STEADY STATE HEAT TRANSFER FROM A DOUBLE RING OF IDENTICAL SPHERES IN A REGULAR ORIENTATION

by

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DEDICATION

To Carolyn whose help, understanding, and encouragement greatly accelerated the completion of this research.

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ABSTRACT

Solutions to Laplace's equation are obtained by the method of reflections for the problem of heat transfer from two parallel rings of spheres arranged in regular polygonal arrays. The mathematical models developed describe the rate of heat transfer and spatial temperature distribution due to an arbitrary number of identical spheres of equal surface temperature correcting Fourier's heat transfer equation for the interference caused by a multiparticle array. Although the method of solution is quite rigorous and can be used to obtain as accurate a solution as desired, only the second reflection was obtained, yielding a first order correction. The model was compared with an exact solution of Laplace's equation in spherical bipolar coordinates for the case of two spheres in space. The accuracy of the model was shown to be related to the density of the array under consideration becoming more reliable with increased dilution of the system.

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NOMENCLATURE

Symbol	Meaning
a	Sphere radius.
A,B	Unknown functions of integration for the second reflection.
$J_0(x)$	Modified Bessel function of order 0.
k	Thermal conductivity of media surrounding the spheres.
$K_0(x), K_1(x)$	Modified Bessel functions of order 0,1 respectively.
$K_{i\tau}(x)$	Modified Bessel function of imaginary order $i\tau$ - a real variable having the integral representation, $K_{i\tau}(x) = \int_0^\infty e^{-x \cosh t} \cos \tau t \, dt.$
n	Number of spheres per regular array.
$P_{(i\tau - 1/2)}(x)$	Legendre's function of imaginary order $(i\tau - 1/2)$.
$Q_b^c(x)$	Legendre's function of order b and rank c.
q	Rate of heat transfer per set of spheres.
$q^{(1)}, q^{(2)}, \ldots q^{(3)}$	Numbered reflections of the rate of heat transfer per set of spheres.
Q	Total rate of heat transfer from the array.
r_s	Distance from sphere center to a point in space.
T	Temperature at a point in space.
T_a	Temperature of the ambient space.
T_s	Temperature at the sphere surface.
x_s, y_s, z_s	Sphere centered Cartesian coordinates.

x_w, y_w, z_w	Wedge centered Cartesian Coordinates.
x_o	Horizontal distance from wedge vertex to sphere center.
z_o	Vertical distance from wedge vertex to sphere center.
∇	Nabla operator.
ρ, ϕ, z_w	Wedge centered cylindrical coordinates.
ϕ_o	Half of the central angle of the wedge unit cell.
λ, τ	Separation constants of Laplace's equation.
$\psi, \psi^{(1)}, \psi^{(2)} \ldots$	Dimensionless temperatures.
ψ_o	Dimensionless temperature evaluated at the sphere center.
α	Coefficient in equation (56) - a function of the number of spheres per array.
β	Coefficient in equation (55) - a function of the number of spheres per array and the aspect ratio, x_o/z_o.
γ	Coefficient in equation (59) - a function of the n number of spheres per array and the aspect ratio, x_o/z_o.

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1. INTRODUCTION

The rate of heat transfer from a single sphere can easily be predicted by Fourier's law of heat transfer. However, the effect that multiple spheres in close proximity have on the rate of heat transfer from individual spheres has never been widely investigated. The work presented herein concerns itself with the development of a mathematical model which describes the rate of heat transfer and the spatial temperature distribution due to the presence of two parallel rings of spheres of uniform surface temperature T_s. The heat transfer model developed corrects Fourier's equation for the interference caused by a multiparticle array. The problem may be treated by considering two spheres located along the midplane of an infinite wedge of an arbitrary central angle. This model represents an interesting problem when the boundary conditions are such that the derivative of temperature normal to the wall is zero (i.e., $\partial T/\partial \phi = 0$ at $\phi = \phi_o$) and that at all points equidistant from the sphere centers, the normal derivative of temperature is zero (i.e., $\partial T/\partial z_w = 0$ at $z_w = 0$).

The solution to the two sphere and wedge problem is identical to the two ring problem and will yield the spatial temperature distribution and rate of heat transfer of two parallel groups of identical spheres arranged in regular planar arrays. In more concrete terms, the model may be used as a first step in the characterization of a packed bed such as a catalytic reactor.

Considering a two ring system, two spheres, one from each of the planar arrays, may be considered to be located within their own wedge-shaped unit cell of central angle $2\phi_o$. ϕ_o, in turn, can be expressed in terms of the number of spheres per ring, n, according to the following relationship:

$$\phi_o = \pi/n. \quad (1)$$

The walls of the unit wedge act as planes of symmetry both for the double layer of regular polygonal arrays and for the resulting temperature distribution. This may be stated in mathematical form as follows:

$$\partial T/\partial \phi = 0 \quad \text{[on the wedge walls]}. \quad (2)$$

Similarly, the plane defined by the equation $z_w = 0$ acts as a plane of symmetry between the arrays and for the resulting temperature distribution. Mathematically, this may be written as,

$$\partial T/\partial z_w = 0 \quad [\text{at } z_w = 0]. \quad (3)$$

2. SUMMARY

Mathematical models were developed for the rate of heat transfer and spatial temperature distribution due to the presence of two parallel rings of identical spheres of equal surface temperature arranged in regular polygonal arrays. Truncation of the solutions to consider only the contributions coming from the second reflection resulted in equations (49) and (59). The heat transfer correction factor, γ, used in equation (59) is obtainable from figures 8-11. Since the higher order reflection terms were neglected, the model presented is valid only for relatively small values of a/x_o and a/z_o. The limitations on these geometric factors are discussed in appendix A.

The heat transfer model obtained by the method of reflections was compared with an exact solution to Laplace's equation for the case of two hot spheres in space. The reflection model compared favorably with the bipolar coordinate solution and, as expected, the accuracy of the reflection solution increased with decreasing values of a/z_o.

3. DEVELOPMENT OF MODEL

The unit cell chosen for the development of the temperature distribution model consists of two spheres of surface temperature T_s, located within an infinite wedge such that a line connecting the the sphere centers would be parallel to the wedge walls (see figures 1 and 2). The temperature field must be a harmonic function, i.e., a solution to Laplace's equation

$$\nabla^2 T = 0 \tag{4}$$

and must also satisfy the boundary conditions. In this case, the boundary conditions are that at the wedge walls the normal derivative of temperature is zero (i.e., $\partial T/\partial\phi = 0$ at $\phi = \phi_o$), at the midplane the normal derivative of temperature is zero (i.e., $\partial T/\partial z_w = 0$ at $z_w = 0$), and the temperature at the sphere surfaces is T_s.

The problem can be solved in terms of a dimensionless temperature, ψ, defined as follows:

$$\psi = (T - T_a)/(T_s - T_a) \tag{5}$$

where T_a is the temperature of the ambient space and T is the temperature of a point in space. Using this definition, the boundary conditions become

A) $\partial\psi/\partial\phi$ (on the wedge walls) = 0; (6)

B) $\partial\psi/\partial z_w$ (at the plane between rings) = 0; (7)

and C) ψ (at the sphere surface) = 1. (8)

Also, according to the definition

$$\psi \text{ (at infinity)} = 0. \tag{9}$$

Rearrangement of equation (5) yields

$$T = (T_s - T_a)\psi + T_a. \tag{10}$$

Using equation (10) in conjunction with Laplace's equation, making the required substitutions and simplifying, one obtains

$$\nabla^2\psi = 0. \tag{11}$$

Hence, ψ is also a solution to Laplace's equation and the problem can be solved in terms of the dimensionless temperature and the appropriate boundary conditions.

The solution to this problem is obtained via use of the method of reflections.[1] This involves obtaining an infinite number of solutions, each solution independently satisfying one or more of the boundary conditions. The resultant sum is a solution which satisfies all of the boundary conditions,

$$\psi = \psi^{(1)}+\psi^{(2)}+\psi^{(3)}+\ldots+\psi^{(\infty)}. \tag{12}$$

Thus, the required solution, ψ, will be an infinite series of individual solutions; the odd numbered solutions satisfy the boundary condition on the sphere surface, while the even numbered solutions satisfy the boundary conditions upon the wedge surface and the surface of the midplane.

It will be the aim of this thesis to obtain up to the second term of this series. The second reflection amounts to a first order

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1. The original reflection technique was developed by Lorenz[9] in conjunction with a problem in fluid mechanics. Haberman [4],[5],[6] has also used this technique in solving the problem of heat transfer from a sphere to a surrounding concentric cylinder.

correction factor on the temperature field produced by the spheres, negating the temperature gradients at the wedge walls and midplane produced by the first reflection.

Due to the dissimilar shapes involved in the problem, wedge-shaped and spherical, no one coordinate system can be used to simultaneously treat both geometries. First, the development of the model, using a spherical coordinate system based upon the upper sphere center as an origin, will be considered. There are certain restrictions upon the first order solution. They are that the solution must be:

A) a harmonic function,

B) equal to 1 at the sphere surface,

and C) a function of r_s alone due to spherical symmetry.

For $\psi^{(1)} = f(r_s)$ only, the well known solution to Laplace's equation in the region outside of the sphere is:

$$\psi^{(1)} = a/r_s. \tag{13}$$

This solution is consistent with the above restrictions since it is a harmonic function whose value is unity at the sphere surface and is a function of r_s only. This solution in conjunction with equation (12), yields

$$\psi = a/r_s + \psi^{(2)} + \psi^{(3)} + \ldots + \psi^{(\infty)}. \tag{14}$$

Truncating after the second reflection term to obtain the first order correction, one obtains:

$$\psi \simeq a/r_s + \psi^{(2)}. \tag{15}$$

The first reflection term sets up a temperature field of concentric spheres of constant temperature equal to a/r_s. These spheres are cut across by the wedge walls as well as by the midplane. The spheres, therefore, set up a temperature distribution on the walls and midplane and, at the same time, set up a thermal gradient perpendicular to these surfaces. However, since the boundary conditions of the problem are that no gradients perpendicular to these surfaces are to exist, the second reflection must cancel the effect of the first reflection. In mathematical terms,

$$\partial\psi^{(2)}/\partial\phi = -\partial\psi^{(1)}/\partial\phi \ , \tag{16}$$

and
$$\partial\psi^{(2)}/\partial z_w = -\partial\psi^{(1)}/\partial z_w . \tag{17}$$

These conditions must hold true only at the wedge walls and the midplane, respectively, and not everywhere else in space. $\psi^{(2)}$ and $\psi^{(1)}$ must be linearly independent solutions to Laplace's equation.

Using Cartesian coordinates one can show that,

$$x_w = x_o + x_s , \tag{18}$$

$$y_w = y_s , \tag{19}$$

$$z_w = z_o + z_s , \tag{20}$$

$$\psi^{(1)} = a/r_s = \frac{a}{\sqrt{x_s^2 + y_s^2 + z_s^2}} \ , \tag{21}$$

$$\psi^{(1)} = \frac{a}{\sqrt{(x_w - x_o)^2 + y_w^2 + (z_w - z_o)^2}} \ , \tag{22}$$

$$\psi^{(1)} = \frac{a}{\sqrt{x_w^2 - 2x_w x_o + x_o^2 + y_w^2 + (z_w - z_o)^2}} \ . \tag{23}$$

At this point, a cylindrical coordinate system, with its origin at the intersection of the wedge center and the plane $z_w = 0$, will be employed. The following relationships apply:

$$x_w = \rho \cos \phi \ , \tag{24}$$

$$y_w = \rho \sin \phi \ , \tag{25}$$

$$x_w^2 + y_w^2 = \rho^2 \ . \tag{26}$$

Making the appropriate substitutions, one obtains

$$\psi^{(1)} = \frac{a}{\sqrt{\rho^2 - 2x_o\rho \cos \phi + x_o^2 + (z_w - z_o)^2}} \ . \tag{27}$$

Differentiating with respect to z,

$$\partial\psi^{(1)}/\partial z_w = \frac{a(z_o - z_w)}{[\rho^2 - 2x_o\rho \cos \phi + x_o^2 + (z_w - z_o)^2]^{3/2}} \ . \tag{28}$$

At the plane between the spheres, this becomes

$$\partial\psi^{(1)}/\partial z_w \ (z_w = 0) = \frac{az_o}{[\rho^2 - 2x_o\rho \cos \phi + x_o^2 + z_o^2]^{3/2}} \ . \tag{29}$$

If $\underline{\rho}$ is defined as

$$\underline{\rho} = (\rho^2 - 2x_o\rho \cos \phi + x_o{}^2)^{1/2} \ , \tag{30}$$

equation (29) reduces to

$$\partial\psi^{(1)}/\partial z_w \ (z_w = 0) = \frac{az_o}{[\underline{\rho}^2 + z^2]^{3/2}} \ . \tag{31}$$

Similarly, differentiation with respect to ϕ yields at the wedge wall

$$\partial\psi^{(1)}/\partial\phi \ (\phi = \phi_o) = \frac{-ax_o\rho \sin\phi_o}{[\rho^2 - 2x_o\rho \cos\phi_o + x_o^2 + (z_w - z_o)^2]^{3/2}} . \quad (32)$$

Transformation analysis indicates that $\psi^{(2)}$ should be in the form

$$\psi^{(2)} = \int_0^\infty \int_0^\infty AK_{i\tau}(\lambda\rho) \cosh(\tau\phi) \cos(\lambda z_w) d\lambda d\tau + \int_0^\infty BJ_0(\lambda\underline{\rho}) e^{-\lambda z_w} d\lambda . \quad (33)$$

The above solution is linearly independent of $\psi^{(1)}$ and is valid everywhere within the wedge. A and B are not constants, but unknown functions of the separation constants, λ and τ. Differentiating $\psi^{(2)}$ with respect to z_w one obtains:

$$\partial\psi^{(2)}/\partial z_w = \int_0^\infty \int_0^\infty AK_{i\tau}(\lambda\phi) \cosh(\tau\phi) [-\lambda \sin(\lambda z_w)] \, d\lambda d\tau + \int_0^\infty BJ_0(\lambda\underline{\rho})(-\lambda) \, (e^{-\lambda z_w}) d\lambda . \quad (34)$$

Evaluation of the derivative at the midplane yields

$$\partial\psi^{(2)}/\partial z_w \ (z_w = 0) = -\int_0^\infty \lambda BJ_0(\lambda\underline{\rho}) \, d\lambda . \quad (35)$$

However, since the normal derivative at the midplane must be equal to zero, the first reflection derivative must cancel the second.

$$\partial\psi^{(2)}/\partial z_w = -\partial\psi^{(1)}/\partial z_w \ [\text{at } z_w = 0] . \quad (17)$$

Substitution of the derivative from equation (35) and (31) respectively results in the following identity:

$$\int_0^\infty \lambda B J_0(\lambda \underline{\rho})\, d\lambda = \frac{a z_o}{[\underline{\rho}^2 + z_o^2]^{3/2}} \quad . \tag{36}$$

It is known from the literature that

$$\int_0^\infty t(e^{-st}) J_v(At)\, dt = r^{-3}(s + vr)(A/R)^v \tag{37}$$[2]

where $r = \sqrt{s^2 + A^2}$ and $R = s + r$.

