

Copyright Warning & Restrictions

The copyright law of the United States (Title 17, United States Code) governs the making of photocopies or other reproductions of copyrighted material.

Under certain conditions specified in the law, libraries and archives are authorized to furnish a photocopy or other reproduction. One of these specified conditions is that the photocopy or reproduction is not to be “used for any purpose other than private study, scholarship, or research.” If a user makes a request for, or later uses, a photocopy or reproduction for purposes in excess of “fair use” that user may be liable for copyright infringement,

This institution reserves the right to refuse to accept a copying order if, in its judgment, fulfillment of the order would involve violation of copyright law.

Please Note: The author retains the copyright while the New Jersey Institute of Technology reserves the right to distribute this thesis or dissertation

Printing note: If you do not wish to print this page, then select “Pages from: first page # to: last page #” on the print dialog screen

The Van Houten library has removed some of the personal information and all signatures from the approval page and biographical sketches of theses and dissertations in order to protect the identity of NJIT graduates and faculty.

INFORMATION TO USERS

This material was produced from a microfilm copy of the original document. While the most advanced technological means to photograph and reproduce this document have been used, the quality is heavily dependent upon the quality of the original submitted.

The following explanation of techniques is provided to help you understand markings or patterns which may appear on this reproduction.

1. The sign or "target" for pages apparently lacking from the document photographed is "Missing Page(s)". If it was possible to obtain the missing page(s) or section, they are spliced into the film along with adjacent pages. This may have necessitated cutting thru an image and duplicating adjacent pages to insure you complete continuity.
2. When an image on the film is obliterated with a large round black mark, it is an indication that the photographer suspected that the copy may have moved during exposure and thus cause a blurred image. You will find a good image of the page in the adjacent frame.
3. When a map, drawing or chart, etc., was part of the material being photographed the photographer followed a definite method in "sectioning" the material. It is customary to begin photoing at the upper left hand corner of a large sheet and to continue photoing from left to right in equal sections with a small overlap. If necessary, sectioning is continued again — beginning below the first row and continuing on until complete.
4. The majority of users indicate that the textual content is of greatest value, however, a somewhat higher quality reproduction could be made from "photographs" if essential to the understanding of the dissertation. Silver prints of "photographs" may be ordered at additional charge by writing the Order Department, giving the catalog number, title, author and specific pages you wish reproduced.
5. PLEASE NOTE: Some pages may have indistinct print. Filmed as received.

Xerox University Microfilms

300 North Zeeb Road
Ann Arbor, Michigan 48106

74-15,105

GÜLÇÜR, Halil Özcan, 1946-
DYNAMICS OF INTERCONNECTED SYSTEMS WITH
PULSE FREQUENCY MODULATORS.

Newark College of Engineering, D.Eng.Sc.,
1973
Engineering, electrical

University Microfilms, A XEROX Company, Ann Arbor, Michigan

DYNAMICS OF INTERCONNECTED SYSTEMS
WITH PULSE FREQUENCY MODULATORS

BY

HALİL ÖZCAN GÜLÇÜR

A DISSERTATION
PRESENTED IN PARTIAL FULFILMENT OF
THE REQUIREMENTS FOR THE DEGREE
OF
DOCTOR OF ENGINEERING SCIENCE
AT
NEWARK COLLEGE OF ENGINEERING

This dissertation is to be used only with due regard to the rights of the author. Bibliographical references may be noted, but passages must not be copied without permission of the College and without credit being given in subsequent written or published work.

Newark, New Jersey

1973

APPROVAL OF DISSERTATION
DYNAMICS OF INTERCONNECTED SYSTEMS
WITH PULSE FREQUENCY MODULATORS
BY
HALİL ÖZCAN GÜLÇÜR
FOR
DEPARTMENT OF ELECTRICAL ENGINEERING
NEWARK COLLEGE OF ENGINEERING

BY
FACULTY COMMITTEE

APPROVED: _____, Chairman

NEWARK, NEW JERSEY

OCTOBER, 1973

ACKNOWLEDGEMENTS

The author wishes to express his sincere debt of gratitude and thanks to Professor A. U. Meyer for his patient and critical guidance and encouragement during the research and the writing of the thesis. He also wishes to thank Professor W. Weissman of The College of Medicine and Dentistry of New Jersey, for his kind help in the part of this work concerning Neurophysiology and for his stimulating discussions.

The advice of Professor G. Peyser has been very fruitful, in particular, that concerning the material of Chapter 3, for which I owe especial thanks to him. The author would also like to thank the other members of the dissertation committee, Professors H. J. Perlis and M. H. Zambuto for their useful comments.

I additionally want to thank to Mrs. Meyer in appreciation of her encouragement, tolerance and hospitality while working in Professor Meyer's home with him, during our many ours of conversation and review of this dissertation.

The fellowship received from the Scientific and Technical Research Council of Turkey (TÜBİTAK) is gratefully acknowledged.

ABSTRACT

The objective of this dissertation is to study the dynamics of systems consisting of interconnections of an arbitrary number of complete-reset pulse frequency modulators (CRPFM's) and linear dynamical subsystems (in general, time-varying, lumped and/or distributed). CRPFM, which represents a generalization of several types of pulse frequency modulators (PFM's), consists of two basic components; a multi-input dynamic element, called the timing-filter (TF) and a threshold device (TD). Whenever the output of the TF reaches a given threshold value the TD generates an impulse and, at the same time, resets all the states of the TF to zero. This dissertation is devoted to two basic aspects of system motion, namely stability of the equilibrium and periodic operation.

Stability is defined in terms of finiteness of the number of pulses emitted by all modulators. This definition of "finite-pulse stability" (FPS) is related to $L_1 \cap L_p$ output stability and implies finite energy expended. An improved Lyapunov-like approach is presented which, however, is difficult to employ for higher order systems. A direct criterion for FPS is given which is not only easy to apply, but also provides bounds on the

number of pulses emitted by each modulator. A comparison is presented between these criteria and previous stability conditions available for special classes of CRPFM systems (e.g., systems with integral PFM or relaxation PFM). In representative examples, the direct FPS criterion yields comparable (or better) stability regions (of parameters).

The second part is devoted to the study of the basic aspects of "periodic" behavior. For multi-modulator PFM systems, the usual concept of periodicity (or almost periodicity) is not meaningful. Therefore, a weaker concept, that of " ϵ_e -near periodicity" is introduced. This notion involves an observation interval (which is usually finite) and a measure of "desired accuracy" or "observation accuracy". Certain necessary and sufficient conditions for the existence of ϵ_e -near periodic motion are presented. For an IPFM system with a time-invariant linear part, a matrix relationship is given, which relates the "period" and the net number of pulses emitted by each modulator over that period to the system parameters.

Periodic behavior is further investigated on a time-discretized approximation of the CRPFM system which reduces to a system containing ideal delays, summing junctions and threshold elements. However, it is still difficult to obtain analytical results from the resulting

(nonlinear) difference equations (except for very short periods of oscillation); nevertheless, these equations can be "linearized" by introduction of extra variables, using Fukunaga's method for nonlinear switching nets. Therefore, classical linear techniques (based on characteristic polynomials and eigenvectors) can be used to obtain information about periodic motion. This approach also applies to McCulloch Pitts type of neural nets and extends existing results on periodic behavior in such networks.

TABLE OF CONTENTS

ACKNOWLEDGEMENTS	ii
ABSTRACT	iii
TABLE OF CONTENTS	vi
LIST OF FIGURES	ix
LIST OF TABLES	xii
CHAPTER 1. INTRODUCTION	1
1.1 General Background, Motivation and Objectives	1
1.2 Pulse Frequency Modulation, Types, Definitions and Classification	6
1.2.1 Introductory remarks.	6
1.2.2 The modulator output relation.	14
1.2.3 Complete-reset pulse frequency modulation (CRPFM).	15
1.2.4 Classification of PFM.	21
1.3 The Neuron and Relation of CRPFM To Neural Modeling	28
1.4 Review of Previous Investigation on PFM	36
1.5 Interconnected System Consisting of CRPFM's and Linear Dynamical Elements: The System Considered in This Dissertation	50
CHAPTER 2. FINITE-PULSE STABILITY OF INTERCONNECTED SYSTEMS WITH COMPLETE-RESET PULSE FREQUENCY MODULATORS	54
2.1 Global Finite-Pulse Stability in PFM Systems	56
2.2 Lyapunov's Second Method	58

2.3	Direct Finite-Pulse Stability Criteria	64
2.3.1	Application to single-loop system with one PFM.	71
2.3.2	Application to systems with more than one PFM.	75
2.4	Frequency Domain Criteria	87
2.5	Conclusions	96
CHAPTER 3.	ON NEARLY PERIODIC MOTION IN INTER-CONNECTED SYSTEMS WITH PULSE FREQUENCY MODULATORS	99
3.1	Introduction	99
3.2	The Concept of ϵ_e -Near Periodicity	101
3.3	Clues From System Stability	106
3.4	Nearly Periodic Motion in PFM Systems	112
3.5	Nearly Periodic Motion in IPFM Systems: The ϵ_e -Period	129
3.6	Conclusions	135
CHAPTER 4.	OSCILLATIONS IN INTERCONNECTED TIME-DISCRETIZED CRPFM SYSTEMS	139
4.1	Introduction	
4.2	System Considerations	140
4.2.1	Time discretization of general CRPFM system.	140
4.2.2	Interconnected system consisting of CRPFM's and ideal delays.	144
4.3	The Autonomous Case, Period of Solutions	151
4.4	Linearization of the System Equations	156
4.5	Determination of Cycles and Periods of the Linearized Net	172

4.6	State Diagram and The Transition Matrix	184
4.7	Conclusions	189
CHAPTER 5.	CONCLUSIONS AND TOPICS FOR FUTURE RESEARCH	191
5.1	Summary of The Results	191
5.2	Suggestions For Future Research	193
APPENDIX A	PROOF OF THEOREM 2.2	197
APPENDIX B	PROOF OF THEOREM 3.3	199
APPENDIX C	PROOF OF THEOREM 3.5	201
APPENDIX D	COMPUTER PROGRAM FOR THE CALCULATION OF THE RESPONSE OF CRPFM SYSTEMS	205
APPENDIX E	COMPUTER PROGRAM FOR LINEARIZING A NONLINEAR NET	222
REFERENCES		232
VITA		244

LIST OF FIGURES

Figure	Title	Page
1.1	Pulse frequency modulator, output pulse sequence and the definition of instantaneous pulse frequency.	7
1.2	A scheme for constructing static pulse frequency modulators. The threshold device (TD) emits a pulse whenever its input signal changes from negative to positive and, at the same time, resets the integrator.	8
1.3	Approximate realization of static PF modulators for slowly-varying inputs.	9
1.4	Integral pulse frequency modulator, (a) block diagram, (b) practical realization (single-signed).	10
1.5	Relaxation pulse frequency modulator (RPFM), (a) block diagram, (b) another practical realization.	12
1.6	Block diagram representation of the complete-reset pulse frequency modulator (CRPFM).	18
1.7	Discrete pulse frequency modulators: (a) Discrete RPFM (also called Discrete Σ PFM) of Shortle and Alexandro (116). (b) Discrete RPFM employed by Monopoli and Wylie.	27
1.8	Diagram of a neuron.	29
1.9	Block diagram of an interconnected system consisting of CRPFM's and linear dynamical subsystems.	51
1.10	An example illustrating the definitions of $t_{i,j}$ and $K_i(t)$.	52
2.1	A simple interconnected PFM system consisting of two RPFM's and an integrator.	62

Figure	Title	Page
2.2	Comparison of stability criteria for the CRPFM system of Fig. 2.1.	62
2.3	(a) RPFM with a first order low-pass filter. (b) Equivalent RPFM system. (c) A CRPFM system in which the dynamics of the LP is much slower than that of the TF. (d) Equivalent system obtained after transformations indicated in Fig. 2.3a and Fig. 2.3.b. Application of Theorem 2.4 to this system yields less conservative results.	74
2.4	A PFM system consisting of the interconnections of two CRPFM's and a time-invariant linear part.	77
2.5	Regions described by inequality (2.14). (a) Example 2.3a, (b) Example 2.3b, and (c) Example 2.3c.	79
2.6	Comparison of stability criteria for the IPFM system of Fig. 2.9. The region with stars is obtained by Theorem 2.5 (Example 2.5, Gelig (44)). Inside of the circular region is the stability region predicted by Theorem 2.4 (Example 2.3b, Gülçür and Meyer (52)).	82
2.7	(a) RPFM with an integrator. (b) Equivalent IPFM system. (c) Equivalent IPFM system of Fig. 2.1.	85
2.8	(a) The RPFM system considered in Section 2.4 2.4. (b) Equivalent system (TE_i is a device that emits a unit impulse whenever the absolute value of its input signal exceeds a threshold value, S_i).	89
2.9	A PFM system consisting of the interconnections of two IPFM's and a time-invariant linear part.	93
3.1	Block diagram of an interconnected system consisting of CRPFM's with linear TF's and linear dynamical subsystems.	101

Figure	Title	Page
3.2	Illustration of Def. 3.1. The function $e(t)$ is ϵ_e^1 -nearly periodic in the time-interval $t \in (0, 2T)$ and ϵ_e^0 -nearly periodic in the time-interval $t \in (0, \infty)$.	103
3.3	Examples of ϵ_e -nearly periodic motion in PFM systems: (a), (b) output waveform of the TF in an IPFM under sinusoidal input; (a) double signed IPFM, (b) single-signed IPFM (note that these waveforms are <u>not</u> periodic in the strict sense). (c) A typical ϵ_e -nearly periodic motion in a PFM system with an almost periodic linear part. (d), (e) Typical ϵ_e -nearly periodic motion in simple CRPFM systems.	105
3.4	Illustration of functions used in Theorem 3.3. (a) Output variable $y(t)$ under ϵ_e -nearly periodic motion. (b) The functions $y_0(t)$ and $y_1(t)$ as defined by (3.7a). (d) $y_0(t)$ and $y_1(t+T)$ shown for comparison. (e) The function $y_1(t+T) - y_0(t)$.	114
3.5	Output waveforms of the system of Example 3.1.	118
3.6	Desired pulse pattern of the pulse generator.	129
4.1	(a) A CRPFM system with time-invariant TF and LP. (b) Time-discretized approximation, using eqs. (4.1)-(4.5).	142
4.2	Block diagram of an interconnected system consisting of m CRPFM's and ideal delays.	143
4.3	State transition diagram of the system considered in Example 4.1.	154
4.4	Operation rules for the Galois field $GF(2)$	160
4.5	(a) Schematic representation of a neural network consisting of interconnections of two excitatory neurons (neuron 1 and neuron 3) and an inhibitory neuron (neuron 2). (b) Block diagram of a simplified model of the same neural network.	161

Figure	Title	Page
4.6	Truth tables of the system of Fig. 4.5b	162
4.7	Truth table for the system of Example 4.1	168
4.8	State transition diagram of an autonomous linear net with a primitive minimum polynomial [$m(\lambda) = p(\lambda) = \lambda^4 + \lambda + 1$].	177
4.9	State transition diagram of an autonomous linear net with an irreducible minimum polynomial [$m(\lambda) = \lambda^4 + \lambda^3 + \lambda^2 + \lambda + 1$].	179
4.10	State transition diagram of an autonomous linear net with $m(\lambda) = p(\lambda) = (\lambda + 1)^2$.	181
4.11	State diagram of the equivalent linear net of the PFM system of Example 4.1. Only the states $\underline{x}_1 - \underline{x}_4$ correspond to the original system; \underline{x}_0 and $\underline{x}_5 - \underline{x}_{15}$ are artificial states.	182
4.12	Illustration of state transitions.	187
4.13	State transition diagram.	188

LIST OF TABLES

Table	Title	Page
2.1	Comparison of The Stability Results For Single-Loop, Single-IPFM (or RPFM) Feed-back System ($A \triangleq MA_0/S$).	72
2.2	Comparison of Stability Theorems For Multi-Modulator PFM Systems.	98
4.1	Prime Factors of $2^n - 1$.	178

CHAPTER 1
INTRODUCTION

1.1 General Background, Motivation and Objectives

Modulation is the process of coding information into a carrier wave by varying some of its characteristics in accordance with a modulation law. In control systems, modulation is used for a variety of reasons; e.g., to adapt to a given mode of controlling power, to utilize given communication channels for some of the signals, to improve noise immunity and accuracy, etc. The carrier wave can be continuous or it can consist of a sequence of pulses. The first case is called continuous wave modulation (CWM) and the second case is called pulse modulation (PM).

Most common forms of CWM are amplitude modulation (AM) and frequency modulation (FM) with a sinusoidal carrier wave. Examples of CWM used in control systems are AC servo systems employing 50 Hz, 60 Hz or 400 Hz sinusoidal carrier AM (63).

With the advance in digital technology, during the early 1950's, pulse modulation has become a subject of increasing interest. A pulse modulator is characterized by the instances of pulse-emission, all in relation to the dynamics of the input signal; this characterization

constitutes the modulation law. Depending on the modulation law, PM can be divided into various groups: pulse-amplitude modulation (PAM), pulse-width modulation (PWM), pulse-position modulation (PPM) and pulse-frequency modulation (PFM) (56, 115). In certain applications it is advantageous to use combinations of the above basic types of PM, e.g. in pulse-code modulation (PCM) (21) and pulse-width-pulse-frequency modulation (PWPFM) (31, 32, 42, 66, 76, 78, 121).

Among the different pulse modulation schemes, PFM is of particular interest because it constitutes the means of information transmission used in the nervous system (6, 46, 64, 84). A pulse frequency modulator is a device that codes information of its input signal into time-intervals and polarities of identical pulses emitted at its output. There is an infinite number of ways by which this coding can be achieved and, not surprisingly, during the relative short time of research activity in this field, many different types of PF modulators have been introduced; they will be reviewed in the next section.

Basically, PFM constitutes a form of relaxation oscillation; most PF modulators can be realized easily by means of simple RC filters and a few relays or solid state threshold devices. There are certain control appli-

cations which favor PFM; for example, control systems employing stepper motors. Among the features of a PFM control system is the fact that it may be designed such that pulses are emitted when needed; this is especially important in applications where control power must be conserved such as in certain spacecraft control systems employing controlling jets. Another feature of PFM is that it has a good degree of noise-immunity (6, 12, 60, 61, 89) as compared to PAM or PWM.

Applications of PFM in control systems have been reported in the following fields.

1. telemetry (108-110),
2. adaptive flight control systems (96),
3. satellite attitude control systems (29, 38),
4. converting continuous signals into proportional pulse frequency for digital processing (37,97),and
5. modelling of neural systems (6,46,85,86,102, 106).

The fact mentioned earlier that information transmission in the nervous system takes place in terms of PFM is a major motivation of research on PFM including the work of this dissertation. There are two equally-significant reasons for this motivation. 1) It is hoped that new understanding could be provided to neuro-physiological system behavior, and 2) since it is believed that biological control sys-

tems have evolved toward optimal states (6), it is anticipated that PFM control systems could provide certain technological advantages, e.g., noise immunity, adaptability, efficiency, etc. Certain aspects of these expectations have already been demonstrated by previous investigators (see Section 1.4).

The bulk of the previous research on PFM is devoted to the study of single-loop PFM feedback systems; relatively little work has been done on multi-loop, multi-modulator PFM systems. However, in order to fully examine the afore-mentioned expectations, a thorough understanding of systems containing several PF modulators is essential. Therefore, the objective of this dissertation is to study the dynamics of systems consisting of interconnections of an arbitrary number of PF modulators and dynamical subsystems.

The scope of this dissertation will be limited to systems containing complete-reset pulse frequency modulators (CRPFM) and linear dynamical subsystems. CRPFM is a generalization of many other known forms of PF modulators and consists of two distinct parts; a multi-input, single-output dynamic element, called the timing filter (TF) which defines the pulse emission instants, and a threshold device (TD) which generates an impulse whenever the output of the TF reaches a given threshold value. A formulation of the CRPFM and the system considered will be given in Sections 1.2.3 and 1.5, respectively.

One of the basic characteristics of a system is its stability. Chapter 2 will be devoted to this important topic; a Lyapunov-like method and a direct method for stability analysis will be presented. Comparison will be made between these methods, including the previously existing methods.

Knowledge of the "periodic" behavior of CRPFM systems can shed light into the manner information is manipulated in the nervous system. For example, reverberatory activity in neural circuits has been suggested as a possible mechanism for short-term memory (see Section 3.1). Thus, a chapter (Chapter 3) is devoted to the basic aspects of "periodic" motion in CRPFM systems. It turns out that for multi-modulator PFM systems, the usual concept of periodicity does not have much meaning. Therefore, a weaker concept of periodicity, that of " ϵ_e -nearly periodicity", will be introduced (Section 3.1) and some basic rules for " ϵ_e -nearly periodic behavior" will be presented.

In order to obtain further insight, this problem will also be studied, in Chapter 4, for a more special system consisting of CRPFM's and ideal delay elements. This special system possesses all the essential properties of neural systems (see Sections 1.3 and 4.2).

1.2 Pulse Frequency Modulation, Types, Definitions and Classification

1.2.1 Introductory remarks. Consider the pulse sequence shown in Fig. 1.1. Let t_1, t_2, \dots denote the instants of pulse-occurrences; the instantaneous frequency of the pulse sequence (64) is defined by

$$f_k = \frac{1}{t_k - t_{k-1}} = \frac{1}{T_k}, \quad t \in [t_k, t_{k+1}) \quad (1.1)^1$$

From this definition, it follows that the instantaneous pulse frequency is a staircase function as shown in Fig. 1.1c.

In "memoriless" or static pulse frequency modulators (SPFM), the instantaneous pulse frequency is a single-valued function of the input signal magnitude at time t_{k-1} or t_k ; in case of the latter:

$$f_k = \frac{1}{t_k - t_{k-1}} = f[e(t_k)] \quad (1.2)$$

¹Here, unidirectional pulses are considered. Certain applications require positive pulses as well as negative pulses; in which case, the value given by eq. (1.1) does not correspond to the usual concept of frequency. Because of this, the quantity defined by eq. (1.1) is sometimes called (instantaneous) pulse-repetition rate. Also, the term pulse-repetition rate modulation is sometimes used as a more precise substitute for PFM.

Eq. (1.2) also means that, after the emission of the $(k-1)$ th pulse, both $(t-t_{k-1})$ and $1/f[e(t)]$ are continuously compared with each other and as soon as both become equal, the next pulse is emitted at $t = t_k$.

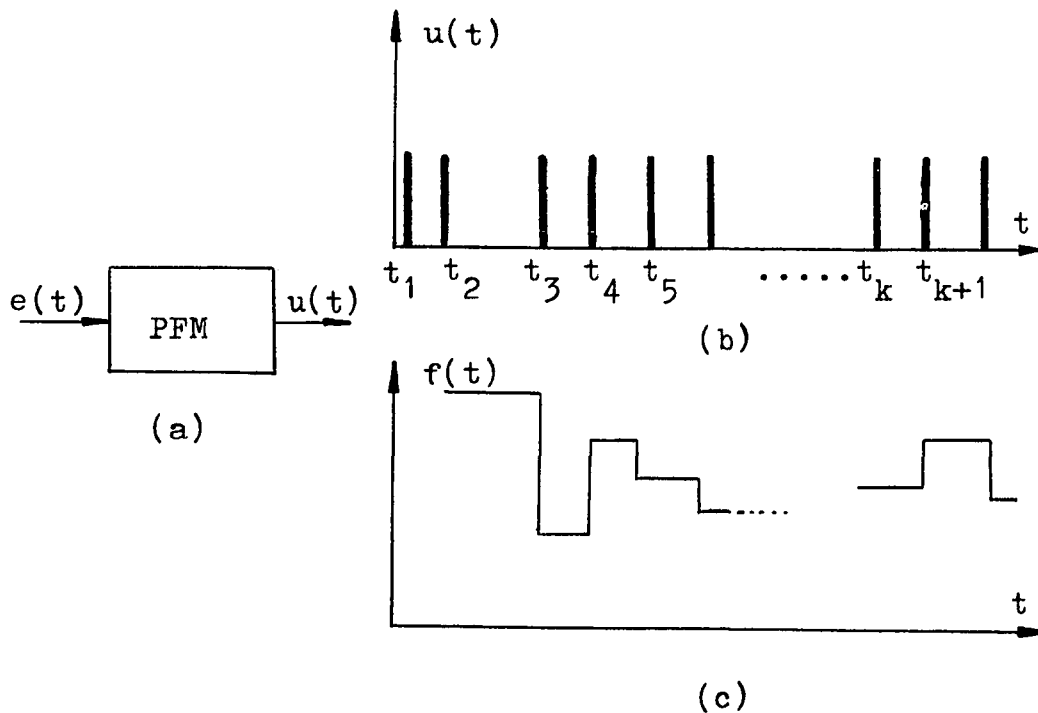


Figure 1.1 Pulse frequency modulator, output pulse sequence and the definition of instantaneous pulse frequency.

In one of the early static pulse frequency modulators the function $1/f(e)$ was given simply by $K \cdot e(t)$, with K being a proportionality constant, Ross, 1949, (110). In that case, the instantaneous pulse frequency is inversely proportional to the input-signal level. Of greater prac-

tical significance; however, are situations where an instantaneous pulse frequency is required that is proportional to the input signal level. This can be accomplished, if the function $f(e)$ is in the form $K \cdot e$. A static PF modulator of this type will be called a linear pulse frequency modulator. (LPFM).

There exist a number of possibilities for realizing static pulse-frequency modulators. One possibility is shown in Fig. 1.2. An integrator is used to generate a

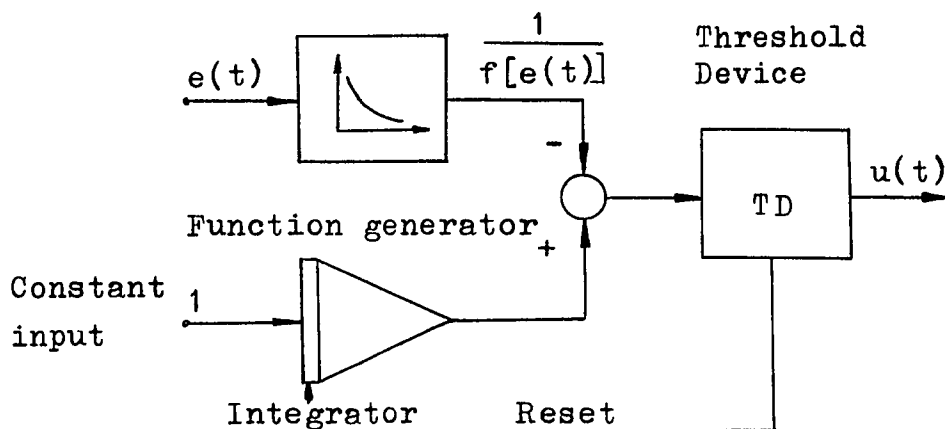


Figure 1.2 A scheme for constructing static pulse frequency modulators. The threshold device (TD) emits a pulse whenever its input signal changes from negative to positive and, at the same time, resets the integrator.

signal proportional to $(t-t_{k-1})$ and a (diode) function generator is used to generate a signal proportional to $1/f[e(t)]$. The difference between these two signals is fed to a threshold device (TD) which emits a pulse as soon

as its input becomes positive and, at the same time, resets the output of the integrator to zero.

Now, assume that the input signal $e(t)$ has a very slow variation with respect to the pulse repetition rate. In that case, the modulator schematic of Fig. 1.2 can be approximated by that of Fig. 1.3, where the integrator

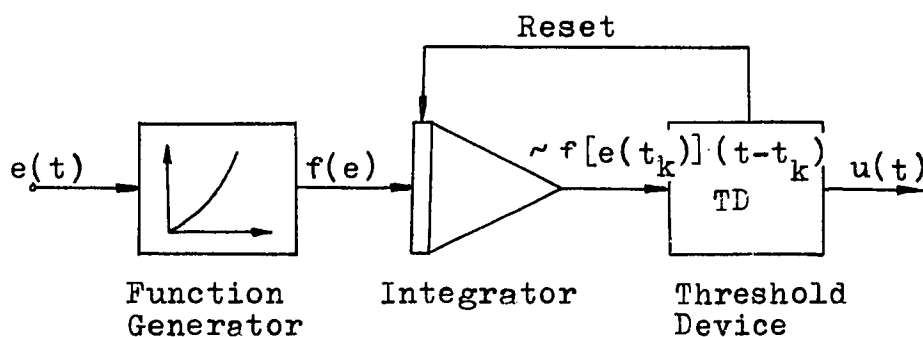


Figure 1.3 Approximate realization of static PF modulators for slowly-varying inputs.

is fed the signal $f[e(t)]$, instead of the constant input 1, and the threshold device (TD) is adjusted such that it emits a pulse whenever its input signal reaches a threshold value of 1. For the LPFM, since $f(e) = K \cdot e$, the function generator is not needed and the final circuit becomes very simple, as shown in Fig. 1.4.

The circuit of Fig. 1.3 approximates the static PFM of Fig. 1.2 for very slow variations of $e(t)$. In general,

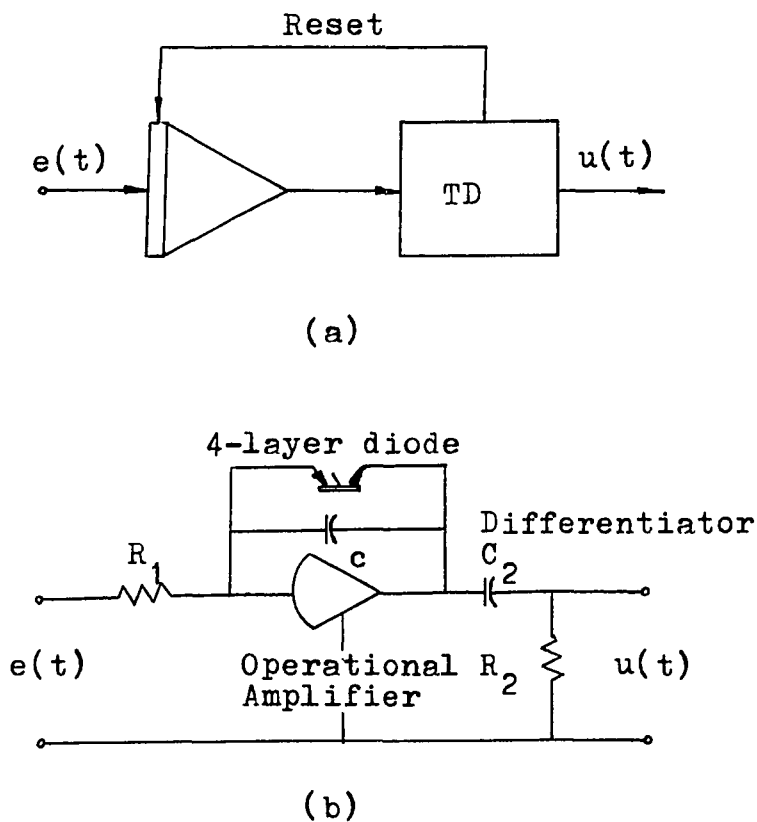


Figure 1.4 Integral pulse frequency modulator,
 (a) block diagram, (b) practical realization
 (single-signed).

however, it represents a different type of modulator in its own right. The same is true for the modulator circuit of Fig. 1.4 in reference to LPFM.

The device shown in Fig. 1.4 integrates its input signal and emits a pulse as soon as it reaches a threshold value, resetting the integrator output to zero at the

same time². This modulator was first defined by Meyer (93) and Li (89) and is called integral pulse frequency modulator (IPFM). For slowly-varying input signals, IPFM produces a pulse train having an instantaneous frequency directly proportional to its input (similar to LPFM)³, furthermore, its ability to smooth-out (through the integration process) any noise superimposed on the input signal, provides it an additional advantage.

A significant difference between the static PF modulators explained previously and IPFM is that, in the latter, the emission of pulses are decided by not only observing the instantaneous value of the input signal, but also its previous values. Therefore, a pulse frequency modulator of this type will be called a dynamic pulse frequency modulator (DPFM).

²In this introductory sub-section, only modulators that emit single-polarity pulses are considered (single-signed PFM). Hence, it is assumed that the signals $f[e(t)]$ and $z(t)$ are nonnegative. When negative pulses are allowed as well as positive pulses, violation of this restriction will not cause in any loss of information if pulse emission instants are determined from $|f[e(t)]|$ or $|z(t)|$ and the sign information is reflected on the output pulses. This second case is called double-signed PFM. For communication applications and for control applications involving stepper motors and/or digital processors, usually single-signed modulators are used. For most control applications, however, double-signed PFM is preferred.

³This fact may be used in designing voltage-to-frequency converters. The circuit of Fig. 1.4b is a fundamental form for many voltage-to-frequency converters.

Another well known DPFM is the relaxation type pulse frequency modulator (RPFM) (90, 93), which is a degeneration of the IPF modulator with a leaky integrator, as shown in Fig. 1.5.

