SIDEWISE FORCE EXERTED ON SLOWLY FALLING SPHERES INSIDE A CIRCULAR CYLINDER

BY

GARY MARK GREENSTEIN

A THESIS

PRESENTED IN PARTIAL FULFILLMENT OF

THE REQUIREMENTS FOR THE DEGREE

OF

MASTER OF SCIENCE IN CHEMICAL ENGINEERING

AT

NEWARK COLLEGE OF ENGINEERING

Newark, New Jersey

September, 1974

APPROVAL OF THESIS

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DEPARTMENT OF CHEMICAL ENGINEERING

NEWARK COLLEGE OF ENGINEERING

BY

FACULTY COMMITTEE

APPROVED: ______________________

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NEWARK, NEW JERSEY

SEPTEMBER, 1974

ACKNOWLEDGEMENTS

I wish to thank the Exxon Research and Engineering Company for the use of its Computer Center in carrying out the numerical evaluations described in this work. I am also most grateful to Dr. T. Greenstein for his advice and constant support. I would also like to thank my devoted wife, Joan, for her patience and encouragement, and for typing this manuscript.

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ABSTRACT

Numerical values are provided for the forces that must be applied to a reference sphere to prevent its sidewise motion when two equal-sized spheres settle through a viscous liquid bounded by a cylindrical tube. These values are presented for two independent set of circumstances:

(1) - the spheres are translating but not rotating

(2) - the spheres are rotating but not translating

Forces have been calculated assuming various distances between the sphere centers and with the line of centers at various angles to the horizontal. The results are discussed for the case where the spheres are both translating and rotating and compared to previous theoretical predictions as well as experimental results.

NOMENCLATURE

a	sphere radius
b	distance from sphere center to cylinder axis
$E(\eta,\beta)$	dimensionless eccentricity function defined by eq. (3.4)
$E(\psi,\beta)$	dimensionless eccentricity function defined by eq. (3.2)
$\vec{F}$	frictional force on particle
$\vec{F}_x$	component of frictional force on particle in sidewise (or radial) direction
$\vec{F}_z$	component of frictional force on particle in z-direction
$\vec{i},\vec{j},\vec{k}$	Cartesian unit vectors
$l(\eta,\beta)$	dimensionless eccentricity function defined in Section 3
$l(\psi,\beta)$	dimensionless eccentricity function defined in Section 3
$N(\eta,\beta)$	dimensionless eccentricity function defined by eq. (4.4)
$N(\psi,\beta)$	dimensionless eccentricity function defined by eq. (4.2)
$n(\eta,\beta)$	dimensionless eccentricity function defined in Section 4
$n(\psi,\beta)$	dimensionless eccentricity function defined in Section
p	dynamic pressure
r	distance between sphere centers when the angle between the line joining their centers and the horizontal is other than 90°
R_o	radius of circular cylinder
$\vec{U},U$	translational velocity and speed, respectively, of sphere center relative to cylinder wall
$\vec{V}$	local fluid velocity
x,y,z	rectangular Cartesian coordinates
z_1	vertical distance between the sphere centers when the angle between the line joining their centers and the horizontal is 90°

β dimensionless eccentricity, b/R_o

ϵ angle from the vertical of the path of the spheres

η dimensionless distance between spheres when the angle between the line joining their centers and the horizontal is 90^o, z_1/R_o

μ fluid viscosity

ψ angle between line joining the sphere centers and the horizontal

$\vec{\Omega} = \vec{i}\Omega_1 + \vec{j}\Omega_2 + \vec{k}\Omega_3$ angular particle velocity

SUPERSCRIPTS

$11S_1$, $12C$, $13S_1$, $23S_1$, $21S_2$, $22C$, $22S_1$ definition follows in Section 2; see also figure 3

1. INTRODUCTION

Prediction of the frictional force in a radial direction on one of two identical particles settling at small Reynolds numbers through a bounded, quiescent, viscous fluid finds application in the fields of rheology and biomedical engineering. Early work by Smoluchowski(1) identified the radial migration effect when two spheres fall in a viscous fluid by utilizing the "method of reflections", but assuming an unbounded medium. The "method of reflections" has been described in detail by Happel and Brenner(2).

A recent investigation by Greenstein and Happel(3) broadened Smoluchowski's work by including wall effects in analyzing the two-sphere problem using the "method of reflections". The latter study, however, was primarily concerned with the frictional forces on the two spheres in their direction of motion. Numerical values for the sidewise forces were not given in their work.

In this work, we have concentrated solely on the frictional forces in the sidewise or radial direction. We have used the "method of reflections" and included wall effects to evaluate numerically the analytical expressions for these sidewise forces.

2. DESCRIPTION OF THE PROBLEM

Let us consider the slow translation or rotation of two equally sized spherical particles in a viscous incompressible fluid confined within an infinitely long circular cylindrical tube. Two independent sets of circumstances were investigated:

(1) the spheres move with arbitrary constant translational velocity ($\vec{U} = -\vec{k}U$) but do not rotate

(2) the spheres rotate with arbitrary constant angular velocity ($\vec{\Omega} = j\Omega_2$) but do not translate

Figures 1 and 2 illustrate the geometric configurations studied. In Figure 1, the spheres are situated at an angle ϕ between the line of centers and the horizontal. The radius of each sphere is a, the cylinder radius is R_o, and the centers of Sphere-1 and Sphere-2 are situated at distances b from the cylinder axis. The distance between the centers of Spheres 1 and 2 is r.

In Figure 2, the spheres are located in the same vertical plane and the sphere centers are displaced from the cylindrical axis by a distance b. Again, the radius of each sphere is a and the cylinder radius is R_o. The spheres are separated by a vertical distance, z_1.

If we consider only translation, we say that the center of each sphere translates with velocity $\vec{U}$ relative to the cylinder wall in the negative z direction, parallel to the cylinder axis ($\vec{U} = -\vec{k}U$). If we consider only rotation, we say that each sphere rotates with angular velocity $\vec{\Omega}$ relative to the cylinder wall about axes parallel to the y-coordinate axis ($\vec{\Omega} = j\Omega_2$). In both cases, at $|z| = \infty$, the fluid is at rest.

The translation and rotation problems can be solved independently, assuming that the fluid motion is governed by the creeping motion and continuity equations:

$$\mu \nabla^2 \vec{V} = \nabla p \qquad (2.1)$$

$$\nabla \cdot \vec{V} = 0 \qquad (2.2)$$

where $\vec{V}$ is the fluid velocity with respect to a coordinate system that moves with Sphere-1, p is the dynamic pressure, and μ the fluid viscosity. The boundary conditions necessary to solve these problems are:

(1) at fluid -solid interfaces there is no relative motion

(2) at $|z| = \infty$ the fluid is at rest

Both of these boundary value problems can be solved by a technique of successive approximations known as the method of reflections (described by Happel and Brenner (2)). Figure 3 outlines a schematic representation of this calculation technique. An initial disturbance is reflected from the boundaries involved and produces smaller and smaller effects with each successive reflection.

The following superscript notation has been adopted to describe the velocity fields

$\vec{V}^{ijS_i}$ and $\vec{V}^{ijC}$ (3) :

i = the sphere at which a disturbance originates

j = the number of times that reflections have occurred to produce this disturbance

S_1 = the latest disturbance reflected from Sphere-1

S_2 = the latest disturbance reflected from Sphere-2

C = the latest disturbance produced by the cylinder wall

As shown in Figure 3, Sphere-2 will disturb the motion of Sphere-1 in two ways:

1) by a direct reflection of its own Stokes field, $\vec{V}^{21S_2}$

2) by the reflection of this field from the cylinder wall and then to Sphere-1, $\vec{V}^{22C}$

Since the equations of motion and boundary conditions are linear, the frictional force $\vec{F}_{II}$ exerted on the reference sphere (Sphere-1) is obtained by adding the frictional forces resulting from each of the individual fields. Hence,

$$\vec{F}_{II} = \vec{F}^{11S_1} + \vec{F}^{13S_1} + \vec{F}^{22S_1} + \vec{F}^{23S_1} + \ldots$$

where $\vec{F}^{ijS_1}$ is the frictional force associated with the field $\vec{V}^{ijS_1}$. This work has considered only the component of $\vec{F}_{II}$ in the radial direction perpendicular to the cylindrical axis, i.e., $\vec{F}_x$.

3. RESULTS FOR THE SIDEWISE FORCE EXERTED ON ONE OF TWO SLOWLY TRANSLATING SPHERES INSIDE A CIRCULAR CYLINDER

If the spheres are only translating, let us consider the system shown in Figure 1. Two equally sized spheres situated at an angle ψ between the line of centers and the horizontal settle through a quiescent fluid in a direction parallel to the cylindrical axis. The frictional force in the radial direction $\vec{F}_x^{23S_1}$, exerted by the fluid on the spheres, due to the wall-sphere interaction effect is given by

$$\vec{F}_x^{23S_1} = \vec{1}6\pi\mu aUE(\psi,\beta)\frac{a}{R_o} + O(\frac{a}{R_o})^3 \qquad (3.1)$$

where

$$E(\psi,\beta) = \frac{3}{2\pi}\, l(\psi,\beta) \qquad (3.2)$$

and ψ is the angle between the line connecting the sphere centers and the horizontal, and β equals b/R_o.

If we consider the system shown in Figure 2, the two spheres are located in the same vertical plane, separated by a distance z_1, and displaced from the cylindrical axis by a given distance as they settle through a quiescent fluid in a direction parallel to the cylindrical axis. In this case, the frictional force in the radial direction $\vec{F}_x^{23S_1}$ due to the wall-sphere interaction effect is given by

$$\vec{F}_x^{23S_1} = \vec{1}6\pi\mu aUE(\eta,\beta)\frac{a}{R_o} + O(\frac{a}{R_o})^3 \qquad (3.3)$$

where

$$E(\eta,\beta) = \frac{3}{2\pi}\, l(\eta,\beta) \qquad (3.4)$$

and $\eta = z_1/R_o$, and $\beta = b/R_o$

Note that the functions $l(\psi,\beta)$ and $l(\eta,\beta)$, as well as $E(\psi,\beta)$ and $E(\eta,\beta)$, are comparable and differ only in the manner in which the first parameter is expressed. The first parameter is necessary to fully define the location of the spheres.

A detailed expression for $l(\psi,\beta)$ or $l(\eta,\beta)$ was derived based upon Greenstein's[4] expression for the $\vec{V}^{22C}$ velocity field. $l(\psi,\beta)$ or $l(\eta,\beta)$ was found to be a complicated function of (ψ,β) or (η,β) involving modified Bessel functions of the first and second kind and requiring numerical evaluation. A high speed computer was employed to accomplish this numerical evaluation and the program is shown in Appendix 1.

Values of $E(\psi,\beta)$ and $l(\psi,\beta)$ vs. β are tabulated in Table 1 for values of ψ equal to 30°, 45°, and 60°. Similarly, values of $E(\eta,\beta)$ and $l(\eta,\beta)$ vs. β are tabulated in Table 2 for values on η ranging from 0.00001 to 20.

Based on previous work by Greenstein[4], it can be shown that, if the spheres are translating parallel to the cylinder axis only:

$$\vec{F}_x^{11S_1} = \vec{F}_x^{13S_1} = \vec{0} \qquad (3.5)$$

$\vec{F}_x^{22S_1}$ is also equal to $\vec{0}$ for the system shown in Figure 2. However, for the system shown in Figure 1, it can be shown that:

$$\vec{F}_x^{22S_1} = \vec{i}6\pi\mu a\left(-\frac{3aU}{8b}\cos^2\psi\sin\psi\right) \qquad (3.6)$$

This expression becomes for values of $\psi = 30^\circ$, 45° and 60°:

$$\begin{array}{ll} \underline{\psi} & \\ 30^\circ & \vec{F}_x^{22S_1} = \vec{i}6\pi\mu a\left(-0.140625\frac{aU}{b}\right) \\ 45^\circ & \vec{F}_x^{22S_1} = \vec{i}6\pi\mu a\left(-0.132582\frac{aU}{b}\right) \\ 60^\circ & \vec{F}_x^{22S_1} = \vec{i}6\pi\mu a\left(-0.0811898\frac{aU}{b}\right) \end{array} \qquad (3.7)$$

The total sidewise force exerted by the fluid on the reference sphere for the system described by Figure 1 can be computed as the sum of $\vec{F}_x^{22S_1}$ and $\vec{F}_x^{23S_1}$:

$$\begin{aligned} \vec{F}_x &= \vec{F}_x^{22S_1} + \vec{F}_x^{23S_1} \\ &= \vec{i}6\pi\mu a\left(-\frac{3}{8}\frac{aU}{b}\cos^2\psi\sin\psi + UE(\psi,\beta)\frac{a}{R_o}\right) + O\left(\frac{a}{R_o}\right)^3 \end{aligned} \qquad (3.8)$$

Values of $E(\psi,\beta)$ can be obtained from Table 1.

The total sidewise force exerted by the fluid on the reference sphere for the system shown in Figure 2 is $\vec{F}_x^{23S_1}$ (given by equation (3.3)). Values of $E(\eta,\beta)$ can be obtained from Table 2.

These sidewise forces cause radial migration when the spheres are freely suspended in the fluid. To prevent the spheres from moving sidewise, a force, $-\vec{F}_x$, equal in magnitude but opposite in direction to $\vec{F}_x$ would have to be exerted on the spheres. Forces resulting from successive reflections have been neglected, since they are of the same order of magnitude as the error term (on the order of $(\frac{a}{R_o})^3$).

4. RESULTS FOR THE SIDEWISE FORCE EXERTED ON ONE OF TWO SLOWLY ROTATING SPHERES INSIDE A CIRCULAR CYLINDER

Now let us again consider the system in Figure 1, but this time with the spheres only rotating. Two equally sized spheres situated at an angle ψ between the line of centers and the horizontal rotate but do not translate in a quiescent fluid. The frictional force in the radial direction $\vec{F}_{x,r}{}^{23S_1}$ due to the wall-sphere interaction effect is given by

$$\vec{F}_{x,r}{}^{23S_1} = -\vec{1}8\pi\mu a^2\Omega_2\left(\frac{a}{R_o}\right)^2 N(\psi,\beta)+O\left(\frac{a}{R_o}\right)^4 \qquad (4.1)$$

where

$$N(\psi,\beta) = \frac{3}{2\pi}\, n(\psi,\beta) \qquad (4.2)$$

and ψ is the angle between the line connecting the sphere centers and the horizontal, and β equals b/R_o.

If we consider the system shown in Figure 2, the two spheres are located in the same vertical plane, separated by a given distance z_1 and displaced from the cylindrical axis by a given distance as they rotate in the y-direction (but do not translate) in a quiescent fluid. The frictional force in the radial direction $\vec{F}_{x,r}{}^{23S_1}$ due to the wall-sphere interaction effect can be expressed as

$$\vec{F}_{x,r}{}^{23S_1} = -\vec{1}8\pi\mu a^2\Omega_2\left(\frac{a}{R_o}\right)^2 N(\eta,\beta)+O\left(\frac{a}{R_o}\right)^4 \qquad (4.3)$$

where

$$N(\eta,\beta) = \frac{3}{2\pi}\, n(\eta,\beta) \qquad (4.4)$$

and $\eta = z_1/R_o$, and $\beta = b/R_o$.

The functions $n(\psi,\beta)$ and $n(\eta,\beta)$, and similarly $N(\psi,\beta)$ and $N(\eta,\beta)$, are directly comparable and differ only in the form of the first parameter necessary to define the location of the spheres.

A detailed expression for $n(\psi,\beta)$ or $n(\eta,\beta)$ was developed based upon Greenstein's[4] expression for the $\vec{V}^{22C}$ velocity field.

As in the translation problem, $n(\psi,\beta)$ or $n(\eta,\beta)$ was determined to be a complicated function of (ψ,β) or (η,β) involving modified Bessel functions of the first and second kind. This function was evaluated on a high speed computer using numerical techniques and the program used is shown in Appendix 2.

Values of $N(\psi,\beta)$ and $n(\psi,\beta)$ vs. β are tabulated in Table 3 at values of ψ equal to 30°, 45° and 60°. Values of $N(\eta,\beta)$ and $n(\eta,\beta)$ vs. β are tabulated in Table 4 for values of η ranging from 0.00001 to 20.

Based on previous work by Greenstein(4), it can be shown that, if the spheres are rotating only:

$$\vec{F}_x{}^{11S_1} = \vec{F}_x{}^{13S_1} = \vec{0} \qquad (4.5)$$

An expression for the values of $\vec{F}_{x,r}{}^{22S_1}$ can also be derived from Greenstein's work. For the geometric system described in Figure 1, this expression is:

$$\vec{F}_{x,r}{}^{22S_1} = \vec{i}6\pi\mu a\left(\frac{a^3\Omega_2}{4b^2}\cos^2\psi\sin\psi\right) \qquad (4.6)$$

which, if evaluated at values of ψ = 30°, 45° and 60°, becomes:

ψ		
30°	$\vec{F}_{x,r}{}^{22S_1} = \vec{i}6\pi\mu a\left(0.09375\ \frac{a^3\Omega_2}{b^2}\right)$	
45°	$\vec{F}_{x,r}{}^{22S_1} = \vec{i}6\pi\mu a\left(0.0883883\ \frac{a^3\Omega_2}{b^2}\right)$	(4.7)
60°	$\vec{F}_{x,r}{}^{22S_1} = \vec{i}6\pi\mu a\left(0.0541265\ \frac{a^3\Omega_2}{b^2}\right)$	

A similar expression for the system shown in Figure 2 is:

$$\vec{F}_{x,r}{}^{22S_1} = \vec{i}6\pi\mu a\left(\frac{a^3\Omega_2}{z_1{}^2}\right) \qquad (4.8)$$

indicating that this component of the sidewise force increases as the spheres are moved closer together.

