \left[ \bar{q}_2^* \left( \frac{\partial \bar{v}}{\partial t} - \frac{\partial \bar{v}^*}{\partial t} \right) \right] d\Omega dt = \text{tr} \left[ \delta \bar{v}'_{2f} q_{2f}^* \right] d\Omega - \int_{t_0}^{t_f} \int_{\Omega} \text{tr} \left[ \delta \bar{v}'_{2f} \frac{\partial \bar{q}_2^*}{\partial t} \right] d\Omega dt. \quad (3.29)$$

Substituting (3.7), (3.8), (3.22) through (3.29) into (3.23) and simplifying, we have,

$$\begin{aligned} J - J^* &\geq \int_{t_0}^{t_f} \int_{\Omega} \left[ \delta \underline{x}' \left[ \frac{(\partial H)}{\partial \underline{x}}^* + (-1)^r \frac{\partial^r}{\partial y^r} \frac{(\partial H)}{\partial \underline{x}_{y^r}}^* + \frac{\partial q_1^*}{\partial t} \right] \right. \\ &\quad \left. + \text{tr} \left[ \delta \bar{v}' \left[ \frac{(\partial H)}{\partial \bar{v}}^* + (-1)^r \frac{\partial^r}{\partial y^r} \frac{(\partial H)}{\partial \bar{v}_{y^r}}^* + \frac{\partial \bar{q}_2^*}{\partial t} \right] \right] \right] d\Omega dt \\ &= 0. \end{aligned} \quad (3.30)$$

in view of (3.5) and (3.6). This completes the proof for Theorem 3.2

### C. Summary

Two Theorems and a Lemma for necessary and sufficient conditions for optimality of problem P of Chapter II were stated. These, indeed, form the basis for the discussion of physical problems and solutions of numerical examples to be presented in subsequent chapters.

For ease of reference, the above Theorems are stated below;

1. Theorem 3.1 : In order that  $\underline{u}^*$  and  $\bar{w}^*$  be optimal for problem P, it is necessary that there exists non-zero  $p \times 1$  vector  $\underline{q}_1(t,y)$  and a non-zero  $n \times p$  matrix  $\bar{q}_2(t,y)$  such that  $\underline{q}_1^*, \bar{q}_2^*, \underline{u}^*, \bar{v}^*$  are a solution of the following system:

$$\left( \frac{\partial \theta}{\partial \underline{x}} \right)^* = \underline{q}_1^* , \quad t=t_f \quad (3.1)$$

$$\left( \frac{\partial \theta}{\partial \bar{v}} \right)^* = \bar{q}_2^* , \quad t=t_f \quad (3.2)$$

$$\left( \frac{\partial H}{\partial \underline{x}_{y^r}} \right)^* = 0 , \quad \Omega = \partial \Omega \quad (3.3)$$

$$\left( \frac{\partial H}{\partial \bar{v}_{y^r}} \right)^* = 0 , \quad \Omega = \partial \Omega \quad (3.4)$$

$$\left( \frac{\partial H}{\partial \underline{x}} \right)^* + (-1)^{r_1} \frac{\partial}{\partial y^r} \left( \frac{\partial H}{\partial \underline{x}_{y^r}} \right)^* + \frac{\partial \underline{q}_1^*}{\partial t} = 0 , \quad (t,y) \in T \times \Omega \quad (3.5)$$

$$\left( \frac{\partial H}{\partial \bar{v}} \right)^* + (-1)^{r_2} \frac{\partial}{\partial y^r} \left( \frac{\partial H}{\partial \bar{v}_{y^r}} \right)^* + \frac{\partial \bar{q}_2^*}{\partial t} = 0 , \quad (t,y) \in T \times \Omega \quad (3.6)$$

$$\left( \frac{\partial H}{\partial \underline{u}} \right)^* = 0, \quad (t, y) \in T \times \Omega \quad (3.7)$$

$$\left( \frac{\partial H}{\partial \underline{w}} \right)^* = 0, \quad (t, y) \in T \times \Omega \quad (3.8)$$

2.Lemma : If the specified initial and boundary conditions given in (2.13) and (2.14) are modified as:

$$\underline{x}(t_0, y) = b_1, \frac{\partial \underline{x}}{\partial y} \Big|_{\partial \Omega} = b_3, \dots, \frac{\partial^{r-1} \underline{x}}{\partial y^{r-1}} \Big|_{\partial \Omega} = b_{r+1} \quad (3.17)$$

$$\text{and } \bar{v}(t_0, y) = 0, \frac{\partial \bar{v}}{\partial y} \Big|_{\partial \Omega} = 0, \dots, \frac{\partial^{r-1} \bar{v}}{\partial y^{r-1}} \Big|_{\partial \Omega} = 0 \quad (3.18)$$

then, the necessary conditions are the same as those stated in Theorem 3.1 of Chapter III, except that equations (3.3) and (3.4) are replaced by,

$$\frac{\partial^{r-1}}{\partial y^{r-1}} \left( \frac{\partial H}{\partial \underline{x}_{yr}} \right)^* = 0, \quad \Omega = \partial \Omega \quad (3.19)$$

$$\frac{\partial^{r-1}}{\partial y^{r-1}} \left( \frac{\partial H}{\partial \bar{v}_{yr}} \right)^* = 0, \quad \Omega = \partial \Omega \quad (3.20)$$

3. Theorem 3.2 : Let  $(\underline{x}^*, \bar{v}^*, \underline{u}^*, \bar{w}^*, \underline{q}_1^*, \bar{q}_2^*)$  be continuous, differentiable and a solution of equations (3.1) through (3.8) and (2.3), (2.13) and (2.14). If the following conditions hold;

(a)  $\theta_f$ , defined in (2.10), is convex in  $\underline{x}_f$  and  $\bar{v}_f$  at  $t=t_f$

(b)  $H$ , defined in (3.9), is convex in  $(\underline{x}, \underline{x}_{y,r}, \bar{v}, \bar{v}_{y,r}, \underline{u}, \bar{w}^*)$

then,  $\underline{u}^*$  and  $\bar{w}^*$ , are minima for problem P.

## CHAPTER IV

### STANDBY REDUNDANT SYSTEMS

In this chapter the mathematical model of a standby redundant system with arbitrary repair rate and non-constant failure rate will be derived. Two problems involving the synthesis of optimum repair rate and low sensitivity design will be formulated. The necessary and sufficient conditions for extremum for both problems are stated. These conditions are established by direct application of the general theory derived in Chapter III.

#### A. Introduction

The application of probability theory to system reliability' analysis and evaluation is well known (8, 71). Of particular interest is the analysis of the reliability of repairable systems using Markov processes. The usual assumptions made in the derivation of the mathematical model include the assumptions of constant failure rate and constant repair rate. These assumptions, which conveniently lighten the mathematical computations, may not be valid for many physical systems (85). For example, the repair rate may not be constant because of maintainability, availability and other factors. In this investigation the assumption of constant repair rate will be relaxed. But the assumption of constant failure rate will be retained. For, this does not appear to be too objectionable (85).

We are primarily concerned in this investigation with the optimization and study of sensitivity in classes of reliability problems repre-

sented mathematically by partial differential equations. Such structures are also referred to as distributed parameter systems.

Standby redundancy, one of the physical systems selected for purposes of illustration, is an important and widely used technique for reliability improvement (3, 85). A designer may find redundancy not only the quickest, easiest and cheapest solution, if the component is available and cheap, but also the only practical solution if the reliability requirement is beyond the state of the art. There are some serious problems which the designer may have to face. The components may not after all be cheap if a large number of redundant structures are required. More importantly, there may be limitations on the designer such as size, weight, power requirements and the need for complex sensing and switching circuitry. On balance, however, designers find it easier to resort to redundancy than to other means of reliability improvement.

The failure rate of the standby system is assumed in this investigation to vary around a nominal value during the operation of the system. This assumption is dictated by the practical necessity of component replacements and repair. Both operations - repair and replacement - do affect the failure characteristics of the system. It is further assumed that a functional can be found which, together with an efficient maintenance of repair rate, may represent the performance of the system cost-wise. Achieving a minimum of this cost functional under conditions of operation near the nominal value of the failure rate is the ideal desirable result.

Many researches have considered the problem of maximizing system reliability (32, 58), but only have lumped or continuous parameter systems



have so far been considered. Recent investigators in this area of lumped parameter reliability optimization include Misra ( 57 ) and Tillman ( 88 ). This treatment of reliability problems as lumped parameter systems does not take into account the repair time in the formulation of the mathematical model. In considering the repair time in the derivation of the mathematical model of the standby system we shall arrive at a model represented by partial differential equations. The theory of low sensitivity design developed earlier will then be applied. It is true that standby redundancy has been extensively explored in the literature, but none of the authors discussed low sensitivity design of these structures ( 3,31,85).

In the next section a mathematical model of the standby system with repair will be derived. In later sections problems involving low sensitivity design will be formulated and solved by means of the theorems established in Chapter III. The solutions are stated in the form of corollaries.

#### B. Mathematical Model for Two-element Standby Redundant System

The derivation of the mathematical model in this section is somewhat different from that of Rau ( 71 ) based on the following assumptions: standby redundant system with both arbitrary failure and repair rates.

Consider a two-element standby redundant system with failure rate  $\lambda(t)$  and repair rate  $u(t,\tau)$  as shown in figure 4.1. It is assumed that there are both perfect switching and sensing and that no warm up time is required.

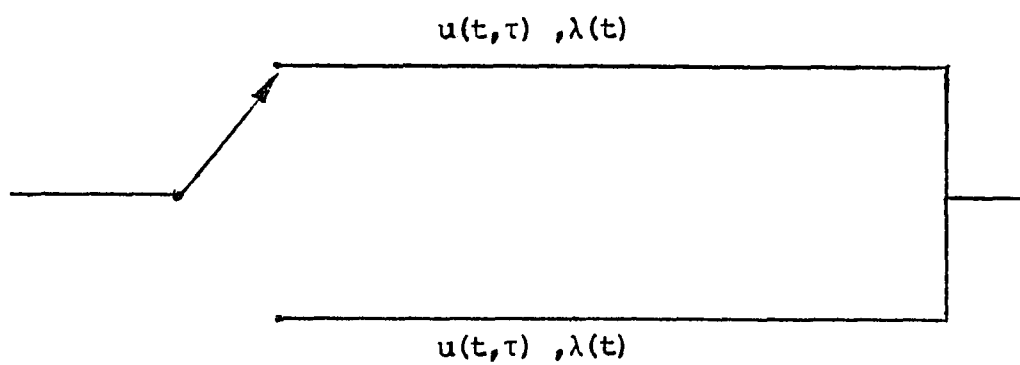


FIGURE 4.1: Two-element Standby Redundant System.

At any time  $t$  the system must be in one of the following states with probability  $p_i(t)$ , ( $i=1,2,3$ ).

State 1: both elements are operable, but only one is operating.

State 2: one element has failed and the other is operating.

State 3: both elements have failed.

Define the probabilities  $\lambda(t)\Delta t$ ,  $u(t,\tau)\Delta t$ , and  $p_2(t,\tau)d\tau$  as follows:

$\lambda(t)\Delta t \cong$  the probability that failure occurs in the interval  $(t, t+\Delta t)$  given that it is working at time  $t$ . (4.1)

$u(t,\tau)\Delta t \cong$  the probability that a failed element is repaired in the interval  $(\tau, \tau+\Delta t)$  given that it has been under repair for  $\tau$  units of time and that it is working at time  $t$ . (4.2)

( from statistical point of view  $u(t,\tau) \rightarrow u(\tau)$  as  $t \rightarrow \infty$ )

$p_2(t,\tau)d\tau \cong$  the probability that the system is in state 2 at time  $t$  and has been there from  $\tau$  to  $\tau + d\tau$  units of time, where  $\tau$  is the time to repair an element after a fault has been detected. (4.3)

Remark 1: It follows from (4.1) that the probability that no failure occurs in the interval  $(t, t+\Delta t)$  is equal to  $1-\lambda(t)\Delta t$ .

Remark 2: Properties of  $p_2(t,\tau)$ .

$$(a) p_2(t,\tau) = 0 \quad \text{for } \tau > t \quad (4.4)$$

$$(b) p_2(t) = \int_0^t p_2(t,\tau)d\tau \quad 0 \leq \tau \leq t \quad (4.5)$$

$$(c) p_2(t,0) = \lambda(t)p_1(t) \quad \text{for } \tau=0, \quad t \geq 0 \quad (4.6)$$

The last property follows from the fact that  $\lambda(t)p_1(t)\Delta t$ , which denotes the probability that the system is in state 1 at time  $t$  and switches into state 2 during the next increment of time  $\Delta t$ , is precisely what is meant by  $p_2(t,0)\Delta t$ .

It is seen that the only ways that the system can be in state 1 at time  $t+\Delta t$  are:

1. The system is in state 1 and no failure occurs during the interval  $(t, t+\Delta t)$ ; or
2. The system was in state 2 at time  $t$ , had been there for  $\tau$  time units, and was repaired during the interval  $(t, t+\Delta t)$ . Thus we must sum or integrate the probability of occurrence of an event for the above second type over all  $\tau$  for  $0 \leq \tau \leq t$ .

hence,

$$p_1(t+\Delta t) = p_1(t)(1-\lambda(t)\Delta t) + \int_0^t p_2(t,\tau)u(t,\tau)d\tau\Delta t \quad (4.7)$$

In order for the system to be in state 2 at time  $t+\Delta t$  and to have been there  $\tau+\Delta t$  time units, the system must be in state 2 at time  $t$ , have been there for  $\tau$  units of time, and no failure and repair occur in the next  $\Delta t$  time units. Thus we have,

$$p_2(t+\Delta t, \tau+\Delta t) = p_2(t, \tau)[1 - u(t, \tau)\Delta t][1 - \lambda(t)\Delta t] \quad (4.8)$$

In order for the system to be in state 3 at time  $t+\Delta t$ , it was either there at time  $t$  or was in state 2 at time  $t$ , had been there  $\tau$  time units, and was not repaired during  $(t, t+\Delta t)$  but yet the remaining component failed in the increment of  $\Delta t$ .

hence,

$$p_3(t+\Delta t) = p_3(t) + \int_0^t p_2(t,\tau) [1 - u(t,\tau)\Delta t] \lambda(t) \Delta t d\tau \quad (4.9)$$

Dividing equations (4.7) through (4.9) by  $\Delta t$  and letting  $\Delta t \rightarrow 0$ , we obtain, respectively,

$$\frac{dp_1(t)}{dt} = -\lambda(t)p_1(t) + \int_0^t p_2(t,\tau)u(t,\tau)d\tau \quad (4.10)$$

$$\frac{\partial p_2(t,\tau)}{\partial \tau} + \frac{\partial p_2(t,\tau)}{\partial t} = -[u(t,\tau) + \lambda(t)]p_2(t,\tau) \quad (4.11)$$

$$\frac{dp_3(t)}{dt} = \lambda(t) \int_0^t p_2(t,\tau)d\tau \quad (4.12)$$

where  $\lambda(t) \neq 0$  is given.

The initial and boundary conditions are given by,

$$p_1(0) = 1. \quad (4.13)$$

$$p_3(0) = 0 \quad (4.14)$$

$$p_2(t,0) = \lambda(t)p_1(t) \quad (4.15)$$

Also the sum of the probabilities equals one,

$$\text{i.e. } p_1(t) + p_2(t) + p_3(t) = 1 \quad (4.16)$$

### C. Formulation, Statement of Problems and Proof of Corollaries

Two problems will be formulated in this section. The first problem involves the synthesis of optimum repair rate for the two-element standby redundant system. The second problem concerns the synthesis of low sensitivity optimum control. In this latter regard, it would be useful to define the following sensitivity and other quantities.

Assuming a constant failure rate  $\lambda$ , but arbitrary repair rate  $u(t, \tau)$

$$v_1(t) = \frac{\partial p_1(t)}{\partial \lambda}, \quad \text{system sensitivity} \quad (4.17)$$

$$v_2(t, \tau) = \frac{\partial p_2(t, \tau)}{\partial \lambda}, \quad \text{state sensitivity} \quad (4.18)$$

$$w(t, \tau) = \frac{\partial u(t, \tau)}{\partial \lambda}, \quad \text{sensitivity control} \quad (4.19)$$

The performance index of interest is defined as the sum of the minimum energy and the unreliability of the system. The minimum energy performance index is given by,

$$J_{ME} = \int_0^T \int_0^t [ (u^2 + p_2^2) / 2. ] d\tau dt, \quad 0 \leq \tau \leq t \leq T \quad (4.20)$$

where  $T$  is the operating time.

Let  $F(t)$  be the failure time distribution with density function  $f(t)$ , then,

$$F(t) = \int_0^t f(x) dx \quad (4.21)$$

is the system's unreliability at time  $t$ . But the reliability of the system,  $R(t)$ , is given by,

$$R(t) = 1 - F(t) \quad (4.22)$$

It follows from (4.21) and (4.22) that

$$F(t) = \int_0^t \left( - \frac{dR}{dt} \right) dt \quad (4.23)$$

and the unreliability at time T is given by,

$$F(T) = \int_0^T \left( - \frac{dR}{dt} \right) dt \quad (4.24)$$

$$\text{but } R(t) = 1 - p_3(t) = p_1(t) + p_2(t) \quad (4.25)$$

hence (4.24) reduces to,

$$\begin{aligned} F(T) &= - \int_0^T \left( \frac{dR}{dt} \right) dt \\ &= \int_0^T \left( \frac{dp_3(t)}{dt} \right) dt \\ &= \int_0^T \int_0^t \lambda p_2(t, \tau) d\tau dt \end{aligned} \quad (4.26)$$

in view of (4.12).

The functional performance index is selected by combining (4.20) and (4.26), the result is,

$$J = \int_0^T \int_0^t \left( \lambda k_1 p_2 + k_2 u^2 / 2. + k_3 p_2^2 / 2. \right) d\tau dt \quad (4.27)$$

where  $k_1$ ,  $k_2$ , and  $k_3$  are weighting factors.

Remark: It is important to note that in the deterministic equivalence of above system, it is assumed that  $p_1$ ,  $p_2$ ,  $p_3$  represent the statistical means of the respective probabilities.

1. Problem 4.1: synthesis of optimum repair rate

Determine optimum  $u^*(t, \tau) \in U$  for  $0 \leq \tau \leq t \leq T$ , and  $k_2 \in R$  which minimize,

$$J(k_2, p_2, u) = \int_0^T \int_0^t (\lambda k_1 p_2 + k_2 u^2/2 + k_3 p_2^2/2) d\tau dt, \quad 0 \leq \tau \leq t \leq T \quad (4.28)$$

subject to

$$\frac{\partial p_2}{\partial t} + \frac{\partial p_2}{\partial \tau} + (u + \lambda)p_2 = 0 \quad (4.29)$$

The initial and boundary conditions are,

$$\frac{dp_1}{dt} + \lambda p_1 = \int_0^t p_2 u d\tau, \quad t \geq 0 \quad (4.30)$$

$$p_1(0) = 1 \quad (4.31)$$

$$p_2(t, 0) = \lambda p_1(t) \quad t \geq 0 \quad (4.32)$$

$$p_3(0) = 0 \quad (4.33)$$

Remark: The constraints (4.12) and (4.16) have been combined in the formulation of  $F(T)$  ( see equation (4.26) ).

The necessary and sufficient conditions for optimality of problem 4.1 are stated in corollaries 4.1 and 4.2 respectively.



(a) Corollary 4.1 : necessary conditions for problem 4.1

In order that the control  $u(t, \tau)$  be optimal for problem 4.1, it is necessary that there exists non-zero continuous function  $q(t, \tau), 0 \leq \tau \leq t \leq T$  which together with  $u(t, \tau)$  satisfy;

$$\frac{\partial q^*(t, \tau)}{\partial t} + \frac{\partial q^*(t, \tau)}{\partial \tau} - [u^*(t, \tau) + \lambda]q^*(t, \tau) - k_1\lambda - k_3 p_2^*(t, \tau) = 0,$$

$$0 < \tau < t < T \quad (4.34)$$

$$q^*(T, \tau) = 0 \quad 0 < \tau < T \quad (4.35)$$

$$k_2 u^*(t, \tau) + q^*(t, \tau) p_2^*(t, \tau) \quad (4.36)$$

and through which the trajectory governed by

$$\frac{\partial p_2^*(t, \tau)}{\partial t} + \frac{\partial p_2^*(t, \tau)}{\partial \tau} + [u^*(t, \tau) + \lambda]p_2^*(t, \tau) = 0, \quad (4.37)$$

is transferred from  $p_2(0, 0)$  to  $p_2(T, T)$  in fixed time  $T$  by the control in the admissible class  $u \in U, 0 \leq \tau \leq t \leq T$ .

Remark: In view of equation (4.32)  $p_2(0, 0) = \lambda$ . Also, for optimal performance  $p_2(T, T)$  must satisfy the relation  $p_2(T, T) \leq \lambda$ . In other words the system is required to be in near state 1 at the end of the operation.

(b) Proof of corollary 4.1

Let  $q(t, \tau)$  be the Lagrange multiplier. From equations (4.28) and (4.29) it is seen that the Hamiltonian,  $H$ , is given by,

$$H \cong \lambda k_1 p_2 + k_2 u^2 / 2 + k_3 p_2^2 / 2 - q \left[ \frac{\partial p_2}{\partial \tau} + (u + \lambda) p_2 \right] \quad (4.38)$$

The necessary conditions stated in corollary 4.1 follow immediately from application of Theorem 3.1 of Chapter III to (4.38).

(c) Corollary 4.2 : sufficient conditions for problem 4.1

Let  $(p^*, u^*, q^*)$  be continuous, differentiable and a solution to equations (4.34) through (4.37). If  $k_2 k_3 - q^2 \gg 0, k_3 \gg 0$ , for  $k_1, k_2$  and  $q$ , not all zero, then  $u^*(t, \tau) \in U$  for  $0 \leq \tau \leq t \leq T$  is optimal for problem 4.1.

(d) Proof of corollary 4.2

We form the matrix,  $\bar{M}$ , of second differentials,  $\sum_{i=1}^3 \sum_{j=1}^3 \frac{\partial^2 H}{\partial x_i \partial x_j}$ , where  $H$  is defined by (4.38),  $\underline{x} = (p_2, u, \frac{\partial p_2}{\partial \tau})$ ,

and

$$\bar{M} = \begin{bmatrix} k_3 & -q & 0 \\ -q & k_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (4.39)$$

The matrix  $\bar{M}$  is clearly positive semi-definite if  $k_3 k_2 \gg q^2, k_3 \gg 0$ , which implies that  $H$  defined by (4.38) is convex in  $(p_2, u, \frac{\partial p_2}{\partial \tau})$ . Corollary 4.2 is immediately established from Theorem 3.2 of Chapter III.

## 2. Synthesis of Low Sensitivity Optimal Repair Rate

The sensitivity equations and their associated initial and boundary conditions are obtained by differentiating equations (4.29) through (4.33) with respect to  $\lambda$ . In view of the definitions of the sensitivity functions given by (4.17) through (4.19), it is seen that the sensitivity equations are given by;

$$\frac{\partial v_2}{\partial t} + \frac{\partial v_2}{\partial \tau} + (u + \lambda)v_2 + (w + 1)p_2 = 0 \quad (4.40)$$

$$\frac{\partial v_1}{\partial t} + \lambda v_1 + p_1 = \int_0^t (v_2 u + p_2 w) d\tau, \quad t \geq 0 \quad (4.41)$$

$$v_1(0) = 0 \quad (4.42)$$

$$v_2(t, 0) = p_1(t) + \lambda v_1(t) \quad t \geq 0 \quad (4.43)$$

$$v_3(0) = 0 \quad (4.44)$$

Let  $k_i$  ( $i=1,2,\dots,6$ ) be weighting constants, then the selected performance index is given by,

$$J = \int_0^T \int_0^t [ (k_1 u^2 + k_2 p_2^2 + k_3 v_2^2 + k_4 w^2) / 2 + k_5 u w + k_6 p_2 v_2 ] d\tau dt, \quad 0 \leq \tau \leq t \leq T \quad (4.45)$$

Problem 4.2 is stated as follows;

### (a) Problem 4.2

Determine the optimal  $(u^*(t, \tau), w^*(t, \tau))$  and  $k_1, k_2 \in \mathbb{R}$  for  $0 \leq \tau \leq t \leq T$ , which transfer the trajectories generated by (4.29) and (4.40)

from  $(p_2(0,0), v_2(0,0))$  to  $(p_2(T,T), v_2(T,T))$  while the functional performance index (4.45) is minimized, subject to (4.30) through (4.33) and (4.41) through (4.44). The constants  $k_i$  ( $i=3,4,5,6$ ) are given.

The necessary and sufficient conditions for problem 4.2 are stated in corollaries 4.3 and 4.4 respectively.

(b) Corollary 4.3 : necessary conditions for problem 4.2

In order that the controls  $u$  and  $w$  be optimal for problem 4.2, it is necessary that there exist non-zero continuous functions  $q_1(t, \tau)$  and  $q_2(t, \tau)$ ,  $0 \leq \tau \leq t \leq T$ , which together with  $u$  and  $w$  satisfy;

$$\frac{\partial q_1^*}{\partial t} + \frac{\partial q_1^*}{\partial \tau} - k_6 v_2^* - q_1^* (u^* + \lambda) - k_2 p_2^* - (w^* + 1) q_2^* = 0 \quad (4.46)$$

$$\frac{\partial q_2^*}{\partial t} + \frac{\partial q_2^*}{\partial \tau} - q_2^* (u^* + \lambda) - k_3 v_2^* - k_6 p_2^* = 0 \quad (4.47)$$

$$k_1 u^* + k_5 w^* + q_1^* p_2^* + q_2^* v_2^* = 0 \quad (4.48)$$

$$k_4 w^* + k_5 u^* + p_2^* q_2^* = 0 \quad (4.49)$$

$$q_1^*(T, \tau) = 0 \quad (4.50)$$

$$q_2^*(T, \tau) = 0 \quad (4.51)$$

and through which the trajectories governed by (4.29) and (4.40) are transferred from  $(p_2(0,0), v_2(0,0))$  to  $(p_2(T,T), v_2(T,T))$  in fixed time  $T$  by the controls in the admissible classes  $u \in U$ ,  $w \in W$  for  $0 \leq \tau \leq t \leq T$

(c) Proof of corollary 4.3

Define  $H$ , the Hamiltonian as follows;

$$H = (k_1 u^2 + k_2 p_2^2 + k_3 v_2^2 + k_4 w^2) / 2 + k_5 u w + k_6 p_2 v_2 - q_1 \left[ \frac{\partial p_2}{\partial \tau} + (u+\lambda) p_2 \right] - q_2 \left[ \frac{\partial v_2}{\partial \tau} + (u+\lambda) v_2 + (w+1) p_2 \right] \quad (4.52)$$

where  $q_1(t, \tau)$  and  $q_2(t, \tau)$  are Lagrange multipliers.