Letting $v = 0$, $t = \lambda$, $s = z_o$, and $A = \underline{\rho}$ yields the following identity after simplification:

$$\frac{z_o}{[\underline{\rho}^2 + z_o^2]^{3/2}} = \int_0^\infty \lambda(e^{-z_o\lambda}) J_0(\lambda\underline{\rho})\, d\lambda \quad . \tag{38}$$

Comparison of equations (36) and (38) yields the following identity:

$$a\int_0^\infty \lambda(e^{-z_o\lambda}) J_0(\lambda\underline{\rho})\, d\lambda = \int_0^\infty \lambda B J_0(\lambda\underline{\rho})\, d\lambda \quad . \tag{39}$$

The constant B can now be obtained by comparing like terms.

$$B = a(e^{-z_o\lambda}) \quad . \tag{40}$$

Substitution for B into equation (33) yields:

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2. Harry Bateman, Tables of Integral Transforms, Vol. 1, p. 182.

$$\psi^{(2)} = \int_0^\infty \int_0^\infty AK_{i\tau}(\lambda\rho)\ \cosh(\tau\phi)\ \cos(\lambda z_w)\ d\lambda d\tau$$

$$+ \int_0^\infty aJ_0(\lambda\underline{\rho})\left(e^{-\lambda(z_w + z_o)}\right)\ d\lambda\ . \tag{41}$$

A search of the literature yields the following Laplace transform:

$$\int_0^\infty J_v(At)(e^{-st})\ dt = r^{-1}\left(A/R\right)^v\ ,^3 \tag{42}$$

where $r = \sqrt{s^2 + A^2}$ and $R = s + r$.

Letting $s = z_w + z_o$, $v = 0$, $t = \lambda$, and $A = \underline{\rho}$ one obtains the following identity:

$$\int_0^\infty J_0(\lambda\underline{\rho})\left(e^{-\lambda(z_w + z_o)}\right)\ d\lambda = \left(\underline{\rho}^2 + (z_w + z_o)^2\right)^{-1/2}\ . \tag{43}$$

Using the definition of $\underline{\rho}$ equation (43) may be rewritten as

$$\int_0^\infty J_0(\lambda\underline{\rho})\left(e^{-\lambda(z_w+z_o)}\right)d\lambda = \left(\rho^2 - 2\rho x_o\ \cos\ \phi + x_o^2 + (z_w + z_o)^2\right)^{-1/2}\ . \tag{44}$$

Substitution of this equation into equation (41), differentiation of $\psi^{(2)}$ with respect to ϕ, and evaluation of the resultant expression at the wedge wall yields:

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3. Bateman, Vol. 1, p. 182.

$$\partial\psi^{(2)}/\partial\phi\ (\phi=\phi_o) = \int_0^\infty \int_0^\infty \tau A K_{i\tau}(\lambda\rho)\ \sinh(\tau\phi_o)\ \cos(\lambda z_w)\ d\lambda d\tau$$

$$- \frac{a\rho x_o \sin\phi_o}{[\rho^2 - 2\rho x_o \cos\phi_o + x_o^2 + (z_w + z_o)^2]^{3/2}} \quad . \tag{45}$$

However, since the normal derivative at the wedge wall must be equal to zero, the first reflection derivative must cancel the second.

$$\partial\psi^{(2)}/\partial\phi = -\partial\psi^{(1)}/\partial\phi\ [\text{at } \phi = \phi_o] \ . \tag{16}$$

Substitution from the proper equations for the derivatives in equation (16) and rearrangement of the resultant equation yields:

$$\int_0^\infty \int_0^\infty \tau A \sinh(\tau\phi_o)\ K_{i\tau}(\lambda\rho)\ \cos(\lambda z_w)\ d\lambda d\tau =$$

$$\frac{a\rho x_o \sin\phi_o}{[\underline{\rho}_o^2 + (z_w - z_o)^2]^{3/2}} + \frac{a\rho x_o \sin\phi_o}{[\underline{\rho}_o^2 + (z_w + z_o)^2]^{3/2}} \quad , \tag{46}$$

where $\underline{\rho}_o^2 = \rho^2 - 2\rho x_o \cos\phi_o + x_o^2$.

Inversion fo the Fourier and Lebedev transforms yields the value of A.[4]

$$A = \frac{8aK_{i\tau}(\lambda x_o)\ \sinh[\tau(\pi - \phi_o)]\ \cos(\lambda z_o)}{\pi^2 \sinh(\tau\phi_o)} \quad . \tag{47}$$

---

4. The details of the transformations are included in appendix C.

Using theis definition of Λ, along with the equality in equation (43), equation (41) may be rewritten as:

$$\psi^{(2)}=\left(\frac{8a}{\pi^2}\right)\int_0^\infty\int_0^\infty K_{i\tau}(\lambda x_o)K_{i\tau}(\lambda\rho)\left(\frac{\sinh[\tau(\pi-\phi_o)]}{\sinh(\tau\phi_o)}\right)\cos(\lambda z_o)\ \cos(\lambda z_w)\ \cosh(\tau\phi)\ d\lambda d\tau$$

$$+\ \frac{a}{\sqrt{\rho^2+(z_w+z_o)^2}}\ . \qquad (48)$$

The approximate temperature field may now be expressed as:

$$\psi\simeq\frac{a}{\sqrt{\rho^2-2x_o\rho\ \cos\ \phi+x_o^2+(z_w-z_o)^2}}+\frac{a}{\sqrt{\rho^2-2x_o\rho\ \cos\ \phi+x_o^2+(z_w+z_o)^2}}$$

$$+\left(\frac{8a}{\pi^2}\right)\int_0^\infty\int_0^\infty K_{i\tau}(\lambda x_o)K_{i\tau}(\lambda\rho)\left(\frac{\sinh[\tau(\pi-\phi_o)]}{\sinh(\tau\phi_o)}\right)\cos(\lambda z_o)\cos(\lambda z_w)\cosh(\tau\phi)\ d\lambda d\tau. \qquad (49)$$

The rate of heat transfer per set of spheres, q, can be expressed as the series;

$$q = q^{(1)} + q^{(2)} + q^{(3)} + \ldots + q^{(\infty)}\ . \qquad (50)$$

Truncating the above series yields the following approximate solution:

$$q \simeq q^{(1)} + q^{(2)} + q^{(3)} + q^{(4)}\ . \qquad (51)$$

This truncated series can be shown from appendix D to be

$$q \simeq 4\pi ka(T_s - T_a)[1 - \psi^{(2)}_{(x_o,0,z_o)}]\ , \qquad (52)$$

where $\psi^{(2)}_{(x_o,0,z_o)}$ refers to $\psi^{(2)}$ evaluated at the sphere center. This will henceforth be refered to as $\psi^{(2)}_o$.

Evaluation of $\psi^{(2)}$ at the sphere center $(\rho=x_o, \phi=0, z_w=z_o)$ yields upon simplification

$$\psi_o^{(2)} = \left(\frac{4a}{\pi^2}\right)\int_0^\infty \int_0^\infty [K_{i\tau}(\lambda x_o)]^2 \, H \, d\lambda d\tau$$

$$+ \left(\frac{4a}{\pi^2}\right)\int_0^\infty \int_0^\infty [K_{i\tau}(\lambda x_o)]^2 \, H \cos(2\lambda z_o) \, d\lambda d\tau + \frac{a}{2z_o} \quad , \qquad (53)$$

where $H = \dfrac{\sinh[\tau(\pi-\phi_o)]}{\sinh(\tau\phi_o)}$.

Integration with respect to λ yields

$$\psi_o^{(2)} = \left(\frac{a}{x_o}\right)\int_0^\infty \left(\frac{H}{\cosh(\tau\pi)}\right) d\tau$$

$$+ \left(\frac{a}{x_o}\right)\int_0^\infty \left(\frac{H}{\cosh(\tau\pi)}\right) P_{(i\tau-1/2)}[1 + 2(z_o/x_o)^2] \, d\tau + a/2z_o \; . \qquad (54)$$

[5]

Ignoring the last two terms in equation (54) makes this solution identical to that proposed for $\psi^{(2)}$ at the sphere center for a single plane of identical spheres arranged in a regular polygonal array.[6] This result is to be expected, given the similar geometries involved.

The second term of equation (54) may be shown to be

$$\left(\frac{a}{x_o}\right)\int_0^\infty \left(\frac{H}{\cosh(\tau\pi)}\right) P_{(i\tau-1/2)}[1 + 2(z_o/x_o)^2] d\tau = \beta(a/z_o), \qquad (55)$$

[7]

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5. Details of the integration are included in appendix E.
6. David Horwat, The Steady State Heat and Temperature Distribution of a Hot Sphere Within an Infinite Wedge, p. 22.
7. The development of this identity is shown in appendix F.

where $\beta = \sum_{r=1}^{r_2} \{1 + \{(x_o/z_o) \sin(\pi r/n)\}^2\}^{-1/2}$

and $r_2 = [n/2] =$ largest integer $\leq n/2$.

Examination of the coefficient in equation (55) indicates that for very small values of x_o/z_o, β reduces to $[n/2]$. This is evident in the graphs of β versus x_o/z_o in figures 3-7.

The first term of equation (54) has been shown in a previous work to be,

$$(a/x_o) \int_0^\infty \left(\frac{H}{\cosh(\tau\pi)}\right) d\tau = (a/x_o)\alpha , \tag{56}$$

where $\alpha = \int_0^\infty \frac{\sinh[(n-1)\tau\pi/n]\, d\tau}{\sinh(\tau\pi/n)\, \cosh(\tau\pi)}$.[8]

The geometric view factor, α, was calculated for various numbers of spheres in the aforementioned work.

Equation (56) may be rewritten as follows:

$$(a/x_o) \int_0^\infty \left(\frac{H}{\cosh(\tau\pi)}\right) d\tau = (z_o/x_o)(a/z_o)\alpha . \tag{57}$$

Hence, equation (54) may be written as:

$$\psi_o^{(2)} = (a/z_o)\gamma, \tag{58}$$

where $\gamma = \{(z_o/x_o)\alpha + \beta + 1/2\}$.

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8. Horwat, p. 22.

Therefore, the final form of the heat transfer equation is,

$$q \simeq 4\pi ka(T_s - T_a)\left(1 - \gamma(a/z_o)\right) , \tag{59}$$

where q is the rate of heat transfer per set of spheres. The rate of heat transfer from the entire array would merely be the rate of heat transfer per set multiplied by the number of spheres per ring, n.

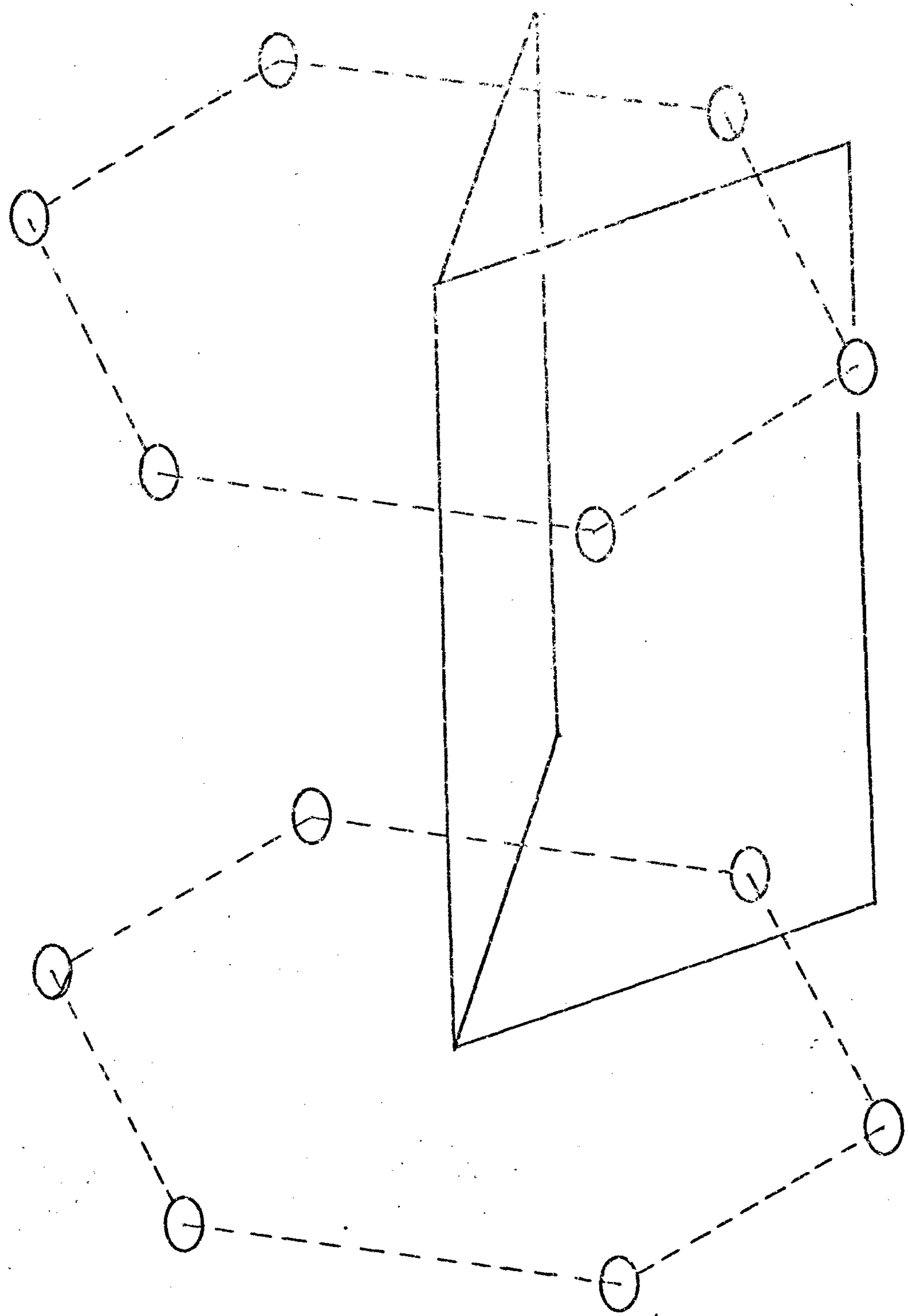
$$Q \simeq 4\pi kan(T_s - T_a)\left(1 - \gamma(a/z_o)\right) . \tag{60}$$

The spatial temperature distribution is modeled by

$$\psi \simeq \frac{a}{\sqrt{\rho^2 - 2x_o\rho \cos \phi + x_o^2 + (z_w - z_o)^2}} + \frac{a}{\sqrt{\rho^2 - 2x_o\rho \cos \phi + x_o^2 + (z_w + z_o)^2}}$$

$$+ \left(\frac{8a}{\pi^2}\right) \int_0^\infty \int_0^\infty K_{i\tau}(\lambda x_o) K_{i\tau}(\lambda\rho) \left(\frac{\sinh[\tau(\pi - \phi_o)]}{\sinh(\tau\phi_o)}\right) \cos(\lambda z_o) \cos(\lambda z_w) \cosh(\tau\phi) \, d\lambda d\tau . \tag{49}$$

This ends the development of the models for heat transfer and spatial temperature distribution from two parallel groups of identical spheres arranged in regular polygonal arrays. There are limitations on the use of the γ coefficient. These are outlined in appendix A.

