THE STEADY STATE HEAT AND TEMPERATURE DISTRIBUTION OF A HOT SPHERE WITHIN AN INFINITE WEDGE

by

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ABSTRACT

This thesis presents a mathematical model of the steady state heat and temperature distributions of a hot sphere located along the midplane of an infinitely long wedge of any arbitrary central angle. The heat and temperature distributions of this geometric configuration are of immense value,since through the use of this model as a wedge shaped unit cell the description of any number of hot spheres, arranged in a regular planar array can be immediately determined.

The method of reflections is used to solve Laplace's equation , $\nabla^2 T=0$, analytically using the sphere and the wedge walls as boundary conditions. Only the second reflection was obtained,yielding a first order correction.

The resulting model of an individual sphere within a wedge,and an arbitrary number of spheres arranged in a regular polygonal planar array were obtained. The regular planar array was tested and compared with known exact solutions of Laplace's equation in Bipolar coordinates [for the solution of two spheres in space] and Toroidal coordinates [for the solution approximating an extremely large number of densely packed spheres in a regular planar array] . The model tested accurately in the comparison with Bipolar coordinates,while the comparison of the developed model with a toroid showed the limitations of a first order correction solution.

ACKNOWLEDGEMENTS

The author wishes to gratefully acknowledge Dr. Ernest Bart who,as a source of consistently sound help,encouragement, and guidance , enabled the completion of this research.

DEDICATION

To Pat

SYMBOLS USED AND THEIR MEANINGS

Symbol	Meaning
a	Sphere radius
A	Unknown function of integration-second reflection
C_1, C_2	Constants of integration for the first reflection
k	Heat transfer coefficient - BTU/hr ft^2
K_0, K_1	Modified Bessel functions of orders 0,1, respectively
$K_{\iota\tau}$	Modified Bessel functions of imaginary order $\iota\tau$
m, n	Integer indices
N	The number of spheres in a regular array
$P_n(x)$	Legendre's functions of order n
$Q_n(x)$	Legendre's functions of order n
Q	Rate of heat transfer
r_s, Φ, ϕ	Sphere centered spherical coordinates
T	Temperature at a point in space
T_1	Temperature at the sphere surface
T_2	Temperature at the wedge walls - fixed
T_{amb}	Temperature of the ambient space
V	Dummy variable of integration - first reflection
x_s, y_s, z_s	Sphere centered Cartesian coordinates
x_w, y_w, z_w	Wedge centered Cartesian coordinates
x_o	Distance from sphere center to wedge vertex
ρ, θ, z	Wedge centered cylindrical coordinates
θ_o	One half of the central angle of the wedge unit cell
λ, τ	Separation constants of Laplace's equation
∇	Nabla operator
$\Psi, \Psi^{(1)}, \Psi^{(2)}$	Normalized temperature variables

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INTRODUCTION

The work presented herein concerns itself with the development of a mathematical model which describes the temperature distribution due to the presence of a hot sphere located along the midplane of an infinite wedge of an arbitrary central angle. Two basic problems are treated; each problem differs only in boundary conditions. The simplest occurs when the wedge walls are held at constant uniform temperature; the second, and far more interesting problem, occurs when boundary conditions at the wedge walls are $dT/d\theta = 0$.

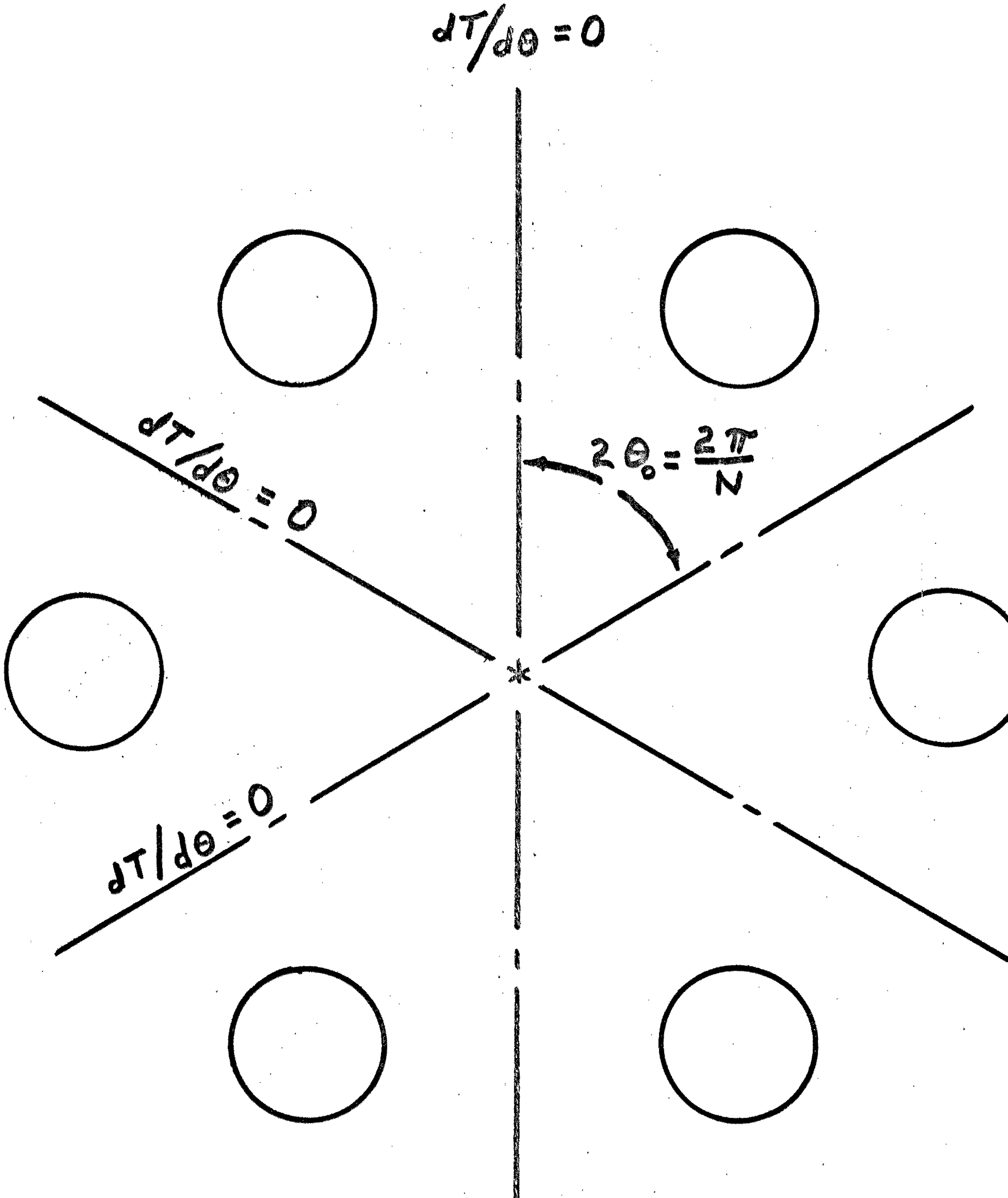
Problems relating to a sphere within a wedge develop when trying to describe large numbers of hot spheres arrange in a regular planar array. Given the array shown in Figure 1., each sphere can be considered to be located within its own particular wedge-shaped unit cell , of central angle $2\theta_o$. θ_o in turn is expressible in terms of the number of spheres, N , according to the following relation:

$$\theta_o = \pi/N$$

The walls of the unit cell prove to be lines of symmetry , both for the regular polygonal array and the resulting temperature distribution. The lines of symmetry within the temperature field are mathematically;

$$dT/d\theta = 0 \text{ [on the wedge walls]}$$

The second, more complex model, simultaneously solving Laplace's equation with boundary conditions of $dT/d\theta = 0$ [on the wedge walls] and $T = T_1$ [on the sphere surface], can be developed from the simpler solution, a sphere within a wedge of uniform surface temperature.



Lines of Symmetry (N=6)
Figure 1.