The necessary conditions follow immediately from application of Theorem 3.1 of Chapter III to (4.52).

(d) Corollary 4.4 : sufficient conditions for problem 4.2

Let  $(p_2^*, v_2^*, u^*, w^*, q_1^*, q_2^*)$  be continuous, differentiable and a solution of equations (4.29) through (4.33), (4.40) through (4.44) and (4.46) through (4.51). If the determinants  $|D_i| \geq 0$  ( $i=1, 2, \dots, 4$ ) for some values of  $k_i$  ( $i=1, 2, \dots, 6$ ),  $q_1$  and  $q_2$ , not all zero, then  $u^* \in U$  and  $w^* \in W$  for  $0 \leq \tau \leq t \leq T$  are optimal for problem 4.2.

where we define  $|D_1| = k_1$  (4.53)

$$|D_2| = \begin{vmatrix} k_1 & -q_1 \\ -q_1 & k_2 \end{vmatrix} \quad (4.54)$$

$$|D_3| = \begin{vmatrix} k_1 & -q_1 & k_5 \\ -q_1 & k_2 & -q_2 \\ k_5 & -q_2 & k_3 \end{vmatrix} \quad (4.55)$$

$$|D_4| = \begin{vmatrix} k_1 & -q_1 & k_5 & q_2 \\ -q_1 & k_2 & -q_2 & k_6 \\ k_5 & -q_2 & k_3 & 0 \\ -q_2 & k_6 & 0 & k_4 \end{vmatrix} \quad (4.56)$$

Proof of corollary 4.4

(e) We form, as usual, the matrix of partials defined by

$$\bar{M} = \sum_{i=1}^6 \sum_{j=1}^6 \frac{\partial^2 H}{\partial x_i \partial x_j} \quad (4.57)$$

Where  $H$  is the Hamiltonian defined by (4.52)

and  $\underline{x} = [ p_2, v_2, u, w, \frac{\partial p_2}{\partial \tau}, \frac{\partial v_2}{\partial \tau} ]$

$$\bar{M} = \begin{bmatrix} k_1 & -q_1 & k_5 & q_2 & 0 & 0 \\ -q_1 & k_2 & -q_2 & k_6 & 0 & 0 \\ k_5 & -q_2 & k_3 & 0 & 0 & 0 \\ -q_2 & k_6 & 0 & k_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (4.58)$$

Since  $||D_5|| = ||D_6|| = 0$ , matrix,  $\bar{M}$ , is clearly positive semi-definite if  $|D_i| \geq 0$  ( $i=1,2,\dots,4$ ), for some  $k_i$  ( $i=1,2,\dots,6$ ),  $q_1, q_2$ , not all zero, where  $|D_i|$  are defined by (4.53) through (4.56). This result implies that  $H$  is convex in  $\underline{x}$ . Corollary 4.4 is immediately established from Theorem 3.2 of Chapter III.

#### D. Summary

The necessary conditions for both problems 4.1 and 4.2 were established by direct application of Theorem 3.1 of Chapter III. The sufficient conditions were obtained by application of the conditions for convexity and Theorem 3.2 of Chapter III. It follows, therefore, that for each of the performance indices (4.28) and (4.45) the extremum, if it exists, is a global minimum.

CHAPTER V

HEAT CONTROL PROCESS

The second class of problems selected for purposes of illustration of the theory derived in Chapter III is represented by the heat equation. This equation finds extensive use in the study of conduction, diffusion and other processes where heat transfer takes place. In spite of its enormous importance, very little work on sensitivity analysis of the heat equation has been reported in the literature ( 83 ).

Problems involving the sensitivity analysis of the heat equation are presented in this chapter. It is assumed that the variations in the parameter - the diffusivity - are small.

A. General Formulation and Boundary Conditions

For simplicity, only the one - dimensional version of the heat equation will be considered. This, in its simplest form, is given by;

$$\frac{\partial T}{\partial t} = a \frac{\partial^2 T}{\partial y^2} + bu \quad (5.1)$$

where

$T(t,y)$  is the temperature in the material for

$$t \in (t_0, t_f) \text{ and } y \in (y_0, y_f)$$

$u(t,y)$  is the heat forcing function or control

'a' is the diffusivity with nominal value,  $a_0$

'b' is a constant

$(t_0, t_f)$  and  $(y_0, y_f)$  are fixed and given intervals.



## 1. Boundary conditions

Boundary conditions for the heat equation (5.1) are extremely important. For, these boundary conditions must be formulated with care in order to insure existence and uniqueness of solutions. In mathematical jargon, the problem must be 'well posed' by suitable choice of boundary conditions.

Since the region under consideration in this problem is finite, the distribution,  $T(0,y)$ , within the material at time  $t = 0$  may be specified. In addition, either the temperature  $T$  or the rate of heat flow,  $\frac{\partial T}{\partial y}$ , across the boundaries ( at both  $y=y_0$  and  $y=y_f$  ) may be prescribed. A general boundary condition is given by,

$$\alpha_1 T(t,y) + \alpha_2 \frac{\partial T(t,y)}{\partial y} = \alpha_3, \text{ at } y=y_0 \text{ and } y=y_f \quad (5.2)$$

where  $\alpha_i$  ( $i = 1,2,3$ ) are constants. (37)

Figure 5.1 exhibits some of these acceptable boundary conditions.

In some problems the forcing function is prescribed at one boundary. This leads to the so-called boundary control formulation. These boundary control problems will not be considered in this investigation. Only distributed control problems will be discussed in this chapter.

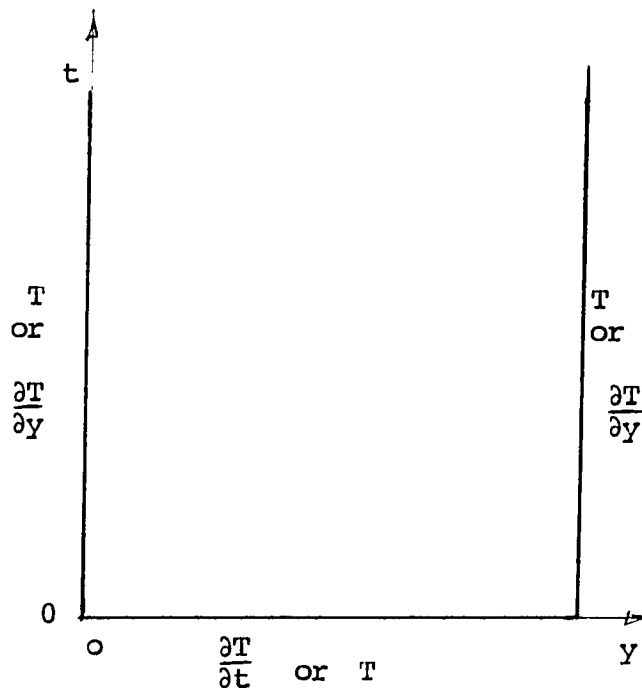


Figure 5.1 : Boundary Conditions For One-dimensional Heat Equation.

## 2. General formulation

The problems posed here involve synthesis of low sensitivity optimal control. We define the following sensitivity quantities;

$$v = \frac{\partial T}{\partial a}, \text{ temperature sensitivity} \quad (5.3)$$

$$w = \frac{\partial u}{\partial a}, \text{ control sensitivity} \quad (5.4)$$

By differentiating (5.1) with respect to 'a', it is seen that,

$$\frac{\partial v}{\partial t} = a \frac{\partial^2 y}{\partial y^2} + \frac{\partial^2 T}{\partial y^2} + bw \quad (5.5)$$

Let the initial and boundary conditions for (5.1) be given by,

$$T(0, y) = \beta_0, \quad y \in (y_0, y_f) \quad (5.6)$$

and either  $T(t, y_0) = \beta_1, \quad t \in (t_0, t_f) \quad (5.7)$

$$T(t, y_f) = \beta_2, \quad t \in (t_0, t_f) \quad (5.8)$$

or  $\left. \frac{\partial T}{\partial y} \right|_{y=y_0} = \beta_3, \quad t \in (t_0, t_f) \quad (5.9)$

$$\left. \frac{\partial T}{\partial y} \right|_{y=y_f} = \beta_4, \quad t \in (t_0, t_f) \quad (5.10)$$

where  $\beta_i$  ( $i = 0, 1, \dots, 4$ ) are constants.

The corresponding initial and boundary conditions for the sensitivity equation (5.5) are obtained by differentiating (5.6) through (5.10).

Hence,

$$v(0,y) = 0 , \quad y \in (y_0, y_f) \quad (5.11)$$

and either

$$v(t, y_0) = 0 , \quad t \in (t_0, t_f) \quad (5.12)$$

$$v(t, y_f) = 0 , \quad t \in (t_0, t_f) \quad (5.13)$$

or

$$\left. \frac{\partial v}{\partial y} \right|_{y=y_0} = 0, \quad t \in (t_0, t_f) \quad (5.14)$$

$$\left. \frac{\partial v}{\partial y} \right|_{y=y_f} = 0, \quad t \in (t_0, t_f) \quad (5.15)$$

The cost functional used in this chapter is defined by;

$$J = \int_{y_0}^{y_f} \int_{t_0}^{t_f} [ (R_1 T^2 + R_2 v^2 + R_3 u^2 + R_4 w^2 )/2. + R_5 Tv + R_6 uw ] dt dy \quad (5.16)$$

where  $R_i$  ( $i=1,2,..6$ ) are weighting constants.

In the problems which follow, we seek to find the set of controls which will minimize  $J$  under the constraints given by (5.1) and (5.5) together with the appropriate initial and boundary conditions.

## B. Problem Statement and Proof of Corollaries

Two problems involving low sensitivity optimal control will be formulated in this section. In the first problem, the two boundaries will be maintained at different temperatures. This is analogous to the heat sink phenomenon. In the second problem both boundaries are insulated. The necessary and sufficient conditions will be stated in the form of corollaries, since they are derived by direct application of Theorems 3.1 and 3.2 of Chapter III.

### 1. Heat sink problem

Problem 5.1 is stated as follows;

#### (a) Problem 5.1

Determine the optimal  $u \in U$  and  $w \in W$  for  $t \in (t_0, t_f)$  and  $y \in (y_0, y_f)$  which transfer the trajectories generated by (5.1) and (5.5) from  $(T(t_0, y_0), v(t_0, y_0))$  to  $(T(t_f, y_f), v(t_f, y_f))$  while the functional performance index given by (5.16) is minimized, subject to the initial and boundary conditions (5.6) through (5.8) and (5.11) through (5.13).

The necessary and sufficient conditions for optimality of problem 5.1 are stated in Corollaries 5.1 and 5.2 respectively.

(b) Corollary 5.1 : necessary conditions for problem 5.1

In order that the controls  $u^* \in U$  and  $w^* \in W$  for  $t \in (t_0, t_f)$  and  $Y \in (Y_0, Y_f)$  be optimal for problem 5.1, it is necessary that there exist non-zero functions  $q_1(t, Y)$  and  $q_2(t, Y)$  for  $t \in (t_0, t_f)$  and  $Y \in (Y_0, Y_f)$  which, together with  $u^*$  and  $w^*$ , satisfy;

$$\frac{\partial q_1^*}{\partial t} + a \frac{\partial^2 q_1^*}{\partial Y^2} + \frac{\partial^2 q_2^*}{\partial Y^2} + R_1 T^* + R_5 v^* = 0, \quad (5.17)$$

$$\frac{\partial q_2^*}{\partial t} + a \frac{\partial^2 q_2^*}{\partial Y^2} + R_2 v^* + R_5 T^* = 0, \quad (5.18)$$

$$R_3 u^* + q_1^* b + R_6 w^* = 0, \quad (5.19)$$

$$R_4 w^* + q_2^* b + R_6 u^* = 0, \quad (5.20)$$

$$q_1^*(t_f, Y) = q_2^*(t_f, Y) = 0, \quad Y \in (Y_0, Y_f) \quad (5.21)$$

$$q_1^*(t, Y_0) = q_2^*(t, Y_0) = 0, \quad t \in (t_0, t_f) \quad (5.22)$$

$$q_1^*(t, Y_f) = q_2^*(t, Y_f) = 0, \quad t \in (t_0, t_f) \quad (5.23)$$

and through which the trajectories governed by (5.1) and (5.5) are transferred from  $(T(t_0, Y_0), v(t_0, Y_0))$  to  $(T(t_f, Y_f), v(t_f, Y_f))$ , where  $(t_0, t_f, Y_0, Y_f)$  are fixed.

(c) Proof of corollary 5.1

Define the Hamiltonian,

$$H = (R_1 T^2 + R_2 v^2 + R_3 u^2 + R_4 w^2) / 2 + R_5 T v + R_6 u w \\ + q_1 \left( a \frac{\partial^2 T}{\partial y^2} + b u \right) + q_2 \left( a \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 T}{\partial y^2} + b w \right) \quad (5.24)$$

Where  $q_1(t, y)$  and  $q_2(t, y)$  are Lagrange multipliers.

Corollary 5.1 follows immediately from application of the necessary conditions in Theorem 3.1 of Chapter III to (5.24).

(d) Corollary 5.2: sufficient conditions for problem 5.1

Let  $(T^*, v^*, u^*, w^*, q_1^*, q_2^*)$  be continuous, differentiable and a solution to equations (5.1) and (5.5) through (5.8), (5.11) through (5.13), and (5.17) through (5.23), if  $R_1 \geq 0$ ,  $R_1 R_2 - R_5^2 \geq 0$ ,  $R_3 \geq 0$ , and  $R_3 R_4 - R_6^2 \geq 0$  for some  $R_i$  ( $i = 1, 2, \dots, 6$ ) not all zero, then  $u^* \in U$  and  $w^* \in W$  for  $t \in (t_0, t_f)$  and  $y \in (y_0, y_f)$  are optimal for problem 5.1

(e) Proof of corollary 5.2

We form the second partials of  $H$  and define the matrix,  $\bar{M}$ , as

$$\bar{M} = \sum_{i=1}^6 \sum_{j=1}^6 \frac{\partial^2 H}{\partial x_i \partial x_j}, \quad \text{where } \underline{x} = (T, v, u, w, \frac{\partial^2 T}{\partial y^2}, \frac{\partial^2 v}{\partial y^2})$$

$$\bar{M} = \begin{bmatrix} R_1 & R_5 & 0 & 0 & 0 & 0 \\ R_5 & R_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & R_3 & R_6 & 0 & 0 \\ 0 & 0 & R_6 & R_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (5.25)$$

The matrix,  $\bar{M}$ , is clearly positive semi-definite for  $R_1 \geq 0$ ,  $R_1 R_2 - R_5^2 \geq 0$ ,  $R_3 \geq 0$ , and  $R_3 R_4 - R_6^2 \geq 0$  for some values of  $R_i$  ( $i=1,2,\dots,6$ ) not all zero. This implies that  $H$  is convex in  $\underline{x}$ . Corollary 5.2 is immediately established from Theorem 3.2 of Chapter III.

## 2. Problem 5.2 : Insulated Boundary Case

Determine the optimal  $u^* \in U$  and  $w^* \in W$  for  $t \in (t_0, t_f)$  and  $y \in (y_0, y_f)$  which transfer the trajectory generated by (5.1) and (5.5) from  $(T(t_0, y_0), v(t_0, y_0))$  to  $(T(t_f, y_f), v(t_f, y_f))$  while the functional performance index given by (5.16) is minimized. The initial and boundary conditions are given by equations (5.6), (5.9), (5.10), (5.11) (5.14) and (5.15).

### (a) Corollary 5.3 : necessary conditions for problem 5.2

The necessary conditions for optimality of problem 5.2 are the same as those stated in Corollary 5.1, except that equations (5.22) and (5.23) are replaced by,

$$\left. \frac{\partial q_1}{\partial y} \right|_{y=y_0} = \left. \frac{\partial q_2}{\partial y} \right|_{y=y_0} = 0 \quad (5.26)$$

$$\text{and} \quad \left. \frac{\partial q_1}{\partial y} \right|_{y=y_f} = \left. \frac{\partial q_2}{\partial y} \right|_{y=y_f} = 0 \quad (5.27)$$



(b) Proof of corollary 5.3

The proof follows immediately from application of the Lemma following Theorem 3.1 of Chapter III.

The sufficient conditions for problem 5.2 are stated in corollary 5.4 below.

(c) Corollary 5.4 : sufficient conditions for problem 5.2

Let  $(T^*, v^*, u^*, w^*, q_1^*, q_2^*)$  be continuous, differentiable and a solution to equations (5.1), (5.5), (5.9) through (5.10), (5.11), (5.14) through (5.23); if  $R_1 \geq 0$ ,  $R_1 R_2 - R_6^2 \geq 0$ ,  $R_3 \geq 0$ ,  $R_3 R_4 - R_6^2 \geq 0$ , for some  $R_i$  ( $i=1, 2, \dots, 6$ ), not all zero, then  $u^* \in U$  and  $w^* \in W$  are optimal for problem 5.2

(d) Proof for corollary 5.4

The proof is the same as that for Corollary 5.2.

C. Summary

Two problems involving heat transfer have been considered in this chapter. The necessary and sufficient conditions were established for each problem. The proofs follow directly from Theorems 3.1 and 3.2 of Chapter III. Since the sufficient conditions were established by means of the theory of convexity it follows that the extremum, if it exists, is a global minimum.

## CHAPTER VI

### NUMERICAL COMPUTATION

In this chapter discrete models and some problems in the setting up of finite difference equations for numerical computation are considered. The necessary conditions for numerical convergence of the solution of the discrete equations will be established. In establishing these conditions, tests for stability and consistency will be used. These tests are treated in detail elsewhere in the literature. (1, 56, 99).

#### A. Techniques for Solving Distributed Parameter Control Problems

It is possible to solve distributed parameter control problems by first establishing a spatially discretized model for both the system's equations and the functional performance index. The result is a set of matrix difference equations. Alternatively, a corresponding set of difference equations may be obtained by time-discretization. After either of the above discretization process, the well-established techniques for solving optimal control problems in continuous systems are then employed. (78).

For some problems it may be better to discretize the space or time parameter after the necessary conditions for optimality have been obtained. This enables us to use the well established techniques for solving two-point boundary value problems in continuous systems.

The solution of the optimal control leads to the solution of two-point boundary value problems for distributed parameter systems.

There are four widely used computational techniques for solving distributed parameter two-point boundary value problems. (48, 78). These are:

1. Techniques based on iteration of the control to improve the performance index. For example, direct search on the performance index and the method of steepest ascent based on the second variation.
  2. Techniques that iterate on the state equations. For example, quasilinearisation.
  3. Techniques that iterate on the boundary conditions while the actual state and co-state equations are retained. For example, shooting methods and invariant imbedding.
- and
4. Techniques based on multi-level and hierarchical methods. (98).

In this investigation, both space and time parameters of the state and co-state equations will be discretized. Algorithms will then be developed for solving these discrete equations by iterating on the state equations and improving the performance index.

Since this study relies on computer methods for solving the PDE, it is important to consider problems such as stability of the numerical solution, convergence and consistency of the discrete equation with the differential equation. Before doing so, some of the discrete models often used to approximate partial differentials will be examined.

### B. Finite Difference and Computational Molecules

The success of any discretization scheme depends on the choice of difference formulae used to approximate the partial derivatives. Only a few of the finite difference formulae will be considered in this section. These will be used in developing discrete approximations of the partial differential equations of interest in this investigation.

$$\text{Let } t = ih = i \Delta t \quad (i = 1, 2, \dots) \quad (6.1)$$

$$y = jk = j \Delta y \quad (j = 1, 2, \dots) \quad (6.2)$$

Taylor's series for  $(u + \Delta t, y)$  about  $(t, y)$ , gives,

$$u(t+\Delta t, y) = u(t, y) + \Delta t \frac{\partial u}{\partial t} + \frac{(\Delta t)^2}{2} \frac{\partial^2 u}{\partial t^2} + \dots \quad (6.3)$$

which upon division by  $\Delta t$ , results in the relation,

$$\frac{\partial u}{\partial t} = (u(t+\Delta t, y) - u(t, y)) / \Delta t - \frac{\Delta t}{2} \frac{\partial^2 u}{\partial t^2} - \dots \quad (6.4)$$

Hence

$$\left. \frac{\partial u}{\partial t} \right|_{i,j} = (u_{i+1,j} - u_{i,j})/h + O_f(h) \quad (6.5)$$

$$\text{where } O_f(h) = -\frac{h}{2} \left( \frac{\partial^2 u}{\partial t^2} \right)_{i,j} - \dots \quad (6.6)$$

Equation (5.5) is the forward difference approximation for the differential  $\frac{\partial u}{\partial t}$  at the grid point  $(i,j)$ .  $O_f(h)$  is the truncation error associated with the forward difference approximation. Evidently, this error approaches zero as  $h$  approaches zero.

Similarly, the backward difference approximation and its associated truncation error,  $O_b(h)$ , are given by,

$$\left. \frac{\partial u}{\partial t} \right|_{i,j} = (u_{i,j} - u_{i-1,j})/h + O_b(h) \quad (6.7)$$

$$\text{where } O_b(h) = \frac{h}{2} \left( \frac{\partial^2 u}{\partial t^2} \right)_{i,j} - \dots \quad (6.8)$$

In a similar fashion, the forward difference approximation for the second partial derivative  $\frac{\partial^2 u}{\partial y^2}$  is given by (2)

$$\left. \frac{\partial^2 u}{\partial t^2} \right|_{i,j} = (u_{i+1,j} - 2u_{i,j} + u_{i-1,j})/k^2 + O(k^2) \quad (6.9)$$

where

$$O_{ff}(k^2) = -\frac{k^2}{12} \left( \frac{\partial^4 u}{\partial y^4} \right)_{i,j} \quad (6.10)$$

Again both  $O_b(h)$  and  $O_{ff}(k^2)$  tend to zero as  $h$  and  $k$  approach zero.

The above are some of the models which may be used in setting up the difference equations. They are first order approximations. Higher order approximations are also available (2). These higher order molecules are more accurate, but they lead to implicit formulae which are more difficult to solve.

### C. Problems in Setting up the Finite Difference Equations

It is again emphasised that the finite difference models set up in the previous sections are only approximate models. Hence, they introduce errors such as truncation, discretization and round-off errors. There are, therefore, inherent problems in the setting up of the finite difference equations. (38).

The first problem in choosing a finite difference formula is stability. When round-off and other errors grow and eventually "swamping" the true solution, numerical instability is said to have occurred. Definitionally, therefore, the system of equations is unstable if, as  $h$  and  $k$  tend to zero, the finite difference solutions at a point or within an interval become unbounded. This phenomenon may be avoided or limited by using smaller interval sizes. There are various sophisticated methods for determining stability (56). They will not be discussed in this investigation. It is worth noting that a useful and simple method

for determination of stability is to test for positiveness of the co-efficients of the various terms in the discrete equation. In other words, a condition is imposed to ensure that the co-efficients do not change sign. Thus, apart from ensuring that the equation has a unique solution, stability ensures, at least in principle, that the growth of round-off errors is bounded. (56).

After ensuring stability of the discrete equation, it is essential to also ensure that the discrete model is consistent with the original differential equation. For, an approximate numerical scheme may converge to the solution of another equation if the discrete approximation is not consistent with the original partial differential equation. The discrete approximation,  $L(u_{i,j}) = 0$ , is said to be consistent with its original equation,  $L(u(t,y)) = 0$ ,

$$\text{if } \lim_{\substack{h \rightarrow 0 \\ k \rightarrow 0}} \left\{ L(u(t,y)) - L(u_{i,j}) \right\} = 0 \quad (6.11)$$

In other words, the finite difference equation is said to be consistent with the differential equation if the local truncation errors tend to zero as  $h$  and  $k$  tend to zero. This is the basic test of consistency which we shall apply to our discrete equations.

Thirdly, there is the problem of convergence. A solution converges if it approaches the true or exact solution as the grid is refined. For properly posed initial value problems, it is well established that for convergence it is necessary and sufficient that the discrete equation be stable and consistent. This is the Lax Equivalence Theorem (2).

In addition to the above problems - stability, consistency and convergence - there are problems such as step size and initial trial solution which must be considered. In selecting the step size, it is often necessary to strike a compromise between accuracy and speed of solution. This is usually done by trial and error methods. In the case of initial trial solution, it is often possible to resort to physical considerations and intuition. (41).



#### D. Stability, Consistency and Estimated Error

In this section the stability, consistency and error will be examined. The heat equation and its associated co-state equations are used to illustrate the techniques involved. In order to express these equations more compactly, define  $U = (T, v, q_1, q_2)'$ . After substituting  $u$  and  $w$  from (5.19) and (5.20) into (5.1) and (5.2) respectively, and setting  $R_5=R_6=0$ , the system of equations in Problem 5.1 reduces to

$$\frac{\partial U}{\partial t} = A_0 \frac{\partial^2 U}{\partial y^2} - C_0 U \quad (6.12)$$

where

$$A_0 = \begin{bmatrix} a & 0 & 0 & 0 \\ 1 & a & 0 & 0 \\ 0 & 0 & -a & -1 \\ 0 & 0 & 0 & -a \end{bmatrix} \quad (6.13)$$

and

$$C_0 = \begin{bmatrix} 0 & 0 & 1/R_3 & 0 \\ 0 & 0 & 0 & 1/R_4 \\ R_1 & 0 & 0 & 0 \\ 0 & R_2 & 0 & 0 \end{bmatrix} \quad (6.14)$$

## 1. Stability

It is proposed in this section to determine the stability conditions for the system of heat equations given by equation (6.12). This equation is similar to the one dimensional parabolic equation. There is a difference in the sense that  $A_0$  is a matrix. Usually  $A_0$  is a scalar.

Among the techniques available for determination of stability of discrete equations are;

- (a) The Matrix Method
- (b) The von Neumann Method
- (c) Brauer's Theorem
- (d) Fourier Method

Use will be made of Brauer's Theorem to determine conditions for stability of (6.12).