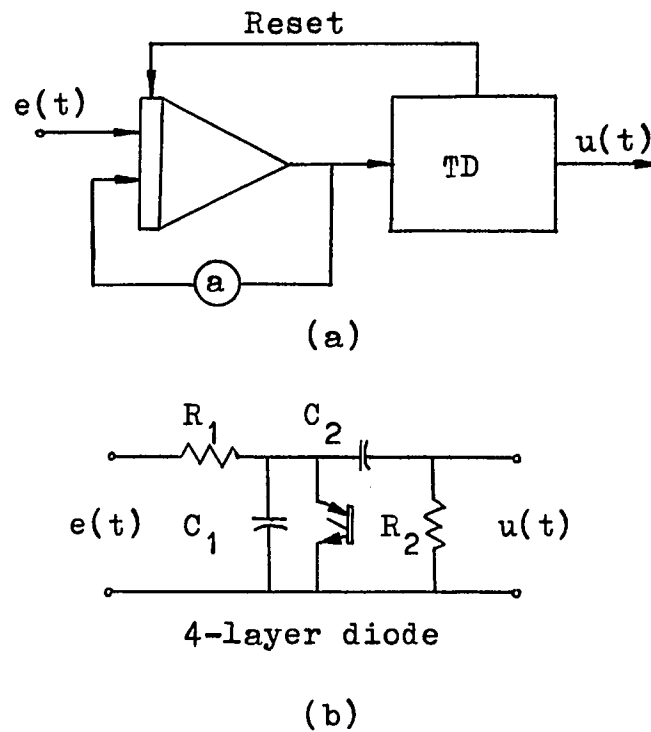


Figure 1.5 Relaxation pulse frequency modulator (RPFM), (a) block diagram, (b) another practical realization.

A more general DPFM scheme is to feed the input signal into a dynamical system with a single output (which will be called a timing filter, (TF) and emit a pulse as soon as the TF-output exceeds a threshold value, S . Immediately following the pulse emission, some or all of the internal states of the TF are reset to fixed values.

In the first case, the DPFM will be called a partial-reset PFM (PRPFM) and in the second case a complete-reset PFM (CRPFM) (51, 52).

PFM is basically an asynchronous form of pulse modulation since the time-interval between successive pulses is used for information coding purposes. This may be an advantage in some applications since it eliminates the need for costly synchronization equipment; however, in other applications, e.g., in application where time multiplexing is an economic necessity, or in certain applications involving digital processing, it may be necessary to assign a clock signal to the output pulses. This form of modulation is called synchronous PM. Pulse amplitude modulation (PAM), pulse width modulation (PWM) and pulse code modulation (PCM) are examples of synchronous PM. However, it is also possible to introduce a clock signal to PFM; in that case the information coding may be performed by counting the number of pulses within each given period (of the clock signal). Such a form of pulse modulation (116, 128) may be called synchronous PFM or discrete PFM.

In this dissertation discussions will be centered mainly on the complete-reset PFM (CRPFM); this is done for the following reasons:

- (a) CRPFM represents a generalization of PF modulators which have proven to be useful for many control applications, namely, the IPFM and the RPFM.
- (b) It resembles the process of impulse generation in the nervous system.
- (c) It can easily be realized using a simple filter (RC, RLC, or active) plus a few discrete-type elements.

CRPFM will be discussed in more detail in sub-section 1.2.3.

1.2.2 The Modulator output relation. The output signal of any pulse frequency modulator, $u(t)$, is defined in terms of a sequence of impulses of equal strength, M and of impulse polarity $b_k = \pm 1$, emitted at time-instances t_k ($k = 1, 2, \dots$), i.e.,

$$u(t) = M \sum_{k=1}^N b_k \delta(t-t_k), \quad 0 \leq t < t_{N+1} \quad (1.3)$$

where $\delta(t)$ is the unit impulse. Eq. (1.3) shall be called the modulator-output relation.

The pulse emission times, t_k and pulse-polarities, b_k follow some given functional relations in terms of the input signal, $e(t)$; i.e., both t_k and b_k are determined

by a modulator input relation for a given PF modulator type.

Note that the modulator-output is defined in terms of impulses rather than some defined waveform. This is done for the sake of generality. Any physical pulse-waveform, say $f(t)$, can be obtained by feeding the modulator output $u(t)$ through a linear filter of transfer function $F(s) = \mathcal{L}\{f(t)\}$. Since this linear filter can be combined with the subsystem following the modulator, the modulator output as given in (1.3) represents a convenient general form.

1.2.3. Complete - reset pulse frequency modulation (CRPFM). Before proceeding to the definition of CRPFM, two well known examples of CRPFM will be discussed, namely, integral pulse frequency modulation (IPFM) and relaxation pulse frequency modulation (RPFM). These two modulators have already been discussed in sub-section 1.2.1 (see Figs. 1.4 and 1.5)⁴.

First, consider integral pulse frequency modulation (IPFM) (64, 90, 94), which is defined such that the input signal $e(t)$ is fed to an integrator whose output, $z(t)$,

⁴Figs. 1.4 and 1.5 represent single-signed IPFM and RPFM, respectively (output-pulses have one polarity only). Here in sub-section 1.2.3, the general case of double-signed PFM will be presented.

is fed to a threshold device (TD), which, whenever $|z(t)|$ reaches a threshold value, S , resets the integrator-output to zero and emits an impulse of strength M , whose polarity is equal to the sign of $z(t)$ just before the impulse-emission. Thus, the functional relations defining t_k and b_k are given by

$$z(t) = \int_{t_{k-1}}^t e(\tau) d\tau, \quad t_{k-1} \leq t < t_k \quad (1.4)$$

$$t_k = \min \{ t \mid t > t_{k-1} \text{ and } |z(t)| \geq S \} \quad (1.5a)$$

and

$$b_k = \text{sgn}[z(t_k^-)] \quad (1.5b)^5$$

IPFM is a simple form of PFM, whose definition was inspired by pulse modulation in the nervous system (Meyer, 1961), though, of course, the relation between IPFM and PFM in the nervous system is very approximate⁶. A process somewhat closer related to PFM in the nervous system, yet still representing a rather gross simplification of the latter, is given in terms of relaxation pulse frequency modulation⁷ (RPFM) (93), defined by

⁵Note from (1.4) that $z(t_k^+) = 0$ for all $k = 1, 2, \dots$

⁶It is possible, however, to represent some other types of PFM in terms of IPFM and dynamic elements (93).

⁷The name relaxation pulse frequency modulation comes from its relation to relaxation oscillation (93). A generalization of RPFM where the first order relation (1.6)

$$z(t) = \int_{t_{k-1}}^t e^{-\frac{t-\tau}{T_R}} e(\tau) d\tau, \quad t_{k-1} < t < t_k \quad (1.6)$$

Eqs. (1.3) and (1.5) remain the same for RPFM. Note that RPFM represents a generalization of IPFM, to which it reduces when $T_R \rightarrow \infty$.

Complete-reset pulse frequency modulation (CRPFM) represents a generalization of the above to the extent that the dynamic element of input $e(t)$ and response $z(t)$ can be of any order (not necessarily of order one as in (1.4) or (1.6)). CRPFM will be described next.

Fig. 1.6 shows the functional block diagram of the CRPFM. It consists of a resettable timing-filter (TF) and a threshold device (TD). The TD is activated by the output signal of the TF, $z(t)$, in such a way that an impulse is emitted whenever $|z(t)|$ exceeds a threshold-level, S ; the polarity of that impulse is equal to the sign of $z(t)$. Furthermore, at the instant of impulse-emission all state variables of the timing filter are reset to zero⁸.

between $e(t)$ and $z(t)$ is not linear as in (1.6) but non-linear, has been defined by Pavlidis and Jury as Σ PFM (101). In the literature, RPFM has also been called "neural PFM" (NPFM) (101).

⁸CRPFM incorporates the features of the "nth order neural trigger" of Pavlidis (102), the "functional pulse

IPFM and RPFM, discussed above, are special cases of CRPFM, where the timing-filter (TF) is of first order. In fact, the modulator output relation (1.3) and the threshold relations (1.5) are valid for (general) CRPFM.

Eqs. (1.4) and (1.6) are the timing-filter equations for IPFM and RPFM, respectively. As illustrated in Fig. 1.6, the timing filter equation for general CRPFM is given by

$$z(t) = \int_{t_{k-1}}^t f[e(\tau), t, \tau] d\tau, \quad t_{k-1} < t < t_k \quad (1.7)$$

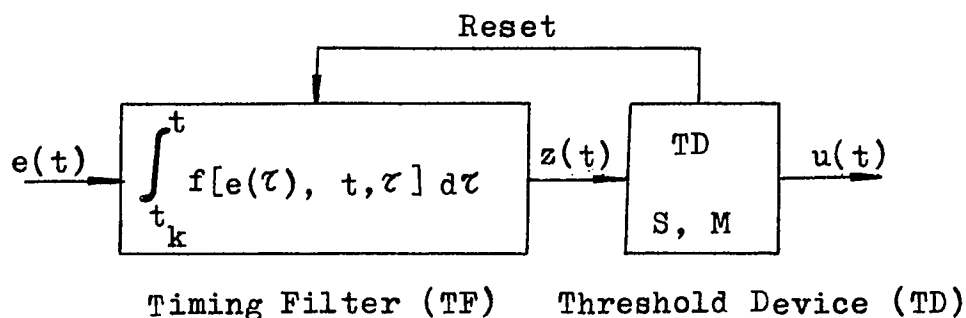


Figure 1.6 Block diagram representation of the complete reset pulse frequency modulator (CRPFM).

frequency modulator" of Jury and Blanchard (65), the "type II pulse modulator" of Skoog and Blankenship (121) and the "pulse frequency modulator of the second kind with complete clearing of the time-marking filter" of Kuntsevich and Chekhovoi (79).

where $e(t)$ represents the modulator-input signal. The function $f[e(\tau), t, \tau]$ is usually of the form

$$f[e(\tau), t, \tau] = g_0(t, \tau) e(\tau) \quad (1.8)$$

where the kernel $g_0(t, \tau)$ is the impulse response of a (usually RC lowpass) single-input, single-output linear dynamic system⁹. For IPFM (see, eq. (1.4)), it is

$$g_0(t, \tau) = 1 \quad (t > \tau) \text{ and for RPFM (see, eq. (1.6)), it is}$$

$$g_0(t, \tau) = e^{-\frac{t-\tau}{T_R}} \quad (t > \tau).$$

In addition to the generalization of the TF from that for IPFM (eq. (1.4)) and RPFM (eq. (1.6)) to that for CRPFM (eq. (1.7)), the threshold relation will be generalized from that given by (1.5) (for both IPFM

⁹Let the timing filter output $z(t)$ be described in terms of the state vector $\underline{x}(t)$ of the TF and input signal $e(t)$ as¹⁰

$$z(t) = \underline{c}^T(t) \left[\underline{\varnothing}(t, t_k) \underline{x}(t_k^+) + \int_{t_k}^t \underline{\varnothing}(t, \tau) \underline{b}(\tau) e(\tau) d\tau \right]$$

$$t_k < t < t_{k+1}$$

where $\underline{\varnothing}(t, \tau)$ is the state transition matrix, $\underline{b}(t)$ is a column vector and $\underline{c}^T(t)$ is a row vector. Since during impulse emission, at time t , the threshold device resets the state to zero, i.e., $\underline{x}(t_k^+) = 0$, the first term of the above equation vanishes. Comparison with eqs. (1.7) and (1.8) gives, therefore

$$g_0(t, \tau) = \underline{c}^T(t) \underline{\varnothing}(t, \tau) \underline{b}(\tau).$$

¹⁰In this dissertation, underlined capital letters, underlined small-case letters, and the superscript, T, will be used to denote, matrices, column vectors and the transpose of a matrix, respectively.

and RPFM) to include a refractory period, T_0 , which is a time interval during which the modulator cannot regenerate another impulse. A refractory period exists in physical PFM such as that in the nervous system (see Section 1.3). With this inclusion the threshold relation for CRPFM becomes:

$$t_k = \min\{t \mid t > t_{k-1} + T_0 \text{ and } |z(t)| \geq S\} \quad \begin{array}{l} \text{for (double-} \\ \text{signed)} \\ \text{CRPFM} \end{array} \quad (1.9)$$

The pulse-intervals, as well as the pulse-polarities, are used as carriers of information. There, exists, however, physical PFM where only pulses with one polarity are emitted (e.g., in the nervous system); such case is referred to as single-signed PFM, in order to distinguish it from the more general case of double-signed PFM (above). For the sake of brevity, the prefix "double-signed" may not be used, i.e., the term "CRPFM" is defined to imply "double-signed CRPFM". However, for single-signed PFM the prefix will be necessary. For single-signed CRPFM, the threshold relation is given by

$$t_k = \min\{t \mid t > t_{k-1} + T_0 \text{ and } z(t) \geq S\} \quad \begin{array}{l} \text{for single-} \\ \text{signed} \\ \text{CRPFM} \end{array} \quad (1.10)$$

A variation of the CRPFM is the partial-reset PFM (PRPFM), in which only some of the internal states of the TF are reset. A particular case of such a scheme is the

output-reset PFM (ORPFM), where only the output of the TF is reset. Note that IPFM and RPFM also belong to this category, since they are of first-order.

If the TF of an ORPFM is linear and of a certain structure, it may be expressible in terms of another ORPFM and a linear subsystem. This would be useful in some analytic studies of dynamic systems containing ORPFM's. Examples of this point will be given in Sections 2.3 and 2.4.

1.2.4 Classification of PFM. The output relation for any pulse frequency modulator has been presented in Section 1.2.2 as eq. (1.3):

$$u(t) = M \sum_{k=1}^N b_k \delta(t-t_k), \quad 0 \leq t < t_{N+1} \quad (1.3)$$

In this sub-section, a classification of PFM will be presented in terms of the dependency of the pulse-emission times, t_k and the pulse-polarities, $b_k = \pm 1$, in terms of the input signal $e(t)$ over $-\infty \leq t < t_k$. In general, this relation may be expressed as

$$t_k = \mathcal{G}_t[t_1, t_2, \dots, t_{k-1}; e(t), t, -\infty \leq t < t_k] \quad (1.11a)$$

and

$$b_k = \mathcal{G}_b[t_1, t_2, \dots, t_{k-1}; e(t), t, -\infty \leq t < t_k] \quad (1.11b)$$

where \mathcal{G}_t and \mathcal{G}_b are functional operators (\mathcal{G}_t is a positive operator¹¹). Depending on the type of these operators PFM may be subdivided into various classes:

(A) Finite-Memory PFM of Order N (FMPFM):

In this case the k th pulse-instant, t_k is a single-valued function of both the previous N pulse-instants, t_{k-1} , t_{k-2} , ..., t_{k-N-1} and the values of the input at these instants, i.e.,

$$t_k = f_t[t_{k-1}, t_{k-2}, \dots, t_{k-N-1}; e(t_k), \dots, e(t_{k-N})] \quad (1.12a)$$

and

$$b_k = f_b[t_{k-1}, t_{k-2}, \dots, t_{k-N-1}; e(t_k), \dots, e(t_{k-N})] \quad (1.12b)$$

(A.1) Special case: Static PFM (SPFM) (N = 0):

$$t_k = t_{k-1} + f_t[e(t_k)] \quad (1.13a)^{12}$$

and

$$b_k = f_b[t_k, e(t_k)] \quad (1.13b)$$

Examples of SPFM:

(A.1a) Ross' SPFM (Ross, 1949)

$$t_k = t_{k-1} + K \cdot e(t_k) \quad (\text{defined for } (1.44a) \text{ nonnegative}$$

and

$$b_k = 1 \quad (\text{and continuous inputs}) \quad (1.14b)$$

¹¹See Willems (129), p.26.

¹²Note that $f_t[e(t_k)] = 1/f[e(t_k)]$, where $f[e(t_k)]$ is as defined by (1.2).

(A.1b) Linear PFM (LPFM)

$$t_k = t_{k-1} + \frac{1}{K e(t_k)} \quad \begin{array}{l} \text{(defined (1.15a)} \\ \text{for} \\ \text{positive} \\ \text{inputs)} \end{array} \quad (1.15a)$$

$$b_k = 1 \quad (1.15b)$$

(A.2) Special case: Finite-memory PFM (FMPFM
of order 1.

Examples of FMPFM of order 1:

(A.2a) PFM of the first type (Kuntsevich and
Chekhovoi, 1967, (75))

$$t_k = t_{k-1} + f[e(t_{k-1})] \quad (1.16a)$$

$$b_k = \begin{cases} 1 & \text{for } e(t_k) > S \\ 0 & \text{for } |e(t_k)| < S, S > 0 \\ -1 & \text{for } e(t_k) > -S \end{cases} \quad (1.16b)$$

(A.2b) Amplitude dependent PFM (Clark and
Noges, 1966, (27))

t_k is given by eq. (1.16a) with

$$f(e) = \begin{cases} T_N - \frac{e}{S} (T_N - T) & \text{for } |e| < S \\ T & \text{for } |e| > S \end{cases} \quad (1.17a)$$

where T and T_N are positive constants and,

$$b_k = \text{sgn } e(t_k) \quad (1.17b)$$

(A.2c) δ -modulation¹³

$$t_k = t_{k-1} + T \quad (1.18a)$$

$$b_k = \text{sgn } e(t_k) - e(t_{k-1}) \quad (1.18b)$$

Note that here the pulse-output is periodic; the modulation affects only the pulse-polarities.

(B) Partial-Reset PFM (PRPFM):

$$t_k = \min\{t \mid t > t_{k-1} + T_0 \text{ and } |z(t)| \geq S\} \quad (1.19a)$$

$$b_k = \text{sgn } z(t_k) \quad (1.19b)$$

where, for a PRPFM with a linear TF, $z(t) = \underline{c}(t)^T \underline{x}(t)$, where the state \underline{x} of the TF consists of two component-vectors, \underline{x}_1 , and \underline{x}_2 , such that \underline{x}_1 is reset to a vector-value $\underline{\alpha}$ during pulse emission, i.e.,

$$\begin{aligned} \underline{x}(t) &= \begin{bmatrix} \underline{x}_1(t) \\ \underline{x}_2(t) \end{bmatrix} = \underline{\phi}(t, t_{k-1}) \begin{bmatrix} \underline{\alpha} \\ \underline{x}_2(t_{k-1}) \end{bmatrix} + \\ &+ \int_{t_{k-1}}^t \underline{\phi}(t, \tau) \underline{b}(\tau) \underline{e}(\tau) d\tau \end{aligned} \quad (1.19c)$$

The other states of the TF represented by \underline{x}_2 , are not reset.

¹³There are many different forms of δ -modulation, which is becoming popular in communication; eqs. (1.18a) and (1.18b) describe one of the early forms (see (1) for more detail).

(B.1) Complete-reset PFM (CRPFM):

CRPFM was introduced in subsection 1.2.3. It may be considered as a special case of partial-reset PFM where $\underline{x}_1 = \underline{x}$, $\underline{x}_2 = 0$ and $\underline{a} = 0$. Therefore, the relations for t_k and b_k are given by (1.19a) and (1.19b), respectively. The expression for $z(t)$ becomes

$$z(t) = \int_{t_{k-1}}^t \underline{c}(t)^T \phi(t, \tau) \underline{b}(\tau) \underline{e}(\tau) d\tau, \quad (1.20a)$$

which, more generally, can be written as

$$z(t) = \int_{t_{k-1}}^t f[e(\tau), t, \tau] d\tau. \quad (1.7)$$

Examples of CRPFM:

(B.1a) Integral PFM (IPFM) (Jones, Meyer and Li, 1961, (64))

$$z(t) = \int_{t_{k-1}}^t e(\tau) d\tau \quad (1.4)$$

(B.1b) Modified IPFM (Bombi and Ciscato, 1967, (11))¹⁴

In this case, eq. (1.19a), defining the impulse-instants is modified to

$$t_k = \min\{t \mid t > t_{k-1} \text{ and } z(t) = f(t - t_{k-1})\} \quad (1.21)$$

i.e., the threshold level is dependent on $(t-t_{k-1})$. The equations for the pulse polarities b_k and the output of the TF, $z(t)$ are the same as those for IPFM.

(B.1c) Relaxation PFM (RPFM) (Meyer, 1961, (93))

$$z(t) = \int_{t_{k-1}}^t e^{-a(t-\tau)} e(\tau) d\tau, \quad (1.6)$$

where a is a constant.

(B.1d) Sigma PFM (Σ PFM)¹⁵ (Pavlidis and Jury, 1965, (101))

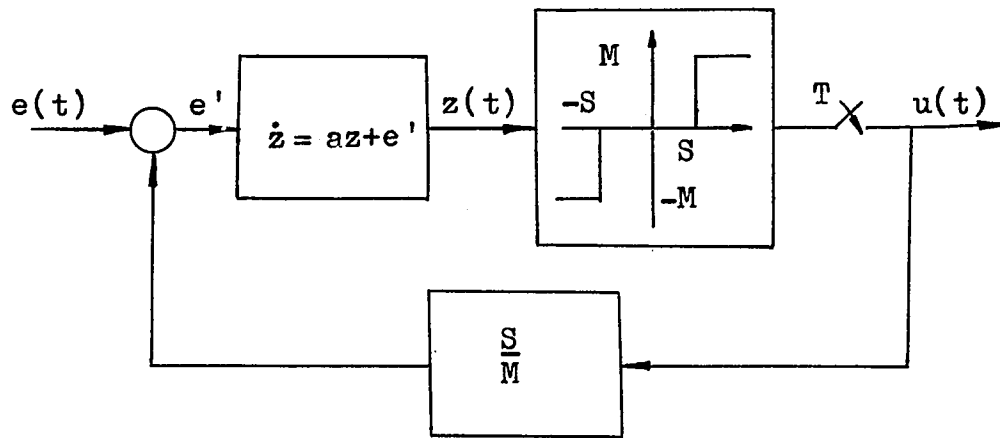
$$z(t) = \int_{t_{k-1}}^t \{e(\tau) - g[z(\tau)]\} d\tau \quad (1.22)$$

(B.1e) Discrete RPFM¹⁶ (Shortle and Alexandro 1966, (116))

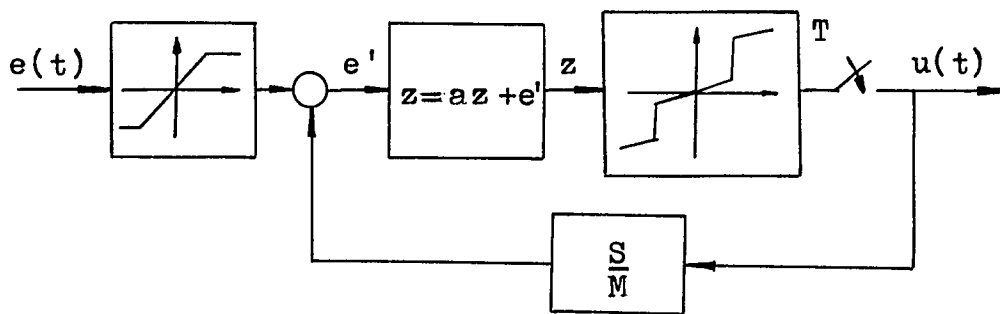
This is a discrete approximation of RPFM (See Fig. 1.7a); output pulses of the modulator are allowed to

¹⁴Based on experimental evidence, Bombi and Ciscato claim that a feedback system employing the modified IPFM can have a better transient response without sacrificing noise immunity (11).

¹⁵Pavlidis later used the same name (i.e., Σ PFM) for any process in which a dynamic system emits an impulse whenever any one of its variable (from a specified group) exceeds a threshold value (in general, time-varying) associated with that variable (103).



(a)



(b)

Figure 1.7 Discrete pulse frequency modulators:
 (a) Discrete RPFM (also called Discrete Σ PFM) of Shortle and Alexandro (116).
 (b) Discrete RPFM employed by Monopoli and Wylie (128).

occur only at discrete intervals of time.

$$t_k = t_{k-1} + T \quad (T \text{ is a fixed sampling interval}) \quad (1.23a)$$

$$z(t_k) = e^{-aT} z(t_{k-1}) - b_{k-1} S + \int_{t_{k-1}}^{t_{k-1}+T} e^{-a(t_k+T-\tau)} e(\tau) d\tau \quad (1.23b)$$

$$b_k = \begin{cases} M & \text{for } z(t_k^-) > S \\ 0 & \text{for } |z(t_k^-)| < S \\ -M & \text{for } z(t_k^-) < -S \end{cases} \quad (1.23c)$$

Shortle and Alexandro call this modulation discrete Σ PFM. Note that the discrete RPFM is also a nonlinear pulse amplitude modulator. A slightly different version of the discrete RPFM has been studied by Wylie (128), in which the dead zone characteristic is modified and a saturation type nonlinearity is connected in series with the modulator (see Fig. 1.7b).

1.3 The Neuron and Relation of CRPFM to Neural Modeling

In Section 1.1 it was mentioned that information transmission in the nervous system may be expressed in terms of PFM. For the benefit of uninitiated reader, it is appropriate to digress slightly and provide some brief explanation about the neuron and its properties.

The fundamental unit of the nervous system is the

neuron (nerve cell). It can be shown that the CRPFM exhibits many of its properties. The objective in this section is to discuss this point, without attempting to give details of a CRPFM model for the neuron.

The neuron, like the other cells has a body with cytoplasm, contains a nucleus and is surrounded by a polarizable membrane¹⁶. Its structure shows a remarkable adaptation to its special task, generally possessing several relatively short projections called dendrites (see Fig. 1.8) that carry impulses to the cell body and a longer projection called an axon¹⁷ that carries impulses to

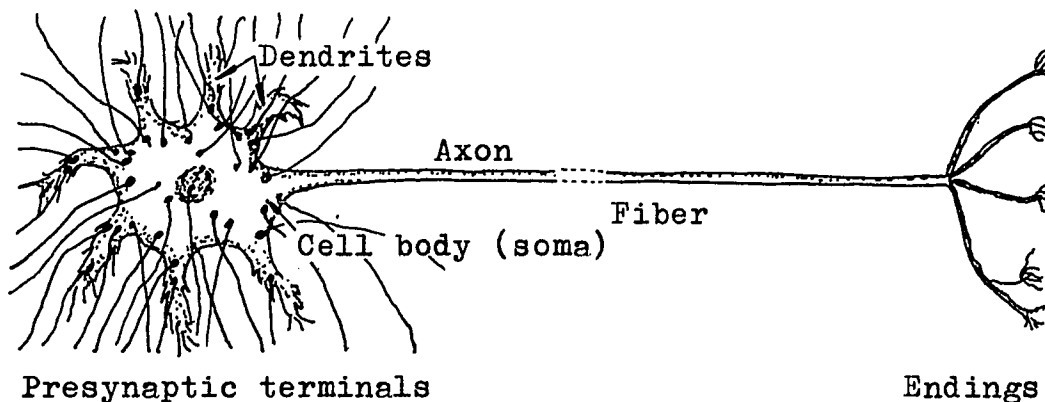


Fig. 1.8 Diagram of a neuron.

¹⁶For more information on the neuron and the nervous system, the reader is referred to (115), (50) or standard textbooks on neurophysiology.

other neurons or neurally activated structures, e.g., muscles and glands. Hundreds of nerve fibrils from other neurons terminate on presynaptic terminals which lie on the dendrites and the cell body (or, soma) at the "synapse".

During the "resting state" the permeability of the cell membrane to sodium ions is low, and the permeability to potassium ions is high; there is a greater concentration of sodium ions in the extracellular fluid and a greater concentration of potassium ions in the intracellular fluid. The equilibrium is maintained by a molecular "ionic pump". The ionic charge distribution is such that the inside of the cell is maintained at a potential of -70mV with respect to the outside.

If there is a suitable external stimulation (electrical, mechanical or chemical), the permeability of the cell membrane to sodium ions temporarily increases. As a result sodium ions rush inside the cell, increasing the somatic potential (membrane potential) up to 30 mV with respect to the extracellular fluid. After the per-

¹⁷An axon is also referred to as a nerve fiber. It can be very long (in man about one meter) and carries impulses to the next neuron. The velocity of conduction increases with fiber diameter and varies from 0.5 m/sec to 120 m/sec.

turbation, the permeability of the cell membrane soon returns to its original state as the extra sodium ions are pumped out. This whole activity lasts about 15 msec and is known as an action potential. Once it is started at any point on the membrane of a normal fiber, the action potential will travel as a depolarizing wave over the entire fiber. The critical potential for this firing makes an "all or nothing" law for neural activity.

Except for sensory neurons, an action potential is usually triggered by stimulation of the presynaptic terminals. A neural impulse arriving at a presynaptic terminal causes automatic "emptying" of some chemical (excitatory transmitter substance) which locally increases the permeability of the cell membrane to sodium ions and sodium ions rush inside the cell, thereby effecting a temporary increase in the somatic potential. This produces the excitatory post synaptic potential, EPSP. If the resulting somatic potential is above a certain level, (the threshold for excitation of the neuron) which is about 10 mV above the resting potential) an action potential is initiated.

The EPSP caused by stimulation of a single synaptic terminal is not sufficient to trigger an action potential, unless the stimulation is continuous. However, the effect

of stimulation of several excitatory presynaptic terminals on the postsynaptic potential is additive and when a sufficient number of excitatory synaptic terminals are excited simultaneously, an action potential is activated. This property is called spatial summation.

The post synaptic potential can also be increased above the threshold value necessary for starting an action potential if a single presynaptic terminal is made to discharge successively. This phenomenon is called temporal summation.

If the EPSP is below the threshold potential, its effect is slowly neutralized. Meanwhile, the excitation of the neuron becomes easier to effect; a neural impulse can be triggered by the addition of smaller number of excitatory discharges. This property is known as facilitation.

Following an impulse emission, the neuron returns to its "resting state" in about 50-200 msec. To initiate an action potential before this time, the post synaptic potential must be increased to a level much greater than the normal threshold value, i.e., the neuron is in the relative refractory state. Immediately following a new impulse emission for about 0.5 msec a new impulse cannot be generated. This period is known as the absolute ref-

ractory period of the neuron.

There are also some synaptic terminals that release inhibitory transmitters which probably cause an increase in the permeability of the cellular membrane to potassium (not to sodium) ions. As a result, potassium ions rush outside of the cell and the postsynaptic potential decreases. Thus, the effect of the inhibitory presynaptic terminals is to lower the somatic potential which in turn means activation of an action potential becomes more difficult.

Although the details of the molecular events taking place in the generation of an action potential in a neuron are still not well understood, specific models based on experimental studies have been developed which account for many observed phenomena (namely, changes in sodium and potassium conductances, all or nothing law, spatial and temporal summation, refractoriness, facilitation, etc.) Hodgkin and Huxley (59)¹⁸. These models are described by highly nonlinear differential equations and their use for the study of the behavior of networks of

¹⁸An excellent survey of neural models is presented in reference (55) which includes most of the important references on the subject up to 1966. Later references can be found in (125).

interconnected neurons appears to be too complicated to be feasible, except for simple networks containing one or two neurons. Therefore, in order to study effectively the behavior of neural networks containing several neurons one has to make reasonable simplifying assumptions. An alternative is the use of simulation studies; this has been done in the past by many investigators but provides only limited insight (55).

The most important characteristic of a neuron is its all-or-nothing response. This has been the basis of "formal" (or binary) neurons first introduced by McCulloch and Pitts (92); specifically, it was assumed that (a) the spatial summation is linear, (b) excitation can be denoted by a positive weight and inhibition by a negative weight, (c) the refractory period is constant and (d) the threshold is time invariant. A neural network consisting of "formal" neurons then becomes essentially a network consisting of interconnections of unit delays and binary elements. Such systems have been studied in automata theory (see Section 4.1). In the model of McCulloch and Pitts, temporal summation, relative refractoriness, facilitation, synaptic and axonal delays have been neglected. A similar formulation has also been used by Caianiello and associates (2, 18, 19, 20, 33).

A more complete neural model can be given in terms of the CRPFM described in the previous section. The CRPFM exhibits many of the properties of the neuron. Note that the CRPFM also has the properties of threshold to excitation and the all-or-nothing response. Spatial summation can easily be accounted for by having a multi-input TF. The output of the TF corresponds to the somatic potential; the TF must be chosen such that "excitatory" stimulation increases the "somatic potential" while "inhibitory" stimulation decreases it. Facilitation and temporal summation is inherent in CRPFM since the effect of any input will continue for some time due to dynamics of the TF. To account for relative refractoriness, a negative feedback to the input of the TF can be used; this was suggested by Pavlidis (102) for his RPFM model for a neuron. By selecting a suitable TF it is also possible to account for accomodation¹⁹.

Some investigators believe that stochastic activity plays an important role in neural behavior and resort to stochastic models (4, 46, 125). This is mainly because in experimental studies neural activity appears to be

¹⁹This means that the neuron is more difficult to excite by slowly varying signals than by relatively fast signals (50), p. 62.

very irregular²⁰. However, a PFM system can also exhibit similar behavior without noise present or random changes in its parameters (102). Therefore, a deterministic neural model which employ a CRPFM is capable of simulating also the spontaneous activity of a neural circuit.

1.4 Review of Previous Investigation on PFM

After the Second World War the subject of nonlinear control has become a very active research area and many new techniques for the analysis of nonlinear systems have been developed. Especially, stability of single-loop nonlinear feedback systems has been studied extensively. PFM systems consisting of a single modulator and a linear plant in a single-loop feedback configuration benefited from these developments; after its introduction into control systems by Meyer and Li (90, 93), most of the new techniques were applied to these systems.

Due to nonlinearity and memory characteristics, PFM control systems are difficult to study analytically. However, the total number of papers on PFM, presently exceeding ninety gives an indication of the activity in this

²⁰An explanation for this spontaneous neural activity is emission of many packets of transmitter substances at a presynaptic terminal upon activation by a neural impulse (125).

research area. The literature on PFM can be divided into four main groups: (i) stability, (ii) periodic motion, (iii) optimal control, and (iv) statistical properties. In the following, each group will be reviewed in chronological order.