<u>DEVELOPMENT OF MODEL - WEDGE WALLS AT CONSTANT TEMPERATURE</u>

The simplest unit cell would consist of a sphere of constant temperature , T_1, located within a wedge of constant wall temperature T_2, as shown in Figure 2. The temperature field must be a harmonic function , <u>ie.</u> a solution to Laplace's equation

$$\nabla^2 T = 0 \qquad (1)$$

and must also be consistent with the boundary conditions. In this case, the satisfaction of the boundary conditions requires that the temperature of the sphere surface be T_1 and the temperature at the wedge walls be T_2.

$$\text{Let } \Psi = (T - T_2)/(T_1 - T_2) \qquad (2)$$

By sudstituting the variable Ψ , defined in equation(2), for the temperature variable ,T, the boundary conditions become normalized in terms of Ψ.

$$\Psi(\text{on the sphere surface}) = 1 \qquad (3)$$

$$\Psi(\text{on the wedge walls}) = 0 \qquad (4)$$

Ψ is also a solution of Laplace's equation in that :

$$\nabla^2 T = 0 \qquad (1)$$

T is related to Ψ by transposing equation (2).

$$T = \Psi(T_1 - T_2) + T_2 \qquad (5)$$

Performing the required substitution and stipulating that (T_1-T_2) be a non-zero fixed constant one obtains:

Sphere and Wedge of Constant Temperature

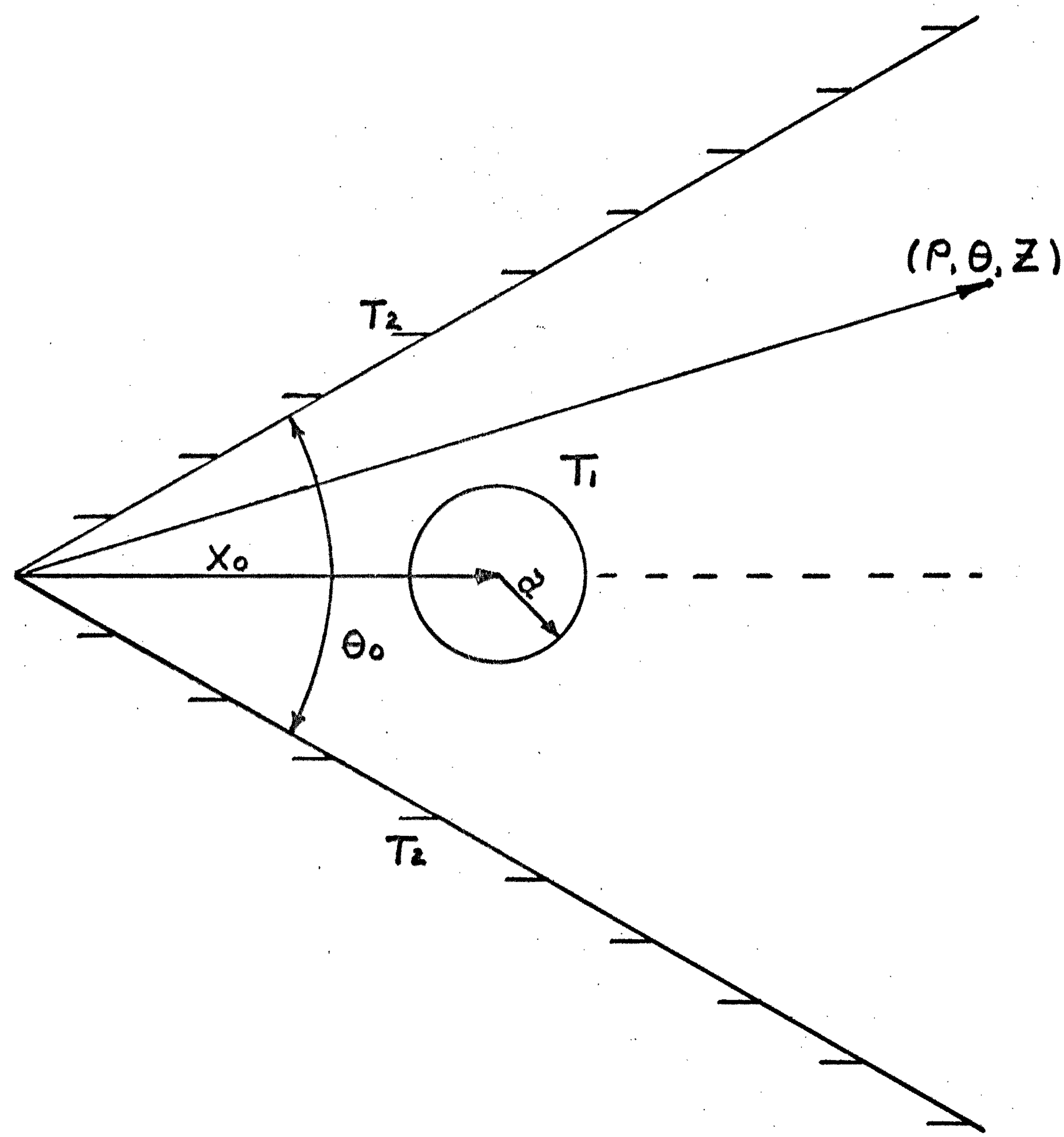


Figure 2

$$\nabla^2 T = (T_1 - T_2)\nabla^2 \Psi = 0 \tag{6}$$

Therefore:

$$\nabla^2 \Psi = 0 \tag{7}$$

We now have reduced the problem into the normalized temperature variable with the appropriate boundary conditions.

The problem inherently possesses two dissimilar geometries, wedge-shaped and spherical. No single coordinate system can be used to simultaneously treat both geometries. The method of reflections must be used as an algorithm . The method involves obtaining an infinite number of solutions, each solution individually being the solution to a boundary condition upon one surface, and adding them. The resultant sum is a solution which satisfies the boundary conditions upon both surfaces. Thus, the required solution Ψ will be built up as an infinite series of individual solutions; the odd numbered solutions satisfy the boundary conditions on the sphere surface and the even numbered solutions satisfy the boundary conditions upon the wedge surface.

$$\Psi = \Psi^{(1)} + \Psi^{(2)} + \Psi^{(3)} + \ldots\ldots \Psi^{(\infty)} \tag{8}$$

The aim of this thesis will be to obtain up to the second term of this reflection series. The second reflection amounts to a first order correction factor, correcting the temperature field of the sphere in accordance with the effect of the wedge walls.

Starting with a sphere of surface temperature $T=T_1$ and a spherical coordinate system based upon the sphere center as an origin, the harmonic function , due to spherical symmetry, will be a function of the spherical radius alone. The boundary restrictions are:

$\Psi^{(1)} =$ A harmonic function

$\Psi^{(1)} = 1.$ [At the sphere surface, ie. $r_s = a$]

$\Psi^{(1)} =$ A function of r_s alone due to spherical symmetry

In spherical coordinates for $\Psi^{(1)} \neq f(\Phi, \phi)$, the well known solution to Laplace's equation in the region exterior to the sphere is:

$$\Psi^{(1)} = a/r_s \tag{9}$$

$\Psi^{(1)}$ is consistant with the boundary conditions since it is a harmonic function, its value at the sphere surface is 1 , and it is a function of r_s alone. It also exhibits the characteristic property that:

$$\underset{x \to \infty}{\text{limit}} [\Psi^{(1)}(x)] = 0 \tag{10}$$

Figure 3 shows a plot of the isotherms of $\Psi^{(1)}$ as a function of r_s expressed as multiples of a .