Consider,

$$\frac{\partial U}{\partial t} = \frac{U_{i+1,j} - U_{i,j}}{h} + O(h) \quad (6.15)$$

and

$$\frac{\partial^2 U}{\partial y^2} = \frac{U_{i,j+1} - 2U_{i,j} + U_{i,j-1}}{k^2} + O(k^2) \quad (6.16)$$

where  $\Delta t = h$  and  $\Delta y = k$

$$\text{and } U = [ u_{i,j}^{(1)}, \dots, u_{i,j}^{(4)} ] \quad (6.17)$$

From (6.12) it is seen that

$$\frac{U_{i+1,j} - U_{i,j}}{h} = A_0 \left[ \frac{U_{i,j+1} - 2U_{i,j} + U_{i,j-1}}{k^2} \right] - C_0 U_{i,j} + O(h+k^2) \quad (6.18)$$

where the small quantity,  $O(h+k^2)$  is the error of the approximation.

Neglecting this small error (6.18) reduces to,

$$\frac{u_{i+1,j} - u_{i,j}}{h} = \frac{A_0}{k^2} [ u_{i,j+1} - 2u_{i,j} + u_{i,j-1} ] - C_0 u_{i,j} \quad (6.19)$$

where  $u_{i,j}$  is the solution of the approximate discrete equation (6.19)

Define  $r = \frac{h}{k^2}$ , and re-arrange (6.19), the result is;

$$u_{i+1,j} = rA_0 u_{i,j-1} + [ I_4 - 2rA_0 - hC_0 ] u_{i,j} + rA_0 u_{i,j+1} \quad (6.20)$$

where  $I_4$  is the 4-dimensional unit matrix.

Now let the prescribed boundary conditions be  $U(t, Y_0) = U(t, Y_f) = 0$ ,

$$\text{then, } u_{i,0} = u_{i,n+1} = 0 \quad (6.21)$$

expanding (6.20) for  $i = 1, 2, \dots, n$  and making use of (6.21), the result is the following tria-diagonal system of equations.

$$\begin{aligned}
u_{i+1,1} &= 0 + [I_4 - 2rA_0 - hC_0]u_{i,1} + rA_0 u_{i,2} \quad \dots \\
u_{i+1,2} &= rA_0 u_{i,1} + rA_0 u_{i,1} + [I_4 - 2rA_0 - hC_0]u_{i,2} + rA_0 u_{i,3} \dots \\
&\cdot \qquad \qquad \qquad \cdot \qquad \qquad \qquad \cdot \qquad \qquad \qquad \cdot \\
&\cdot \qquad \qquad \qquad \cdot \qquad \qquad \qquad \cdot \qquad \qquad \qquad \cdot \\
u_{i+1,n} &= \cdot \qquad \qquad \qquad \cdot \qquad \qquad \qquad \cdot rA_0 u_{i,n-1} + [I_4 - 2rA_0 - hC_0]u_{i,n}
\end{aligned}
\tag{6.22}$$

The above system of equations may be written in the form,

$$u_{i,j+1} = Au_{i,j} \tag{6.23}$$

where

$$A = \begin{bmatrix}
[I_4 - 2rA_0 - hC_0] & rA_0 & \cdot & \cdot \\
rA_0 & [I_4 - 2rA_0 - hC_0] & rA_0 & \cdot \\
\cdot & & & \\
\cdot & & & \cdot rA_0 [I_4 - 2rA_0 - hC_0]
\end{bmatrix}$$

(6.24)

A is a  $4n \times 4n$  square matrix.

and  $u_i = [u_{i,1}, u_{i,2}, \dots, u_{i,n}]$

Now, stability of the finite difference equation (6.23) is ensured if all the eigenvalues of matrix A are, in absolute value, less than or equal to 1 (2) . A simple method of estimating the eigenvalues of matrix A is by applying Brauer'- Theorem. This Theorem states that if  $a_s$  is the diagonal term in the s-row of a square matrix and  $P_s$  is the sum of the moduli of the terms in the s-row, excluding the diagonal term, then all the eigenvalues,  $\lambda$  . must lie inside or on the boundary of the circle,

$$| \lambda - a_s | = P_s \quad (6.25)$$

From (6.24), it is seen that  $\lambda$  is greatest for one of the rows in the second block of matrices in the second main row of (6.24). Hence the row of elements of interest,  $A_2$ , is given by;

$$A_2 = [ rA_0 \quad ( I_4 - 2rA_0 - hC_0 ) \quad rA_0 \quad . \quad . \quad . ] \quad (6.26)$$

a Assuming typical values,  $R_i = 1$  ( $i=1,..6$ ), then substituting (6.13) and (6.14) into (6.26), leads to

$$A_2 = \begin{bmatrix} ra & 0 & 0 & 0 & (1-2ra) & 0 & -h & 0 & ra & 0 & 0 & 0 & . & . & . \\ r & ra & 0 & 0 & -2r & (1-2ra) & 0 & -h & r & ra & 0 & 0 & . & . & . \\ 0 & 0 & -ra & -r & -h & 0 & (1+2ra) & 2r & 0 & 0 & -ra & -r & . & . & . \\ 0 & 0 & 0 & -ra & 0 & -h & 0 & (1+2ra) & 0 & 0 & 0 & -ra & . & . & . \end{bmatrix} \quad (6.27)$$

For convenience, listed below are values of  $a_s$  and  $P_s$  for the four rows of  $A_2$ ,

Row No.	$a_s$	$P_s$	
1.	(1-2ra)	(2ra + h )	
2.	(1-2ra)	(2ra+h+4r )	
3.	(1+2ra)	(2ra+h+4r)	
4.	(1+2ra)	(2ra + h )	(6.28)

Evidently, the largest estimate of the eigenvalues may be obtained by considering elements in rows 2 or 3 of (6.28).

Consider the elements of row 2, and let  $\lambda_2$  be the possible eigenvalue associated with that row. Then by Braur's Theorem,

$$|\lambda_2 - (1-2ra)| \leq (2ra + h + 4r) \quad (6.29)$$

which implies that

$$-(2ra+h+4r) \leq [\lambda_2 - (1-2ra)] \leq (2ra+h+4r) \quad (6.30)$$

hence the two possible values of  $\lambda_2$  are given by,

$$\lambda_{21} = 1-4ra-h-4r \quad (6.31)$$

$$\text{and } \lambda_{22} = 1+h+4r \quad (6.32)$$

$$\text{for stability } |\lambda_{21}| \leq 1 \text{ and } |\lambda_{22}| \leq 1 \quad (6.33)$$

hence from (6.31),

$$-1 \leq (1-4ra-h-4r) \leq 1 \quad (6.34)$$

the inequality (6.34) results in the conditions;

$$r \leq (2-h)/4(a+1) \quad (6.35)$$

and  $r \geq -h/4(a+1) \quad (6.36)$

The second condition (6.35) is obviously meaningless, hence it is ignored.

In a similar fashion (6.32) and (6.33) imply,

$$-1 \leq 1 + h + 4r \leq 1 \quad (6.37)$$

from which it is seen that;

$$r \geq -(2 + h)/4 \quad (6.38)$$

and  $r \leq -h/4 \quad (6.39)$

Again, the conditions (6.38) and (6.39) are meaningless, since  $r = h/k^2$  is a positive number.

Similarly, the possible eigenvalues from row 3,  $\lambda_3$ , are given by;

$$-(2ra+h+4r) < [ \lambda_3 - (1+2ra) ] < (2ra+h+4r) \quad (6.40)$$

which results in the eigenvalues,

$$\lambda_{31} = 1 - h - 4r \quad (6.41)$$

and  $\lambda_{32} = 1+4ar+h+4r \quad (6.42)$

$$\text{For stability, } |\lambda_{31}| < 1 \quad \text{and} \quad |\lambda_{32}| < 1 \quad (6.43)$$

From (6.41) and (6.42) we obtain the conditions,

$$-1 < 1-h-4r < 1 \quad (6.44)$$

$$\text{i.e.} \quad r < (2-h)/4 \quad (6.45)$$

$$\text{and} \quad r < -h/4 \quad (6.46)$$

The second condition (6.46) is valueless.

Again from (6.42) and (6.43) it is seen that

$$-1 < 1+4ar+h+4r < 1 \quad (6.47)$$

which results in the conditions,

$$r > -(2+h)/4(a+1) \quad (6.48)$$

$$\text{and} \quad r < -h/4(a+1) \quad (6.49)$$

Again, conditions (6.48) and (6.49) are valueless.

From (6.35) and (6.45) it is seen that the more severe condition is given by (6.45),

$$\text{i.e.} \quad r < (2-h)/4(a+1), \text{ for } h < 2 \quad (6.50)$$

Inequality (6.50) is therefore the condition for stability of the discrete equation (6.20)



Now, for typical values,  $h=.02$ ,  $k=.5$  and  $(a)_{\max} = 2$ ,  
 we obtain  $r = h/k^2 = .08$  (6.51)

But from (6.45),  $r \leq (2-h)/4(a+1)$

$= .17$ , in view of the above typical values.

$> h/k^2 = .08$

Hence the choice of grid net lengths satisfy the stability  
 condition (6.49).

With both stability condition and consistency requirement satisfied,  
 the discrete solution necessarily converges. (2) • This latter condition,  
 consistency, is considered in the next sub-section.

## 2. Consistency

$$\text{Consider } \frac{\partial U}{\partial t} = A_0 \frac{\partial^2 U}{\partial y^2} - C_0 U \quad (6.21)$$

subject to the initial and boundary conditions,

$$U(0,y) = 0, \quad 0 \leq y \leq 1 \quad (6.52)$$

$$\text{and } U(t,0) = U(t,1) = 0, \quad t \geq 0 \quad (6.53)$$

The explicit difference approximation to (6.21) was obtained in the previous section and is given by,

$$u_{i+1,j} = [ I_4 - 2rA_0 - hc_0 ] u_{i,j} + rA_0 [ u_{i,j+1} + u_{i,j-1} ] \quad (6.21)$$

where  $\Delta t = h$ ,  $\Delta y = k$  and  $r = h/k^2$

Now, denote the exact solution of (6.12) by  $U$  and the exact solution of the finite difference equation (6.21) by  $u$ , then the error,  $e$ , is

$$e = U - u \quad (6.54)$$

At the mesh points,

$$u_{i,j} = U_{i,j} - e_{i,j}, \quad u_{i+1,j} = U_{i+1,j} - e_{i+1,j}, \text{ etc.}, \quad (6.55)$$

Substituting (6.55) into (6.21) leads to

$$\begin{aligned} e_{i+1,j} = & [ I_4 - 2rA_0 - hc_0 ] e_{i,j} + rA_0 [ e_{i,j+1} + e_{i,j-1} ] \\ & + U_{i+1,j} - [ I_4 - 2rA_0 - hc_0 ] U_{i,j} \\ & - rA_0 [ U_{i,j+1} + U_{i,j-1} ] \end{aligned} \quad (6.56)$$

Applying the Extended Mean Value Theorem for continuous functions,

$$U_{i,j+1} = U_{i,j} + k \left( \frac{\partial U}{\partial y} \right)_{i,j} + \frac{k^2}{2} \frac{\partial^2 U}{\partial y^2}(t, y + \theta_1 k) \quad (6.57)$$

$$U_{i,j-1} = U_{i,j} - k \left( \frac{\partial U}{\partial y} \right)_{i,j} + \frac{k^2}{2} \frac{\partial^2 U}{\partial y^2}(t, y - \theta_2 k) \quad (6.58)$$

$$U_{i+1,j} = U_{i,j} + h \frac{\partial U}{\partial t}(t + \theta_3 h, y) \quad (6.59)$$

where  $0 < \theta_1 < 1$ ,  $0 < \theta_2 < 1$ , and  $0 < \theta_3 < 1$

Substituting (6.57) through (6.59) into (6.56) leads to,

$$\begin{aligned} e_{i+1,j} = & [ I_4 - 2rA_0 - hC_0 ] e_{i,j} + rA_0 [ e_{i,j+1} + e_{i,j-1} ] \\ & + h \frac{\partial U}{\partial t}(t + \theta_3 h, y) + hC_0 U_{i,j} - \frac{rk^2}{2} A_0 \left[ \frac{\partial^2 U}{\partial y^2}(t, y + \theta_1 k) \right. \\ & \left. + \frac{\partial^2 U}{\partial y^2}(t, y - \theta_2 k) \right] \end{aligned} \quad (6.60)$$

Applying the Intermediate Value Theorem,

$$\frac{1}{2} \left[ \frac{\partial^2 U}{\partial y^2}(t, y + \theta_1 k) + \frac{\partial^2 U}{\partial y^2}(t, y - \theta_2 k) \right] = \frac{\partial^2 U}{\partial y^2}(t, y + \theta_4 k) \quad (6.61)$$

where  $-1 < \theta_4 < 1$

hence (6.60) reduces to

$$\begin{aligned} e_{i+1,j} = & [ I_4 - 2rA_0 - hC_0 ] e_{i,j} + rA_0 [ e_{i,j+1} + e_{i,j-1} ] \\ & + h \left[ \frac{\partial U}{\partial t}(t + \theta_3 h, y) + C_0 U_{i,j} - A_0 \frac{\partial^2 U}{\partial y^2}(t, y + \theta_4 k) \right] \end{aligned} \quad (6.62)$$

Let  $E_i$  denote the norm of the maximum error along the  $i$  th time-row and  $M$  be defined by

$$M = \left\| \left| \frac{\partial U(t+\theta_3 h, y)}{\partial t} + C_0 U_{i,j} - A_0 \frac{\partial^2 U(t, y+\theta_4 k)}{\partial y^2} \right| \right\| \quad (6.63)$$

$$\text{If } \beta = \left\| \left| I_4 - 2rA_0 - hC_0 \right| + 2r \left| A_0 \right| \right\| \leq 1 \quad (6.64)$$

then,

$$\begin{aligned} \left\| \left| e_{i+1,j} \right| \right\| &\leq \left\| \left| \left[ I_4 - 2rA_0 - hC_0 \right] \right| \right\| \cdot \left\| \left| e_{i,j} \right| \right\| \\ &\quad + r \cdot \left\| \left| A_0 \right| \right\| \left[ \left\| \left| e_{i,j+1} \right| \right\| + \left\| \left| e_{i,j-1} \right| \right\| \right] + hM \\ &= \beta E_i + hM \end{aligned} \quad (6.65)$$

Since (6.65) is true for all  $i$ , the following is also true,

$$E_{i+1} \leq \beta E_i + hM = \beta^2 E_{i-1} + hM(1 + \beta) = \beta^3 E_{i-2} + hM(1 + \beta + \beta^2) \quad (6.66)$$

$$\text{Hence, } E_i \leq \beta^i E_0 + hM (1 + \beta + \beta^2 + \dots + \beta^{i-1}) \quad (6.67)$$

Initially  $u$  and  $U$  are the same, therefore  $E_0 = 0$

hence,

$$E_i \leq hM (1 + \beta + \beta^2 + \dots + \beta^{i-1}) \quad (6.68)$$

when  $k$  tends to zero,  $h = rk^2$  also tends to zero and  $M$  tends to

$$\left( \frac{\partial U}{\partial t} - A_0 \frac{\partial^2 U}{\partial y^2} + C_0 U \right)_{i,j}$$

But  $U$  is a solution of the differential equation (6.12), hence the limiting value of  $M$  and therefore  $E_i$  is zero. This proves that  $u$  converges to  $U$  as  $k$  tends to zero, and hence establishes the consistency of the discrete approximation.

### 3. Estimated Error

Consider the equation

$$\frac{\partial U}{\partial t} - A_0 \frac{\partial^2 U}{\partial y^2} + C_0 U = 0 \quad (6.12)$$

The above equation was approximated by the model

$$\frac{U_{i+1,j} - U_{i,j}}{h} - A_0 [U_{i,j+1} - 2U_{i,j} + U_{i,j-1}] / k^2 + C_0 U_{i,j} = 0 \quad (6.19)$$

Applying the Extended Mean Value Theorem,

$$\begin{aligned} U_{i,j+1} = & U_{i,j} + k \left( \frac{\partial U}{\partial y} \right)_{i,j} + \frac{k^2}{2!} \left( \frac{\partial^2 U}{\partial y^2} \right)_{i,j} + \frac{k^3}{3!} \left( \frac{\partial^3 U}{\partial y^3} \right)_{i,j} \\ & + \frac{k^4}{4!} \frac{\partial^4 U}{\partial y^4}(t, y + \theta_1 k) \end{aligned} \quad (6.69)$$

Similarly,

$$\begin{aligned} U_{i,j-1} = & U_{i,j} - k \left( \frac{\partial U}{\partial y} \right)_{i,j} + \frac{k^2}{2!} \left( \frac{\partial^2 U}{\partial y^2} \right)_{i,j} - \frac{k^3}{3!} \left( \frac{\partial^3 U}{\partial y^3} \right)_{i,j} \\ & + \frac{k^4}{4!} \frac{\partial^4 U}{\partial y^4}(t, y - \theta_2 k) \end{aligned} \quad (6.70)$$

where  $0 < \theta_1' < 1$  and  $0 < \theta_2' < 1$

Adding (6.69) and (6.70) and simplifying gives

$$[U_{i,j+1} + U_{i,j-1} - 2U_{i,j}] / k^2 = \frac{\partial^2 U}{\partial y^2} + \frac{k^2}{12} \frac{\partial^4 U}{\partial y^4}(t, y + \theta_3' k) \quad (6.71)$$

where  $-1 < \theta_3' < 1$

$$\text{also, } U_{i+1,j} = U_{i,j} + h \left( \frac{\partial U}{\partial t} \right)_{i,j} + \frac{h^2}{2} \frac{\partial^2 U(t+\theta'_4 h, y)}{\partial t^2}$$

$$\text{i.e. } [U_{i+1,j} - U_{i,j}] / h = - \frac{\partial U}{\partial t} + \frac{h}{2} \frac{\partial^2 U(t+\theta'_4 h, y)}{\partial t^2} \quad (6.72)$$

$$0 < \theta'_4 < 1$$

The truncation error  $T$  is obtained by substituting (6.71) and (6.72) into (6.19) and subtracting (6.19) from (6.12). The result is (2)

$$T = - \frac{h}{2} \frac{\partial^2 U(t+\theta'_4 h, y)}{\partial t^2} + A_0 \frac{k^2}{12} \frac{\partial^4 U(t, y+\theta'_3 k)}{\partial y^4} + C_0 (U - U_{i,j}) \quad (6.73)$$

Clearly the last term in (6.73) is due to computer round-off error. This term is very small in an average computer (2). Thus, neglecting this small error and taking the norm of (6.73) gives

$$\|T\| = \frac{h}{2} \left\| \frac{\partial^2 U}{\partial t^2} \right\| + \frac{k^2}{12} \|A_0\| \left\| \frac{\partial^4 U}{\partial y^4} \right\| \quad (6.74)$$

The second and fourth differentials in (6.74) are approximated by,

$$\frac{\partial^2 U(t+\theta'_4 h, y)}{\partial t^2} = [u_{i+\theta'_4+1,j} - 2u_{i+\theta'_4,j} + u_{i+\theta'_4-1,j}] / h^2 \quad (6.75)$$

$$\begin{aligned} \frac{\partial^4 U(t, y+\theta'_3 k)}{\partial y^4} = & [u_{i,j+2+\theta'_3} - 4u_{i,j+1+\theta'_3} + 6u_{i,j+\theta'_3} \\ & - 4u_{i,j-1+\theta'_3} + u_{i,j-2+\theta'_3}] / k^4 \end{aligned} \quad (6.76)$$

With both  $\theta'_3$  and  $\theta'_4$  equal to their respective maximum and

minimum values the above differentials are computed for Example 7.5 using the typical values  $R_1 = R_2 = R_3 = R_4 = 1$ ,  $a_0 = .5$ ,  $h = .02$  and  $k = .4$ . The results are

$$\left\| \frac{\partial^2 U}{\partial t^2} \right\| = .76 \quad (6.77)$$

$$\left\| \frac{\partial^4 U}{\partial y^4} \right\| = 1.32 \quad (6.78)$$

$$\text{Also } \|A_0\| = a_0 + 1 = 1.5 \quad (6.79)$$

Hence the estimated error is

$$\begin{aligned} \|T\| &= \frac{h}{2} (.76) + \frac{k^2}{12} (1.5)(1.32) \\ &= .033 \end{aligned} \quad (6.80)$$

Considering the fact that only first order approximations were used in the derivation of the discrete model, the above error is reasonable.

The above error is the truncation error, which is different from the total error given by equation ( 6.68 ),

### E. Summary

The Stability and Consistency were established for the heat conduction system of equations. It was also established that the truncation error was of the order .033, which is reasonable. There is no doubt, however, that this error may be improved by using higher order discrete models. Unfortunately, these higher order discrete models lead to the solution of implicit equations which are very difficult to solve.



CHAPTER VII

NUMERICAL EXAMPLES

Numerical solutions for the classes of distributed parameter problems discussed in Chapters IV and V are presented in this chapter. As far as possible attempts will be made to formulate examples of practical significance. The appropriate plots for various state and sensitivity quantities are also presented. The first set of examples involves synthesis of optimum and low sensitivity design for the two-element standby redundant system. The second set of examples relates to the problems of heat transfer discussed in Chapter V.

A. Stand-by Redundant System

1. Example 7.1 :synthesis of optimum repair rate

For problem 4.1 of Chapter IV, it is required to determine  $u^*$  and  $k_2$ . Numerical quantities given are;

$$T = 1. \tag{7.1}$$

$$k_1 = .5 \tag{7.2}$$

$$k_3 = 10 \tag{7.3}$$

$$\lambda_0 = .1 \tag{7.4}$$

$$\text{and } p_1(t) = .9\exp(-.6\lambda t) + .1\exp(-4.6\lambda t) , \tag{7.5}$$

$$0 \leq \tau \leq t \leq T$$

(a) Results

After applying Corollary 4.1 of Chapter IV and substituting  $u(t,\tau)$  from (4.36) into (4.34) and (4.37) it is seen that;

$$\frac{\partial q^*}{\partial t} + \frac{\partial q^*}{\partial \tau} - q^*(\lambda - q^*p_2/k_2) - k_1\lambda - k_3p_2^* = 0, \quad (7.6)$$

$$\frac{\partial p_2^*}{\partial t} + \frac{\partial p_2^*}{\partial \tau} + p_2^*(\lambda - q^*p_2^*/k_2) = 0 \quad (7.7)$$

subject to the boundary conditions (4.35), (4.30) through (4.33).

Equations (7.6) and (7.7) are solved with the aid of their characteristic curves. Note that these characteristics cross the  $t$ -axis and are inclined to it at an angle of  $45^\circ$ . A few of these curves are shown in figure 7.1.1. By writing an explicit discrete formula for, say,  $p_2$ , the values of  $p_2$  along a characteristic may be found if any point on it is known. The equations are so discretized as to ensure that the solutions for  $p_2$  and  $q$  progress in the direction of the arrows shown in the appropriate diagram. This effectively ensures that boundary conditions (4.32) and (4.35) are taken into account. Selecting  $\Delta t = \Delta \tau = h$ , the discrete versions of (7.6) and (7.7) are;

$$q_{i,j+1}^* = q_{i-1,j}^* + hq_{i,j}^* [\lambda + u_{i,j}^* + hk_1\lambda + hk_3(p_2^*)_{i,j}], \quad (7.8)$$

$$(p_2^*)_{i-1,j} = (p_2^*)_{i,j+1} + h(p_2^*)_{i,j} [u_{i,j}^* + \lambda] \quad (7.9)$$

$$\text{where } u_{i,j}^* = -q_{i,j}^*(p_2^*)_{i,j}/k_2 \quad (7.10)$$

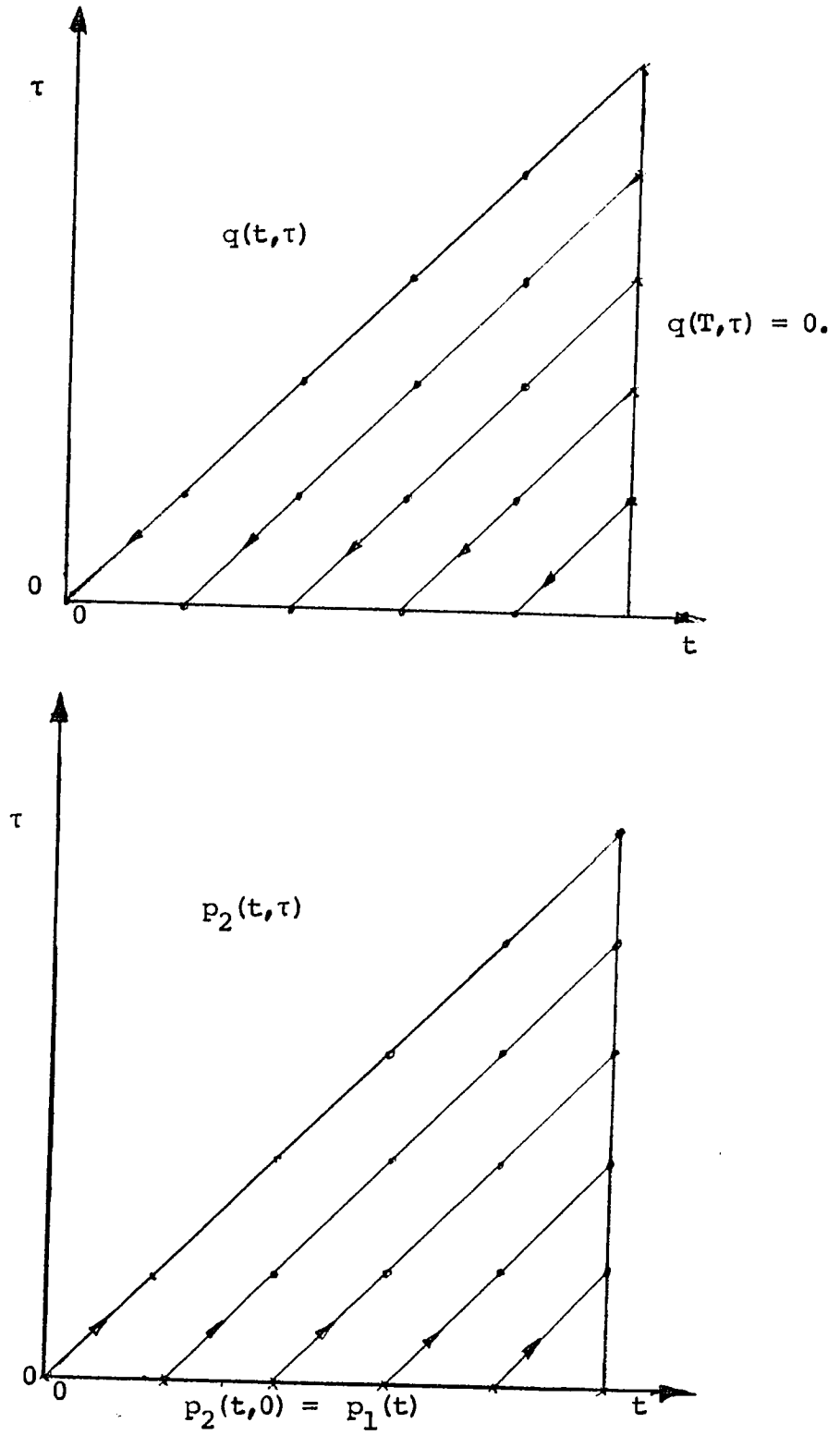


FIGURE 7.1.1 Characteristics and computing grids for problem 7.1

The algorithm for solving (7.8) and (7.9) may be summarized as follows;

- (i) Discretize (7.6) and (7.7) using first order forward and backward discrete models to obtain equations (7.8) and (7.9) above.
- (ii) Starting from the last characteristic, compute  $p_2$  and  $q$  on the grid points and in the direction of their respective arrows.
- (iii) Calculate  $u$  from (7.10)
- (iv) Select  $k_2$  and repeat steps (ii) and (iii) until (4.30) is satisfied.