The total sidewise force on a reference sphere for either of the geometries discussed is the sum of $\vec{F}_{x,r}{}^{22S_1}$ and $\vec{F}_{x,r}{}^{23S_1}$.

Forces resulting from successive reflections have been neglected, since they are of the same order of magnitude as the error term (on the order of $(\frac{a}{R_o})^4$).

5. QUALITATIVE DISCUSSION OF RESULTS WHEN SPHERES ARE TRANSLATING AND ROTATING

Let us examine in a qualitative manner the total sidewise force on one sphere when the spheres are translating and rotating. By comparing equation (3.1) with equation (4.1) and equation (3.3) with equation (4.3), it is clear that the value of $\vec{F}_{x,r}{}^{23}S_1$ due to rotation is much smaller than the value of $\vec{F}_x{}^{23}S_1$ due to translation. This is true because the value of a/R_o is quite small. As a matter of fact, $\vec{F}_{x,r}{}^{23}S_1$ due to rotation is of the order of magnitude $(\frac{a}{R_o})^2$, whereas the order of magnitude of $\vec{F}_x{}^{23}S_1$ due to translation is $(\frac{a}{R_o})$. Hence, when the spheres are translating and rotating, the value of $\vec{F}_{x,r}{}^{23}S_1$ due to rotation can be neglected in computing the total sidewise force.

The $\vec{F}_{x,r}{}^{22}S_1$ component of the sidewise force due to rotation however, cannot be neglected in computing the total sidewise force on the reference sphere when the spheres are translating and rotating. In particular, for the system shown in Figure 2, the value of $\vec{F}_x{}^{22}S_1$ due to translation is zero, whereas the value of $\vec{F}_{x,r}{}^{22}S_1$ due to rotation must be computed from equation (4.8). It should be noted, however, that the value of $\vec{F}_{x,r}{}^{22}S_1$ computed from equation (4.8) is of the order of $(\frac{a}{z_1})^2$ which is still an order of magnitude less than the value of $\vec{F}_x{}^{23}S_1$ due to translation. For the system described by Figure 1, the value of $\vec{F}_x{}^{22}S_1$ due to translation is of the order of $(\frac{a}{b})$ whereas the value of $\vec{F}_{x,r}{}^{22}S_1$ due to rotation is of the order of $(\frac{a}{b})^2$. Again, the sidewise component due to rotation is of a lower order of magnitude.

In summary, it can be shown that when two equally sized spheres situated at an angle ψ between the line of centers and the horizontal translate and rotate through a quiescent fluid, the sidewise force on the reference sphere is influenced much more strongly by the translation of the spheres than by the rotation of the spheres. This conclusion is based on a qualitative inspection of the orders of magnitude of the sidewise components of the frictional forces.

6. COMPARISON OF THEORY WITH OTHER THEORETICAL AND EXPERIMENTAL RESULTS

The problem of the motion of two spheres in an infinite medium was first investigated by Smoluchowski[1][5]. In his work, Smoluchowski used the method of reflections but did not consider wall effects. In addition, the spheres were considered to be only translating. Based on expressions derived for the component of force tending to diminish the resistance in the direction of motion along the line connecting the centers of the two spheres, Smoluchowski developed the relationship

$$\sin \epsilon = \frac{3}{4}\frac{a}{r}\left(1 - \frac{3}{2}\frac{a}{r}\right)\cos\psi\ \sin\psi \qquad (6.1)$$

for the angle ϵ through which the spheres would be deflected from the vertical towards the line of centers. In other words, the spheres would exhibit motion in a radial direction as well as in a vertical direction.

By not considering wall effects, Smoluchowski only considered those components of the frictional force referred to in this paper as $\vec{F}^{11S}1$ and $\vec{F}^{22S}1$. As shown in Figure 3, the other components of the frictional force on Sphere-1 result from wall-sphere interactions, which Smoluchowski did not consider. Equation (6.1) can be derived from Greenstein's[4] equation for the $\vec{V}^{21S}2$ velocity field by making a number of approximations. This derivation is shown in Appendix 3.

It should be noted that Smoluchowski would predict radial migration only for the system shown in Figure 1. For the geometry in Figure 2, Smoluchowski's ϵ would be zero. This is totally consistent with the derivation presented in Section 3 of this paper which states that $\vec{F}_x^{11S_1} = \vec{0}$ in both cases and that $\vec{F}_x^{22S_1}$ exists for the system shown in Figure 1 but is equal to $\vec{0}$ for the system in Figure 2.

From the results presented in this paper, the total sidewise force ($\vec{F}_x$) on the reference sphere can be calculated. This could be coupled with the appropriate value for the total vertical force ($\vec{F}_z$) on the sphere based on Greenstein's[4] expressions to obtain an angle of deflection, ϵ, which would include the effect of wall-sphere interactions. This would therefore extend Smoluchowski's work and result in more theoretically correct values of ϵ.

Eveson, Hall and Ward[6] carried out a series of experiments to study the motion of two spheres settling through a quiescent fluid bounded by a circular cylinder. When the spheres were positioned as in Figure 1, it was found that the paths of the spheres were deflected from the vertical by a small angle. The experimental values for these angles were stated to be in agreement with Smoluchowski's relationship (equation (6.1)), although numerical values were not published.

Since the values of ϵ tend to be quite small, the difference between equation (6.1) and the method proposed in this paper to obtain ϵ may have been beyond the accuracy of Eveson, Hall and Ward's experiments. This would explain the statement that experimental values are in agreement with Smoluchowski's relationship. Nevertheless, the importance of Eveson, Hall and Ward's experiments was to demonstrate that radial migration does occur when two spheres settle through a quiescent fluid bounded by a circular cylinder, as discussed in this paper.

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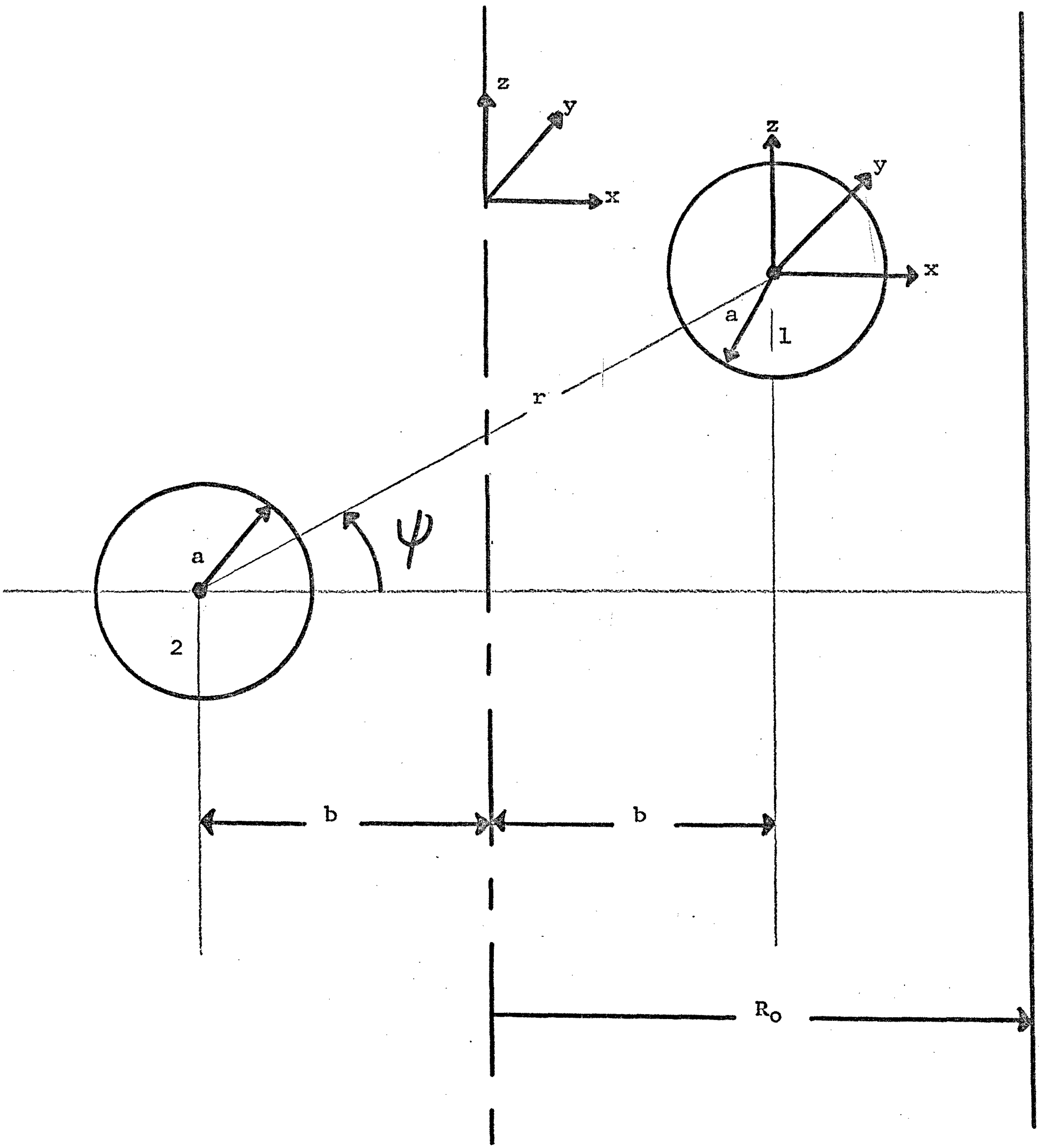

Figure 1 - Two spheres situated at an angle ψ between the line of centers and the horizontal

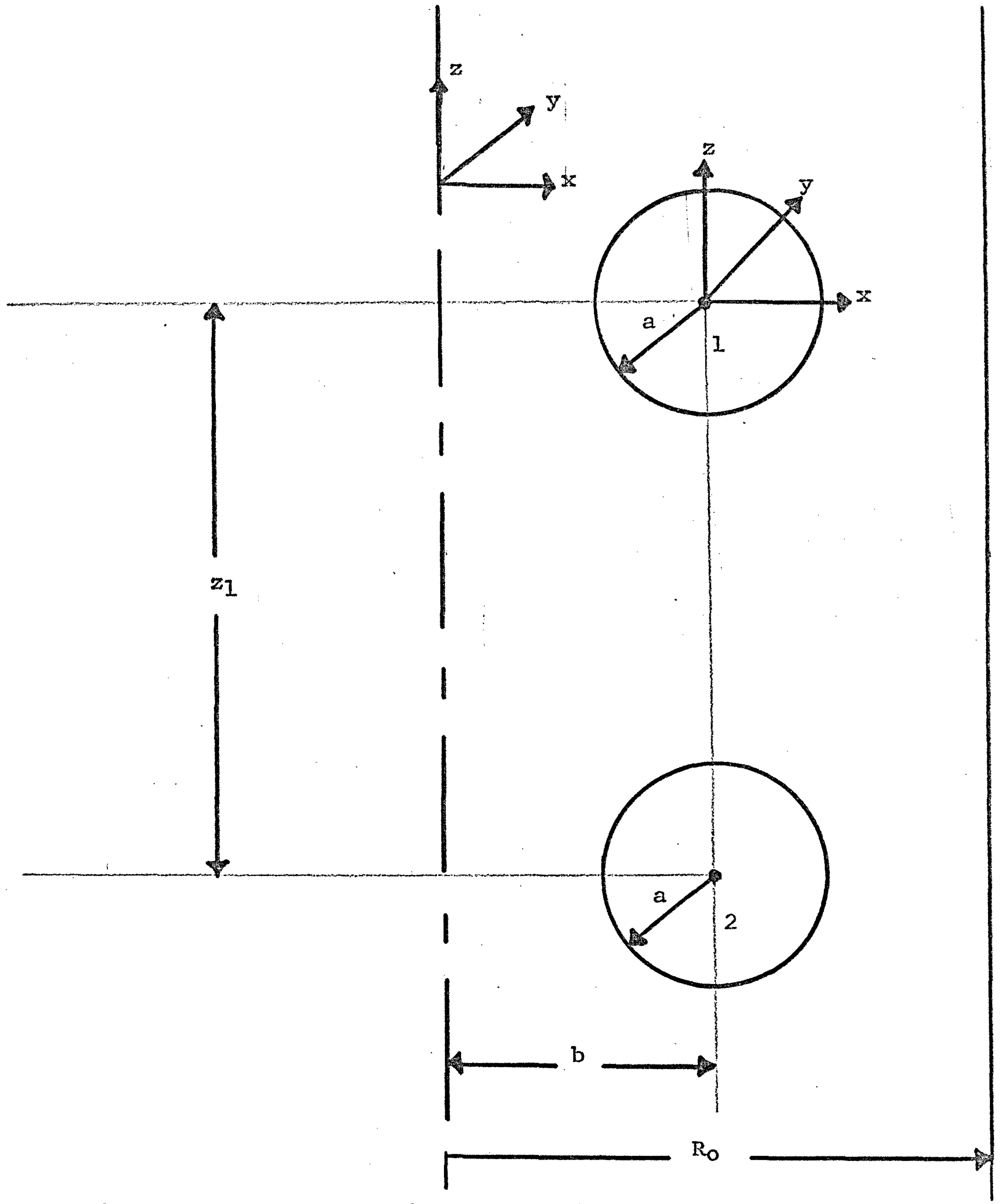

Figure 2 - Two spheres in same vertical plane and displaced from the axis of a circular cylinder by a given distance

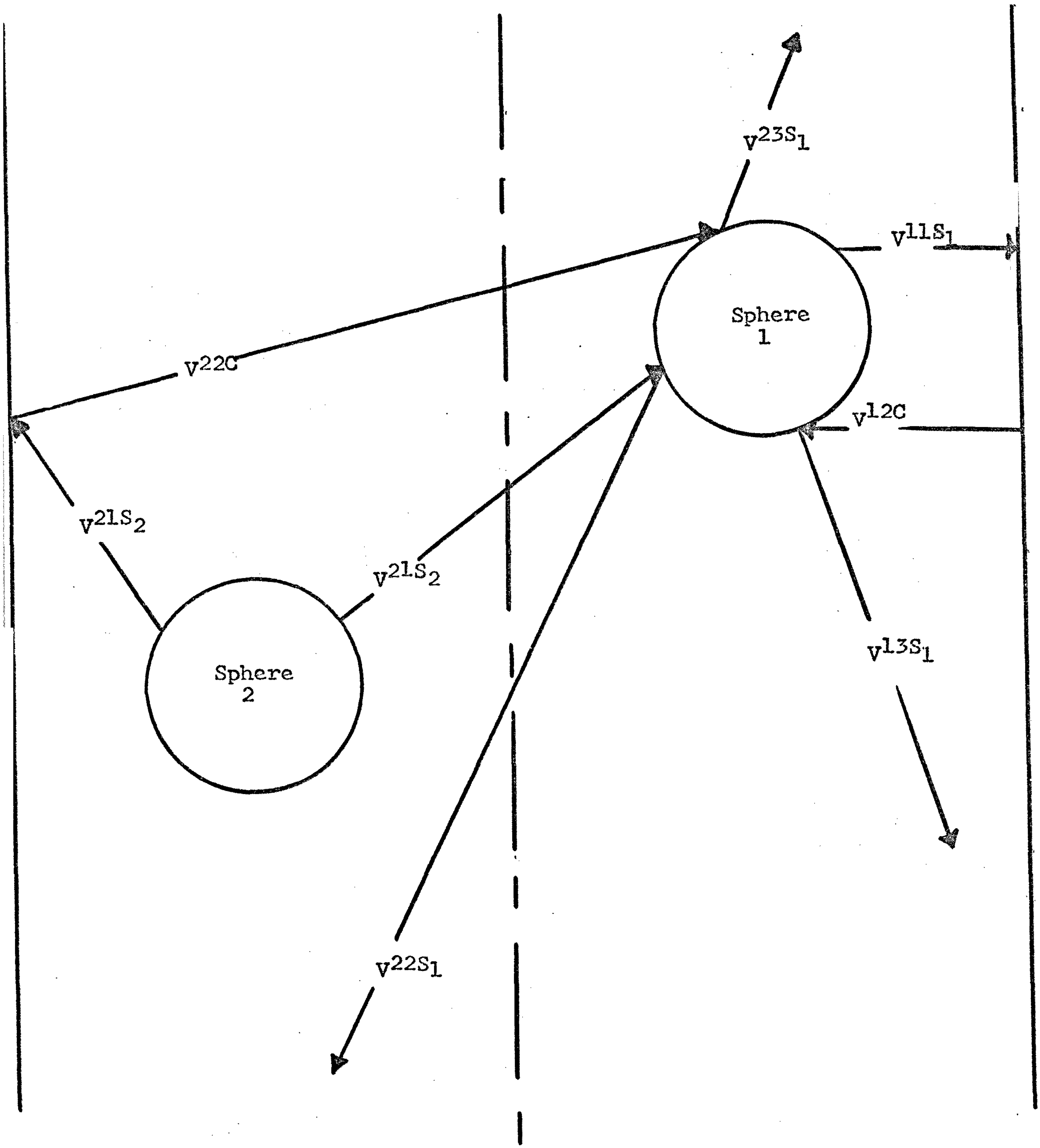

Figure 3 - Schematic representation of velocity fields

TABLE 1

$I(\psi, \beta)$ AND $E(\psi, \beta)$ VS. β FOR VARIOUS VALUES OF ψ

β	$I(\psi=30°, \beta)$	$E(\psi=30°, \beta)$	$I(\psi=45°, \beta)$	$E(\psi=45°, \beta)$
0.05	-0.00229853	-0.00109747	-0.00397989	-0.00190026
0.10	-0.00928425	-0.00443290	-0.0160484	-0.00766256
0.15	-0.0211509	-0.0100988	-0.0363889	-0.0173744
0.20	-0.0381173	-0.0181997	-0.0649391	-0.0310061
0.25	-0.0602832	-0.0287831	-0.100969	-0.0482091
0.30	-0.0873666	-0.0417145	-0.142730	-0.0681484
0.35	-0.1186239	-0.0566387	-0.187597	-0.0895707
0.40	-0.153010	-0.0730568	-0.232452	-0.110988
0.45	-0.189158	-0.0903162	-0.273979	-0.130815
0.50	-0.224951	-0.107406	-0.309214	-0.147639
0.55	-0.258674	-0.123508	-0.336020	-0.160438
0.60	-0.288304	-0.137655	-0.353282	-0.168680
0.65	-0.312224	-0.149076	-0.360951	-0.172342
0.70	-0.329146	-0.157155	-0.360018	-0.171896
0.75	-0.338382	-0.161566	-0.352182	-0.168154
0.80	-0.339953	-0.162316	-0.339547	-0.162122
0.85	-0.334593	-0.159756	-0.369575	-0.176459