1.4.1 Stability. A majority of the research on PFM is devoted to the important area of stability. One of the basic approaches used in most of these works is Lyapunov's second method (22, 23, 25, 27, 38, 72, 75-80, 103, 104). This will be discussed first.

Farrenkopf, et. al. (38) were the first to use Lyapunov's second method in stability studies of PFM control systems. For a satellite attitude control system consisting of a plant with double integration and an IPFM, they applied a discrete version of a Lyapunov theorem given by LaSalle and Lefshetz (83) (based on Okamura and Yoshizawa's work) using a quadratic Lyapunov function, and showed that

(i) the system is asymptotically stable in the large to a set U_0 , enclosing the equilibrium condition, and

(ii) all ultimate states of the system must eventually be within a set V_0 , enclosing the origin of the state-space (implying nonexistence of higher-order limit cycles).

Clark and Noges (27) extended this work to include inner bounds to limit cycle motion and applied the result to obtain both inner and outer bounds in a single-loop amplitude dependent PFM system (see Section 1.2.4), using a quadratic Lyapunov function.

Pavlidis (103, 104) extended Lyapunov's direct method for the investigation of stability of a class of discontinuous dynamical systems -which he defined as a generalization of PFM systems- by selecting a positive definite function which was constant or decreasing along the trajectories of the system when no pulses are emitted (to check whether the emission of pulses will stop in finite time) and decreasing during pulse emission (to check whether the system will come to a prescribed region).

Jury and Blanchard (65) used a theorem similar to that of Farrenkopf et. al. (38) to study asymptotical stability in the Lagrange sense of IPFM control systems.

In the aforementioned publications, sufficient conditions for stability were stated in terms of conditions on the Lyapunov functions; some simple examples were included for demonstration of the theorems, but no method for constructing of a Lyapunov function, allowing direct estimation of stability regions in the parameter space of the system was presented²¹. Kuntsevich and Chekhovoi

(75), again using a discrete version of the Lyapunov theorem of LaSalle and Lefshetz, obtained such a method for a single-loop system containing a PFM of the first type (see Section 1.2.4) which they defined as a modulator in which the pulse frequency is a function of the discrete values of the error signal. However, their method required several complex manipulations, making it impossible to analyze the stability condition in a general form.

In a later paper, Kuntsevich and Chekhovoi (76) utilizing a system containing two modulators, demonstrated how the method of the previous paper could be extended to multi-modulator systems. This was followed by another paper by Chekhovoi (22) in which the stability conditions of (75) were presented in a more manageable form.

King-Smith and Cumpston (72) used Lyapunov's second method with a quadratic Lyapunov function to determine boundedness of motion in a single-loop IPFM feedback system with a stationary linear element and showed that the boundedness of motion depended on the stability of the equivalent linear system²².

²¹Pavlidis (103) has presented certain results concerning stability of single-loop PFM systems; however, Kuntsevitch and Chekhovoi (79, 25), using an example, show that it is erroneous.

²²This was shown previously also by Meyer (93); he demonstrated that as the input to an IPFM becomes very

A similar result was later obtained by Chekhovoi (23) for a more general PFM system in which a hysteresis type nonlinearity was assumed to precede a PF modulator such that the pulse frequency was bounded. For a PFM system with an asymptotically stable linear part, the motion was shown to be bounded. This result includes a previous frequency domain stability condition of Gelig (41) as a special case. The results of (22) were later extended to RPFM systems (79).

Varadarajan used the same theorem employed earlier by Kuntsevitch and Chekhovoi (75) to determine the conditions such that the state trajectory of a single-loop ORPFM feedback system with an asymptotically stable linear, time-invariant TF and plant will enter into a region in which the modulator cannot fire. The condition obtained is the same given previously by Pavlidis (103) (see also footnote 21).

Kuntsevitch and Chekhovoi recently published extensions of their work on stability with certain improve-

large, it can be replaced by an equivalent linear gain of M/S , where M is the impulse strength and S is the threshold of the modulator. Therefore, if the linear part of the system is asymptotically stable and if the equivalent linear system obtained by replacing the IPFM by its equivalent linear gain is also asymptotically stable, then the motion will be bounded.

ments; Kuntsevitch (80) for single-loop feedback systems with pulse-width modulation, or with "PFM of the first kind" (see Section 1.2.4), and Chekhovoi for single-loop CRPFM systems.

In general, Lyapunov methods are difficult to use, especially for higher order systems. An alternative is Popov's frequency domain method. Popov's theorem (or most of its generalizations) cannot be applied directly to PFM systems because of the fact that a PF modulator generates pulses having a variable sampling interval which is a function of the input signal. Dymkov (35) was first to apply Popov's theorem to a single-loop RPFM feedback system by representing the RPFM in the form of an equivalent relay system having a hysteresis type nonlinearity. Essentially the same result was independently obtained by Monopoli and Wylie (95)²³

Gelig (41), following steps similar to that used in the derivation of Popov's theorem, gave a frequency domain stability criteria for a more general PFM system containing an hysteresis type nonlinearity in series with a PF modulator, such that the pulse frequency was bounded. In

²³In (95) the modulator is defined as a modified form of RPFM in such a way that the restrictions of Popov's theorem are satisfied.

(41), he employed the same approach, for a system containing several "pulse elements" to derive frequency domain stability criteria. These "pulse elements" were introduced for modelling pulse frequency modulators (type I or ORPFM, discussed in Section 1.2.4) or pulse-width modulators; however no consideration was given to the pulse emission law.

Popov's theorem was extended by Tyskin for nonlinear sampled data systems. In order to apply this extension to PFM systems, Shortle and Alexandro (116) defined a discrete approximation to an RPFM (see Section 1.2.4) such that it had an equivalent representation in terms of a dead zone nonlinearity and a PAM (sampler). Later, Kan and Jury (68) made an attempt to apply Popov's theorem to RPFM systems, directly; however, as a result of a certain transformation used in the process, the modulator lost its resetting property²⁴. Chekhovoi's attempt at the same problem was successful; he used Yakubovitch's extension of Popov's theorem for systems with hysteresis type nonlinearities (24).

In this area, Gelig recently published three interes-

²⁴This was pointed out by Kuntsevich in a private communication to the authors of (68).

ting papers; in the first (43), he presented frequency-domain stability criteria for a single-loop RPFM feedback system, in the second paper (44), the results of the first paper were generalized to multimodulator systems. In the third paper (45), systems with "PFM of the first type" were considered. The method used in these papers is essentially the same employed in his previous work (41, 42)²⁵.

Some investigators have used a different approach and made direct use of the basic functional properties of the system equations to obtain stability conditions. Among them, Skoog and Blankenship (121), determined a simple and useful condition for BIBO stability of a single-loop CRPFM feedback system with a linear stationary plant, based on a theorem of Zames (130). The condition is of the form:

$$\frac{M\tau}{S} \int_0^{\infty} |g(t)| dt < 1$$

where, $g(t)$ is the impulse response of the linear plant, S is the threshold of the modulator and M and τ represent the amplitude and the duration of the pulses, respectively.

²⁵A summary of Gelig's results is presented in Section 2.3.

(the modulator was assumed to emit rectangular pulses). This result was also obtained independently by Meyer and generalized by Guy [(48), p. 47] to a feedback system containing two IPFM's in a single feedback loop.

Interestingly, another independent investigator, Kan (69), (for an IPFM feedback system) also obtained the same condition using a somewhat different approach.

1.4.2 Periodic motion in PFM systems. Due to the abundance of different possible modes of periodic motions peculiar to PFM systems²⁶ this topic has even been given some consideration. For periodic motion in single-loop IPFM systems where the modulator emits equally spaced pulses of equal polarity, Meyer (93) obtained a closed form expression for the period and investigated its stability by linearizing the system about this motion. He also extended this work to cases where the pulse pattern was more complex and obtained certain necessary conditions for the existence of periodic motion. These conditions have been verified by King-Smith and Cumpston (71) using an independent approach. Some of those

²⁶For example, even a single loop system containing an IPFM and a plant with a single integration possesses infinite number of different limit cycle oscillations (38, 93).

results have also been reported by Varadarajan and Pai (127).

A practical method for studying periodic motion is the describing function method which is also useful for stability analysis. It has been applied to single-loop PFM systems by a number of investigators.

Li (90), by studying the periodicity of output pulse distribution under sinusoidal excitation, derived the describing function of the IPFM from the fundamental component of the Fourier series of the periodic pulse train and applied describing function methods for stability analysis. Pavlidis and Jury (101) instead of assuming a sinusoidal wave as an input to the modulator, assumed a square wave and determined a "quasi-describing function" from the ratio of the output fundamental sinusoid to the input fundamental component of the square wave.

Dymkov, in (36) compared the describing function method and the quasi-describing function method of Jury and Pavlidis and argued that for high order linear plants, the output would resemble a sinusoid rather than a square wave, and recommended the standard describing function technique.

Guy (48) calculated the second and third harmonic content of the single IPFM describing function and showed

that it contained high magnitudes at the lower numbered pulse-patterns; he warned against the use of this method when low numbered pulse-patterns are predicted, unless the linear plant provides exceptionally good low pass filtering. He also calculated the resultant compound describing function for an IPFM feedback system consisting of two modulators and two linear elements in a single loop.

1.4.3 Optimal control. Optimization of PFM systems was first considered by Pavlidis (105). For a single-loop feedback system with a PFM as an error modifier, he used some heuristic arguments and concluded that for the minimum time problem the control function $r(t)$ is of the form $\pm R$, and for the minimum fuel problem -although non-unique- is of the form $0, \pm R$, where R is a constant and the admissible controls are such that $|r(t)| < R$.

Other investigators in this field considered only open-loop control problems. In this case, the objective is to find the (optimal) pulse-instants and pulse-polarities of a series of PFM pulses (control input to the plant), such that a certain function of the final states of the plant (performance index) is optimized. In particular, Onyshko (92), assuming a linear system of the form

$$\dot{\underline{x}} = \underline{A} \underline{x} + f(u) \text{ with a performance index } J = \underline{c}^T \underline{x}(t_f),$$

used a modified Maximum Principle and Dynamic Programming for the synthesis of optimal control, by restricting the pulse-instants to discrete times kT ($k = 0, 1, 2, \dots$), where t_f denotes the final time, \underline{c} is a constant column vector and T is a sampling interval.

Stoep (122), following Onyshko, also restricted the control pulses to discrete times and, considered a performance index consisting the weighted sum of a quadratic terminal state error and fuel consumption. Using an enumerative technique, he determined the optimal performance index. For the same system, he also considered a more general mode of operation in which the control is only magnitude-limited (to the pulse amplitude) and determined the optimal performance index for this mode. For special cases, he demonstrated that the difference between the two values of the performance index is very small.

Onyshko and Noges (99) gave a modified Maximum Principle applicable to open-loop PFM systems with linear plants operating over a finite time interval. For the same problem, Lermentov and Noges (87) presented a geometrical method for determining the regions of initial state (admissible regions) from which the system state could be carried to the origin within a specified time. Lermentov (88) also determined the gradient of a cost function of

the form $J = f(\underline{x}(t_f))$ for PFM control inputs to be employed in numerical optimization methods and using a numerical example demonstrated the result to be identical with that obtained by application of the modified Maximum Principle.

1.4.4 Statistical properties. Although it is the predicted high degree of noise immunity²⁷ that aroused first interests in PFM, due to complexity arising from the inherent nonlinearity of these systems, it is difficult to obtain conclusive analytical results.

Li, in a chapter in his doctoral dissertation (90), discussed the immunity to channel noise for IPFM telemetry by considering an additive, discretized transmission channel noise consisting of independent and identically distributed pulses with zero-mean and Gaussian-amplitude distribution; the signal noise was measured at the receiving end in terms of the number of false pulses per unit time per unit frequency. Bombi and Ciscato (12), studied the problem of jitter in a relatively simple situation of constant input signal and additive Gaussian

²⁷ For example, an IPFM is capable of averaging out a high frequency noise of sufficiently small amplitude during each instantaneous pulse period.

noise (additive to the signal input to the PFM); they discussed the conditions under which the probability density of the jitter in IPFM output pulses is quasi-Gaussian and calculated the power spectrum of the output signal.

Hutchinson et. al. (60,61), calculated the autocorrelation function and the spectral density function of the output of an IPFM and an RPFM for a zero-mean, stationary and normally distributed magnitude unit-white noise input with constant spectral density.

In an interesting, physiologically-oriented paper, Bayly (6), using spectral analysis techniques, demonstrated simple low-pass filtering to be an effective means of demodulating PFM signals and a multichannel system consisting of IPFMs for demodulation and low-pass filters for demodulation to be capable of improving the signal-to-noise distortion ratio over that possible on any one of the channels alone and argued these to be the reasons of Nature's using PFM. Spectral analysis of IPFM was also developed independently by Lee (84, 85).

1.5 Interconnected System Consisting of CRPFM's and
Linear Dynamic Elements: The System Considered
in This Dissertation

The system considered in this dissertation is shown in Fig. 1.9. The PFM block contains m CRPFM's ($m = 1, 2, \dots$). The modulator output vector, $\underline{u}(t)$ is applied to a linear dynamical subsystem (LP) whose impulse response matrix is $\underline{G}(t, \tau)$. A combination of the output vector of the linear part LP, $\underline{y}(t)$ and an external input vector $\underline{r}(t)$ is fed to the modulator block. $\underline{y}_0(t)$ is the initial condition response vector of the LP which could also include disturbances.

Let $t_{i,j}$ be the instant at which the i th modulator emits its j th pulse and let $K_i(t)$ denote the total number of pulses emitted by the i th modulator prior to time t (see Fig. 1.10 for an illustration of these definitions). The operation of the system is given by the relation for the i th timing filter output signal (leading to the i th TD):

$$y_i(t) = y_{0i}(t) + \sum_{j=1}^m \sum_{k=1}^{K_j(t)} \text{sgn}[z_j(t_{j,k}^-)] g_{ij}(t, t_{j,k}),$$

$$t_{i, K_i(t)} < t < t_{i, K_i(t)+1} \quad (1.24a)$$

and

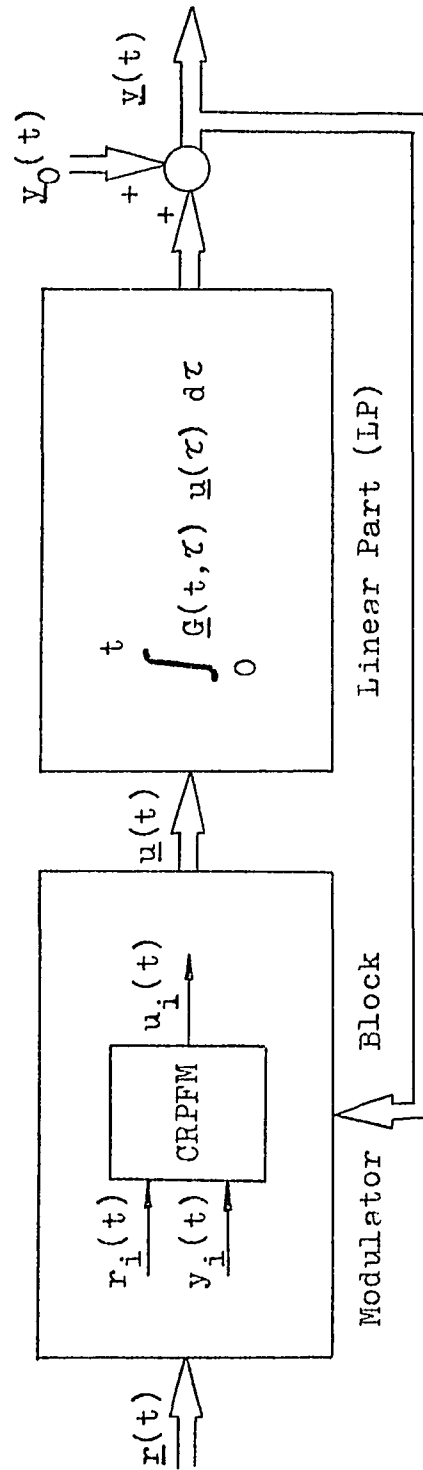


Figure 1.9 Block diagram of an interconnected system consisting of CRPFM's and linear dynamical subsystems.

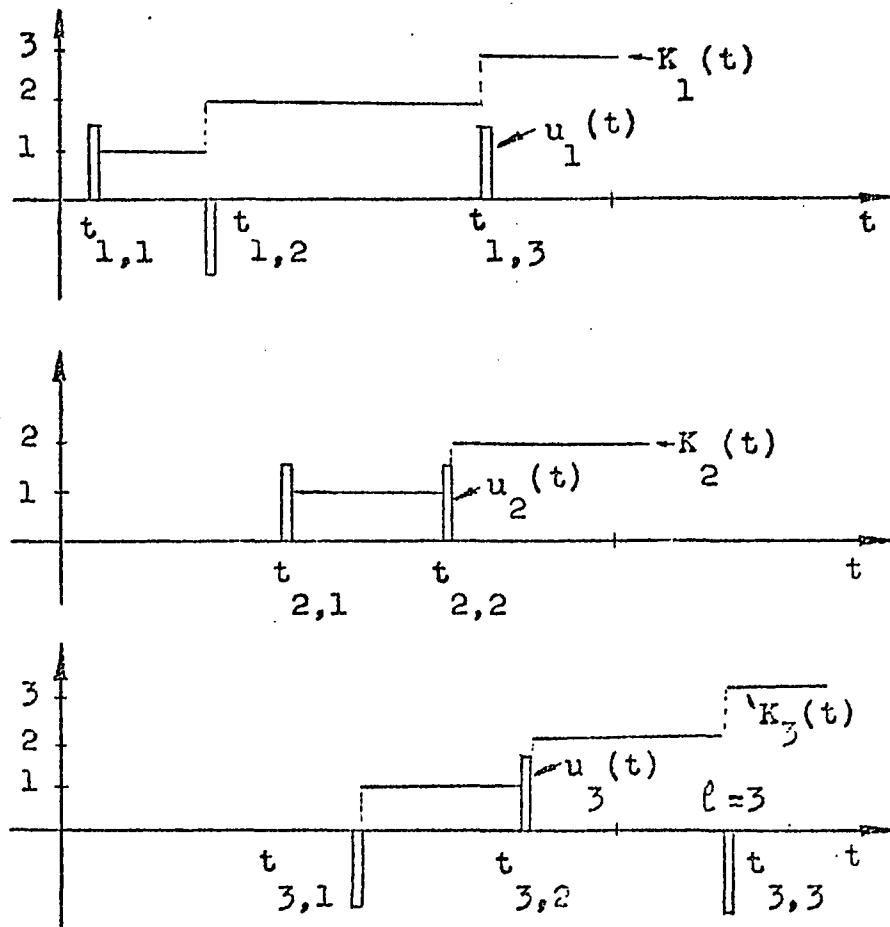


Figure 1.10 An example illustrating the definitions of $t_{i,j}$ and $K_i(t)$.

$$z_i(t) = \int_{t_{i,K_i}(t)}^t f_i[r_i(\tau), y_i(\tau), t, \tau] d\tau, \quad (i = 1, \dots, m)$$

$$t_{i,K_i}(t) < t < t_{i,K_i}(t)+1 \quad (1.24b)^{28}$$

Consider some fixed time t , and let t^f denote the firing time of the next impulse (after time t) that may be emitted by any one of the m modulators; it is given by

$$t^f = \min_{i=1, \dots, m} \{ t \mid t > t_{i,K_i}(t) + T_{0,i} \text{ and } |z_i(t)| \geq S_i \}$$

(1.24c)

The identification number of the modulator that has fired at $t = t^f$ is then

$$\ell = \{ i = 1, 2, \dots, m \mid t^f > t_{i,K_i}(t) + T_{0,i} \text{ and } |z_i(t^f)| \geq S_i \}$$

(1.24d)

Thus,

$$t_{\ell, K_\ell}(t) + 1 = t^f \quad (1.24e)$$

Equations (1.24a)-(1.24e) are the basic equations governing the operation of the PFM system of Fig. 1.9.

²⁸Note that, for the i th modulator ($i = 1, \dots, m$), comparison of eq. (1.24b) with (1.7) yields

$$f_i[e(t), t, \tau] = f_i[r_i(\tau), y_i(\tau), t, \tau] .$$

CHAPTER 2

FINITE-PULSE STABILITY OF INTERCONNECTED SYSTEMS
WITH COMPLETE-RESET PULSE FREQUENCY MODULATORS*

Stability in PFM systems have previously been discussed by several investigators using various approaches, namely, Lyapunov's second method (22, 23, 25, 27, 72, 75-80, 103, 104), frequency domain method (Popov's method or its generalizations) (24, 35, 41-45, 68, 95, 116), functional analysis approaches (41-45, 69, 121) and linearization techniques (36, 48, 89, 90, 93, 101). Most of these works were, however, restricted to systems containing one or two modulators and only few results, have so far, been presented for multi-modulator systems.

The objective of this chapter is to present stability criteria for the CRPFM system discussed in Section 1.4 which contains an arbitrary (finite) number of CRPFM's. Stability is defined in terms of upper bounds on the number of pulses emitted by each modulator. This definition of finite-pulse stability has physical meaning in that the number of pulses emitted from a modulator is a measure of energy spent by that modulator during the

* Part of this chapter was presented at the 1972 Allerton Conference on Circuit and System Theory, Monticello, Ill. and published in IEEE Transactions on Automatic Control, vol. AC-18, no. 4, August 1973, pp. 378-392.

operation of the system. Not surprisingly, the concept of finite-pulse stability is related to $L_1 \cap L_p$ output stability¹.

For a special case of the CRPFM system considered in this work, for an RPFM system² containing several relaxation type PF modulators Gelig (44) recently obtained frequency domain stability criteria. Apart from Gelig, Pavlidis (103) and Kuntsevitch and Chekhovoi (77) also considered stability in multi-modulator PFM systems, both using Lyapunov's second method; however, neither of these papers presented procedures that permit direct estimation of parameter-regions sufficient for stability.

In this chapter, first a Lyapunov method will be discussed. Then, an approach will be presented by which upper bounds are determined for the number of pulses emitted by each modulator. Finiteness of these bounds for all modulators constitutes finite-pulse stability. Sufficient conditions are established for finite-pulse stabi-

¹See Section 2.1.

²In this work the name RPFM system refers to the basic configuration of Fig. 1. 9, in which all the modulators are of the relaxation type. Since an IPFM is a special case of an RPFM and since a system containing ORPFM's can be transformed into a system containing only RPFM's, the same name will sometimes refer also to an IPFM system or an ORPFM system.

lity. Gelig's frequency domain method is also discussed and the results are compared with respect to effectiveness in terms of size of parameter-regions sufficient for stability, generality (in terms of classes of applicable systems) and ease of application.

2.1 Global Finite-Pulse Stability in PFM Systems

The number of pulses emitted from a modulator is a measure of the energy spent by that modulator during the operation of the system. Therefore, the stability of a PFM system can be related to this variable, which leads to the following definition:

Definition 2.1: A PFM system is called globally finite-pulse stable (GFPS) if for every set of initial conditions and for every input $\underline{x}(t) \in L_1[0, \infty)$ ³ the number of pulses emitted by each modulator remain finite as $t \rightarrow \infty$.

Clearly, after all modulators have ceased firing, the plant will remain without input and its motion can be studied independently by standard methods. The following lemma relates the above definition to the concept of L_p output stability³.

³This is a convenient, standard mathematical notation which stands for the collection of all the measurable functions $x(t)$ which map the interval $[0, \infty)$ into the real line $(-\infty, \infty)$ such that the integral

$$\|x(t)\|_p \triangleq \left(\int_0^{\infty} |x(t)|^p dt \right)^{1/p}$$

is finite.

Lemma 2.1: If the PFM system of Fig. 1.10 is GFPS and if the components $g_{ij}(t, \tau)$ of the impulse response matrix $\underline{g}(t, \tau)$ as well as the components $y_{0i}(t)$ of the initial condition response vector $\underline{y}_0(t)$ for every set of initial conditions are all in $L_p[0, \infty)$, then each component $y_i(t)$ of the output vector $\underline{y}(t)$ is in $L_p[0, \infty)$.
Proof: Consider the i th component of the output vector of the system

$$y_i(t) = y_{0i}(t) + \sum_{j=1}^m M_j \sum_{k=1}^{K_j(t)} \text{sgn}[z_j(t_{j,k}^-)] g_{ij}(t, t_{j,k}) \quad (i = 1, 2, \dots, m) \quad (2.1)$$

Applying Minkowski's inequality (triangle inequality in L_p -spaces) to eq. (2.1) yields

$$\|y_i(t)\|_p \leq \|y_{0i}(t)\|_p + \sum_{j=1}^m M_j \sum_{k=1}^{K_j(\infty)} \|g_{ij}(t, t_{j,k})\|_p \quad (i = 1, 2, \dots, m) \quad (2.2)$$

From this inequality it immediately follows that when $K_j(\infty) < \infty$ then $y_{0,i}, g_{ij} \in L_p[0, \infty)$ implies $y_i \in L_p[0, \infty)$.

$\|x(t)\|_p$ is known as the L_p -norm of the function $x(t)$. The space $L_p[0, \infty)$ is defined as the collection of all measurable functions which are bounded on $[0, \infty)$. The integration is not necessarily restricted to the positive real line $[0, \infty)$ but can be any subset of the set of real numbers.

For a discussion of system stability in terms of L_p -spaces see, for example, Willems (129).

2.2 Lyapunov's Second Method

Let the continuous part (plant)⁴ be of order n_p and the timing filters of modulators \underline{i} be of order n_{ti} ($i = 1, 2, \dots, m$). Let

$$\underline{x}_p(t) = [x_{p,1}(t), x_{p,2}(t), \dots, x_{p,n_p}(t)]^T \quad (2.3a)$$

be the state vector of the LP and let

$$\underline{x}_t^i(t) = [x_{t,1}^i(t), x_{t,2}^i(t), \dots, x_{t,n_{ti}}^i(t)]^T \quad (i = 1, 2, \dots, m) \quad (2.3b)$$

be the state vector of the TF of the i th modulator ($i = 1, 2, \dots, m$). Let the combined state vector of the total system be denoted by

$$\underline{x}(t) = \begin{bmatrix} \underline{x}_p(t) \\ \vdots \\ \underline{x}_t^1(t) \\ \vdots \\ \underline{x}_t^2(t) \\ \vdots \\ \vdots \\ \underline{x}_t^m(t) \end{bmatrix} \quad (2.3c)$$

Let \underline{w}_j^0 be a possible state \underline{x} occurring immediately after impulse emission of the j th modulator. Let \underline{w}_j^1 be the

⁴Actually, linearity of the plant is not required for application of Lyapunov's method.

state \underline{x} reached immediately after modulator j fires the next time. These definitions imply that \underline{w}_j^0 and \underline{w}_j^1 both belong to the set

$$\mathcal{W}_j = \left\{ \underline{w} \mid \underline{x}_p \in E^{np}; \quad \underline{x}_t^i \in \{ \underline{x}_t \in E^{nt_i} \mid |z_i(t)| < S_i, \right. \\ \left. t_{j,k} < t < t_{j,k+1}, i \neq j, (i = 1, \dots, m); \quad \underline{x}_t^j = 0, \right. \\ \left. t = t_{j,k}, t_{j,k+1}, \dots \right\} \quad (2.4)$$

The following theorem holds:

Theorem 2.1: If there exists a positive scalar function $V(\underline{x})$ and a constant $\epsilon > 0$ such that for all $j = 1, \dots, m$ and for every $\underline{w}_j^0 \in \mathcal{W}_j$ and every $\underline{w}_j^1 \in \mathcal{W}_j$,

$$V(\underline{w}_j^0) - V(\underline{w}_j^1) > \epsilon \quad (2.5a)$$

then the CRPFM system of Fig. 1.9 is GFPS.

Proof: Consider the j th modulator. Note that (2.5a) implies

$$V[\underline{x}(t_{j,k}^+)] - V[\underline{x}(t_{j,k+1}^+)] > \epsilon$$

from which the following inequality is obtained

$$V[\underline{x}(t_{j,K_j}^+)] < V[\underline{x}(t_{j,0}^+)] - K_j \epsilon$$

where K_j represents the total number of impulses emitted by the j th modulator as $t \rightarrow \infty$. Assume that $V[\underline{x}(t_{j,0}^+)]$ is finite. Then K_j must also be finite, otherwise the

above inequality yields $V[\underline{x}(t_{j,K_j}^+)] < -\infty$, which is a contradiction since V is a positive function.

(QED).

Condition (2.5a) of Theorem 1 can be replaced by stronger conditions, such as the following:

$$V(\underline{w}_j^0) - V(\underline{w}_j^1) > \epsilon \quad (2.5b)$$

where \underline{w}_j^1 is the state immediately after emission of the next impulse following emission of modulator j (Note that, after firing of modulator j , the next impulse may be emitted by any of the modulators, not necessarily by modulator j).

A still stronger condition is the following:

$$\left. \begin{array}{l} \frac{d}{dt} V[\underline{x}(t)] \leq 0 \text{ for } t \in (t_0, t_1), \quad \forall \underline{x}(t_0) \in \mathcal{W}_j \\ V[\underline{x}(t_1^-)] - V[\underline{x}(t_1^+)] > \epsilon \end{array} \right\} \quad (2.5c)$$

where t_0 represents an emission-time of modulator j and t_1 is the time of emission of the next impulse (by any modulator) after time t_0 . Theorem 1 with condition (2.5c) corresponds essentially to one presented by Pavlidis (103)⁵

⁵Pavlidis' theorem (103) includes also the requirement that $V(\underline{x}) = 0$ for \underline{x} within some region inside the region for which no impulse emission is possible.

Example 2.1: Consider the simple interconnected PFM system consisting of two RPFM's and an integrator, shown in Fig. 2.1. Note that $x_t^1(t) = z_1(t)$, $x_t^2(t) = z_2(t)$, $\underline{x} = [x_p, x_t^1, x_t^2]^T$,

$$\omega_1 = \{ [x_p, 0, z_2]^T \mid -\infty < x_p < +\infty, |z_2| < S_2 \}$$

and

$$\omega_2 = \{ [x_p, z_1, 0]^T \mid -\infty < x_p < +\infty, |z_1| < S_1 \}$$

Let $V(\underline{x}) = x_p^2$. Consider the first modulator, let $\underline{x}(0) = \underline{x}_1^0 = [x_p^0, 0, z_2^0]^T$. Assume that the next impulse of the system is emitted also by the first modulator.

The output of its TF is

$$z_1(t) = \frac{1 - e^{-a_1 t}}{a_1} x_p^0$$

In this case, noting that $z_1(t)$ and x_p^0 have the same signs,

$$x_p^1 = x_p^0 + M_1 \operatorname{sgn} x_p^0$$

Condition (2.5a) of Theorem 2.1 requires

$$(x_p^0)^2 - (x_p^1)^2 > \epsilon,$$

or,

$$M_1(2|x_p^0| + M_1) < -\epsilon$$

For $M_1 > 0$, the above inequality becomes $2|x_p^0| + M_1 < 0$,

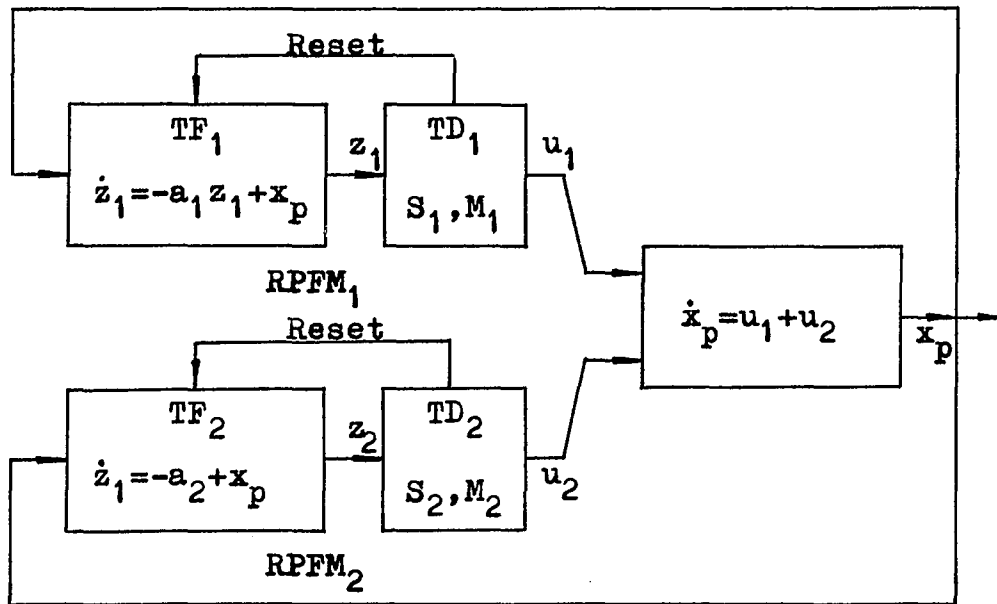


Figure 2.1. A simple interconnected PFM system consisting of two RPFM's and an integrator.