The plots for  $p_2^*$  and  $u^*$  are shown in figure 7.1.2. The computer program, OPRATE, is listed in Appendix B-1.

The computed value for  $k_2$  is .45.

It is seen from figure 7.1.2 that  $p_2^*$  is fairly constant over the operating period. The optimal repair rate,  $u^*$ , however, decreases steadily until it reaches zero at the end of the operation.

From the computer results, the greatest absolute value of  $q$  is .67.

Clearly,  $k_2 k_3 - q^2 \leq 0$  which is the condition for convexity stated in Corollary 4.1. Hence, the optimum obtained in this example is a minimum.

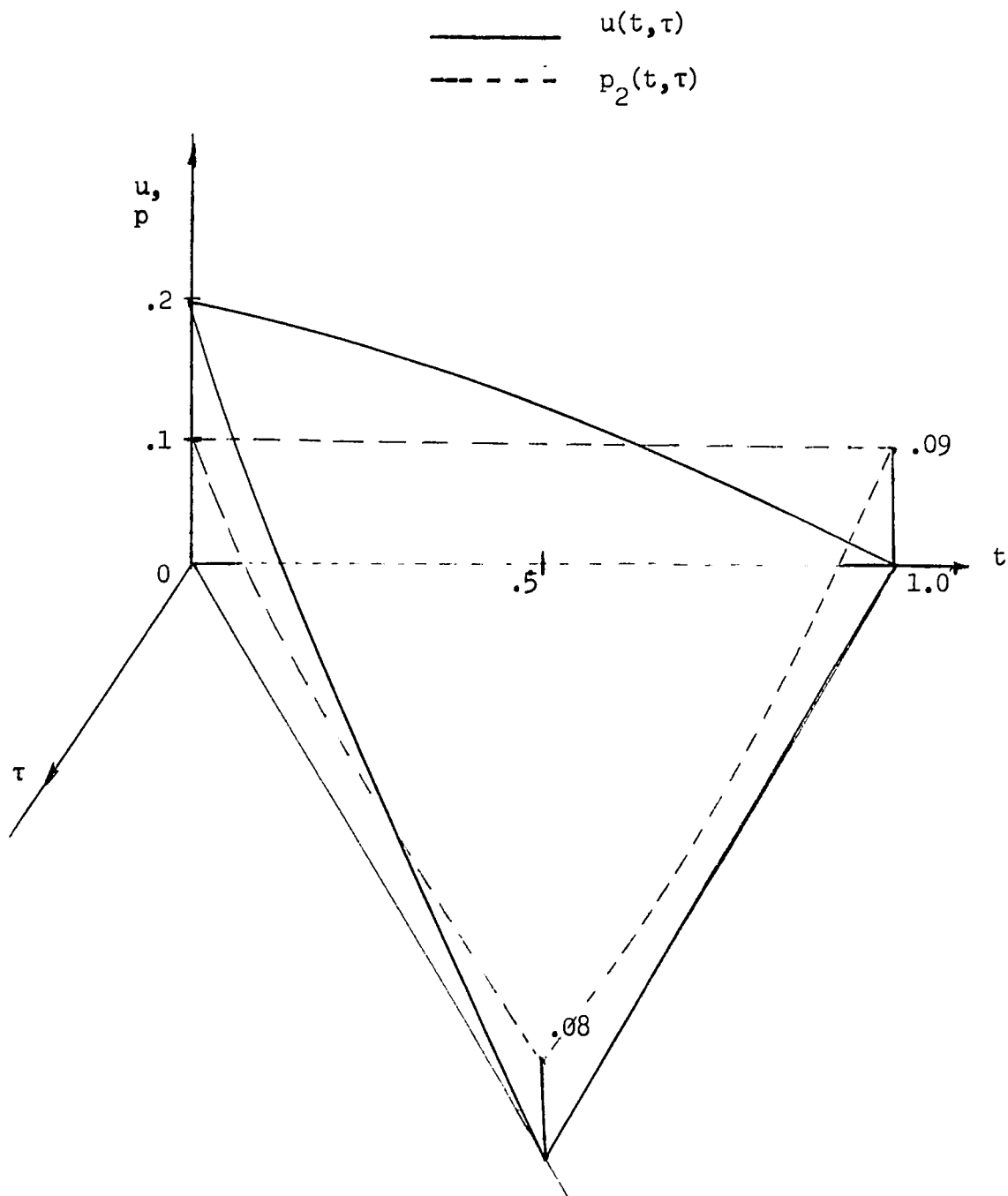


FIGURE 7.1.2 : Optimum State and Control for Problem 7.1

## 2. Sensitivity optimization for redundant systems

Three examples based on problem 4.2 of Chapter IV are presented in this section. It is proposed to determine both  $u^*$  and  $w^*$  and ( for some cases ) the relative sensitivity of the performance index. The relative or comparative sensitivity,  $S_R$  , of the performance index,  $J$ , is defined (79) by;

$$S_R = \frac{J - J^*}{J^*} \quad (7.11)$$

where  $J^*$  is the value of the performance index at the optimal point and  $J$ , is the value of the performance index at the non-optimal points. The system, it is noted, is not optimal for system parameters different from the nominal value,  $\lambda_0$ .

Remark : In computing the value of the performance index,  $J$  , it is important to note that all quantities within the double integral signs must appear as magnitudes or absolute quantities. This is necessary, as explained by Sage (79), in order to eliminate the undesirable cancelling effects of quantities which may be negative during the computation.

### (a) Example 7.2

For problem 4.2 of Chapter IV, it is required to find  $u^*$ ,  $w^*$ ,  $p_2^*$ ,  $v_2^*$ ,  $k_1$  and  $k_2$  given that,

$$k_3 = 1 \quad (7.12)$$

$$k_4 = 1 \quad (7.13)$$

$$k_5 = 1 \quad (7.14)$$

$$k_6 = 1 \quad (7.15)$$

$$\lambda_0 = .05 \quad (7.16)$$

$$\text{also } p_1(t) = .5 + .5\exp(-\lambda t) - .5\lambda t\exp(-\lambda t) \quad (7.17)$$

$$T = 1. \quad (7.18)$$

From Corollary 4.3 and the following definitions,

$$k_7 = k_1/D \quad (7.19)$$

$$k_8 = k_5/D \quad (7.20)$$

$$k_9 = k_4/D \quad (7.21)$$

$$\text{where } D = k_1k_4 - k_5^2 \quad (7.22)$$

it can be shown that,

$$\frac{\partial q_1^*}{\partial t} + \frac{\partial q_1^*}{\partial \tau} - v_2^* - q_1^*(u^* + \lambda) - k_2 p_2^* - (w^* + 1) = 0 \quad (7.23)$$

$$\frac{\partial q_2^*}{\partial t} + \frac{\partial q_2^*}{\partial \tau} - q_2^*(u^* + \lambda) - v_2^* - p_2^* = 0 \quad (7.24)$$

$$u^* = k_8 p_2^* q_2^* - k_9 q_1^* p_2^* - k_9 q_2^* v_2^* \quad (7.25)$$

$$w^* = k_8 q_1^* p_2^* + k_8 q_2^* v_2^* - k_7 p_2^* q_2^* \quad (7.26)$$

In addition, equations (4.40) through (4.44), (4.29) and (4.30) must be satisfied at the optimal point. The corresponding boundary conditions for the above co-state equations are;

$$q_1^*(T, \tau) = 0 \quad (7.27)$$

$$q_2^*(T, \tau) = 0 \quad (7.28)$$

where  $0 \geq \tau \geq t \geq T$

The discretized version of the state and co-state equations (4.29), (4.30), (7.23) and (7.24) are,

$$(v_2^*)_{i-1,j} = (v_2^*)_{i,j+1} + h(u_{i,j}^* + \lambda)(v_2^*)_{i,j} + h(w_{i,j}^* + 1)(p_2^*)_{i,j} \quad (7.29)$$

$$(p_2^*)_{i-1,j} = (p_2^*)_{i,j+1} + h(u_{i,j}^* + \lambda)(p_2^*)_{i,j} \quad (7.30)$$

$$(q_1^*)_{i,j+1} = (q_1^*)_{i-1,j} + h(v_2^*)_{i,j} + h(q_1^*)_{i,j} [u_{i,j}^* + \lambda] + h k_2 (p_2^*)_{i,j} + (q_2^*)_{i,j} [w_{i,j}^* + 1] h \quad (7.31)$$

$$(q_2^*)_{i,j+1} = (q_2^*)_{i-1,j} + h(q_2^*)_{i,j} [u_{i,j}^* + \lambda] + (v_2^*)_{i,j} h + h(p_2^*)_{i,j} \quad (7.32)$$

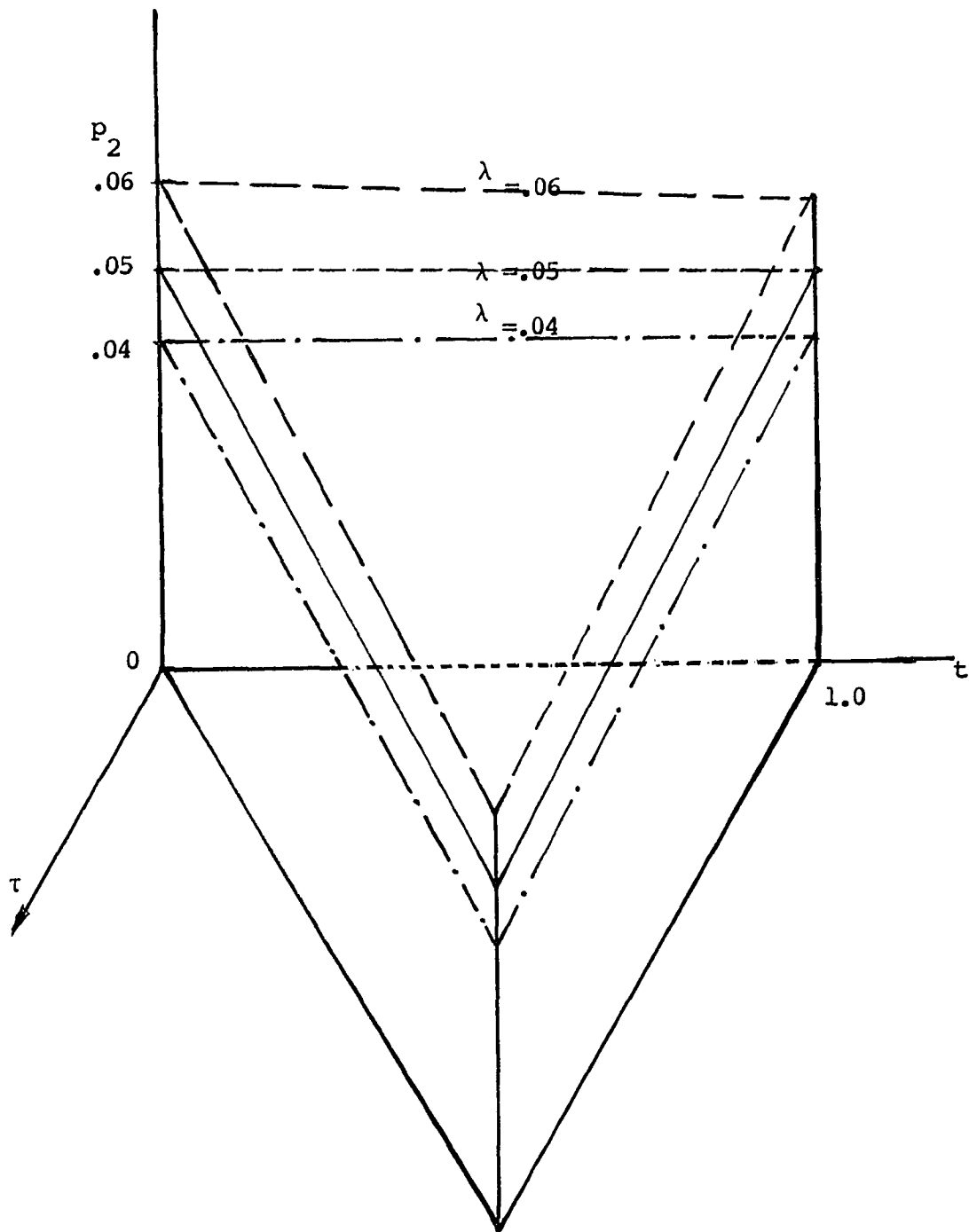
where  $u^*$  and  $w^*$  are defined by (7.25) and (7.26) respectively.

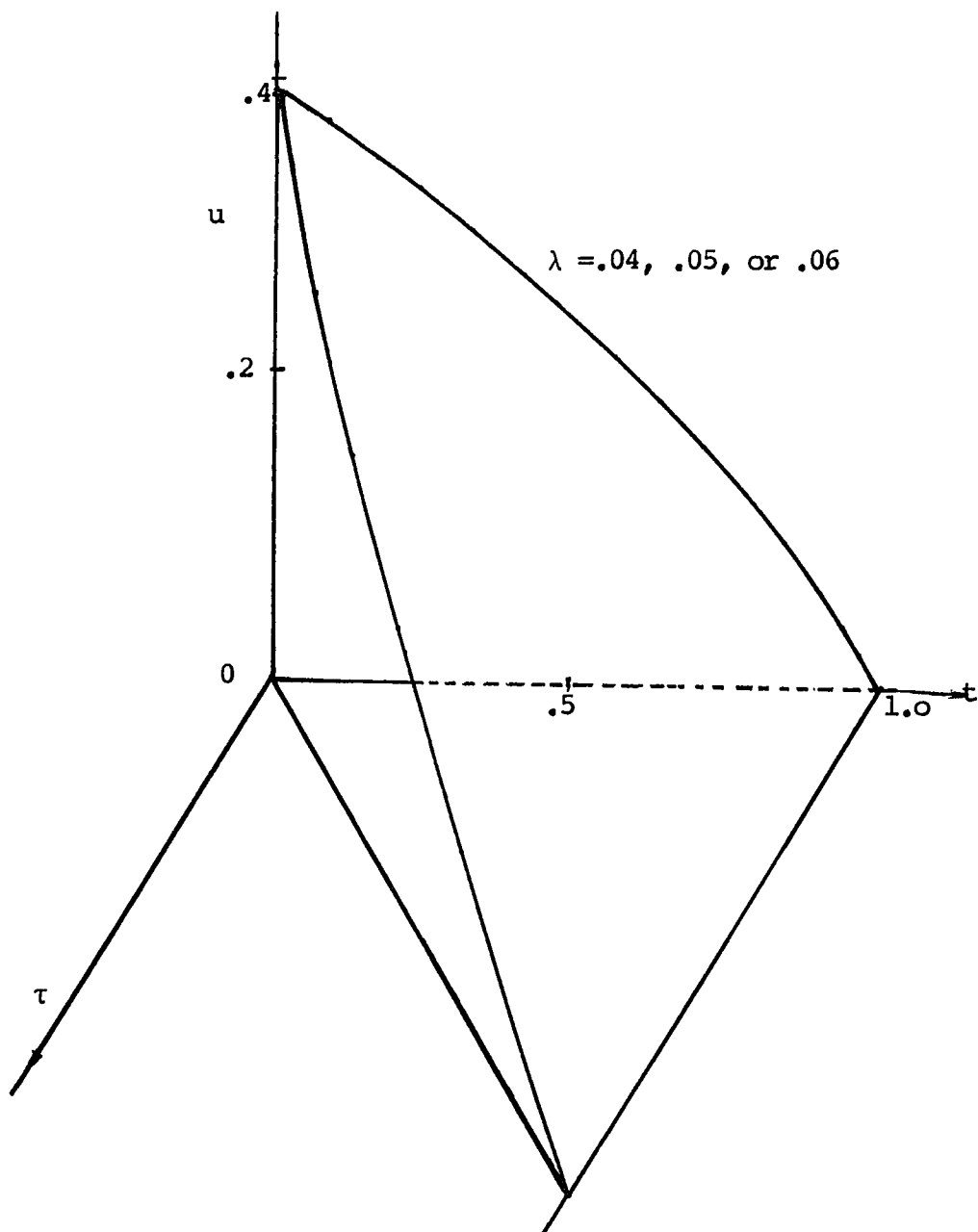
#### (b) Results

It is noted that the above equations are similar to those obtained for Example 7.1. Hence the algorithm for solving the above equations is also similar to that developed for solving Example 7.1. The plots for  $p_2^*$ ,  $u^*$ ,  $v_2^*$ , and  $w^*$  are shown in figures 7.2.1 through 7.2.4 for some values of  $\lambda$ . The computer program, LOWSEN, is listed in Appendix B-2.

The computed values for the constants are,  $k_1 = 1.7$  and  $k_2 = .9$ . From the results, it is seen that  $u^*$  is least sensitive to changes in the failure rate. Both  $w^*$  and  $p_2^*$  vary noticeably with parameter variations. The state sensitivity,  $v_2^*$ , varies only slightly with changes in  $\lambda$ .



FIGURE 7.2.1 : Plot of  $p_2$  for Problem 7.2

FIGURE 7.2.2 : Plot of  $u(t, \tau)$  for Problem 7.2

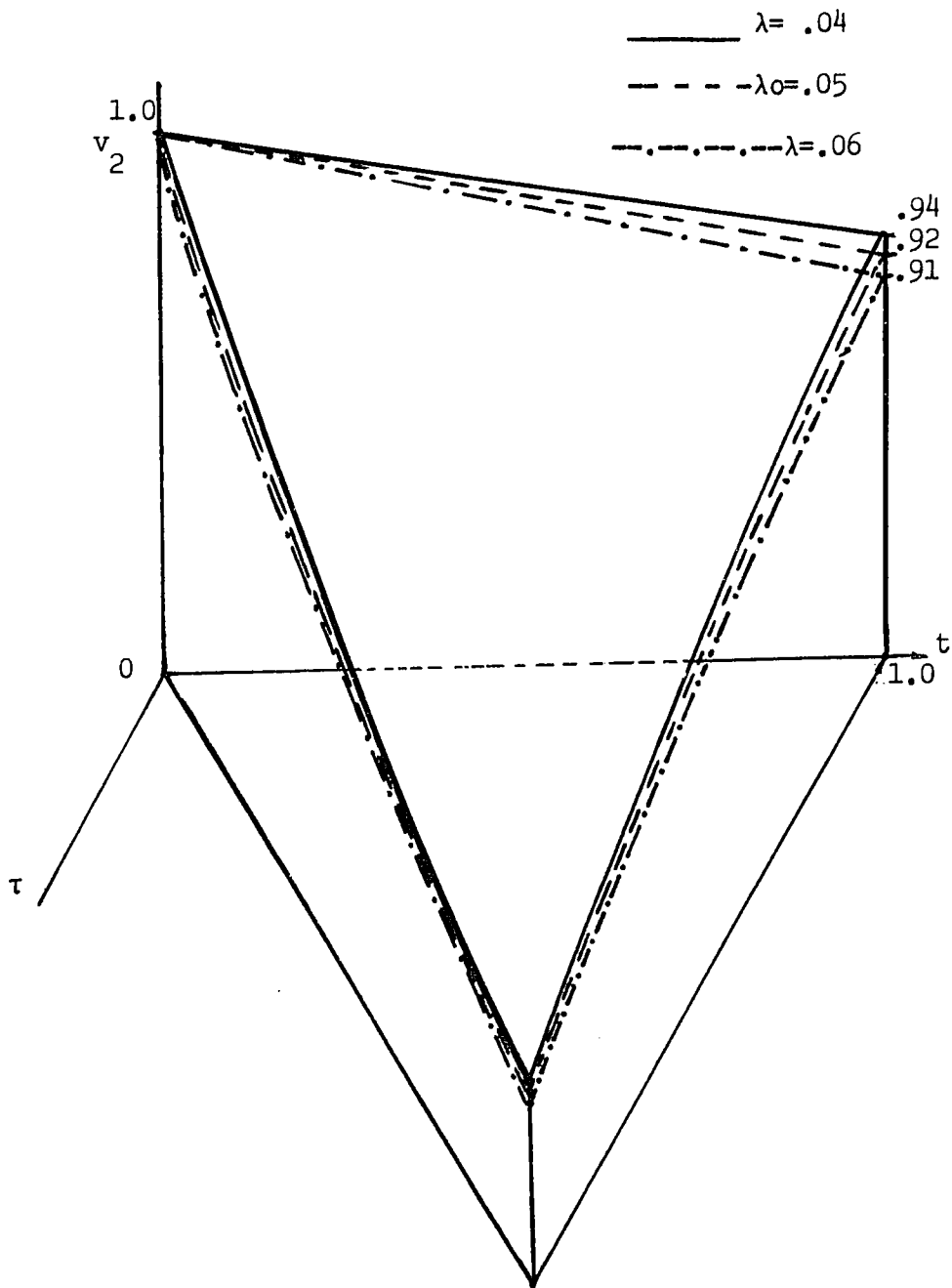
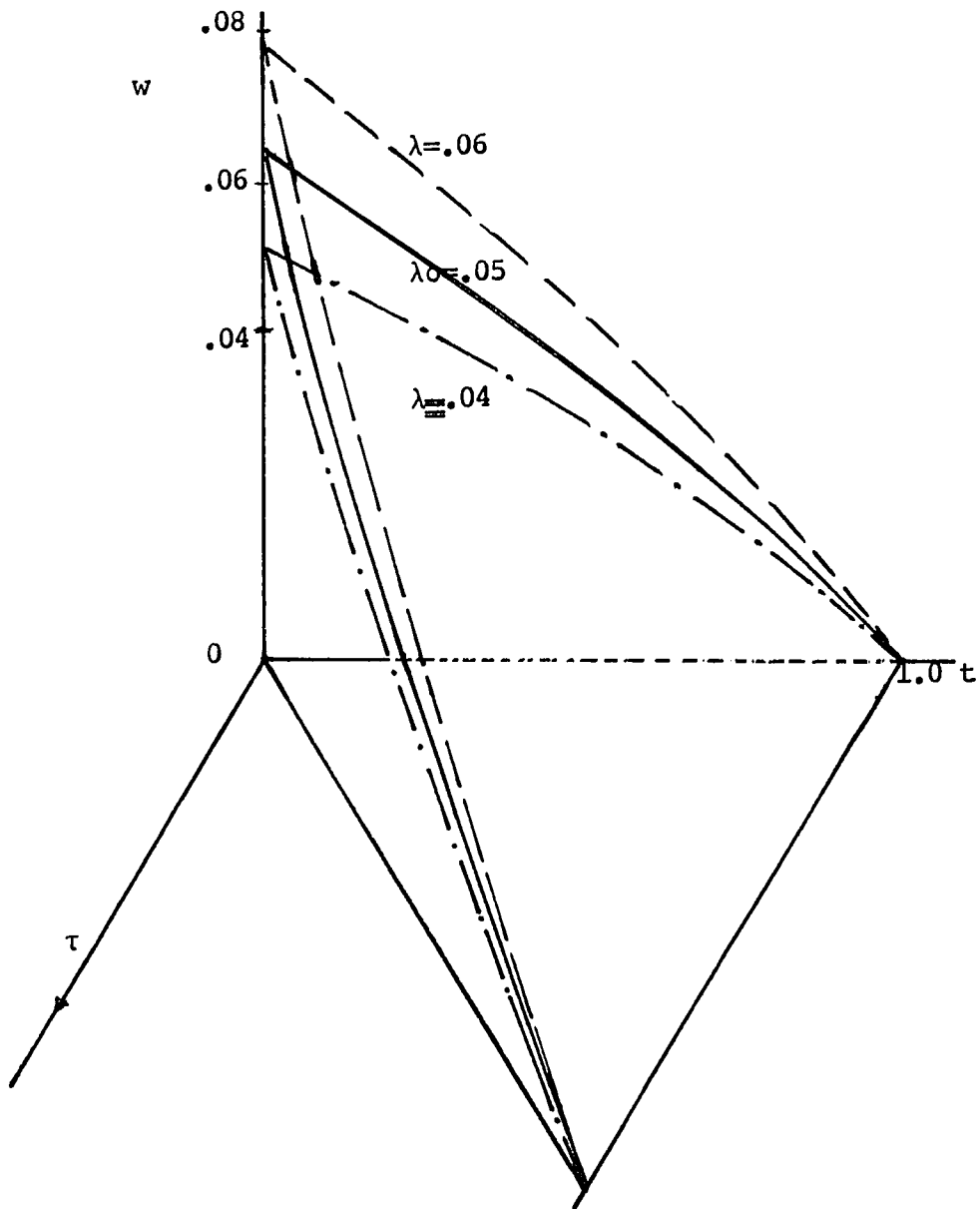