β	$I(\psi=60°, \beta)$	$E(\psi=60°, \beta)$
0.05	-0.00688600	-0.00328782
0.10	-0.0275898	-0.0131731
0.15	-0.0614273	-0.0293294
0.20	-0.105708	-0.0504716
0.25	-0.155181	-0.0740933
0.30	-0.203116	-0.0969805
0.35	-0.243321	-0.116177
0.40	-0.271718	-0.129736
0.45	-0.287007	-0.137035
0.50	-0.290352	-0.138633
0.55	-0.284464	-0.135822
0.60	-0.272576	-0.130146
0.65	-0.257639	-0.123013
0.70	-0.241873	-0.115486
0.75	-0.226672	-0.108228
0.80	-0.212713	-0.101563
0.85	-0.200199	-0.0955879

TABLE 2

$I(\eta, \beta)$ AND $E(\eta, \beta)$ VS. β FOR VARIOUS VALUES OF η

β	10^3x $I(\eta=.00001, \beta)$	10^3x $E(\eta=.00001, \beta)$	$I(\eta=0.01, \beta)$	$E(\eta=0.01, \beta)$
0.01	———	———	-0.000396754	-0.000189436
0.1	-0.00401767	-0.00191830	-0.00401491	-0.00191698
0.2	-0.00834350	-0.00398373	-0.00833766	-0.00398094
0.3	-0.0133963	-0.00639626	-0.0133866	-0.00639164
0.4	-0.0198913	-0.00949738	-0.0198761	-0.00949015
0.5	-0.0292634	-0.0139723	-0.0292391	-0.0139607
0.6	-0.0449186	-0.0214470	-0.0448753	-0.0214264
0.7	-0.0766169	-0.0365819	-0.0765204	-0.0365358
0.8	-0.162935	-0.0777957	-0.162591	-0.0776315
0.85	-0.278494	-0.132971	-0.277579	-0.132534

β	$I(\eta=0.05, \beta)$	$E(\eta=0.05, \beta)$	$I(\eta=0.1, \beta)$	$E(\eta=0.1, \beta)$
0.01	-0.00197782	-0.000944339	-0.00391872	-0.00187105
0.1	-0.0200127	-0.00955535	-0.0396417	-0.0189275
0.2	-0.0415487	-0.0198380	-0.0822317	-0.0392628
0.3	-0.0666740	-0.0318345	-0.131745	-0.0629034
0.4	-0.0989042	-0.0472253	-0.194869	-0.0930429
0.5	-0.145253	-0.0693532	-0.284721	-0.135944
0.6	-0.222218	-0.106101	-0.431302	-0.205932
0.7	-0.376244	-0.179643	-0.714598	-0.341196
0.8	-0.783428	-0.374059	-1.40060	-0.668740
0.85	-1.30630	-0.623715	-2.15616	-1.02949

β	$I(\eta=0.2, \beta)$	$E(\eta=0.2, \beta)$	$I(\eta=0.4, \beta)$	$E(\eta=0.4, \beta)$
0.01	-0.00754993	-0.00360483	-0.0130292	-0.00622098
0.1	-0.0762977	-0.0364295	-0.131169	-0.0626287
0.2	-0.157749	-0.0753195	-0.267873	-0.127900
0.3	-0.251124	-0.119903	-0.416489	-0.198859
0.4	-0.367302	-0.175374	-0.584808	-0.279225
0.5	-0.526337	-0.251307	-0.781093	-0.372944
0.6	-0.767524	-0.366466	-1.00452	-0.479625
0.7	-1.17378	-0.560441	-1.20476	-0.575233
0.8	-1.86931	-0.892533	-1.14872	-0.548474
0.85	-2.25217	-1.07533	-0.876416	-0.418458

TABLE 2
(cont.)

β	$I(\eta=0.6,\beta)$	$E(\eta=0.6,\beta)$	$I(\eta=0.8,\beta)$	$E(\eta=0.8,\beta)$
0.01	-0.0153857	-0.00734613	-0.0148619	-0.00709601
0.1	-0.154035	-0.0735461	-0.147828	-0.0705827
0.2	-0.308977	-0.147526	-0.290474	-0.138691
0.3	-0.464524	-0.221794	-0.420571	-0.200808
0.4	-0.616680	-0.294443	-0.525693	-0.251000
0.5	-0.751776	-0.358946	-0.584588	-0.279120
0.6	-0.832850	-0.397657	-0.565223	-0.269874
0.7	-0.780844	-0.372826	-0.436937	-0.208622
0.8	-0.502578	-0.239963	-0.219556	-0.104831
0.85	-0.297600	-0.142093	-0.114174	-0.0545139

β	$I(\eta=1.0,\beta)$	$E(\eta=1.0,\beta)$	$I(\eta=1.2,\beta)$	$E(\eta=1.2,\beta)$
0.01	-0.0125061	-0.00597122	-0.00947619	-0.00452454
0.1	-0.123574	-0.0590022	-0.0930529	-0.0444295
0.2	-0.237817	-0.113549	-0.175655	-0.0838689
0.3	-0.331828	-0.158436	-0.237019	-0.113168
0.4	-0.391901	-0.187119	-0.266462	-0.127226
0.5	-0.401882	-0.191884	-0.255590	-0.122035
0.6	-0.348185	-0.166246	-0.203542	-0.0971848
0.7	-0.234392	-0.111914	-0.124259	-0.0593294
0.8	-0.101027	-0.0482367	-0.0484453	-0.0231309
0.85	-0.0488484	-0.0233234	-0.0374640	-0.0178877

β	$I(\eta=1.4,\beta)$	$E(\eta=1.4,\beta)$	$I(\eta=1.6,\beta)$	$E(\eta=1.6,\beta)$
0.01	-0.00659991	-0.00315122	-0.00428087	-0.00204396
0.1	-0.0644487	-0.0307720	-0.0416026	-0.0198638
0.2	-0.119604	-0.0571067	-0.0760840	-0.0363274
0.3	-0.156798	-0.0748654	-0.0973271	-0.0464702
0.4	-0.169142	-0.0807594	-0.101455	-0.0484413
0.5	-0.153770	-0.0734195	-0.0883161	-0.0421678
0.6	-0.114801	-0.0548135	-0.0626521	-0.0299142
0.7	-0.0652537	-0.0311563	-0.0336713	-0.0160768
0.8	-0.0236983	-0.0113151	-0.0115458	-0.00551271
0.85	-0.0216344	-0.0103297	-0.00502704	-0.00240023

TABLE 2
(cont.)

β	$I(\eta=2.0,\beta)$	$E(\eta=2.0,\beta)$	$I(\eta=2.5,\beta)$	$E(\eta=2.5,\beta)$
0.01	-0.00149058	-0.000711700	-0.000265598	-0.000126814
0.1	-0.0143710	-0.00686162	-0.00253374	-0.00120977
0.2	-0.0256571	-0.0122503	-0.00437855	-0.00209060
0.3	-0.0315330	-0.0150559	-0.00508548	-0.00242814
0.4	-0.0310896	-0.0148442	-0.00460517	-0.00219881
0.5	-0.0252129	-0.0120383	-0.00329796	-0.00157466
0.6	-0.0164206	-0.00784026	-0.00177841	-0.000849129
0.7	-0.00797553	-0.00380804	-0.000620513	-0.000296273
0.8	-0.00241961	-0.00115531	-0.0000800211	-0.0000382073

β	10^3x $I(\eta=3.0,\beta)$	10^3x $E(\eta=3.0,\beta)$	10^3x $I(\eta=3.5,\beta)$	10^3x $E(\eta=3.5,\beta)$
0.01	-0.00769027	-0.00367183	-0.0168686	-0.00805417
0.1	-0.0645746	-0.0308321	-0.164753	-0.0786638
0.2	-0.0631400	-0.0301471	-0.305920	-0.146066
0.3	-0.0295674	-0.0141174	-0.400714	-0.191327
0.4	-0.180887	-0.0863674	-0.431750	-0.206146
0.5	-0.313453	-0.149663	-0.393039	-0.187662
0.6	-0.352302	-0.168212	-0.296252	-0.141450
0.7	-0.273958	-0.130806	-0.173153	-0.0826745
0.8	-0.132197	-0.0631196	-0.0666206	-0.0318090

β	10^3x $I(\eta=4.0,\beta)$	10^3x $E(\eta=4.0,\beta)$	10^6x $I(\eta=8.0,\beta)$	10^6x $E(\eta=8.0,\beta)$
0.01	-0.00848402	-0.00405082	-0.000486454	-0.000232265
0.1	-0.0820588	-0.0391802	-0.00118708	-0.000566790
0.2	-0.147932	-0.0706325	-0.000945605	-0.000451493
0.3	-0.185286	-0.0884676	-0.00617888	-0.00295020
0.4	-0.188229	-0.0898728	-0.000446258	-0.000213073
0.5	-0.159769	-0.0762842	-0.00162187	-0.000774384
0.6	-0.112096	-0.0535220	-0.00355178	-0.00169585
0.7	-0.0604338	-0.0288550	-0.00434382	-0.00207402
0.8	-0.0216215	-0.0103235	-0.0206333	-0.00985169

β	10^6x $I(\eta=20,\beta)$	10^6x $E(\eta=20,\beta)$
0.01	-0.0000990008	-0.0000472694
0.1	-0.000923821	-0.000441092
0.2	-0.00146969	-0.000701723
0.3	-0.00362426	-0.00173046
0.4	-0.00348456	-0.00166375
0.5	-0.00690809	-0.00329837
0.6	-0.00641325	-0.00306210
0.7	-0.0115289	-0.00550462
0.8	-0.00121437	-0.000579818

TABLE 3

$n(\psi,\beta)$ AND $N(\psi,\beta)$ VS. β FOR VARIOUS VALUES OF η

β	$n(\psi=30°,\beta)$	$N(\psi=30°,\beta)$	$n(\psi=45°,\beta)$	$N(\psi=45°,\beta)$	$n(\psi=60°,\beta)$	$N(\psi=60°,\beta)$
0.05	0.202471	0.0966727	0.348894	0.166584	0.595091	0.284135
0.10	0.408763	0.195170	0.693459	0.331102	1.12953	0.539308
0.15	0.622293	0.297123	1.02791	0.490792	1.55069	0.740399
0.20	0.845613	0.403750	1.34385	0.641641	1.82234	0.870102
0.25	1.08034	0.515825	1.63037	0.778445	1.93221	0.922561
0.30	1.32699	0.633591	1.87459	0.895049	1.89340	0.904032
0.35	1.58413	0.756365	2.06234	0.984693	1.73975	0.830670
0.40	1.84802	0.882365	2.18116	1.04143	1.51550	0.723601
0.45	2.11408	1.00940	2.22273	1.06127	1.26412	0.603572
0.50	2.37177	1.11324	2.18438	1.04297	1.01997	0.486998
0.55	2.60902	1.24572	2.07126	0.988952	0.804948	0.384334
0.60	2.80744	1.34045	1.89557	0.905066	0.629114	0.300380
0.65	2.94702	1.40710	1.67505	0.799776	0.493532	0.235644
0.70	3.00770	1.43607	1.42992	0.682737	0.393672	0.187965
0.75	2.97364	1.41981	1.17982	0.563323	0.322461	0.153964
0.80	2.83966	1.35584	0.941555	0.449559	0.272430	0.130076
0.85	2.61165	1.24697	0.727704	0.347453	0.236986	0.113153

TABLE 4

$n(\eta, \beta)$ AND $N(\eta, \beta)$ VS. β FOR VARIOUS VALUES OF η

β	10^3x $n(\eta=.00001, \beta)$	10^3x $N(\eta=.00001, \beta)$	$n(\eta=0.01, \beta)$	$N(\eta=0.01, \beta)$
0.1	-0.0380197	-0.0181531	-0.0379953	-0.0181414
0.2	-0.0482421	-0.0230339	-0.0482104	-0.0230188
0.3	-0.0697369	-0.0332969	-0.0696893	-0.0332742
0.4	-0.113109	-0.0540056	-0.113027	-0.0539666
0.5	-0.207279	-0.0989686	-0.207116	-0.0988907
0.6	-0.446028	-0.212963	-0.445624	-0.212699
0.7	-1.23239	-0.588422	-1.23096	-0.587740
0.8	-5.38621	-2.57173	-5.37607	-2.56688
0.85	-15.7216	-7.50649	-15.6763	-7.48486

β	$n(\eta=0.05, \beta)$	$N(\eta=0.05, \beta)$	$n(\eta=0.1, \beta)$	$N(\eta=0.1, \beta)$
0.1	-0.189603	-0.0905286	-0.376882	-0.179948
0.2	-0.240489	-0.114825	-0.477484	-0.227982
0.3	-0.347420	-0.165881	-0.688482	-0.328726
0.4	-0.562942	-0.268785	-1.11232	-0.531093
0.5	-1.02997	-0.491775	-2.02544	-0.967077
0.6	-2.20982	-1.05511	-4.30795	-2.05689
0.7	-6.06717	-2.89686	-11.6098	-5.54323
0.8	-26.0465	-12.4363	-47.4312	-22.6467
0.85	-74.1781	-35.4174	-126.007	-60.1640

β	$n(\eta=0.2, \beta)$	$N(\eta=0.2, \beta)$	$n(\eta=0.4, \beta)$	$N(\eta=0.4, \beta)$
0.1	-0.735556	-0.351202	-1.33664	-0.638200
0.2	-0.927698	-0.442943	-1.65732	-0.791313
0.3	-1.32768	-0.633921	-2.30645	-1.10125
0.4	-2.12057	-1.01250	-3.53213	-1.68647
0.5	-3.79065	-1.80990	-5.91249	-2.82300
0.6	-7.79962	-3.72404	-10.8854	-5.19737
0.7	-19.6231	-9.36934	-22.2343	-10.6161
0.8	-67.1287	-32.0516	-49.3058	-23.5418
0.85	-145.065	-69.2633	-72.5444	-34.6374

TABLE 4
(cont.)

β	$n(\eta=0.6,\beta)$	$N(\eta=0.6,\beta)$	$n(\eta=0.8,\beta)$	$N(\eta=0.8,\beta)$	$n(\eta=1.0,\beta)$	$N(\eta=1.0,\beta)$
0.1	-1.72009	-0.821282	-1.87375	-0.894649	-1.84040	-0.878724
0.2	-2.07976	-0.993013	-2.19939	-1.05013	-2.09488	-1.00023
0.3	-2.77897	-1.32686	-2.80357	-1.33860	-2.54516	-1.21522
0.4	-4.01145	-1.91532	-3.78998	-1.80958	-3.22609	-1.54034
0.5	-6.15745	-2.93996	-5.31759	-2.53896	-4.16849	-1.99031
0.6	-9.92209	-4.73745	-7.57105	-3.61491	-5.35452	-2.55660
0.7	-16.4018	-7.83127	-10.5831	-5.05305	-6.61903	-3.16035
0.8	-26.1424	-12.4821	-13.6925	-6.53769	-7.51153	-3.58649
0.85	-31.1948	-14.8944	-14.6842	-7.01120	-7.59212	-3.62497

β	$n(\eta=1.2,\beta)$	$N(\eta=1.2,\beta)$	$n(\eta=1.4,\beta)$	$N(\eta=1.4,\beta)$	$n(\eta=1.6,\beta)$	$N(\eta=1.6,\beta)$
0.1	-1.68612	-0.805064	-1.47360	-0.703590	-1.24898	-0.596346
0.2	-1.86398	-0.889985	-1.58706	-0.757767	-1.31568	-0.628190
0.3	-2.16468	-1.03356	-1.77082	-0.845506	-1.41926	-0.677645
0.4	-2.58768	-1.23552	-2.01222	-0.960766	-1.54631	-0.738306
0.5	-3.11486	-1.48724	-2.28406	-1.09056	-1.67440	-0.799467
0.6	-3.68642	-1.76014	-2.53589	-1.21080	-1.77065	-0.845424
0.7	-4.16918	-1.99064	-2.68917	-1.28398	-1.79377	-0.856460
0.8	-4.34316	-2.07370	-2.64677	-1.26374	-1.70306	-0.813153
0.85	-4.23562	-2.02236	-2.52540	-1.20579	-1.60689	-0.767233

β	$n(\eta=2.0,\beta)$	$N(\eta=2.0,\beta)$
0.1	-0.859034	-0.410158
0.2	-0.876634	-0.418562
0.3	-0.901297	-0.430338
0.4	-0.926101	-0.442181
0.5	-0.941909	-0.449728
0.6	-0.938415	-0.448060
0.7	-0.906276	-0.432715
0.8	-0.840154	-0.401144
0.85	-0.794591	-0.379389

TABLE 4
(cont.)