$$\Psi = \Psi(1) + \Psi(2) + \Psi(3) + \ldots \Psi(\infty) \tag{8}$$

$$\Psi = a/r_s + \Psi^{(2)} + \Psi^{(3)} + \ldots \Psi^{(\infty)} \tag{11}$$

Truncating after the second reflection term to obtain

Isotherms of the First Reflection

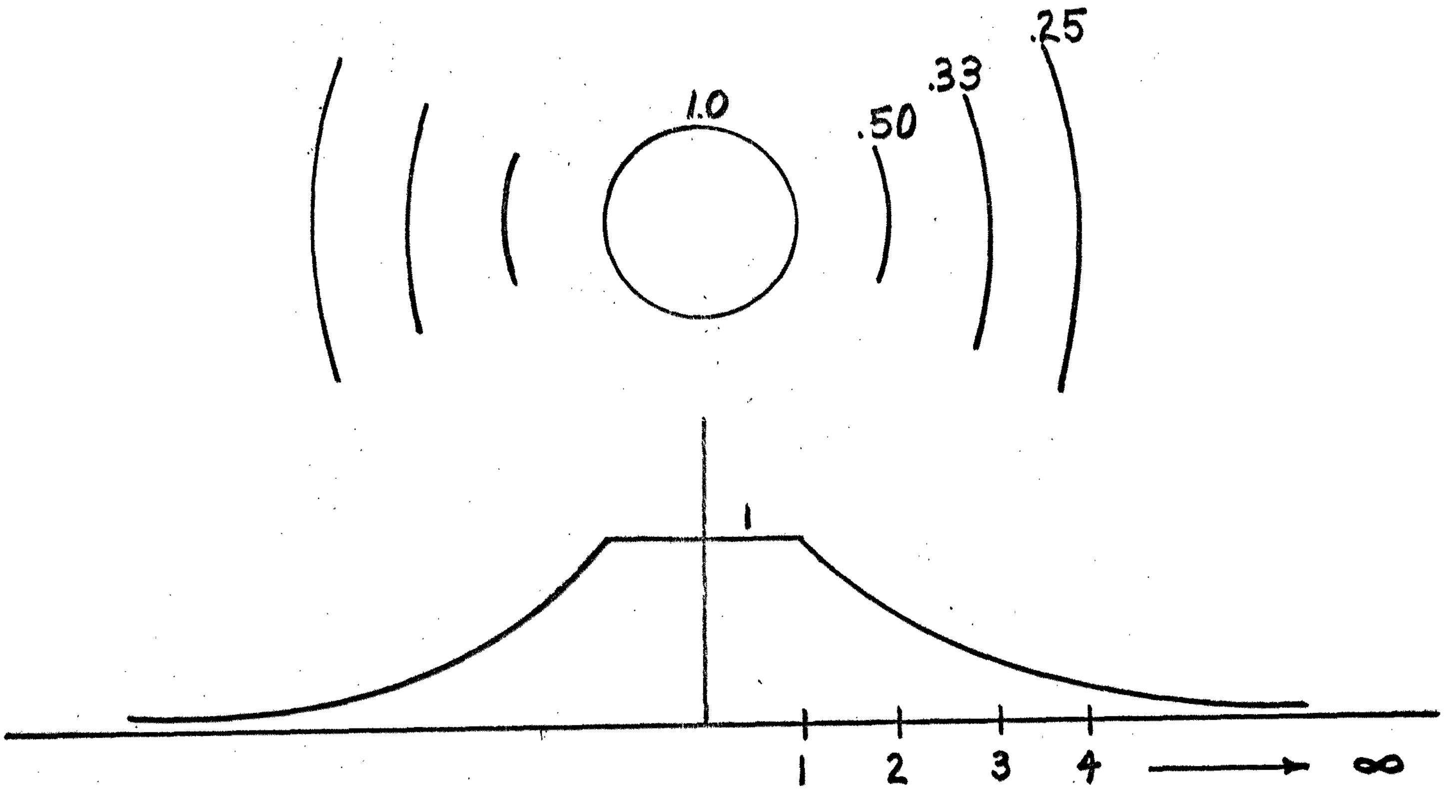


Figure 3

a first order correction ,

$$\Psi \simeq a/r_s + \Psi^{(2)} \quad . \qquad (12)$$

$\Psi^{(1)}$ based upon sphere surface boundary conditions sets up a temperature field of concentric spheres of constant temperature of value a/r_s . These concentric spheres are cut across by the walls of the wedge , maintained at constant ,uniform temperature. The hot sphere sets up a temperature distribution on the wedge walls. However, since the boundary conditions at the wedge walls require that the wall temperature expressed in terms of Ψ be zero, the second reflection must cancel off the effect of the first reflection, shown in figure 4.

$\Psi^{(2)} = -\Psi(1)$ [To satisfy that $\Psi=0$ on the wedge walls] This condition must be satisfied only upon the wedge surface and not everywhere else in space. $\Psi^{(2)}$ and $\Psi^{(1)}$ must be linearly independant solutions to Laplace's equation . $\Psi^{(2)}$ must also be a harmonic function.

From geometry, in cartesian coordinates:

$$x_w = x_o + x_s \qquad (13)$$

$$y_w = y_s \qquad (14)$$

$$z_w = z_s \qquad (15)$$

$$\Psi^{(1)} = a/r_s = \frac{a}{\sqrt{x_s^2 + y_s^2 + z_s^2}} \qquad (16)$$

Intersection of Wedge and First Reflection

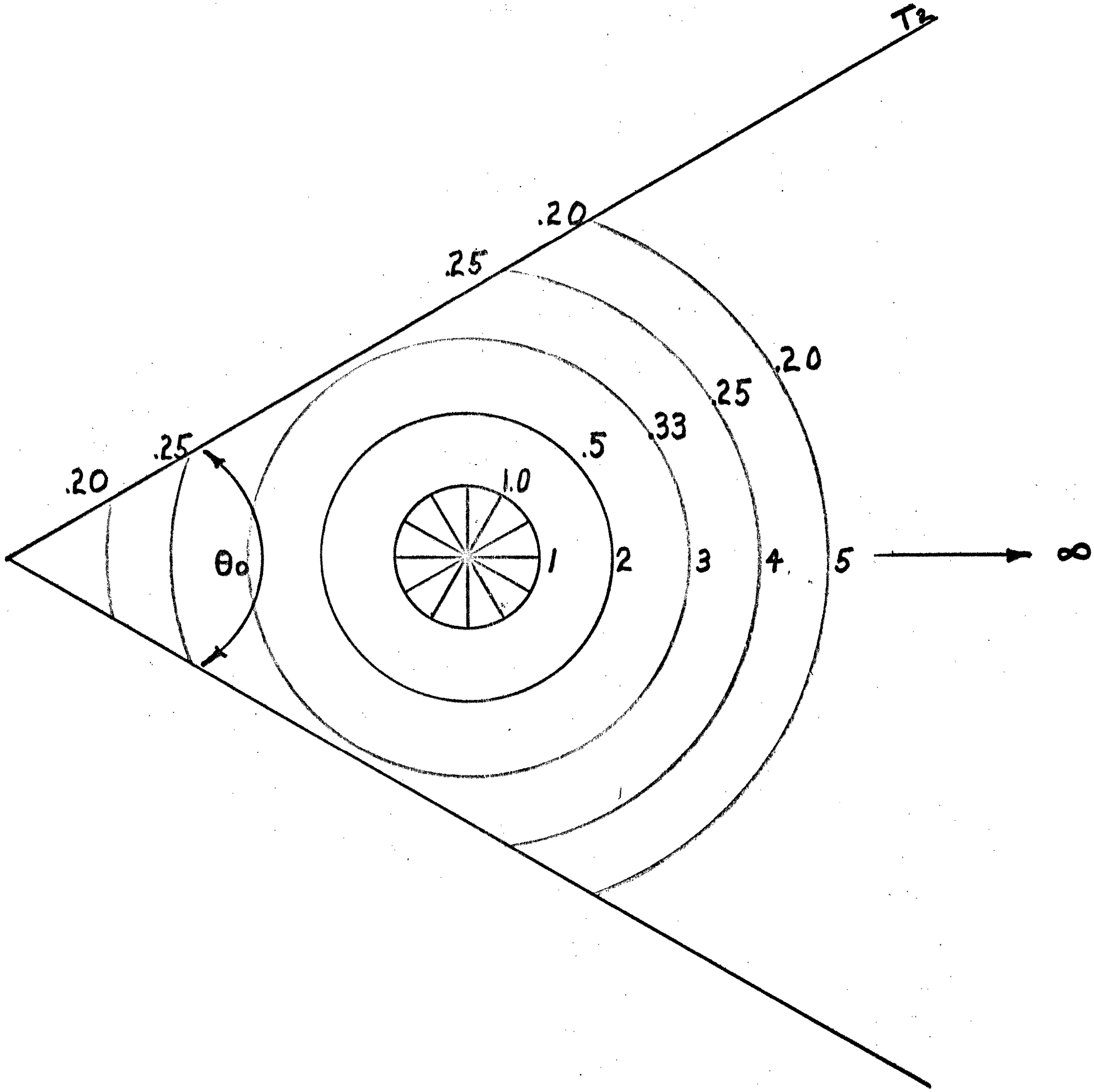


Figure 4

$$\psi^{(1)} = \frac{a}{\sqrt{(x_w - x_\circ)^2 + y_w^2 + z_w^2}} \tag{17}$$

$$\psi^{(1)} = \frac{a}{\sqrt{x_w^2 - 2x_w x_\circ + x_\circ^2 + y_w^2 + z_w^2}} \tag{18}$$

Shifting to cylindrical coordinates with origin at the wedge center

$$x_w = \rho \cos(\theta) \tag{19}$$

$$y_w = \rho \sin(\theta) \tag{20}$$

$$x_w^2 + y_w^2 = \rho^2 \tag{21}$$

$$\psi^{(1)} = \frac{a}{\sqrt{\rho^2 - 2x_\circ \rho \cos(\theta) + x_\circ^2 + z_w^2}} \tag{22}$$

$\psi^{(2)}$ must be a harmonic and equal to $-\psi^{(1)}$ at the wedge walls. Transform analysis indicates that the form of $\psi^{(2)}$ should be:

$$\psi^{(2)} = \int_0^\infty\int_0^\infty A \cosh(\tau\theta)\ K_{i\tau}(\lambda\rho)\ \cos(\lambda z_w)\ d\lambda d\tau \tag{23}$$

The above solution is valid everywhere within the domain bounded by the wedge , <u>ie</u>. $\infty > \rho \geq 0$, $\infty > z_w > -\infty$.
The constant A is really not a constant but an unknown function of the separation constants λ and τ. A can not be a function of the variables ρ, z_w , or θ.At the wedge walls when $\theta = \theta_\circ$,

$$\psi^{(2)} = -\psi^{(1)} = \frac{-a}{\sqrt{\rho^2 - 2x_\circ \rho \cos(\theta_\circ) + x_\circ^2 + z_w^2}} \tag{24}$$

$$\int_0^\infty\int_0^\infty A \cosh(\tau\theta_\circ) K_{i\tau}(\lambda\rho)\cos(\lambda z_w)\, d\lambda\, d\tau = \frac{-a}{\sqrt{\rho^2 - 2x_\circ\rho\cos(\theta_\circ) + x_\circ^2 + z_w^2}} \quad (25)$$

Inverting the z_w transform yields:

$$\int_0^\infty A \cosh(\tau\theta_\circ)\, K_{i\tau}(\lambda\rho)\, d\tau = -2a/\pi\, K_\circ(\sqrt{\rho^2 - 2x_\circ\rho\cos(\theta_\circ) + x_\circ^2}\;\lambda) \quad (26)$$

However:

$$\int_0^\infty K_{i\tau}(\lambda\rho)\, K_{i\tau}(\lambda x_\circ)\cosh[\tau(\pi-\theta_\circ)]\, d\tau = \pi/2\, K_\circ(\lambda\sqrt{\rho^2 - 2x_\circ\rho\cos(\theta_\circ) + x_\circ^2}) \quad (27)$$