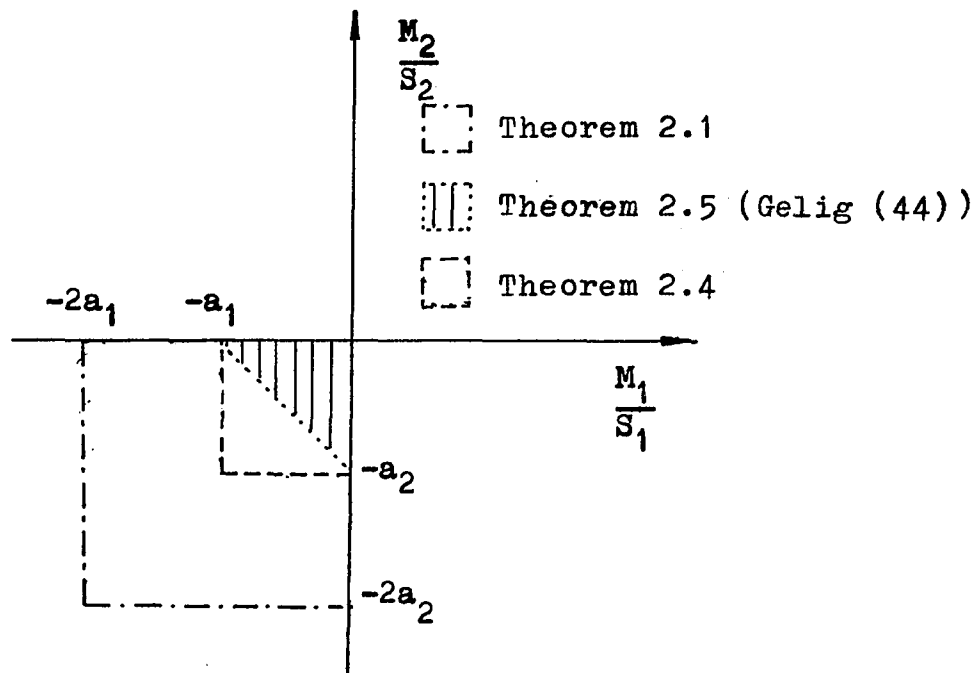


Figure 2.2 Comparison of stability criteria for the CRPFM system of Fig. 2.1.

which is not possible. However, for $M_1 < 0$, it yields $2|x_p^0| + M_1 > 0$. But, $|x_p^0| > a_1 S_1$ (otherwise no pulse emission would have taken place). Therefore, M_1 must be selected such that

$$0 > M_1 > -2a_1 S_1. \quad (2.6a)$$

Similarly, for the second modulator, the same argument gives

$$0 > M_2 > -2a_2 S_2 \quad (2.6b)$$

Now assume that successive impulses of the system are emitted by the second modulator, before the first modulator starts firing again. Let $t_{2,j}$ be the instant when the second modulator emits its j th impulse, after $t = 0$. Then, for $0 < t < t_{2,j}$,

$$z_2(t) = \left(z_2^0 - \frac{x_p^0}{a_2} \right) e^{-a_2 t} + \frac{x_p^0}{a_2}.$$

Again the sign of the impulse emitted is the same as the sign of x_p^0 . Thus, $x_p(t_{2,1}^+) = x_p^0 + M_2 \operatorname{sgn} x_p^0$; therefore since, by (2.6b), $M_2 < 0$, it is:

$$[x_p(t_{2,1}^+)]^2 < (x_p^0)^2 \quad (2.7)$$

Generalizing (2.7) from $t_{2,0}^+$ to $t_{2,j}^+$ yields:

$$[x_p(t_{2,j+1}^+)]^2 < [x_p(t_{2,j}^+)]^2, \quad (j=1,2,\dots,N-1) \quad (2.8)$$

By a similar argument used in obtaining (2.7), if the next impulse of the system is emitted by the first modulator again, at $t = t_{1,1}$,

$$(x_p^1)^2 = [x_p(t_{1,1}^+)]^2 < [x_p(t_{2,v}^+)]^2 \quad (2.9)$$

Combining (2.7), (2.8) and (2.9),

$$(x_p^1)^2 < (x_p^0)^2 \quad (2.10)$$

Clearly, the same argument is valid also for the second modulator. Thus, if M_1 and M_2 are selected in accordance with relations (2.6a) and (2.6b), respectively, all the conditions of Theorem 1 will be satisfied and the system under consideration will be GFPS.

Note that, in this case, conditions (2.5b) and (2.5c) are also satisfied. The stability region, determined by inequalities (2.6a) and (2.6b) is shown in Fig. 2.2.

2.3 Direct Finite-Pulse Stability Criteria

The number of impulses, $K_i(t)$ emitted by the i th modulator (prior to time t) in a PFM system ($i = 1, \dots, m$) represents a measure of the energy spent by the corresponding modulator in the interval $[0, t)$. Therefore, it is desirable to estimate this number directly, without solving the system equations. In the subsequent develop-

ments an upper bound for $K_i(t)$ will be determined for the CRPFM system of Fig. 1.9. Existence of these bounds for all modulators and for $t \rightarrow \infty$ implies GFPS.

In this section the following conditions are assumed to be satisfied:

Condition 1: $y_{0i}(\cdot), r_i(\cdot) \in L_1[0, t)$, for $i = 1, \dots, m$,

Condition 2: there exist functions $g'_{ij}(\cdot) \in L_1[0, t)$ such that $\forall t_1, t_2 \in [0, t)$ and $i, j = 1, \dots, m$,

$$|g_{ij}(t_1, t_2)| \leq |g'_{ij}(t_1 - t_2)|, \text{ and}$$

Condition 3: there exist finite nonnegative constants α_i and β_i such that $\forall t_1, t_2 \in [0, t)$ and $i = 1, \dots, m$,

$$|f_i[r_i, y_i, t_1, t_2]| \leq \alpha_i |r_i| + \beta_i |y_i|.$$

Let $\underline{k}(t)$ and $\underline{y}(t)$ be m -dimensional column vectors with elements $K_i(t)$ and

$$v_i(t) = \frac{1}{S_i} \int_0^t [\alpha_i |r_i(\tau)| + \beta_i |y_{0i}(\tau)|] d\tau \quad (2.11)$$

$(i = 1, \dots, m)$

respectively, and let $\underline{H}(t)$ and $\underline{H}'(t)$ be $m \times m$ matrices whose elements in the i th row and j th column are

$$h_{ij}(t) \triangleq \int_0^t |g'_{ij}(\tau)| d\tau, \quad (i, j = 1, \dots, m) \quad (2.12)$$

and

$$h'_{ij}(t) \triangleq \beta_i \frac{M_j}{S_i} h_{ij}(t), \quad (i, j = 1, \dots, m) \quad (2.13)$$

respectively. The following fundamental theorem is useful for estimating the upper bounds for the number of pulses emitted by each modulator:

Theorem 2.2: If Conditions 1-3 are satisfied then the vector $\underline{k}(t)$ of the number of pulses emitted by the modulators prior to time t satisfies the following matrix inequality

$$[\underline{I} - \underline{H}'(t)] \underline{k}(t) \leq \underline{v}(t) \quad (2.14)$$

The proof of Theorem 2.2 is given in Appendix A.

For GFPS it is only necessary to show that a sum containing the number of firings of all modulators with positive coefficients remains finite as $t \rightarrow \infty$. Thus,

Theorem 2.3: If Conditions 1-3 are satisfied as $t \rightarrow \infty$ and if the matrix $\underline{P} [\underline{I} - \underline{H}'_{\infty}]$ has a row with all positive elements, where \underline{P} is a nonnegative matrix⁶ and $\underline{H}'_{\infty} = \lim_{t \rightarrow \infty} \underline{H}'(t)$, then the CRPFM system of Fig. 1.10 is GFPS.

Proof: Let $\underline{P} [\underline{I} - \underline{H}'_{\infty}] = [\gamma_{ij}]$. Premultiplying both sides of inequality (2.14) (as $t \rightarrow \infty$) by \underline{P} yields the inequality

⁶A matrix is called nonnegative if and only if it has no negative elements. Nonnegative matrices play an important role in various fields like mathematical economics, theory of games, linear programming, etc., and have been extensively studied, Bellman (7), Gantmacher (40), Lancaster (81).

$$\delta_{i1}K_1 + \dots + \delta_{im}K_m \leq p_{i1}v_1 + \dots + p_{im}v_m < \infty \quad (2.15)$$

$$(i = 1, \dots, m)$$

Since $p_{ij} > 0$ and $\delta_{ij} > 0$ for some i , finiteness of the above inequality implies $K_j < \infty$ for $j = 1, 2, \dots, m$.

(QED).

Inequality (2.14) restricts the vector $\underline{k}(t)$, which represents the number of impulses emitted by the modulators to a certain region. When the matrix $[\underline{I} - \underline{H}'(t)]^{-1}$ exists and is nonnegative⁶, this region is finite. In this case inequality (2.14) can be transformed into

$$\underline{k}(t) \leq \text{In}\left\{[\underline{I} - \underline{H}'(t)]^{-1}\underline{v}(t)\right\} \quad (2.16)$$

where the notation $\text{In}\{\cdot\}$ stands for the integer part of the corresponding vector. Inequality (2.16) determines the upper bounds of the number of pulse emissions as $t \rightarrow \infty$. Let

$$\underline{k}_\infty \triangleq \text{In}\left\{\lim_{t \rightarrow \infty} [\underline{I} - \underline{H}'(t)]^{-1}\underline{v}(t)\right\} \quad (2.17a)$$

then \underline{k}_∞ is the required upper bound, since

$$\underline{k}(t) \leq \underline{k}_\infty, \quad \forall t \in [0, \infty) \quad (2.17b)$$

The above result may be stated in terms of the following theorem:

Theorem 2.4: If Conditions 1-3 are satisfied as $t \rightarrow \infty$ and if the matrix $[\underline{I} - \underline{H}'_\infty]^{-1}$ is nonnegative then the CRPFM system of Fig. 1.9 is GFPS.

For large systems it may be cumbersome to invert the matrix $[\underline{I} - \underline{H}'_{\infty}]$. The lemma to be stated next provides means to avoid this inversion.

Lemma 2.2: If the spectral radius⁷ $\lambda(\underline{A})$ of a nonnegative matrix \underline{A} is smaller than unity, then the matrix $[\underline{I} - \underline{A}]^{-1}$ exists and is nonnegative.

The proof of this lemma follows from the identity:

$$[\underline{I} - \underline{A}]^{-1} = \underline{I} + \underline{A} + \underline{A}^2 + \underline{A}^3 + \dots \quad (2.18)$$

provided $\lambda(\underline{A}) < 1$ (see Barnett and Storey, p. 60).

Since the matrix \underline{A} is nonnegative, so is \underline{A}^2 , \underline{A}^3 , ... and their sum; hence the proof.

Lemma 2.2. and Theorem 2.3 yield the following corollary:

Corollary 2.1: If Conditions 1-3 are satisfied as $t \rightarrow \infty$ and if $\lambda(\underline{H}'_{\infty}) < 1$, then the CRPFM system of Fig. 1.9 is GFPS.

With the present computer technology, it is not a very difficult task to calculate the eigenvalues of a matrix⁸. However, the following lemma eliminates this

⁷Spectral radius of a matrix is defined as the magnitude of the largest eigenvalue, i.e., if λ_i ($i = 1, \dots, m$) are the eigenvalues of the matrix \underline{A} , then $\lambda(\underline{A}) = \max_i |\lambda_i|$

⁸If the coefficients of the characteristic polynomial $P(\lambda) = |\lambda \underline{I} - \underline{H}'_{\infty}|$ are known, the condition $\lambda(\underline{H}'_{\infty}) < 1$ can be checked using the Routh-Hurwitz Criterion on $P[(r+1)/(r-1)]$. However, this method is not recommended for large systems.

need, in most cases.

Lemma 2.3: If \underline{A} is an $m \times m$ nonnegative matrix and if

$$\sum_{j=1}^m a_{ij} < 1, \quad i = 1, \dots, m,$$

then the matrix $[\underline{I} - \underline{A}]^{-1}$ is also nonnegative.

The proof of Lemma 2.3 follows from Gersgorin's theorem⁹ and Lemma 2.2. Matrices satisfying conditions of Lemma 2.3 are known as Minkowski-Leontieff matrices¹⁰.

Corollary 2.1. Lemma 2.3 and Theorem 2.3 lead to the following corollary.

Corollary 2.2: If Conditions 1-3 are satisfied (for $t \rightarrow \infty$) and

$$\sum_{j=1}^m h'_{ij}(\infty) = \frac{\beta_i}{s_i} \sum_{j=1}^m \int_0^{\infty} |M_j g'_{ij}(\tau)| d\tau < 1,$$

for $i = 1, 2, \dots, m$, then the CRPFM system described in Section 1.4 is GFPS.

For example, when applied to a single-loop, single

⁹This is a very useful theorem for obtaining bounds on eigenvalues and states that every eigenvalue of a matrix \underline{A} lies at least in one of the disks

$$|z - a_{ij}| \leq \sum_{\substack{j=1 \\ j \neq i}}^m a_{ij}$$

(Lancaster, p. 226).

¹⁰Theorem 6 of (Lancaster, p. 288) is very close to Lemma 2.3.

RPFM (or IPFM) system with a time-invariant linear element, then either Theorem 2.3 or Corollary 2.2 imposes the condition

$$\frac{M}{S} \|g_1\| < 1 \quad (2.19)$$

where, $\|g\|_1 = \int_0^{\infty} |g(t)| dt$. If there are two RPFM's (or IPFM's) in a single-loop (i.e., $g_{ii}(t) = 0$, $i = 1, 2$) then Theorem 2.3 yields

$$\frac{M_1}{S_1} \frac{M_2}{S_2} \|g_{12}\|_1 \cdot \|g_{21}\|_1 < 1 \quad (2.20)$$

Inequality (2.19) was obtained independently and almost simultaneously by Skoog and Blankenship (121), Kan (69) and (2.20) by Guy (48).

It is important to note that, although Corollary 2.2 is easier to apply than Corollary 2.1, which is in turn easier to apply than Theorems 2.4 and 2.3, they are not equivalent. Therefore, it is recommended to use Corollary 2.2 first and, if it fails, to refer to Corollary 2.1 and then to Theorems 2.4 and 2.3.

In case the linear part is time-invariant and all the elements of its impulse response matrix $\underline{G}(t)$ do not change sign, the matrix $\underline{H}(t)$ becomes the step response matrix and the limit $\lim_{t \rightarrow \infty} \underline{H}(t)$ can be evaluated easily from

$$\lim_{t \rightarrow \infty} \underline{H}(t) = \left| \lim_{s \rightarrow 0} \frac{\underline{G}(s)}{s} \right| \quad (2.21)$$

where $\underline{G}(s) \triangleq \int_0^{\infty} e^{-st} \underline{G}(t) dt$. The bars $|\cdot|$ are used to infer that the absolute values of each element of the corresponding matrix is to be taken. If only some elements of the impulse response matrix do not change sign, it is still possible to use the same formula for those elements¹¹.

It should be noted that all results of this section apply to systems containing single-signed modulators, as well as double-signed modulators. This is due to the absolute value operations used in the derivation of Theorem 2.2 (see Appendix A), which also cause invariance of the results with respect to the "sign" of the feedback. In fact, this is not very surprising since the sign of the feedback can be controlled by the signs of the modulator output pulses.

2.3.1 Application to single-loop system with one PFM. In order to provide the reader with a basis for comparison, Table 2.1 is presented which summarizes some of the previous stability results applied to simple configurations containing single IPFM or RPFM (24, 25, 43, 69, 72, 79, 12). The tests developed in the pre-

¹¹It is very easy to determine $h_{ij}(t)$, experimentally. All the necessary equipment is an integrator preceded by an absolute value circuit. Exciting the system by a pulse with very short duration and measuring the output from the integrator gives $h_{ij}(t)$, directly.

TABLE 2.1
 Comparison of The Stability Results For Single Loop,
 Single-IPFM (or RPFM) Feedback System ($A \triangleq MA_0/S$).

	$G(s)$	Frequency Criteria [22], [43]	Lyapunov Methods [72], [79]	Condition (2.19) (121), (69)
	$\frac{A_0}{s+a_1}$	$-a_1 < A < 0$ [24] $-a_1 < A < a_1$ [43]	$-a_1 < A < a_1$ [72], [79]	$ A < a_1$
$\frac{1}{s}$ (IPFM)	$\frac{A_0}{(s+a_1)(s+a_2)}$	empty set [24]	$0 < A < 0.015$ [72]	$ A < 0.02$
	$\frac{A_0(s+b_1)}{(s+a_1)(s+a_2)}$	$-\frac{a_1 a_2}{b_1} < A < 0$ [43] $a_1 a_2 < b_1(a_1 + a_2)$	$0 < A < \frac{a_1 a_2}{b_1}$ [79] $a_1 a_2 < b_1(a_1 + a_2)$	$ A < \frac{a_1 a_2}{b_1}$ $ b_1 > a_1 > a_2$
	$\frac{A_0}{s+a_1}$	$-a_1 < A < a_0$ [24] $-a_1 < A < a_0 + 1/q$ [43] $q = \{(a_0 + a_1)^2 + 4a_0^2\}^{\frac{1}{2}} - a_0 - a_1$	$-$	$ A < a_1$
$\frac{1}{s+a_0}$ (RPFM)	$\frac{A_0}{(s+0.1)(s+0.2)}$ ($a_0=0.5$)	$-0.02 < A < 0.09$ [24]	$-0.39 < A < 0.018$ [79]	$ A < 0.02$

vious section (which coincide with (69) and (121), in the single-modulator case) are easier to apply and give better results in most cases, except when the time constant of the RPFM is appreciably smaller than that of the linear part.

In certain cases it may be possible to transform the CRPFM under consideration to obtain larger parameter-regions (sufficient) for stability. An example, which is indicated by Table 2.1 and which is frequently encountered is the case where the TF's have time-constants significantly smaller than that of the LP. Clearly, the stability region obtained by direct application of condition (2.19) for the RPFM system considered at the bottom of Table 2.1 is rather conservative. By a simple transformation, however, the effectiveness of the same condition can be improved significantly; this will be demonstrated by the following example.

Example 2.2: Consider the single-loop RPFM feedback system of Fig. 2.3c, which was also used as an example system in Table 2.1 (last entry of the table). The transfer function of the plant and the TF are

$$G(s) = \frac{A_0}{(s+0.1)(s+0.2)} \quad (2.22a)$$

and

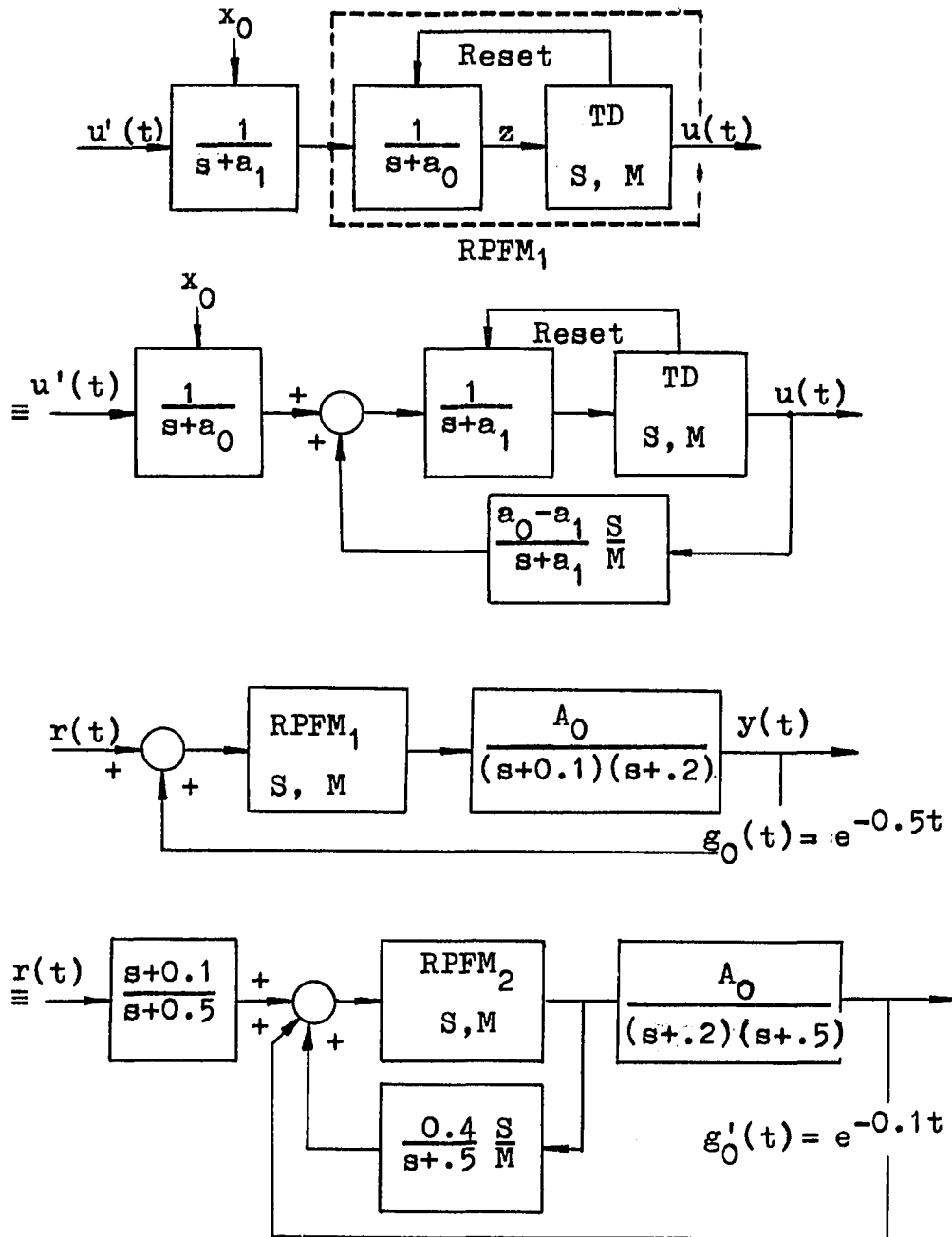


Figure 2.3 (a) RPFM with a first order low-pass filter.
 (b) Equivalent RPFM system.
 (c) A CRPFM system in which the dynamics of the LP is much slower than that of the TF.
 (d) Equivalent system obtained after transformations indicated in Fig. 2.3a and Fig. 2.3b. Application of Theorem 2.4 to this system yields less conservative results.

$$G_0(s) = \frac{1}{s+a_0}, \quad a_0 > 0 \quad (2.22b)$$

respectively. For GFPS, direct application of condition (2.19) yielded

$$|A| < 0.02 \quad (2.23)$$

By simple block diagram manipulations this system can be transformed into the equivalent form shown in Fig.

2.3d. The transfer functions of the plant and the TF of the equivalent system are

$$G'(s) = \frac{A_0}{(s+0.2)(s+0.5)} + \frac{0.4}{s+0.5} \frac{S}{M} \quad (2.24a)$$

and

$$G'_0(s) = \frac{1}{s+0.1}, \quad (2.24b)$$

respectively. Since the impulse response of the equivalent system does not also change sign, $\|g\|_1$ can be easily evaluated from eq. (2.21). Thus, (2.19) yields

$$\left| 10A + \frac{4}{5} \right| < 1,$$

or,

$$-0.18 < A < 0.02, \quad (2.25)$$

where, $A \triangleq A_0 \frac{S}{M}$. Clearly, condition (2.25) is significantly less conservative than condition (2.23).

2.3.2 Application to systems with more than one PFM. The objective of this section is to stress the

meaning of matrix inequality (2.14). Three examples will be presented to demonstrate the regions described by this inequality. As the first example, a PFM system containing an IPFM with a memoryless nonlinearity and an RPFM is selected with parameters such that this inequality does not provide any information. In the second example, by slightly modifying some of the parameters of the system (thresholds and pulse-strengths of the modulators, in this particular case), boundedness of the number of pulse-emissions are guaranteed. The third example will be presented to show that, even though the conditions of Theorem 2.3 are not satisfied, inequality (2.14) can still provide information relating boundedness of the number of firings of one modulator to that of the other.

For the purposes of Examples 2.3a-2.3c the system of Fig. 2.4, containing one IPFM preceded by a nonlinearity and one RPFM is considered. It is not difficult to see that Conditions 1-3 (as $t \rightarrow \infty$) are satisfied and that $\alpha_i = \beta_i = 1$ ($i = 1, 2$). From eqs. (2.11)-(2.13), the matrix \underline{H}'_{∞} is easily computed as

$$\underline{H}'_{\infty} = \begin{bmatrix} 3 |M_1/S_1| & 5 |M_2/S_2| \\ 2 |M_1/S_2| & 4 |M_2/S_2| \end{bmatrix} \quad (2.26)$$

The eigenvalues of the matrix \underline{H}'_{∞} depend on the values

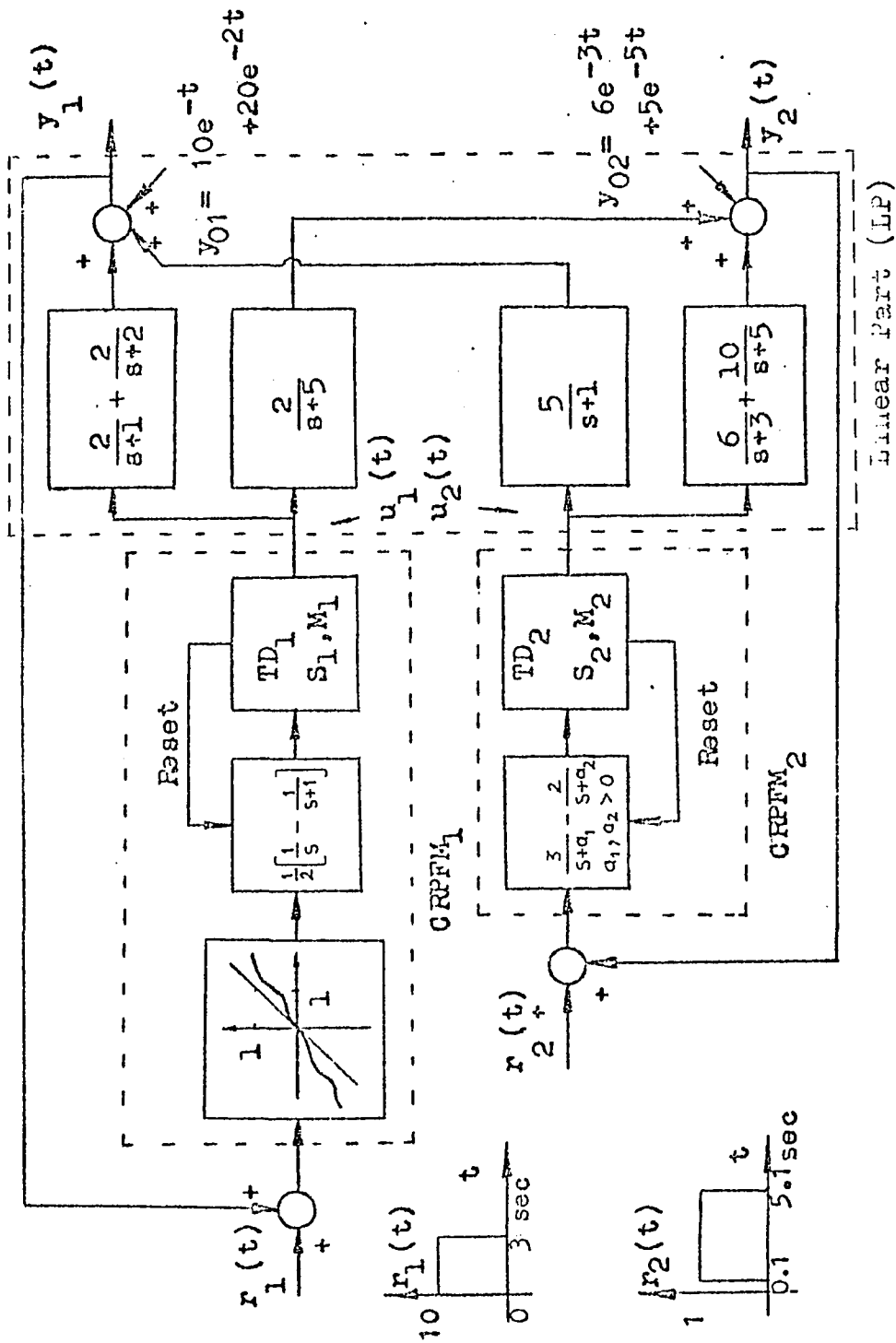


Figure 2.4 A PFM system consisting of the interconnections of two CRPFM's and a time-invariant linear part.

M_1 , S_1 , M_2 and S_2 (pulse strengths and thresholds of the modulators).

Example 2.3a (Case Where $\lambda(\underline{H}'_{\infty}) > 1$): For $M_1 = S_1 = 5$ and $M_2 = S_2 = 1$ (2.22) gives

$$\underline{H}'_{\infty} = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}$$

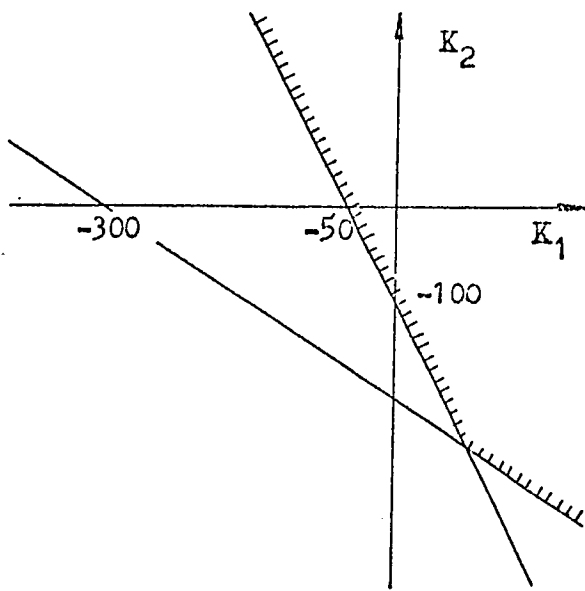
The eigenvalues of this matrix are 2 and 5. Since 1 is not an eigenvalue, the matrix $[\underline{I} - \underline{H}'_{\infty}]^{-1}$ exists and is

$$[\underline{I} - \underline{H}'_{\infty}]^{-1} = \begin{bmatrix} -3/4 & 1/4 \\ 1/2 & -1/2 \end{bmatrix}$$

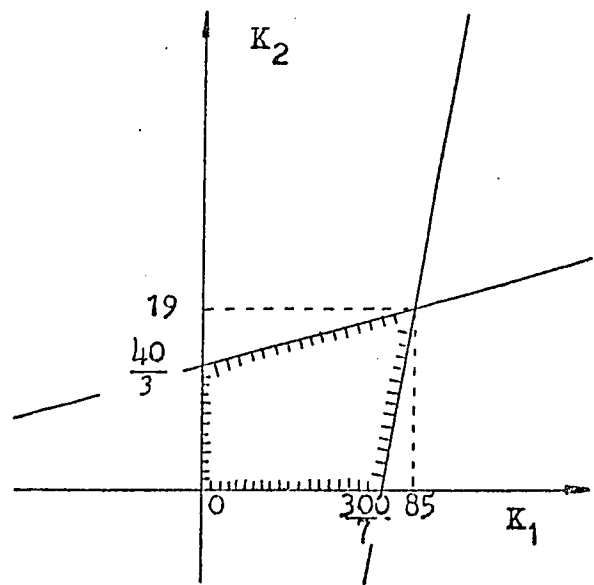
However, it contains negative elements and therefore inequality (2.17b) is not applicable. Nevertheless, inequality (2.14) is still valid and is illustrated in Fig. 2.5a. Clearly, in this case it also does not provide any information.

Although the theory developed in this section cannot establish instability, the system of this example does not appear to be stable.

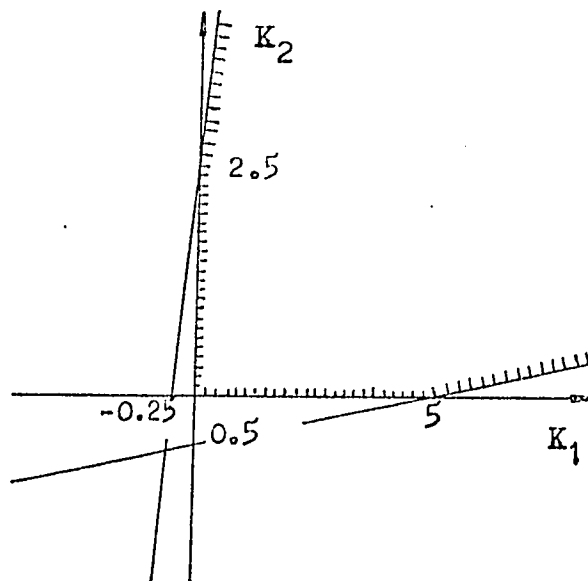
Example 2.3b (Case Where $\lambda(\underline{H}'_{\infty}) < 1$): By either increasing the thresholds or decreasing the pulse strengths, the eigenvalues of the matrix \underline{H}'_{∞} can be brought into the unit circle. For $S_1 = S_2 = 1$ and $M_1 = M_2 = 0.1$,



(a)



(b)



(c)

Figure 2.5 Regions described by inequality (2.14).