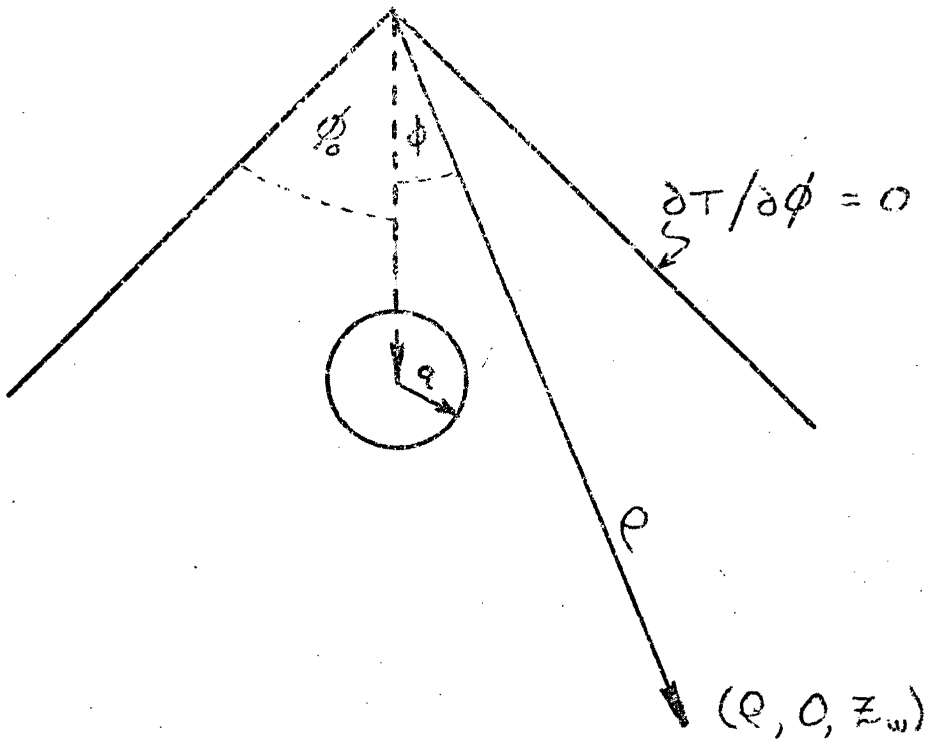


PARALLEL ARRAYS OF SPHERES
SHOWING UNIT WEDGE FOR N=6

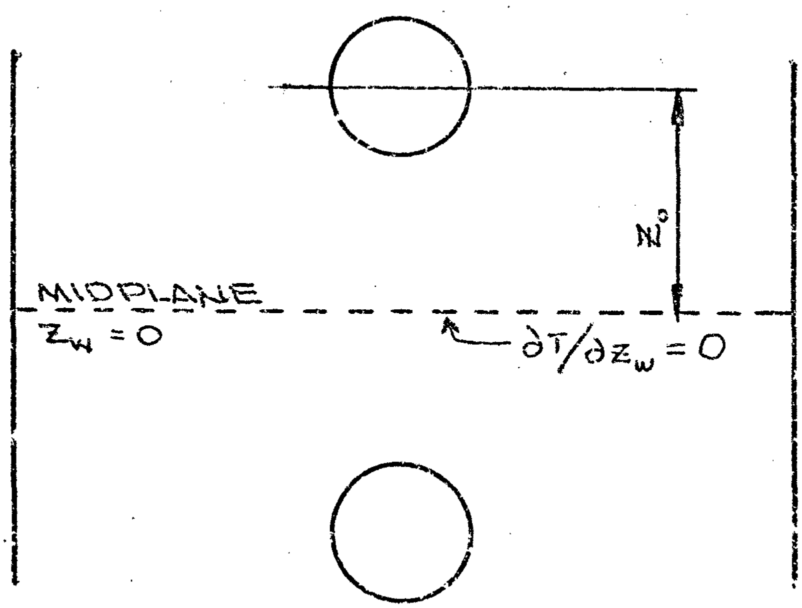
FIGURE 1

FIGURE 2

UNIT WEDGE AND SPHERES



TOP VIEW



FRONT VIEW

4. RESULTS AND CONCLUSIONS

The coefficient β in equation (55) was calculated for various values of the number of spheres per ring, n, and the aspect ratio, x_o/z_o, using a Hewlitt-Packard programmable calculator. The results are presented in Table 1 and figures 3-7. For the trivial case of n = 1, it can easily be shown that $\beta = 0$ for all values of x_o/z_o.

The heat transfer correction factor, γ, in equations (58) and (59) was calculated for various values of n and x_o/z_o. The results are included in Table 2 and figures 8-11. For n = 1, it can be shown that $\gamma = 0.5$ for all values of x_o/z_o.

Observe that in equation (59) the sign of the correction term, $-\gamma(a/z_o)$, is negative. Thus, increasing the value of γ or a/z_o has the effect of reducing q. Table 2 shows that for a specified value of a/z_o, increasing the number of spheres per ring results in a lower value of q, all other things being equal. However, the total heat transfer increases with increasing n since the term in brackets in equation (60) always decreases more slowly than the increase in n. Thus, the greater the number of spheres per array, the greater the total rate of heat transfer, all other things being equal, but the efficiency of each sphere as a source is diminished.

The models developed are, of necessity, not rigorous in describing the behavior of packed beds since the particles in such beds do not form a regular array such as that treated in this report, nor are they identical in shape or size. The models are, however, a first step in the attempt to characterize heat transfer in a packed bed. Future developments along this line would include the development of higher order reflection terms(perhaps including a general equation for the higher order terms), making it possible to solve the problem for concentrated systems. Experimental verification of the models is also in order but this is a task much more easily said than done.

<u>TABLE 1</u>

β Coefficient for Various Values of n and x_o/z_o

n \ x_o/z_o	0.01	0.02	0.1	0.5	1.0	2.0	5.0	10	50	100
2	0.99950	0.99980	0.99504	0.89443	0.70711	0.44721	.196116	0.09950	0.01999	0.01000
3	0.99996	0.99985	0.99627	0.91766	0.75593	0.50000	0.22502	0.11471	0.02309	0.01155
4	1.99993	1.99970	1.99255	1.83724	1.52360	1.02456	0.46828	0.23953	0.04827	0.02414
5	1.99994	1.99975	1.99378	1.86252	1.58672	1.11328	0.52791	0.27229	0.05503	0.02752
6	2.99990	2.99960	2.99006	2.78223	2.35746	1.65432	0.79252	0.41033	0.08305	0.04154
7	2.99991	2.99965	2.99130	2.80752	2.42120	1.75039	0.86741	0.45350	0.09213	0.04609
8	3.99988	3.99950	3.98757	3.72724	3.19206	2.29463	1.14304	0.59997	0.12210	0.06109
9	3.99989	3.99955	3.98881	3.75253	3.25582	2.39206	1.22593	0.65009	0.13287	0.06648
10	4.99985	4.99940	4.98509	4.67225	4.02669	2.93687	1.50736	0.80235	0.16433	0.08222
20	9.99973	9.99890	9.97266	9.39728	8.19982	6.14998	3.38783	1.90836	0.40611	0.20353
50	24.9994	24.9974	24.9354	23.5724	20.7192	15.7895	9.07152	5.41020	1.27688	0.64520
100	49.9987	49.9949	49.8732	47.1975	41.5849	31.8555	18.5450	11.2705	2.89714	1.49261
500	249.994	249.975	249.376	236.199	208.510	160.383	94.3327	58.1537	16.3737	9.04057
1000	499.987	499.950	498.755	472.450	417.167	321.042	189.067	116.758	33.2374	18.5760
5000	2499.94	2499.75	2493.78	2362.46	2086.42	1606.32	946.944	585.589	168.147	94.8601
10000	4999.87	4999.50	4987.57	4724.98	4172.99	3212.91	1894.29	1171.63	336.784	190.215

GRAPH OF β VS. X_0/Z_0 FOR N = 2, 4, 6 AND 8.