β	$n(\eta=2.5,\beta)$	$N(\eta=2.5,\beta)$	$n(\eta=3.0,\beta)$	$N(\eta=3.0,\beta)$
0.1	-0.539317	-0.257505	-0.360533	-0.172142
0.2	-0.539883	-0.257775	-0.359113	-0.171464
0.3	-0.539550	-0.257616	-0.356560	-0.170245
0.4	-0.536645	-0.256229	-0.352671	-0.168388
0.5	-0.529326	-0.252734	-0.347304	-0.165825
0.6	-0.516047	-0.246394	-0.340477	-0.162566
0.7	-0.496125	-0.236882	-0.332475	-0.158745
0.8	-0.470220	-0.224513	-0.323896	-0.154649
0.85	-0.455688	-0.217575	-0.319655	-0.152624

β	$n(\eta=3.5,\beta)$	$N(\eta=3.5,\beta)$	$n(\eta=4.0,\beta)$	$N(\eta=4.0,\beta)$
0.1	-0.259014	-0.123670	-0.196686	-0.0939108
0.2	-0.258116	-0.123241	-0.196255	-0.0937047
0.3	-0.256634	-0.122534	-0.195558	-0.0933720
0.4	-0.254609	-0.121567	-0.194633	-0.0929305
0.5	-0.252117	-0.120377	-0.193530	-0.0924035
0.6	-0.249285	-0.119025	-0.192309	-0.0918208
0.7	-0.246296	-0.117598	-0.191041	-0.0912154
0.8	-0.243380	-0.116206	-0.189795	-0.0906206
0.85	-0.242030	-0.115561	-0.189200	-0.0903361

β	$n(\eta=8.0,\beta)$	$N(\eta=8.0,\beta)$	$n(\eta=20,\beta)$	$N(\eta=20,\beta)$
0.1	-0.0491615	-0.0234729	-0.00785602	-0.00375097
0.2	-0.0491443	-0.0234647	-0.00785557	-0.00375076
0.3	-0.0491157	-0.0234510	-0.00785485	-0.00375041
0.4	-0.0490758	-0.0234320	-0.00785381	-0.00374992
0.5	-0.0490251	-0.0234077	-0.00785250	-0.00374929
0.6	-0.0489633	-0.0233782	-0.00785089	-0.00374852
0.7	-0.0488908	-0.0233436	-0.00784897	-0.00374761
0.8	-0.0488083	-0.0233042	-0.00784676	-0.00374655
0.85	-0.0487623	-0.0232823	-0.00784571	-0.00374605