(a) Example 2.3a,

(b) Example 2.3b, and

(c) Example 2.3c.

$$\underline{H}'_{\infty} = \begin{bmatrix} 0.3 & 0.5 \\ 0.04 & 0.4 \end{bmatrix} \quad \text{and} \quad \underline{v}_{\infty} = \begin{bmatrix} 50 \\ 8 \end{bmatrix}$$

The eigenvalues are inside the unit circle and

$$[\underline{I} - \underline{H}'_{\infty}]^{-1} = \begin{bmatrix} 3/2 & 5/4 \\ 1/10 & 7/4 \end{bmatrix}.$$

Since all the elements of this matrix are nonnegative, inequality (2.17b) is valid. Let

$$r_1(t) = \begin{cases} 10 & \text{for } 0 < t < 3 \\ 0 & \text{elsewhere} \end{cases}, \quad (2.27a)$$

and,

$$r_2(t) = \begin{cases} 1 & \text{for } 0.1 < t < 5.1 \\ 0 & \text{elsewhere} \end{cases} \quad (2.27b)$$

and let

$$\underline{y}_0(t) = \begin{bmatrix} 10e^{-t} + 20e^{-2t} \\ 6e^{-3t} + 5e^{-5t} \end{bmatrix} \quad (2.27c)$$

From (2.11), for $t \rightarrow \infty$,

$$\underline{v}_{\infty} = \begin{bmatrix} 50 \\ 8 \end{bmatrix} \quad (2.28)$$

Thus, (2.17) yields

$$\underline{k}_{\infty} = \begin{bmatrix} 85 \\ 19 \end{bmatrix} \quad (2.29)$$

i.e., the first modulator stops after firing at most 85

emissions and the second modulator after at most 19 emissions. It is interesting to compare this bound with the actual numbers of total emissions obtained from a simulation of the system which, for the given conditions, yielded

$$\underline{k}_{\infty}(\text{Actual}) = \begin{bmatrix} 82 \\ 17 \end{bmatrix} \quad (2.30)$$

Thus, for this example, the bounds on \underline{k}_{∞} obtained from (2.17a) are rather close to the actual values¹².

The matrix $[\underline{I} - \underline{H}'_{\infty}]^{-1}$ will be nonnegative if the following inequalities are satisfied:

$$1 - 3|A_1| > 0, \quad (2.31a)$$

$$1 - 4|A_2| > 0, \quad (2.31b)$$

and

$$1 - 3|A_1| - 4|A_2| + 10|A_1A_2| > 0 \quad (2.31c)$$

where $A_1 \triangleq M_1/S_1$ and $A_2 \triangleq M_2/S_2$. If A_1 and A_2 are chosen in accordance with the above relations, the system will be GFPS. This region is plotted in Fig. 2.6 (inside of the circular region).

Example 2.3c (Case Where $\lambda(\underline{H}'_{\infty}) > 1$): Let

¹²Most simulation studies of simple CRPFM systems yielded a good agreement with inequality (2.16).

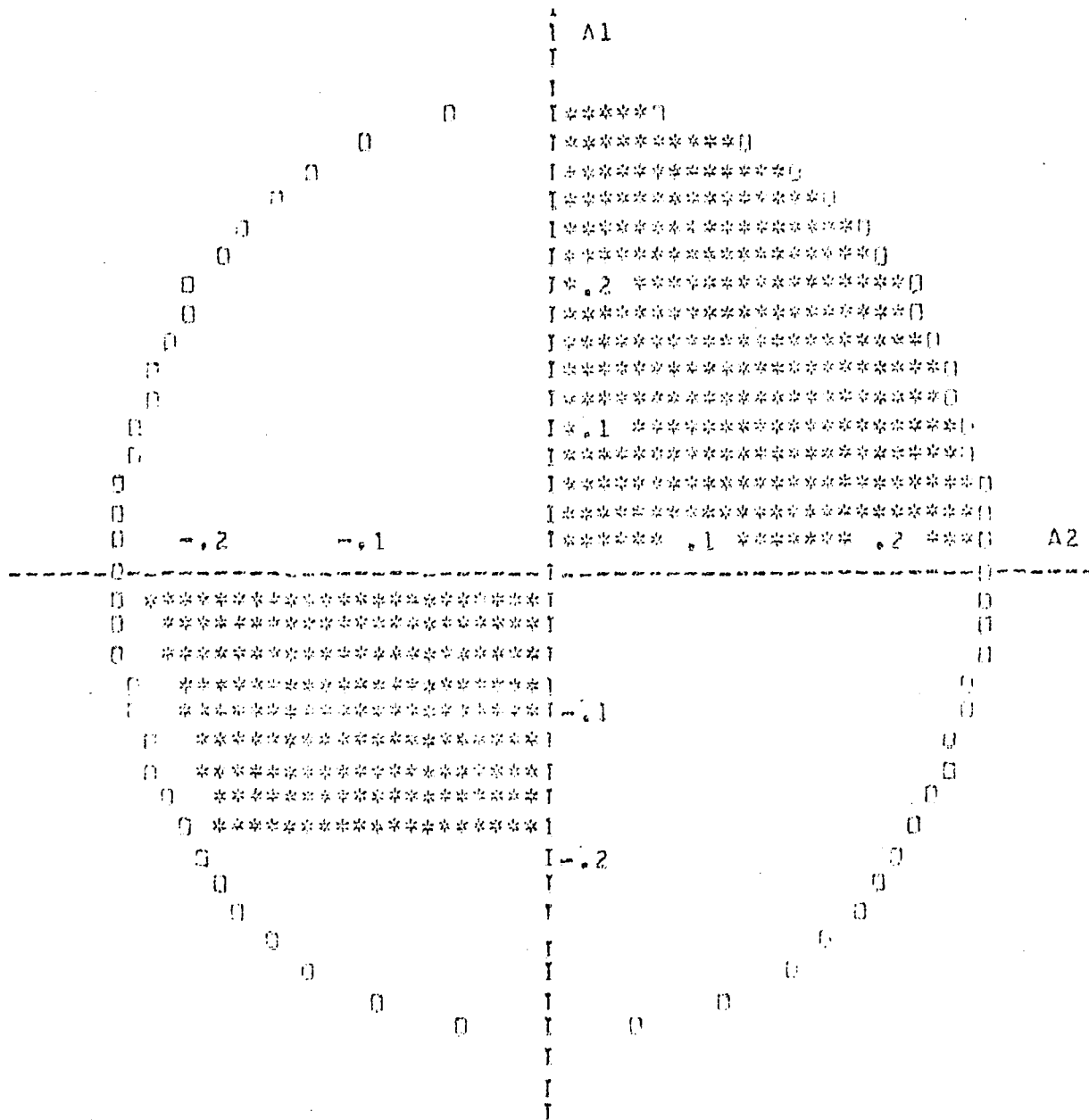


Figure 2.6 Comparison of stability criteria for the IPFM system of Fig. 2.9. The region with stars is obtained by Theorem 2.5 (Example 2.5, Gelig (44)). Inside of the circular region is the stability region predicted by Theorem 2.4 (Example 2.3b, Gülçür and Meyer (52)).

$$\underline{H}'_{\infty} = \begin{bmatrix} 0.8 & 2.0 \\ 4.0 & 0.6 \end{bmatrix} \quad \text{and} \quad \underline{y}_{\infty} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

The eigenvalues of this matrix are greater than 1 in magnitude and the matrix $[\underline{I} - \underline{H}'_{\infty}]^{-1}$ contains negative terms. Thus, inequality (2.16) (as $t \rightarrow \infty$) is not applicable. However, inequality (2.14) gives

$$\begin{bmatrix} 0.2 & -2 \\ -4 & 0.4 \end{bmatrix} \underline{k}_{\infty} \leq \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad (2.32)$$

The region described by this inequality is shown in Fig. 2.5c. As in Example 2.3a, stability cannot be established for this case. The only information gained from Fig. 2.5c is that if $K_1 \rightarrow \infty$ then $K_2 \rightarrow \infty$ and vice-versa, which means that any system instability is associated with continued firing of both modulators.

Satisfaction of Conditions 1-3 (as $t \rightarrow \infty$) essentially requires the linear part of the system to be asymptotically stable and the input signals to be absolutely integrable. In cases, where the linear part contains integration and/or the input signals contain constant parts, GFPS may still exist, provided the TF's provide sufficient filtering. In these cases, Theorem 2.2 and Theorem 2.3 cannot be applied directly. However, it may be possible to transform the system under consideration in such a way that the conditions of these theorems will be satisfied.

A frequent occurrence is an RPFM preceded by an integrator (see Fig. 2.7a). It can be replaced by an IPFM subsystem as shown in Fig. 2.7b (see Meyer (93), for other possible transformations).

Example 2.4: Consider again the RPFM system of Fig. 2.1 which was treated in Example 2.1. In this case Theorem 2.3 and Theorem 2.4 cannot be applied directly because the LP contains an integrator. ($g(\cdot) \notin L_1[0, \infty)$). However, this system can easily be transformed into the form shown in Fig. 2.7c. The equivalent IPFM system of Fig. 2.7c has the impulse response matrix

$$\underline{G}(t) = \begin{bmatrix} (1 + a_1 \frac{S_1}{M_1}) e^{-a_1 t} & e^{-a_1 t} \\ e^{-a_2 t} & (1 + a_2 \frac{S_2}{M_2}) e^{-a_2 t} \end{bmatrix} \quad (2.33)$$

Thus, relation (2.14) of Theorem 2.2 yields

$$\begin{bmatrix} 1 - \left| 1 + \frac{M_1}{a_1 S_1} \right| (1 - e^{-a_1 t}) & - \left| \frac{M_2}{a_2 S_1} \right| (1 - e^{-a_1 t}) \\ - \left| \frac{M_1}{a_2 S_2} \right| (1 - e^{-a_2 t}) & 1 - \left| 1 + \frac{M_2}{a_2 S_2} \right| (1 - e^{-a_2 t}) \end{bmatrix} \underline{k}(t) \leq \underline{v}(t) \quad (2.34)$$

The matrix $[\underline{I} - \underline{H}'(t)]^{-1}$ is nonnegative if

$$-1 < \frac{M_1}{a_1 S_1} < 0 \quad \text{and} \quad -1 < \frac{M_2}{a_2 S_2} < 0.$$

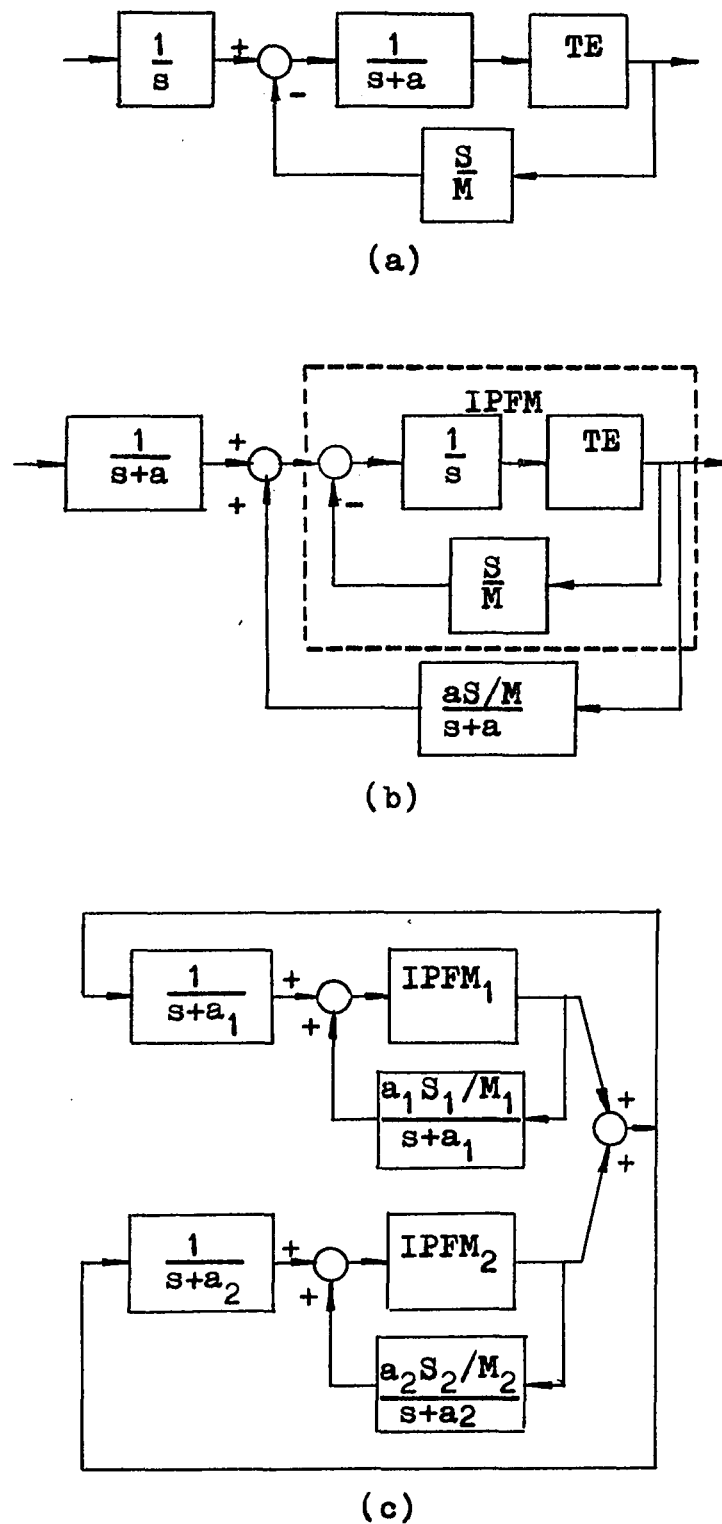


Figure 2.7 (a) RPFM with an integrator,
 (b) Equivalent IPFM system,
 (c) Equivalent IPFM system for the RPFM system
 of Fig. 2.1.

In this case inequality (2.16) is applicable and, since $\underline{y}(t)$ is finite $\underline{k}(t)$ is finite. This region is shown in Fig. 2.2.

To summarize, in this section upper bounds on the number of pulses emitted by each modulator during the operation of a CRPFM system were determined. Such number is indicative of the amount of energy spent by the corresponding modulator and thus the upper bound of the number of pulses emitted by all modulators represents a measure of stability. Sufficient conditions under which this number is finite were established and were shown to depend on nonnegativity of a certain matrix.

The features of the results of this section are the following.

1) Generality. The conditions apply to PFM systems containing distributed and/or lumped linear parts; the timing filters are allowed to include nonlinearities. The number of loops are not limited to one; the modulators are quite general (not restricted to IPFM or RPFM) and can be single-signed or double signed.

2) Simplicity. Once the \underline{H}'_{∞} matrix is known, it is relatively easy to apply the stability conditions.

Direct application of Theorems 2.3-2.4 or Corollaries 2.1-2.2 require all linear plants to be asymptotically

stable and all input signals to be absolutely integrable. In some cases where the linear plants contain integration and/or the input signals contain d-c parts, global finite-pulse stability may still exist; stability conditions for these cases can be obtained by transforming the system using simple block diagram manipulations (see Example 2.4). Similar transformations can also be used to obtain less conservative results (see Example 2.2).

Application of the results to a single-loop, single-modulator system gave a condition which was previously obtained (69, 121) and examples yielded stability regions comparable (often better) to those obtained by other methods (such as described in (24, 43, 72, 79)). The same was found to be true in comparison with a recent frequency domain stability criterion for interconnected systems (44). This will be discussed next.

2.4 Frequency Domain Criteria

Gelig (44), in a recent paper, obtained frequency-domain stability criteria for a PFM system consisting of m -relaxation type pulse frequency modulators (RPFM's) and a time-invariant linear part. To provide ground for comparison, in this section, a summary of his results is given. The results are also applied to the systems considered in the previous sections.

The system considered by Gelig is a special case of the general CRPFM system of Fig. 1.9 (see Fig. 2.8a); the function $f_i(\)$ is

$$f_i[r_i(\tau), y_i(\tau), t, \tau] = e^{-a_i(t-\tau)} [r_i(\tau) + y_i(\tau)] \quad (2.35)$$

($i = 1, \dots, m$)

where a_i is a positive constant. Also since the LP is time-invariant,

$$g_{ij}(t, \tau) = g_{ij}(t - \tau), \quad (i, j = 1, \dots, m) \quad (2.36)$$

It is convenient to transform this system into the form shown in Fig. 2.8b. Let $\underline{g}(s)$ be the Laplace transform of the impulse-response matrix of the linear part of the system, i.e., $\underline{g}(s) \triangleq \int_0^{\infty} e^{-st} \underline{g}(t) dt$. Let $\underline{\mathcal{G}}(s)$ be an $m \times m$ matrix whose element in the i th row and the j th column is defined by

$$\mathcal{G}_{ij}(s) = \frac{1}{s + a_i} [s_i \delta_{ij} - M_j \mathcal{G}_{ij}(s)] \quad (2.37)$$

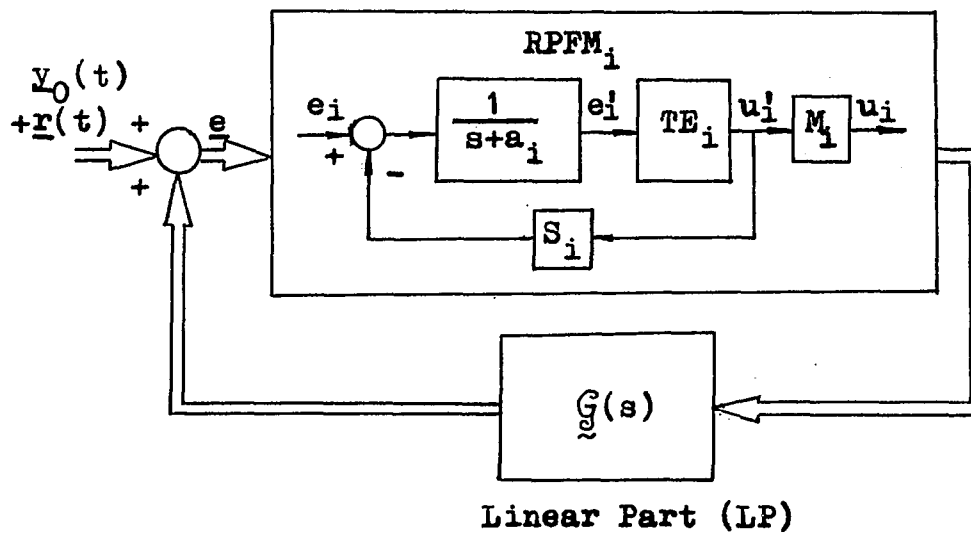
($i, j = 1, \dots, m$)

where δ_{ij} is Kronecker's symbol ($\delta_{ij} = 1$ for $i = j$, $\delta_{ij} = 0$ for $i \neq j$).

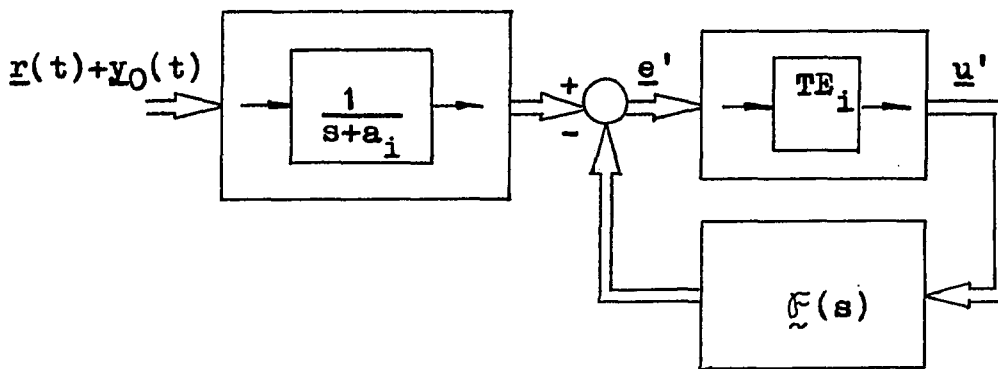
It is assumed that the matrix $\underline{\mathcal{G}}(s)$ can be represented in the form

$$\underline{\mathcal{G}}(s) = \underline{\chi}(s) + \frac{1}{s} \underline{R} \quad (2.38)^{13}$$

where $\underline{\chi}(s)$ is analytic for $\text{Re } s \geq 0$ (i.e., all singularities in the l.h.p.) and \underline{R} is a constant $m \times m$ matrix.



(a)



(b)

Figure 2.8 (a) The RPFM system considered in Section 2.4.
 (b) Equivalent system (TE_i is a device that emits a unit impulse whenever the absolute value of its input signal exceeds a threshold value, S_i)

Furthermore, it is assumed that $\underline{r}'(t) = \underline{r}(t) + \underline{y}_0(t)$ (input + initial condition response) is bounded and that $\underline{r}'(t) \rightarrow 0$ as $t \rightarrow \infty$. If $\underline{\mathcal{G}}(s)$ has a pole at $s = 0$, $\underline{r}'(t)$ could contain a constant part, however, the transient part must vanish as $t \rightarrow \infty$.

Let \underline{R}' be an $m \times m$ matrix with the element in the i th row and the j th column defined by

$$r'_{ij} = \begin{cases} r_{ij}, & \text{if } \underline{\mathcal{G}}(s) \text{ has a pole at } s = 0, \\ \lim_{s \rightarrow \infty} s \underline{g}_{ij}(s) \cdot M_j, & \text{otherwise} \end{cases} \quad (2.39)$$

$$\underline{F}_0 = \lim_{s \rightarrow \infty} s \underline{\chi}(s) \quad (2.40)$$

¹³The case where all the poles of the matrix $\underline{\mathcal{G}}(s)$ have negative real parts is called the noncritical case in the Russian literature. If there is a simple pole at the origin but all the other poles are in the l.h.p., this case is called the simplest critical case. Gelig (44), considers also the case where the matrix $\underline{\mathcal{G}}(s)$ contains simple poles on the imaginary axis and can be represented in the form

$$\underline{\mathcal{G}}(s) = \underline{\chi}(s) + \frac{1}{s} \underline{R} + \sum_{i=1}^q \frac{1}{s^2 + \omega_i^2} [\underline{A}_i + \underline{B}_i]$$

where \underline{R} , \underline{A}_i , \underline{B}_i are constant $m \times m$ matrices and $\underline{\chi}(s)$ is analytic for $\text{Re } s \geq 0$. If the matrix $\underline{\mathcal{G}}(s)$ has poles at $s = +j\omega_i$ ($i = 1, \dots, q$), the term $\underline{r}'(t)$ is allowed to contain terms of the form $\underline{\alpha}_i \sin \omega_i t + \underline{\beta}_i \cos \omega_i t$, where $\underline{\alpha}_i$ and $\underline{\beta}_i$ are constant vectors.

$$\underline{m}(s) = [\underline{T} + s\underline{\theta}] \underline{x}(s) - \underline{\theta} \underline{F}_0 \quad (2.41)$$

and

$$\underline{S}_0 = \lim_{s \rightarrow \infty} s \underline{m}(s) \quad (2.42)$$

where \underline{T} and $\underline{\theta}$ are $m \times m$ diagonal matrices with elements τ_i and θ_i ($i = 1, 2, \dots, m$), respectively.

The following theorem was proved by Gelig (44).

Theorem 2.5 (Gelig, (44)): If there exist constants $\tau_i > 0$ and $\theta_i \geq 0$ ($i = 1, 2, \dots, m$) satisfying the conditions:

$$(1) \quad \tau_1 s_1 - \theta_1 (r'_{11} - s_1 a_1) > 0,$$

$$\tau_i s_i - \theta_i (r'_{ii} - s_i a_i) - 2\theta_i \sum_{j=1}^{i-1} |r'_{ij}| > 0$$

$$(i = 1, \dots, m)$$

$$(2) \quad \underline{S}_0 = \underline{S}_0^T,$$

$$(3) \quad \text{Re}[\underline{m}(j\omega) + \underline{m}^T(j\omega)] \text{ is a positive semi-definite}$$

matrix, and

$$(4) \quad \underline{T} \underline{R} = \underline{R}^T \underline{T} \text{ is a positive definite matrix,}$$

then the RPFM system of Fig. 2.8a is GFPS¹⁴.

¹⁴If the matrix $\underline{G}(s)$ contains simple poles at $s = +j\omega_i$, ($i = 1, \dots, q$) then in addition to the conditions 1-4 of Theorem 2.5, the following conditions must be satisfied (see footnote no. 13)

$$\underline{A}_i^{-1} \underline{B}_i = \underline{B}_i \underline{A}_i^{-1}; \quad \underline{T} \underline{A}_i = \underline{A}_i^T \underline{T} \text{ is a p.d. matrix,}$$

$\underline{T} \underline{B}_i = \underline{B}_i^T \underline{T}$ is a p.s.d. matrix, and

$$\underline{\theta} = \frac{1}{\omega_i^2} \underline{T} \underline{A}_i^{-1} \underline{B}_i \quad (i = 1, 2, \dots, q).$$

Example 2.5: Consider the PFM system shown in Fig. 2.9, containing two IPFM's and a LP with the following impulse transfer matrix:

$$\underline{G}(s) = \begin{bmatrix} \left(\frac{2}{s+1} + \frac{2}{s+2}\right) A_1 & \frac{5}{s+1} A_2 \\ \frac{2}{s+5} A_1 & \left(\frac{6}{s+3} + \frac{10}{s+5}\right) A_2 \end{bmatrix}$$

where A_1 and A_2 are parameters to be determined such that GFPS is assured.

This system has the same structure as that of the system of Fig. 2.4 considered earlier in Example 2.3, with the exception of the modulators, which are IPFM's here. Gelig's theorem is not applicable to the system of Fig. 2.4. Note that stability conditions (2.31), which were obtained for the system of Fig. 2.4 by application of the results of Section 2.3, are also the same for the system considered in this example.

From (2.37) and (2.38),

$$\underline{X}(s) = \begin{bmatrix} \left(\frac{2}{s+1} + \frac{1}{s+2}\right) A_1 & \frac{5}{s+1} A_2 \\ \frac{2}{5(s+5)} A_1 & \left(\frac{2}{s+3} + \frac{2}{s+5}\right) A_2 \end{bmatrix} \quad (2.43)$$

and

$$\underline{R} = \begin{bmatrix} (1-3A_1) & -5A_2 \\ -\frac{2}{5} A_1 & (1-4A_2) \end{bmatrix} \quad (2.44)$$

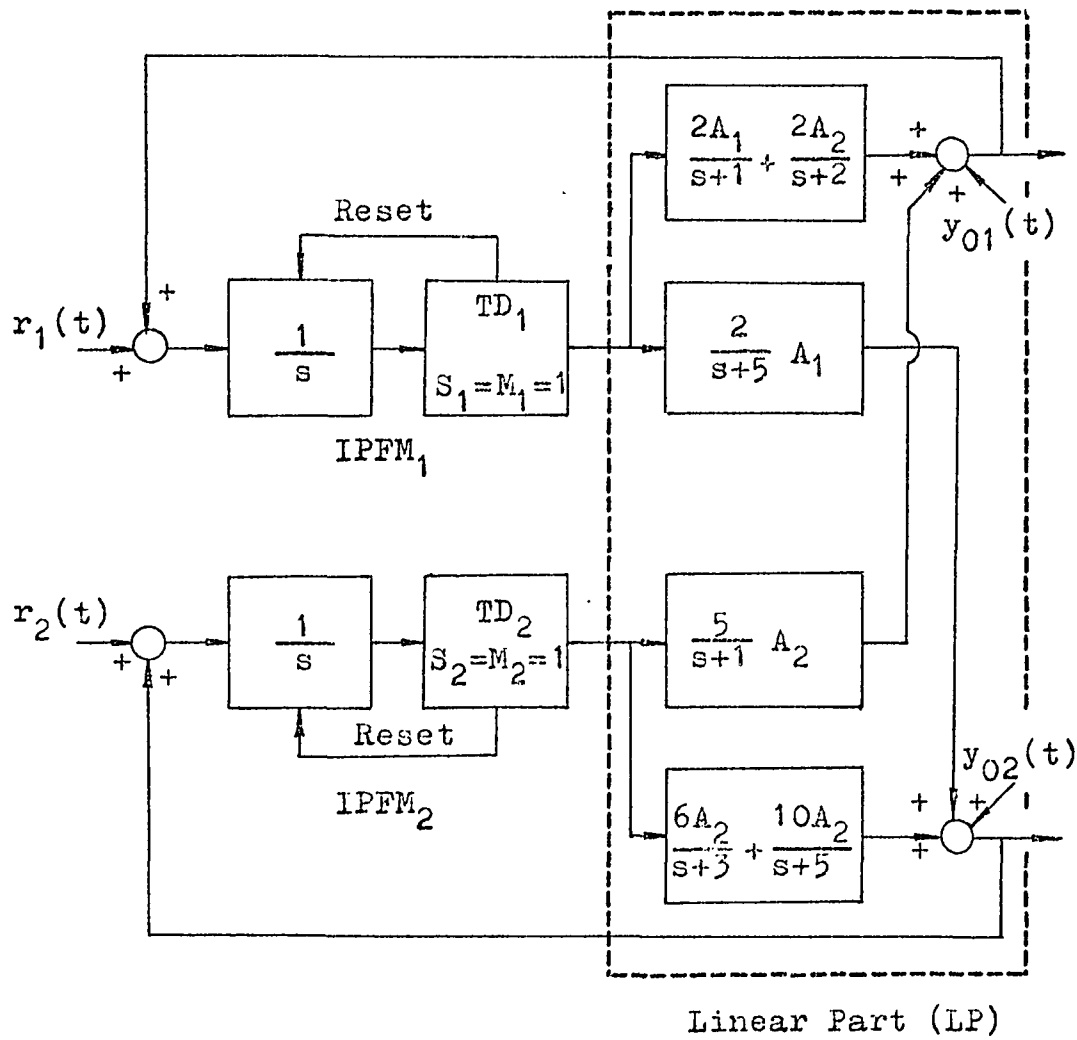


Figure 2.9 A PFM system consisting of the interconnections of two IPFM's and a time-invariant linear part.

From (2.41),

$$\underline{m}(s) = \left[\begin{array}{cc} \left(2 \frac{\tau_1^{-\theta_1}}{s+1} + \frac{\tau_1^{-2\theta_1}}{s+2} \right) A_1 & 5 \frac{\tau_1^{-\theta_1}}{s+1} A_1 \\ 2 \frac{\tau_2^{-5\theta_2}}{s+5} A_1 & 2 \left(\frac{\tau_2^{-3\theta_2}}{s+3} + \frac{\tau_2^{-5\theta_2}}{s+5} \right) A_2 \end{array} \right] \quad (2.45)$$

Condition (4) of Theorem 2.5 gives

$$\tau_2 = \frac{25A_2}{2A_1} \tau_1 \quad (2.46)$$

and

$$1 - 4A_2 - 3A_1 + 10 A_1 A_2 > 0 \quad (2.47)$$

Since $\tau_1 > 0$ and $\tau_2 > 0$, (2.46) will not be satisfied if $A_1 A_2 < 0$. Thus, Theorem 2.5 fails for $A_1 A_2 < 0$. Condition (2) yields

$$\theta_2 = \frac{5A_2}{2A_1} \theta_1 \quad (2.48)$$

Equations (2.46), (2.48) and condition (1) of Theorem 2.5 yield

$$\tau_1 + 4A_1 \theta_1 > 0, \quad (2.49a)$$

and

$$\tau_1 + (3.2A_2 - 0.8|A_1|) \theta_1 > 0 \quad (2.49b)$$

An alternative to conditions (2.49a) and (2.49b) can be obtained by changing the numbering of the modulators.

Thus,

$$\tau_1 + 3.2A_2\theta_1 > 0 \quad (2.50a)$$

and

$$\tau_1 + (4A_1 - 10|A_2|)\theta_1 > 0 \quad (2.50b)$$

Finally, the frequency condition (condition (3) of Theorem 2.5) yields

$$A_1 \left(\frac{\tau_1 - \theta_1}{1 + \omega^2} + \frac{\tau_1 - 2\theta_1}{4 + \omega^2} \right) > 0, \quad -\infty < \omega < \infty \quad (2.51a)$$

and

$$2 \left(\frac{\tau_1 - \theta_1}{1 + \omega^2} + \frac{\tau_1 - 2\theta_1}{4 + \omega^2} \right) \left(\frac{15\tau_1 - 9\theta_1}{9 + \omega^2} + \frac{25\tau_1 - 25\theta_1}{25 + \omega^2} \right) - 5(\tau_1 - \theta_1)^2 \left(\frac{1}{1 + \omega^2} + \frac{5}{25 + \omega^2} \right)^2 \geq 0 \quad (2.51b)$$

$$-\infty < \omega < \infty$$

Now, the problem is reduced to that of selecting $\tau_1 > 0$ and $\theta_1 > 0$, such that relations (2.47), (2.49) or (2.50) (2.51a) and (2.51b) are satisfied. The stability region in the A_1 - A_2 plane as defined by these relations is shown in Fig. 2.6 (region with stars)¹⁵.

Example 2.6: Now, consider the same system considered in Example 2.1. For $a_1 = a_2 = a$ ($0 < a < \frac{1}{2}$), Theorem 2.5 yields

¹⁵The stability region was obtained by a computer program which used Sturm's test to check the frequency condition (2.51b).

$$\frac{M_2}{S_2} + \frac{M_1}{S_1} + a \geq 0, \quad M_1 \leq 0 \quad \text{and} \quad M_2 \leq 0 \quad (2.52)$$

For this case, the stability region determined from Theorem 2.5 is shown in Fig. 2.2 (region with horizontal shading).