By comparing like terms one can conclude:

$$A = \frac{-4a\cosh[\tau(\pi-\theta_\circ)]\, K_{i\tau}(\lambda x_\circ)}{\pi^2 \cosh(\tau\theta_\circ)} \quad (28)$$

$$\psi^{(2)} = \int_0^\infty\int_0^\infty \frac{\cosh[\tau(\pi-\theta_\circ)]}{\cosh(\tau\theta_\circ)} K_{i\tau}(\lambda x_\circ)\cosh(\tau\theta) K_{i\tau}(\lambda\rho)\cos(\lambda z_w)\, d\lambda\, d\tau\, [-4a/\pi^2] \quad (29)$$

The approximate temperature field may now be expressed as:

$$\psi \simeq a/\sqrt{\rho^2 - 2x_\circ\rho\cos(\theta_\circ) + x_\circ^2 + z_w^2}$$
$$-4a/\pi^2 \int_0^\infty\int_0^\infty \frac{\cosh[\tau(\pi-\theta_\circ)]}{\cosh(\tau\theta_\circ)} K_{i\tau}(\lambda x_\circ)\cosh(\tau\theta) K_{i\tau}(\lambda\rho)\cos(\lambda z_w)\, d\lambda\, d\tau \quad (30)$$

Q, the rate of heat transfer, can be expressed as the series;

$$Q = Q^{(1)} + Q^{(2)} + Q^{(3)} + \ldots Q^{(\infty)} \quad (31)$$

Truncating the above series to form a first order correction;

$$Q \simeq Q^{(1)} + Q^{(2)}$$

This truncated series can be shown, from Appendix A , to be equal to :

$$Q \simeq 4\pi k a(T_1 - T_2)\, [1 + \psi^{(2)}\{x_\circ, 0, 0\}] \quad (32)$$

This law is analogous to Faxen's law, used primarily hydrodynamics of low Reynold's numbers. At the sphere center $(x_\circ,0,0)$, $\Psi^{(2)}$ is defined by:

$$\Psi^{(2)}\{x_\circ,0,0\} = -a/x_\circ \int_\circ^\infty \frac{\cosh[\tau(\pi-\theta_\circ)]\ d\tau}{\cosh(\tau\theta_\circ)\ \cosh(\tau\pi)} \qquad (33)$$

$$Q \simeq 4\pi ka(T_1-T_2)\left[\ 1-a/x_\circ \int_\circ^\infty \frac{\cosh[\tau(\pi-\theta_\circ)]\ d\tau}{\cosh(\tau\theta_\circ)\ \cosh(\pi\tau)}\right] \qquad (34)$$

Equation (33) is obtained by evaluating equation (29) at $\rho=x_\circ,\theta=0,z_W=0$.

$$\Psi^{(2)}\{x_\circ,0,0\} = -4a/\pi^2 \int_\circ^\infty\int_\circ^\infty \frac{\cosh[\tau(\pi-\theta_\circ)]}{\cosh(\tau\theta_\circ)}[K_{\iota\tau}(\lambda x_\circ)]^2\ d\lambda d\tau \qquad (35)$$

$$\Psi^{(2)}\{x_\circ,0,0\} = -a/x_\circ \int_\circ^\infty \frac{\cosh[\tau(\pi-\theta_\circ)]}{\cosh(\tau\theta_\circ)\ \cosh(\pi\tau)} P_{\iota\tau-1/2}(1)\ d\tau \qquad (36)$$

Equation (36) is obtained by inverting the λ transform within equation (35). Also due to its conical nature $P_{\iota\tau-1/2}(1) = 1$ for all values of τ. Therefore;

$$\Psi^{(2)}\{x_\circ,0,0\} = -\ a/x_\circ \int_\circ^\infty \frac{\cosh[\tau(\pi-\theta_\circ)]\ d\tau}{\cosh(\tau\theta_\circ)\ \cosh(\pi\tau)} \qquad (33)$$

This completes the development of the models of heat transfer rate and temperature distribution for a hot sphere within the walls of a wedge maintained at constant temperature.

This solution leads to the presentation of a more theoretically interesting problem, the problem of a hot sphere in a wedge of boundary conditions $dT/d\theta = 0$. This corresponds to the unit cell to be used in the analysis of a large number of hot spheres arranged in a regular planar array. A solution to this problem involves the identical differential equation as before, namely Laplace's equation; the boundary conditions are now modified.

$$\Psi[\text{at the sphere surface}] = 1$$

$$d\Psi^{(1)}/d\theta\ [\text{at the wedge walls}] = -d\Psi^{(2)}/d\theta\ [\text{at the wedge walls}]$$

$\Psi^{(1)}$ remains the same as in the previous problem.

$$\Psi^{(1)} = \frac{-a}{\sqrt{\rho^2 - 2x_o\rho\cos(\theta) + x_o^2 + z_w^2}} \tag{37}$$

$$\frac{d\Psi^{(1)}}{d\theta} = \frac{-ax_o\rho\sin(\theta)}{\left[\sqrt{\rho^2 - 2x_o\rho\cos(\theta) + x_o^2 + z_w^2}\right]^3} \tag{38}$$

$\Psi^{(2)}$ will be of the same form as in the previous problem.

$$\Psi^{(2)} = \int_0^\infty\!\!\int_0^\infty A\cosh(\tau\theta)\,K_{i\tau}(\lambda\rho)\cos(\lambda z_w)\,d\lambda d\tau \tag{39}$$

$$d\Psi^{(2)}/d\theta = \int_0^\infty\!\!\int_0^\infty A\tau\sinh(\tau\theta)K_{i\tau}(\lambda\rho)\cos(\lambda z_w)\,d\lambda d\tau \tag{40}$$

Equation (40) is obtained from equation (39) by performing the indicated differentiation with respect to θ. The boundary conditions state that the derivatives with respect to the variable θ must cancel each other only at the wedge walls. $\{\theta = \pm\theta_o\}$

$$\int_0^\infty\!\!\int_0^\infty A\tau\sinh(\tau\theta_o)K_{i\tau}(\lambda\rho)\cos(\lambda z_w)\,d\lambda d\tau = \frac{ax_o\rho\sin(\theta_o)}{(\rho^2 - 2x_o\rho\cos(\theta_o) + x_o^2 + z_w^2)^{3/2}} \tag{41}$$

Inverting the λ transform,

$$\int_0^\infty A\tau \sinh(\tau\theta_\circ) K_{i\tau}(\lambda\rho)\, d\tau = 2/\pi \int_0^\infty \frac{a\rho x_\circ \sin(\theta_\circ) \cos(\lambda z_w)\, dz_w}{(\rho^2 - 2x_\circ\rho\cos(\theta_\circ) + x_\circ^2 + z_w^2)^{3/2}} \quad (42)$$

Evaluating the cosine transform with respect to z_w.

$$\int_0^\infty A\tau \sinh(\tau\theta_\circ)\ K_{i\tau}(\lambda\rho)\, d\tau = \frac{2ax_\circ\lambda\rho\sin(\theta_\circ) K_1(\lambda\sqrt{\rho^2 - 2x_\circ\rho\cos(\theta_\circ) + x_\circ^2})}{\pi\sqrt{\rho^2 - 2x_\circ\rho\cos(\theta_\circ) + x_\circ^2}} \quad (43)$$

To solve for the value of A , the τ transform must be inverted, and a final relation must be derived . Given,

$$\int_0^\infty K_{i\tau}(\lambda x_\circ) K_{i\tau}(\lambda\rho) \cosh[\tau(\pi-\theta)]\, d\tau = \pi/2\ K_0(\lambda\sqrt{\rho^2 - 2x_\circ\rho\cos(\theta) + x_\circ^2}) \quad (44)$$

$$\frac{d}{d\theta} \int_0^\infty K_{i\tau}(\lambda x_\circ) K_{i\tau}(\lambda\rho) \cosh[\tau(\pi-\theta)]\, d\tau = \frac{d}{d\theta}\ \pi/2\ K_0(\lambda\sqrt{\rho^2 - 2x_\circ\rho\cos(\theta) + x_\circ^2}) \quad (45)$$

Taking the indicated derivative with respect to θ , the following equalities develop:

$$\int_0^\infty A\tau \sinh(\tau\theta_\circ) K_{i\tau}(\lambda\rho)\ d\tau = \frac{2a\lambda\rho x_\circ \sin(\theta_\circ) K_1(\lambda\sqrt{\rho^2 - 2x_\circ\rho\cos(\theta_\circ) + x_\circ^2})}{\pi\sqrt{\rho^2 - 2x_\circ\rho\cos(\theta_\circ) + x_\circ^2}} \quad (46)$$

$$= 4a/\pi^2 \int_0^\infty \tau K_{i\tau}(\lambda x_\circ) K_{i\tau}(\lambda\rho) \sinh[\tau(\pi-\theta_\circ)]\ d\tau \quad (47)$$

From these two equalities the value of A can be determined by comparing like terms. One can conclude that the value of A is;

$$A = \frac{4a \sinh[\tau(\pi-\theta_\circ)]\ K_{i\tau}(\lambda x_\circ)}{\pi^2 \sinh(\tau\theta_\circ)} \quad (48)$$