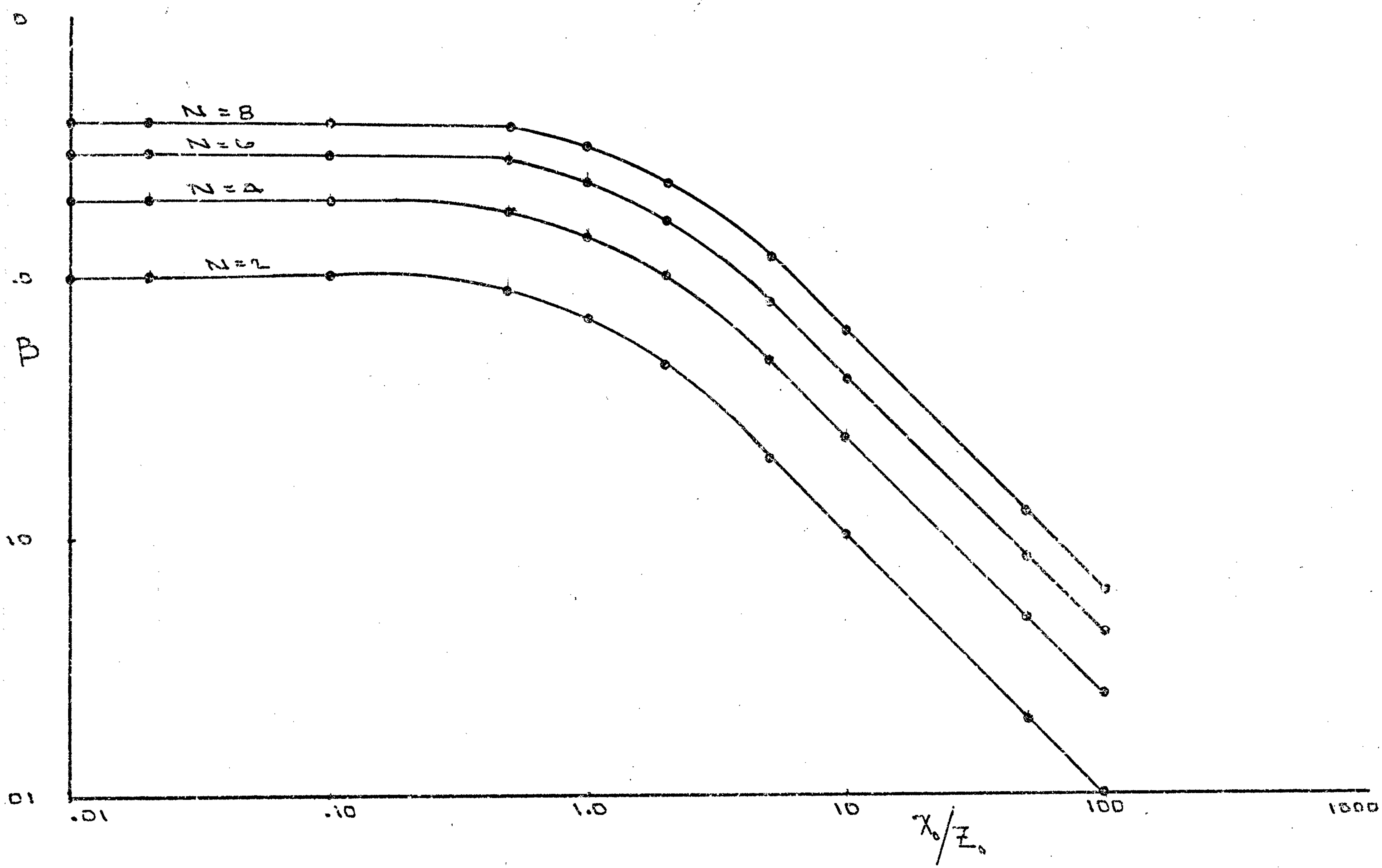


FIGURE 3

GRAPH OF β VS. X_0/Z_0 FOR N=3,5,7 AND 9

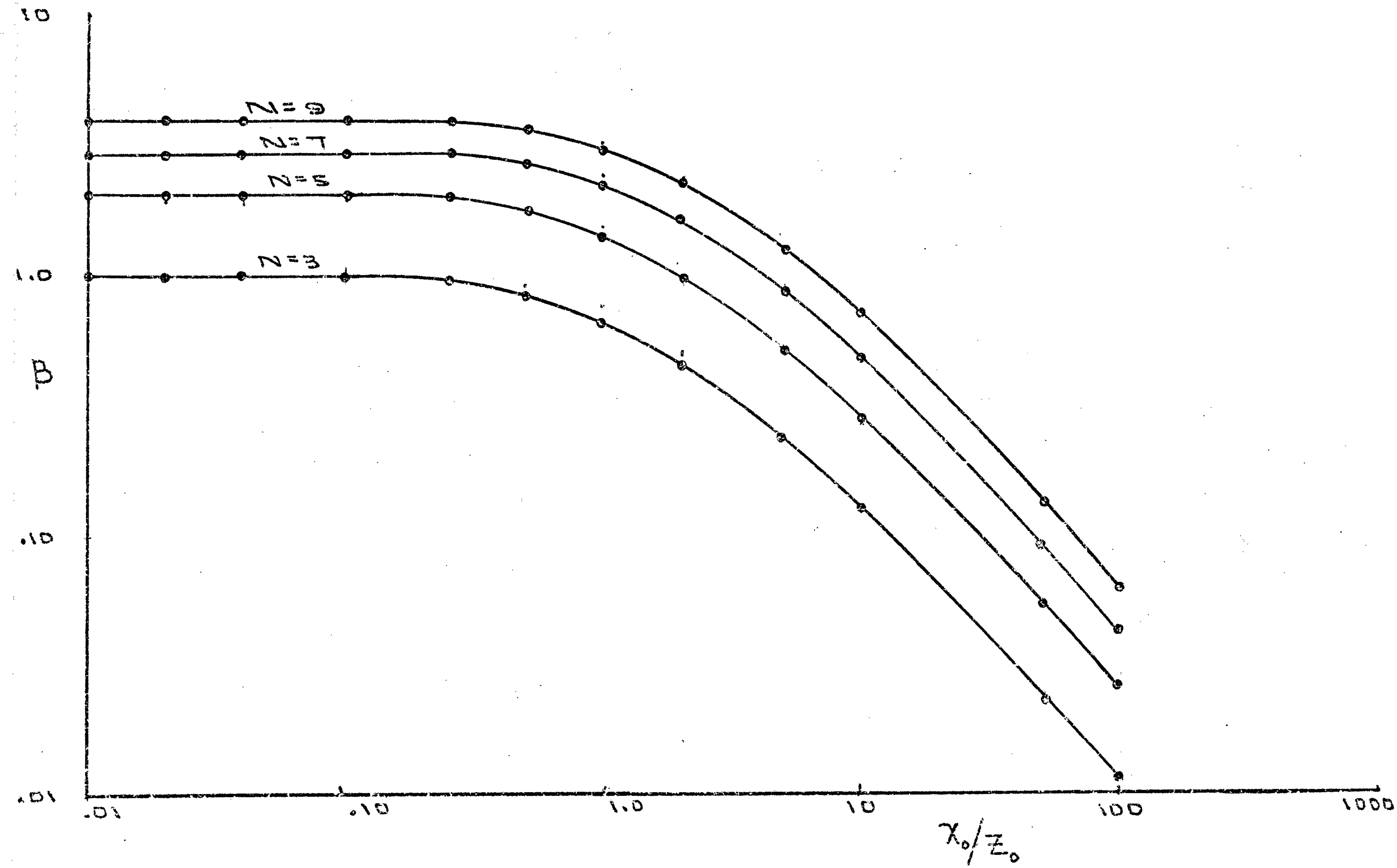


FIGURE 4

GRAPH OF β VS. x_0/z_0 FOR N=10,20 AND 50.