APPENDIX 1

COMPUTER PROGRAM USED TO EVALUATE $I(\psi,\beta)$ AND $E(\psi,\beta)$ OR $I(\eta,\beta)$ AND $E(\eta,\beta)$

```
C     MAIN PROGRAM                                                             TG 0002
      IMPLICIT REAL*8(A-H,O-Z)                                                DP    DP
      READ (5,1111)           MAXJ, MAXN, JUNCT, PMAX, DELPO, TESTI,           TG 0003
     1TESTS                                                                    TG 0004
 1111 FORMAT(I2, 3X,I2, 2X, I1, 9X, 4F10.8)                                    TG 0005
      READ (5,33) NN                                                           TG 0006
   33 FORMAT(I5)                                                               TG 0007
      DO 2222 I=1,NN                                                           TG 0008
      READ (5,200)          XSUBI, YSUBI, ZSUBI, XSUBJ, YSUBJ, ZSUBJ           TG 0009
  200 FORMAT(6F10.9)                                                           TG 0010
 2222 V=CVEL(XSUBI, YSUBI, ZSUBI, XSUBJ, YSUBJ, ZSUBJ, MAXJ, PMAX,             TG 0011
     1DELPO, MAXN, TESTI, TESTS, JUNCT)                                        TG 0012
 1028 STOP                                                                     TG 0013
      END                                                                      TG 0014
C     PROGRAM COMPUTES FBESI                                                   TG 0015
      FUNCTION FBESI(N,X)                                                      TG 0016
      IMPLICIT REAL*8(A-H,O-Z)                                                DP    DP
      PSUM = 1.0                                                               TG 0017
      TEMP1 = 1.0                                                              TG 0018
      TEMP2 = 1.0                                                              TG 0019
      TEMP3 = 1.0                                                              TG 0020
      K = 0                                                                    TG 0021
    1 K=K+1                                                                    TG 0022
      TEMP1 = TEMP1 *((0.5* X)**2 )                                            TG 0023
      PK=K                                                                     TG 0024
      TEMP2 = TEMP2 * PK                                                       TG 0025
      AK = N + K                                                               TG 0026
      TEMP3 = TEMP3 * AK                                                       TG 0027
      TERM = TEMP1 / (TEMP2 * TEMP3)                                           TG 0028
      RATIO = TERM/PSUM                                                        TG 0029
      DIF = RATIO - .1D-09                                                     DP 0030
      PSUM = PSUM + TERM                                                       TG 0031
      IF (DIF) 2,2,1                                                           TG 0032
    2 FBESI = PSUM                                                             TG 0033
      RETURN                                                                   TG 0034
      END                                                                      TG 0035
C     PROGRAM COMPUTES FBESK                                                   TG 0036
      FUNCTION FBESK(N,X)                                                      TG 0037
      IMPLICIT REAL*8(A-H,O-Z)                                                DP    DP
      PSUM=1.0                                                                 TG 0038
      TEMP1=1.0                                                                TG 0039
      TEMP2 = 1.0                                                              TG 0040
      TEMP3 = 1.0                                                              TG 0041
      TEMP4=1.0                                                                TG 0042
      K=0                                                                      TG 0043
    1 K= K + 1                                                                 TG 0044
      TEMP1 = TEMP1*((0.5*X)**2)                                               TG 0045
      PK=K                                                                     TG 0046
      TEMP2 = TEMP2 * PK                                                       TG 0047
      BK=N-K                                                                   TG 0048
      TEMP3 = TEMP3 * BK                                                       TG 0049
      TEMP4 = -TEMP4                                                           TG 0050
      TERM = (TEMP4 * TEMP1 )/(TEMP2*TEMP3)                                    TG 0051
      RATIO = DABS (TERM)/PSUM                                                 DP 0052
      DIF = RATIO - .1D-09                                                     DP 0053
      PSUM=PSUM + TERM                                                         TG 0054
      IF (K-N+1) 2,3,3                                                         TG 0055
    2 IF (DIF) 3,3,1                                                           TG 0056
    3 FBESK= 0.5 * PSUM                                                        TG 0057
```

```
      END
C     CVEL FUNCTION SUBPROGRAM
      FUNCTION CVEL (XSUBI, YSUBI, ZSUBI, XSUBJ, YSUBJ, ZSUBJ, MAXJ,
     1PMAX,DELPO,MAXN, TESTI, TESTS, JUNCT)
      IMPLICIT REAL*8(A-H,O-Z)
C     NON-EXECUTABLE STATEMENTS
      DIMENSION CINT(200), ACINT(200)
   11 FORMAT ( 48H1        SPHERE GENERATING CYLINDER VELOCITY FIELD/
     156H0                          XSUBI                 YSUBI           ZS
     2UBI/1H ,13X, E15.8, 4X, E15.8, 4X, E15.8// 46H0  POSITION AT WHICH
     3FIELD STRENGTH IS MEASURED/ 56H0                 XSUBJ            YSU
     4BJ                ZSUBJ/1H , 13X, E15.8, 4X, E15.8, 4X,E15.8// 59H0
     5     SPHERE AND FIELD POSITIONS IN CYLINDRICAL COORDINATES/ 55H0
     6            RSUBI              RSUBJ                PHI/1H , 13X, E15.
     78, 4X, E15.8, 4X, E15.8//35H0 COEFFICIENTS OF FILON INTEGRATION/55
     8H0             ALPHA                 BETA              GAMMA/1H ,13X,
     9E15.8,4X, E15.8, 4X, E15.8)
   12 FORMAT(1H0/27H0    BESSEL FUNCTIONS FOR N=I2/  4H0Q1=E15.8, 2X,
     1  8H IN(Q1)=E15.8, 2X, 9H IN1(Q1)=E15.8,2X, 8H KN(Q1)=E15.8/
     2                    4H0Q2=E15.8,2X, 8H IN(Q2)=E15.8,2X,9H IN1(Q2)=
     3 E15.8/ 4H0Q3=E15.8, 2X, 8H IN(Q3)=E15.8, 9H IN1(Q3)=E15.8)
   13 FORMAT( 45H0      NUMERICAL EVALUATION OF INTEGRAL FOR N= I2,
     1 16H BY FILON METHOD/1H0)
   15 FORMAT( 48H0      INTEGRAND=G(P)*COSF(P*Z)       P=A(DELP)B=E15.8,
     111H(,E15.8,1H),E15.8//( 8H          7E16.8))
   16 FORMAT(1H0/7H0CESUM=E15.8,8H  COSUM=E15.8, 7H CINTG=E15.8,
     18H  TINTG=E15.8/ 12H0      TOPINT=E15.8, 7H RECTA=E15.8, 10H   RATIO
     2I=E15.8)
   17 FORMAT(1H0/22H0    TOTAL INTEGRAL (N=I2, 2H)=E15.8, 19H  SUM OF INT
     1EGRALS=E15.8,               9H  RATIOS=E15.8)
   18 FORMAT(47H0  INTEGRAND=HSUBN(P)*COSF(P*Z)      P=A(DELP)B=E15.8,
     111H(,E15.8,1H),E15.8//( 8H           7E16.8))
   19 FORMAT(1H0/10H0    CESUM=E15.8, 8H  COSUM=E15.8, 9H   CINTG=E15.8,
     1 9H    TINTG=E15.8/ 12H0      TOPINT=E15.8,   8H  RECTA=E15.8,
     2  8H RATIOI=E15.8)
  101 FORMAT(30H0CYLINDER VELOCITY FIELD AFTER  I3, 14H INTEGRATIONS=
     1E15.8)
  102 FORMAT(1H0/28H0   FBESSEL FUNCTIONS FOR N=I2/6H0  Q1=E15.8,2X,
     1 9H FIN(Q1)=E15.8,11H  FIN1(Q1)=E15.8,  9H FKN(Q1)=E15.8/
     2                   5H0 Q2=E15.8, 9H FIN(Q2)=E15.8, 11H  FIN1(Q2)=
     3 E15.8 /4H0Q3=E15.8, 9H FIN(Q3)=E15.8,10H FIN1(Q3)=E15.8)
C     THIS SECTION COMPUTES FILON COEFFICIENTS, ALPHA, BETA, GAMMA
      RSUBI = DSQRT (XSUBI * XSUBI + YSUBI * YSUBI)
      RSUBJ = DSQRT (XSUBJ * XSUBJ + YSUBJ * YSUBJ)
      IF (XSUBI) 1,2,3
    2 IF(YSUBI) 4,5,6
    5 PHII = 0.
      GO TO 7
    6 PHII = 3.14159265/2.
      GO TO 7
    4 PHII =-3.14159265/2.
      GO TO 7
    1 PHII = DATAN (YSUBI/XSUBI) + 3.14159265
      GO TO 7
    3 PHII = DATAN (YSUBI/XSUBI)
    7 IF(XSUBJ) 8,9,10
    9 IF(YSUBJ) 20, 21, 22
   21 PHIJ= 0.
      GO TO 23
   22 PHIJ= 3.14159265/2.
```

```
      GO TO 23                                                          TG 0119
   20 PHIJ = - 3.14159265/2.                                            TG 0120
      GO TO 23                                                          TG 0121
    8 PHIJ = DATAN (YSUBJ/XSUBJ) + 3.14159265                           DP 0122
      GO TO 23                                                          TG 0123
   10 PHIJ = DATAN (YSUBJ/XSUBJ)                                        DP 0124
   23 PHI = PHII - PHIJ                                                 TG 0125
      Z = ZSUBI - ZSUBJ                                                 TG 0126
      AZ = DABS (Z)                                                     DP 0127
      DELP = DELPO/(1.0 + DSIN(AZ *DELPO))                              DP 0128
      THETA = Z * DELP                                                  TG 0129
      IF(THETA) 24, 25, 24                                              TG 0130
   25 ALPHA = 0.                                                        TG 0131
      BETA = 2./3.                                                      TG 0132
      GAMMA =4./3.                                                      TG 0133
      GO TO 26                                                          TG 0134
   24 THETA2 = THETA * THETA                                            TG 0135
      THETA3 = THETA * THETA2                                           TG 0136
      STHETA = DSIN(THETA)                                              DP 0137
      CTHETA = DCOS(THETA)                                              DP 0138
      SINCOS = STHETA * CTHETA                                          TG 0139
      ALPHA =(THETA2 + THETA*SINCOS-2.*STHETA*STHETA)/ THETA3           TG 0140
      BETA =2.*(THETA*(1.+ CTHETA*CTHETA)-2.*SINCOS)/THETA3             TG 0141
      GAMMA = 4.*(STHETA-THETA* CTHETA)/THETA3                          TG 0142
   26 GO TO (27,28,27),JUNCT                                            TG 0143
   27 WRITE (6,11)          XSUBI, YSUBI, ZSUBI, XSUBJ, YSUBJ, ZSUBJ,   TG 0144
     1RSUBI, RSUBJ, PHI, ALPHA, BETA, GAMMA                             TG 0145
C     THIS SECTION COMPUTES THE INTEGRALS FOR N=0 (1) MAXN              TG 0146
   28 MAXNP= MAXN +1                                                    TG 0147
      MAXJP =MAXJ +1                                                    TG 0148
      MAXJP1 = MAXJP-1                                                  TG 0149
      TNNTG = 0.                                                        TG 0150
      DO 31 NP=1, MAXNP                                                 TG 0151
      N = NP-1                                                          TG 0152
      FN =N                                                             TG 0153
      A=0.                                                              TG 0154
      TINTG = 0.                                                        TG 0155
      IF(N) 29, 29, 30                                                  TG 0156
C     INTEGRAL FOR N=0                                                  TG 0157
   29 DO 32 JP=1, MAXJP                                                 TG 0158
      J=JP-1                                                            TG 0159
      DJ=J                                                              TG 0160
      P=A + DJ*DELP                                                     TG 0161
      IF(P) 40, 41, 40                                                  TG 0162
   41 CINT(JP) = 0.0                                                  GG    1
      FGOFP=0.0                                                       GG   1A
      GO TO 32                                                          TG 0164
C     CALL BESSEL FUNCTIONS                                             TG 0165
   40 NPLUS1 = N                                                        TG 0166
      Q1 =P                                                             TG 0167
      Q2 =P*RSUBI                                                       TG 0168
      Q3=P*RSUBJ                                                        TG 0169
      CALL BESI ( Q1, NPLUS1, BIN, IER)                                 TG 0170
      MPLUS1 = NPLUS1 + 1                                               TG 0171
      CALL BESI ( Q1, MPLUS1, BIN1, IER)                                TG 0172
      CALL BESK ( Q1, NPLUS1, BKN, IER)                                 TG 0173
      CALL BESK ( Q1, MPLUS1, BKN1, IER)                                TG 0174
      IF(Q2) 43, 44, 43                                                 TG 0175
   44 PINFI = 1.                                                        TG 0176
      [illegible] =0.                                                   TG 0177
      GO TO 45                                                          TG 0178
```

```
43 CALL BESI ( Q2, NPLUS1, BINRI, IER)
   CALL BESI ( Q2, MPLUS1, BIN1RI, IER)
45 IF(Q3) 46, 47, 46
47 BINRJ =1.
   BIN1RJ = 0.
   GO TO 48
46 CALL BESI ( Q3, NPLUS1, BINRJ, IER)
   CALL BESI ( Q3, MPLUS1, BIN1RJ, IER)
48 GO TO (49, 49, 50)   ,JUNCT
50 WRITE (6,12)          N, Q1, BIN, BIN1, BKN, Q2, BINRI,
  1BIN1RI, Q3, BINRJ, BIN1RJ
C     COMPUTES INTEGRAND CINT(J)
49 A1 = BINRI/BIN
   A3 = BIN1RI/BIN1
   A4 = BIN1RJ/BIN1
   A6 =A3*A4
   A7 =A1 * A4
   A9 = BIN/BIN1
   A10 = A9*A9
   A11 = A10- 2.* A9/P -1.
   A12 = BIN1/BIN
   A13 = BKN*BIN1
   A18 = BINRJ/BIN
   A26 = -A7*A12
   A27 = RSUBI*A6*A12
   A28 = RSUBJ*A18*A1*A9
   A29 = -RSUBJ*RSUBI*A3*A9*A18
   A30=(A26+A27+A28+A29)/A11
   A31=P*A13*A4
   A32=-A1*A12
   A33=A12*A3*RSUBI
   A34=A31*(A32+A33)
   A35=P*A1
   A36=RSUBJ*BKN*BIN*A18
   A37=-BKN1*BIN1*A4
   A38=A35*(A36+A37)
   A39=-2.0/(P*A11)
   A40=-RSUBI*A3
   A41=A1+A40
   A42=A4*A41
   A43=A39*A42
53 GOFP=A30+A34+A38+A43
   IF (JP-1) 1053,1053,1054
1053 FGOFP=GOFP
1054 CINT(JP) = GOFP*DSIN(P*Z)
32 CONTINUE
   CESUM =0.
   COSUM = 0.
    DO 33 JP=1,MAXJP,2
33 CESUM = CESUM + CINT (JP)
   CESUM = CESUM -( CINT(1)+ CINT(MAXJP))/2.
   DO 34 JP=2,MAXJP1,2
34 COSUM = COSUM + CINT(JP)
   IF (THETA) 54, 55, 54
55 CINTG =DELP*(BETA*CESUM + GAMMA*COSUM)
   BJ=MAXJ
   B= A + BJ *DELP
   GO TO 56
54 BJ=MAXJ
   B= A + BJ *DELP
   SINZA =DSIN(Z*A)
```

```
      SINZB =DSIN(Z*B)                                                  DP 0217
      COSZA =DCOS(Z*A)                                                  DP 0218
      COSZB =DCOS(Z*B)                                                  DP 0219
      CINTG = DELP*(-ALPHA*(CINT(MAXJP)*COSZB/SINZB - FGOFP*COSZA)     GG   20
     1        + BETA*CESUM + GAMMA*COSUM)                               GG   21
   56 TINTG = TINTG + CINTG                                             TG 0222
      TNNTG = TNNTG + CINTG                                             TG 0223
      DO 35 JP=1, MAXJP                                                 TG 0224
   35 ACINT(JP) = DABS(CINT(JP))                                        DP 0225
      TOPINT = 0.                                                       TG 0226
      DO 37 JP = 1, MAXJP                                               TG 0227
      IF (TOPINT -ACINT(JP)) 60, 37, 37                                 TG 0228
   60 TOPINT = ACINT(JP)                                                TG 0229
   37 CONTINUE                                                          TG 0230
      BJ=MAXJ                                                           TG 0231
      RECTA= DELP * BJ*TOPINT*2.                                        TG 0232
      RATIOI = RECTA /DABS(TNNTG)                                       DP 0233
      GO TO (61, 62, 61), JUNCT                                         TG 0234
   61 WRITE (6,13)N                                                     TG 0235
      WRITE (6,15)              A, DELP, B, (CINT(JP), JP=1, MAXJP)     TG 0236
      WRITE (6,16)             CESUM, COSUM, CINTG, TINTG, TOPINT, RECTA, TG 0237
     1RATIOI                                                            TG 0238
   62 A=B                                                               TG 0239
      BJ=MAXJ                                                           TG 0240
      B= A + BJ *DELP                                                   TG 0241
      IF (B- PMAX) 63, 63, 64                                           TG 0242
   63 IF(TESTI - RATIOI) 29, 29, 64                                     TG 0243
   64 CYL = TINTG                                                       TG 0244
      CYLPHI = TINTG                                                    TG 0245
      IF(RSUBI * RSUBJ) 31, 65, 31                                      TG 0246
C     INTEGRAL FOR  N=1,2,3,...                                         TG 0247
   30 DO 38 JP=1, MAXJP                                                 TG 0248
      J= JP-1                                                           TG 0249
      DJ=J                                                              TG 0250
      P= A+DJ*DELP                                                      TG 0251
      IF(P) 66, 67, 66                                                  TG 0252
   67 CINT(JP) = 0.0                                                    GG   22
      FHOFP=0.0                                                         GG   22A
      GO TO 38                                                          TG 0254
   66 IF (N-10) 68, 69, 69                                              TG 0255
   69 IF (P/FN - 0.2) 70, 70, 68                                        TG 0256
C     CALL BESSEL FUNCTIONS                                             TG 0257
   68 NPLUS1 = N                                                        TG 0258
      Q1 = P                                                            TG 0259
      Q2 = P*RSUBI                                                      TG 0260
      Q3 =P*RSUBJ                                                       TG 0261
      CALL BESI ( Q1, NPLUS1, BIN, IER)                                 TG 0262
      MPLUS1 = NPLUS1 + 1                                               TG 0263
      CALL BESI ( Q1, MPLUS1, BIN1, IER)                                TG 0264
      CALL BESK ( Q1, NPLUS1, BKN, IER)                                 TG 0265
      CALL BESI ( Q2, NPLUS1, BINRI, IER)                               TG 0266
      CALL BESI ( Q2, MPLUS1, BIN1RI, IER)                              TG 0267
      CALL BESI ( Q3, NPLUS1, BINRJ, IER)                               TG 0268
      CALL BESI ( Q3, MPLUS1, BIN1RJ, IER)                              TG 0269
      CALL BESK(Q1,MPLUS1,BKN1,IER)
      GO TO (71, 71, 72), JUNCT                                         TG 0270
   72 WRITE (6,12)              N, Q1, BIN, BIN1, BKN, Q2, BINRI,       TG 0271
     1BIN1RI, Q3, BINRJ, BIN1RJ                                         TG 0272
C     COMPUTES INTEGRAND CINT(J)                                        TG 0273
   71 A1 = P*P                                                          TG 0274
      A2 = A1 *A1                                                       TG 0275
```

```
      A3 = P * A1
      A4 = BIN1 / BIN
      A5 = A4/P
      A6 = A4 *A4
      A7 = BINRI /BIN
      A8 = BINRJ / BIN
      A12= BIN1RI/BIN
      A9 =A12*RSUBI
      A10 = BIN1RJ/BIN
      A11 = A7 * A8
      A14 = A10 * A7
      A15= A12 * A8
      A16 = A4* A14
      A17 = A9*A10
      A18 = - A3 * A4 * A6
      A19 = A5 * A5
      A20 = A2 * (A5 - A19 * (2. + 3. * FN))
      A22 = A1 * (FN -2.*FN * FN * A5 - 4. * FN * A5)
      A23 = A18 + A20
      A24 =  A23 + A22
      A27 = BKN * BIN
      A58=RSUBJ*A11
      A59=-A16+A17+A58
      A60=P*A59
      A61=-A4*A11/RSUBJ
      A62=A15*RSUBI/RSUBJ
      A63=FN*(A61+A14+A62)
      A64=2.0*FN*FN*A11/(P*RSUBJ)
      A65=A27*(A60+A63+A64)
      A66=-P*BKN1*BIN*A14
      A67=-FN*BKN1*BIN*A11/RSUBJ
      A68=A66+A67
      A69=-RSUBI*RSUBJ*A15
      A70=A58*A4
      A71=A17*A4
      A72=-A16*A4
      A73=A3*A4*(A72+A71+A70+A69)/A24
      A74=BIN/BIN1
      A75=(-2.0*A16+A61*A4+A62*A4+2.0*A17+A58+A69*A74)
      A76=FN*A1*A4*A75/A24
      A77=(-A14+A61+A62+A17*A74)
      A78=FN*FN*P*A4*A77/A24
      A79=A17-A4*A14
      A80=2.0*A1*A4*A79/A24
      A81=2.0*FN*P*A4*A77/A24
      HOFP=A65+A68+A73+A76+A78+A80+A81
      IF(JP-1) 1055,1055,1056
 1055 FHOFP = HOFP
 1056 CINT(JP) = HOFP*DSIN(P*Z)
      GO TO 38
C     CALL FBESSEL FUNCTIONS
   70 Q1 = P
      Q2 = P* RSUBI
      Q3=P*RSUBJ
      FIN = FBESI(N,Q1)
      FINRI=FBESI(N,Q2)
      IF(Q2-Q3) 73, 74, 73
   74 FINRJ = FINRI
      GO TO 75
   73 FINRJ = FBESI (N,Q3)
   75 FIN1 = FBESI (N+1,Q1)
```

```
      FIN1RI = FBESI (N+1, Q2)                                      TG 0318
      IF(Q2 - Q3) 76,77,76                                          TG 0319
   77 FIN1RJ = FIN1RI                                               TG 0320
      GO TO 78                                                      TG 0321
   76 FIN1RJ = FBESI (N+1, Q3)                                      TG 0322
   78 FKN = FBESK (N, Q1)                                           TG 0323
      FKN1 = FBESK(N+1,Q1)
      GO TO (79, 79, 80), JUNCT                                     TG 0324
   80 WRITE (6,102)           N, Q1, FIN, FIN1, FKN, Q2, FINRI,     TG 0325
     1FIN1RI, Q3, FINRJ, FIN1RJ                                     TG 0326
C     COMPUTES INTEGRAND CINT(J)                                    TG 0327
   79 A1 = RSUBI**2                                                 TG 0328
      A2 =1./RSUBJ                                                  TG 0329
      A3 =RSUBI**N                                                  TG 0330
      A4 =RSUBJ **N                                                 TG 0331
      A5 = FINRI /FIN                                               TG 0332
      A6 = FINRJ /FIN                                               TG 0333
      A7 = FIN1RI/FIN                                               TG 0334
      A8 = FIN1RJ / FIN                                             TG 0335
      A9 = FIN1 / FIN                                               TG 0336
      A10 = A9 *A9                                                  TG 0337
      A11 = A5 * A6                                                 TG 0338
      A12 = A7 *A8                                                  TG 0339
      A13 = A7 * A6                                                 TG 0340
      A14 = A8 * A5                                                 TG 0341
      A15 = FKN *FIN                                                TG 0342
      A19 = P*P                                                     TG 0346
      A20 = A19 * A19                                               TG 0347
      A21 = A19 * A20                                               TG 0348
      A22 = N+1                                                     TG 0349
      A23 = A22 * A22                                               TG 0350
      A24 = A22 * A23                                               TG 0351
      A25 = - A21 *A9 * A10/(8.*A24)                                TG 0352
      A26 = A20*(0.5*A9/A22 - 0.5 *A10/A23 -0.75*FN*A10/A23)        TG 0353
      A27 = A19*(- 2.*FN*A9/A22 +(FN*FN*(FIN -FIN1) +FN* FIN)/(A22*FIN)) TG 0354
      AA27 = A25 + A26                                              TG 0355
      A28 = AA27 + A27                                              TG 0356
      A29 = A3*A4                                                   TG 0357
      A34=-A9*A14
      A35=A1*A12
      A36=A34+A35
      A37=RSUBJ*A29
      A38=P*A19*A37/(4.*A23)                                        GG   48
      A39=A38*A36
      A40=A2*A29
      A41=FN*P*A40/(2.*A22)                                         GG   49
      A42=-A9*A11
      A43=A1*A13
      A44=RSUBJ*RSUBJ*A14
      A45=A41*(A42+A43+A44)
      A46=P*A37                                                     GG   50
      A47=A46*A11                                                   GG   51
      A48=2.*FN*FN*A40*A11/P                                        GG   52
      A49=A15/FN
      A50=A49*(A39+A45+A47+A48)
      A51=-2.*FKN1*FIN*(A46*A14/(2.*A22)+FN*A40*A11/P)              GG   53
      A52=A21*A46*A10*A36/(16.*A24*A22*A28)
      A53=1./A9
      A54=FN*P*A20*A40*A10/(8.*A24*A28)                             GG   54
      A55=A43+A42
      A56=RSUBJ*RSUBJ*A53*A35
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```
      A57=A55-2.*A44+2.*A56
      A58=A54*A57
      A59=P*A20*A37*A9/(4.*A23*A28)                                     GG   55
      A60=A35/A22                                                       GG   56
      A61=A34/A22                                                       GG   57
      A62=A59*(A60+A61-A55)                                             GG   58
      A63=FN+FN*P*A19*A40*A9/(4.*A23*A28)                               GG   59
      A64=-A44+A55+A56                                                  GG   60
      A65=A63*A64                                                       GG   61
      A66=FN*P*A19*A40*A9/(2.*A22*A28)                                  GG   62
      A67=A56/A22                                                       GG   63
      A68=A55/A22                                                       GG   64
      A69=A44/A22                                                       GG   65
      A70=RSUBJ*RSUBJ*A11                                               GG   66
      A71=A1*RSUBJ*RSUBJ                                                GG   67
      A72=A71*A13*A53                                                   GG   68
      A73=A66*(A67+A68-A69+A70-A72)                                     GG   69
      HOFP=A50+A51+A52+A58+A62+A65+A73                                  GG  71A
      IF(JP-1) 1057,1057,1058                                           GG   72
 1057 FHOFP=HOFP                                                        GG   73
 1058 CINT(JP) = HOFP*DSIN(P*Z)                                         GG   74
   38 CONTINUE                                                          TG 0364
      CESUM =0.                                                         TG 0365
      COSUM =0.                                                         TG 0366
      DO 39 JP=1, MAXJP,2                                               TG 0367
   39 CESUM = CESUM + CINT(JP)                                          TG 0368
      CESUM = CESUM -(CINT(1) + CINT(MAXJP))/2.                         TG 0369
      DO 81 JP =2, MAXJP1, 2                                            TG 0370
   81 COSUM = COSUM + CINT(JP)                                          TG 0371
      IF (THETA) 82, 83, 82                                             TG 0372
   83 CINTG = DELP *(BETA*CESUM + GAMMA * COSUM)                        TG 0373
      BJ=MAXJ                                                           TG 0374
      B= A + BJ *DELP                                                   TG 0375
      GO TO 84                                                          TG 0376
   82 BJ=MAXJ                                                           TG 0377
      B= A + BJ *DELP                                                   TG 0378
      SINZA =DSIN(Z*A)                                                  DP 0379
      SINZB =DSIN(Z*B)                                                  DP 0380
      COSZA =DCOS(Z*A)                                                  DP 0381
      COSZB =DCOS(Z*B)                                                  DP 0382
      CINTG = DELP*(-ALPHA*(CINT(MAXJP)*COSZB/SINZB - FHOFP*COSZA)      GG   75
     1       . + BETA*CESUM + GAMMA*COSUM)                              GG   76
   84 TINTG = TINTG + CINTG                                             TG 0385
      TNNTG = TNNTG + CINTG                                             TG 0386
      DO 85 JP = 1, MAXJP                                               TG 0387
   85 ACINT(JP) = DABS(CINT(JP))                                        DP 0388
      TOPINT = 0.                                                       TG 0389
      DO 86 JP=1,MAXJP                                                  TG 0390
      IF(TOPINT-ACINT(JP))87,86,86                                      TG 0391
   87 TOPINT = ACINT (JP)                                               TG 0392
   86 CONTINUE                                                          TG 0393
      BJ=MAXJ                                                           TG 0394
      RECTA=DELP*BJ         *TOPINT*(1.+2. * RSUBI*RSUBJ)               TG 0395
      RATIOI = RECTA /DABS(TNNTG)                                       DP 0396
      GO TO (88,89,88), JUNCT                                           TG 0397
   88 WRITE (6,13) N                                                    TG 0398
      WRITE (6,18)          A, DELP, B, (CINT(JP), JP=1,MAXJP)          TG 0399
      WRITE (6,19)          CESUM, COSUM, CINTG, TINTG, TOPINT,         TG 0400
     1RECTA, RATIOI                                                     TG 0401
   89 A=B                                                               TG 0402
      BJ=MAXJ                                                           TG 0403
```

```
      B= A + BJ *DELP                                                   TG 0404
      IF(B-PMAX)90,90,91                                                TG 0405
   90 IF(TESTI-RATIOI)30,30,91                                          TG 0406
   91 CYLL = CYL                                                        TG 0407
      CYL = CYL +2.*TINTG                                               TG 0408
      CYLPHI = CYLPHI + 2.0*DCOS(FN*PHI)*TINTG                          DP 0409
      RATIOS = 2.0*(1.0+4.0*RSUBI*RSUBJ)*DABS(TINTG/CYLL)               DP 0410
      GO TO (92,93, 92),JUNCT                                           TG 0411
   92 WRITE (6,17)           N, TINTG, CYL, RATIOS                      TG 0412
   93 IF(TESTS-RATIOS)31,65,65                                          TG 0413
   31 CONTINUE                                                          TG 0414
C     COMPUTES CVEL                                                     TG 0415
   65 CVEL = 3./(2.*3.14159265)* CYLPHI
      PROD = (1.-RSUBI)*CVEL                                           GG   77
      PRODD = (1.-RSUBI)*CYLPHI                                        GG   78
      WRITE (6,11)          XSUBI, YSUBI, ZSUBI, XSUBJ, YSUBJ, ZSUBJ,   TG 0418
     1RSUBI, RSUBJ, PHI, ALPHA, BETA, GAMMA                             TG 0419
      WRITE (6,1234)             PROD, PRODD                           GG   79
 1234 FORMAT (6H0PROD=E15.8,3X,7H PRODD=E15.8)                         GG   80
      WRITE (6,555)          TNNTG,TINTG,CYLPHI,CYLL,CYL,N,             TG 0422
     1RATIOS,B,RATIOI,CVEL,PROD,RSUBI                                  GG   81
  555 FORMAT(1H0/7H0TNNTG=E15.8,3X,  7H TINTG=E15.8,3X,  8H CYLPHI=E15.8 TG 0424
     1/    6H0CYLL=E15.8,3X, 5H CYL=E15.8,3X, 3H N=I2/ 8H0RATIOS=E15.8,  TG 0425
     23X, 3H B=E15.8, 3X, 8H RATIOI=E15.8/ 6H0CVEL=E15.8,5X,            TG 0426
     36H PROD=E15.9,3X,7H RSUBI=E15.8)                                 GG   82
      WRITE (6,685)          MAXJ,MAXN,JUNCT,DELP,TESTI,TESTS,PMAX      TG 0428
  685 FORMAT(1H0/6H0MAXJ=I2,3X, 6H MAXN=I2,3X, 7H JUNCT=I1/ 6H0DELP=E15. TG 0429
     18,3X, 7H TESTI=E15.8,3X, 7H TESTS=E15.8,3X, 6H PMAX=E15.8)        TG 0430
      GO TO (94,95, 94), JUNCT                                          TG 0431
   94 WRITE (6,101)             N, CVEL                                 TG 0432
   95 RETURN                                                            TG 0433
C     THIS PROGRAM COMPUTES EOFBETA
      END                                                               TG 0434
C                                                                       TG 0435
C     .................................................................. TG 0436
C                                                                       TG 0437
C        SUBROUTINE BESK                                                TG 0438
C                                                                       TG 0439
C           COMPUTE THE K BESSEL FUNCTION FOR A GIVEN ARGUMENT AND ORDER TG 0440
C                                                                       TG 0441
C        USAGE                                                          TG 0442
C           CALL BESK(X,N,BK,IER)                                       TG 0443
C                                                                       TG 0444
C        DESCRIPTION OF PARAMETERS                                      TG 0445
C           X  -THE ARGUMENT OF THE K BESSEL FUNCTION DESIRED           TG 0446
C           N  -THE ORDER OF THE K BESSEL FUNCTION DESIRED              TG 0447
C           BK -THE RESULTANT K BESSEL FUNCTION                         TG 0448
C           IER-RESULTANT ERROR CODE WHERE                              TG 0449
C              IER=0  NO ERROR                                          TG 0450
C              IER=1  N IS NEGATIVE                                     TG 0451
C              IER=2  X IS ZERO OR NEGATIVE                             TG 0452
C                                                                       TG 0453
C        REMARKS                                                        TG 0454
C           N MUST BE GREATER THAN OR EQUAL TO ZERO                     TG 0455
C           X SHOULD BE LESS THAN 60 TO AVOID EXCEEDING THE RANGE OF    TG 0456
C           THE MACHINE                                                 TG 0457
C                                                                       TG 0458
C        SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED                  TG 0459
C           NONE                                                        TG 0460
C                                                                       TG 0461
C        METHOD                                                         TG 0462
```

```
      A=.57721566 +DLOG(B)
      C=B*B
      IF(N-1)37,43,37
C
C     COMPUTE K0 USING SERIES EXPANSION
C
   37 G0=-A
      X2J=1.
      FACT=1.
      HJ=.0
      DO 40 J=1,6
      RJ=1./FLOAT(J)
      X2J=X2J*C
      FACT=FACT*RJ*RJ
      HJ=HJ+RJ
   40 G0=G0+X2J*FACT*(HJ-A)
      IF(N)43,42,43
   42 BK=G0
      RETURN
C
C     COMPUTE K1 USING SERIES EXPANSION
C
   43 X2J=B
      FACT=1.
      HJ=1.
      G1=1./X+X2J*(.5+A-HJ)
      DO 50 J=2,8
      X2J=X2J*C
      RJ=1./FLOAT(J)
      FACT=FACT*RJ*RJ
      HJ=HJ+RJ
   50 G1=G1+X2J*FACT*(.5+(A-HJ)*FLOAT(J))
      IF(N-1)31,52,31
   52 BK=G1
      RETURN
      END
C
C     ...................................................................
C
C        SUBROUTINE BESI
C
C        PURPOSE
C           COMPUTE THE I BESSEL FUNCTION FOR A GIVEN ARGUMENT AND ORDER
C
C        USAGE
C           CALL BESI(X,N,BI,IER)
C
C        DESCRIPTION OF PARAMETERS
C           X  -THE ARGUMENT OF THE I BESSEL FUNCTION DESIRED
C           N  -THE ORDER OF THE I BESSEL FUNCTION DESIRED
C           BI -THE RESULTANT I BESSEL FUNCTION
C           IER-RESULTANT ERROR CODE WHERE
C              IER=0 NO ERROR
C              IER=1 N IS NEGATIVE
C              IER=2 X IS NEGATIVE
C
C        REMARKS
C           N MUST BE GREATER THAN OR EQUAL TO ZERO
C           X SHOULD BE LESS THAN 60 TO AVOID EXCEEDING THE RANGE OF
C           THE MACHINE
C
```

```
C          SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED
C             NONE
C
C          METHOD
C             COMPUTES ZERO ORDER AND FIRST ORDER BESSEL FUNCTIONS USING
C             SERIES APPROXIMATIONS AND THEN COMPUTES N-TH ORDER FUNCTION
C             USING RECURRENCE RELATION.
C
C     ...................................................................
C
      SUBROUTINE BESI(X,N,BI,IER)
      IMPLICIT REAL*8(A-H,O-Z)
C
C     CHECK FOR ERRORS IN N AND X AND EXIT IF ANY ARE PRESENT
C
      IER=0
      BI=1.0
      IF(N)150,15,10
   10 IF(X)160,20,20
   15 IF(X)160,17,20
   17 RETURN
C
C     DEFINE TOLERANCE
C
   20 TOL=1.D-6
C
C     IF ARGUMENT GT 12 AND GT N, USE ASYMPTOTIC FORM
C
      IF(X-12.)40,40,30
   30 IF(X-FLOAT(N))40,40,110
C
C     COMPUTE FIRST TERM OF SERIES AND SET INITIAL VALUE OF THE SUM
C
   40 XX=X/2.
      FACTN=1.
      IF(N-1)70,70,50
   50 DO 60 I=2,N
      FI=I
   60 FACTN=FACTN*FI
   70 TERM=(XX**N)/FACTN
      BI=TERM
      XX=XX*XX
C
C     COMPUTE UP TO 30 TERMS, STOPPING WHEN ABS(TERM) LE ABS(SUM OF TERM
C     TIMES TOLERANCE
C
      DO 90 K=1,30
      IF(DABS(TERM)-DABS(BI*TOL))100,100,80
   80 FK=K*(N+K)
      TERM=TERM*(XX/FK)
   90 BI=BI+TERM
C
C     RETURN BI AS ANSWER
C
  100 RETURN
C
C     X GT 12 AND X GT N, SO USE ASYMPTOTIC APPROXIMATION
C
  110 FN=4*N*N
      XX=1./(8.*X)
      TERM=1.
```

```
      BI=1.                                                             TG 0644
      DO 130 K=1,30                                                     TG 0645
      IF(DABS(TERM)-DABS(TOL*BI))140,140,120                            DP 0646
  120 FK=(2*K-1)**2                                                     TG 0647
      TERM=TERM*XX*(FK-FN)/FLOAT(K)                                     TG 0648
  130 BI=BI+TERM                                                        TG 0649
  140 PI=3.141592653                                                    TG 0650
      BI=BI*DEXP(X)/DSQRT(2.0*PI*X)                                     DP 0651
      GO TO 100                                                         TG 0652
  150 IER=1                                                             TG 0653
      GO TO 100                                                         TG 0654
  160 IER=2                                                             TG 0655
      GO TO 100                                                         TG 0656
      END                                                               TG 0657
```