2.5 Conclusions

In this chapter global finite-pulse stability (GFPS) in PFM systems is considered. Two different approaches are presented. The first approach is based on Lyapunov's second method and the second approach is a direct approach involving careful application of inequalities to the equations describing the system. A summary of a recent frequency-domain criterion of Gelig (44), applicable only to RPFM systems, is also included for comparison purposes.

The Lyapunov approach could provide effective stability criteria, but it is difficult to apply; especially for higher-order systems.

Gelig's frequency response criterion is restricted to relaxation pulse frequency modulation (RPFM) systems with a time-invariant linear parts. It can handle "critical cases" where the LP has simple poles on the imaginary axis or at the origin. However, in order to obtain a good parameter-region sufficient for stability, it requi-

res selection of two arbitrary parameters for each modulator of the system, subject to frequency-domain conditions and other inequality constraints.

The Direct GFPS criterion is the simplest to apply and at the same time, provides bounds on the number of pulses emitted by each modulator. It is also applicable to more general systems; the timing filters are allowed to include nonlinearities, the LP can be time-varying and the modulators are not restricted to IPFM or RPFM. It cannot handle the "critical cases" directly. However, it is usually possible to transform the system in such a way that the criteria will be applicable (see, e.g., Example 2.4). Comparative examples yielded greater stability regions of parameters from the direct GFPS criteria than from (an optimal application of) Gelig's frequency-domain criteria.

A summary of comparison of the three methods is given in Table 2.2.

TABLE 2.2
Comparison of Stability Theorems
For Multi-Modulator PFM Systems

Theorem	2.1	2.3 and 2.4	2.5 Gelig (44)
Type of modulator	CRPFM	CRPFM (the TF can be nonlinear)	RPFM only
Type of the LP	linear, nonlinear, time-var.	linear, lumped distributed, can be time-varying	linear, lumped or distributed, time-invariant only
Restrictions on the LP	finite order.	the LP must be asymptotically stable. This can be relaxed (e.g., Example 2.4).	all the elements of the transfer matrix $\bar{G}(s)$ of the equivalent system must be analytical for $\text{Re } s > 0$. $\bar{G}(s)$ can contain simple poles at $s=0$ and on the imaginary axis.
Restrictions on the input signals	not explicit.	all input signals must be absolutely integrable. In some cases this condition can be relaxed (see Example 2.4). or sinusoidal terms with frequency ω_i) but the transient parts must vanish as $t \rightarrow \infty$.	all the input signals must be bounded and must go to zero as $t \rightarrow \infty$. If the transfer function matrix $\bar{G}(s)$ of the LP has a pole at $s=0$ (and/or $s=j\omega_i$) they can contain constant parts (and/or sinusoidal terms with frequency ω_i) but the transient parts must vanish as $t \rightarrow \infty$.
Effectiveness of Theorem: (how conservative?)	depends on choice of Lyapunov function	gives better results if the TF's have time constants that are small compared to those of the LP.	depends on choice of $2m$ auxiliary parameters, which can be determined by optimization process to provide max. parameter region.
Ease of application	most difficult, especially for high order systems.	easiest.	difficult if optimal stability region (of parameters) is desired.

CHAPTER 3

ON NEARLY PERIODIC MOTION IN INTERCONNECTED
SYSTEMS WITH PULSE FREQUENCY MODULATORS3.1 Introduction

The motion of a PFM system can be classified into the following groups:

1. Finite-pulse stable motion (all the modulators of the system stop firing in finite time),
2. unstable motion (the input signals of the modulators become very large; pulse frequencies increase until some part of the system is saturated or deformed),
3. nearly periodic motion (the input signals of the modulators repeat -within reasonable bounds- periodically), and
4. non-periodic motion (motion which is not nearly periodic).

Sufficient conditions for global finite-pulse stability were presented in the previous chapter. In this chapter, the objective is to study periodic motion in PFM systems. Owing to the abundance of different possible modes of periodic motion, peculiar to these systems, this topic has been discussed even in the earliest publications (89, 93), and has been given a considerable amount of

attention (29, 36, 48, 101, 103, 127). The abundance of periodic modes, in some way, is not unexpected; in fact reverberatory activity in neural circuits has long been suggested as a possible mechanism of instantaneous memory¹.

Previous investigations of this subject have been restricted to single-loop systems². Unlike single-loop, single-modulator systems, however, multi-modulator systems cannot, in practice, have pure periodic motion. This is true for most high order physical systems³ and is related to the fact that there are only a countable number of rational numbers.

Clearly, a weaker concept of periodicity is necessary; this will be given in the following section.

In this chapter, The CRPFM system of Fig. 3.1 is considered, in which all the TF's are linear. This system is slightly less general than the system of Fig. 1.9.

¹Instantaneous (short-term, temporary) memory "refers to one's ability to recall tremendous amounts of information from one second to the next or from minute to minute", Guyton (50), p. 722.

²In (103) Pavlidis describes a Lyapunov method which is applicable also to multi-loop systems.

³Consider, for example, a fourth order linear, time-invariant, conservative system. It has a general solution of the form $a_1 \cos \omega_1 t + b_1 \sin \omega_1 t + a_2 \cos \omega_2 t + b_2 \sin \omega_2 t$, which is periodic, if and only if ω_1 and ω_2 are commensurable, i.e., ω_1/ω_2 is a rational number, Hahn (54).

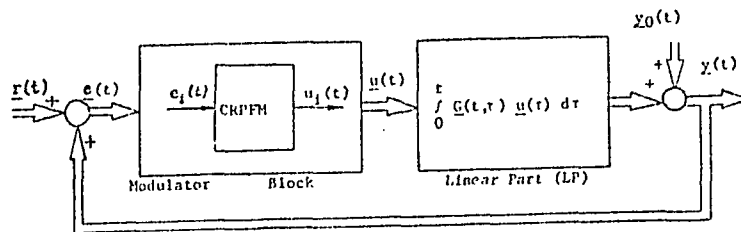


Figure 3.1 Block diagram of an interconnected system consisting of CRPFM's with linear TF's and linear dynamical subsystems.

However, some of the results to be presented are directly extendable to the system of Fig. 1.9.

3.2 The Concept of ϵ_e -Near Periodicity

One possibility of defining a "weak" period is to translate all the pulse-instants, $t_{i,k}$ by a number $T > 0$, and to compare the translated points with the original points; if the translated points are in the vicinity of the original points, possibly with a small number of exceptions, the number T could be considered as a "weak" period of the system. This definition would be difficult to employ; however, the following definition has the same implications and, at the same time, is easier to handle.

Definition 3.1 Given $\epsilon_e \geq 0$ and $a > 0$, the motion of a PFM system will be called ϵ_e -nearly periodic (ϵ_e -n.p.) in the interval $t \in (0, a]$, if

- (1) impulse emission of at least one modulator continues during $0 < t < \infty$ (i.e., does not stop),
and
- (2) there exists a number $T \in (0, a]$, such that the input vector of the modulators $\underline{e}(t)$, satisfies the relation

$$\|\underline{e}(t+T) - \underline{e}(t)\| \leq \epsilon_e, \quad \forall t \in (0, a] \quad (3.1)^4$$

The interval $t \in (0, a]$ will be called the observation interval and the smallest number, T , satisfying (3.1) will be called the ϵ_e -period of the motion. This definition is illustrated in Fig. 3.2.

The above concept differs from that of almost periodicity, introduced by Bohr, about half a century ago. Let X be a Banach space⁵ and let $\|\underline{x}\|$ denote the corresponding norm of $\underline{x} \in X$.

Definition 3.2a A subset \mathcal{S} of the set of real numbers is called relatively dense if there exists a number $\ell > 0$ (inclusion length), such that any interval of length ℓ contains at least one number of \mathcal{S} Besicovitch (8) .

⁴In certain cases it may be advantageous to replace condition (3.1) by

$$\|\mathcal{H}[\underline{e}(t+T)] - \mathcal{H}[\underline{e}(t)]\| \leq \epsilon_e, \quad t \in (0, a],$$

where \mathcal{H} is an appropriate linear functional.

⁵This is a complete, normed vector space (129), e.g., L_p -spaces (see footnote no. 3, p. 56).

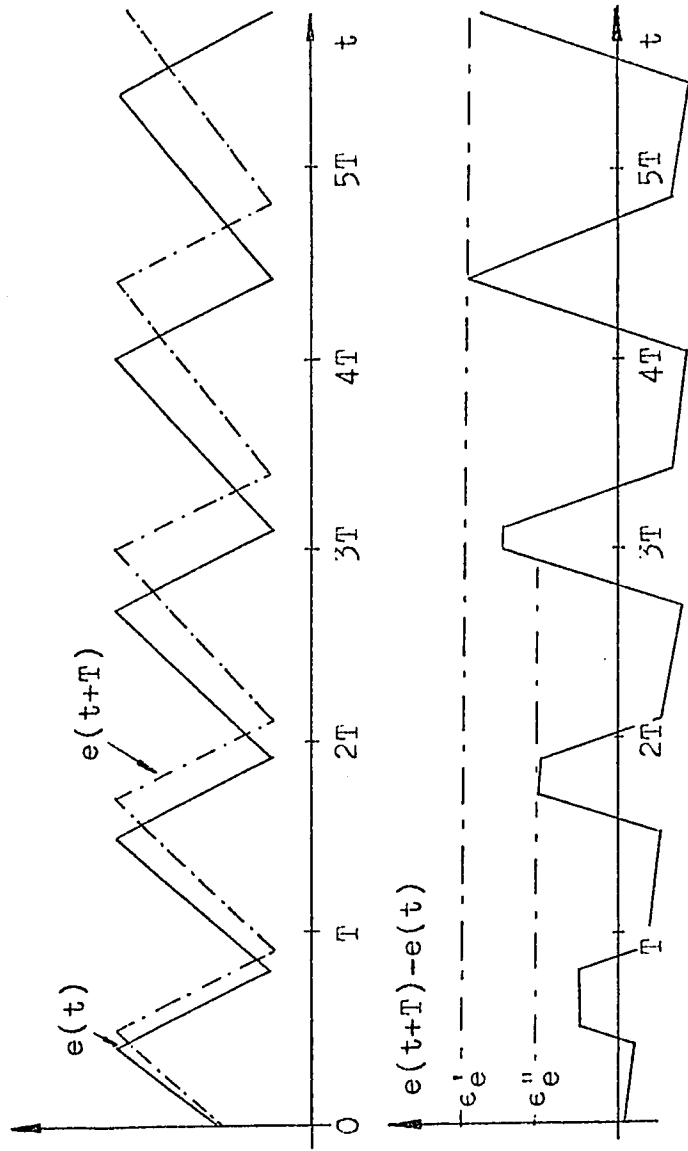


Figure 3.2 Illustration of Def. 3.1. The function $e(t)$ is ϵ' -nearly periodic in the time-interval $t \in (0, 2T)$ and ϵ'' -nearly periodic in the time-interval $t \in (0, \infty)$.

Definition 3.2b A continuous vector function $\underline{f}(t) \in X$ is called almost periodic⁶ if to every $\epsilon > 0$ there corresponds a relatively dense set $\{T\}_\epsilon$ such that

$$\|\underline{f}(t+T) - \underline{f}(t)\| \leq \epsilon, \quad \forall T \in \{T\}_\epsilon.$$

The class of almost periodic functions contains all functions $\underline{f}(t)$ constructed by summing a finite number of terms of the form $\underline{a}_i \cos(\omega_i t + \theta_i)$, where ω_i and θ_i are constants and \underline{a}_i is a constant vector. It can be shown that almost periodicity is invariant with respect to operations of addition, multiplication, (in most cases) division and differentiation, integration and other limiting processes and that to any almost periodic function corresponds a "Fourier series" type of general trigonometric series (3), (8).

Among the differences between ϵ_e -near periodicity and almost periodicity are the facts that the latter concept requires continuity⁷ and an infinite observation

⁶See Besicovitch (8), Amerio and Prouse (3) and (for a survey of other equivalent definitions), Fink (39).

⁷The conditions of Def. 3.2b can be relaxed to a certain degree by associating a linear functional \mathcal{H} with the function $\underline{f}(t)$ and requiring $\mathcal{H}[\underline{f}(t)]$ to be almost periodic for all linear functionals of a dual space X^* (this is known as weak almost periodicity) (3).

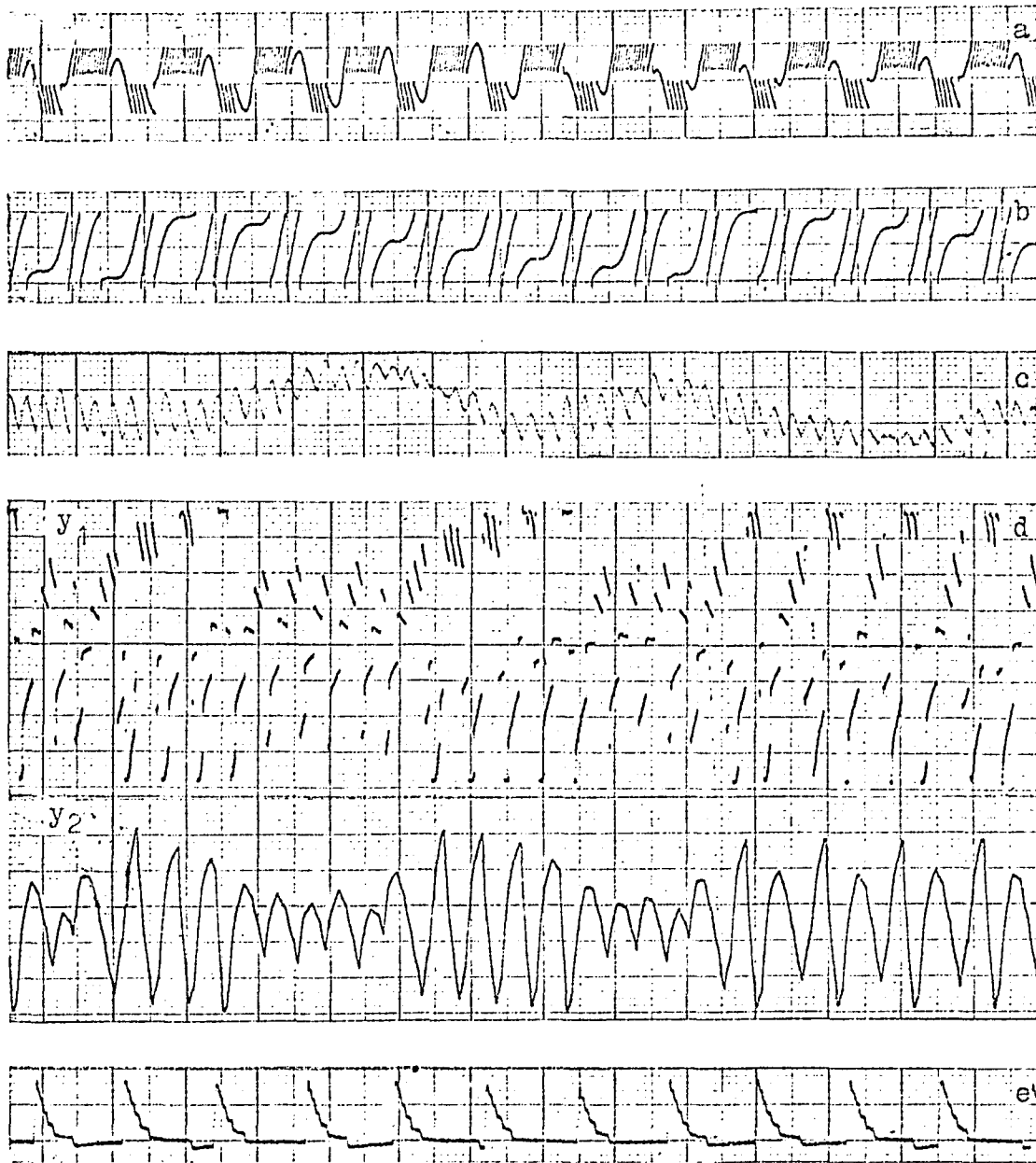


Figure 3.3 Examples of $\epsilon\epsilon$ -nearly periodic motion in PFM systems: (a), (b) output waveform of the TF in an IPFM under sinusoidal input; (a) double-signed IPFM, (b) single-signed IPFM (note that these waveforms are not periodic in the strict sense). (c) A typical $\epsilon\epsilon$ -nearly periodic motion in a PFM system with an almost periodic linear part. (d), (e) Typical $\epsilon\epsilon$ -nearly periodic motion in simple CRPFM systems.

interval. In a PFM system the vector $\underline{e}(t)$ may not be continuous. Moreover, it has been observed that motion which appears to be periodic over a reasonable time may change erratically after some time (see Section 3.5). However, in a practical situation, observation of the system may not be continued indefinitely. Furthermore, the measuring equipment used in the observation has some accuracy limitations, which must also be taken into consideration (this, in a loose way, corresponds to ϵ_e of Def. 3.1). Therefore, under proper conditions, one might conclude that a motion satisfying Def. 3.1 is "periodic".

It should be clear that Def. 3.1 makes sense only for "small" values of ϵ_e ; this value must be selected properly for the system under consideration, according to the accuracy requirements. For example, for certain cases the value $\epsilon_e \cong 0.01 \sup_{0 < t < \infty} \|\underline{e}(t)\|$ might be satisfactory.

3.3 Clues From System Stability

Knowledge about stability of the equilibrium can provide valuable clues to the study of oscillatory behavior. Therefore, before proceeding to the main result of this chapter (to be presented in the next section), certain relevant stability conditions and their implications with respect to periodic motion will be discussed.

Under certain conditions, the pulse frequencies of

the modulators keep on increasing. This motion, defined by continued increase in pulse frequency of any modulator, will be called uncontained motion. Conversely, the absence of uncontained motion will be called contained motion and the corresponding property, namely, that for every set of initial conditions the pulse frequencies will be bounded during $0 < t < \infty$ will be referred to as containment of the PFM system. In other words, containment means that for any given interval of length T , the number of impulses emitted by each modulator is uniformly bounded.

The containment of a PFM system can easily be tested: It can be shown that as the input signal to a CRPFM becomes very large, it can be replaced by a constant gain. Therefore, the test consists of replacing all the modulators with linear gains and determining whether the equivalent linear system is stable.

In order to prove the above assertion, consider the emission instant of the $(k+1)$ st pulse, t_{k+1} . From eqs. (1.5a), (1.5b), (1.7) and (1.8), it follows that

$$\int_{t_k}^{t_{k+1}} g_0(t_{k+1}, \tau) e(\tau) d\tau = b_{k+1} S \quad (3.2)$$

The mean value theorem of calculus gives

$$g_0(t_{k+1}, \xi_{k+1}) e(\xi_{k+1})(t_{k+1} - t_k) = b_{k+1} S \quad (3.3)$$

where $t_k \leq \xi_{k+1} \leq t_{k+1}$. Now, consider that the output of the CRPFM is connected to a linear element whose impulse response is $g(t, \tau)$; let $y(t)$ denote the output of the linear element. It is

$$\frac{y(t)}{e(t)} = \frac{M}{S} \sum_{k=1}^{\infty} g_0(t_k, \xi_k) g(t, t_k) \frac{e(\xi_k)}{e(t)} (t_k - t_{k-1}) \quad (3.4)$$

Since the input signal $e(t)$ is assumed to be (uniformly) large, the ratio $e(\tau)/e(t)$ is (uniformly) bounded for all $\tau < t$. Therefore, application of Duhamel's theorem to (3.4) yields

$$\frac{y(t)}{e(t)} \sim \frac{M}{Se(t)} \int_0^{\infty} g_0(\tau, \tau) g(t, \tau) e(\tau) d\tau, \quad (3.5)$$

i.e., if $g_0(\tau, \tau) = g_0$ (constant)⁸, the effect of the CRPFM, for large inputs, is equivalent to a linear gain of Mg_0/S ⁹.

In a real system, the pulse frequencies will be bounded due to saturation and/or presence of refractory period. This corresponds to saturation of the equivalent gains Mg_0/S .

⁸This is always true if the TF is time-invariant.

⁹For a single-signed CRPFM this approach gives a linear gain of Mg_0/S for nonnegative input signals. When the input is negative, no pulse is emitted and the gain switches to zero.

The above result leads to the following theorem.

Theorem 3.1 Consider the CRPFM system of Fig. 3.1; assume that the TF's are such that $g_{0i}(t,t) = g_{0i}$ (constant) $\forall t > 0$ and $i = 1, 2, \dots, m$. Then a necessary condition for the motion to be contained is that the equivalent system obtained by replacing all the modulators with linear gains of $M_i g_{0i} / S_i$ ($i = 1, \dots, m$) be asymptotically stable.

The containment from the equivalent linear system can be investigated using any of the conventional stability methods. A special system of interest is the case in which the LP is time-invariant and finite-dimensional; this case is treated in the following corollary.

Corollary 3.1 Consider the CRPFM system of Fig. 3.1; assume that all the TF's are time-invariant and that the LP is also time-invariant and is described by the equations

$$\dot{\underline{x}}(t) = \underline{A} \underline{x}(t) + \underline{B} \underline{u}(t) \text{ and } \underline{y}(t) = \underline{C} \underline{x}(t)$$

Let \underline{S} be the $m \times m$ diagonal matrix whose elements are

$$s_{ii} = M_i g_{0i}(0) / S_i, \quad (i = 1, \dots, m)$$

Then, a necessary condition for the motion to be contained is that all the eigenvalues of the matrix $\underline{A} + \underline{B} \underline{S} \underline{C}$ have negative real parts.

The uncontained motion does not exclude the possible existence of nearly periodic motion. In a PFM system ϵ_e -n.p. motion might be present, even though the conditions of Theorem 3.1 (or Corollary 3.1) are not satisfied; however, large perturbations will render the motion to "run away", i.e., to be uncontained. Therefore, knowledge of containment is useful.

Clearly, "containment" of motion in a PFM system represents a necessary condition for global finite-pulse stability (see Section 2.1, p. 56) which denotes the property that every set of initial conditions results in motion where each modulator emits a finite number of impulses during $0 < t < \infty$. If a PFM system is "contained" but not GFPS, then the motion will "keep on going"; this class of motion where at least one modulator does not stop firing as $t \rightarrow \infty$ will be called continued impulse emission. The class of "continued impulse emission" includes periodic motion and non-periodic motion. Moreover, the n.p. motions (Def. 3.1) of the usual interest are of the class of continued impulse emission.

Sufficient conditions for global finite-pulse stability in CRPFM systems were presented in Chapter 2. These conditions, in their negated forms, are also necessary conditions for the existence of continued impulse emission

in CRPFM systems; they are summarized in Theorem 3.2 and Corollary 3.2.

Theorem 3.2 Consider the CRPFM system of Fig. 3.1. If the following conditions are satisfied

- (1) $y_{0i}(t)$ and $r_i(t)$ are absolutely integrable in the interval $(0, \infty)$ ($i = 1, 2, \dots, m$),
- (2) there exist absolutely integrable functions $g'_{ij}(t)$ such that $|g_{ij}(t, \tau)| \leq |g'_{ij}(t-\tau)|$ ($i, j = 1, \dots, m$), and
- (3) $|g_{0i}(t, \tau)| \leq \delta_i$, where δ_i are finite constants ($i = 1, \dots, m$),

then, for the existence of continued impulse emission, it is necessary that the matrix

$$\left[\delta_{ij} - \frac{M_j}{S_i} \delta_i \int_0^{\infty} |g'_{ij}(t)| dt \right]^{-1}$$

contain at least one negative element¹⁰.

Corollary 3.2 If the conditions (1)-(3) of Theorem 3.2 are satisfied then, for the existence of continued impulse emission, for the CRPFM system of Fig. 3.1, it is necessary that

¹⁰ δ_{ij} is the Kronecker's delta; $\delta_{ij} = \begin{cases} 1 & \text{for } i = j, \\ 0 & \text{for } i \neq j. \end{cases}$

(1) at least one eigenvalue of the matrix

$$\left[\delta_i \frac{M_j}{S_i} \int_0^{\infty} |g'_{ij}(t)| dt \right]$$

be outside of the unit circle in the complex plane, and

(2) for at least one $i = 1, 2, \dots, m$,

$$\sum_{j=1}^m \delta_i \frac{M_j}{S_i} \int_0^{\infty} |g'_{ij}(t)| dt > 1 .$$

3.4 Nearly Periodic Motion in PFM Systems

In this section, two (upper) bounds will be presented for ϵ_e , such that for a given period T and a given observation interval $t \in (0, a]$, the motion is ϵ_e -n.p. (i.e., Def. 3.1 is satisfied). The first bound is applicable to more general cases; however, it can be difficult to obtain conservative values, since this usually requires numerical techniques. The second bound is especially useful if the impulse response on the LP is "almost periodic" (e.g., the LP has poles only on the imaginary axis) or the LP contains poles very close to the imaginary axis. Before presenting these results, certain relevant notation will be introduced.

Let $g_j(t, \tau)$ denote the j th column of the impulse

response matrix of the LP, $\underline{g}(t, \tau)$; ℓ_k denote the identification number of the modulator emitting the k th pulse of the system, t_k denote the emission time of this k th pulse and b_k denote its polarity. Then, the output vector, $\underline{y}(t)$, is given by

$$\underline{y}(t) = \underline{y}_0(t) + \sum_{k=1}^N M_{\ell_k} b_k \underline{g}_{\ell_k}(t, t_k), \quad 0 < t < t_{N+1} \quad (3.6)$$

This is an alternative to expression (1.24a).

Let $\bar{\underline{y}}_n(t)$ be the output vector of the PFM system obtained by disconnecting all the modulators for $t > nT$, and let

$$\underline{y}_n(t) \triangleq \begin{cases} \underline{y}(t) & \text{for } t \leq nT \\ \bar{\underline{y}}_n(t) & \text{for } t > nT \end{cases}, \quad n = 0, 1, 2, \dots \quad (3.7a)$$

Also, let

$$\underline{\zeta}_n(t) \triangleq \underline{y}_{n+1}(t) - \underline{y}_n(t) \quad (3.7b)$$

The vector function $\underline{\zeta}_n(t)$, to be called the modified forced response, represents the zero-initial condition response of the continuous part of the system to an input that is applied only during $nT \leq t < (n+1)T$ and is equal to that generated by the modulators during this interval when the modulators are connected (see Fig. 3.4).

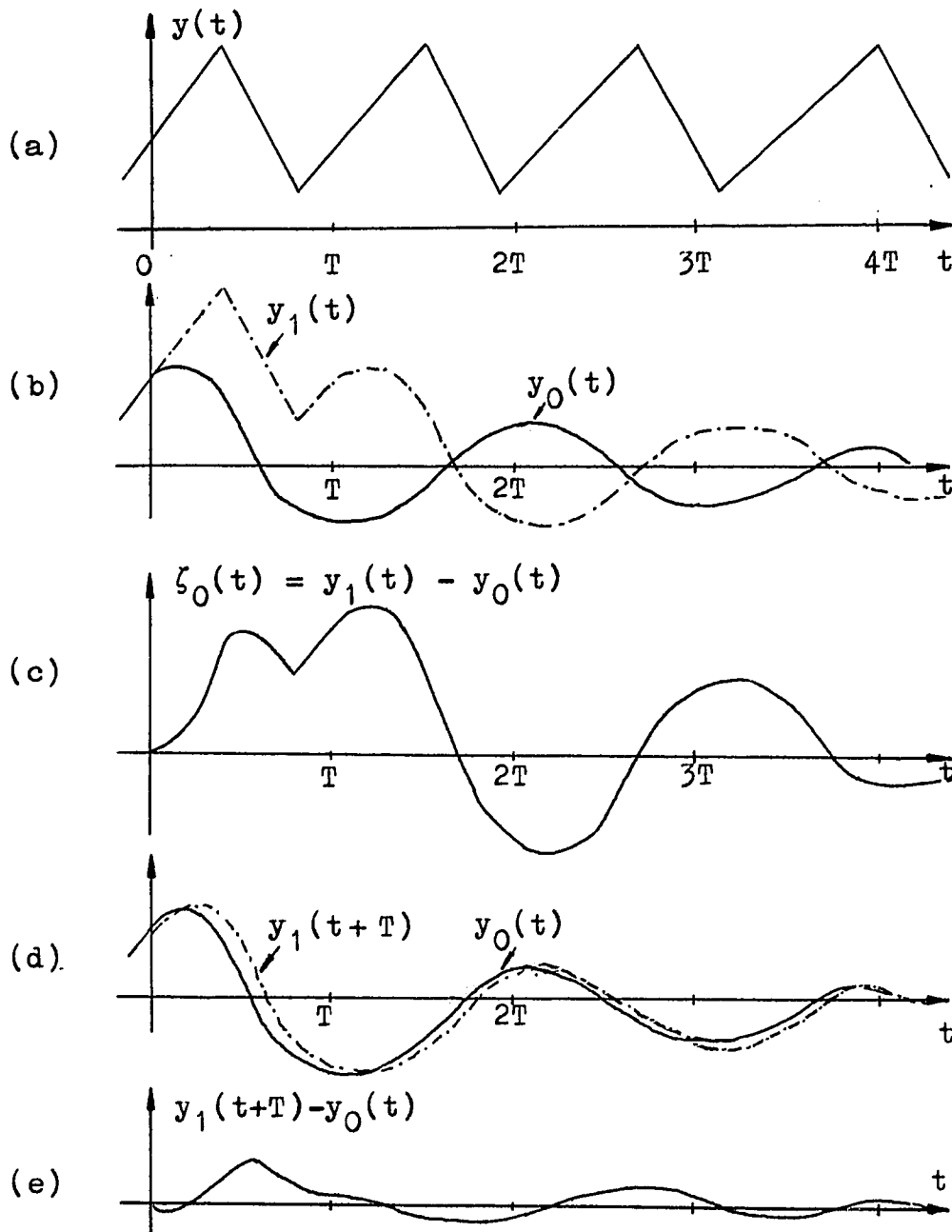


Figure 3.4 Illustration of functions used in Theorem 3.3.
 (a) Output variable $y(t)$ under ϵ -nearly periodic motion.
 (b) The functions $y_0(t)$ and $y_1(t)$ as defined by (3.7a).
 (c) The function $\zeta_0(t)$ as defined by (3.7b).
 (d) $y_0(t)$ and $y_1(t+T)$ shown for comparison.
 (e) The function $y_1(t+T) - y_0(t)$.

With the background presented above, the following theorem can now be stated.

Theorem 3.3 Consider the CRPFM system of Fig. 3.1. For a given $T > 0$ and a given $a > 0$, let

$$\epsilon_r = \sup_{0 < t \leq a} \|\underline{x}(t+T) - \underline{x}(t)\| \quad (3.8a)$$

and let

$$\epsilon_0 = \sup_{0 < t \leq T} \|\underline{y}_1(t+T) - \underline{y}_0(t)\| \quad (3.8b)$$

If there exists a $\sigma \geq 0$ and an

$$\epsilon_e \geq \epsilon_r + \epsilon_0 (1 + \sigma)^{\left[\frac{a}{T}\right] + 1} \quad (3.9a)^{11}$$

such that for every two initial condition responses, $\underline{y}_0^1(t)$ and $\underline{y}_0^2(t)$, satisfying

$$\rho = \sup_{0 < t \leq T} \|\underline{y}_0^1(t) - \underline{y}_0^2(t)\| \leq \epsilon_e \quad (3.9b)$$

the corresponding modified forced responses satisfy

$$\sup_{0 < t \leq T} \|\underline{z}_0^1(t) - \underline{z}_0^2(t)\| \leq \rho\sigma \quad (3.9c)$$

then the motion of the system is ϵ_e -n.p. with the ϵ_e -period T , in the given observation interval $t \in (0, a]$.

The proof of Theorem 3.3 is presented in Appendix B.

¹¹The square brackets $[\cdot]$ denotes integer part.

The following corollary is a simple extension of Theorem 3.3.

Corollary 3.3 Consider the CRPFM system of Fig. 3.1. For a given $T > 0$ and a given $a > 0$, if there exists a $\sigma' \geq 0$ and an

$$\epsilon_e \geq \epsilon_r + \epsilon_0 (\sigma')^{\left[\frac{a}{T}\right] + 1} \quad (3.9a')$$

such that for every two initial condition responses, $y_0^1(t)$ and $y_0^2(t)$, satisfying (3.9b), the corresponding modified responses satisfy

$$\sup_{0 < t \leq T} \| y_0^1(t) + \underline{x}_0^1(t) - y_0^2(t) - \underline{x}_0^2(t) \| \leq \rho \sigma' \quad (3.9c')$$

then the motion of the system is ϵ_e -n.p. with the ϵ_e -period T , in the given observation interval $t \in (0, a]$, where ϵ_r and ϵ_0 are given by (3.8a) and (3.8b), respectively.

The upper bound for ϵ_e provided by Theorem 3.3 (or Corollary 3.3) might be large, depending on the value of σ (or σ') which satisfies conditions (3.9a)-(3.9c) [or (3.9a'), (3.9b) and (3.9c')]. If the value of σ (or σ') is not much greater than the minimum σ (σ_{\min}) satisfying conditions (3.9), this might mean that the motion will rapidly degenerate and after a certain time will have a completely different pattern. However, if $\sigma \gg \sigma_{\min}$, Theorem 3.3 does not provide any useful infor-

mation, since the bound furnished by Theorem 3.3. is much larger than $\sup_{0 < t \leq a} \|\underline{e}(t+T) - \underline{e}(t)\|$. Therefore, σ must be carefully determined. In general, without resort to numerical techniques, it may be difficult to obtain conservative values.