Having the value of A, $\psi^{(2)}\{x_\circ, 0, 0\}$ develops to be:

$$\psi^{(2)}\{x_\circ, 0, 0\} = \int_0^\infty\int_0^\infty A\ K_{i\tau}(\lambda x_\circ)\ d\lambda\, d\tau \quad (49)$$

$$\psi^{(2)}\{x_o,0,0\} = \frac{4a}{\pi^2}\int_0^\infty\int_0^\infty \frac{\sinh[\tau(\pi-\theta_o)]\, K_{i\tau}(\lambda x_o)K_{i\tau}(\lambda x_o)\,d\lambda\, d\tau}{\sinh(\tau\theta_o)} \quad (50)$$

Inverting the λ transform as before,

$$\psi^{(2)}\{x_o,0,0\} = a/x_o \int_0^\infty \frac{\sinh[\tau(\pi-\theta_o)]\, d\tau}{\sinh(\tau\theta_o)\cosh(\pi\tau)} \quad (51)$$

The model for heat transfer is now:

$$Q = 4\pi ka(T_1-T_2)\left[1 - a/x_o \int_0^\infty \frac{\sinh[\tau(\pi-\theta_o)]\, d\tau}{\sinh(\tau\theta_o)\cosh(\pi\tau)}\right] \quad (52)$$

For N spheres arranged in a regular polygon, each individual sphere can be considered to be enclosed in a wedge of central angle θ_o , where $\theta_o = \pi/N$. The heat transfer rate per sphere is:

$$Q = 4\pi ka(T_1-T_{amb})\left[1 - a/x_o \int_0^\infty \frac{\sinh[(\{N-1\}/N)\pi\tau]\, d\tau}{\sinh[\pi\tau/N]\cosh(\pi\tau)}\right] \quad (53)$$

The rate of heat transfer from the entire array would merely be the rate of heat transfer per sphere , equation (53), multiplied by the number of spheres , N .

The temperature distribution is modeled by,

$$\psi \approx a/\sqrt{\rho^2-2x_o\rho\cos(\theta)+x_o^2+z_w^2}$$
$$- 4a/\pi^2 \int_0^\infty\int_0^\infty \frac{\sinh[\tau(\pi-\theta_o)]K_{i\tau}(\lambda x_o)}{\sinh(\tau\theta_o)}\cosh(\tau\theta)K_{i\tau}(\lambda\rho)\cos(\lambda z_w)\,d\lambda\, d\tau \quad (54)$$

Summarizing the results for a hot sphere within the boundaries of a wedge shaped unit cell:

For a wedge of fixed wall temperature, the heat transfer rate Q is:

$$Q \simeq 4\pi k a(T_1 - T_2)\left[1 - a/x_\circ \int_\circ^\infty \frac{\cosh[\tau(\pi-\theta_\circ)]\, d\tau}{\cosh(\tau\theta_\circ)\cosh(\pi\tau)} \right]$$

For a wedge of boundary conditions $dT/d\theta = 0$ at the walls.

$$Q \simeq 4\pi k a(T_1 - T_{amb})\left[1 - a/x_\circ \int_\circ^\infty \frac{\sinh[(\{N-1\}/N)\pi\tau]\, d\tau}{\sinh(\pi\tau/N)\ \cosh(\pi\tau)} \right]$$

Where Q is the heat transfer rate per sphere and N is the number of spheres arranged in the regular planar array.

For a single sphere in space, the central angle of the wedge is 180°. The formula in this case degenerates to:

$$Q = 4\pi k a(T_1 - T_{amb})$$

This is known to be the correct solution to the heat transfer rate of a single sphere in space. For two spheres in space, the equation yields:

$$Q = 4\pi a k(T_1 - T_{amb})\ [\ 1.\ -\ a/2x_\circ]$$

since the value of the integral yields:

$$\int_\circ^\infty d\tau/\cosh(\pi\tau) = 1/2$$

The results of this study are shown in Appendix 2. A final regarding the accuracy of the formula appears in Appendix 3 . In Appendix 3 the formula is used to approximate a toroid by allowing the number of spheres to become large. In the case of two spheres in space, the above solution compares most favorably with the answer derived from bipolar coordinates. In the attempt to approximate a toroid, the solution is limited by the $a/x_\circ$ value, as shown in Appendix 3.

SUMMARY

In summary, a mathematical solution to Laplace's equation was developed for a sphere in a wedge type unit cell. Two types of boundary conditions were considered:a wedge of fixed uniform wall remperature, and a wedge along whose walls the derivative of temperature with respect to a change in the central angle was zero. This latter model was used to describe an array of hot spheres in space arranged in a regular planar array. The model was tested and proved accurate in all cases for one and two spheres. From a comparison with the bipolar coordinate solution to Laplace's equation, the accuracy of the first order correction model was shown to be related to a/x_o. In an attempt to compare the model with a toroidal coordinate solution the number of spheres was allowed to increase and the inter-sphere spacing was permitted to decrease until all the spheres were tangent. It was found through computer analysis that the value of the geometric view factor;

$$\int_0^\infty \frac{\sinh[(\{N-1\}/N)\pi\tau]\ d\tau}{\sinh(\tau\theta/N)\ \cosh(\tau\pi)}$$

increased much faster than the decrease in the value of $[\ a/x_o\ ]_{max.}$ with the number of spheres. Thus with the spheres touching the first order correction model was inaccurate and higher order terms in the reflection series would be needed to achieve accuracy in this case.

Future advances along these lines would be the development of higher order terms in the reflection series, allowing the solution to the problem of a large quantity of spheres touching or similar concentrated systems. The reflection technique may provide a method of simultaneously solving the creeping motion equation and the equation of continuity within the boundaries of a wedge-like unit cell. The resulting model would then be an effective model of sedimentation.

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APPENDIX A

Proof of equation (32)

The rate of heat transfer ,Q, is expressible as a series similar in form to the series developed for the temperature , T.

$$Q = Q^{(1)} + Q^{(2)} + Q^{(3)} + Q^{(4)} \ldots + Q^{(\infty)} \qquad \text{(A-1)}$$

The form of $Q^{(j)}$ is developed from the definition of Q.

$$Q = -k \int (d\ \text{Area}) \cdot dT/dr_s \qquad \text{(A-2)}$$

$$Q^{(j)} = ka^2(T_1 - T_{amb}) \int_{\Phi=0}^{2\pi} \int_{\phi=0}^{\pi} [d\Psi^{(j)}/dr_s]_{r=a} \sin(\phi)\, d\phi\, d\Phi \qquad \text{(A-3)}$$

The variable T is replaced in the definition (A-2) by its equivalent in terms of Ψ, and the resultant equation is integrated over the sphere surface. The form of $[d\Psi^{(j)}/dr_s]_{r=a}$ in equation (A-3) is presently known in wedge centered cylindrical coordinates. In order to perform the necessary integration, the function $[\, d\Psi^{(j)}/dr_s]_{r=a}$ must be translated to a sphere centered spherical coordinate system. To translate the function to spherical coordinates it must be expressed as a series.

[For even numbered reflections]

$$\Psi^{(j)} = \sum_{n=m}^{\infty} \sum_{m=0}^{\infty} B_{m,n}^{(m)}\, r_s^{n} \cos(m\Phi)\, P_n^m(\cos(\phi)) \qquad \text{(A-4)}$$

[For odd numbered reflections]

$$\Psi^{(j)} = \sum_{n=m}^{\infty} \sum_{m=0}^{\infty} C_{m,n}^{(m)}\, r_s^{(-n-1)} \cos(m\Phi)\, P_n^m(\cos(\phi)) \qquad \text{(A-5)}$$

Taking the derivative of equations(A-4) and (A-5), and evaluating these functions at the sphere surface.

$$[d\Psi/dr_s]_{r_s=a} = \sum_{n=m}^{\infty}\sum_{m=0}^{\infty} B_{n,m}^{(m)}(n)a^{n-1}\cos(m\Phi)P_n^m(\cos(\phi)) \quad \text{(A-6)}$$

[even reflection]

$$[d\Psi/dr_s]_{r_s=a} \quad \sum_{n=m}^{\infty}\sum_{m=0}^{\infty} -C_{n,m}^{(m)}(n+1)a^{-n-2}\cos(m\Phi)P_n^m(\cos(\phi)) \quad \text{(A-7)}$$

[odd reflection]

Integrating these derivatives over the sphere surface.

$$Q^{(j=\text{even})} = ka^2(T_1-T_{amb})\int_{\Phi=0}^{2\pi}\int_{\phi=0}^{\pi}\sum_{n=m}^{\infty}\sum_{m=0}^{\infty} B_{n,m}^{(m)}na^{n-1}\cos(m\Phi)P_n^m(\cos\phi))\sin(\phi)d\phi d\Phi \quad \text{(A-8)}$$

$$Q^{(j=\text{odd})} = -ka^2(T_1-T_{amb})\int_{\Phi=0}^{2\pi}\int_{\phi=0}^{\pi}\sum_{n=m}^{\infty}\sum_{m=0}^{\infty} C_{n,m}^{(m)}(n+1)a^{-n-2}\cos(m\Phi)P_n^m(\cos\phi))\sin(\phi)\,d\phi d\Phi \quad \text{(A-9)}$$

By examination of the integrals several terms can be eliminated.

$$\int_{\Phi=0}^{2\pi}\cos(m\Phi)\,d\Phi \quad \begin{matrix} = 0\ [\text{ for } m\neq 0\ ] \\ = 2\pi[\text{ for } m=0\ ] \end{matrix}$$