FIGURE 7.2.3 : Plot of  $v_2$  for Problem 7.2

FIGURE 7.2.4 : Plot of  $w$  for Problem 7.2

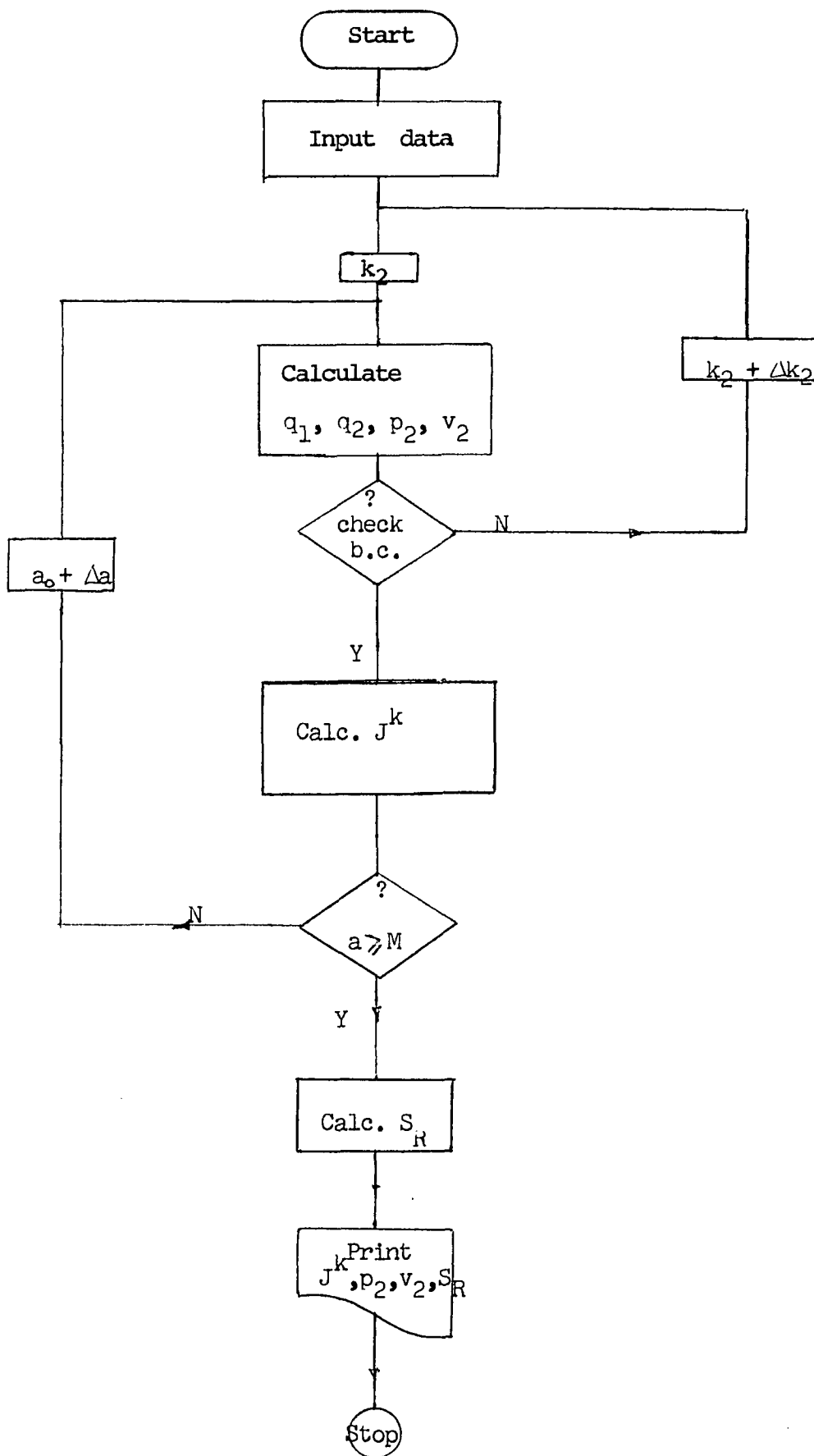


FIGURE 7.2.5 : FLOW CHART FOR PROBLEM 7.2

(c) Example 7.3

This example is essentially the same as Example 7.2, with the following modifications;

$$\lambda_0 = .5 \quad (7.33)$$

$$k_3 = .7 \quad (7.34)$$

$$k_4 = .5 \quad (7.35)$$

$$k_5 = 0 \quad (7.36)$$

$$k_6 = .5 \quad (7.37)$$

it is required to find,

(1)  $k_1$  and  $k_2$

(2) the relative sensitivity curves for

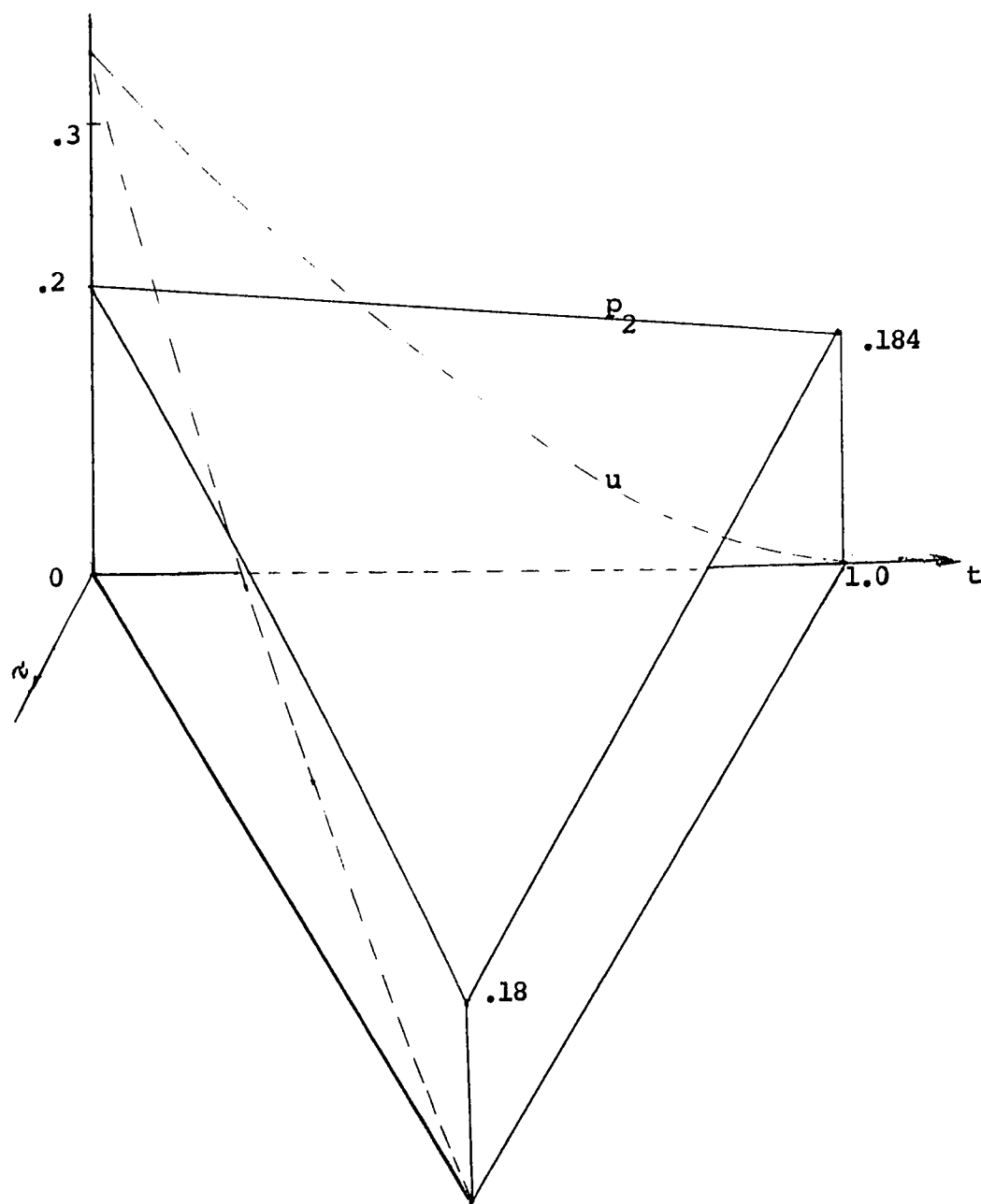
$$k_1 = .6, .9$$

and (the plots for  $p_2^*$ , and  $u^*$ )

(d) Results

The results are plotted in figures 7.3.1 and 7.3.2, the computed values for the constants are  $k_1 = .62$  and  $k_2 = 1.1$

It is seen from figure 7.3.2 that the relative sensitivity curves are nearly the same for small parameter changes. It is, therefore, concluded that the system is insensitive to small changes in  $k_1$ .

FIGURE 7.3.1 : Plot of  $p_2$  and  $u$  for Problem 7.3

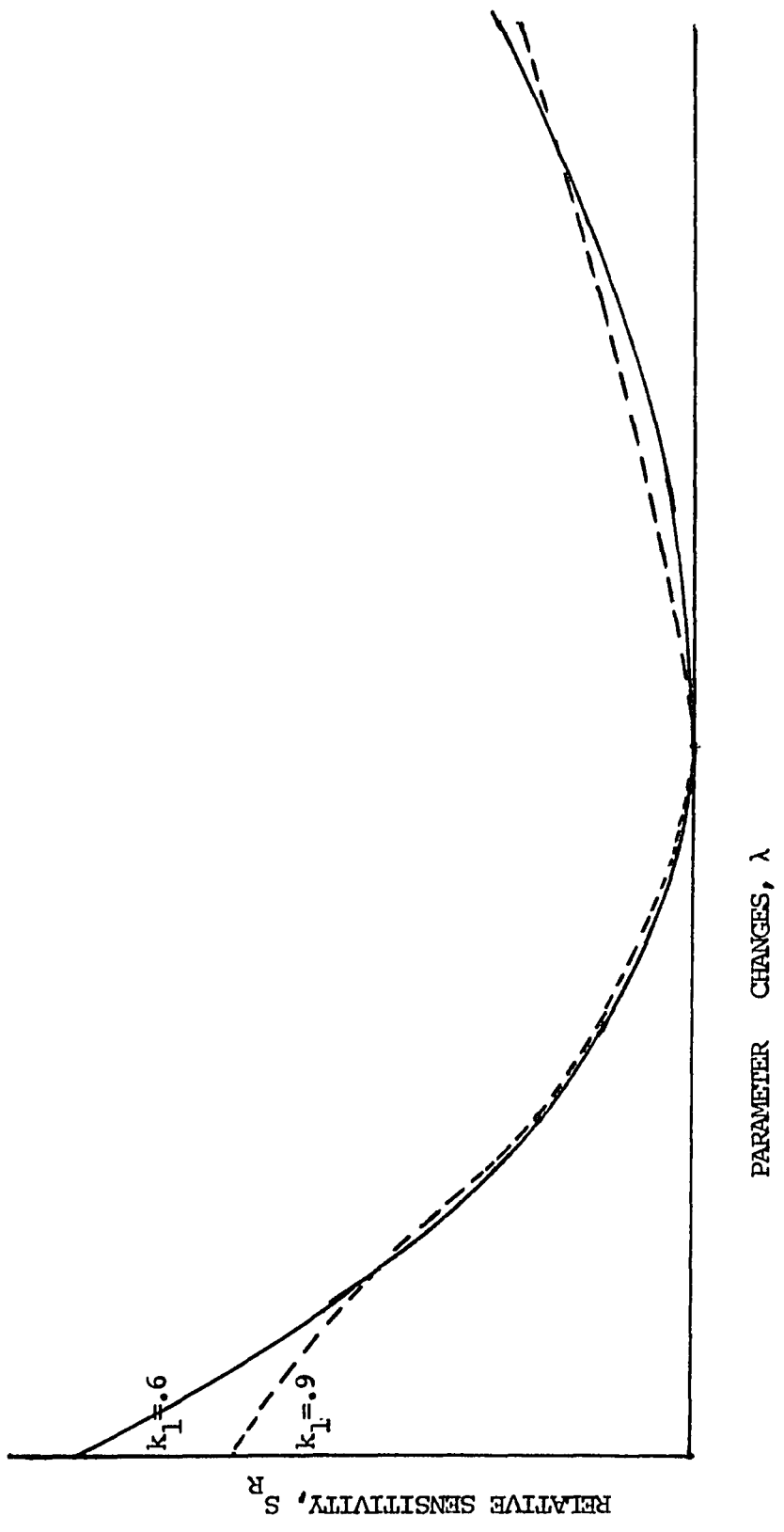


FIGURE 7.3.2 : Plot of Relative Sensitivity



(e) Example 7.4

This example is the same as Example 7.2, with the following modifications,

$$\lambda_0 = .2 \quad (7.38)$$

$$p_1(t) = .5\exp(-\gamma t) + .5\exp [ -(2-\gamma) t ]$$

(7.39)

$$k_2 = 1 \quad (7.40)$$

$$k_3 = .7 \quad (7.41)$$

$$k_4 = .5 \quad (7.42)$$

$$k_5 = 0 \quad (7.43)$$

$$k_6 = .5 \quad (7.44)$$

It is required to determine;

- (1)  $\gamma$  and  $k_1$
  - (2) the relative sensitivity curve
- and (3) the plots for  $p_2^*$  and  $u^*$

(f) Results

The computed values for the constants are  $\gamma = .72$  and  $k_1 = 1.5$ . The graphs for  $p_2^*$  and  $u^*$  are shown in figure 7.4.1 and the relative sensitivity curve is shown in figure 7.4.2

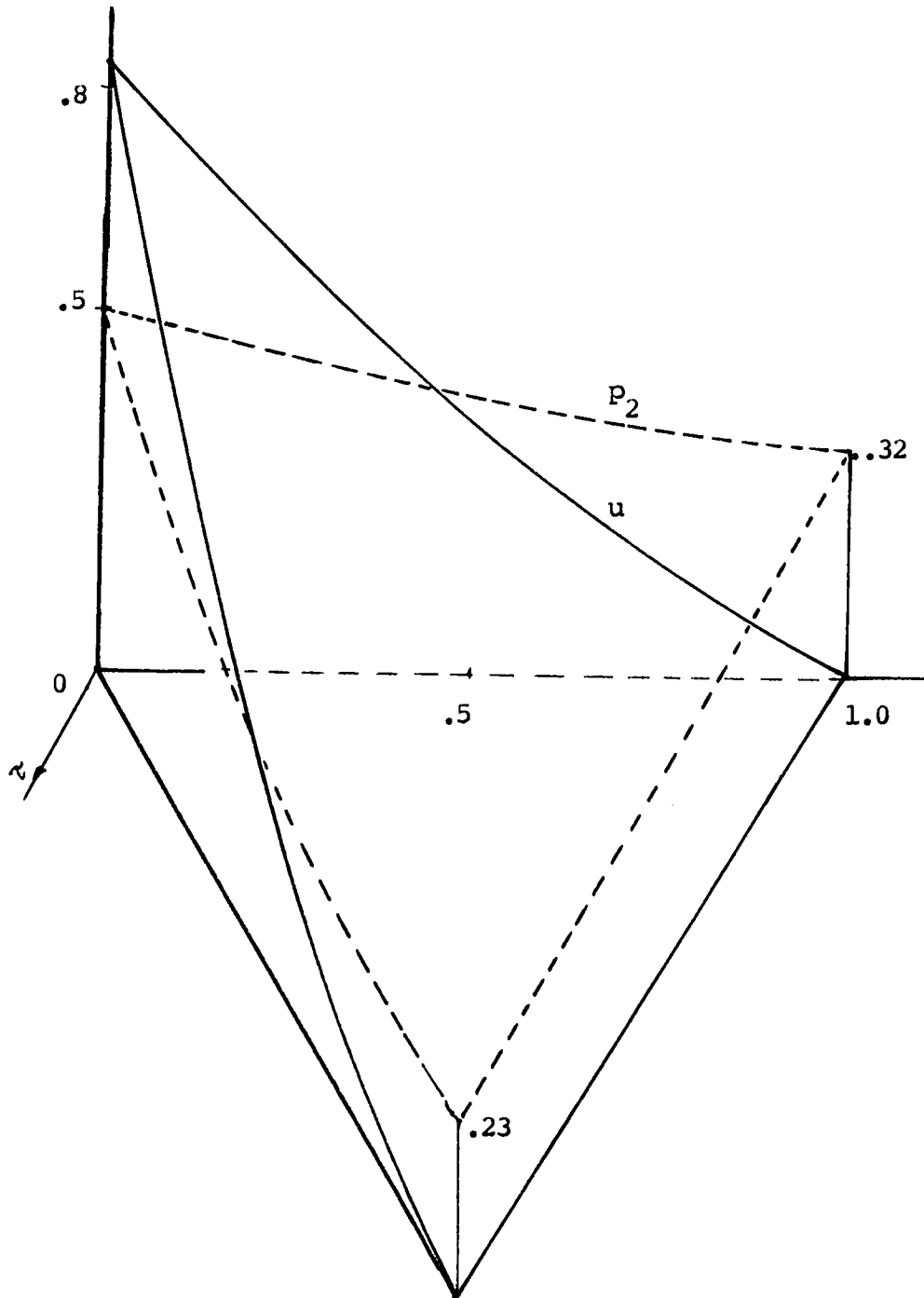


FIGURE 7.4.1 : Plot of  $p_2$  and  $u$  for Example 7A

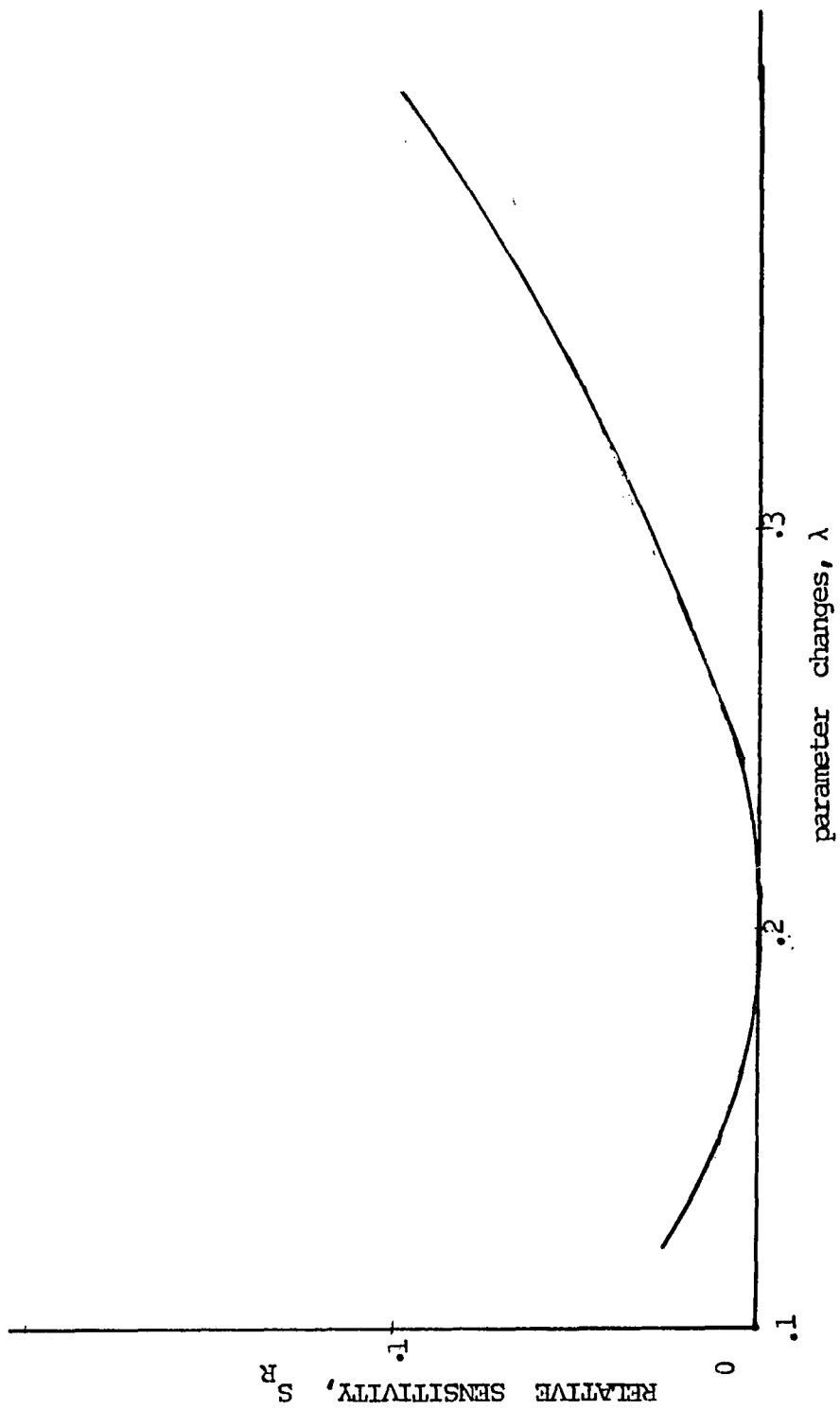


FIGURE 7.4.2 : Plot of Relative Sensitivity vs. parameter changes.

## B. Heat Transfer Examples

The importance of the heat conduction equation has already been noted. Poor heat conduction in solids may lead to catastrophic failure, general degradation, and hence adversely affect the performance of the device. One of the factors which may affect the transfer of heat is variation in the value of the diffusivity of the material. Diffusivity variations are not exactly known analytically. This is not unexpected since diffusivity is a function of three parameters - thermal conductivity ( $\kappa$ ), density ( $\rho$ ) and the specific heat ( $c$ ). Indeed, each of these parameters may vary in its own peculiar manner ( 21, 27 ).

The diffusivity,  $a$ , is related to the above three parameters through the relation,

$$a = \frac{\kappa}{c\rho} \quad \text{cm/sec.} \quad (7.45)$$

Practical values for this parameter range from zero to about 4.0. Three examples based on the two problems discussed in Chapter V will be presented in the following sections. As usual, it is assumed that changes in diffusivity are uniform and small.

1. Example 7.5

This example is based on the heat sink problem presented in Chapter V. It is required to determine  $T^*$  and  $v^*$  for problem 5.1, given the following numerical values;

$$b = 1 \quad (7.46)$$

$$R_2 = cR_1 \quad (7.47)$$

$$R_4 = cR_3 \quad (7.48)$$

$$R_5 = R_6 = 0 \quad (7.49)$$

$$T(0,y) = v(0,y) = 0 \quad (7.50)$$

$$T(t,0) = T(t,2.5) = 0 \quad (7.51)$$

$$v(t,0) = v(t,2.5) = 0 \quad (7.52)$$

where  $R_1$  and  $R_3$  are given constants.

The following costate equations and their associated boundary conditions follow immediately from application of Corollary 5.1 of Chapter V;

$$\frac{\partial q_1}{\partial t} = \frac{\partial^2 q_1}{\partial y^2} + \frac{\partial^2 q_2}{\partial y^2} - R_1 T \quad (7.53)$$

$$\frac{\partial q_2}{\partial t} = a \frac{\partial^2 q_2}{\partial y^2} - R_2 v \quad (7.54)$$

$$u = -q_1/R_3 \quad (7.55)$$

$$w = -q_2/R_4 \quad (7.56)$$

$$q_1(t,0) = q_2(t,0) = 0, \quad t \in (0,1) \quad (7.57)$$

$$q_1(1,y) = q_2(1,y) = 0, \quad y \in (0,2.5) \quad (7.58)$$

$$q_1(t,2.5) = q_2(t,2.5) = 0, \quad t \in (0,1) \quad (7.59)$$

(a) Results

After substituting  $u$  and  $w$  from equations (7.55) and (7.56) into the state equations (5.1) and (5.5) the resulting equations, together with the co-state equations are discretized to obtain the approximate difference equations. The programming algorithm for the solution of the two-point boundary value problem in discretized version is summarized as following;

- (1) Guess values  $T^0, v^0, J^0$
- (2) Compute  $q_1^1, q_2^1$  by solving backwards the discretized co-state equations.
- (3) Using  $q_1^1$  and  $q_2^1$ , find  $T^1$  and  $v^1$  by solving forwards the discretized state equations.
- (4) Compute  $J^1$  from  $T^1, v^1, q_1^1, q_2^1$ .
- (5) Repeat steps (2) to (4) above until  $J^k$  meets the specified criterion  $|J^{k+1} - J^k| < \epsilon$ , for given small positive number  $\epsilon$ .

With  $\Delta t = .04, \Delta y = .5, \epsilon = 10^{-3}$ , good results were obtained after an average of about 7 iterations. The flow chart for digital computation is shown in figure 7.5.1 The program is listed in Appendix B-3 as routine LHSINK.

The optimal solutions  $T^*$  and  $v^*$  are plotted in figure 7.5.2 and 7.5.3 respectively for  $R_1=R_3=1, c=.5$  and  $a_0=.5$ . Typical values are tabulated in Table 7.5. For a material intended to function as a heat sink, a fairly high temperature gradient at  $x=2.5$  is desirable. Unfortunately as the temperature gradient increases the sensitivity also increases. Hence there is a need for compromise between those two quantities. The effect of weighting factors  $R_1$  and  $R_2$  on the value of the performance index is shown in figure 7.5.4. An increase in  $R_1$  relative to  $R_2$  does not improve the shape of the curves, which should ideally be flat over a wide range of parameter changes.