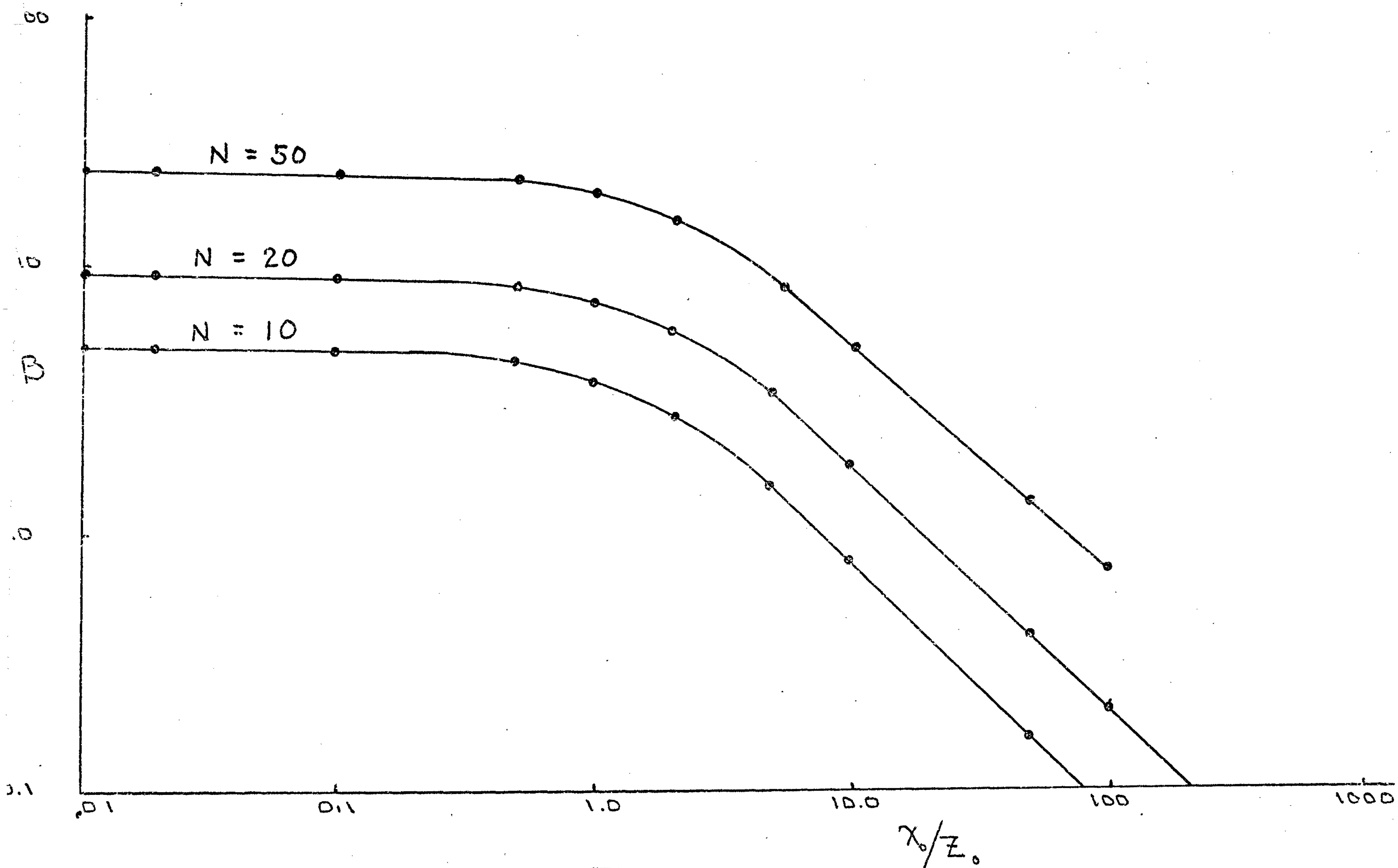


FIGURE 5

GRAPH OF β VS. x_o/Z_o FOR N = 100 AND 500

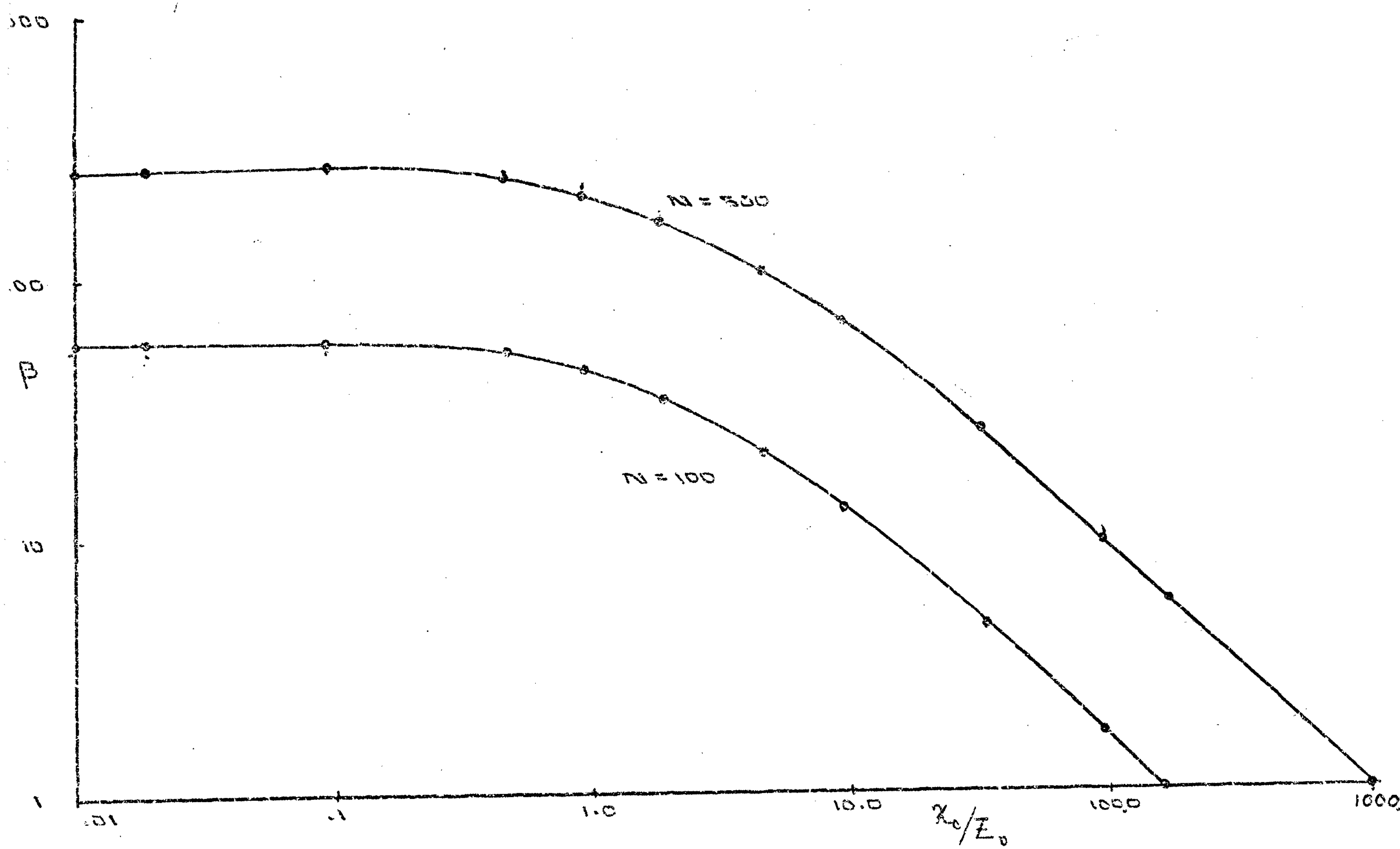


FIGURE 6

GRAPH OF B VS. x_0/z_0 FOR N = 1,000, 5,000 AND 10,000

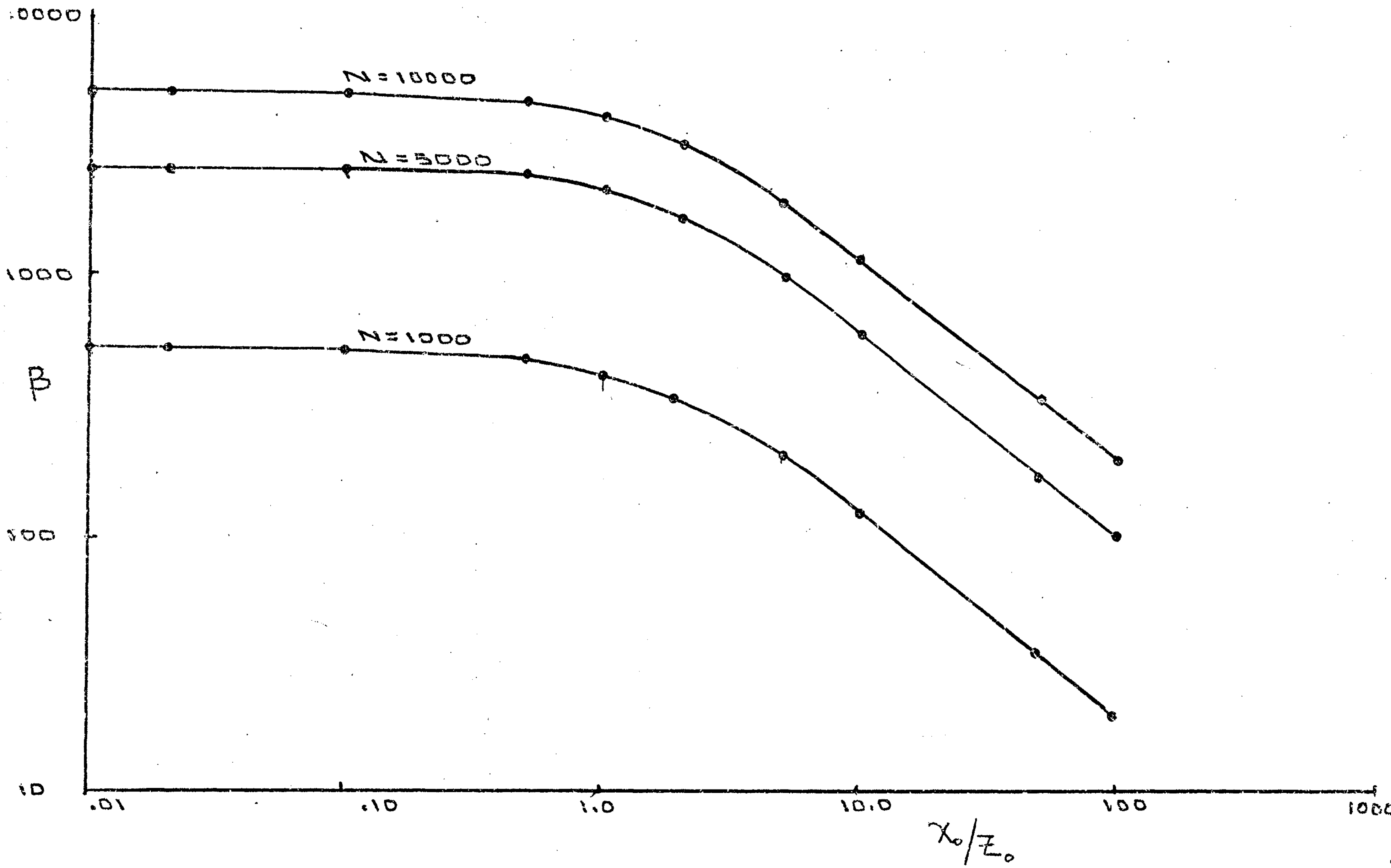


FIGURE 7

TABLE 2

γ Coefficient for Various Values of n and x_o/z_o

n \ x_o/z_o	0.01	0.02	0.1	0.5	1.0	2.0	5.0	10	50	100
2	51.4999	26.5000	6.49504	2.39443	1.70711	1.19721	0.79612	0.64950	0.53000	0.51500
3	116.970	59.2349	13.0433	3.72706	2.41063	1.57735	0.95596	0.73018	0.54618	0.52309
4	193.921	98.2102	21.6347	6.16566	3.93781	2.48167	1.35112	0.93095	0.58655	0.54328
5	277.776	140.138	30.0214	7.86804	4.83948	2.98966	1.57847	1.04757	0.61009	0.55505
6	368.970	186.235	40.0371	10.5916	6.51216	3.98167	2.02346	1.27580	0.65615	0.57809
7	464.453	233.976	49.5866	12.5266	7.53073	4.55516	2.28931	1.41445	0.68432	0.59218
8	565.473	284.986	60.5849	15.4467	9.30179	5.59950	2.76499	1.66094	0.73430	0.61718
9	669.466	336.983	70.9854	17.5519	10.4055	6.21689	3.05586	1.81505	0.76586	0.63298
10	777.990	391.744	82.7341	20.6221	12.2516	7.29932	3.55234	2.07484	0.81883	0.65948
20	1997.40	1003.95	209.163	49.6353	28.5688	16.5845	7.86163	4.39526	1.30349	0.90222
50	6451.54	3238.52	668.039	152.593	85.4796	48.4198	22.4236	12.3362	3.06209	1.78781
100	15109.1	7579.79	1556.23	348.870	192.671	107.648	49.1622	26.8291	6.40886	3.49840
500	101158.	50704.5	10340.7	2254.86	1218.09	665.423	296.649	159.562	37.0553	19.6314
1000	224381.	112440.	22887.3	4950.55	2656.47	1440.94	637.327	341.138	78.5134	41.4640
5000	1378050	690275.	140049.	29873.9	15842.4	8484.57	3698.54	1961.64	444.257	232.915
10000	2976730	1490865	302161.	64160.1	33890.8	18072.1	7838.25	4143.86	931.630	487.888

GRAPH OF γ VS. X_0/Z_0 FOR N = 2,4,6,8 AND 10.

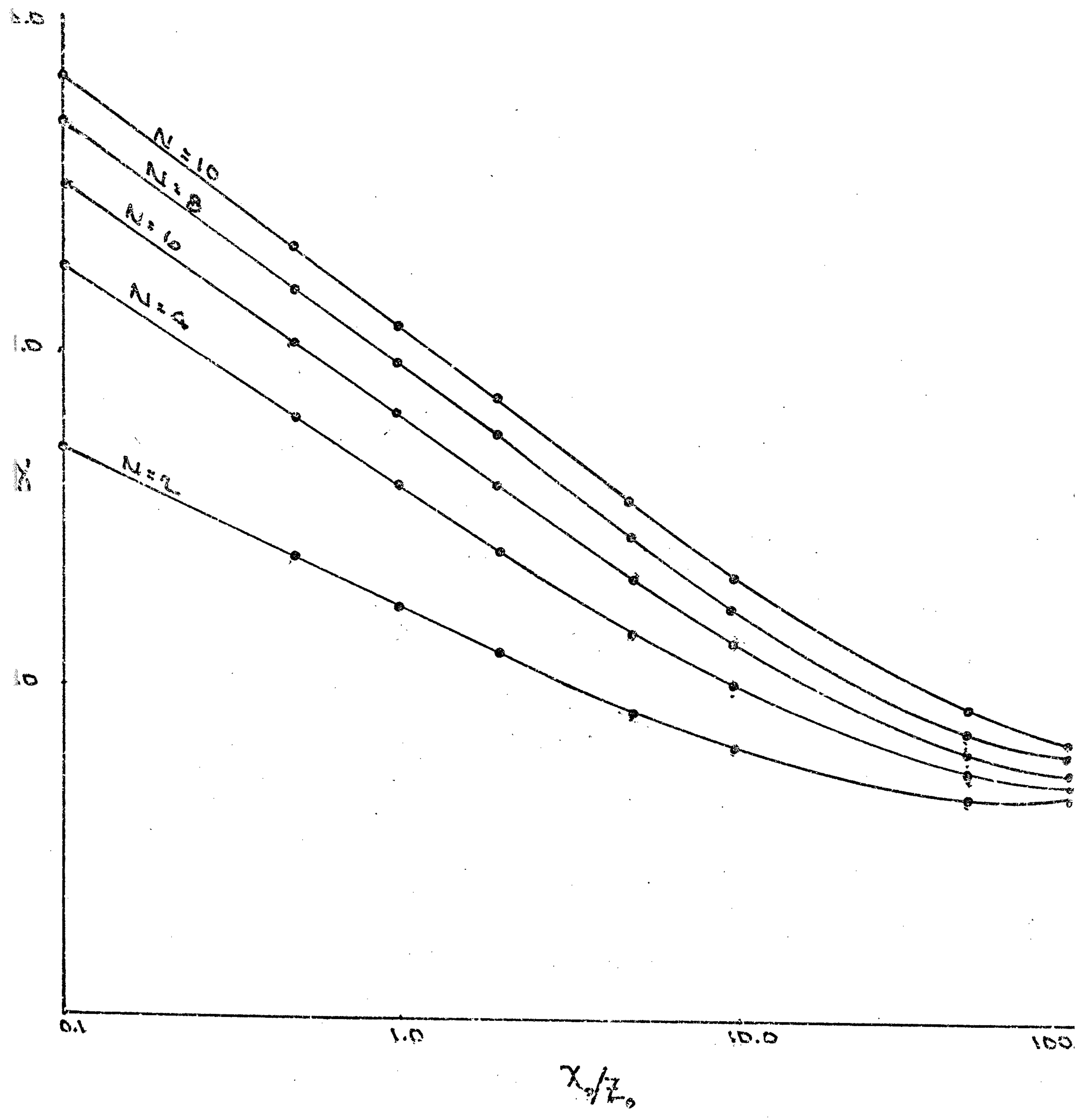


FIGURE 8

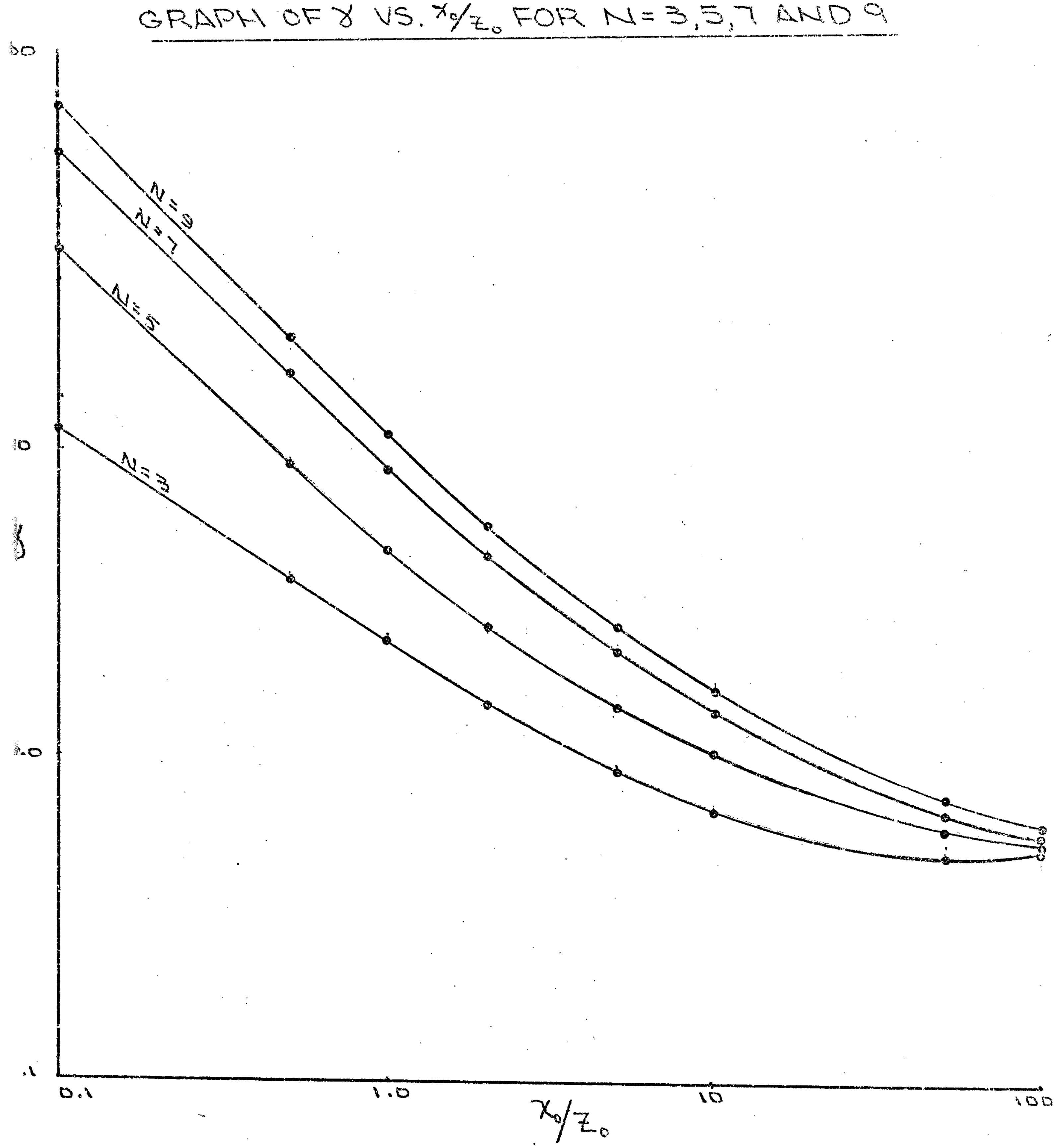


FIGURE 9

GRAPH OF γ VS. x_0/z_0 FOR N = 20, 50 AND 100.