Thus:

$$Q^{(j=\text{even})} = 2\pi ka^2(T_1-T_{amb})\int_0^{\pi}\sum_{n=1}^{\infty} B_{0,n}^{(m)}\,n\,a^{n-1}P_n(\cos(\phi))\sin(\phi)d\phi \quad \text{(A-10)}$$

$$Q^{(j=\text{odd})} = -2\pi ka^2(T_1-T_{amb})\int_0^{\pi}\sum_{n=0}^{\infty} C_{0,n}^{(m)}(n+1)a^{-n-2}P_n(\cos(\phi))\sin(\phi)d\phi \quad \text{(A-11)}$$

but

$$\int_0^{\pi}P_n(\cos(\phi))\sin(\phi)\,d\phi \quad \begin{matrix} = [\ 0 \text{ for } n\neq 0\ ] \\ = [\ 2 \text{ for } n=0\ ] \end{matrix}$$

Therefore;

$$Q^{(j=\text{even})} = 0 \quad \text{(A-12)}$$

$$Q^{(j=\text{odd})} = -4\,k(T_1-T_{amb})\,C_{0,0}^{(m)} \quad \text{(A-13)}$$

The rate of heat transfer is merely the sum of the odd terms in Q^j. The boundary conditions used with the reflection method indicate that, in general, at the sphere surface.

$$\Psi^{(\text{next odd})}[a,\phi,\Phi] = -\Psi^{(\text{even})}[a,\phi,\Phi] \qquad (A-14)$$

$$\Psi^{(\text{next odd})} = \sum_{n=m}^{\infty}\sum_{m=0}^{\infty} C_{m,n}^{(m)} \frac{\cos(m\Phi)\, P_n^m(\cos(\phi))}{a^{n+1}} \qquad (A-15)$$

$$= \sum_{n=m}\sum_{m=0} B_{m,n}^{(m)}\, a^n \cos(m\Phi)\, P_n^m(\cos(\phi)) \qquad (A-16)$$

One can conclude :

$$C_{m,n}^{(m)} = -B_{m,n}^{(m)}\, a^n\, a^{n+1} \qquad (A-17)$$

For n=m=0,

$$C_{0,0}^{(m)} = -B_{0,0}^{(m)}\, a \qquad (A-18)$$

but

$$C_{0,0}^{(m)} = -B_{0,0}^{(m)}\, a = -a\,\Psi^{(2m)}[0,0,0] \qquad (A-19)$$

The final summary indicates:

$$Q^{(2m)} = 0 \qquad (A-20)$$

$$Q^{(2m+1)} = 4\pi ak(T_1 - T_{amb})\,\Psi^{(2m)}[0,0,0] \qquad (A-21)$$

$$Q = \sum_{0}^{\infty} Q^{(2m+1)} = 4\pi ak(T_1 - T_{amb})[1 - \sum_{m=1}^{\infty}\Psi^{(2m)}[0,0,0] \qquad (A-22)$$

Where $\Psi^{(2m)}[0,0,0]$ refers to $\Psi^{(2m)}$ evaluated at the sphere center, or $\Psi^{(2m)}[x_o,0,0]$ which refers to the same position except that wedge centered coordinates are used to express location.

APPENDIX B

Comparison with the exact Bipolar Coordinate Solution

As was indicated by equations(57) and (58) the first order solution for two spheres in space is:

$$Q = 4\ ak(T_1 - T_{amb})\ [\ 1 - a/2x_\circ\ ]$$

This type of geometry is identical to the solution of Laplace's equation in bipolar coordinates . The comparison with bipolar coordinates shows that the truncation of higher order terms in the reflection series leaves an error This error approaches zero as the higher order terms of the reflection series become less significant. The first order correction solution will approach the bipolar coordinate solution as $a/x_\circ$ approaches very small numbers.This result is similar to the effect of linearizing a power series by the truncation of terms higher than order 2 and limiting the argument to small values. The first order correction appears to be consistent with the bipolar solution within computer accuracy. The comparison is shown in Table 1 . The computer program from which this comparison was derived follows table 1.

TABLE 1

A comparison with the exact Bipolar coordinate solution

a/x_o	Q - 1st order	Q -Bipolar	% Error
.1000	.9523866	.9500000	2.5×10^{-1}
.0100	.9950249	.9950000	2.5×10^{-3}
.0010	.9995002	.9995000	2.5×10^{-5}
.0001	.9999500	.9999500	2.5×10^{-7}

```
$JOB              DAVID HORWAT
C       DAVID HORWAT ***THESIS***
C       BIPOLAR SOLUTION AND COMPARISON
        IMPLICIT REAL*8(A-H,O-Z)
 8888    FORMAT ('1')
        PRINT 8888
        DO 57 LG = 1,9
        READ , AX
        N = -1
        SUM = 0.
        SERIES = 1.
        Y = AX/2.
  51    N = N+1
        RN = N
        BO =DLOG(1./AX+DSQRT(AX**(-2) - 1. ))
        TERM =DEXP ((RN+.500)*(-BO))/COSH((RN+.500)*BO)
        SUM = SUM+TERM
        IF(TERM - 1.0E-30) 52,51,51
  52    SUM =DSQRT(AX**(-2) - 1.)*SUM
        K = 1
        SERIES = SERIES + (Y**K)*(-1.**K)
        ERROR = SUM - SERIES
        PERCNT = ERROR/SUM*100.
        PRINT 53
  53    FORMAT ('0',8X,'A/X VALUE',9X,'SUM',12X,'SERIES' ,
     $ 12X,' % ERROR ')
        ERROR = PERCNT
        PRINT 59 ,LG,AX,SUM,SERIES,ERROR
  59    FORMAT (' ',I5,6(E15.7,2X))
        Z =(1. - SUM   )/AX
        PRINT 54 , Z
        ZZ = .5
        PRINT 55,ZZ
  54    FORMAT ('0 THE K VALUE OF THE BIPOLAR COORDINATE SOLUTION IS : ',F8.6
     $ F8.6)
  55    FORMAT('  THE K VALUE OF OUR FIRST REFLECTION IS: ',F8.6 )
        ERR = Z-ZZ
        IF (Z) 58,57,58
  58    PCNT = ERR/Z*100.
        PRINT 56 , ERR,PCNT
  56    FORMAT (' THE ERROR BETWEEN K VALUES IS: ',F9.5,3X,'THE PERCENT ERR
     $RROR IS: ',F9.6)
  57    CONTINUE
        PRINT 8888
        STOP
        END

        REAL FUNCTION COSH*8(Z)
        IMPLICIT REAL*8(A-H,O-Z)




        COSH = ( DEXP(Z) + DEXP(-Z))*.500000000
        RETURN
        END
```

	A/X VALUE	SUM	SERIES	% ERROR
1	0.1000000D 00	0.9523866D 00	0.9500000D 00	0.2505951D 00

THE K VALUE OF THE BIPOLAR COORDINATE SOLUTION IS : 0.476134
THE K VALUE OF OUR FIRST REFLECTION IS: 0.500000
THE ERROR BETWEEN K VALUES IS: -0.02387 THE PERCENT ERROR IS: -5.012530

	A/X VALUE	SUM	SERIES	% ERROR
2	0.1000000D-01	0.9950249D 00	0.9950000D 00	0.2500062D-02

THE K VALUE OF THE BIPOLAR COORDINATE SOLUTION IS : 0.497512
THE K VALUE OF OUR FIRST REFLECTION IS: 0.500000
THE ERROR BETWEEN K VALUES IS: -0.00249 THE PERCENT ERROR IS: -0.500013

	A/X VALUE	SUM	SERIES	% ERROR
3	0.1000000D-02	0.9995002D 00	0.9995000D 00	0.2500001D-04

THE K VALUE OF THE BIPOLAR COORDINATE SOLUTION IS : 0.499750
THE K VALUE OF OUR FIRST REFLECTION IS: 0.500000
THE ERROR BETWEEN K VALUES IS: -0.00025 THE PERCENT ERROR IS: -0.050000

	A/X VALUE	SUM	SERIES	% ERROR
4	0.1000000D-03	0.9999500D 00	0.9999500D 00	0.2500000D-06

THE K VALUE OF THE BIPOLAR COORDINATE SOLUTION IS : 0.499975
THE K VALUE OF OUR FIRST REFLECTION IS: 0.500000
THE ERROR BETWEEN K VALUES IS: -0.00002 THE PERCENT ERROR IS: -0.005000

	A/X VALUE	SUM	SERIES	% ERROR
5	0.1000000D-04	0.9999950D 00	0.9999950D 00	0.2500009D-08

THE K VALUE OF THE BIPOLAR COORDINATE SOLUTION IS : 0.499998
THE K VALUE OF OUR FIRST REFLECTION IS: 0.500000
THE ERROR BETWEEN K VALUES IS: -0.00000 THE PERCENT ERROR IS: -0.000500

	A/X VALUE	SUM	SERIES	% ERROR
6	0.1000000D-05	0.9999995D 00	0.9999995D 00	0.2495366D-10

THE K VALUE OF THE BIPOLAR COORDINATE SOLUTION IS : 0.500000
THE K VALUE OF OUR FIRST REFLECTION IS: 0.500000
THE ERROR BETWEEN K VALUES IS: -0.00000 THE PERCENT ERROR IS: -0.000050