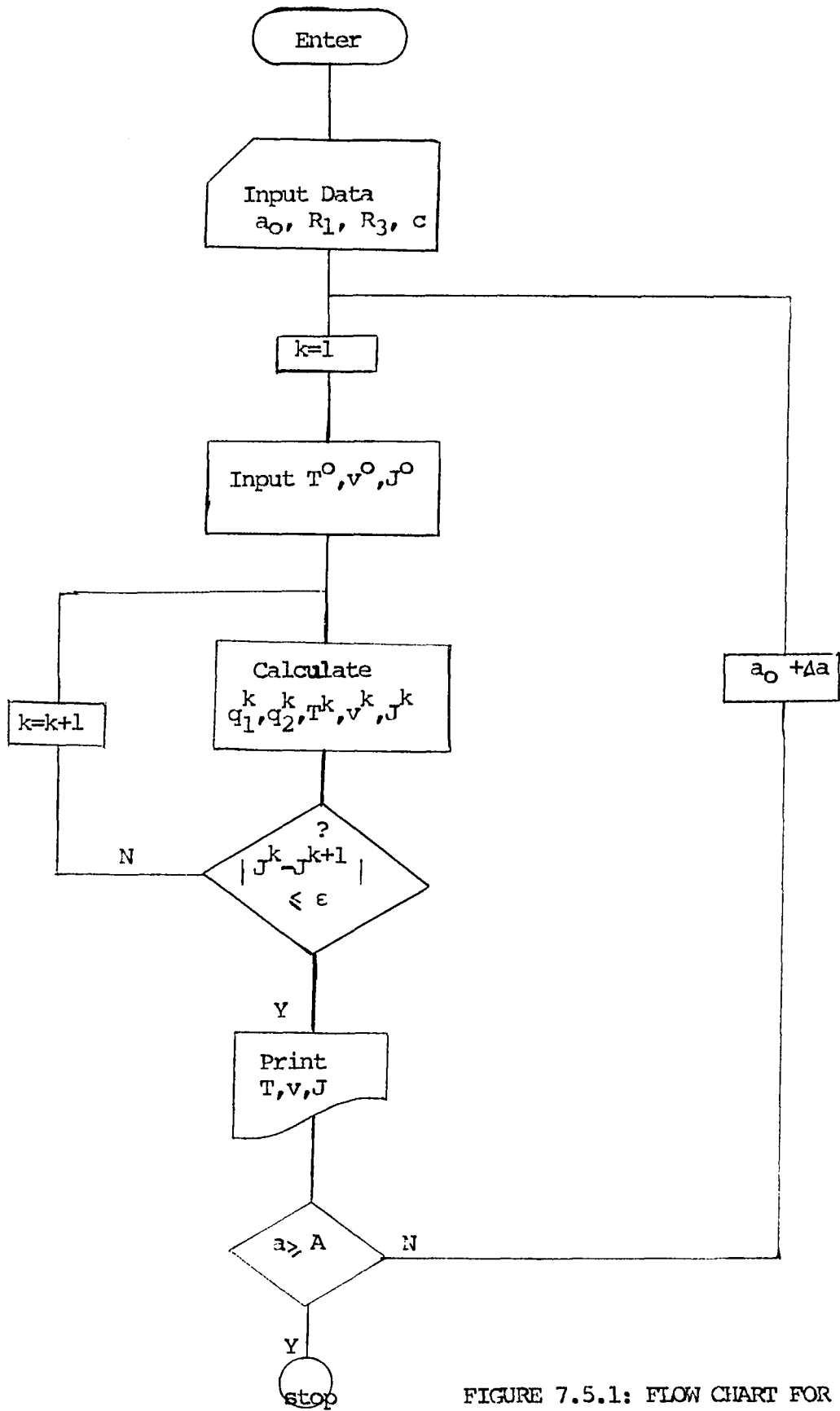
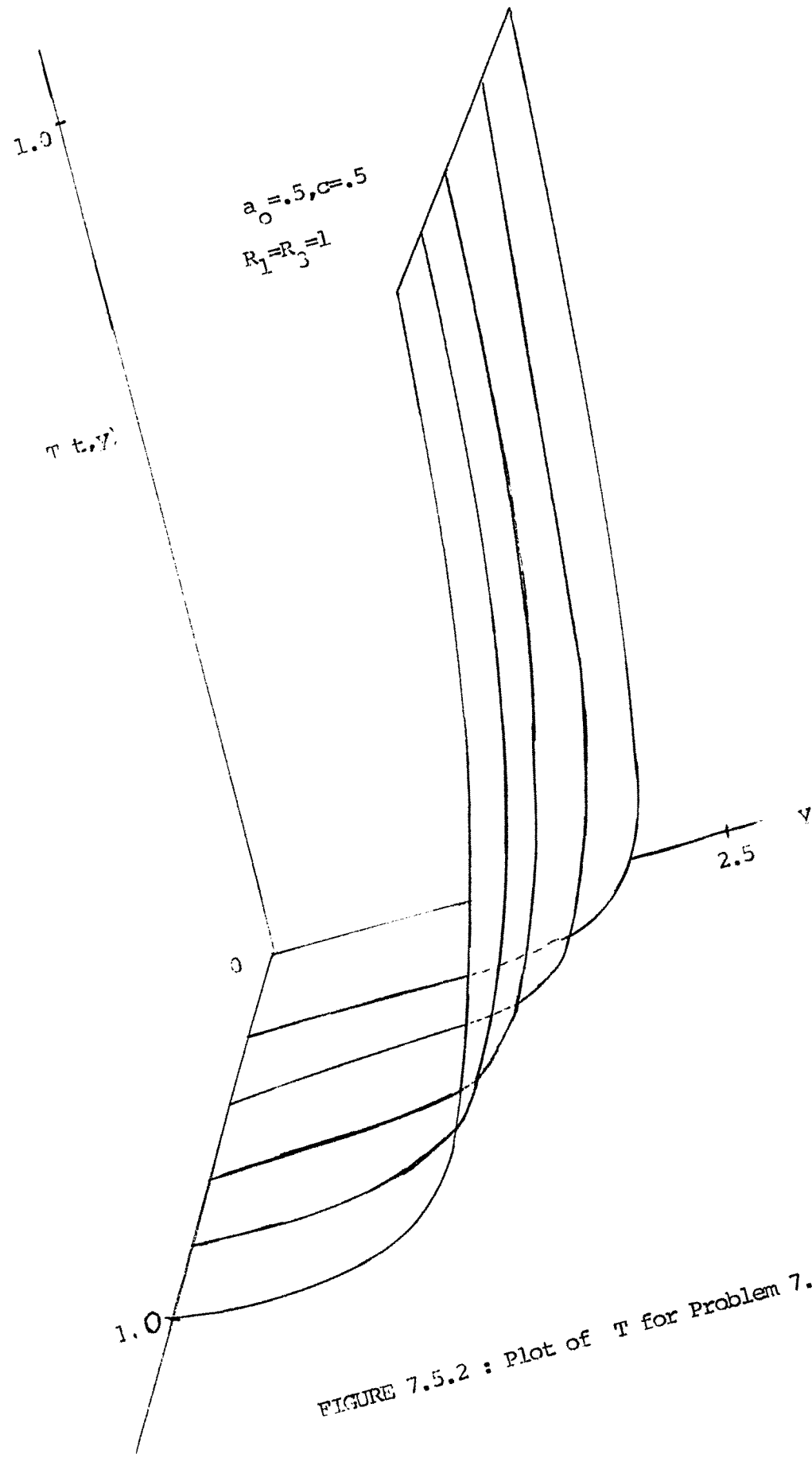
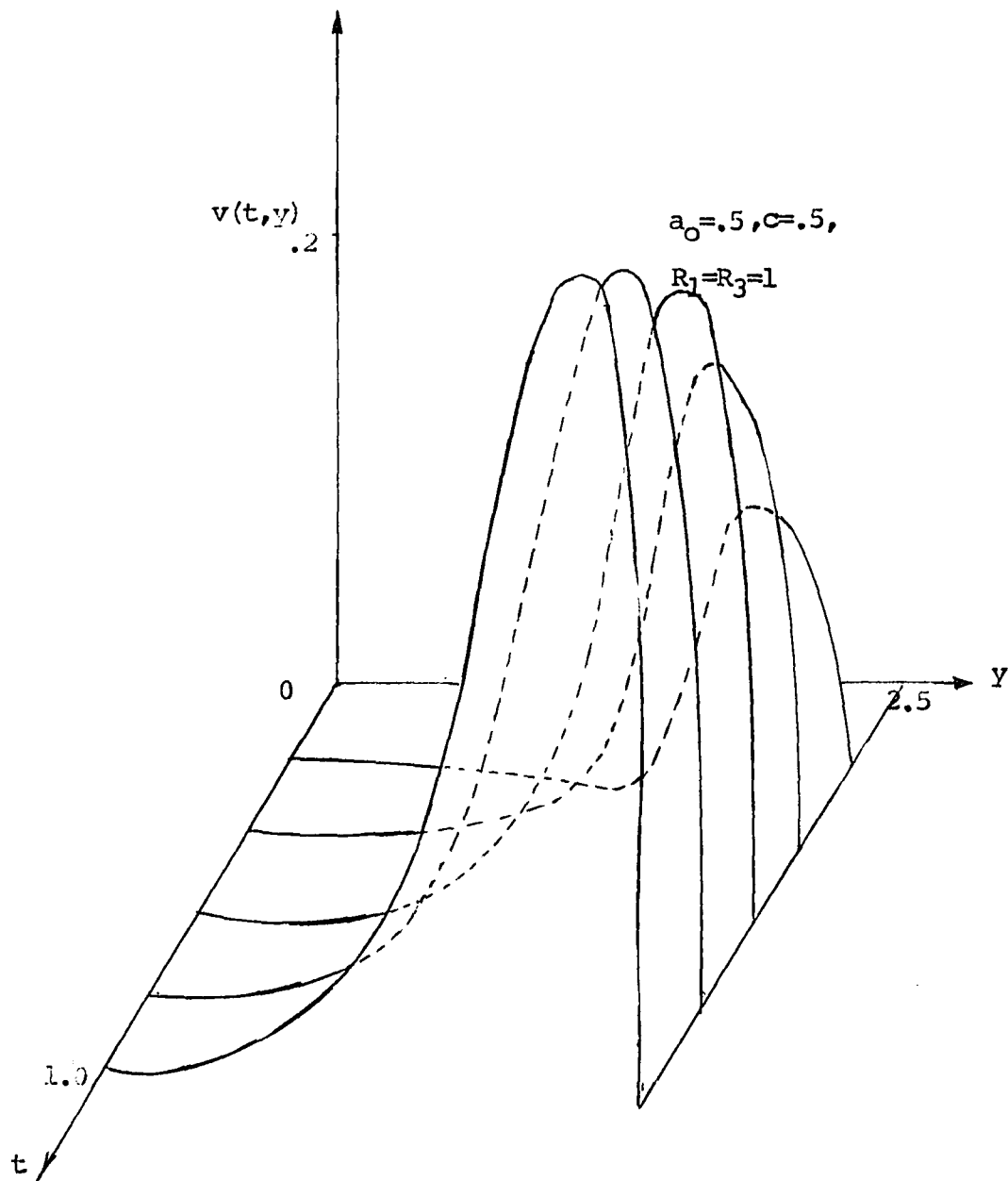


FIGURE 7.5.1: FLOW CHART FOR  
EXAMPLE 7.5

FIGURE 7.5.2 : Plot of  $T$  for Problem 7.5



FIGURE 7.5.3 : Plot of  $v$  for Problem 7.5

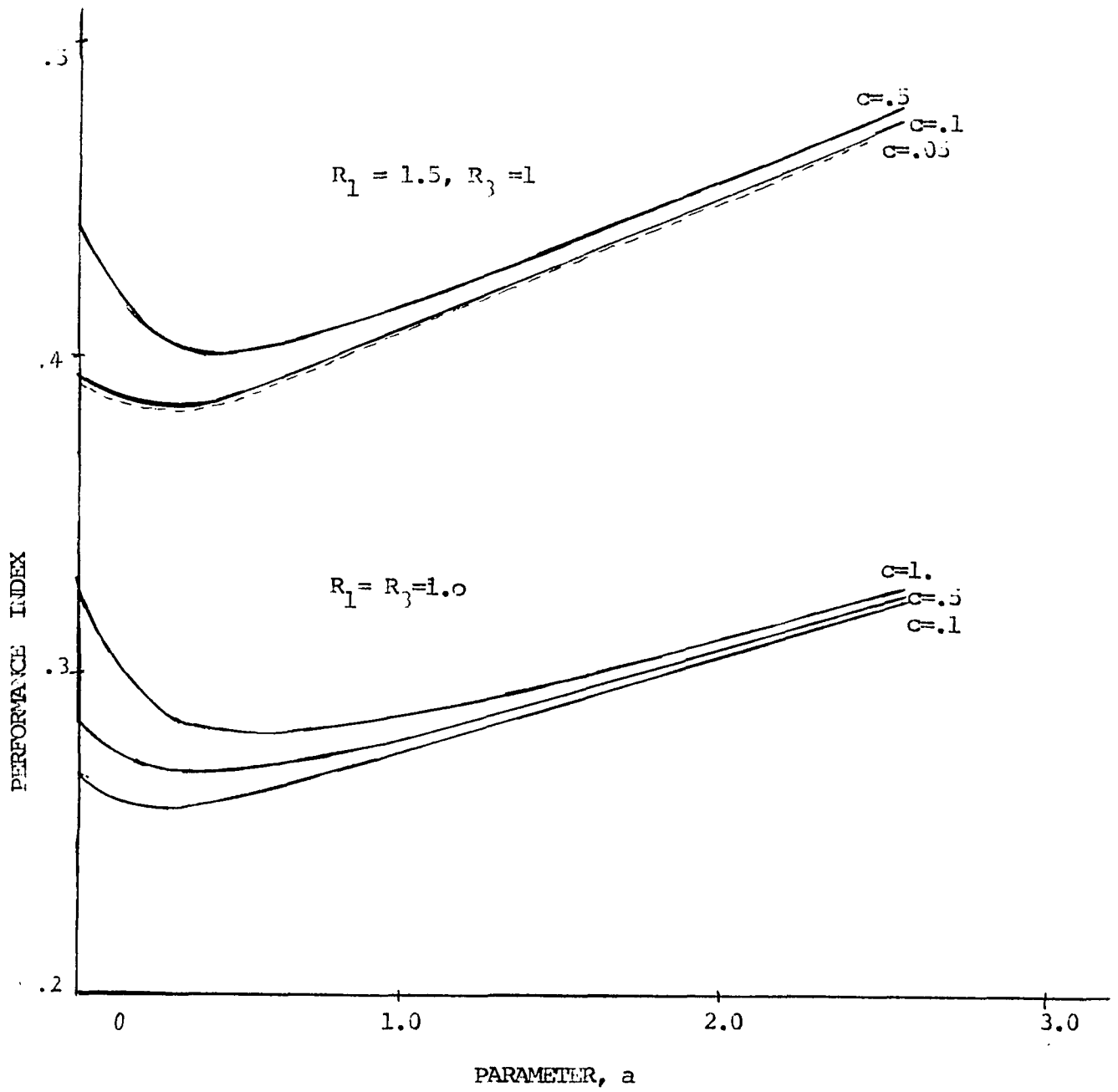


FIGURE 7.5.4 : Plot of Performance Index Variations

J	a	Tempt. grad. at $y=2.5$	Peak Sensitivity	
			+ve	-ve
.281	.01	.43	.46	.021
.270	.51	.35	.36	.003
.280	1.01	.28	.21	.003
.294	1.51	.28	.14	.003
.308	2.01	.41	.11	.002
.321	2.51	.20	.09	.002
.334	2.01	.18	.08	.001

Table 7.5: Variation of Temperature gradient and sensitivity for Example 7.5

Since  $R_5=R_6=0$  and  $R_i$  ( $i=1,2, 4$ ) are all positive it is obvious that the sufficient conditions stated in Corollary 5.2 are satisfied. Hence the solutions obtained in this example are the required minima.

## 2. Example 7.6

This example is the same as Example 7.5 except that it is here assumed that parameter variations have no effect on the control, u. i.e.  $w=0$ . This assumption is widely used by authors who investigate low sensitivity in continuous systems. (50).

### (a) Results

The results for  $T$  and  $v$  are shown in Figure 7.6.1 and 7.6.2 for  $R_1=R_3=1$ ,  $c=.5$  and  $a_0=.5$ . The variations of the performance index with respect to parameter changes are shown in figure 7.6.3. These curves appear to be similar to the corresponding curves for Example 7.5. But the Table showing the variations of temperature gradient and sensitivity with parameter changes show some increase in both the temperature gradient and the corresponding sensitivity. The pattern of behavior is the same hence there is no advantage in this example in including an additional term,  $w$ .

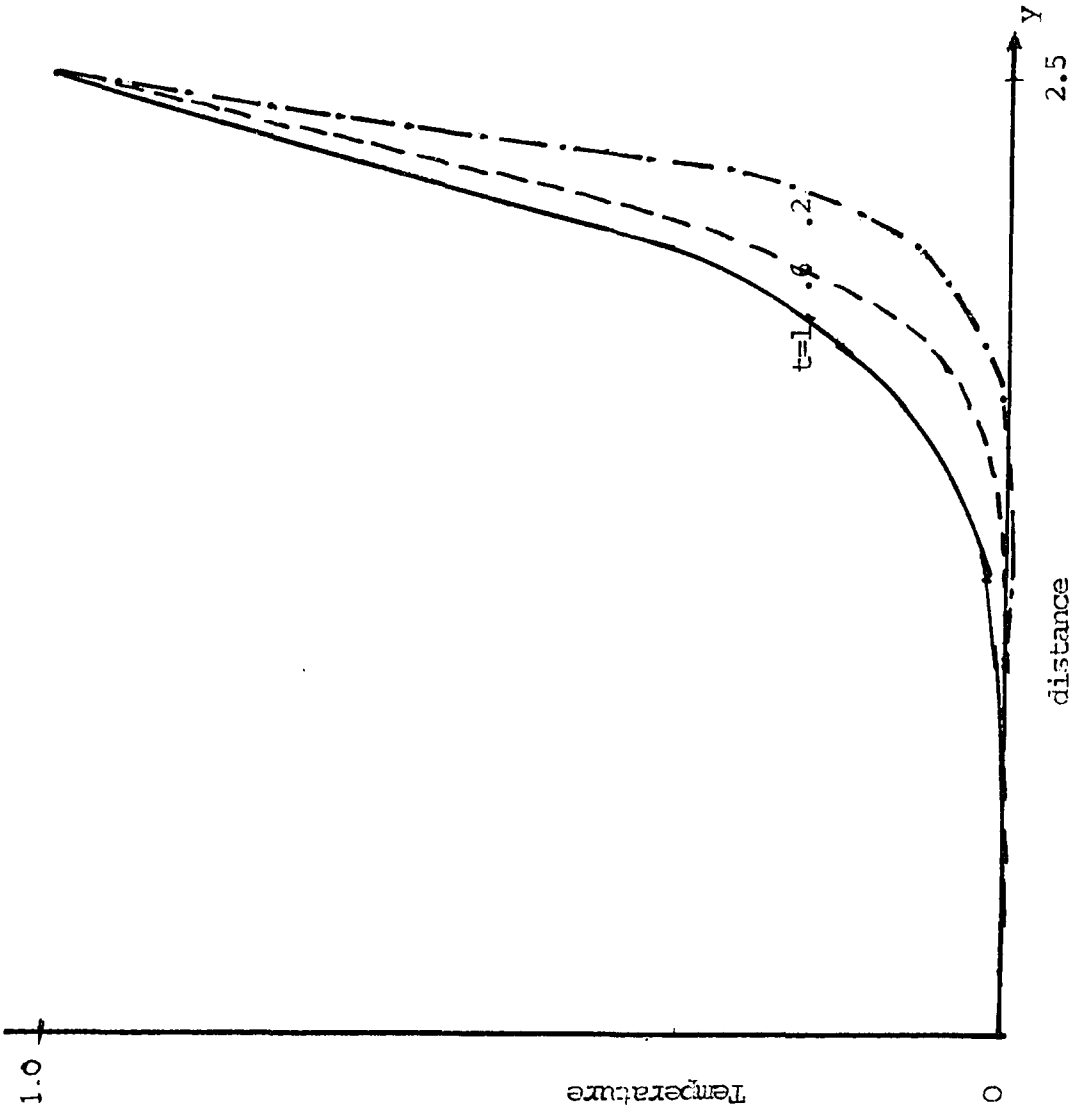


FIGURE 7.6.1 : Plot of T for Problem 7.6

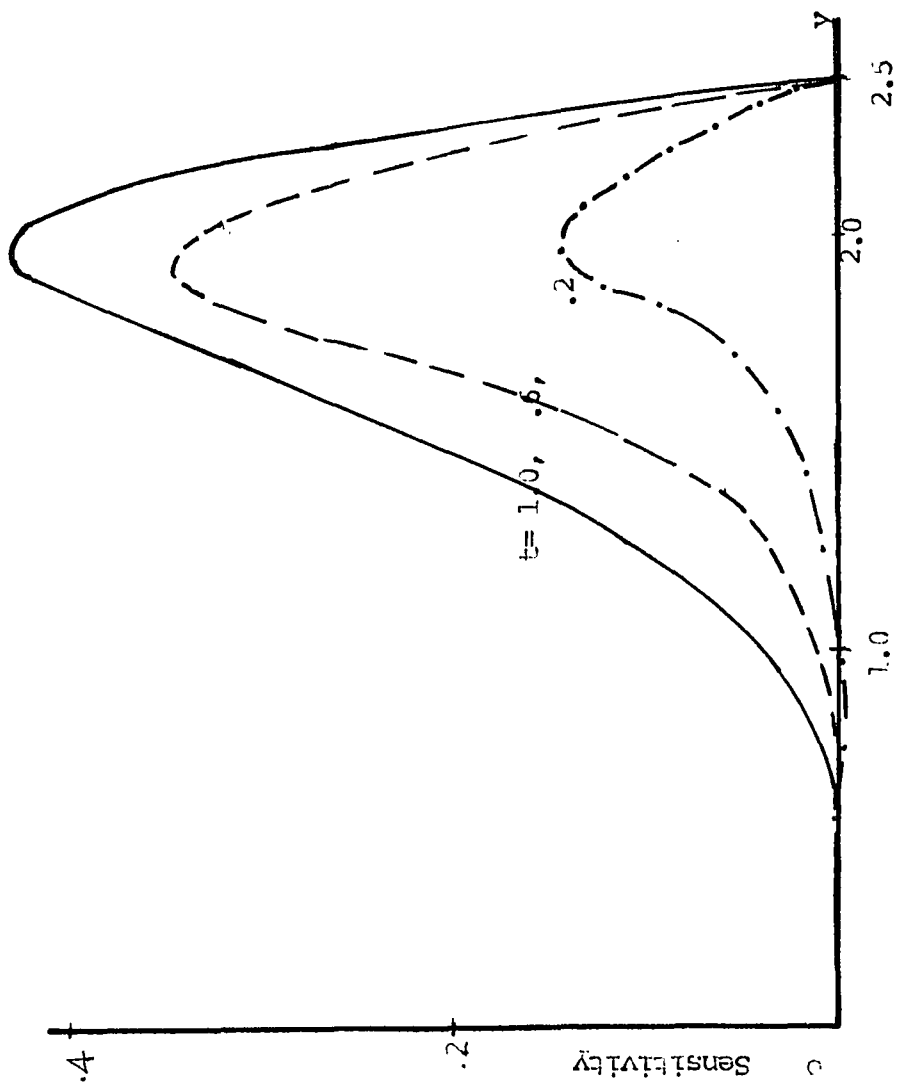


FIGURE 7.6.2 : Plot of  $v$  for Problem 7.6

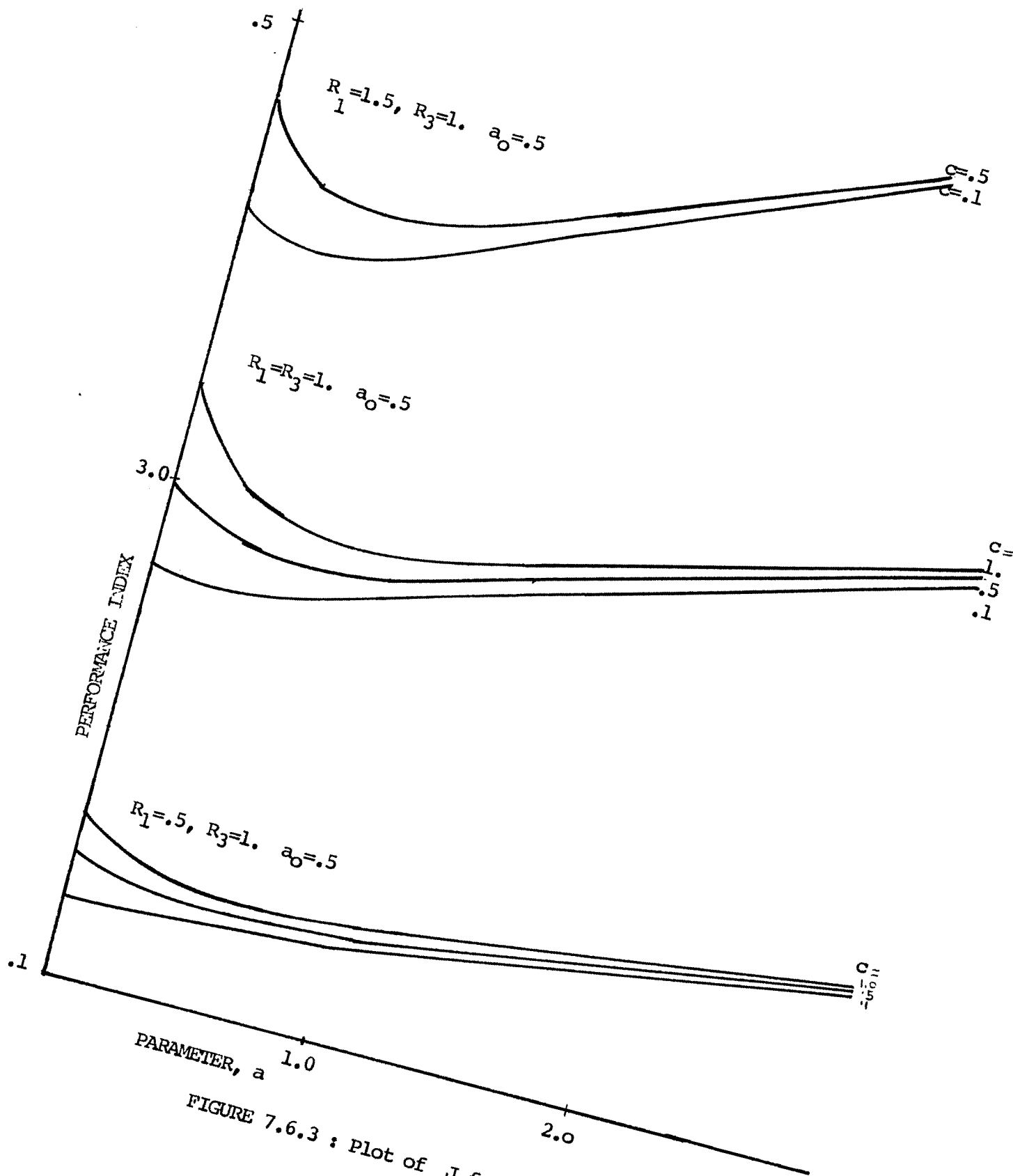


FIGURE 7.6.3 : Plot of  $J$  for Problem 7.6



J	a	Tempt. grad. at $y = 2.5$	Peak Sensitivity	
			+ve	-ve
.287	.01	1.75	.74	.037
.273	.51	1.42	.43	.002
.281	1.01	1.16	.24	.002
.294	1.51	1.00	.16	.000
.308	2.01	0.88	.13	.000
.322	2.51	0.78	.11	.000
.334	3.01	0.72	.09	.000