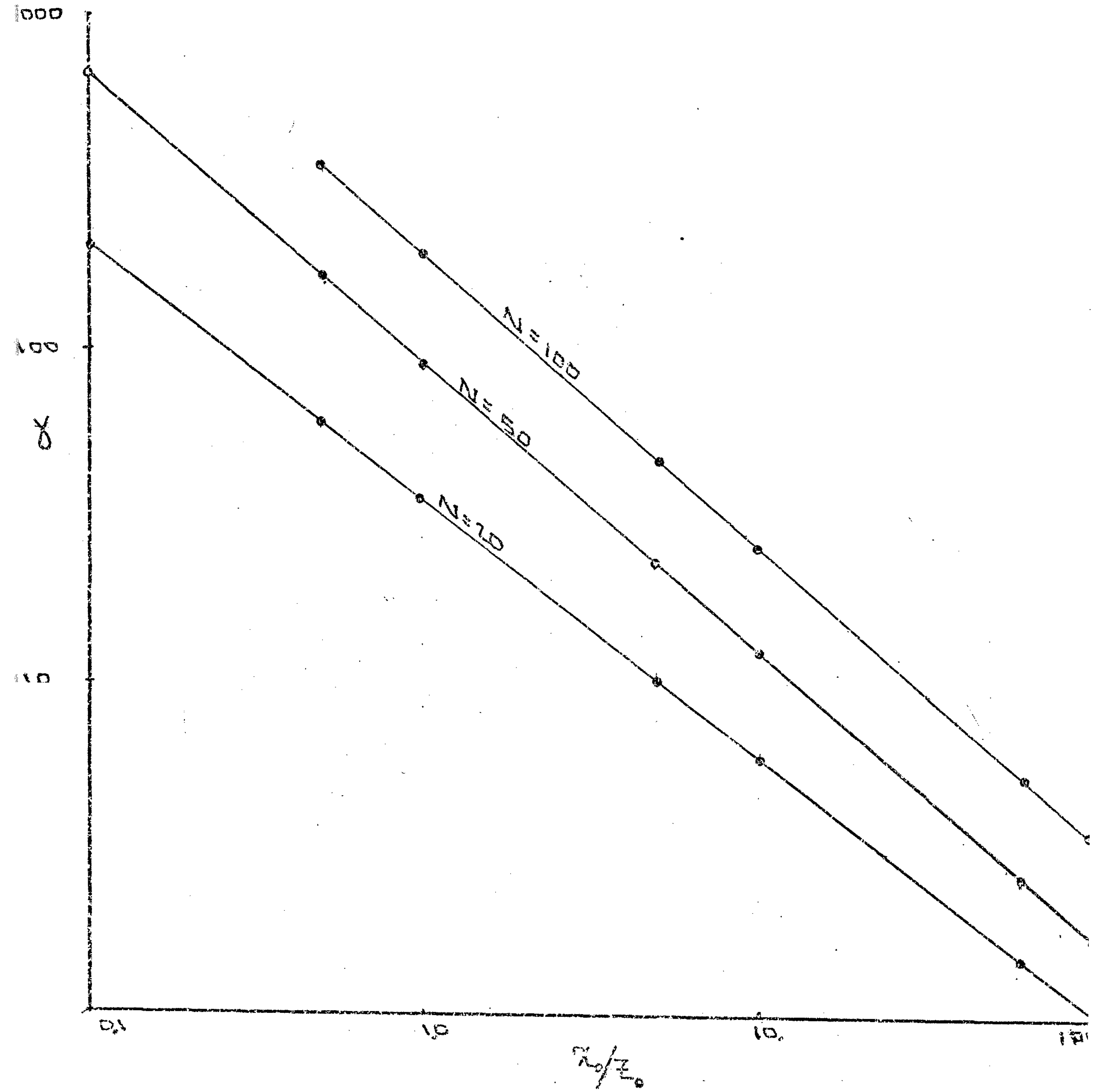


FIGURE 10

GRAPH OF γ VS. x/z_0 FOR N = 500, 1,000, 5,000 AND 10,000

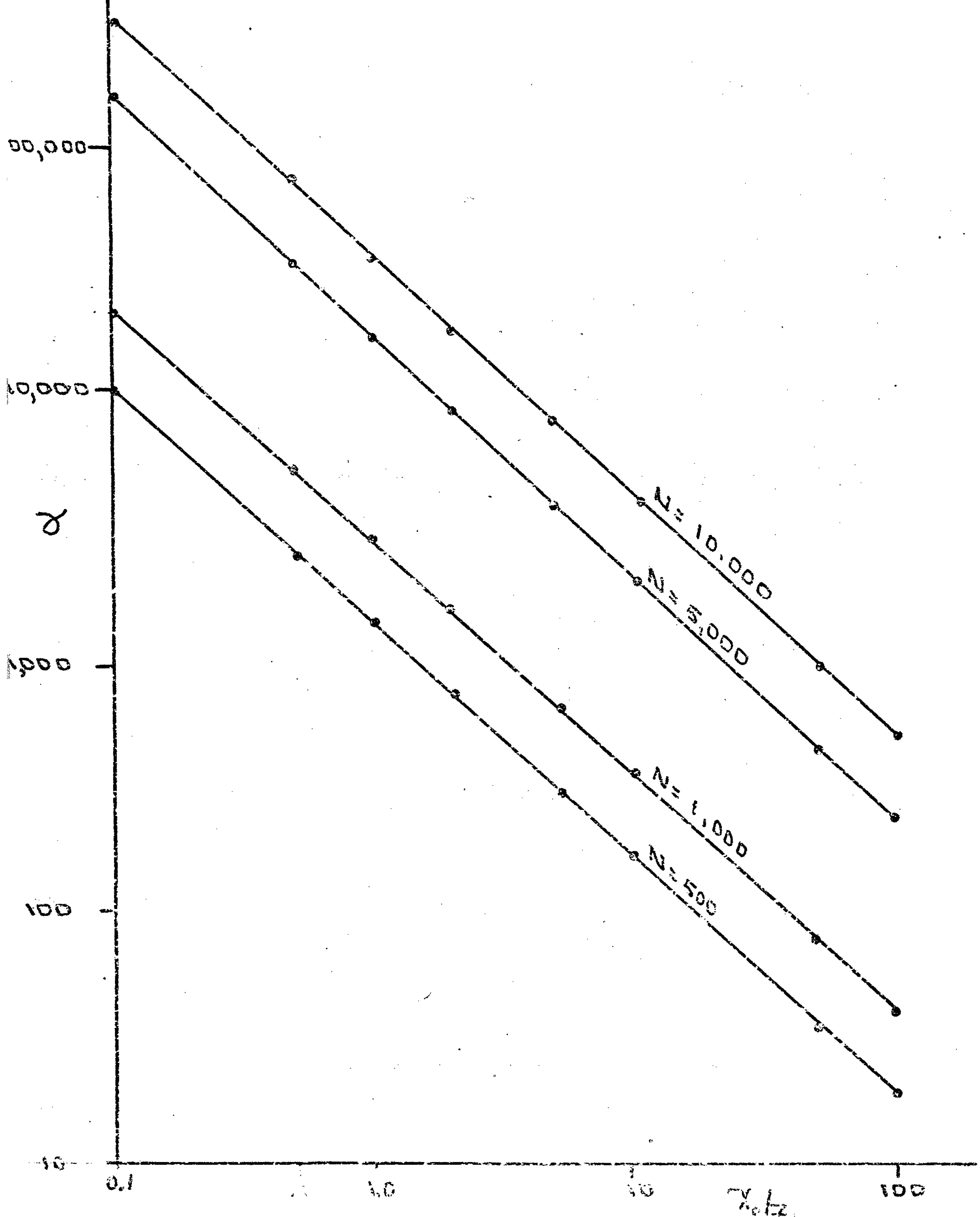


FIGURE 11

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APPENDIX A

Limitations on a/x_o and a/z_o

The system consisting of two planes of spheres contains geometric limitations on the variables a/x_o and a/z_o . The maximum value of a/z_o is fixed by the contact of one sphere from each of the planar arrays. Thus, for the spheres in contact,

$$(a/z_o)_{max} = 1. \qquad \text{(A-1)}$$

Also, the value of a/x_o is fixed geometrically by the tangency of the spheres to the planes of the wedge. This condition corresponds to each sphere in the ring touching both adjacent ones. It may be shown that the value of a/x_o corresponding to this case is:

$$(a/x_o)_{max} = \sin(\pi/n) \ . \qquad \text{(A-2)}$$

It is important not to confuse the geometric limitations on a/x_o and a/z_o with those imposed upon the mathematical solution as a result of the deletion of the higher ordered reflections. Since the higher order reflections have been neglected in the development of this model, the solution presented is valid only for relatively small values of a/x_o and a/z_o. This is due to the fact that the reflection solution is a power series in increasing powers of a/x_o and a/z_o. Therefore, for small values of a/x_o and a/z_o, the higher order terms become small and neglecting them is justified. It has already been shown that there are geometric limitations on the on the values of a/x_o and a/z_o. Thus, the solutions are most valid when $a/x_o << (a/x_o)_{max}$ and $a/z_o << (a/z_o)_{max}$. As a rule of thumb, a/x_o should not exceed $0.1(a/x_o)_{max}$ and a/z_o should be less than $0.1(a/z_o)_{max}$.

The aspect ratio, x_o/z_o, may be calculated from the geometric factors according to the following equation:

$$x_o/z_o = \frac{a/z_o}{a/x_o}. \tag{A-3}$$

Since the aspect ratio is dependent on the geometric factors, the values of x_o/z_o that may validly be used are fixed by equation (A-3). Once a/z_o is chosen, the values of x_o/z_o that may be used are,

$$(x_o/z_o)_{min} \leqq x_o/z_o \leqq \infty \tag{A-4}$$

where $(x_o/z_o)_{min} = \frac{10(a/z_o)}{(a/x_o)_{max}}$. (A-5)

APPENDIX B

Comparison of Reflection Solution to Bipolar Coordinate Solution

For the case of two identical spheres in space, an exact solution in spherical bipolar coordinates may be obtained. Values of γ were obtained by Bart and Horwat[a] for comparison with the value of γ obtained via the method of reflections. For the latter case, it can be shown that $\gamma = 0.5$ for all values of a/z_o . For values of $a/z_o = 0.1, 0.05, 0.01$, the bipolar solution was shown to yield values of 0.476133, 0.48780, and 0.49751, respectively. The improvement in agreement with decreasing a/z_o is caused by the increased validity of neglecting the higher order reflection terms.

a. Ernest N. Bart and David W. Horwat, Solutions to Laplace's Equation for (1) a Sphere in a Wedge and (2) Transport from an Arbitrary Number of Spheres in a Planar Array, pp. 20-21.

APPENDIX C

Proof of Equation (47)

The three terms in equation(46) may be labelled E, F, and G, respectively. From a search of the pertinent literature the following cosine transform is obtained,

$$g(p) = \int_0^\infty (x^{\pm\mu}) K_\mu (Ax) \cos(xp)\, dx = \left(\frac{\sqrt{\pi}}{2}\right)(2A)^{\pm\mu}\,\Gamma\left(\pm\mu + \frac{1}{2}\right)(p^2+A^2)^{\mp\mu-1/2} \quad \text{[a]} \quad \text{(C-1)}$$

If the real part of μ is greater than $-1/2$, the upper sign must be used, whereas if it is less than $+1/2$, the lower sign is used.

Letting $x = \lambda$, $\mu = 1$, $p = z_w - z_o$, and $A = \rho_o$ one obtains the following identity:

$$g(p) = \frac{(\pi/2)\rho_o}{[(z_w - z_o)^2 + \rho_o^2]^{3/2}} = \int_0^\infty \lambda K_1(\rho_o\lambda) \cos[\lambda(z_w - z_o)]\, d\lambda. \quad \text{(C-2)}$$

By comparing terms in F and g(p) one can conclude that:

$$F = \left(\frac{2a\rho x_o \sin\phi_o}{\pi\rho_o}\right) g(p) . \quad \text{(C-3)}$$

Substitution for g(p) from equation (C-2) yields:

$$F = \left(\frac{2a\rho x_o \sin\phi_o}{\pi\rho_o}\right) \int_0^\infty \lambda K_1(\rho\lambda) \cos[\lambda(z_w - z_o)]\, d\lambda. \quad \text{(C-4)}$$

Similarly, allowing p in equation (C-1) to be $z_w + z_o$, one obtains the following identity:

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a. Bateman, Vol. 1, p. 49.

$$G = \left(\frac{2a\rho x_o \sin\,\phi_o}{\pi\rho_o}\right)\int_0^\infty \lambda K_1(\rho_o\lambda)\cos[\lambda(z_w + z_o)]\;d\lambda. \qquad \text{(C-5)}$$

Substitution of the above identities into equation (46) yields:

$$E = \left(\frac{2a\rho x_o \sin\,\phi_o}{\pi\rho_o}\right)\int_0^\infty \lambda K_1(\rho_o\lambda)\{\cos[\lambda(z_w - z_o)] + \cos[\lambda(z_w + z_o)]\}\;d\lambda. \qquad \text{(C-6)}$$

However,

$$\cos[\lambda(z_w - z_o)] + \cos[\lambda(z_w - z_o)] = 2\cos(\lambda z_w)\cos(\lambda z_o). \qquad \text{(C-7)}$$

Hence, equation (C-6) may be written as:

$$E = \left(\frac{4a\rho x_o \sin\,\phi_o}{\pi\rho_o}\right)\int_0^\infty \lambda K_1(\rho_o\lambda)\;\cos(\lambda z_w)\;\cos(\lambda z_o)\;d\lambda. \qquad \text{(C-8)}$$

From the literature one may obtain the following identity:[b]

$$K_o(\lambda\sqrt{\rho^2 - 2\rho x_o\;\cos\,\phi + x_o^2}\,) = \left(\frac{2}{\pi}\right)\int_0^\infty K_{i\tau}(\lambda\rho)K_{i\tau}(\lambda x_o)\;\cosh[\tau(\pi-\phi)]\;d\tau. \qquad \text{(C-9)}$$

Taking the derivative with respect to ϕ and letting $y = \lambda\sqrt{\rho^2 - 2\rho x_o\;\cos\,\phi + x_o^2}$, the left hand side (LHS) of equation (C-9) becomes,

$$\text{LHS} = \frac{\partial[K_o(y)]}{\partial\phi} = \left(\frac{\partial y}{\partial\phi}\right)\left(\frac{\partial[K_o(y)]}{\partial y}\right) = -\left(\frac{\partial y}{\partial\phi}\right)K_1(y). \qquad \text{(C-10)}$$

$$\text{LHS} = -\left(\frac{\lambda\rho x_o\;\sin\,\phi}{\sqrt{\rho^2 - 2\rho x_o\;\cos\,\phi + x_o^2}}\right)K_1(y)\;. \qquad \text{(C-11)}$$

---

b. F. Oberhettinger and T.P. Higgins, Tables of Lebedev, Mehler, and Generalized Mehler Transforms, p. 3.

Evaluation of equation (C-11) at the wedge wall yields the following:

$$\mathrm{LHS} = -\left(\frac{\lambda\rho x_o \sin\phi_o}{\rho_o}\right) K_1(\lambda\rho_o). \tag{C-12}$$

Differentiation of equation (C-9) with respect to ϕ yields for the right hand side (RHS) the following:

$$\mathrm{RHS} = -\left(\frac{2}{\pi}\right)\int_0^\infty \tau K_{i\tau}(\lambda\rho)K_{i\tau}(\lambda x_o)\ \sinh[\tau(\pi-\phi_o)]\ d\tau. \tag{C-13}$$

Equating equations (C-12) and (C-13), one obtains after rearrangement:

$$K_1(\lambda\rho_o) = \left(\frac{2\rho_o}{\pi\lambda\rho x_o \sin\phi_o}\right)\int_0^\infty \tau K_{i\tau}(\lambda\rho)K_{i\tau}(\lambda x_o)\ \sinh[\tau(\pi-\phi_o)]\ d\tau. \tag{C-14}$$

Using this definition of $K_1(\lambda\rho_o)$, equation (C-8) may be equated to the original definition of E in equation (46).

$$\left(\frac{8a}{\pi^2}\right)\int_0^\infty\int_0^\infty \tau K_{i\tau}(\lambda\rho)K_{i\tau}(\lambda x_o)\ \sinh[\lambda(\pi-\phi_o)]\ \cos(\lambda z_w)\ \cos(\lambda z_o)\ d\tau d\lambda$$
$$= \int_0^\infty\int_0^\infty \tau A K_{i\tau}(\lambda\rho)\ \sinh(\tau\phi\ )\ \cos(\lambda z_w)\ d\lambda d\tau. \tag{C-15}$$

Comparing like terms, one may conclude that

$$A = \frac{8aK_{i\tau}(\lambda x_o)\ \sinh[\tau(\pi-\phi_o)]\ \cos(\lambda z_o)}{\pi^2\ \sinh(\tau\phi_o)}\quad . \tag{47}$$

APPENDIX D

Proof of Equation (52)

The rate of heat transfer per set of spheres, q, is expressible in series form,

$$q = q^{(1)} + q^{(2)} + q^{(3)} + \ldots + q^{(\infty)} \; . \qquad \text{(D-1)}$$

The form of $q^{(m)}$ is developed from the Fourier equation of heat transfer.

$$dq/dA = -k(\partial T/\partial r_s), \qquad \text{(D-2)}$$

$$q^{(j)} = -\int_0^{2\pi}\int_0^{\pi} k(\partial T/\partial r_s)_a \; a^2 \sin\theta \; d\theta d\phi \; , \qquad \text{(D-3)}$$

$$\psi^{(j)} = (T - T_a)/(T_s - T_a) \; , \qquad \text{(5)}$$

$$(\partial\psi^{(j)}/\partial r_s)_a = (T_s - T_a)^{-1}(\partial T/\partial r_s)_a \; . \qquad \text{(D-4)}$$

Substituting for $(\partial T/\partial r_s)_a$ in equation (D-3), one obtains,

$$q^{(j)} = -(T_s - T_a)ka^2 \int_0^{2\pi}\int_0^{\pi} (\partial\psi^{(j)}/\partial r_s)_a \sin\theta \; d\theta d\phi \; . \qquad \text{(D-5)}$$

The solution to Laplace's equation in spherical coordinated is, for even numbered reflections.

$$\psi^{(2j)} = \sum_{i=0}^{\infty} \sum_{m=0}^{i} \{r_s^i \; A_{m,2j}^{(i)} \cos(m\phi) \; P_i^m(\mu)\} \; . \qquad \text{(D-6)}$$

For odd numbered reflections,

$$\psi^{(2j+1)} = \sum_{i=0}^{\infty} \sum_{m=0}^{i} \{r_s^{(-i-1)} \; B_{m,2j+1}^{(i)} \cos(m\phi) \; P_i^m(\mu)\} \; , \qquad \text{(D-7)}$$

where $\mu = \cos\theta$.