	A/X VALUE	SUM	SERIES	% ERROR
7	0.1000000D-06	0.1000000D 01	0.1000000D 01	0.4468648D-12

THE K VALUE OF THE BIPOLAR COORDINATE SOLUTION IS : 0.500000
THE K VALUE OF OUR FIRST REFLECTION IS: 0.500000
THE ERROR BETWEEN K VALUES IS: -0.00000 THE PERCENT ERROR IS: -0.000009

	A/X VALUE	SUM	SERIES	% ERROR
8	0.1000000D-07	0.1000000D 01	0.1000000D 01	0.3247402D-12

THE K VALUE OF THE BIPOLAR COORDINATE SOLUTION IS : 0.500000
THE K VALUE OF OUR FIRST REFLECTION IS: 0.500000
THE ERROR BETWEEN K VALUES IS: -0.00000 THE PERCENT ERROR IS: -0.000065

	A/X VALUE	SUM	SERIES	% ERROR
9	0.1000000D-08	0.1000000D 01	0.1000000D 01	0.5509482D-12

THE K VALUE OF THE BIPOLAR COORDINATE SOLUTION IS : 0.499994
THE K VALUE OF OUR FIRST REFLECTION IS: 0.500000

APPENDIX C

A comparison with a toroidal coordinate solution

A third comparison can exist for which the accurate and exacting closed form solution to Laplace's equation are known. A large number of spheres, all tangent to each other can be used to approximate a toroid. For N spheres touching, as shown in figure 5 , the a/x_o value is related to the number of spheres ,N, by:

$$[\,a/x_o\,]_{max.} = \sin(\pi/N)$$

To be larger than this value of a/x_o, would imply the crushing of spheres into each other.

Comparing the results for a first order correction and the toroidal solution, an immense error is noted which grows with an increase in the number of spheres. These results are depicted in Table 2 . This error is due to the concentrated nature of this system. When spheres tend to touch each other the higher order terms are extremely significant and their truncation leads to a large error. For proper accuracy:

$$[a/x_o] \,/\, [a/x_o]_{max.} \ll 1$$

and;

$$a/x_o \int_0^{\infty} \frac{\sinh[(\{N-1\}/N)\pi\tau]\; d\tau}{\sinh\,[\tau\theta/N]\,\cosh(\pi\tau)} < 1$$

The accuracy of the first order correction is dependent upon the a/x_o values, and until the higher order terms of this reflection series are developed or until accurate closed form solutions to Laplace's equation are developed for spheres in regular polygonal arrays a precise and accurate error analysis is impossible. However combining the results of the computer comparisons with bipolar and toroidal coordinates one can speculate that the percentage of error might be of the form:

$$\%\ \text{error} = 25[a/x_o\ /\ (a/x_o)_{\text{max.}}]^2$$

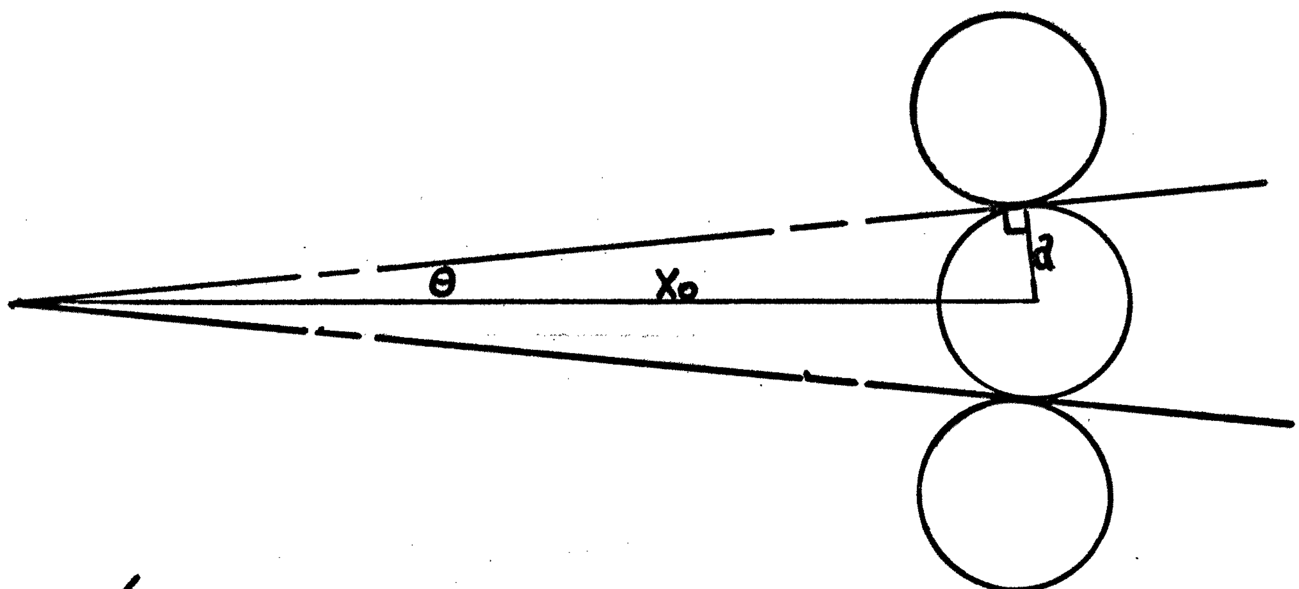
The computer program from which the data in table 2 is derived follows table 2 .

TABLE 2

A comparison with a toroidal coordinate solution

Spheres	Q - 1st order	Q - toroidal	% error
4	.7071066	.5099413	169.3 %
10	.3090169	.3243985	527.6 %
50	.0627904	.2070628	1565. %
1000	.00314158	.1275128	4833. %