APPENDIX 2

COMPUTER PROGRAM USED TO EVALUATE m(ψ,β) AND M(ψ,β) OR m(η,β) AND M(η,β)

```
C      MAIN PROGRAM                                                      TG 0002
       IMPLICIT REAL*8(A-H,O-Z)                                         DP    DP
       READ (5,1111)           MAXJ, MAXN, JUNCT, PMAX, DELPO, TESTI,    TG 0003
      1TESTS                                                             TG 0004
 1111 FORMAT(I2, 3X,I2, 2X, I1, 9X, 4F10.8)                              TG 0005
       READ (5,33) NN                                                    TG 0006
   33 FORMAT(I5)                                                         TG 0007
       DO 2222 I=1,NN                                                    TG 0008
       READ (5,200)           XSUBI, YSUBI, ZSUBI, XSUBJ, YSUBJ, ZSUBJ   TG 0009
  200 FORMAT(6F10.9)                                                     TG 0010
 2222 V=CVEL(XSUBI, YSUBI, ZSUBI, XSUBJ, YSUBJ, ZSUBJ, MAXJ, PMAX,       TG 0011
      1DELPO, MAXN, TESTI, TESTS, JUNCT)                                 TG 0012
 1028 STOP                                                               TG 0013
       END                                                               TG 0014
C      PROGRAM COMPUTES FBESI                                            TG 0015
       FUNCTION FBESI(N,X)                                               TG 0016
       IMPLICIT REAL*8(A-H,O-Z)                                         DP    DP
       PSUM = 1.0                                                        TG 0017
       TEMP1 = 1.0                                                       TG 0018
       TEMP2 = 1.0                                                       TG 0019
       TEMP3 = 1.0                                                       TG 0020
       K = 0                                                             TG 0021
     1 K=K+1                                                             TG 0022
       TEMP1 = TEMP1 *((0.5* X)**2 )                                     TG 0023
       PK=K                                                              TG 0024
       TEMP2 = TEMP2 * PK                                                TG 0025
       AK = N + K                                                        TG 0026
       TEMP3 = TEMP3 * AK                                                TG 0027
       TERM = TEMP1 / (TEMP2 * TEMP3)                                    TG 0028
       RATIO = TERM/PSUM                                                 TG 0029
       DIF = RATIO - .1D-09                                              DP 0030
       PSUM = PSUM + TERM                                                TG 0031
       IF (DIF) 2,2,1                                                    TG 0032
     2 FBESI = PSUM                                                      TG 0033
       RETURN                                                            TG 0034
       END                                                               TG 0035
C      PROGRAM COMPUTES FBESK                                            TG 0036
       FUNCTION FBESK(N,X).                                              TG 0037
       IMPLICIT REAL*8(A-H,O-Z)                                         DP    DP
       PSUM=1.0                                                          TG 0038
       TEMP1=1.0                                                         TG 0039
       TEMP2 = 1.0                                                       TG 0040
       TEMP3 = 1.0                                                       TG 0041
       TEMP4=1.0                                                         TG 0042
       K=0                                                               TG 0043
     1 K= K + 1                                                          TG 0044
       TEMP1 = TEMP1*((0.5*X)**2)                                        TG 0045
       PK=K                                                              TG 0046
       TEMP2 = TEMP2 * PK                                                TG 0047
       BK=N-K                                                            TG 0048
       TEMP3 = TEMP3 * BK                                                TG 0049
       TEMP4 = -TEMP4                                                    TG 0050
       TERM = (TEMP4 * TEMP1 )/(TEMP2*TEMP3)                             TG 0051
       RATIO = DABS (TERM)/PSUM                                          DP 0052
       DIF = RATIO - .1D-09                                              DP 0053
       PSUM=PSUM + TERM                                                  TG 0054
       IF (K-N+1) 2,3,3                                                  TG 0055
     2 IF (DIF) 3,3,1                                                    TG 0056
     3 FBESK= 0.5 * PSUM                                                 TG 0057
```

```
      RETURN                                                            TG 0058
      END                                                               TG 0059
C     CVEL FUNCTION SUBPROGRAM                                          TG 0060
      FUNCTION CVEL (XSUBI, YSUBI, ZSUBI, XSUBJ, YSUBJ, ZSUBJ, MAXJ,    TG 0062
     1PMAX,DELPO,MAXN, TESTI, TESTS, JUNCT)                             TG 0063
      IMPLICIT REAL*8(A-H,O-Z)                                         DP    DP
C     NON-EXECUTABLE STATEMENTS                                         TG 0064
      DIMENSION CINT(200), ACINT(200)                                   TG 0065
   11 FORMAT ( 4PH1      SPHERE GENERATING CYLINDER VELOCITY FIELD/     TG 0066
     166HO                        XSUBI                 YSUBI        ZS TG 0067
     2UBI/1H ,13X, E15.8, 4X, E15.8, 4X, E15.8// 46HO  POSITION AT WHICH TG 0068
     3FIELD STRENGTH IS MEASURED/ 56HO                  XSUBJ        YSU TG 0069
     4BJ                 ZSUBJ/1H , 13X, E15.8, 4X, E15.8, 4X,E15.8// 59HO  TG 0070
     5    SPHERE AND FIELD POSITIONS IN CYLINDRICAL COORDINATES/ 55HO   TG 0071
     6           RSUBI               RSUBJ                PHI/1H , 13X, E15. TG 0072
     78, 4X, E15.8, 4X, E15.8//35HO COEFFICIENTS OF FILON INTEGRATION/55 TG 0073
     8HO              ALPHA                BETA               GAMMA/1H ,13X, TG 0074
     9E15.8,4X, E15.8, 4X, E15.8)                                       TG 0075
   12 FORMAT(1HO/27HO    BESSEL FUNCTIONS FOR N=I2/  4HOQ1=E15.8, 2X,   TG 0076
     1  8H IN(Q1)=E15.8, 2X, 9H IN1(Q1)=E15.8,2X, 8H KN(Q1)=E15.8/      TG 0077
     2                    4HOQ2=E15.8,2X, 8H IN(Q2)=E15.8,2X,9H IN1(Q2)=  TG 0078
     3 E15.8/ 4HOQ3=E15.8, 2X, 8H IN(Q3)=E15.8, 9H IN1(Q3)=E15.8)       TG 0079
   13 FORMAT( 45HO      NUMERICAL EVALUATION OF INTEGRAL FOR N= I2,     TG 0080
     1 16H BY FILON METHOD/1HO)                                         TG 0081
   15 FORMAT( 48HO       INTEGRAND=G(P)*COSF(P*Z)        P=A(DELP)B=E15.8, TG 0082
     11H(,E15.8,1H),E15.8//( 8H          7E16.8))                       TG 0083
   16 FORMAT(1HO/7HOCESUM=E15.8,8H  COSUM=E15.8, 7H CINTG=E15.8,        TG 0084
     18H  TINTG=E15.8/ 12HO     TOPINT=E15.8, 7H RECTA=E15.8, 10H   RATIO TG 0085
     2I=E15.8)                                                          TG 0086
   17 FORMAT(1HO/22HO   TOTAL INTEGRAL (N=I2, 2H)=E15.8, 19H  SUM OF INT TG 0087
     1EGRALS=E15.8,              9H  RATIOS=E15.8)                      TG 0088
   18 FORMAT(47HO  INTEGRAND=HSUBN(P)*COSF(P*Z)       P=A(DELP)B=E15.8, TG 0089
     11H(,E15.8,1H),E15.8//( 8H          7E16.8))                       TG 0090
   19 FORMAT(1HO/10HO    CESUM=E15.8, 8H  COSUM=E15.8, 9H   CINTG=E15.8, TG 0091
     1 9H    TINTG=E15.8/ 12HO     TOPINT=E15.8,  8H  RECTA=E15.8,      TG 0092
     2  8H RATIOI=E15.8)                                                TG 0093
  101 FORMAT(30HOCYLINDER VELOCITY FIELD AFTER  I3, 14H INTEGRATIONS=   TG 0094
     1E15.8)                                                            TG 0095
  102 FORMAT(1HO/28HO    FBESSEL FUNCTIONS FOR N=I2/6HO  Q1=E15.8,2X,   TG 0096
     1 9H FIN(Q1)=E15.8,11H  FIN1(Q1)=E15.8,  9H FKN(Q1)=E15.8/         TG 0097
     2                     5HO Q2=E15.8, 9H FIN(Q2)=E15.8, 11H  FIN1(Q2)= TG 0098
     3 E15.8 /4HOQ3=E15.8, 9H FIN(Q3)=E15.8,10H FIN1(Q3)=E15.8)         TG 0099
C     THIS SECTION COMPUTES FILON COEFFICIENTS, ALPHA, BETA, GAMMA      TG 0100
      RSUBI = DSQRT (XSUBI * XSUBI + YSUBI * YSUBI)                     DP 0101
      RSUBJ = DSQRT (XSUBJ * XSUBJ + YSUBJ * YSUBJ)                     DP 0102
      IF (XSUBI) 1,2,3                                                  TG 0103
    2 IF(YSUBI) 4,5,6                                                   TG 0104
    5 PHII = 0.                                                         TG 0105
      GO TO 7                                                           TG 0106
    6 PHII = 3.14159265/2.                                              TG 0107
      GO TO 7                                                           TG 0108
    4 PHII =-3.14159265/2.                                              TG 0109
      GO TO 7                                                           TG 0110
    1 PHII = DATAN (YSUBI/XSUBI) + 3.14159265                           DP 0111
      GO TO 7                                                           TG 0112
    3 PHII = DATAN (YSUBI/XSUBI)                                        DP 0113
    7 IF(XSUBJ) 8,9,10                                                  TG 0114
    9 IF(YSUBJ) 20, 21, 22                                              TG 0115
   21 PHIJ= 0.                                                          TG 0116
      GO TO 23                                                          TG 0117
   22 PHIJ= 3.14159265/2.                                               TG 0118
```

```
      GO TO 23                                                          TG 0119
   20 PHIJ = - 3.14159265/2.                                            TG 0120
      GO TO 23                                                          TG 0121
    8 PHIJ = DATAN (YSUBJ/XSUBJ) + 3.14159265                           DP 0122
      GO TO 23                                                          TG 0123
   10 PHIJ = DATAN (YSUBJ/XSUBJ)                                        DP 0124
   23 PHI = PHII - PHIJ                                                 TG 0125
      Z = ZSUBI - ZSUBJ                                                 TG 0126
      AZ = DABS (Z)                                                     DP 0127
      DELP = DELPO/(1.0 + DSIN(AZ *DELPO))                              DP 0128
      THETA = Z * DELP                                                  TG 0129
      IF(THETA) 24, 25, 24                                              TG 0130
   25 ALPHA = 0.                                                        TG 0131
      BETA = 2./3.                                                      TG 0132
      GAMMA =4./3.                                                      TG 0133
      GO TO 26                                                          TG 0134
   24 THETA2 = THETA * THETA                                            TG 0135
      THETA3 = THETA * THETA2                                           TG 0136
      STHETA = DSIN(THETA)                                              DP 0137
      CTHETA = DCOS(THETA)                                              DP 0138
      SINCOS = STHETA * CTHETA                                          TG 0139
      ALPHA =(THETA2 + THETA*SINCOS-2.*STHETA*STHETA)/ THETA3           TG 0140
      BETA =2.*(THETA*(1.+ CTHETA*CTHETA)-2.*SINCOS)/THETA3             TG 0141
      GAMMA = 4.*(STHETA-THETA* CTHETA)/THETA3                          TG 0142
   26 GO TO (27,28,27),JUNCT                                            TG 0143
   27 WRITE (6,11)             XSUBI, YSUBI, ZSUBI, XSUBJ, YSUBJ, ZSUBJ, TG 0144
     1RSUBI, RSUBJ, PHI, ALPHA, BETA, GAMMA                             TG 0145
C     THIS SECTION COMPUTES THE INTEGRALS FOR N=0 (1) MAXN              TG 0146
   28 MAXNP= MAXN +1                                                    TG 0147
      MAXJP =MAXJ +1                                                    TG 0148
      MAXJP1 = MAXJP-1                                                  TG 0149
      TNNTG = 0.                                                        TG 0150
      DO 31 NP=1, MAXNP                                                 TG 0151
      N = NP-1                                                          TG 0152
      FN =N                                                             TG 0153
      A=0.                                                              TG 0154
      TINTG = 0.                                                        TG 0155
      IF(N) 29, 29, 30                                                  TG 0156
C     INTEGRAL FOR N=0                                                  TG 0157
   29 DO 32 JP=1, MAXJP                                                 TG 0158
      J=JP-1                                                            TG 0159
      DJ=J                                                              TG 0160
      P=A + DJ*DELP                                                     TG 0161
      IF(P) 40, 41, 40                                                  TG 0162
   41 CINT(JP)=0.0                                                     GG    1R
      FGDEP=0.0                                                        GG    1A
      GO TO 32                                                          TG 0164
C     CALL BESSEL FUNCTIONS                                             TG 0165
   40 NPLUS1 = N                                                        TG 0166
      Q1 =P                                                             TG 0167
      Q2 =P*RSUBI                                                       TG 0168
      Q3=P*RSUBJ                                                        TG 0169
      CALL BESI ( Q1, NPLUS1, BIN, IER)                                 TG 0170
      MPLUS1 = NPLUS1 + 1                                               TG 0171
      CALL BESI ( Q1, MPLUS1, BIN1, IER)                                TG 0172
      CALL BESK ( Q1, NPLUS1, BKN, IER)                                 TG 0173
      CALL BESK ( Q1, MPLUS1, BKN1, IER)                                TG 0174
      IF(Q2) 43, 44, 43                                                 TG 0175
   44 BINRI = 1.                                                        TG 0176
      BIN1RI =0.                                                        TG 0177
      GO TO 45                                                          TG 0178
```

```
43 CALL BESI ( Q2, NPLUS1, BINRI, IER)                          TG 0179
   CALL BESI ( Q2, MPLUS1, BIN1RI, IER)                          TG 0180
45 IF(Q3) 46, 47, 46                                             TG 0181
47 BINRJ =1.                                                     TG 0182
   BIN1RJ = 0.                                                   TG 0183
   GO TO 48                                                      TG 0184
46 CALL BESI ( Q3, NPLUS1, BINRJ, IER)                           TG 0185
   CALL BESI ( Q3, MPLUS1, BIN1RJ, IER)                          TG 0186
48 GO TO (49, 49, 50) , JUNCT                                    TG 0187
50 WRITE (6,12)            N, Q1, BIN, BIN1, BKN, Q2, BINRI,     TG 0188
  1BIN1RI, Q3, BINRJ, BIN1RJ                                     TG 0189
C     COMPUTES INTEGRAND CINT(J)                                 TG 0190
49 A1=BINRI/BIN                                                 GG    2R
   A2=BINRJ/BIN                                                 GG    3R
   A3=BIN1RI/BIN1                                               GG    4R
   A4=BIN1RJ/BIN1                                               GG    5R
   A6=A3*A4                                                     GG    6R
   A8=A2*A3                                                     GG    7R
   A9=BIN/BIN1                                                  GG    8R
   A10=A9*A9                                                    GG    9R
   A11=A10-2.0*A9/P-1.0                                         GG   10R
   A12=BIN1/BIN                                                 GG   11R
   A13=BKN*BIN1                                                 GG   12R
   A26=-A6*A12                                                  GG   13R
   A27=RSUBJ*A8*A9                                              GG   14R
   A28=(A26+A27)/A11                                            GG   15R
   A29=-P*A13*A12*A6                                            GG   16R
   A30=-2.0/(P*A11)                                             GG   17R
   A31=A30*A6                                                   GG   18R
53 GOFP=A28+A29+A31                                             GG   19R
   IF (JP-1) 1053,1053,1054                                     GG   17
1053 FGOFP=GOFP                                                 GG   18
1054 CINT(JP) = GOFP*DSIN(P*Z)                                  GG   19
32 CONTINUE                                                      TG 0201
   CESUM =0.                                                     TG 0202
   COSUM = 0.                                                    TG 0203
    DO 33 JP=1,MAXJP,2                                           TG 0204
33 CESUM = CESUM + CINT (JP)                                     TG 0205
   CESUM = CESUM -( CINT(1)+ CINT(MAXJP))/2.                     TG 0206
   DO 34 JP=2,MAXJP1,2                                           TG 0207
34 COSUM = COSUM + CINT(JP)                                      TG 0208
   IF (THETA) 54, 55, 54                                         TG 0209
55 CINTG =DELP*(BETA*CESUM + GAMMA*COSUM)                        TG 0210
   BJ=MAXJ                                                       TG 0211
   B= A + BJ *DELP                                               TG 0212
   GO TO 56                                                      TG 0213
54 BJ=MAXJ                                                       TG 0214
   B= A + BJ *DELP                                               TG 0215
   SINZA =DSIN(Z*A)                                              DP 0216
   SINZB =DSIN(Z*B)                                              DP 0217
   COSZA =DCOS(Z*A)                                              DP 0218
   COSZB =DCOS(Z*B)                                              DP 0219
   CINTG = DELP*(-ALPHA*(CINT(MAXJP)*COSZB/SINZB - FGOFP*COSZA) GG   20
  1        + BETA*CESUM + GAMMA*COSUM)                          GG   21
56 TINTG = TINTG + CINTG                                         TG 0222
   TNNTG = TNNTG + CINTG                                         TG 0223
   DO 35 JP=1, MAXJP                                             TG 0224
35 ACINT(JP) = DABS(CINT(JP))                                    DP 0225
   TOPINT = 0.                                                   TG 0226
   DO 37 JP = 1, MAXJP                                           TG 0227
   IF (TOPINT -ACINT(JP)) 60, 37, 37                             TG 0228
```

```
 60 TOPINT = ACINT(JP)                                                   TG 0229
 37 CONTINUE                                                             TG 0230
    BJ=MAXJ                                                              TG 0231
    RECTA= DELP * BJ*TOPINT*2.                                           TG 0232
    RATIOI = RECTA /DABS(TNNTG)                                          DP 0233
    GO TO (61, 62, 61), JUNCT                                            TG 0234
 61 WRITE (6,13)N                                                        TG 0235
    WRITE (6,15)          A, DELP, B, (CINT(JP), JP=1, MAXJP)            TG 0236
    WRITE (6,16)         CESUM, COSUM, CINTG, TINTG, TOPINT, RECTA,      TG 0237
   1RATIOI                                                               TG 0238
 62 A=B                                                                  TG 0239
    BJ=MAXJ                                                              TG 0240
    B= A + BJ *DELP                                                      TG 0241
    IF (B- PMAX) 63, 63, 64                                              TG 0242
 63 IF(TESTI - RATIOI) 29, 29, 64                                        TG 0243
 64 CYL = TINTG                                                          TG 0244
    CYLPHI = TINTG                                                       TG 0245
    IF(RSUBI * RSUBJ) 31, 65, 31                                         TG 0246
C   INTEGRAL FOR  N=1,2,3,...                                            TG 0247
 30 DO 38 JP=1, MAXJP                                                    TG 0248
    J= JP-1                                                              TG 0249
    DJ=J                                                                 TG 0250
    P= A+DJ*DELP                                                         TG 0251
    IF(P) 66, 67, 66                                                     TG 0252
 67 CINT(JP)=0.0                                                        GG   20R
    FHOFP=0.0                                                           GG   22A
    GO TO 38                                                             TG 0254
 66 IF (N-10) 68, 69, 69                                                 TG 0255
 69 IF(P/FN-0.2) 68,68,68                                               GG   93R
C   CALL BESSEL FUNCTIONS                                                TG 0257
 68 NPLUS1 = N                                                           TG 0258
    Q1 = P                                                               TG 0259
    Q2 = P*RSUBI                                                         TG 0260
    Q3 =P*RSUBJ                                                          TG 0261
    CALL BESI ( Q1, NPLUS1, BIN, IER)                                    TG 0262
    MPLUS1 = NPLUS1 + 1                                                  TG 0263
    CALL BESI ( Q1, MPLUS1, BIN1, IER)                                   TG 0264
    CALL BESK ( Q1, NPLUS1, BKN, IER)                                    TG 0265
    CALL BESI ( Q2, NPLUS1, BINRI, IER)                                  TG 0266
    CALL BESI ( Q2, MPLUS1, BIN1RI, IER)                                 TG 0267
    CALL BESI ( Q3, NPLUS1, BINRJ, IER)                                  TG 0268
    CALL BESI ( Q3, MPLUS1, BIN1RJ, IER)                                 TG 0269
    CALL BESK(Q1,MPLUS1,BKN1,IER)
    GO TO (71, 71, 72), JUNCT                                            TG 0270
 72 WRITE (6,12)            N, Q1, BIN, BIN1, BKN, Q2, BINRI,            TG 0271
   1BIN1RI, Q3, BINRJ, BIN1RJ                                            TG 0272
C   COMPUTES INTEGRAND CINT(J)                                           TG 0273
 71 A1=P*P                                                              GG   21R
    A2=A1*A1                                                            GG   22R
    A3=P*A1                                                             GG   23R
    A4=BIN1/BIN                                                         GG   24R
    A5=A4/P                                                             GG   25R
    A6=A4*A4                                                            GG   26R
    A7=BINRI/BIN                                                        GG   27R
    A8=BINRJ/BIN                                                        GG   28R
    A9=BIN1RI/BIN                                                       GG   29R
    A10=A9*RSUBI                                                        GG   30R
    A11=BIN1RJ/BIN                                                      GG   31R
    A12=A7*A8                                                           GG   32R
    A13=A11*A7                                                          GG   33R
    A14=A9*A8                                                           GG   34R
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```
      A15=P*A4*FN                                         GG   35R
      A16=A4*A13                                          GG   36R
      A17=A10*A11                                         GG   37R
      A18=-A3*A4*A6                                       GG   38R
      A19=A5*A5                                           GG   39R
      A20=A2*(A5-A19*(2.+3.*FN))                          GG   40R
      A22=A1*(FN-2.*FN*FN*A5-4.*FN*A5)                    GG   41R
      A23=A18+A20                                         GG   42R
      A24=A23+A22                                         GG   43R
      A27=BKN*BIN                                         GG   44R
      A30=A14/A15                                         GG   45R
      A31=-FN*FN*A30/RSUBJ                                GG   46R
      A32=A12/A15                                         GG   47R
      A33=-2.0*FN*FN*FN/(P*RSUBI*RSUBJ)                   GG   48R
      A34=A32*A33                                         GG   49R
      A35=A9*A11                                          GG   50R
      A36=-P*A35                                          GG   51R
      A37=-FN*A14/RSUBJ                                   GG   52R
      A38=-2.0*FN*A13/RSUBI                               GG   53R
      A39=A33/FN                                          GG   54R
      A40=A39*A12                                         GG   55R