Under certain ideal conditions a PFM system might possess a pure periodic motion. However, in Section 1.5 it will be shown that even slightest parameter perturbations can change this motion drastically. Nevertheless, after small parameter changes, the motion may still look like the unperturbed motion, at least for a while, i.e., the motion may be ϵ_e -n.p. In this case, Theorem 3.3 can be used to estimate, for example, tolerances of the system parameters to assure an ϵ_e -n.p. motion of a given accuracy in some given interval.

Theorem 3.3 and the notation introduced in this section will be illustrated by an example. However, first the following useful concept is presented.

Definition 3.3 Given $\epsilon_0 \geq 0$ and $a > 0$, an initial condition response, $\underline{y}_0(t)$ of a PFM system will be called an ϵ_0 -proper initial condition response (ϵ_0 -PICR) in the interval $t \in (0, a]$, if there exists a number, $T > 0$ such that the relation

$$\|\underline{y}_1(t+T) - \underline{y}_0(t)\| \leq \epsilon_0, \quad \forall t \in (0, a] \quad (3.10)$$

is satisfied.

Example 3.1 Consider a two modulator IPFM system with a time-invariant LP having the following impulse response matrix

$$\underline{G}(t) = \begin{bmatrix} t & -t \\ t & -t \end{bmatrix}, \quad t > 0 \quad (3.11a)$$

Let $\underline{r}(t) = \underline{0}$, $S_1 = S_2 = 4$ and $M_1 = M_2 = 4$. Also, let the initial condition response be

$$\underline{y}_0(t) = \begin{bmatrix} 2t+1 \\ 2t+1 \end{bmatrix} \quad (3.11b)$$

With this initial condition response, the motion of the system is as shown in Fig. 3.5. Note that the motion is

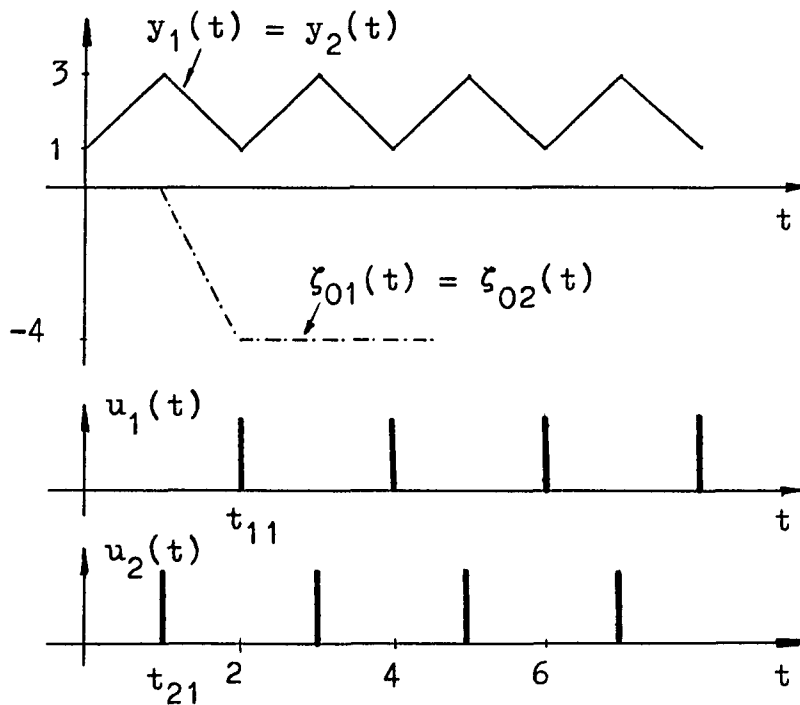


Figure 3.5 Output waveforms of the system of Example 3.1.

periodic with a period $T = 2$ secs and that $y_0(t)$ is a 0-PICR (i.e., $\epsilon_0 = 0$). For this systems, the modified forced response is given by

$$\underline{\xi}_0(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \xi_0(t) \quad (3.11c)$$

where,

$$\xi_0(t) \triangleq \begin{cases} 0 & \text{for } t < 0 \\ -4t+4 & \text{for } 1 \leq t \leq 2 \\ -4 & \text{for } t > 2 \end{cases} \quad (3.11d)$$

Now, assume that the thresholds of the modulators S_1 and S_2 are perturbed by infinitesimal amounts δS_1 and δS_2 , respectively. Let δt_{11} and δt_{21} be the corresponding infinitesimal changes in the pulse-emission instants. These quantities can be calculated by considering the threshold relations.

Let

$$y_0(t) = (c_0 t + c_1) \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad (3.12)$$

The threshold relation for the first modulator is

$$\int_0^{t_{11}} (c_0 t + c_1) dt - 4 \int_{t_{21}}^{t_{11}} (t - t_{21}) dt = S_1 \quad (3.13a)$$

The threshold relation for the second modulator is

$$z_2(0) + \int_0^{t_{21}} (c_0 t + c_1) dt = S_2 \quad (3.13b)$$

From (3.13a) one can obtain

$$2\delta c_0 + 2\delta c_1 + \delta t_{11} + 4\delta t_{21} = \delta S_1 \quad (3.14a)$$

Similarly, (3.13b) yields

$$3\delta t_{21} + \frac{1}{2}\delta c_0 + \delta c_1 = \delta S_2 \quad (3.14b)$$

Assuming $\delta c_0 = \delta c_1 = 0$ (i.e., no perturbation in the initial condition response), (3.14a) and (3.14b) can be combined to obtain

$$\delta t_{11} = \delta S_1 - \frac{4}{3} \delta S_2 \quad (3.15a)$$

and

$$\delta t_{21} = \frac{1}{3} \delta S_2 \quad (3.15b)$$

Using (3.7b), (3.8b) and (3.11b)-(3.11d) and the norm

$$\left\| \begin{bmatrix} a \\ b \end{bmatrix} \right\| = |a| + |b|, \text{ one can now obtain}$$

$$\begin{aligned} \epsilon_0 &= \sup_{0 < t \leq 2} \|y_1(t+T) - y_0(t)\| = 8|\delta t_{21} - \delta t_{11}| \\ &= 8 \left| \frac{5}{3} \delta S_2 - \delta S_1 \right| \end{aligned} \quad (3.16)$$

As an example, let $\delta S_1 = \delta S_2 = 0.0001$; (3.16) then yields

$$\epsilon_0 = 5.3 \cdot 10^{-4}$$

Now, consider two different initial condition response vectors $\underline{y}_0^1(t) = (c_0^1 t + c_1^1) \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\underline{y}_0^2(t) = (c_0^2 t + c_1^2) \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ that are in the vicinity of the 0-PICR given by (3.11b). Let $c_0^1 - c_0^2 \triangleq \delta c_0$ and $c_1^1 - c_1^2 \triangleq \delta c_1$, the resulting infinitesimal differences in the pulse emission instants can be obtained from (3.14a) and (3.14b). Since the threshold values are assumed to remain unchanged (i.e., $\delta S_1 = \delta S_2$), (3.14b) yields

$$\delta t_{21} = - \left(\frac{1}{2} \delta c_0 + \delta c_1 \right) \quad (3.17a)$$

Similarly, (3.14a) and (3.15a) yield

$$\delta t_{11} = - \frac{4}{3} \delta c_0 - \frac{2}{3} \delta c_1 \quad (3.17b)$$

Using (3.9c), (3.11c)-(3.11d), (3.17a) and (3.17b), the following relation can be obtained

$$\begin{aligned} \sup_{0 < t \leq 2} \|\underline{z}_0^1(t) - \underline{z}_0^2(t)\| &= 8 \left\{ \max \left| \delta t_{21} \right|, \left| \delta t_{11} - \delta t_{21} \right| \right\} \\ &= \frac{4}{3} \max \left\{ \left| 7\delta c_0 + 2\delta c_1 \right|, \left| \delta c_0 + 2\delta c_1 \right| \right\} \end{aligned} \quad (3.18a)$$

But, from (3.9b) and (3.12),

$$\rho = \sup_{0 < t \leq 2} \|\underline{y}_0^1(t) - \underline{y}_0^2(t)\| = 4 \left| \delta c_0 + \delta c_1 \right| \quad (3.19)$$

From (3.18) and (3.16), it is not difficult to see that condition (3.9c) of Theorem 3.3 will be satisfied for

$\sigma \geq \frac{7}{3}$. Therefore, for an observation interval of ten periods, (3.9a) yields

$$\epsilon_e \leq \left(1 + \frac{7}{3}\right)^{10} = 2.03 \times 10^5 \epsilon_0 \quad (3.20)$$

Now, assume that an ϵ_e -n.p. motion is required such that $\epsilon_e \leq 0.1$ for an observation interval of ten periods. From (3.20), it is seen that this requirement will be fulfilled if $\epsilon_0 \leq 4.9 \times 10^{-5}$. From (3.16), it follows that if $|\delta S_1| < 0.4 \times 10^{-4}$ and $|\delta S_2| < 0.6 \times 10^{-4}$, then $\epsilon_e \leq 0.1$.

For comparison purposes the system considered was also studied by simulation. A digital simulation of the system yielded $\epsilon_e = 0.16$ in an observation interval of 20 secs, after a small perturbation in the threshold values (S_1 and S_2 both changed from 4.000 to 4.001). This result is much smaller than the bound given by (3.20). However, (3.20) is applicable to a larger parameter region. For this particular case, one can calculate $\sigma' = 1.8$ and $\epsilon_0 = 4 \times 10^{-4}$. Therefore, (3.9a') yields $\epsilon_e \leq 0.22$. This bound has the same order of magnitude as the actual value.

For the special case of a CRPFM system with a LP whose impulse response is almost periodic or contains terms of the form $\underline{f}_i(t) = e^{-a_i t} \underline{f}'_i(t)$ ($i = 1, 2, \dots, q$), where a_i 's are small positive constants and $\underline{f}'_i(t)$'s are

periodic vector functions of t , it is possible to find another useful upper bound for ϵ_e . Before determining this bound, first note that if

$$\|\underline{n}(t)\| \triangleq \|\underline{f}'_i(t+T) - \underline{f}'_i(t)\| \leq \epsilon_{f_i}$$

and $a_i T \ll 1$, then

$$\begin{aligned} \|\underline{f}_i(t+T) - \underline{f}_i(t)\| &= \|e^{-a_i(t+T)} \underline{f}'_i(t+T) - e^{-a_i t} \underline{f}'_i(t)\| \\ &\leq \|(1 - e^{-a_i T}) \underline{f}'_i(t) + e^{-a_i T} \underline{n}(t)\| \\ &< a_i T \|\underline{f}'_i(t)\| + \epsilon_{f_i}. \end{aligned} \quad (3.21)$$

Now, consider the input vector to the modulator block; using (3.6) and applying the triangle inequality, the following inequality can be obtained:

$$\begin{aligned} \|\underline{e}(t+T) - \underline{e}(t)\| &\leq \|\underline{x}(t+T) - \underline{x}(t)\| + \|\underline{y}_0(t+T) - \underline{y}_0(t)\| \\ &\quad + \sum_{j=1}^N M_{\ell_j} \|\underline{g}_{\ell_j}(t+T, t_j) - \underline{g}_{\ell_j}(t, t_j)\| \end{aligned} \quad (3.22)$$

$0 < t \leq a$

Let

$$\epsilon_r = \sup_{0 < t \leq a} \|\underline{x}(t+T) - \underline{x}(t)\|, \quad (3.23a)$$

$$\epsilon_{y_0} = \sup_{0 < t \leq a} \|\underline{y}_0(t+T) - \underline{y}_0(t)\| \quad (3.23b)$$

and let

$$\epsilon_g = \sup_{\substack{0 < t < a \\ 0 < \tau < a \\ i=1, \dots, m}} \|M_i \underline{g}_i(t+T, \tau) - \underline{g}_i(t, \tau)\| \quad (3.23c)$$

From inequality (3.30), it follows that

$$\|\underline{e}(t+T) - \underline{e}(t)\| \leq \epsilon_r + \epsilon_{y_0} + N\epsilon_g \quad (3.24)$$

i.e., for $\epsilon_e = \epsilon_r + \epsilon_{y_0} + N\epsilon_g$, the motion is ϵ_e -n.p. in the interval $t \in (0, a]$. This result is stated as a lemma.

Lemma 3.1 Let ν be the average number of pulses emitted in an interval of length T , then the motion of the CRPFM system of Fig. 3.1 is ϵ_e -n.p. in the interval $t \in (0, a]$, where $\epsilon_e \geq \epsilon_r + \epsilon_{y_0} + \nu([\frac{a}{T}] + 1)\epsilon_g$ with ϵ_r , ϵ_{y_0} and ϵ_g as defined in (3.23a) (3.23b) and (3.23c), respectively.

Lemma 3.1 is directly applicable to PFM systems with almost periodic LP's and inputs, where in a given, large-enough (finite) observation interval $(0, a]$, it is possible to find a T such that the values ϵ_r , ϵ_{y_0} and ϵ_g are arbitrarily small¹². This consideration yields the following Corollary.

Corollary 3.4 If both the input vector, $\underline{r}(t)$, and the impulse response matrix of the LP, $\underline{g}(t, \tau)$ are almost periodic¹³, then for a large enough (finite) observation

¹²See Besicovitch, Theorem 11, p. 5.

interval $(0, a]$ and for any ϵ_e , the motion of the CRPFM system of Fig. 3.1 is ϵ_e -n.p.

The following example illustrates the above corollary.

Example 3.2 Consider a two-modulator CRPFM system with a constant input and a time-invariant LP, having the following impulse-response matrix

$$\underline{G}(t) = \begin{bmatrix} \cos \omega t & -\frac{1}{\omega} \sin \omega t \\ \sin \omega t & \cos \omega t \end{bmatrix} \quad (3.25)$$

Let $M_1 = M_2 = 1$. Note that, in this case, $\epsilon_r = 0$ and $\underline{G}(t+T) = \underline{G}(t)$, where $T = 2\pi/\omega$. Clearly, if $\epsilon_{y_0} = 0$, then the motion will be ϵ_e -n.p. for any $\epsilon_e > 0$ with period $t = 2\pi/\omega$.

Example 3.3 In order to demonstrate the applicability of Lemma 3.1 to PFM systems where the impulse response of the LP contains lightly damped terms, consider again the system treated in Example 3.2. Assume, however, that the impulse response matrix is given by

$$\underline{G}(t) = e^{-bt} \begin{bmatrix} \cos \omega t - \frac{b}{\omega} \sin \omega t & -\frac{1}{\omega} \sin \omega t \\ \frac{\omega^2 + b^2}{\omega} \sin \omega t & \cos \omega t + \frac{b}{\omega} \sin \omega t \end{bmatrix} \quad (3.26)$$

¹³i.e., all the elements of the matrix $\underline{G}(t, \tau)$ are almost periodic.

where b is a positive scalar such that $b \ll \omega/2\pi$. Note that for $b = 0$, this impulse response matrix reduces to that of (3.25).

Eq. (3.26) yields

$$\mathbf{g}_1(t) = e^{-bt} \begin{bmatrix} \cos \omega t - \frac{b}{\omega} \sin \omega t \\ \frac{\omega^2 + b^2}{\omega} \sin \omega t \end{bmatrix} \quad \text{and} \quad \mathbf{g}_2(t) = e^{-bt} \begin{bmatrix} -\frac{1}{\omega} \sin \omega t \\ \cos \omega t + \frac{b}{\omega} \sin \omega t \end{bmatrix}$$

Therefore, using (3.21) and the norm $\| [x_1, x_2]^T \| = [x_1^2 + x_2^2]^{\frac{1}{2}}$, one can easily obtain

$$\| \mathbf{g}_1(t+T) - \mathbf{g}_1(t) \| \leq bT \left(1 + \frac{b^2}{\omega^2}\right)^{\frac{1}{2}} + \frac{\omega^2 + b^2}{\omega}$$

and

$$\| \mathbf{g}_2(t+T) - \mathbf{g}_2(t) \| \leq bT \frac{1}{\omega} + \left(1 + \frac{b^2}{\omega^2}\right)^{\frac{1}{2}}.$$

Thus,

$$\epsilon_g \leq bT \frac{\omega^2 + b^2}{\omega} + \left(1 + \frac{b^2}{\omega^2}\right)^{\frac{1}{2}} \cong b(2\pi + T)$$

Application of Lemma 3.1 finally yields

$$\epsilon_e \leq \nu \left(\left\lceil \frac{\epsilon_g}{T} \right\rceil + 1 \right) \epsilon_g \cong \nu b(2\pi + T) \left(\left\lceil \frac{\epsilon_g}{T} \right\rceil + 1 \right).$$

So far, conditions for the existence of ϵ_e -n.p. motion were considered and upper bounds for ϵ_e were determined. It is also important to obtain an expression for the ϵ_e -period of the motion. However, this is analytically a very difficult task and will be carried out, in the next section, only for the IPFM system.

3.5 Nearly Periodic Motion in IPFM Systems: The

ϵ_e -Period

In this section a special CRPFM system, namely, an IPFM system is considered. The basic configuration of this system is as shown in Fig. 3.1; however, all the modulators are assumed to be integral type PFM's only. The following theorem gives a matrix relationship which relates the ϵ_e -period and the net number of pulses emitted by each modulator over that period to the system parameters.

Theorem 3.4 Consider an IPFM system with a time-invariant linear part. Assume that the conditions of Theorem 3.3 are satisfied. Furthermore, assume also that there exist positive constants B_0 , B_g , a_0 and a_g , such that

$$\|M_{i1}g_i(t)\| \leq B_g e^{-a_g t}, \quad t > 0, \quad i = 1, \dots, m \quad (3.27a)$$

and

$$\|y_0(t)\| \leq B_0 e^{-a_0 t}, \quad t > 0. \quad (3.27b)$$

Then, the ϵ_e -period of the motion satisfies the matrix relation

$$\underline{P} \underline{q} = T \underline{r}_0 + \underline{\psi} \quad (3.28)$$

where, \underline{P} is an $m \times m$ matrix whose elements p_{ij} are defined by

$$p_{ij} = h_{ij}(0) + \begin{cases} S_i & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases}, \quad (3.29a)^{14}$$

$$h_{ij}(t) = \int M_j g_{ij}(t) dt \quad (\text{indefinite integral}) \quad (3.29b)$$

\underline{r}_0 and \underline{q} are m -dimensional column vectors, whose elements are defined by

$$q_i = \begin{array}{l} \text{(the number of positive pulses)} \\ - \text{(the number of negative pulses)} \\ \text{emitted by the } i\text{th modulator in} \\ \text{the interval } (0, T]. \end{array} \quad (3.29c)$$

and

$$r_{0i} = \frac{1}{T} \int_0^T r_i(t) dt. \quad (3.29d)$$

$\underline{\psi}$ is an m -dimensional column vector which depends on the deviation of ϵ_e -n.p. motion from pure periodic motion such that

$$\|\underline{\psi}\| \leq \left[\epsilon_0 \frac{(1+\sigma)^N}{\sigma} + N\epsilon_r \right]^T + \frac{B_0}{a_0} e^{-a_0 NT} + \frac{B_g}{a_g} e^{-a_g NT}, \quad (3.29e)$$

$$N = \left[\frac{a}{T} \right] + 1 \quad (3.29f)$$

and

$$\epsilon_e \leq \epsilon_r + \epsilon_0 (1+\sigma)^N \quad (3.29g)$$

¹⁴If the linear part of the system is described by the matrix equations $\dot{\underline{x}} = \underline{A} \underline{x} + \underline{B} \underline{u}$, $\underline{y} = \underline{C} \underline{x}$, then

$$\underline{H}(0) = \underline{C} \underline{A}^{-1} \underline{B}.$$

The proof of Theorem 3.4 is given in Appendix C.

For a single-loop, single-modulator IPFM system and for $\epsilon_e \rightarrow 0$, Theorem 3.4 yields the relation

$$T = \left| \frac{g}{r_0} [S + h(0)] \right| \quad (e_e = 0) \quad (3.30)$$

This result was previously obtained by Meyer (93) and has later been verified by King-Smith and Cumpston (71) and Varadarajan and Pai (127).

The following example illustrates utility of Theorem 3.3.

Example 3.4 A multiple output pulse generator is to be designed to provide the periodic waveforms shown in Fig. 3.6.

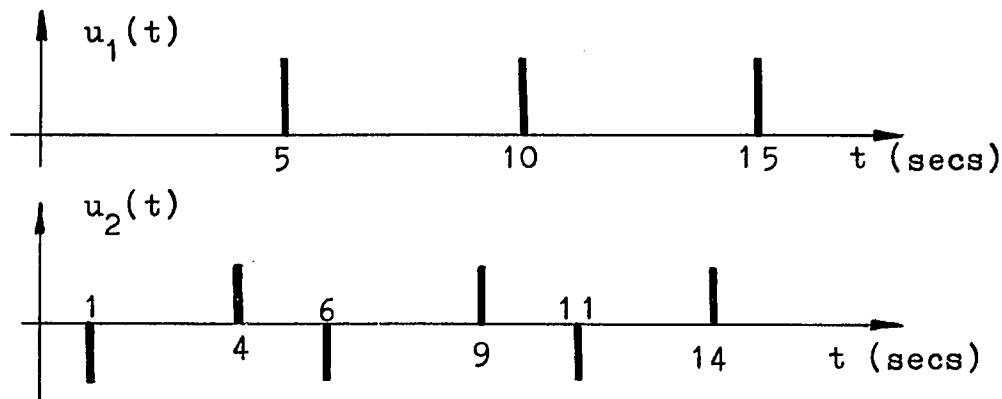


Figure 3.6 Desired pulse pattern of the pulse generator.

An IPFM system containing two IPFM's and a second

order time-invariant linear part with a constant input is a good candidate for the job. In order to facilitate the design, some of the parameters of the system can be chosen arbitrarily: Let $S_1 = 1$, $S_2 = 1$, $M_1 = 1$ and $M_2 = 1$, and let the LP be described by the equations

$$\dot{\underline{x}}(t) = \underline{A} \underline{x}(t) + \underline{B} \underline{u}(t)$$

$$\underline{y}(t) = \underline{C} \underline{x}(t)$$

where,

$$\underline{A} = \begin{bmatrix} -a_1 & 0 \\ 0 & -a_2 \end{bmatrix}, \quad \underline{B} = \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix}, \quad \text{and} \quad \underline{C} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

For containment of the system, the matrix $\underline{A} + \underline{B}\underline{S}\underline{C}$ must possess no eigenvalues in the r.h.p. (Corollary 3.1). This condition is easily satisfied if $a_1 > -1$ and $a_2 > -1$.

From (3.29b) one can easily obtain (see footnote no. 14)

$$\underline{H}(0) = \underline{C} \underline{A}^{-1} \underline{B} = \begin{bmatrix} 1/a_1 & 0 \\ b/a_2 & 1/a_2 \end{bmatrix} \quad (3.31a)$$

Substitution of (3.31) into (3.29a) yields

$$\underline{P} = \underline{I} + \underline{H}(0) = \begin{bmatrix} 1/a_1 + 1 & 0 \\ b/a_2 & 1/a_2 + 1 \end{bmatrix} \quad (3.31b)$$

From (3.29e), it follows that as $\epsilon_e \rightarrow 0$, $\underline{P} \rightarrow 0$. There-

fore, for $\epsilon_e = 0$, (3.28) and (3.31b) give

$$\underline{q} = \underline{P}^{-1} \underline{r}_0 T = \begin{bmatrix} \frac{a_1 r_{01}}{a_1 + 1} \\ 1 \\ \frac{1}{a_2 + 1} \left(\frac{-b r_{01} a_1}{a_1 + 1} + a_2 r_{02} \right) \end{bmatrix} T \quad (3.32)$$

Since the period is $T = 5$ secs, and $\underline{q} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, (3.32) yields

$$q_1 = 1 = \frac{5r_{01}}{a_1 + 1} a_1 \quad (3.33a)$$

and,

$$q_2 = 0 = \frac{1}{a_2 + 1} \left(-b \frac{5r_{01} a_1}{a_1 + 1} + 5a_2 r_{02} \right) \quad (3.33b)$$

(3.33a) and (3.33b) can be combined to obtain

$$b = 5 a_2 r_{02} \quad (3.34)$$

The initial condition response vector, $\underline{y}_0(t)$ is given by

$$\underline{y}_0(t) = \underline{C} e^{\underline{A}t} \underline{x}(0) = \begin{bmatrix} x_{01} e^{-a_1 t} \\ x_{02} e^{-a_2 t} \end{bmatrix} \quad (3.35)$$

The output vector is

$$\underline{y}_1(t) = \underline{y}_0(t) + \begin{bmatrix} -e^{-a_1(t-5)} \\ -e^{-a_2(t-1)} - e^{-a_2(t-4)} - b e^{-a_2(t-5)} \end{bmatrix} \quad (3.36)$$

The proper initial condition of the system can be obtained from (3.10), for $\epsilon_0 = 0$, giving

$$x_{01} e^{-5a_1} - 1 + x_{01} = 0 \quad (3.37)$$

and

$$-x_{02} e^{-5a_2} + e^{-4a_2} - e^{-a_2} - b + x_{02} = 0 \quad (3.38)$$

(3.37) yields

$$x_{01} = \frac{1}{1 - e^{-5a_1}} \quad (3.39)$$

The threshold relation for the first modulator is

$$\int_0^5 [r_{01} - x_{01} e^{-a_1 t}] dt = 1 \quad (3.40)$$

Similarly, the threshold relations for the second modulator are

$$\int_0^4 [r_{02} - x_{02} e^{-a_2 t} + e^{-a_2(t-1)}] dt = 1 \quad (3.41)$$

and

$$\int_0^6 [r_{02} - x_{02} e^{-a_2 t} + e^{-a_2(t-1)} - e^{-a_2(t-4)}] dt - b \int_0^6 e^{-a_2(t-5)} dt = -1 \quad (3.42)$$

Substitution of (3.39) into (3.40) yields

$$5 r_{01} = 1 + \frac{1}{a_1}$$

Thus, one can select $r_{01} = 1$ and $a_1 = \frac{1}{4}$.

(3.34) and (3.38) yield

$$x_{02} = \frac{5a_2 r_{02} + e^{-a_2} - e^{-4a_2}}{1 - e^{-5a_2}} \quad (3.43)$$

Elimination of r_{02} from (3.41) and (3.42) and substitution of (3.34) and (3.43) into the resulting equation yields an equation containing only one unknown, a_2 . This equation can be solved for a_2 , yielding

$$a_2 = 0.5748298$$

Hence,

$$r_{02} = -0.06405757$$

From (3.34)

$$b = -0.184108$$

and from (3.43)

$$x_{02} = 0.2950206$$

At $t = 0^+$ the integrator of the first modulator is reset to zero. However, the output of the second integrator is

$$z_2(0) = -1 - \int_0^1 (r_{02} - x_{02} e^{-a_2 t}) dt = -0.71155822.$$

Now, assume that the matrix \underline{P}^{-1} exists and let

$$\underline{p} = \underline{P}^{-1} \underline{x}_0 \quad (3.44)$$

Furthermore, assume also that the components of the vector \underline{p} are rational numbers, i.e., $|p_i| = N_i/D_i$, where N_i and D_i are integers ($i = 1, \dots, m$) and that the vector \underline{q} has no zero component. Then, for $\epsilon_g \rightarrow 0$, the period T is an integer multiple of the number

$$T_0 = \frac{\text{LCM}(D_1, D_2, \dots, D_m)}{\text{GCD}(N_1, N_2, \dots, N_m)} \quad (3.45)^{15}$$

The number T_0 will be called the the elementary period. Unlike linear systems, the period of oscillations in PFM systems (or, nonlinear systems, in general) could also depend on the initial conditions. If $\underline{x}_0 \neq \underline{0}$, it is seen that, for this special case the possible periods of oscillation (under different initial conditions) are quantized, such that they are multiples of the elementary period, T_0 . If $\underline{x}_0 = \underline{0}$, then the number of positive pulses will be equal to the number of negative pulses emitted by each modulator.

¹⁵LCM(.., .., ..) and GCD(.., .., ..) stand for least common multiple and greatest common divisor, respectively, e.g.,
 $\text{LCM}(3, 6, 15) = \text{LCM}(3, 2 \cdot 3, 3 \cdot 5) = 2 \cdot 3 \cdot 5 = 30,$
 $\text{GCD}(3, 6, 15) = 3.$

From (3.45) it is not difficult to see that any slight perturbation in either the system parameters or the input will yield a completely different period of motion, provided p_i ($i = 1, \dots, m$) remain rational after the perturbation. If any component of the vector \underline{p} becomes irrational, then $T \rightarrow \infty$. This point is illustrated in the following example.

Example 3.5 Consider again the system treated in Example 3.4. Assume that the parameters a_2 , b and the input $r_2(t)$ are perturbed slightly (from the values calculated in Example 3.4), such that $a_2 = 0.575$, $r_2(t) = -0.064$ and $b = -0.1841$. Substitution of these values into (3.45) gives $p_1 = N_1/D_1 = 1/5$, and $p_2 = N_2/D_2 = \frac{1}{78750}$. Therefore, for $\epsilon_e \rightarrow 0$, (3.42) yields

$$T_0 = \frac{\text{LCM}(5, 78750)}{\text{GCD}(1, 1)} = 78750 .$$

For very small perturbations, it is reasonable to assume that there will not be a noticeable change in the motion. What explanation can be given to this "discrepancy"?

The answer lies in ϵ_e ; it can be related to measurement error and has a small but nonzero value. The vector $\underline{\psi}$ in (3.28) is an arbitrary vector. It can be selected such that condition (3.29e) is satisfied. In this case, (3.28) yields a number T_0 (the elementary period)

such that the ϵ_e -period T is an integer multiple of T_0 . For small perturbations, since $\underline{\psi}$ is arbitrary, this elementary period will be independent of the parameters of the system or the inputs.

Note that the term $\underline{\psi}$ in (3.28) must be such that it can neutralize the effect of parameter perturbations. In order to elucidate this consider again the system of Example 3.4.

Example 3.6 Let $\underline{\psi}' = \underline{P}^{-1} \underline{\psi}$, then for the system of Example 3.4, (3.28) yields

$$\underline{a} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \underline{p} T + \underline{\psi}' = \begin{bmatrix} 1 + \psi'_2 \\ \frac{5}{78750} + \psi'_2 \end{bmatrix}$$

In order to cancel the effect of parameter perturbations, $\underline{\psi}'$ must be selected as

$$\underline{\psi}' = \underline{P}^{-1} \underline{\psi} = \begin{bmatrix} 0 \\ -5 \\ \frac{5}{78750} \end{bmatrix}$$

Therefore,

$$\underline{\psi} = \underline{P} \underline{\psi}' = \begin{bmatrix} 0 \\ 23 \\ (315)^2 \end{bmatrix} \quad (3.46)$$

From equations (3.29e), (3.29g) and (3.46), one can see that for $\epsilon_e > 2 \times 10^{-4}$, the motion could be considered as ϵ_e -n.p.

3.6 Conclusions

This chapter was concerned with the basic aspects of periodic behavior in multi-modulator PFM systems. Since motion in a PFM system is not necessarily (purely) periodic or almost periodic, a weaker concept, that of ϵ_e -nearly periodic motion (ϵ_e -n.p.m.) was introduced. This notion has been defined in terms of a given accuracy within which the motion could be considered "periodic" in a given observation interval.

For CRPFM systems sufficient conditions were given such that the motion would not be finite-pulse stable (i.e., the modulators will not stop firing in finite time) or uncontained (i.e., the pulse frequencies will not keep on increasing). The first set of conditions constitutes a basic necessary condition for the existence of ϵ_e -n.p. motion, while violation of the latter, for large perturbations, means the motion will "run away".

Two upper bounds were presented for ϵ_e , such that for a given period and a given observation interval, the motion will be ϵ_e -n.p. The first bound is applicable to more general cases; however, in certain cases, it can be much larger than the actual value. The second bound is especially useful if the impulse response matrix of the LP is almost periodic (e.g., a finite-dimensional time-

invariant LP, having a transfer matrix with all its poles on the imaginary axis) or contains only lightly damped periodical terms.

For IPFM systems with time invariant LP's a matrix relationship was presented, relating the period of motion and the net number of pulses emitted by each modulator over that period. This relation clearly demonstrates the difference between periodic behavior of single-modulator and multi-modulator systems: Pure periodic motion, in the latter, is possible only in the "ideal" case when all the components of a certain vector of system parameters are rational numbers. Practically, however, pure periodic motion or approximately periodic motion may look alike because of measurement inaccuracy. Therefore, both measurement (or observation) accuracy and the observation interval must be considered in investigations of periodic behavior.

For some sets of system parameters, it is also possible that the motion is not ϵ_e -n.p., except for unreasonably large values of ϵ_e . In this case the motion will have a random appearance, such as has been observed in experimental studies of neural activity. Thus, the results of this work might offer clues in the research on "random" activity in the nervous system.

CHAPTER 4

OSCILLATIONS IN INTERCONNECTED TIME-DISCRETIZED CRPFM
SYSTEMS4.1 Introduction

In the previous chapter certain aspects of periodic behavior of CRPFM systems were considered. However, many interesting problems were left unsolved, such as the determination of the possible period(s) of motion and prediction of possible pulse patterns for given sets of system parameters. Only a partial answer to this question was given for the special case of an IPFM system.

To obtain further results, a different approach is used in this chapter; namely, time discretized approximation of the CRPFM system. Such approximations are in fact utilized in numerical computations of the system response (see Appendix D).