Tangency of N Spheres



$a/X_0 = \sin(\theta)$

$\theta = \pi/N$

$(a/X_0)_{max} = \sin(\pi/N)$

figure 5

```
*** WATFIV VERSION 1.3 *** JOB=002    DAVID HORWAT                    74/148 21:39:09 *** WATFIV ***
```


```
*** WATFIV VERSION 1.3 *** JOB=002    DAVID HORWAT                    74/148 21:39:09 *** WATFIV ***

*** WATFIV VERSION 1.3 *** JOB=002    DAVID HORWAT                    74/148 21:39:09 *** WATFIV ***

*** WATFIV VERSION 1.3 *** JOB=002    DAVID HORWAT                    74/148 21:39:09 *** WATFIV ***
```


```
$JOB             DAVID HORWAT
C       DAVID HORWAT   ***THESIS***
C       TOROIDAL SOLUTION VS. FIRST REFLECTION
        COMMON N,Z
        PRINT 1000
        READ, RNUM
        PI = 3.1415926535
        AX = SIN(PI/RNUM)
        Z = 1./AX
        FACTOR = 2./PI*SQRT(Z**2 -1.)
        SUM = 0.
        N = -1
        E = 1.
 61     N = N+1
        RN = N
        QA = Q(RN-.5000)
        PA = P(RN-.500000   )
        TERM = E*QA/PA
        PB = P ( RN + .50000)
        QB = Q ( RN + .500000)
        CHECK = PB * QA - QB * PA
        THEO = 1./ ( RN + .50000)
        ER = CHECK - THEO
        PCER = ER / THEO * 100.
        PRINT 1000
        PRINT 802 , CHECK,THEO,ER,PCER
 802    FORMAT(' WRONSKIAN ACTUAL ',E15.7,'WRONSKIAN THEORETICAL ',E15.7
     $ ,// '  ERROR',E15.7,'  PERCENT ERROR ', E15.7)
        PRINT 1000
        E = 2.
        SUM = SUM + TERM
        IF ( TERM - 1.0E-30 ) 62,61,61
 62     SUM = SUM*FACTOR/RNUM
        NUM = 0.000
        N = RNUM
        DELTA = .00001
        X2 = 0.
        FX0 = FLOAT(N-1)
        INDEX = 0.00
        AREA = 0.000
 1      CONTINUE
        X0 = X2
        X1 = X0 + DELTA
        X2 = X1 + DELTA
        IF(INDEX) 5,6,5
 5      FX0 = FX2
 6      FX1 = FUNC(X1)
        FX2 = FUNC(X2)
        INDEX = INDEX + 1

        TERM = (FX0 + 4.*FX1 + FX2)/3.*DELTA
        AREA = AREA + TERM
        DELTA = DELTA*1.25
```

```
      IF ( TERM - 1.0E-8) 2,2,1
2     PRINT 1000
1000  FORMAT (' ')
      PRINT 3,N,AREA,TERM
3     FORMAT ( ' THE NUMBER OF SPHERES IS: ',I4,' THE INTEGRAL IS:',F15
     $.8,' TOLERANCE',E15.8)
      SERIES = 1.0
      Y = AX*AREA
      K = 1
      SERIES = SERIES + (Y**K)*(-1.**K)
      ERROR = SUM - SERIES
      PERCNT = ERROR/SUM*100.
53    PRINT , K,AX,SUM,SERIES,ERROR,PERCNT
      Z =(1. - SUM   )/AX
      PRINT 54 , Z
      ZZ = AREA
      PRINT 55,ZZ
54    FORMAT ('0 THE K VALUE OF THE TOROID  COORDINATE SOLUTION IS : ',F8.6)
     $ E 15.6)
55    FORMAT('  THE K VALUE OF OUR FIRST REFLECTION IS: ',E 15.6 )
      ERR = Z-ZZ
      PCNT = ERR/Z*100.
57    PRINT 56 , ERR,PCNT
56    FORMAT (' THE ERROR BETWEEN K VALUES IS: ',F8.3,3X,'THE PERCENT ERR
     $RROR IS: ',E 15.6 )
      STOP
      END

      FUNCTION FUNC (X)
      COMMON N , QAX
      Z =N
      ARG = 3.1415926535*X/FLOAT(N)
      Y = ARG
      FUNC = TANH(Z*Y)/TANH(Y)-1.
      RETURN
      END

      FUNCTION TANH(X)
      IF(X-25.)2,2,3
2     TANH = (EXP(X)-EXP(-X))/(EXP(X)+EXP(-X))
      GO TO 4
3     TANH = 1.
4     RETURN
      END

      FUNCTION FACTN(MK)
      K = 1
      IF (MK) 2,2,3
3     DO 1 L = 1,MK
1     K = K*L
2     FACTN = K
      RETURN
      END

      FUNCTION PSI1(K)
      SUM = -.57721566
      IF(K) 2,2,3




3     DO 1 L = 1,K
1     SUM = SUM + 1./FLOAT(L)
```

```
      2     PSI1 = SUM
            RETURN
            END

            FUNCTION PSI2(K)
            SUM2 = 0.0
            SUM = -.57721566 - 2.*ALOG(2.)
            IF ( K ) 2,2,3
      3     KA = 2*K-1
            DO 1 L = 1,KA, 2
      1     SUM2 = SUM2 + 1./FLOAT(L)
      2     PSI2 = SUM + 2.*SUM2
            RETURN
            END

            FUNCTION Q(X)
            COMMON N,Z
            A = N
            RN = N
            NRN = X + .51
            N = NRN
            RN = NRN
            DELTA =ALOG(Z+SQRT(Z*Z-1.))
            FACTOR = 3.1415926535       *EXP(-DELTA*(RN+.5000))
            K = -1
            SUM = 0.
      1     K = K +1
            RK = K
            NUM = 1
            NQ = 2*N+2*K-1
            IF (NQ) 5,5,7
      7     DO 3 L = 1,NQ,2
      3     NUM = NUM * L
      5     NZ = 2*K-1
            IF (NZ) 8,8,9
      9     DO 4 L = 1,NZ,2
      4     NUM = NUM*L
      8     TERM = FLOAT(NUM)/(2.**(N+2*K))/FACTN(N+K)/FACTN(K)*EXP(-2.*RK*DELTA)
            LTA)
            SUM = SUM + TERM
            IF ( TERM - 1.0E-10) 2,1,1
       2    Q = SUM*FACTOR
            PRINT 99 , N,Z,Q
     99     FORMAT('    Q ',I2,'-1/2(',F6.1,') = ',E12.6)
            N = A
            RETURN
            END

            FUNCTION P(X)
            COMMON N,Z
            SUM1 = 0.00
            A = N
            RN2 = N
            NRN2 = X + .51
            N = NRN2
            RN2 = NRN2
            DELTA =ALOG(Z+SQRT(Z*Z-1.))
            FCTOR2 = EXP(-DELTA*(RN2+.5000))/3.14159265

            IF(N) 1,2,1
      1     CONTINUE
            FACTOR =        EXP(-DELTA*(RN2...
```

```
      DO 3 JG = 1,N
      K = JG-1
      RK = K
      DEN = 1.
      IA = 2*K+1
      IB = 2*N-2*K-1
      IF ( IB ) 12,12,11
 11   DO 6 L = IA,IB,2
 6    DEN = DEN*FLOAT(L)
 12   TERM = FACTN(N-K-1)/FACTN(K)*EXP(-2.*RK*DELTA)*2.**(N-2*K)/DEN
 3    SUM1 = SUM1 + TERM
      SUM1 = SUM1*FACTOR
 2    SUM2 = 0.00
      K = -1
 4    K=K+1
      RK = K
      TERM2 = EXP(-2.*RK*DELTA)*(2.*DELTA+PSI1(K)-PSI2(K)+PSI1(K+N)-PSI
     $ 2(K+N))
      NUM = 1
      IC = 2*N + 2*K-1
      IF ( IC ) 13,13,10
 10   DO 7 L = 1,IC,2
 7    NUM = NUM*L
 13   ID = 2*K-1
      IF ( ID ) 14,14,15
 15   DO 8 L = 1,ID,2
 8    NUM = NUM*L
 14   TERM = FLOAT(NUM)/FACTN(K+N)/FACTN(K)/2.**(N+2*K)*TERM2
      SUM2 = SUM2 + TERM
      IF ( TERM - 1.0E-10) 5,4,4
 5    SUM2 = SUM2*FCTOR2
      P = SUM1 + SUM2
      PRINT 9,N,Z,P
 9    FORMAT('    P ',I2,'-1/2(',F6.1,') = ',E12.6)
      N = A
      RETURN
      END

$ENTRY

 Q  0-1/2( 318.3) = 0.124512E 00
 P  0-1/2( 318.3) = 0.197876E 00
 P  1-1/2( 318.3) = 0.160629E 02
 Q  1-1/2( 318.3) = 0.977915E-04

WRONSKIAN ACTUAL   0.1999993E 01WRONSKIAN THEORETICAL   0.2000000E 01

 ERROR -0.6675720E-05  PERCENT ERROR  -0.3337860E-03

 Q  1-1/2( 318.3) = 0.977915E-04
 P  1-1/2( 318.3) = 0.160629E 02
 P  2-1/2( 318.3) = 0.681723E 04
 Q  2-1/2( 318.3) = 0.115208E-06

WRONSKIAN ACTUAL   0.6666647E 00WRONSKIAN THEORETICAL   0.6666666E 00

 ERROR -0.1966953E-05  PERCENT ERROR  -0.2950430E-03
```

Q 2-1/2(318.3) = 0.115208E-06

```
  P  2-1/2( 318.3) = 0.681723E 04
  P  3-1/2( 318.3) = 0.347198E 07
  Q  3-1/2( 318.3) = 0.150809E-09

WRONSKIAN ACTUAL   0.3999990E 00WRONSKIAN THEORETICAL   0.4000000E 00

 ERROR -0.9536743E-06  PERCENT ERROR  -0.2384186E-03

  Q  3-1/2( 318.3) = 0.150809E-09
  P  3-1/2( 318.3) = 0.347198E 07
  P  4-1/2( 318.3) = 0.189455E 10
  Q  4-1/2( 318.3) = 0.207278E-12

WRONSKIAN ACTUAL   0.2857141E 00WRONSKIAN THEORETICAL   0.2857143E 00

 ERROR -0.1788139E-06  PERCENT ERROR  -0.6258488E-04

  Q  4-1/2( 318.3) = 0.207278E-12
  P  4-1/2( 318.3) = 0.189455E 10
  P  5-1/2( 318.3) = 0.107210E 13
  Q  5-1/2( 318.3) = 0.293035E-15

WRONSKIAN ACTUAL   0.2222220E 00WRONSKIAN THEORETICAL   0.2222222E 00

 ERROR -0.2384186E-06  PERCENT ERROR  -0.1072884E-03

  Q  5-1/2( 318.3) = 0.293035E-15
  P  5-1/2( 318.3) = 0.107210E 13
  P  6-1/2( 318.3) = 0.620466E 15
  Q  6-1/2( 318.3) = 0.421939E-18

WRONSKIAN ACTUAL   0.1818179E 00WRONSKIAN THEORETICAL   0.1818181E 00

 ERROR -0.2384186E-06  PERCENT ERROR  -0.1311302E-03

  Q  6-1/2( 318.3) = 0.421939E-18
  P  6-1/2( 318.3) = 0.620466E 15
  P  7-1/2( 318.3) = 0.364618E 18
  Q  7-1/2( 318.3) = 0.615436E-21

WRONSKIAN ACTUAL   0.1538460E 00WRONSKIAN THEORETICAL   0.1538461E 00

 ERROR -0.1788139E-06  PERCENT ERROR  -0.1162291E-03


THE NUMBER OF SPHERES IS: 1000 THE INTEGRAL IS:  2239.59000000 TOLERANCE 0.00000000E 00
           1    0.3141587E-02    0.1275128E 00   -0.6035866E 01    0.6163378E 01    0.4833535E 04

 THE K VALUE OF THE TOROID  COORDINATE SOLUTION IS :    0.277722E 03
 THE K VALUE OF OUR FIRST REFLECTION IS:    0.223959E 04
THE ERROR BETWEEN K VALUES IS: ********   THE PERCENT ERROR IS:   -0.706415E 03

CORE USAGE       OBJECT CODE=    9912 BYTES,ARRAY AREA=      60 BYTES,TOTAL AREA AVAILABLE=   63584  BYTES

DIAGNOSTICS        NUMBER OF ERRORS=       0, NUMBER OF WARNINGS=       0, NUMBER OF EXTENSIONS=       0

COMPILE TIME=     0.59 SEC,EXECUTION TIME=     0.50 SEC,  WATFIV - VERSION 1 LEVEL 3 MARCH  1971      DATE=  74/148
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