Table 7.6: Variation of Temperature gradient and sensitivity for Example 7.6

### 3. Example 7.7

This is essentially the same as Example 7.5 with the following modifications;

(i) The boundaries are insulated, i.e.

$$\left. \frac{\partial T}{\partial Y} \right|_{Y_0=0} = \left. \frac{\partial T}{\partial Y} \right|_{Y_f=2} = 0 \quad (7.60)$$

The following numerical values are also given,

$$a_0 = .5 \quad (7.61)$$

$$t_0 = 0 \quad (7.62)$$

$$t_f = 0 \quad (7.63)$$

#### (a) Results

The plots of  $T$  and  $v$  are shown in figure 7.7.1 and 7.7.2 respectively. The effect of  $R_1$  and  $R_3$  and parameter changes on the performance index is shown in figure 7.7.3. The performance index curve for this problem is far superior to that for other examples,

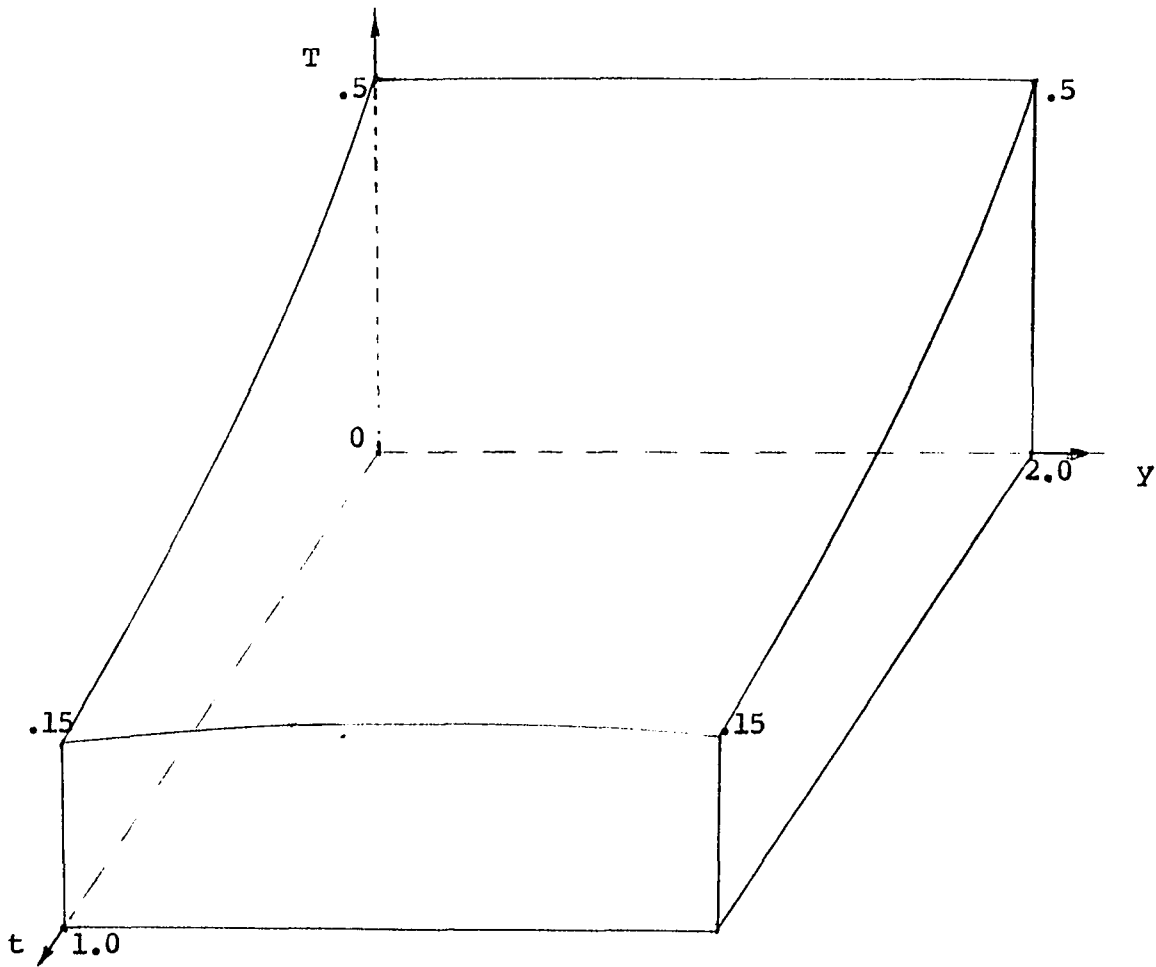


FIGURE 7.7.1 : Plot of Temperature for Example 7.7

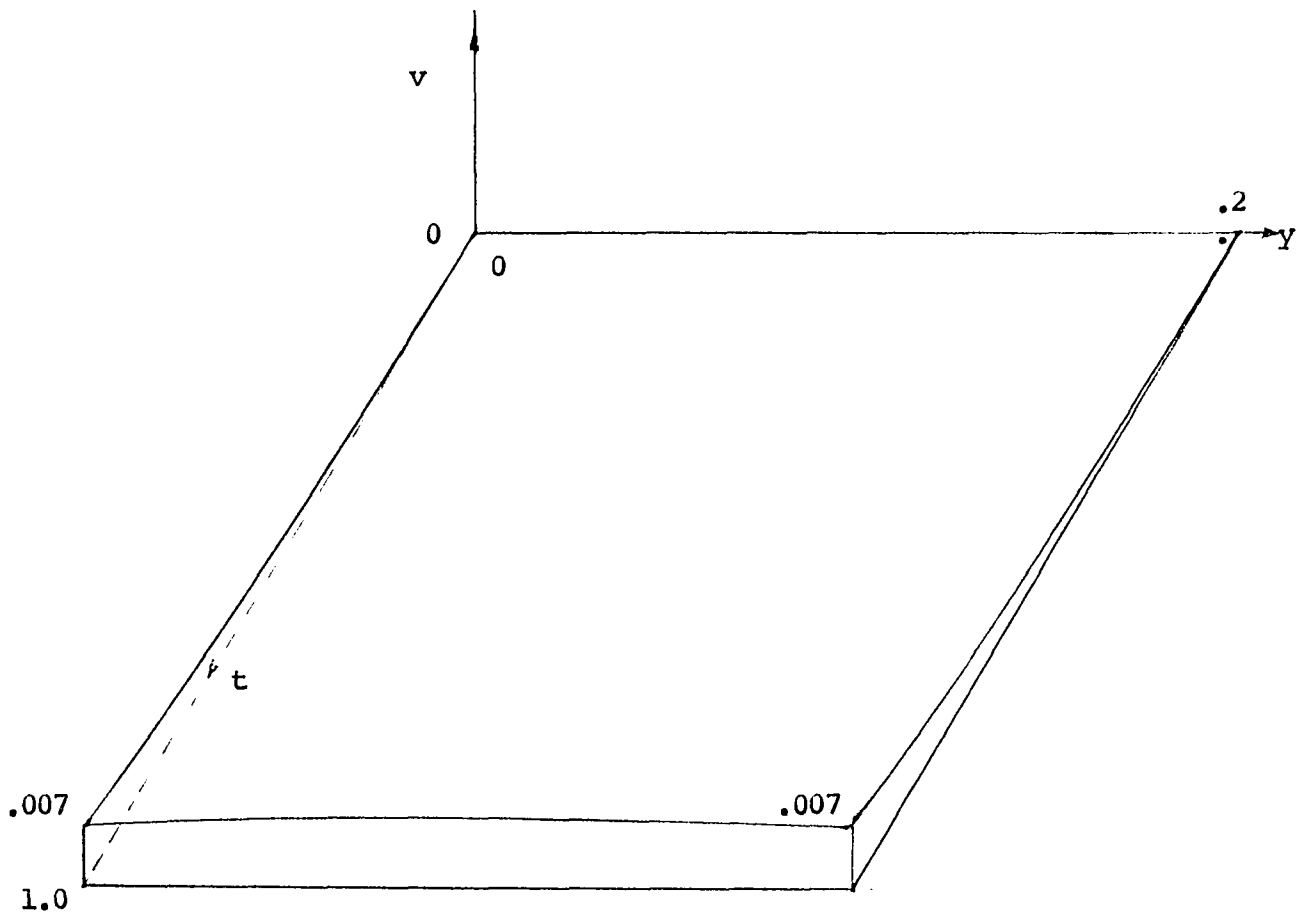


FIGURE 7.7.2 : Plot of Sensitivity for Problem 7.2

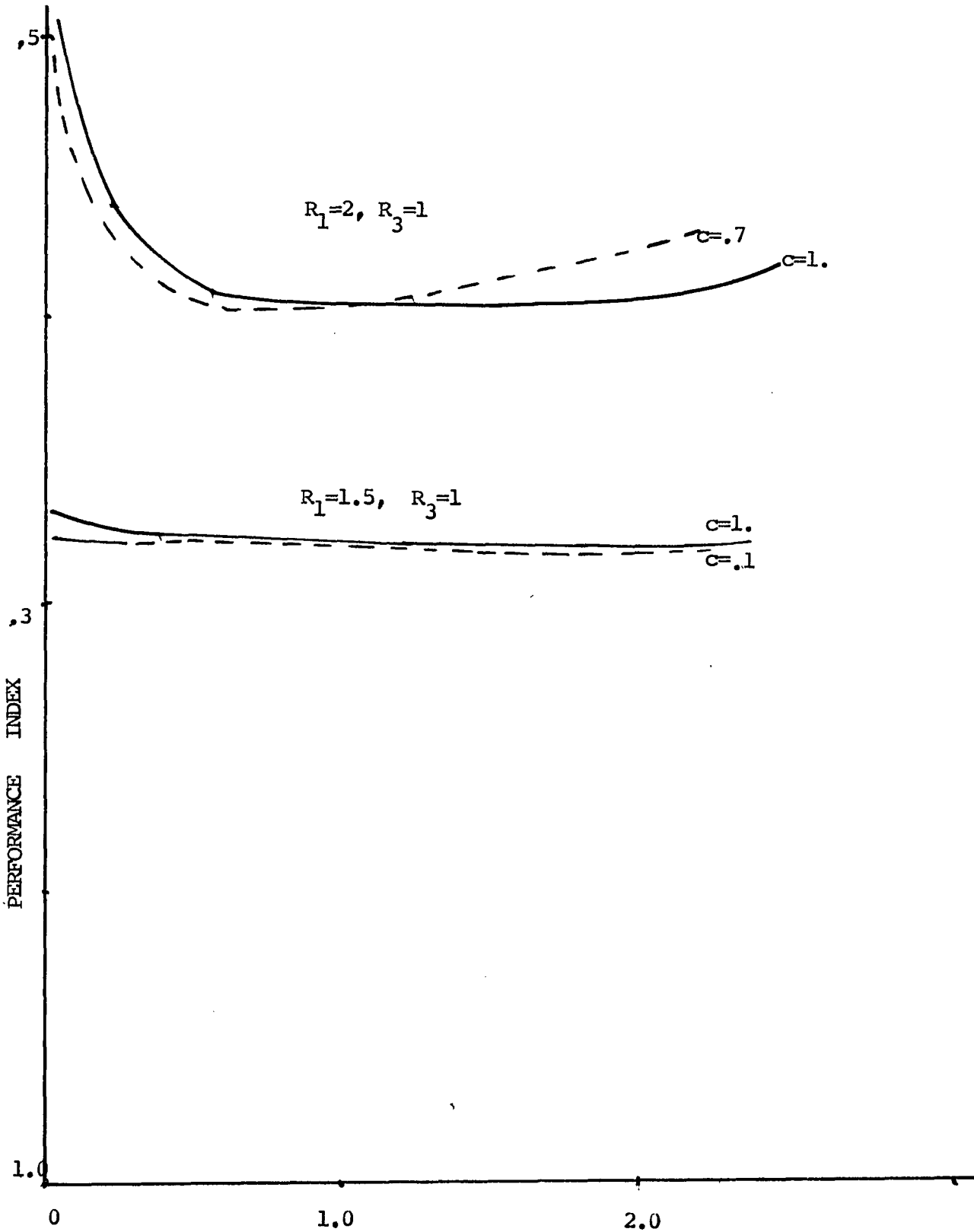


FIGURE 7.7.3 : Plot of Performance Index for Example 7.7

### C. Comment on the Numerical Results

The numerical results presented in this chapter have demonstrated the applicability of the theory derived in Chapter III. These results suggest the following logical approach to sensitivity problems;

1. Selection of an attribute to characterize the system.
2. Derivation of the appropriate mathematical model for the system, taking into account all parameters of interest in the design of the system.
3. Selection of an appropriate performance index, and hence an augmented index involving both the state and sensitivity functions.
4. Derivation of the sensitivity equation.
5. Optimization of the performance index subject to both the state and sensitivity equations.

It is only after the above steps have been taken that the detail compromise or trade-offs may be called into play. The second stage of optimization, therefore, aims at a finer definition of the range of compromises consistent with competing design criteria. This second stage may not be necessary in all cases.

It is again emphasized that there may not be any significant advantage in including the control sensitivity,  $w$ , in this analysis. Other researches (50) have expressed similar doubt in connection with continuous system sensitivity synthesis. The fact remains, however, that since the control vector is firmly under the control of the

desinger, uninfluenced by parameter variations, its elimination can be justified on practical grounds.

#### D. Conclusion

Numerical solutions for the problems formulated in Chapters IV and V have been presented. These solutions demonstrate the practical applications of the theory derived in Chapter III, to the two classes of reliability problems under investigation. It is noted that the boundary condition requirement imposed by (4.30) seriously restricts the choice of failure model for the stand-by system. Also, the classes of problems are confined to those whose mathematical models are included in the model formulated in Chapter II.



## CHAPTER VIII

### CONCLUSIONS

The main objective of this dissertation is to extend the theory of synthesis of low sensitivity optimal control to classes of problems known as distributed parameter systems. After review of prior work in the area of discrete, continuous and distributed parameter system sensitivity, it was noted that while extensive research papers exist on problems of discrete and continuous system sensitivity, there are only a few papers devoted to distributed parameter system sensitivity. In view of the fact that there is no general theory for partial differential equations, a partial differential equation was proposed for the classes of systems of interest in this research. This model formed the basis for the development of the distributed parameter sensitivity theory for the design of low sensitivity optimal control.

Specifically, the following are the summaries of this research:

(i) The necessary and sufficient conditions were established for the vector-matrix partial differential equation constraints. This, in control theory, is an extension of the vector Maximum Principle. The development of the theory presented in both Chapters II and III is very general. By following similar reasoning, similar theories can be derived for other partial differential equations with little modifications.

(ii) The technique of low sensitivity design for distributed parameter optimal control was demonstrated by applying the general theory to two common physical systems in reliability - the standby system and the heat conduction equation. Before doing so, the mathematical model for the standby system was derived. The derivation of optimum maintenance of failure rate and minimum variations of performance index was presented.

(iii) Recognizing that closed form solutions are not often available for distributed parameter systems, numerical algorithms were developed for the solution of practical problems. Since accuracy is of great importance, the explicit models developed were examined to ensure consistency and stability. Indeed, a model calculation was performed to determine the error for the heat conduction equation.

(iv) By means of a series of examples, both the general theory and the numerical techniques were applied to practical examples. In particular, it was demonstrated that in the area of low sensitivity design, optimization is essentially the pursuit of compromises among sets of given design criteria. In addition this compromise must necessarily be pursued in more than one stage. It may thus be desirable to optimize an augmented cost function in order to determine the range of acceptable compromises. A second optimization will then be needed to determine the exact compromise required to satisfy the design criteria. This demonstrates that unlike other optimization problems, low

sensitivity design is a multiple stage operation.

Probably the most important conclusion of this research is that it is practically feasible to apply a complex theory such as low sensitivity design to reliability problems. In addition we have demonstrated the feasibility of extending sensitivity studies to distributed parameter systems and thereby laid the foundation for application to the other areas of interest in chemical engineering, circuitry, and power systems.

CHAPTER IXAREAS FOR FUTURE RESEARCH

There is no doubt that in a virtually unexplored area such as synthesis of low sensitivity control in distributed parameter systems, there are several areas for future research. The following suggested areas are severely limited in order to conform with the underlying philosophy of this research, i.e., that any theory must lend itself to practical applications.

In this research, only open-loop control was considered. It would therefore be useful to study the case of closed-loop control for the distributed parameter system and, if possible, compare these to ~~open~~ -loop solutions. The reader must be forewarned that the mathematics involved may prove oppressive. For example, the determination of the lagrange multipliers for the simple distributed linear regulator problem leads to the solution of Riccati Partial differential equations. These equations are hard to solve.

It was realized early in this research that very little work has been done on the problem of existence of solutions. It would certainly be useful to investigate this important problem of existence together with the related question of "well posing" of problems. These are not of small significance. Since distributed parameter problems are difficult to solve, it would certainly be useful to be assured through

the use of suitable theorems that the solution in fact exists. This will save time and also inject order into an apparently chaotic area. And more importantly, it will accelerate application of the theory to practical problems.

In this investigation, only the calculus of variations was considered. A vast area of research exists in the study of other optimization techniques, such as dynamic programming, Ritz and gradient methods. These are the so-called direct methods.

There is no doubt that other areas in reliability suggest themselves as useful areas for research. It is recalled that the main problem in reliability, however, is the derivation of the mathematical model. In this regard, it may be useful to explore other models such as higher order Markov processes and semi-Markov chains for the reliability analysis. These other models may probably turn out to be more difficult. But if they enable the use of wider classes of failure models for the standby and other redundant systems to be used, the effort may be worthwhile.

Finally, since repair policy is important in repairable systems, it may be interesting to investigate the design of low sensitivity systems for various repair policies. This indeed is a desirable and direct extension of the present work.

## APPENDIX A

A-1 The Euler Lagrange Equations

The necessary conditions for extremum of the functional,

$$J(\underline{x}) = \int_{t_0}^{t_f} F(\underline{x}, \underline{x}_t, t) dt \quad \text{A-1.1}$$

are given by;

$$\frac{\partial F^*}{\partial \underline{x}} - \frac{d}{dt} \frac{\partial F^*}{\partial \underline{x}_t} = 0 \quad \text{A-1.2}$$

$$\frac{\partial F^*}{\partial \underline{x}_t} = 0, \text{ for } t = t_0, t_f \quad \text{A-1.3}$$

where  $(t_0, t_f)$  are given,  $\underline{x}_t = \frac{d\underline{x}(t)}{dt}$  and  $(.)^*$  represents  $(.)$  evaluated at the extremum.

A -2 The Euler Ostrogradski Equations

For a multiple integral given by;

$$J(\underline{x}(t, \tau)) = \int_{\Delta} F(\underline{x}, \underline{x}_t, \underline{x}_\tau, t, \tau) dt d\tau \quad \text{A-2.1}$$

the conditions for extremum are given by;

$$\frac{\partial F^*}{\partial \underline{x}} - \frac{\partial}{\partial t} \frac{\partial F^*}{\partial \underline{x}_t} - \frac{\partial}{\partial \tau} \frac{\partial F^*}{\partial \underline{x}_\tau} = 0 \quad \text{A-2.2}$$

$$\frac{\partial F^*}{\partial \underline{x}_t} = 0, \text{ } t = t_0, t_f \quad \text{A-2.3}$$

$$\frac{\partial F^*}{\partial \underline{x}_\tau} = 0, \text{ } t = t_0, t_f \quad \text{A-2.4}$$

### A-3. Conditions for Convexity

The necessary conditions for convexity of  $f(\underline{x})$  is summarized in Theorem A-3(77)

#### Theorem A-3

If  $f(x_1, x_2, \dots, x_n)$  is differentiable and convex, then

$$f(\underline{x}) - f(\underline{x}^*) \geq \nabla f^*(\underline{x} - \underline{x}^*) \quad \text{A-3.1}$$

where  $\underline{x} = [x_1, x_2, \dots, x_n]$

$$\nabla f^* = \left[ \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n} \right] \Bigg|_{\underline{x}=\underline{x}^*}$$

and  $(.)^*$  denotes  $(.)$  evaluated at the extremum.

### A-4 Quadratic Test for Convexity

If  $f(x_1, x_2, \dots, x_n)$  is twice differentiable and continuous function in an open convex set  $D$ , it is convex in  $D$  if and only if the quadratic form (77),

$$\sum_{i=1}^n \sum_{j=1}^n \frac{\partial^2 f}{\partial x_i \partial x_j} \quad \text{A-4.1}$$

is positive semi-definite for every point  $\underline{x}$  in  $D$ .

APPENDIX B  
COMPUTER PROGRAMS



```

C
C
C      B=1      SUBROUTINE  OPRATE
C
      DIMENSION P1(21),P2(21,21),P3(21),P(21),Q1(21,21),
1U(21,21),AT(21),X(21),COST(21)
      TC=0
      H=.05
      L=21
      ALD=.1
      PRINT 272,ALD
      DO55M=1,L
      AT(M)=FLOAT(M-1)*H
      P1(M)=.9*EXP(-.6*ALD*AT(M))+.1*EXP(-4.6*ALD*AT(M))
      P2(M,1)=ALD*P1(M)
55  CONTINUE
      AK1=.5
      AK3=10.
      DO233KL=5,500,5
      AK2=FLOAT(KL)/100
      DO66N=1,L
      Q1(L,N)=0.
66  CONTINUE
      IL=2*(L-1)
      DO30IK=L,IL
      IP=2*L-IK
      IQ=IK
      DO40K=L,IQ
      I=2*L-K
      J=IQ=K+1
      U(I,J)=-P2(I,J)*Q1(I,J)/AK2
      Q1(I=1,J)=Q1(I,J+1)-H*(U(I,J)+ALD)*Q1(I,J)-H*AK1*ALD-H*AK3*P2(I,J)
40  CONTINUE
      I=0
      J=0
      DO20I=IP,L
      J=J+1
      P2(I,J+1)=P2(I-1,J)-H*(U(I,J)+ALD)*P2(I,J)
20  CONTINUE
30  CONTINUE
C      SUBPRNG FOR U(I,J)
      DO58J=1,L
      DO57I=J,L
      U(I,J)=-P2(I,J)*Q1(I,J)/AK2
57  CONTINUE
58  CONTINUE
C      SUBPROGRAM  CALC. P1 , P2, P3.
      P(1)=0.
      DO75I=2,L
      SUM1=0.
      DO76J=2,I
      SUM1=SUM1+H*P2(I,J)
76  CONTINUE
      P(I)=SUM1

```

```

75 CONTINUE
   DD13M=1,L
   P3(M)=1.-P1(M)-P(M)
13 CONTINUE
C SUBPRNG, TO CHECK BOUNDARY COND.
  PRINT222
  M=1
  X(1)=(P1(2)-P1(1))/H+ALD*P1(1)-P2(1,1)*U(1,1)*H
  PRINT10,M,X(1)
   DD68I=2,L
   SUM2=0.
   DD69J=2,I
   SUM2=SUM2+H*P2(I,J)*U(I,J)
69 CONTINUE
  X(I)=(P1(I)-P1(I=1))/H +ALD*P1(I)-SUM2
68 CONTINUE
  SUM3=0.
   DD37IX=1,L
37 SUM3=SUM3+ABS(X(IX))
   SUM3=SUM3/51.
   PRINT10,KL,SUM3
   SUMJ=0.
   DD421I=1,L
   DD422J=1,I
   SUMJ=SUMJ+H*H*(.5*AK2*U(I,J)*U(I,J)+ALD*AK1*ABS(P2(I,J))
   1+.5*AK3*P2(I,J)*P2(I,J))
422 CONTINUE
421 CONTINUE
   PRINT414
   IC=IC+1
   PRINT401,SUMJ,ALD
C END OF SUBPRNGS.
  PRINT 21
   DD70JJ=1,L,5
   J=L=JJ+1
   PRINT100,(P2(I,J),I=1,L,5)
70 CONTINUE
   PRINT808
   DD707JJ=1,L,5
   J=L=JJ+1
   PRINT100,(Q1(I,J),I=1,L,5)
707 CONTINUE
   PRINT19
   DD59JJ=1,L,5
   J=L=JJ+1
   PRINT100,(U(I,J),I=1,L,5)
59 CONTINUE
   PRINT11
   DD4M=1,L,5
   4 PRINT10,M,P1(M),P(M),P3(M)
   IF(SUM3=,002)77,77,233
233 CONTINUE
77 PRINT303,AK2
272 FORMAT(////////,45X,' RESULTS FOR LAMDA=',F7.4,/)

```

```
100 FORMAT(2X,20F10.4)
10  FORMAT(5X,I4,3F10.4)
808 FORMAT(///, ' VALUES FOR Q(I,J) ')
11  FORMAT(///, ' TIME P1(T) P2(T) P3(T) ')
19  FORMAT(///, ' VALUES FOR U(I,J) ')
222 FORMAT(///, ' TIME VS. ERROR ')
21  FORMAT(///, ' VALUES FOR P2(I,J) ')
401 FORMAT(15X,F9.5,10X,F6.4)
414 FORMAT(///,15X, ' COST LAMDA ')
303 FORMAT(///,45X, ' VALUE OF K2=',F5.3)
STOP
END
```

```

C
C
C      B-2      SUBROUTINE  LOWSEN
C
      DIMENSION P1(51),P2(51,51),A(51,51),Q2(51,51),B(51,51),P3(51),
      1S1(51,51),Q1(51,51),C(51,51),D(51,51),AT(51),P(51),
      2U(51,51),X(51),PH(51,51),COST(20)
      L=51
      H=.02
      Y=.5
      IC=0
      DD133KL=1,10
      ALD=FLOAT(KL)/100.
      DD11=1,L
      DD2J=1,L
      P2(I,J)=0.
      S1(I,J)=0.
      U(I,J)=0.
      PH(I,J)=0.
      2  CONTINUE
      1  CONTINUE
      PRINT 272,ALD
      DD55M=1,L
      AT(M)=FLOAT(M-1)*H
      P1(M)=.5+.5*EXP(-ALD*AT(M))- .5*ALD*AT(M)*EXP(-ALD*AT(M))
      P2(M,1)=ALD*P1(M)
      S1(M,1)=P1(M)- .5*ALD*AT(M)*EXP(-ALD*AT(M))*
      1.5*ALD*AT(M)*2.*EXP(-ALD*AT(M))
      55  CONTINUE
      AK1=1.7
      AK2=.9
      AK3=1.
      AK4=1.
      AK5=1.
      AK6=1.
      DN=AK1*AK4-AK5*AK5
      AK7=AK1/DN
      AK8=AK5/DN
      AK9=AK4/DN
      DD66N=1,L
      Q1(L,N)=0.
      Q2(L,N)=0.
      66  CONTINUE
      IL=2*(L-1)
      DD30IK=L,IL
      IP=2*L-1K
      IQ=IK
      DD40K=L,IQ
      I=2*L-K
      J=IQ-K+1
      A(I,J)=AK8*P2(I,J)*Q2(I,J)-AK9*Q1(I,J)*P2(I,J)-AK9*Q2(I,J)*S1(I,J)
      1+ALD
      B(I,J)=AK8*P2(I,J)*Q2(I,J)+A(I,J)
      C(I,J)=1.-AK7*P2(I,J)*Q2(I,J)+AK8*Q1(I,J)*P2(I,J)
      D(I,J)=(AK8*Q2(I,J)*S1(I,J)+1.-AK7*P2(I,J)*Q2(I,J))*Q2(I,J)
      Q1(I-1,J)=Q1(I,J+1)-H*(B(I,J)*Q1(I,J)+AK6*S1(I,J)+AK2*P2(I,J)+
      1D(I,J))
      Q2(I-1,J)=Q2(I,J+1)-H*(A(I,J)*Q2(I,J)+AK3*S1(I,J)+AK6*P2(I,J))
      40  CONTINUE