      A41=A27*(A31+A34+A36+A37+A38+A40)                   GG   56R
      A42=BKNI*BIN                                        GG   57R
      A43=FN*P*A30/RSUBJ                                  GG   58R
      A44=2.0*FN*FN*A32/(RSUBI*RSUBJ)                     GG   59R
      A45=A42*(A43+A44)                                   GG   60R
      A46=A3*A4/A24                                       GG   61R
      A47=-A4*A35                                         GG   62R
      A48=RSUBJ*A14                                       GG   63R
      A49=A46*(A47+A48)                                   GG   64R
      A50=FN*A1*A4/A24                                    GG   65R
      A51=A4*A14                                          GG   66R
      A52=-A51/RSUBJ                                      GG   67R
      A53=-2.0*A16/RSUBI                                  GG   68R
      A54=-2.0*RSUBJ*A12/RSUBI                            GG   69R
      A55=A50*(A52+A53-A35+A54)                           GG   70R
      A56=FN*FN*P*A4/A24                                  GG   71R
      A57=-2.0*A4*A12                                     GG   72R
      A58=A57/(RSUBI*RSUBJ)                               GG   73R
      A59=-2.0*A13/RSUBI                                  GG   74R
      A60=A56*(A58+A59)                                   GG   75R
      A61=2.0*A1*A4/A24                                   GG   76R
      A62=-A61*A35                                        GG   77R
      A63=2.0*FN*P*A4/A24                                 GG   78R
      A64=-A14/RSUBJ                                      GG   79R
      A65=A63*(A64+A59)                                   GG   80R
      A66=2.0*FN*FN/A24                                   GG   81R
      A67=A66*(-A64+A58)                                  GG   82R
      A68=2.0*FN*FN*FN/A24                                GG   83R
      A69=-A30/RSUBJ                                      GG   84R
      A70=2.0*A12/P                                       GG   85R
      A71=A70/(RSUBI*RSUBJ)                               GG   86R
      A72=A68*(A69+A71)                                   GG   87R
      A73=2.0*FN*FN*FN*FN/A24                             GG   88R
      A74=-2.0*A32/P                                      GG   89R
      A75=A74/(RSUBI*RSUBJ)                               GG   90R
      A76=A73*A75                                         GG   91R
      HOFP=A41+A45+A49+A55+A60+A62+A65+A67+A72+A76        GG   92R
      IF(JP-1) 1055,1055,1056                             GG   44
 1055 FHOFP = HOFP                                        GG   45
 1056 CINT(JP) = HOFP*DSIN(P*Z)                           GG   46
```

```
      A45=A41*(A42+A43+A44)
      A46=P*A37                                                         GG   50
      A47=A46*A11                                                       GG   51
      A48=2.*FN*FN*A40*A11/P                                            GG   52
      A49=A15/FN
      A50=A49*(A39+A45+A47+A48)
      A51=-2.*FKP1*FIN*(A46*A14/(2.*A22)+FN*A40*A11/P)                  GG   53
      A52=A21*A46*A10*A36/(16.*A24*A22*A28)
      A53=1./A9
      A54=FN*P*A20*A40*A10/(8.*A24*A28)                                 GG   54
      A55=A43+A42
      A56=RSUBJ*RSUBJ*A53*A35
      A57=A55-2.*A44+2.*A56
      A58=A54*A57
      A59=P*A20*A37*A9/(4.*A23*A28)                                     GG   55
      A60=A35/A22                                                       GG   56
      A61=A34/A22                                                       GG   57
      A62=A59*(A60+A61-A55)                                             GG   58
      A63=FN*FN*P*A19*A40*A9/(4.*A23*A28)                               GG   59
      A64=-A44+A55+A56                                                  GG   60
      A65=A63*A64                                                       GG   61
      A66=FN*P*A19*A40*A9/(2.*A22*A28)                                  GG   62
      A67=A56/A22                                                       GG   63
      A68=A55/A22                                                       GG   64
      A69=A44/A22                                                       GG   65
      A70=RSUBJ*RSUBJ*A11                                               GG   66
      A71=A1*RSUBJ*RSUBJ                                                GG   67
      A72=A71*A13*A53                                                   GG   68
      A73=A66*(A67+A68-A69+A70-A72)                                     GG   69
      HDFP=A50+A51+A52+A58+A62+A65+A73                                  GG   71A
      IF(JP-1) 1057,1057,1058                                           GG   72
 1057 FHDFP=HDFP                                                        GG   73
 1058 CINT(JP) = HDFP*DSIN(P*Z)                                         GG   74
   38 CONTINUE                                                          TG 0364
      CESUM =0.                                                         TG 0365
      COSUM =0.                                                         TG 0366
      DO 39 JP=1, MAXJP,2                                               TG 0367
   39 CESUM = CESUM + CINT(JP)                                          TG 0368
      CESUM = CESUM -(CINT(1) + CINT(MAXJP))/2.                         TG 0369
      DO 81 JP =2, MAXJP1, 2                                            TG 0370
   81 COSUM = COSUM + CINT(JP)                                          TG 0371
      IF (THETA) 82, 83, 82                                             TG 0372
   83 CINTG = DELP *(BETA*CESUM + GAMMA * COSUM)                        TG 0373
      BJ=MAXJ                                                           TG 0374
      B= A + BJ *DELP                                                   TG 0375
      GO TO 84                                                          TG 0376
   82 BJ=MAXJ                                                           TG 0377
      B= A + BJ *DELP                                                   TG 0378
      SINZA =DSIN(Z*A)                                                  DP 0379
      SINZB =DSIN(Z*B)                                                  DP 0380
      COSZA =DCOS(Z*A)                                                  DP 0381
      COSZB =DCOS(Z*B)                                                  DP 0382
      CINTG = DELP*(-ALPHA*(CINT(MAXJP)*COSZB/SINZB - FHDFP*COSZA)      GG   75
     1        + BETA*CESUM + GAMMA*COSUM)                               GG   76
   84 TINTG = TINTG + CINTG                                             TG 0385
      TNNTG = TNNTG + CINTG                                             TG 0386
      DO 85 JP = 1, MAXJP                                               TG 0387
   85 ACINT(JP) = DABS(CINT(JP))                                        DP 0388
      TOPINT = 0.                                                       TG 0389
      DO 86 JP=1,MAXJP                                                  TG 0390
      IF(TOPINT-ACINT(JP))87,86,86                                      TG 0391
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 87 TOPINT = ACINT (JP)                                                   TG 0392
 86 CONTINUE                                                              TG 0393
    BJ=MAXJ                                                               TG 0394
    RECTA=DELP*BJ          *TOPINT*(1.+2. * RSUBI*RSUBJ)                  TG 0395
    RATIOI = RECTA /DABS(TNNTG)                                           DP 0396
    GO TO (88,89,88), JUNCT                                               TG 0397
 88 WRITE (6,13) N                                                        TG 0398
    WRITE (6,18)             A, DELP, B, (CINT(JP), JP=1,MAXJP)           TG 0399
    WRITE (6,19)             CESUM, COSUM, CINTG, TINTG, TOPINT,          TG 0400
   1RECTA, RATIOI                                                         TG 0401
 89 A=B                                                                   TG 0402
    BJ=MAXJ                                                               TG 0403
    B= A + BJ *DELP                                                       TG 0404
    IF(B-PMAX)90,90,91                                                    TG 0405
 90 IF(TESTI-RATIOI)30,30,91                                              TG 0406
 91 CYLL = CYL                                                            TG 0407
    CYL = CYL +2.*TINTG                                                   TG 0408
    CYLPHI = CYLPHI + 2.0*DCOS(FN*PHI)*TINTG                              DP 0409
    RATIOS = 2.0*(1.0+4.0*RSUBI*RSUBJ)*DABS(TINTG/CYLL)                   DP 0410
    GO TO (92,93, 92),JUNCT                                               TG 0411
 92 WRITE (6,17)             N, TINTG, CYL, RATIOS                        TG 0412
 93 IF(TESTS-RATIOS)31,65,65                                              TG 0413
 31 CONTINUE                                                              TG 0414
C      COMPUTES CVEL                                                      TG 0415
 65 CVEL =  3./(2.*3.14159265)* CYLPHI
    PROD = (1.-RSUBI)*CVEL                                               GG    77
    PRODD = (1.-RSUBI)*CYLPHI                                            GG    78
    WRITE (6,11)            XSUBI, YSUBI, ZSUBI, XSUBJ, YSUBJ, ZSUBJ,     TG 0418
   1RSUBI, RSUBJ, PHI, ALPHA, BETA, GAMMA                                 TG 0419
    WRITE (6,1234)             PROD, PRODD                               GG    79
1234 FORMAT (6H0PROD=E15.8,3X,7H PRODD=E15.8)                            GG    80
    WRITE (6,555)            TNNTG,TINTG,CYLPHI,CYLL,CYL,N,               TG 0422
   1RATIOS,B,RATIOI,CVEL,PROD,RSUBI                                      GG    81
555 FORMAT(1H0/7H0TNNTG=E15.8,3X,  7H TINTG=E15.8,3X,  8H CYLPHI=E15.8 TG 0424
   1/    6H0CYLL=E15.8,3X, 5H CYL=E15.8,3X, 3H N=I2/ 8H0RATIOS=E15.8,   TG 0425
   23X, 3H B=E15.8, 3X, 8H RATIOI=E15.8/ 6H0CVEL=E15.8,5X,               TG 0426
   36H PROD=E15.8,3X,7H RSUBI=E15.8)                                     GG    82
    WRITE (6,685)            MAXJ,MAXN,JUNCT,DELP,TESTI,TESTS,PMAX        TG 0428
685 FORMAT(1H0/6H0MAXJ=I2,3X, 6H MAXN=I2,3X, 7H JUNCT=I1/ 6H0DELP=E15. TG 0429
   18,3X, 7H TESTI=E15.8,3X, 7H TESTS=E15.8,3X, 6H PMAX=E15.8)            TG 0430
    GO TO (94,95, 94), JUNCT                                              TG 0431
 94 WRITE (6,101)              N, CVEL                                    TG 0432
 95 RETURN                                                                TG 0433
C      THIS PROGRAM COMPUTES EOFBETA
    END                                                                   TG 0434
C                                                                         TG 0435
C      .......................................................... TG 0436
C                                                                         TG 0437
C         SUBROUTINE BESK                                                 TG 0438
C                                                                         TG 0439
C            COMPUTE THE K BESSEL FUNCTION FOR A GIVEN ARGUMENT AND ORDER TG 0440
C                                                                         TG 0441
C         USAGE                                                           TG 0442
C            CALL BESK(X,N,BK,IER)                                        TG 0443
C                                                                         TG 0444
C         DESCRIPTION OF PARAMETERS                                       TG 0445
C            X  -THE ARGUMENT OF THE K BESSEL FUNCTION DESIRED             TG 0446
C            N  -THE ORDER OF THE K BESSEL FUNCTION DESIRED                TG 0447
C            BK -THE RESULTANT K BESSEL FUNCTION                          TG 0448
C            IER-RESULTANT ERROR CODE WHERE                               TG 0449
C               IER=0  NO ERROR                                           TG 0450
```

```
C             IER=1  N IS NEGATIVE                                        TG 0451
C             IER=2  X IS ZERO OR NEGATIVE                                TG 0452
C                                                                         TG 0453
C        REMARKS                                                          TG 0454
C           N MUST BE GREATER THAN OR EQUAL TO ZERO                       TG 0455
C           X SHOULD BE LESS THAN 60 TO AVOID EXCEEDING THE RANGE OF      TG 0456
C           THE MACHINE                                                   TG 0457
C                                                                         TG 0458
C        SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED                    TG 0459
C           NONE                                                          TG 0460
C                                                                         TG 0461
C        METHOD                                                           TG 0462
C           COMPUTES ZERO ORDER AND FIRST ORDER BESSEL FUNCTIONS USING    TG 0463
C           SERIES APPROXIMATIONS AND THEN COMPUTES N TH ORDER FUNCTION   TG 0464
C           USING RECURRENCE RELATION.                                    TG 0465
C           RECURRENCE RELATION AND POLYNOMIAL APPROXIMATION TECHNIQUE    TG 0466
C           AS DESCRIBED BY A.J.M.HITCHCOCK,@POLYNOMIAL APPROXIMATIONS    TG 0467
C           TO BESSEL FUNCTIONS OF ORDER ZERO AND ONE AND TO RELATED      TG 0468
C           FUNCTIONS@, M.T.A.C., V.11,1957,PP.86-88, AND G.N. WATSON,    TG 0469
C           @A TREATISE ON THE THEORY OF BESSEL FUNCTIONS@, CAMBRIDGE     TG 0470
C           UNIVERSITY PRESS, 1958, P. 62                                 TG 0471
C                                                                         TG 0472
C     ..................................................................  TG 0473
C                                                                         TG 0474
      SUBROUTINE BESK(X,N,BK,IER)                                         TG 0475
      IMPLICIT REAL*8(A-H,O-Z)                                          DP    DP
      DIMENSION T(12)                                                     TG 0476
      BK=.0                                                               TG 0477
      IF(N)10,11,11                                                       TG 0478
   10 IER=1                                                               TG 0479
      RETURN                                                              TG 0480
   11 IF(X)12,12,20                                                       TG 0481
   12 IER=2                                                               TG 0482
      RETURN                                                              TG 0483
   20 IER=0                                                               TG 0484
      IF(X-1.)36,36,25                                                    TG 0485
   25 A=DEXP(-X)                                                          DP 0486
      B=1./X                                                              TG 0487
      C=DSQRT(B)                                                          DP 0488
      T(1)=B                                                              TG 0489
      DO 26 L=2,12                                                        TG 0490
   26 T(L)=T(L-1)*B                                                       TG 0491
      IF(N-1)27,29,27                                                     TG 0492
C                                                                         TG 0493
C     COMPUTE KO USING POLYNOMIAL APPROXIMATION                           TG 0494
C                                                                         TG 0495
   27 G0=A*(1.25331414-.15666418*T(1)+.088111278*T(2)-.091390954*T(3)     TG 0496
     2+.13445962*T(4)-.22998503*T(5)+.37924097*T(6)-.52472773*T(7)        TG 0497
     3+.55753684*T(8)-.42626329*T(9)+.21845181*T(10)-.066809767*T(11)     TG 0498
     4+.009189383*T(12))*C                                                TG 0499
      IF(N)20,28,29                                                       TG 0500
   28 BK=G0                                                               TG 0501
      RETURN                                                              TG 0502
C                                                                         TG 0503
C     COMPUTE K1 USING POLYNOMIAL APPROXIMATION                           TG 0504
C                                                                         TG 0505
   29 G1=A*(1.2533141+.46999270*T(1)-.14685830*T(2)+.12804266*T(3)        TG 0506
     2-.17364316*T(4)+.28476181*T(5)-.45943421*T(6)+.62833807*T(7)        TG 0507
     3-.66322954*T(8)+.50502386*T(9)-.25813038*T(10)+.078800012*T(11)     TG 0508
     4-.010824177*T(12))*C                                                TG 0509
      IF(N-1)20,30,31                                                     TG 0510
```

```
   30 BK=G1
      RETURN
C
C     FROM K0,K1 COMPUTE KN USING RECURRENCE RELATION
C
   31 DO 35 J=2,N
      GJ=2.*(FLOAT(J)-1.)*G1/X+G0
      G0=G1
   35 G1=GJ
      BK=GJ
      RETURN
   36 B=X/2.
      A=.57721566 +DLOG(B)
      C=B*B
      IF(N-1)37,43,37
C
C     COMPUTE K0 USING SERIES EXPANSION
C
   37 G0=-A
      X2J=1.
      FACT=1.
      HJ=.0
      DO 40 J=1,6
      RJ=1./FLOAT(J)
      X2J=X2J*C
      FACT=FACT*RJ*RJ
      HJ=HJ+RJ
   40 G0=G0+X2J*FACT*(HJ-A)
      IF(N)43,42,43
   42 BK=G0
      RETURN
C
C     COMPUTE K1 USING SERIES EXPANSION
C
   43 X2J=B
      FACT=1.
      HJ=1.
      G1=1./X+X2J*(.5+A-HJ)
      DO 50 J=2,8
      X2J=X2J*C
      RJ=1./FLOAT(J)
      FACT=FACT*RJ*RJ
      HJ=HJ+RJ
   50 G1=G1+X2J*FACT*(.5+(A-HJ)*FLOAT(J))
      IF(N-1)31,52,31
   52 BK=G1
      RETURN
      END
C
C     ..................................................................
C
C        SUBROUTINE BESI
C
C        PURPOSE
C           COMPUTE THE I BESSEL FUNCTION FOR A GIVEN ARGUMENT AND ORDER
C
C        USAGE
C           CALL BESI(X,N,BI,IER)
C
C        DESCRIPTION OF PARAMETERS
C           X  -THE ARGUMENT OF THE I BESSEL FUNCTION DESIRED
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C            N  -THE ORDER OF THE I BESSEL FUNCTION DESIRED                  TG 0572
C            BI -THE RESULTANT I BESSEL FUNCTION                             TG 0573
C            IER-RESULTANT ERROR CODE WHERE                                  TG 0574
C               IER=0 NO ERROR                                               TG 0575
C               IER=1 N IS NEGATIVE                                          TG 0576
C               IER=2 X IS NEGATIVE                                          TG 0577
C                                                                            TG 0578
C        REMARKS                                                             TG 0579
C           N MUST BE GREATER THAN OR EQUAL TO ZERO                          TG 0580
C           X SHOULD BE LESS THAN 60 TO AVOID EXCEEDING THE RANGE OF         TG 0581
C           THE MACHINE                                                      TG 0582
C                                                                            TG 0583
C        SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED                       TG 0584
C           NONE                                                             TG 0585
C                                                                            TG 0586
C        METHOD                                                              TG 0587
C           COMPUTES ZERO ORDER AND FIRST ORDER BESSEL FUNCTIONS USING       TG 0588
C           SERIES APPROXIMATIONS AND THEN COMPUTES N-TH ORDER FUNCTION      TG 0589
C           USING RECURRENCE RELATION.                                       TG 0590
C                                                                            TG 0591
C     ..................................................................     TG 0592
C                                                                            TG 0593
      SUBROUTINE BESI(X,N,BI,IER)                                            TG 0594
      IMPLICIT REAL*8(A-H,O-Z)                                             DP    DP
C                                                                            TG 0595
C     CHECK FOR ERRORS IN N AND X AND EXIT IF ANY ARE PRESENT                TG 0596
C                                                                            TG 0597
      IER=0                                                                  TG 0598
      BI=1.0                                                                 TG 0599
      IF(N)150,15,10                                                         TG 0600
   10 IF(X)160,20,20                                                         TG 0601
   15 IF(X)160,17,20                                                         TG 0602
   17 RETURN                                                                 TG 0603
C                                                                            TG 0604
C     DEFINE TOLERANCE                                                       TG 0605
C                                                                            TG 0606
   20 TOL=1.D-6                                                              DP 0607
C                                                                            TG 0608
C     IF ARGUMENT GT 12 AND GT N, USE ASYMPTOTIC FORM                        TG 0609
C                                                                            TG 0610
      IF(X-12.)40,40,30                                                      TG 0611
   30 IF(X-FLOAT(N))40,40,110                                                TG 0612
C                                                                            TG 0613
C     COMPUTE FIRST TERM OF SERIES AND SET INITIAL VALUE OF THE SUM          TG 0614
C                                                                            TG 0615
   40 XX=X/2.                                                                TG 0616
      FACTN=1.                                                               TG 0617
      IF(N-1)70,70,50                                                        TG 0618
   50 DO 60 I=2,N                                                            TG 0619
      FI=I                                                                   TG 0620
   60 FACTN=FACTN*FI                                                         TG 0621
   70 TERM=(XX**N)/FACTN                                                     TG 0622
      BI=TERM                                                                TG 0623
      XX=XX*XX                                                               TG 0624
C                                                                            TG 0625
C     COMPUTE UP TO 30 TERMS, STOPPING WHEN ABS(TERM) LE ABS(SUM OF TERM     TG 0626
C     TIMES TOLERANCE                                                        TG 0627
C                                                                            TG 0628
      DO 90 K=1,30                                                           TG 0629
      IF(DABS(TERM)-DABS(BI*TOL))100,100,80                                  DP 0630
   80 FK=K*(N+K)                                                             TG 0631
```

```
      TERM=TERM*(XX/FK)                                                 TG 0632
   90 BI=BI+TERM                                                        TG 0633
C                                                                       TG 0634
C     RETURN BI AS ANSWER                                               TG 0635
C                                                                       TG 0636
  100 RETURN                                                            TG 0637
C                                                                       TG 0638
C     X GT 12 AND X GT N, SO USE ASYMPTOTIC APPROXIMATION               TG 0639
C                                                                       TG 0640
  110 FN=4*N*N                                                          TG 0641
      XX=1./(8.*X)                                                      TG 0642
      TERM=1.                                                           TG 0643
      BI=1.                                                             TG 0644
      DO 130 K=1,30                                                     TG 0645
      IF(DABS(TERM)-DABS(TOL*BI))140,140,120                            DP 0646
  120 FK=(2*K-1)**2                                                     TG 0647
      TERM=TERM*XX*(FK-FN)/FLOAT(K)                                     TG 0648
  130 BI=BI+TERM                                                        TG 0649
  140 PI=3.141592653                                                    TG 0650
      BI=BI*DEXP(X)/DSQRT(2.0*PI*X)                                     DP 0651
      GO TO 100                                                         TG 0652
  150 IER=1                                                             TG 0653
      GO TO 100                                                         TG 0654
  160 IER=2                                                             TG 0655
      GO TO 100                                                         TG 0656
      END                                                               TG 0657
```