Taking the derivatives of equations (D-6) and (D-7), respectively, and evaluating the resultant functions at the sphere surface, one obtains:

$$(\partial\psi^{(2j)}/\partial r_s)_a = \sum_{i=0}^{\infty} \sum_{m=0}^{i} \{i a^{(i-1)} A_{m,2j}^{(i)} \cos(m\phi) P_i^m(\mu)\} , \quad \text{(D-8)}$$

$$(\partial\psi^{(2j+1)}/\partial r_s)_a = \sum_{i=0}^{\infty} \sum_{m=0}^{i} \{-(i+1) a^{(-i-2)} B_{m,2j+1}^{(i)} \cos(m\phi) P_i^m(\mu)\} . \quad \text{(D-9)}$$

Substitution of equations (D-8) and (D-9) into equation (D-5) yields the following:

$$q^{(2j)} = -ka^2(T_s-T_a) \int_0^{2\pi} \int_0^{\pi} \left(\sum_{i=0}^{\infty} \sum_{m=0}^{i} (i a^{(i-1)} A_{m,2j}^{(i)} \cos(m\phi) P_i^m(\mu)) \sin\theta \, d\theta d\phi \right), \quad \text{(D-10)}$$

$$q^{(2j+1)} =$$

$$ka^2(T_s-T_a) \int_0^{2\pi} \int_0^{\pi} \left(\sum_{i=0}^{\infty} \sum_{m=0}^{i} \{(i+1) a^{(-i-2)} B_{m,2j+1}^{(i)} \cos(m\phi) P_i^m(\mu) \} \sin\theta \right) d\theta d\phi . \quad \text{(D-11)}$$

By making use of the following identity, equations (D-10) and (D-11) may be simplified.

$$\int_0^{2\pi} \cos(m\phi)\, d\phi = \begin{cases} 0 & (\text{for } m \neq 0) \\ 2\pi & (\text{for } m = 0) \end{cases} . \quad \text{(D-12)}$$

Hence,

$$q^{(2j)} = -2\pi ka^2(T_s-T_a) \int_0^{\pi} \left(\sum_{i=0}^{\infty} \{i a^{(i-1)} A_{0,2j}^{(i)} P_i(\mu) \} \sin\theta \right) d\theta , \quad \text{(D-13)}$$

$$q^{(2j+1)} = 2\pi ka^2(T_s-T_a) \int_0^{\pi} \left(\sum_{i=0}^{\infty} \{(i+1) a^{(-i-2)} B_{0,2j+1}^{(i)} P_i(\mu)\} \sin\theta \right) d\theta . \quad \text{(D-14)}$$

However,

$$\int_0^{\pi} P(\mu) \sin\theta \, d\theta = \begin{cases} 0 \text{ (for } i \neq 0) \\ 2 \text{ (for } i = 0) \ . \end{cases} \qquad \text{(D-15)}$$

Therefore,

$$q^{(2j)} = 0 \ , \qquad \text{(D-16)}$$

$$q^{(2j+1)} = 4\pi k(T_s - T_a) \, B^{(0)}_{0,2j+1} \ . \qquad \text{(D-17)}$$

Hence, the rate of heat transfer is the sum of the odd terms, $\sum_{j=0}^{\infty} q^{(2j+1)}$. The boundary conditions indicate that, in general,

$$\psi^{(2j+1)} = -\psi^{(2j)} \quad \text{(at the sphere surface)} \ . \qquad \text{(D-18)}$$

Using this relationship the following identity may be obtained,

$$\sum_{i=0}^{\infty} \sum_{m=0}^{i} a^{(-i-1)} \, B^{(i)}_{m,2j+1} \cos(m\phi) \, P^m_i(\mu) = -\sum_{i=0}^{\infty} \sum_{m=0}^{i} a^i A^{(i)}_{m,2j} \cos(m\phi) \, P^m_i(\mu) \ . \qquad \text{(D-19)}$$

Comparing terms, one may conclude

$$a^i A^{(i)}_{m,2j} = -a^{(-i-1)} \, B^{(i)}_{2j+1} \qquad \text{(D-20)}$$

It can be shown that for m=i=0, this simplifies to the following:

$$B^{(0)}_{0,2j+1} = -aA^{(0)}_{0,2j} \ . \qquad \text{(D-21)}$$

It can also be shown that

$$A^{(0)}_{0,2j} = \psi^{(2j)} \quad \text{(at the sphere center)} \ . \qquad \text{(D-22)}$$

Therefore,

$$B^{(0)}_{0,2j+1} = -a\psi_o^{(2j)} \quad , \tag{D-23}$$

where $\psi_o^{(2j)}$ is the dimensionless temperature term evaluated at the sphere center. Hence,

$$q^{(2j+1)} = -4\pi ka(T_s - T_a)\psi_o^{(2j)} \quad . \tag{D-24}$$

Equation (D-1) may be approximated as

$$q \simeq q^{(1)} + q^{(2)} + q^{(3)} + q^{(4)}. \tag{D-25}$$

Since the even numbered terms are zero, this may be rewritten as

$$q \simeq q^{(1)} + q^{(3)} \quad . \tag{D-26}$$

This is equivalent to

$$q \simeq -4\pi ka(T_s - T_a)[\psi_o^{(0)} + \psi_o^{(2)}] \quad . \tag{D-27}$$

The first term may be found since,

$$\psi^{(0)} = -\psi^{(1)} = -(a/r_s) \quad \text{(at the sphere surface)} \quad . \tag{D-28}$$

Hence,

$$\psi^{(0)} = -1 \quad \text{(at the sphere surface)} \quad . \tag{D-29}$$

However, $\psi^{(0)}$ is not a function of r_s. Therefore,

$$\psi_o^{(0)} = -1 \quad . \tag{D-30}$$

Equation (D-27) may now be written as

$$q \simeq 4\pi ka(T_s - T_a)[1 - \psi_o^{(2)}] \quad . \tag{52}$$

APPENDIX E

Proof of Equation (54)

$$\psi_o = \left(\frac{4a}{\pi^2}\right)\int_0^\infty\int_0^\infty [K_{i\tau}(\lambda x_o)]^2\ H\ d\lambda d\tau$$
$$+ \left(\frac{4a}{\pi^2}\right)\int_0^\infty\int_0^\infty [K_{i\tau}(\lambda x_o)]^2\ H \cos(2\lambda z_o)\ d\lambda d\tau + a/2z_o\ . \tag{53}$$

From Gradshetyn and Ryzhik[a] the following identity may be obtained:

$$\int_0^\infty K_\nu(ax)\ K_\nu(bx)\ \cos(cx)\ dx = \left(\frac{\pi^2}{4\sqrt{ab}}\right)\sec(\nu\pi)\ P_{(\nu-1/2)}[(a^2+b^2+c^2)/2ab]\ . \tag{E-1}$$

Letting $x=\lambda, a=b=x_o$, $c=0$, and $\nu=i\tau$, the first term of equation (53) may be simplified. Similarly, letting $x=\lambda$, $a=b=x_o$, $c=2z_o$, and $\nu=i\tau$, the second term may be reduced. The result of inverting the transforms is that equation (53) may be written as:

$$\psi_o^{(2)} = \left(\frac{a}{x_o}\right)\int_0^\infty\left(\frac{H}{\cosh(\tau\pi)}\right)\{P_{(i\tau-1/2)}(1)\}\ d\tau$$
$$+ \left(\frac{a}{x_o}\right)\int_0^\infty\left(\frac{H}{\cosh(\tau\pi)}\right)\{P_{(i\tau-1/2)}[1+2(z_o/x_o)^2]\}\ d\tau + a/2z_o\ . \tag{E-2}$$

However, it may be shown that $P_{(i\tau-1/2)}(1) = 1$ for all values of τ. Therefore,

$$\psi_o^{(2)} = \left(\frac{a}{x_o}\right)\int_0^\infty\left(\frac{H}{\cosh(\tau\pi)}\right)d\tau$$
$$+ \left(\frac{a}{x_o}\right)\int_0^\infty\left(\frac{H}{\cosh(\tau\pi)}\right)\{P_{(i\tau-1/2)}[1+2(z_o/x_o)^2]\}\ d\tau + a/2z_o\ . \tag{54}$$

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a. I. S. Gradshetyn and I. M. Ryzhik, Tables of Integrals, Series and Products, p.732.

APPENDIX F

Proof of Equation (55)

A search of the literature yields the following:

$$P_\nu[\cosh \alpha] = (2/\pi)\cot\{(\nu+1/2)\pi\} \int_\alpha^\infty \frac{\sinh\{(\nu+1/2)\theta\}\, d\theta}{\sqrt{2\cosh\theta - 2\cosh\alpha}} \, .^{a} \tag{F-1}$$

Defining α such that

$$\sinh(\alpha/2) = z_o/x_o \, . \tag{F-2}$$

it may be shown that

$$1 + 2(z_o/x_o)^2 = 1 + 2\sinh^2(\alpha/2) = \cosh\alpha \, . \tag{F-3}$$

Therefore,

$$P_{(i\tau-1/2)}[1 + 2(z_o/x_o)^2] = P_\nu[\cosh\alpha] \, . \tag{F-4}$$

where $\nu = i\tau-1/2$ and $\alpha = 2\text{arcsinh}(z_o/x_o)$. Using Lebedev's identity and simplifying, one may conclude that:

$$P_{(i\tau-1/2)}[1+2(z_o/x_o)^2] = (\sqrt{2}/\pi)\left(\frac{\cosh(\tau\pi)}{\sinh(\tau\pi)}\right)\int_\alpha^\infty \frac{\sin(\tau\theta)\, d\theta}{\sqrt{\cosh\theta - \cosh\alpha}} \, . \tag{F-5}$$

Using this identity, it may be shown that:

$$I = \left(\frac{\sqrt{2}a}{\pi x_o}\right) \int_\alpha^\infty \int_0^\infty \left(\frac{\cosh(\tau\phi_o)}{\sinh(\tau\phi_o)} - \frac{\cosh(\tau\pi)}{\sinh(\tau\pi)}\right) \frac{\sin(\tau\theta)\, d\theta}{\sqrt{\cosh\theta - \cosh\alpha}} \, . \tag{F-6}$$

where I is the second term of equation (54).

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a. N. N. Lebedev, Special Functions and Their Applications, p. 173.

From Gradshteyn and Ryzhik[b], it is found that:

$$\operatorname{csch}(\tau\phi_o) = 2\sum_{k=0}^{\infty} e^{-(2k+1)\tau\phi_o} = \frac{1}{\sinh(\tau\phi_o)} \quad . \tag{F-7}$$

Therefore, using equation (F-7), one obtains

$$\coth(\tau\phi_o) = \frac{\cosh(\tau\phi_o)}{\sinh(\tau\phi_o)} = \left(\frac{e^{\tau\phi_o} + e^{-\tau\phi_o}}{2}\right)\left(2\sum_{k=0}^{\infty} e^{-(2k+1)\tau\phi_o}\right) \quad . \tag{F-8}$$

Rearrangement and simplification yields:

$$\coth(\tau\phi_o) = \sum_{k=0}^{\infty} e^{-2k\tau\phi_o} + \sum_{k=0}^{\infty} e^{-2(k+1)\tau\phi_o} . \tag{F-9}$$

Further manipulation results in the following:

$$\coth(\tau\phi_o) = 1 + 2\sum_{k=1}^{\infty} e^{-2k\tau\phi_o} \quad . \tag{F-10}$$

Similarly, it can be shown that:

$$\frac{\cosh(\tau\pi)}{\sinh(\tau\pi)} = 1 + 2\sum_{k=1}^{\infty} e^{-2k\tau\pi} \quad . \tag{F-11}$$

Using the identities in equation (F-10) and (F-11), equation (F-6) may be rewritten as:

$$I = \frac{2\sqrt{2}a}{\pi x_o}\int_{\alpha}^{\infty}\int_{0}^{\infty}\left(\sum_{k=1}^{\infty} e^{-2k\tau\phi_o}\right)\frac{\sin(\tau\theta)\, d\tau d\theta}{\sqrt{\cosh\theta - \cosh\alpha}}$$
$$- \frac{2\sqrt{2}a}{\pi x_o}\int_{\alpha}^{\infty}\int_{0}^{\infty}\left(\sum_{k=1}^{\infty} e^{-2k\tau\pi}\right)\frac{\sin(\tau\theta)\, d\tau d\theta}{\sqrt{\cosh\theta - \cosh\alpha}} \quad . \tag{F-12}$$

This is equivalent to the following:

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b. Gradshteyn and Ryzhik, p.23.

$$I = \frac{2\sqrt{2}a}{\pi x_o} \int_{\alpha}^{\infty} \sum_{k=1}^{\infty} \left(\int_0^{\infty} e^{-2k\tau\phi_o} \frac{\sin(\tau\theta)\, d\tau}{\sqrt{\cosh\theta - \cosh\alpha}} \right) d\theta$$

$$- \frac{2\sqrt{2}a}{\pi x_o} \int_{\alpha}^{\infty} \sum_{k=1}^{\infty} \left(\int_0^{\infty} e^{-2k\tau\pi} \frac{\sin(\tau\theta)\, d\tau}{\sqrt{\cosh\theta - \cosh\alpha}} \right) d\theta \; . \qquad \text{(F-13)}$$

Inverting the Laplace transforms, the following relationship is obtained:

$$I = \frac{2\sqrt{2}a}{\pi x_o} \int_{\alpha}^{\infty} \{ \sum_{k=1}^{\infty} \left(\frac{\theta}{4k^2\phi_o^2 + \theta^2} \right) \} \frac{d\theta}{\sqrt{\cosh\theta - \cosh\alpha}}$$

$$- \frac{2\sqrt{2}a}{\pi x_o} \int_{\alpha}^{\infty} \{ \sum_{k=1}^{\infty} \left(\frac{\theta}{4k^2\pi^2 + \theta^2} \right) \} \frac{d\theta}{\sqrt{\cosh\theta - \cosh\alpha}} \; . \qquad \text{(F-14)}$$

Algebraic manipulation of equation (F-14) yields:

$$I = \frac{2\sqrt{2}a}{\pi x_o} \int_{\alpha}^{\infty} \{ \sum_{k=1}^{\infty} \left(\frac{\theta}{(2\phi_o)^2[k^2 + (\theta/2\phi_o)^2]} \right) \} \frac{d\theta}{\sqrt{\cosh\theta - \cosh\alpha}}$$

$$- \frac{2\sqrt{2}a}{\pi x_o} \int_{\alpha}^{\infty} \{ \sum_{k=1}^{\infty} \left(\frac{\theta}{(2\pi)^2\,[k^2 + (\theta/2\pi)^2]} \right) \} \frac{d\theta}{\sqrt{\cosh\theta - \cosh\alpha}} \; . \qquad \text{(F-15)}$$

From the literature, the following relationship is obtained:

$$\coth(\pi x) = (1/\pi x) + (2x/\pi) \sum_{k=1}^{\infty} (x^2 + k^2)^{-1} . \qquad \text{(F-16)}$$

[c]

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c. Gradshetyn and Ryzhik, p.36.