```

```

DD20I=IP,L
J=J+1
P2(I,J+1)=P2(I-1,J)-H*A(I,J)*P2(I,J)
S1(I,J+1)=S1(I-1,J)-H*(B(I,J)*S1(I,J)+C(I,J)*P2(I,J))
20 CONTINUE
30 CONTINUE
C SUBPRG FOR U(I,J)
PRINT19
DD58J=1,L
DD57I=J,L
U(I,J)=AK8*P2(I,J)*Q2(I,J)-AK9*Q1(I,J)*P2(I,J)-AK9*Q2(I,J)*S1(I,J)
57 CONTINUE
58 CONTINUE
DD59JJ=1,L,5
J=L-JJ+1
PRINT100, (U(I,J),I=1,L,5)
59 CONTINUE
C SUBPRG. TO CALC PH(I,J)
PRINT111
DD101J=1,L
DD102I=J,L
PH(I,J)=-AK7*P2(I,J)*Q2(I,J)+AK8*Q1(I,J)*P2(I,J)
1+AK8*Q2(I,J)*S1(I,J)
102 CONTINUE
101 CONTINUE
DD103KK=1,L,5
J=L-KK+1
PRINT100, (PH(I,J),I=1,L,5)
103 CONTINUE
C SUBPROGRAM CALC. P1 , P2, P3.
PRINT11
P(I)=0.
DD75I=2,L
SUM1=0.
DD76J=2,I
SUM1=SUM1+H*P2(I,J)
76 CONTINUE
P(I)=SUM1
75 CONTINUE
DD13M=1,L
P3(M)=1.-P1(M)-P(M)
13 CONTINUE
DD4M=1,L,5
4 PRINT10,M,P1(M),P(M),P3(M)
C SUBPRG TO CHECK BOUNDARY COND.
PRINT 222
M=1
X(I)=(P1(2)-P1(1))/H+ALD*P1(1)-P2(1,1)*U(1,1)*H
PRINT10,M,X(1)
DD68I=2,L
SUM2=0.
DD69J=2,I
SUM2=SUM2+H*P2(I,J)*U(I,J)
69 CONTINUE
X(I)=(P1(I)-P1(I-1))/H +ALD*P1(I)-SUM2
68 CONTINUE
SUM3=0.
DD321X=1,51
32 SUM3=SUM3+ABS(X(IX))
SUM3=SUM3/51

```

C END OF SUBPROGS.

```

PRINT414
SUMJ=0.
DO421I=1,L
DO422J=1,I
SUMJ=SUMJ+H**2.*( .5*AK1*ABS(U(I,J))*ABS(U(I,J))+
1.5*AK2*P2(I,J)*P2(I,J)+.5*AK3*ABS(S1(I,J))*ABS(S1(I,J))+
2.5*AK4*ABS(PH(I,J))*ABS(PH(I,J))+AK5*ABS(U(I,J))*ABS(PH(I,J))+
3AK6*P2(I,J)*ABS(S1(I,J)))
422 CONTINUE
421 CONTINUE
PRINT401,SUMJ,ALD
PRINT 21
DO70JJ=1,L,5
J=L-JJ+1
PRINT100,(P2(I,J),I=1,L,5)
70 CONTINUE
PRINT27
DO71JK=1,L,5
J=L-JK+1
71 PRINT100,(S1(I,J),I=1,L,5)
IC=IC+1
COST(IC)=SUMJ
133 CONTINUE
PRINT323
DO313IX=1,IC
RSEN=COST(IX)/COST(3)-1.
313 PRINT10,IX,RSEN,COST(IX)
272 FORMAT('1',//,45X,' RESULTS FOR LAMDA=',F6.3,/)
111 FORMAT(////,' VALUES FOR PH(I,J)')
100 FORMAT(4X,21F9.3)
10 FORMAT(13,3F10.3)
11 FORMAT('1',' TIME P1(T) P2(T) P3(T) ')
19 FORMAT(//,' VALUES FOR U(I,J)')
222 FORMAT(////,' TIME VS ERROR ')
21 FORMAT('1',//,' VALUES FOR P2(I,J)')
27 FORMAT(////,' VALUES FOR S1(I,J)')
401 FORMAT(15X,F9.5,10X,F6.4)
414 FORMAT(////,15X,' COST LAMDA ',/)
323 FORMAT(////,' LAMX10 RSEN COST ')
33 STOP
END

```

C  
C  
C  
C  
C  
C

B-3 SUBROUTINE LNSINK

DIMENSION T(33,9),V(33,9),Q1(33,9),Q2(33,9),U(33,9)  
 1,W(33,9),SUM(27),AL(11),PI(11)  
 N=5  
 B=1.  
 AF=.5  
 R1=1.  
 R3=1.  
 R2=AF\*R1  
 R4=AF\*R3  
 DX=.5  
 DT=.02  
 R=DT/DX\*DX  
 KX=0  
 BETAD=1.  
 IL=N+1  
 JL=5\*N+1  
 IL1=IL-1  
 JL1=JL-1  
 IL2=IL-2  
 JL2=JL-2

C

C INPUT BC AND IC FOR T,V,Q1 AND Q2

DO235 J=1, JL  
 T(J,1)=0.  
 V(J,1)=0.  
 T(J,IL)=BETAD  
 V(J,IL)=0.  
 Q1(J,1)=0.  
 Q2(J,1)=0.  
 Q1(J,IL)=0.  
 Q2(J,IL)=0.

235 CONTINUE  
 DO231 I=1, IL  
 T(1,I)=0.  
 V(1,I)=0.  
 Q1(JL,I)=0.  
 Q2(JL,I)=0.

231 CONTINUE  
 DO22 LL=10,2010,500  
 A=FLOAT(LL)/1000.  
 PRINT 252,A  
 KK=0  
 DO100 M=1,30  
 IF(M.GT.1) GO TO 50

C

C FIRST PASS CONDITIONS

DO18 I=2, IL1  
 DO19 J=2, JL  
 V(J,I)=.1  
 T(J,I)=.1

19 CONTINUE  
 18 CONTINUE  
 50 CONTINUE

C

```

      DO41JJ=1,JL1
      DO42I=2,IL1
      Q1(J,I)=Q1(J+1,I)+A*R*(Q1(J+1,I+1)-2.*Q1(J+1,I)+Q1(J+1,I-1))
      3+R*(Q2(J+1,I+1)-2.*Q2(J+1,I)+Q2(J+1,I-1))+DT*T(J,I)*R1
      Q2(J,I)=Q2(J+1,I)+A*R*(Q2(J+1,I+1)-2.*Q2(J+1,I)+Q2(J+1,I-1))+
      4DT*V(J,I)*R2
42  CONTINUE
41  CONTINUE
      DO45J=1,JL1
      DO46I=2,IL1
      T(J+1,I)=T(J,I)+A*R*(T(J,I+1)-2.*T(J,I)+T(J,I-1))-B*B*DT*Q1(J,I)/
      1R3
      V(J+1,I)=V(J,I)+A*R*(V(J,I+1)-2.*V(J,I)+V(J,I-1))+R*(T(J,I+1)
      2-2.*T(J,I)+T(J,I-1))
46  CONTINUE
45  CONTINUE
      DO101J=1,JL
      DO102I=1,IL
      U(J,I)=-Q1(J,I)*B/R3
102 CONTINUE
101 CONTINUE
C
C  CALC. PERF. INDEX
      COSTF=0.
      DO44J=1,JL
      DO55I=1,IL
      COSTF=COSTF+DT*DX*.5*(R1*ABS(T(J,I)*T(J,I))
      1+R2*ABS(V(J,I)*V(J,I))+R3*ABS(U(J,I)*U(J,I)))
55  CONTINUE
44  CONTINUE
      PI(M)=COSTF
      PRINT10,PI(M)
      IF(M.LT.2)GO TO 100
      DIFF=ABS(PI(M-1)-PI(M))
      IF(DIFF.LT. .01)GO TO 30
100  KK=KK+1
      30  KX=KX+1
      AL(KX)=A
      SUM(KX)=PI(KK+1)
C
      PRINT212
      DO77JJ=1,JL,5
      J=JL-JJ+1
77  PRINT200,(T(J,I),I=1,IL)
      PRINT222
      DO88JK=1,JL,5
      J=JL-JK+1
88  PRINT200,(V(J,I),I=1,IL)
      PRINT 818
      DO81JU=1,JL,5
      J=JL-JU+1
81  PRINT200,(U(J,I),I=1,IL)
22  CONTINUE
      PRINT110
      DO79KI=1,KX
79  PRINT10,AL(KI),SUM(KI)
212  FORMAT(//, ' VALUES FOR T(J,I)')
222  FORMAT(///, ' VALUES FOR V(J,I) ')
818  FORMAT(//, ' VALUES FOR U(J,I)')
200  FORMAT(5X,12F9.4)

```



```
10  FORMAT(5X,2F15.4)
252  FORMAT('1',20X, ' RESULTS FOR ALPHA=',F5.2)
      STOP
      END
      END
```

C  
C

REFERENCES

1. Abbott, M.B., An introduction to the method of characteristics, American Elsevier, New York, 1966
2. Ames, W.F., Numerical Methods for Partial Differential Equations, Barnes & Noble Inc., New York, 1969
3. ARINIC Research Corporation, Reliability Engineering, Prentice Hall, Englewood Cliffs, New Jersey, 1964
4. ASME, Control of Distributed Parameter Systems, Symposium, 1969 Joint Automatic Control Conference, University of Colorado, Boulder, Colorado, Aug. 1969
5. Bandler, J.W. and R.E. Seviara, "Wave Sensitivities of Networks", I.E.E.E. Trans. Microwave Theory and Techniques, Vol. MTT-20, pp.138-147, Feb. 1972
6. Belove, C., "Sensitivity Function in Variability Analysis", I.E.E.E. Trans. Reliability, Vol. R-15, No.2, 1966
7. Bennett A.W. and A.P. Sage, "Discrete System Parameter and Sampling Interval Sensitivity", Proc. First Princeton Conference on Information Sciences and Systems, Princeton, New Jersey, 1964
8. Billington, R., Power System Reliability Evaluation, Gordon and Breach, Science Publishers, New York, 1970
9. Bingulac, S.P., "Simultaneous Generation of the Second-order Sensitivity Functions", I.E.E.E. Trans. Aut. Control, Vol. AC-11, pp.563-566, July 1966
10. Bode, H.W., Network Analysis and Feedback Design, Van Nostrand Co. Inc., Princeton, New Jersey, 1945
11. Budurka, W.J., "Sensitivity Constrained Optimal Control Systems", IBM Journal, Vol. 11, pp.427-435, July 1967
12. Burbank, H., "Optimal Control and Identification Using Stochastic Theory", Doctoral Dissertation, New Jersey Institute of Technology, Newark, New Jersey, June 1975
13. Butkovskii, A.G., "Dynamic Accuracy of Electrical and Mechanical Circuits", Akademia, Nank USSR, Moscow, 1958
14. Butkovskii, A.G., "Maximum Principle for Optimal Systems with Distributed Parameters", Automation and Remote Control, Vol.22 pp.1156-1169, 1961
15. Butkovskii, A.G., "The Broadened Principle of the Maximum for Optimal Control Problems", Automation and Remote Control, Vol.24, pp.292-304, 1964

16. Butkovskii, A.G., Theory of Optimal Control of Distributed Parameter Systems, American Elsevier, New York, 1969
17. Butkovskii, A.G. and A.I. Egorov and K.A. Lurie, "Optimal Control of Distributed Systems ( A Survey of Soviet Publications )", Vol.6 No.3, pp.437-478, 1968, SIAM Journal of Control.
18. Chaudhuri, S.P., "Optimal Control Computational Techniques for Distributed Parameter Systems", Ph.D. Dissertation, Florida University at Gainesville, 1967
19. Clark, J.M.C., "Sensitivity of Optimal Control Systems to Measurement Error", in Sensitivity Methods in Control Systems, Radanovic(ed.), Pergamon Press, New York, 1964
20. Courant, R. and Hilbert, D., Methods of Mathematical Physics, Vol. 1., Interscience Publishers, New York, 1962
21. Crank, J., The Mathematics of Diffusion, The Clarendon Press, Oxford, 1956
22. Cruz, J.B. and W.P. Perkins, "A New Approach to the Sensitivity Problem in Multivariable Feedback System Design" I.E.E.E. Trans. Automatic Control, AC-9, pp.216-223, July 1964
23. Cruz, J.B. and T.N. Trick, (Ed.), Proc. Fifth Annual Allerton Conference on Circuit and System Theory, University of Illinois, Urbana, Oct. 1967
24. Cruz, J.B., Feedback Systems, McGraw-Hill Book Co., New York, 1971
25. Cruz, J.B., Sensitivity Analysis, Dowden, Hutchinson & Ross, Inc., Pa., 1973
26. Davis, J.M. and W.R. Perkins, "Comparison Sensitivity of Distributed Parameter Systems", I.E.E.E. Trans. Automatic Control, AC-17, pp.100-105, Feb. 1972
27. Dekker, A.J., Solid State Physics, Prentice-Hall Inc., Englewood Cliffs, New Jersey, 1957
28. Dorato, P., " On Sensitivity of Optimal Control Systems", I.E.E.E. Trans. Automatic Control, Vol.AC-8, pp.256-257, July 1963
29. Dorato, P. and A. Kestenbaum, " Application of Game Theory to the Sensitivity Design of Systems with Optimal Controller Structures", Proc. Third Annual Allerton Conference on Circuit and System Theory, University of Illinois, Urbana, Oct. 1963
30. Elsgolc, L.E., Calculus of Variations, Pergamon Press Ltd., London, 1962

31. Epstein, B. and T. Hosford, "Reliability of some Two-unit Redundant Systems", Proc. Sixth National Symp. Reliability Quality Control, pp.469-476, 1960
32. Fan, L.T., C.L.Hwang and F.A.Tilyman, "A Sequential Simplex Pattern Search Solution to Production Planning Problems", A.I.E.E. Vol.1 No.3, pp.267-272, 1969
33. Gabasov, R. and F.M. Kirillova, "High Order Necessary Conditions For Optimality", SIAM Journal of Control, Vol.10, No.1. pp.127-168, Feb., 1972
34. Gembicki F.W. and Y.Y. Haimes, "Approach to Performance and Sensitivity Multiobjective Optimization: The Goal Attainment Method", I.E.E.E Trans. Aut. Cont., Vol. AC-20, pp.769-771, Dec. 1975
35. Gonzales, R.L., "Control System Synthesis to Minimize Multiparameter Sensitivity", Coordinated Science Lab., University of Illinois, Report R-312, Urbana, Illinois, Aug. 1966
36. Guardabassi, G., A. Locattelli and S. Rinaldi, "On the Insensitivity of the Terminal Conditions in Open Loop Control Systems", I.E.E.E. Trans. Aut. Cont., Vol. AC-13, No.4 pp.442-443, Aug. 1968
37. Hildebrand, F.B., Advanced Calculus for Applications, Prentice Hall Inc., New Jersey, 1963
38. Hildebrand, F.B., Introduction to Numerical Analysis, McGraw-Hill Book Co., New York, 1956
39. Holtzman, J.M. and S. Horing, "The Sensitivity of Terminal Conditions of Optimal Control Systems to Parameter Variations", I.E.E.E. Trans. Aut. Cont., AC-10, pp.420-426, 1965
40. Horowitz, I.M., Synthesis of Feedback Systems, Academic Press, New York, 1963
41. Isaacson, E. and H.B. Keller, Analysis of Numerical Methods, John Wiley and Sons Inc., New York, 1966
42. Kahne, S.J., "Low Sensitivity Design of Optimal Linear Control Systems", Proc. 4th Annual Allerton Conf. on Circuit and System Theory, University of Illinois, Urbana, Oct. 1966
43. Kim, M. and S.H. Gajwani, "A Variational Approach to Optimum Distributed Parameter Systems", I.E.E.E. Trans. Aut. Cont., Vol. AC-13, pp.191-193, 1968
44. King, R.E., "Sensitivity Analysis of a Class of Differential Difference Systems", Int. J. Control, Vol.5, No.6, pp.583-588, 1967
45. Koivo, A.J., "Performance Sensitivity of Dynamical Systems", Proc. I.E.E. (UK), Vol.117, No.4, pp.825-830, April 1970

46. Koivuniemi, A.J. and P. Kruh, "Sensitivity in an Optimal Control System with Distributed Parameters", 5th Annual Allerton Conf. on Circuit and System Theory, University of Illinois, Urbana, Oct. 1967
47. Kokotovic, P.V. and R.S. Rutman, "Sensitivity of Automatic Control Systems (Survey)", Automation and Remote Control, Vol. 26, pp. 727-729, 1965
48. Kolb, R., "Optimal Control of Linear Distributed Parameter Systems", Ph.D. Dissertation, Montana State University, Electrical Engng., 1966
49. Kreindler, E., "Synthesis of Flight Control Systems subject to Vehicle Parameter Variations", Air Force Flight Dynamics Laboratory, Report AFFDL-TR-66-209, Wright-Patterson Base, Ohio, 1967
50. Kreindler, E., "On Minimization of Trajectory Sensitivity", Int. J. Cont., Vol. 8, No. 1, pp. 89-96, 1968
51. Lee, E.S., Quasilinearization and Invariant Imbedding, Academic Press, New York, 1968
52. Lindorff, D.P., "Sensitivity in Sampled Data Systems", IRE Trans. on Aut. Cont., Vol. AC-8, pp. 120-124, April 1963
53. Lurie, K.A., "The Mayer-Bolza Problem for Multiple Integrals and the Optimization of the Performance of the System with Distributed Parameters", J. App. Math. and Mech., Vol. 27, pp. 1284-1299, 1963
54. Mankin, J.B., jr. "A Study of Control System Sensitivity", Doctoral Dissertation, University of Tennessee, Dec. 1969
55. Medanic, J., "Three Segment Method in the Sensitivity Design of Control Systems", 5th Annual Allerton Conf. on Circuit and System Theory, University of Illinois, Urbana, Oct. 1967
56. Mikhlin, S.G. and K.L. Smolitskiy, Approximate Methods for Solution of Differential and Integral Equations, American Elsevier Publishing Co. Inc., New York, 1967
57. Misra, K.B., "Reliability Optimization of a Series-Parallel System-Part I: Lagrange Multiplier Approach; Part II: Maximum Principle Approach", I.E.E.E. Trans. Reliability, Vol. R-21, No. 4, pp. 230-238, Nov. 1972
58. Moskowitz, F. and J.B. McLean, "Some Reliability Aspects of System Design", IRE Trans. on Rel. and Qual. Cont., PROC-8, pp. 7-35, Sept. 1956
59. Ngo, N.T., "Sensitivity of Automatic Control Systems (Review)", Automatica, No. 5, pp. 53-82, May 1971

60. Pagurek, P., "Sensitivity of the Performance of Optimal Control Systems to Plant Parameter Variations", I.E.E.E. Trans. Aut. Cont., Vol.AC-10, pp.178-180, April 1965
61. Pars, L.A., Introduction to the Calculus of Variations, John Wiley and Sons Inc., New York, 1962
62. Perkins, W.R. and J.B.Cruz, "The Parameter Variation Problem in State Feedback Control Systems", Trans. ASME, J.Basic Engrng. Ser.D. Vol.87, pp.120-124, March 1965
63. Perlis, H.J and E.M.Duckworth, "Trajectory Sensitivity in a Class of Distributed Water Quality Control Systems", Proc. of the 13th. J.A.C.Conf., Aug.1972, Stanford University, Palo Alto, California
64. Porter, W.A., "Sensitivity Problems in Distributive Systems", Int. J.Control, Vol.5 No.5, pp.393-412, 1967
65. Porter, W.A., "On the Reduction of Sensitivity of Multivariate Systems", Vol.5, No.1, pp.1-9, 1967, Int.J.Cont.
66. Porter, W.A., "Sensitivity in Distributed Feedback Systems", Int.J.Control, Vol.5. pp.413-423, 1967
67. Porter, W.A., "Minimizing System Sensitivity through Feedback", J. of Franklin Inst., Vol.286, pp.225-240, Sept. 1968
68. Porter, W.A., "Sensitivity Problems in Distributive Systems", Int. J.Cont., Vol.5, No.5, pp.393-412, 1967
69. Radanovic, L. (Ed.), Sensitivity Methods in Control Theory, Pergamon Press Ltd., London, 1966
70. Ralston, A. and H.S.Wilf (Ed.), Mathematical Methods for Digital Computers, Wiley and Sons, New York, 1960
71. Rau, J.G., Optimization and Probability in Systems Engineering, Van Nostrand, Reinhold Co., New York, 1970
- 72.. Rissana, J.J., "Performance Deterioration of Optimum Systems" I.E.E.E. Trans. Aut. Cont. Vol.AC-11, pp.530-532, July 1966
73. Roberts, S.M. and Shipman, Two-Point Boundary Value Problems: Shooting Methods, Elsevier, New York, 1972
74. Robinson, A.C., A Survey of Optimal Control of Distributed Parameter Systems, Aerospace Research Laboratory, 69-0171, Nov. 1969
75. Rohrer, R.A. and M.Sobral, "Sensitivity considerations in Optimal System Design", I.E.E.E. Trans. on Aut. Cont., Vol.AC-10, pp.45-48, Jan. 1965

76. Ross, S.M., Introduction to Probability Models, Academic Press, New York, 1972
77. Saaty, T.L. and J.Bram, Non-linear Mathematics, McGraw-Hill Inc., New York, 1964
78. Sage, A.P. and S.P. Chaudhuri, "Gradient and Quasi-linearization Computational Techniques for Distributed Parameter Systems", Int.J.Control, Vol.6, No.1, pp.81-98, 1967
79. Sage, A.P., Optimum System Control, Prentice-Hall Inc., Englewood Cliffs, New Jersey, 1968
80. Sakawa, Y., "Solution of an Optimal Control Problem in a Distributed Parameter System", I.E.E.E. Trans Aut. Cont., Vol. AC-9, pp.420-426, Oct. 1964
81. Sanathan, C.K. and D.C. Reddi, "Modal Analysis and Sensitivity Reduction in Distributed Parameter Systems", Proc. 6th. Annual Allerton Conf. on Circuit and System Theory, pp.617-618, 1968
82. Sawaragi, Y. and K. Inoue and K. Asai, "Synthesis of Open-Loop Optimal Control with Zero Sensitive Terminal Constraints", Automatica, Vol.5, pp.389-394, 1969
83. Seinfeld, J.H., "Sensitivity in a Class of Distributed Parameter Control Systems", AIChE J., Vol.15, NO.1, pp. 57-63, 1969
84. Sheldon, R., Applied Probability Models with Optimization Applications, Holden-Day, San Francisco, 1970
85. Shooman, M.L., Probabilistic Reliability: An Engineering Approach, McGraw-Hill, New York, 1968
86. Siljak, D.D. and R.C. Dorf, "On the Minimization of Sensitivity in Optimal Control Systems", Proc. 3rd. Annual Allerton Conf. on Circuit and System Theory, University of Illinois, Urbana, pp.225-229, Oct. 1965
87. Sobral, M., "Sensitivity in Optimal Control Systems", Proc. I.E.E.E. Vol.56.No.10, pp.1644-1651, 1968
88. Tillman, F.A. and J. Littschwager, "Integer Programming Formulation of Constrained Reliability Problems", Management Science, Vol.13, pp 887-899, July 1967
89. Tomovic, R. and G.A. Bekey, "Adaptive Sampling Based on Amplitude Sensitivity", I.E.E.E. Trans. Aut. Cont., Vol.AC-11,
90. Tomovic, R., "Modern Sensitivity Analysis", I.E.E.E. Convention Record, Part 6, pp.81-86, March 1965

91. Tomovic, R., Sensitivity in Dynamical Systems, McGraw-Hill Book Co., New York, 1963
92. Tomovic, R. and M.Vukobratovic, General Sensitivity Theory, Elsevier Publishing Co., New York, 1972
93. Von Rosenberg, D.U., Methods for the Numerical Solution of Partial Differential Equations, American Elsevier Publishing Co., Inc. New York, 1969
94. Wang, P.K.C., "Control of Distributed Parameter Systems" Advances in Control Systems: Theory and Applications, C.T.Leondes (Ed), Academic Press, Vol.1. 1964, pp.75-172
95. Wang, P.K.C. and F.Tung, "Optimal Control of Distributed Parameter Systems", Trans. ASME J. Basic Engineering, Vol.86, pp.67-79, 1964
96. Weinstock, R., Calculus of Variations with applications to Physics and Engineering, McGraw Hill Book Co. Inc., New York, 1952
97. White, R.C., "Sensitivity and State Variable Feedback", Nuclear Engineering Dept., University of Arizona, 1967
98. Wismer, D.A., "Optimal Control of Distributed Parameter Systems Using Multilevel Techniques", Doctoral Dissertation, University of California at Los Angeles, 1966
99. Wismer, D.A., Optimization Methods for Large-Scale Systems, McGraw-Hill Book Co., New York, 1971
100. Yeh, H.H. and J.T.Tou, "Optimum Control of a Class of Distributed Parameter Systems", I.E.E.E. Trans. Aut. Control, Vol.AC-12, No.1. pp29-37, 1967
101. Zelen, M., Statistical Theory of Reliability, University of Wisconsin Press, Madison, 1963



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