APPENDIX 3

DERIVATION OF EQUATION (6.1) FROM GREENSTEIN'S EXPRESSION FOR $\vec{V}^{21}S_2$ VELOCITY FIELD

Greenstein[4] provides the following expression for the $\vec{V}^{21}S_2$ velocity field if only translation is considered:

$$\vec{V}^{21}S_2 = \vec{i}\left(\frac{1}{2} Ar^{-1} \sin\psi\cos\psi\right) + \vec{k}\left(Ar^{-1}\sin^2\psi + \frac{A}{2} r^{-1}\cos^2\psi\right) \qquad \text{(A.3.1)}$$

where

$$A = \frac{3}{2} aU \qquad \text{(A.3.2)}$$

Therefore:

$$\vec{F}_x{}^{22}S_1 = 6\pi\mu a(\vec{V}_x{}^{21}S_2) = 6\pi\mu a\left(\frac{3U}{4}\frac{a}{r}\sin\psi\cos\psi\right) \qquad \text{(A.3.3)}$$

and

$$\vec{F}_z{}^{22}S_1 = 6\pi\mu a(\vec{V}_z{}^{21}S_2) = 6\pi\mu a\left(\frac{3U}{2}\frac{a}{r}\sin^2\psi + \frac{3U}{4}\frac{a}{r}\cos^2\psi\right) \qquad \text{(A.3.4)}$$

Also, from Stokes law:

$$F_z{}^{11}S_1 = 6\pi\mu aU \qquad \text{(A.3.5)}$$

Computing ϵ, the angle of deflection:

$$\tan\epsilon = \frac{\vec{F}_x{}^{II}}{\vec{F}_z{}^{II}} = \frac{6\pi\mu aU\left(\frac{3}{4}\frac{a}{r}\sin\psi\cos\psi\right)}{6\pi\mu aU\left(1+\frac{3}{2}\frac{a}{r}\sin^2\psi + \frac{3}{4}\frac{a}{r}\cos^2\psi\right)} \qquad \text{(A.3.6)}$$

The denominator of equation (A.3.6) can be simplified to:

$$1+\frac{3}{2}\frac{a}{r}(\sin^2\psi+\cos^2\psi) - \frac{3}{4}\frac{a}{r}\cos^2\psi = 1+\frac{3}{2}\frac{a}{r} - \frac{3}{4}\frac{a}{r}\cos^2\psi \qquad \text{(A.3.7)}$$

Hence,

$$\tan\epsilon = \frac{\frac{3}{4}\frac{a}{r}\sin\psi\cos\psi}{1+\frac{3}{2}\frac{a}{r} - \frac{3}{4}\frac{a}{r}\cos^2\psi} \qquad \text{(A.3.8)}$$

Dividing the denominator into the numerator, we obtain:

$$\tan\epsilon = \frac{3}{4}\frac{a}{r}\sin\psi\cos\psi - \frac{9}{8}\frac{a^2}{r^2}\sin\psi\cos\psi + \frac{9}{16}\frac{a^2}{r^2}\sin\psi\cos^3\psi + \ldots \qquad \text{(A.3.9)}$$

Neglecting the third term on the right hand side of equation (A.3.9) and subsequent higher order terms, we obtain:

$$\tan\epsilon \approx \frac{3}{4}\frac{a}{r}\left(1 - \frac{3}{2}\frac{a}{r}\right)\sin\psi\cos\psi \qquad \text{(A.3.10)}$$

Furthermore, since ϵ is small, $\tan\epsilon$ is approximately equal to $\sin\epsilon$. Therefore,

$$\sin\epsilon \approx \frac{3}{4}\frac{a}{r}\left(1 - \frac{3}{2}\frac{a}{r}\right)\sin\psi\cos\psi \qquad \text{(A.3.11)}$$

which is the equation (6.1), given by Smoluchowski.