By letting x in equation (F-16) be equal to $\theta/2\phi_o$, the first term of equation (F-15) may be simplified to the following:

$$\sum_{k=1}^{\infty}\left[\frac{\theta}{(2\phi_o)^2[k^2 + (\theta/2\phi_o)^2]}\right] = (\pi/4\phi_o)\coth(\pi\theta/2\phi_o) - 1/2\theta \ . \qquad \text{(F-17)}$$

Similarly, if $x = \theta/2\pi$ the second term of equation (F-15) may be reduced.

$$\sum_{k=1}^{\infty}\left[\frac{\theta}{(2\pi)^2[k^2 + (\theta/2\pi)^2]}\right] = (1/4)\coth(\theta/2) - 1/2\theta \ . \qquad \text{(F-18)}$$

Using equations (F-17) and (F-18), a simplified version of equation (F-15) may be written.

$$I = \frac{a}{\sqrt{2}\pi x_o}\int_{\alpha}^{\infty}[(\pi/\phi_o)\coth(\pi\theta/2\phi_o) - \coth(\theta/2)]\,\frac{d\theta}{\sqrt{\cosh\theta - \cosh\alpha}} \ . \qquad \text{(F-19)}$$

Using equation (F-10), it may be shown that

$$\coth(\pi\theta/2\phi_o) = 1 + 2\sum_{k=1}^{\infty} e^{-k\pi\theta/\phi_o} \ , \qquad \text{(F-20)}$$

and also that

$$\coth(\theta/2) = 1 + 2\sum_{k=1}^{\infty} e^{-k\theta} \ . \qquad \text{(F-21)}$$

Using these relationships, the second term of equation (54) may be expressed as follows:

$$I = \frac{a}{\sqrt{2}\pi x_o \propto} \int^{\infty} \{ \left(\frac{\pi}{\phi_o} - 1\right) + \left(\frac{2\pi}{\phi_o}\right) \sum_{k=1}^{\infty} e^{-k\pi\theta/\phi_o} - 2 \sum_{k=1}^{\infty} e^{-k\theta} \} \frac{d\theta}{\sqrt{\cosh\theta - \cosh\propto}} . \qquad \text{(F-22)}$$

It is obvious that equation (F-22) may be written as the sum of three integrals. Using Laplace transforms, these integrals may be simplified. From Bateman[d], the following identity is obtained:

$$\int_b^{\infty} e^{-pt} [\cosh t - \cosh b]^{\nu-1} dt = -i\sqrt{2/\pi}\, e^{\nu\pi i} \Gamma(\nu) [\sinh b]^{\nu-1/2} Q_{(p-1/2)}^{(1/2-\nu)} [\cosh b] . \qquad \text{(F-23)}$$

Letting $b = \propto$, $t = \theta$, $\nu = 1/2$, and $p = 0$, an expression which is proportional to the first integral of equation (F-22) is obtained. The result is:

$$\text{First integral} = C(-i\sqrt{2} e^{\pi i/2} Q_{(-1/2)} [\cosh \propto]) , \qquad \text{(F-24)}$$

where $C = \left(\frac{\pi}{\phi_o} - 1\right)$.

Letting $p = k\pi/\phi_o$, an expression which is equivalent to the second integral is obtained.

$$\text{Second integral} = \left(\frac{-2\sqrt{2}\pi i}{\phi_o}\right) e^{\pi i/2} \left(\sum_{k=1}^{\infty} Q_{(k\pi/\phi_o - 1/2)} [\cosh \propto]\right) . \qquad \text{(F-25)}$$

Letting $p = k$, the third integral is obtained.

$$\text{Third integral} = 2\sqrt{2} i e^{\pi i/2} \left(\sum_{k=1}^{\infty} Q_{(k-1/2)} [\cosh \propto]\right) . \qquad \text{(F-26)}$$

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d. Bateman, Vol. 1, p. 164.

Combining equations (F-24), (F-25), and (F-26), the following form of equation (F-22) is obtained:

$$I = \left(\frac{a}{\sqrt{2}\pi x_o}\right)\left(\left(\frac{\phi_o-\pi}{\phi_o}\right) i\sqrt{2}e^{\pi i/2}\, Q_{(-1/2)}[\cosh \alpha] - \left(\frac{2\sqrt{2}\pi i}{\phi_o}\right) e^{\pi i/2}\left(\sum_{k=1}^{\infty} Q_{(k\pi/\phi_o - 1/2)}[\cosh \alpha]\right) + 2\sqrt{2}ie^{\pi i/2}\left(\sum_{k=1}^{\infty} Q_{(k-1/2)}[\cosh \alpha]\right)\right) . \qquad \text{(F-27)}$$

Adopting the following definition,

$$\varepsilon_k = \begin{cases} 1 \text{ (for } k = 0) \\ 2 \text{ (for } k \neq 0), \end{cases}$$

and replacing π/ϕ_o with n, the number of spheres per ring, equation (F-27) may be reduced to the following:

$$I = \frac{a}{\sqrt{2}\pi x_o} \{i\sqrt{2}e^{\pi i/2} \sum_{k=0}^{\infty} \varepsilon_k Q_{(k-1/2)}(\cosh \alpha) - ni\sqrt{2}e^{\pi i/2} \sum_{k=0}^{\infty} \varepsilon_k Q_{(nk-1/2)}(\cosh \alpha)\}. \qquad \text{(F-28)}$$

From Magnus et al[e], the following relationship is obtained:

$$\sum_{m=0}^{\infty} \varepsilon_k Q^{\mu}_{(m-1/2)}(z) \cos(m\nu) = e^{i\pi\mu}\sqrt{\pi/2}\, \Gamma(1/2+\mu)(z^2-1)^{\mu/2}(z-\cos \nu)^{(-\mu-1/2)} . \qquad \text{(F-29)}$$

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e. W. Magnus, F. Oberhettinger, and R. P. Soni, Formulas and Theorms for the Special Functions of Mathematical Physics, p. 182.

Letting $\mu = 0$, $z = \cosh \alpha$, and $\nu = 0$, one obtains the following identity:

$$\sum_{m=0}^{\infty} \varepsilon_m Q_{(m-1/2)}(\cosh \alpha) = (\pi/\sqrt{2})(\cosh \alpha - 1)^{-1/2} . \qquad \text{(F-30)}$$

From Magnus et al[f] the following relationship may be obtained:

$$\sum_{k=0}^{\infty} \varepsilon_k \cos(k\nu)\, Q^{\mu}_{(kj-1/2)}(z) =$$

$$e^{i\pi\mu}\left(\frac{1}{\sqrt{2\pi}}\right) \Gamma\left(\frac{1}{2} + \mu\right)\left(\frac{\pi}{j}\right)(z^2-1)^{\mu/2} \sum_{r=r_1}^{r_2} \{z-\cos\left(\frac{2\pi r+\nu}{j}\right)\}^{(-\mu-1/2)} , \qquad \text{(F-31)}$$

where $r_1 = -[j/2 + \nu/2\pi]$,

$r_2 = [j/2 - \mu/2\pi]$,

and $[x]$ is the largest integer $\leq x$.

Letting $\nu = 0$, $j = n$, $z = \cosh \alpha$, and $\mu = 0$, the following result is obtained:

$$\sum_{k=0}^{\infty} \varepsilon_k Q_{(nk-1/2)}(\cosh \alpha) = \left(\frac{\pi}{\sqrt{2n}}\right) \sum_{r=r_1}^{r_2} \{[\cosh \alpha - \cos(2\pi r/n)]^{-1/2}\}, \qquad \text{(F-32)}$$

where $r_1 = -r_2 = -[n/2]$

f. Magnus, Oberhettinger, and Soni, p. 182.

Substitution of equations (F-30) and (F-32) into equation (F-28) and simplification yields

$$I = \left(\frac{a}{\sqrt{2}x_o}\right)\{ \sum_{r=r_1}^{r_2} \{[\cosh \alpha - \cos(2\pi r/n)]^{-1/2}\} - [\cosh \alpha - 1]^{-1/2}\}. \quad \text{(F-33)}$$

However,

$$\cosh \alpha = 1 + 2(z_o/x_o)^2 . \quad \text{(F-3)}$$

Therefore,

$$(\cosh \alpha - 1)^{-1/2} = \{2(z_o/x_o)\}^{-1/2} = \frac{x_o}{\sqrt{2}z_o} . \quad \text{(F-34)}$$

Also, it can be shown that

$$[\cosh \alpha - \cos(2\pi r/n)]^{-1/2} = \left(\frac{x_o}{\sqrt{2}z_o}\right)\{1 + [(x_o/z_o) \sin(\pi r/n)]^2\}^{-1/2} . \quad \text{(F-35)}$$

Using equations (F-34) and (F-35) in equation (F-33), one obtains:

$$I = (a/2z_o)\{ \sum_{r=r_1}^{r_2} ([1 + \{(x_o/z_o) \sin(\pi r/n)\}^2]^{-1/2}) - 1\}. \quad \text{(F-36)}$$

By symmetry, this is equivalent to

$$I = (a/z_o) \{ \sum_{r=1}^{r_2} [1 + \{(x_o/z_o) \sin(\pi r/n)\}^2]^{-1/2}\}. \quad \text{(F-37)}$$

The second term of equation (54) may now be written as

$$(a/x_o)\int_0^\infty \left(\frac{H}{\cosh(\tau\pi)}\right) P_{(i\tau-1/2)}[1 + 2(z_o/x_o)^2]\, d\tau = (a/z_o)\beta , \quad (55)$$

where $\beta = \sum_{r=1}^{r_2} [1 + \{(x_o/z_o) \sin(\pi r/n)\}^2]^{-1/2}$,

and $r_2 = [n/2]$.

APPENDIX G

Sample Problem

Determine the rate of heat transfer per array and the total rate of heat transfer for two parpllel arrays of two spheres each. The spheres have a radius of one inch and a surface temperature of 200 degrees F. The surrounding medium is air at 70 degrees F. x_o = 50 inches and z_o = 20 inches. Repeat the problem for n = 10.

$$k_{air} = 0.015 \frac{\text{BTU}}{\text{hr ft}^2 \text{ (deg. F/ft.)}}$$

$$q \simeq 4\pi ka(T_s - T_a)[1 - \gamma(a/z_o)] \quad (59)$$

a = 1 inch
T_s = 200 deg. F
T_a = 70 deg. F
x_o = 50 inches
z_o = 20 inches

for n = 2,

$$(a/x_o)_{max} = \sin(\pi/n) \quad (A\text{-}2)$$

$$(a/x_o)_{max} = \sin(\pi/2) = 1.0 \quad (G\text{-}1)$$

$$a/x_o = 1/50 = 0.02 < 0.1(a/x_o)_{max} \quad (G\text{-}2)$$

$$x_o/z_o = 50/20 = 2.5 \quad (G\text{-}3)$$

$$a/z_o = 1/20 = 0.05 < 0.1(a/z_o)_{max} \quad (G\text{-}4)$$

Since the values of the geometric factors are small the model is valid.

for $n = 2$ and $x_o/z_o = 2.5$, $\gamma = 1.1$ (figure 8)

$$q_{(n=2)} = 4\pi(0.015)(1/12)(200 - 70)[1 - 1.1(0.05)] \quad \text{(G-5)}$$

$$q_{(n=2)} = 1.9297 \ \text{BTU/hr.-sphere} \quad \text{(G-6)}$$

$$Q_{(n=2)} = nq_{(n=2)} = 2(1.9297 \ \text{BTU/hr.}) = 3.8594 \ \text{BTU/hr.} \quad \text{(G-7)}$$

for $n = 10$ and $x_o/z_o = 2.5$, $\gamma = 6$ (figure 8)

$$q_{(n=10)} = 4\pi(0.015)(1/12)(200 - 70)[1 - 6(0.05)] \quad \text{(G-8)}$$

$$q_{(n=10)} = 1.4294 \ \text{BTU/hr.-sphere} \quad \text{(G-9)}$$

$$Q_{(n=10)} = nq_{(n=10)} = 10(1.4294 \ \text{BTU/hr.}) = 14.294 \ \text{BTU/hr.} \quad \text{(G-10)}$$

From this example it can be seen that as the number of spheres is increased, the total rate of heat transfer is also increased, but the efficiency of each sphere as a source decreases.