It is, however, still difficult to obtain analytical results from the resulting (nonlinear) difference equations (except for oscillations having very short periods). This difficulty can be reduced by "linearization" of these equations by introduction of extra variables, using Fukunaga's method (32) for nonlinear switching nets. In this case, classical linear techniques (based on characte-

ristic polynomials and eigenvectors) can be used to obtain information about periodic motion.

The analysis presented in this chapter is exact for an important class of CRPFM systems, namely that where the LP's consist of interconnections of unit delays and summing junctions. Since McCulloch-Pitts type of neural networks constitute a subclass of this class, the results are also applicable to such networks.

4.2 System Considerations

In this chapter, a time-discretization of the CRPFM system of Fig. 3.1 is considered. It is assumed that the LP is time invariant.

4.2.1 Time discretization of general CRPFM system.

The discretization interval should be carefully selected; a large value can result in serious errors, while a small value means the the dimensions of the approximate system might become very large (as in the case of a LP containing time-delay). In general, it should be selected smaller than the smallest pulse period (i.e., the minimum distance between two successive pulses) expected.

To illustrate the time discretization, consider the general system of Fig. 3.1, with m CRPFM's, a time-invariant LP and time-invariant TF's. It can be represented as

shown in Fig. 4.1a. The time-discretization implies the assumption that pulses may occur only at time-instants kT ; $k = 0, 1, 2, \dots$. This applies to both input $\underline{r}(t)$ and modulator-output $\underline{u}(t)$; i.e.,

$$\underline{r}(t) = \sum_{k=0}^{\infty} \underline{r}^*(k) \delta(t-kT) \quad (4.1)$$

and

$$\underline{u}(t) = \sum_{k=0}^{\infty} \underline{u}^*(k) \delta(t-kT) \quad (4.2)$$

where, $\underline{r}^*(k)$ and $\underline{u}^*(k)$ are the strengths of the impulses of $\underline{r}(t)$ and $\underline{u}(t)$ respectively, at $t = kT$. For convenience, let $\underline{x}_t(k) = \underline{x}_t(kT)$, $\underline{x}_p(k) = \underline{x}_p(kT)$, etc. The equations governing the system of Fig. 4.1a for $k = kT$ are

Linear part (LP):

$$\underline{x}_p(k+1) = \underline{\phi}_p(T) \left[\underline{x}_p(k) + \underline{B}_p \underline{u}^*(k) \right] \quad (4.3a)$$

$$\underline{y}(k) = \underline{C}_p \underline{x}_p(k) \quad (4.3b)$$

Threshold element (TE):

$$\left. \begin{array}{l} \text{If } z_i(k) \leq -S_i \text{ then } u_i^*(k) = -M_i; \underline{x}_{ti}^+(k) = 0 \\ \text{If } |z_i(k)| < S_i \text{ then } u_i^*(k) = 0; \underline{x}_{ti}^+(k) = \underline{x}_{ti}(k) \\ \text{If } z_i(k) > S_i \text{ then } u_i^*(k) = M_i; \underline{x}_{ti}^+(k) = 0 \end{array} \right\} (4.4)$$

($i = 1, \dots, m$)

where $\underline{x}_{ti}^+(k)$ represents the state of the TF of the i th modulator at $t = kT^+$.

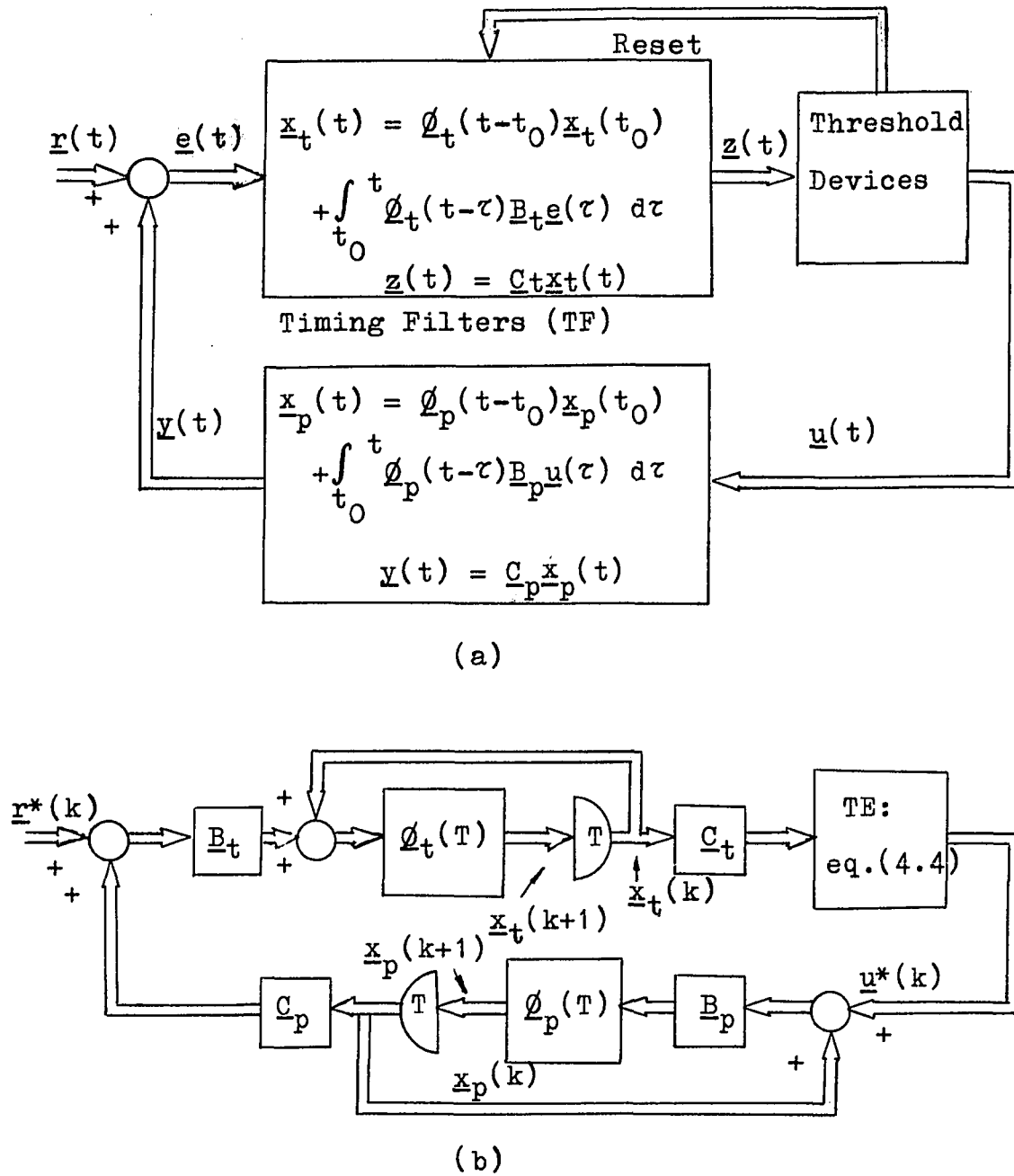


Figure 4.1 (a) A CRPFM system with time-invariant TF and LP.
 (b) Time-discretized approximation, using eqs. (4.1)-(4.5).

Timing filter (TF):

$$\underline{x}_t(k+1) = \mathcal{Q}_t(T) \left[\underline{x}_t^+(k) + \left(\underline{B}_t \underline{r}^*(k) + \underline{y}(k) \right) \right] \quad (4.5a)$$

$$\underline{z}(k) = \underline{C}_t \underline{x}_t(k) \quad (4.5b)$$

A case of special interest is the CRPFM system of Fig. 4.2, where the LP consists of ideal time-delays only. Such a system may be used as an approximation of a neural network. It is especially suited for time-discretization which can approach an exact representation of the system when the discretization-interval, T , is chosen as a certain sub-multiple of the time-delays, T_i ($i = 1, \dots, m$) (provided that the input consists of impulses occurring at intervals kT). This will be discussed next.

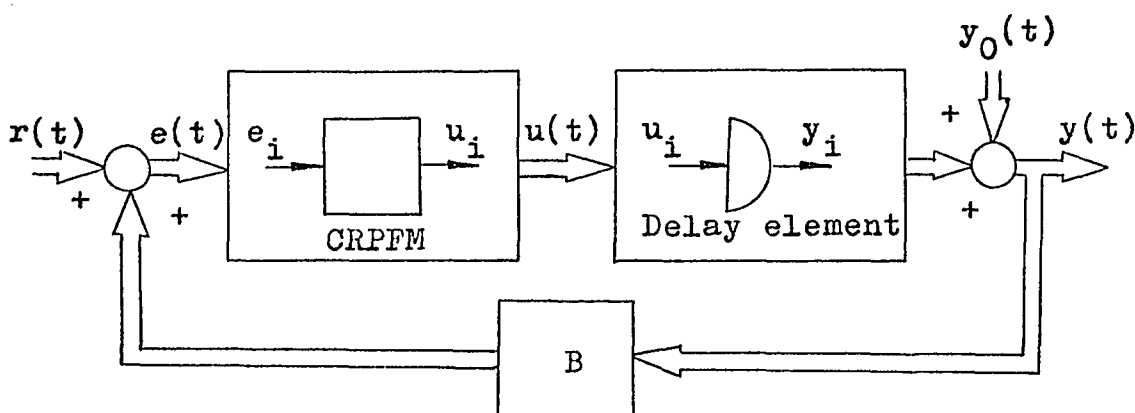


Figure 4.2 Block diagram of an interconnected system consisting of m CRPFM's and ideal delays.

4.2.2 Interconnected system consisting of CRPFM's and ideal delays. In this subsection the system of Fig. 4.2 is considered. The block containing the delay elements is assumed to be described by the equations

$$y_i(t+T_i) = K_i u_i(t) + y_{0i}(t+T_i) \quad (4.6)$$

$$(i = 1, \dots, m)$$

where, T_i and K_i are constants representing the delay times and gains, respectively ($i = 1, \dots, m$). The input vector, $\underline{e}(t)$ to the modulator block is given by

$$\underline{e}(t) = \underline{B} \underline{y}(t) + \underline{r}(t) \quad (4.7)$$

where, \underline{B} is an $m \times m$ matrix.

It is assumed that all motion is of the form of impulses; however, each modulator may emit impulses of different strengths. Thus, the input to the i th modulator, $e_i(t)$ may be expressed in the form

$$e_i(t) = \sum_{j=1}^{\infty} c_{ij} \delta(t - \tau_{ij}) \quad (4.8)$$

where c_{ij} is the impulse strength of $e_i(t)$ at time τ_{ij} .

It is further assumed that each modulator emits an impulse immediately after it receives an impulse; this implies that

$$|c_{ij} g_{0i}(\tau_{ij}, \tau_{ij})| \geq S_i, \quad \begin{matrix} (i = 1, \dots, m, \\ j = 1, 2, \dots) \end{matrix} \quad (4.9)$$

A CRPFM system satisfying the above assumptions shall be said to be in pulse mode operation¹.

With the above assumption, the output of the ith modulator is given by

$$u_i(t) = M_i \sum_{j=1}^{\infty} \text{sgn}[c_{ij}g_{0i}(\tau_{ij}, \tau_{ij})] \delta(t - \tau_{ij}) \quad (4.10)$$

where,

$$\text{sgn}(\zeta) \triangleq \begin{cases} -1 & \text{for } \zeta < 0 \\ 0 & \text{for } \zeta = 0 \\ 1 & \text{for } \zeta > 0 \end{cases} \quad (4.11)$$

Let it be further assumed that

$$g_{0i}(t, t) > 0 \quad \forall t \geq 0 \quad (4.12)$$

This assumption implies little loss of generality since a negative sign can be taken up by M_i . Moreover, since for a timing filter of the form $\dot{\underline{x}}(t) = \underline{A}(t)\underline{x}(t) + \underline{b}(t)\underline{u}(t)$; $\underline{z}(t) = \underline{c}^T(t)\underline{x}(t)$, it is $g_{0i}(t, t) = \underline{c}^T(t)\underline{b}(t)$, if $\underline{c}^T(t)\underline{b}(t)$ does not change sign, then (4.12) can be imposed. This, for example, is true for a time-invariant TF.

With assumption (4.12), (4.10) can be written as

¹For the developments to follow, the modulator can actually be a different type of PF modulator. However, it has to satisfy the assumption that the strength of an incoming impulse is such that it regenerates another ~~imp~~ impulse.

$$u_i(t) = M_i \sum_{j=1}^{\infty} \text{sgn}[c_{ij}] \delta(t - \tau_{ij}) \quad (4.13)$$

For time-discretization, select a discretization-interval, T , such that $T_i = m_i T$, where m_i ($i = 1, \dots, m$) is a positive interval. This interval T must be chosen small enough to ensure that the pulse-emission times τ_{ij} are also multiples of T . The time-discretization consists of consideration of the impulse-strengths of the signals at instants kT . It will also be assumed that the input, $\underline{r}(t) = \sum_{k=0}^{\infty} \underline{r}^*(k) \delta(t - kT)$, will consist of impulses of strengths $\underline{r}^*(k)$ at instant kT . Moreover, the initial condition response, $\underline{y}_0(t)$, is assumed to be given in terms of impulse-trains within the delay times, expressible as

$$\underline{y}_{0i}(t) = \sum_{\ell=0}^{m_i-1} \underline{y}_{0i}^*(\ell) \delta(t - \ell T), \quad (i=1, \dots, m) \quad (4.14)$$

Further, let $\underline{u}^*(k)$ and $\underline{y}^*(k)$ be the impulse-strengths of $\underline{u}(t)$ and $\underline{y}(t)$, respectively at $t = kT$. Then, for pulse mode operation [condition (4.9)], (4.13) becomes:

$$u_i^*(k) = M_i \text{sgn}[e_i^*(k)] \quad (4.15)$$

The initial-condition response given in (4.14) defines $y_i(t)$ from $0 \leq t < T_i$. For $t \geq T_i = m_i T$, $y_i(t)$ will be

given in terms of $y_i^*(k+m_i)$ ($k = 0, 1, 2, \dots$). From (4.6), (4.15) and Fig. 4.2:

$$y_i^*(k+m_i) = K_i M_i \operatorname{sgn} [\underline{B} \underline{y}^*(k) + \underline{r}^*(k)]_i \quad (4.16)$$

The notation $[\underline{x}]_i$ denotes the *i*th element of the vector \underline{x} .

Since it is clear that only impulse-strengths are considered, the superscripts (*) will be dropped for the rest of this chapter. Furthermore, for notational simplicity, certain previously used symbols will be redefined in this chapter to denote different variables. Dropping the asterisk, the above equation becomes:

$$y_i(k+m_i) = K_i M_i \operatorname{sgn} [\underline{B} \underline{y}(k) + \underline{r}(k)]_i \quad (4.17a)$$

(for double-signed CRPFM's) ($i = 1, \dots, m$)

A similar relation can be given for the system of Fig. 4.2 with single-signed CRPFM's; in this case it is

$$y_i(k+m_i) = K_i M_i \mu [\underline{B} \underline{y}(k) + \underline{r}(k)]_i \quad (4.17b)$$

(for single-signed CRPFM's) ($i = 1, \dots, m$)

where $\mu(x)$ denotes the unit step function, defined such that

$$\mu(x) = \begin{cases} 1 & \text{for } x > 0 \\ 0 & \text{for } x \leq 0 \end{cases} \quad (4.17c)$$

Equations (4.17) may be unified as

$$y_i(k+m_i) = K_i M_i \rho[B \underline{y}(k) + \underline{r}(k)]_i \quad (4.18a)$$

$$(i = 1, \dots, m)$$

where

$$\rho(x) = \begin{cases} \text{sgn}(x) & \text{for double-signed CRPFM's} \\ \mu(x) & \text{for single-signed CRPFM's} \end{cases} \quad (4.18b)$$

Equations (4.18a) represent m scalar difference equations of orders m_1, m_2, \dots, m_m , respectively. They can also be represented in terms of $\sum_{i=1}^m m_i$ first-order difference equations which form a vector-difference equation of order

$$n = \sum_{i=1}^m m_i \quad (4.19)$$

Let

$$x_{ij}(k) \triangleq \frac{1}{M_i K_i} y_i(k+j-1), \quad \begin{array}{l} i = 1, \dots, m, \\ j = 1, \dots, m_i \end{array} \quad (4.20)$$

This implies that

$$x_{ij}(k+1) = x_{i,j+1}(k), \quad \begin{array}{l} i = 1, \dots, m, \\ j = 1, \dots, m_i - 1 \end{array} \quad (4.21a)$$

and, from (4.18a)

$$x_{i,m_i}(k+1) = \rho \left[\sum_{j=1}^{\infty} b_{ij} M_j K_j x_{j1}(k) + r_i(k) \right], \quad (4.21b)$$

$$(i = 1, \dots, m)$$

Let

$$\underline{x}_i(k) \triangleq \begin{bmatrix} x_{i1}(k) \\ x_{i2}(k) \\ \vdots \\ x_{im_i}(k) \end{bmatrix} \quad \text{and} \quad \underline{r}_i(k) \triangleq \begin{bmatrix} 0 \\ 0 \\ \vdots \\ r_i(k) \end{bmatrix} \quad (4.22a)$$

(i=1, \dots, m)

and

$$\underline{x}(k) \triangleq \begin{bmatrix} \underline{x}_1(k) \\ \underline{x}_2(k) \\ \vdots \\ \underline{x}_m(k) \end{bmatrix} \quad \text{and} \quad \underline{r}'(k) \triangleq \begin{bmatrix} \underline{r}_1(k) \\ \underline{r}_2(k) \\ \vdots \\ \underline{r}_m(k) \end{bmatrix} \quad (4.22b)$$

Eqs. (4.21) represent the new set of state difference equations in terms of the n-dimensional state vector $\underline{x}(k)$. Because of the definition of $\rho(\cdot)$ and its use in eqs. (4.21b), it follows that

$$x_{ij}(k) = \rho[x_{ij}(k)], \quad i = 1, \dots, m, \quad j = 1, \dots, m_i \quad (4.23)$$

Therefore, eqs. (4.21) can be brought into matrix form as

$$\underline{x}(k+1) = \rho[\underline{D} \underline{x}(k) + \underline{r}'(k)] \quad (4.24)$$

where

$$\rho \underline{x} \triangleq [\rho(x_1), \rho(x_2), \dots, \rho(x_n)]^T, \quad (4.25)$$

$$\underline{D} \triangleq \begin{bmatrix} \underline{D}_{11} & \underline{D}_{12} & \dots & \underline{D}_{1m} \\ \dots & \dots & \dots & \dots \\ \underline{D}_{m1} & \underline{D}_{m2} & & \underline{D}_{mm} \end{bmatrix}, \quad (4.26a)$$

and

$$\underline{D}_{ii} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ b_{ii} M_i K_i & 0 & 0 & \dots & 0 & 0 \end{bmatrix} \quad (4.26b)$$

(i = 1, \dots, m)

and

$$\underline{D}_{ij} = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ b_{ij}K_jM_j & 0 & 0 & \dots & 0 & 0 & 0 \end{bmatrix} \begin{array}{l} \downarrow \\ m_i \\ \downarrow \\ (i, j = 1, \dots, m; \\ i \neq j) \\ \downarrow \\ m_j \end{array} \quad (4.26c)$$

In other words,

$$[\underline{D}_{ij}]_{k,\ell} = \begin{cases} 1 & \text{if } \ell = k+1; k = 1, 2, \dots, m_i-1; j=i \\ b_{ij}K_jM_j & \text{if } k = m_i, \ell = 1 \\ 0 & \text{otherwise} \end{cases} \quad (4.26d)$$

(k = 1, \dots, m_i; \ell = 1, \dots, m_j)

Equations similar to (4.24) have been used to describe McCulloch-Pitts type neural nets [Landahl and Runge (82), Caianiello et.al. (18, 19, 20)]. A special case treated by Caianiello et.al (19) for neural nets assumes $\underline{r}'(k)$ to be constant such that

$$\underline{r}'(k) = -\frac{1}{2} \underline{D} \underline{1}, \quad \underline{1} \triangleq [1, 1, \dots, 1]^T \quad (4.27a)$$

In this case the transformation

$$\hat{\underline{x}}(k) = 2 \underline{x}(k) - \underline{1} \quad (4.27b)$$

reduces (4.24), for the single-signed system [i.e., for $\rho(\cdot) = \mu(\cdot)$], to the form

$$\hat{\underline{x}}(k) = \overline{\text{sgn}} \quad \underline{D} \hat{\underline{x}}(k) \quad \begin{array}{l} \text{(for single-signed} \\ \text{CRPFM's and for} \\ \underline{r}'(k) = -\frac{1}{2} \underline{D} \underline{1}) \end{array} \quad (4.27c)$$

where

$$\overline{\text{sgn}}(x) = \begin{cases} 1 & \text{for } x > 0 \\ -1 & \text{for } x \leq 0 \end{cases} \quad (4.27d)$$

4.3 The Autonomous Case, Period of Solutions

For the case $\underline{x}'(k) = \underline{0}$ eq. (4.24) for the double-signed system becomes

$$\underline{x}(k+1) = \text{sgn } \underline{D}[\underline{x}(k)] \quad (4.28a)$$

or,

$$\hat{\underline{y}}(k+1) = \underline{D} \text{sgn } \hat{\underline{y}}(k), \text{ where } \hat{\underline{y}}(k) \triangleq \underline{D} \underline{x}(k) \quad (4.28b)$$

In this section some properties of eqs. (4.28) will be presented.

Consider the vector-difference equation

$$\underline{x}(k+1) = \mathcal{O}[\underline{x}(k)] \quad (4.28c)$$

where $\mathcal{O}(\cdot)$ denotes some operator [e.g, for (4.28a), it is $\mathcal{O}(\cdot) = \text{sgn } \underline{D}(\cdot)$]. Let

$$\mathcal{O}^k(\cdot) = \underbrace{\mathcal{O}(\mathcal{O}(\mathcal{O}(\dots)))}_{k \text{ times}}$$

Then, the solution of (4.28c) is

$$\underline{x}(k) = \mathcal{O}^k[\underline{x}(0)]$$

A state \underline{x} is called a cyclic state if there exists an integer k such that $\underline{x} = \mathcal{O}^k \underline{x}$. If no such k can be found, then the state is called a transient state. Let k_0 be the value of the smallest k satisfying $\underline{x} = \mathcal{O}^k \underline{x}$; the sequence $\mathcal{O} \underline{x}, \mathcal{O}^2 \underline{x}, \dots, \mathcal{O}^{k_0} \underline{x} = \underline{x}$ is called a cycle. The constant k_0 is called the period (or, sometimes the length) of the cycle. A cycle of period = 1 is called a simple cycle. A simple cycle formed by the zero state

$\underline{x} = [0, 0, \dots, 0]^T$ is called a trivial cycle. If for every possible state \underline{x} , $\sigma(\underline{x}) \neq \underline{x}_f$, then \underline{x}_f is called a first state. A sequence of transient states $\underline{x}_f, \sigma \underline{x}_f, \dots, \sigma^k \underline{x}_f$, generated by a first state \underline{x}_f is called a transient chain (13).

In shift register designs short cycles are, generally, not desirable. However, short cycles might have biological significance (Kauffman (70) relates existence of short cycles to genetic stability). The following lemma is concerned with short cycles, namely, cycles consisting of one or two states.

Lemma 4.1 Let \underline{x}' and \underline{x}'' be the solutions of the equations $\underline{x} = \text{sgn } \underline{D} \underline{x}$ and $\underline{x} = -\text{sgn } \underline{D} \underline{x}$, respectively. Then, the state \underline{x}' will form a simple cycle by itself, and the states \underline{x}'' and $-\underline{x}''$ will form a cycle of period 2.

Proof: Let $\underline{x}(0) = \underline{x}'$, then from (4.28b), $\underline{x}(1) = \text{sgn } \underline{D} \underline{x}'$. But $\underline{x}' = \text{sgn } \underline{D} \underline{x}'$, thus $\underline{x}(1) = \underline{x}'$; $\underline{x}(2) = \underline{x}'$, Similarly, with $\underline{x}(0) = \underline{x}''$, $\underline{x}(1) = \text{sgn } \underline{D} \underline{x}'' = -\underline{x}''$, $\underline{x}(2) = \underline{x}''$, $\underline{x}(3) = -\underline{x}''$, Therefore, \underline{x}' forms a cycle of period 1 and \underline{x}'' and $-\underline{x}''$ form a cycle of period 2.

The existence of the solution of the equation $\underline{x} = \pm \text{sgn } \underline{D} \underline{x}$ is not obvious. Consider the eigenvalues of \underline{D} . Let \underline{x}^+ denote an eigenvector corresponding to a positive eigenvalue, λ^+ , and let \underline{x}^- denote an eigenvector

corresponding to a negative eigenvalue, λ^- . Then, $\underline{D} \underline{x}^+ = \lambda^+ \underline{x}^+$, or $\text{sgn } \underline{D} \underline{x}^+ = \text{sgn } \underline{x}^+$. Thus, if $\underline{x}^+ = \text{sgn } \underline{x}^+$, then $\underline{x}^+ = \underline{x}^+$. Similarly, if $\underline{x}^- = \text{sgn } \underline{x}^-$, then $\underline{x}^- = \underline{x}^-$. This leads to the following corollary:

Corollary 4.1 If the matrix \underline{D} has an eigenvector \underline{x}^+ corresponding to a positive eigenvalue, such that $\underline{x}^+ = \text{sgn } \underline{x}^+$, then \underline{x}^+ will form a simple cycle. If \underline{D} has an eigenvector \underline{x}^- corresponding to a negative eigenvalue, such that $\underline{x}^- = \text{sgn } \underline{x}^-$, then \underline{x}^- and $-\underline{x}^-$ will form a cycle of period 2.

Example 4.1 To illustrate Lemma 4.1, Corollary 4.1 and some of the related notations and definitions, consider the system

$$\underline{x}(k+1) = \overline{\text{sgn}} \begin{bmatrix} 3 & 6 \\ 1 & -2 \end{bmatrix} \underline{x}(k)$$

The eigenvector corresponding to the positive eigenvalue $\lambda = 4$ is of the form $\underline{x}^+ = \begin{bmatrix} 6 \\ 1 \end{bmatrix} \alpha$, and the eigenvector corresponding to the negative eigenvalue $\lambda = -3$ is the form $\underline{x}^- = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \beta$, where α and β are arbitrary scalars. In this case, $\underline{x}^+ \neq \text{sgn } \underline{x}^+$, thus, there is no simple cycle. However, for $\beta = 1$, $\text{sgn } \underline{x}^- = \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \underline{x}^-$. Hence, $\underline{x}'' = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Note that

$$\text{sgn} \begin{bmatrix} 3 & 6 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = -\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

This means that the states $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ will form a cycle of period 2. The remaining states are $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ -1 \end{bmatrix}$; they are transient states. From the first, the net goes to the state $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and from the second to $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$. This knowledge can conveniently be displayed using the diagram shown in Fig. 4.3. Such diagrams are called state transition diagrams.

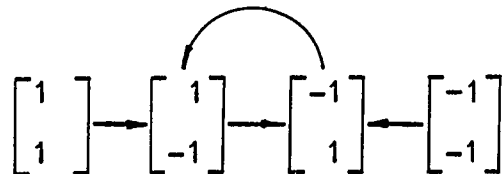


Figure 4.3 State transition diagram of the system considered in Example 4.1.

From Corollary 4.1 and Example 4.1 one might intuitively reach the conclusion that somehow the rank of the matrix \underline{D} and the cycle lengths are related; the smaller the rank, the shorter become the cycle lengths. This point was explored by Caianiello and Accardi. The following theorem is due to Caianiello (19, 20).

Theorem 4.1 [Caianiello (19)] If \underline{D} is of rank 1, then the system described by (4.28) can only have a period of

$$(1) \quad 1 \text{ if } c_2 > 0, \text{ or}$$

$$(2) \quad 2 \text{ if } c_2 < 0,$$

where $c_2 = \underline{b}^T \text{sgn } \underline{a}$ and $\underline{D} = \underline{a} \underline{b}^T$; \underline{a} and \underline{b} are n -dimensional constant vectors.

Proof: Consider (4.28b). Since the matrix \underline{D} is of rank 1, it has the form $\underline{a} \underline{b}^T$, where \underline{a} and \underline{b} are n-dimensional column vectors. For $k = 0$, (4.28b) gives

$$\hat{\underline{y}}(1) = \underline{a} \underline{b}^T \operatorname{sgn} \hat{\underline{y}}(0)$$

Let $\underline{b}^T \operatorname{sgn} \hat{\underline{y}}(0) = c_0$ (a scalar), For $k = 1$,

$$\begin{aligned} \hat{\underline{y}}(2) &= \underline{a} \underline{b}^T \operatorname{sgn} \hat{\underline{y}}(1) = \underline{a} \underline{b}^T \operatorname{sgn} (c_0 \underline{a}) \\ &= \underline{a} \underline{b}^T (\operatorname{sgn} c_0) (\operatorname{sgn} \underline{a}) \end{aligned}$$

For $k = 2$,

$$\hat{\underline{y}}(3) = \underline{a} \underline{b}^T \operatorname{sgn} \hat{\underline{y}}(2) = \underline{a} \underline{b}^T \operatorname{sgn} \underline{a} \underline{b}^T (\operatorname{sgn} c_0) \operatorname{sgn} \underline{a}.$$

Let $\operatorname{sgn} c_0 = c_1$ and $\underline{b}^T \operatorname{sgn} \underline{a} = c_2$ (a scalar), $\hat{\underline{y}}(3)$ becomes

$$\hat{\underline{y}}(3) = \underline{a} \underline{b}^T (\operatorname{sgn} \underline{a}) (\operatorname{sgn} c_1) (\operatorname{sgn} c_2) = \underline{a} c_1 c_2.$$

For $k = 4$, (4.19b) yields

$$\hat{\underline{y}}(4) = \underline{a} \underline{b}^T \operatorname{sgn} \hat{\underline{y}}(3) = c_1 c_2 \underline{a}.$$

Thus, from an inductive reasoning, one can deduce that

$$\hat{\underline{y}}(k) = \begin{cases} \underline{a} c_1 c_2 & \text{for } k = 2, 4, \dots, \\ \underline{a} c_1 c_2 & \text{for } k = 3, 5, \dots \end{cases}$$

For $c_2 > 0$, this means $\underline{y}(k) = \underline{a} c_1 c_2$ ($k = 2, 3, 4, \dots$), i.e., only a simple cycle is possible. However, for $c_2 < 0$, there will be a cycle of period 2.

(QED)

The same conclusion also follows from Corollary 4.1.

Since \underline{D} is of rank 1, it is similar to a matrix which has a nonzero element in a diagonal position as its only entry. Thus, it has only one nonzero eigenvalue and any vector in E^n is an eigenvector corresponding to this eigenvalue. This means that Corollary 4.1 will be satisfied for each possible n -vector whose elements are +1 or -1. If the nonzero eigenvalue is positive, there will be only simple cycles. If it is negative then there will be only cycles of period 2.

The following theorem, which considers the case when the coupling matrix \underline{D} has rank K was proved by Accardi (2).

Theorem 4.2 Accardi (2) If the matrix \underline{D} has rank K , then the number N of the admissible states of the system described by (4.28) and the maximum possible cycle period is such that

$$N \leq 2^n - 2^{n-K+1} + 2.$$

4.4 Linearization of the System Equations

The approach used in Section 4.3 did not prove to be very successful mainly because the operator \mathcal{O} was non-linear (i.e., in general, $\mathcal{O}(c_1 \underline{x}_1 + c_2 \underline{x}_2) \neq c_1 \mathcal{O} \underline{x}_1 + c_2 \mathcal{O} \underline{x}_2$, where c_1 and c_2 are scalars and \underline{x}_1 and \underline{x}_2 are n -vectors). Is it possible, then to find an equivalent

linear system of equations which will adequately describe the systems considered in Section 4.2?

To answer this question first consider eqs. (4.24):

$$\underline{x}(k+1) = \text{sgn} [\underline{D} \underline{x}(k) + \underline{r}'(k)] \quad (\text{double-signed system})$$

and

$$\underline{x}(k+1) = \mu[\underline{D} \underline{x}(k) + \underline{r}'(k)] \quad (\text{single-signed system})$$

describing the systems considered in this chapter. From the above equations, it is not difficult to see that for the double-signed system, the elements of the vector $\underline{x}(k)$ can only take on the values +1, 0 or -1 and for the single-signed system 0 and 1 are the only possible values², i.e., only a finite number of states are possible.

Since only a finite number of states are permissible, it is possible to find an equivalent linear system by introducing extra variables. For binary switching nets such a technique was described by Fukunaga (132), in a short technical note. Some of the consequences of this linearization was later worked out by da Fonseca and

²The variable $y(k)$ corresponds to the pulse strengths. For the double-signed system, +1 represents a positive pulse and -1 represents a negative pulse. For a single-signed system +1 represents the presence of a pulse and 0 represents its absence. In the double-signed case one can also use the symbol 0 to denote the absence of a pulse.

McCulloch (131). In general, this technique can be extended to any autonomous net defined with respect to a finite field³.

A field is an algebraic system consisting of two operators and their inverses (e.g, addition and its inverse, subtraction; multiplication and its inverse, division). The field of real numbers and the field of complex numbers are examples of infinite fields which are used in the analysis of continuous systems. When the variables of the system under consideration is restricted

³A field is an algebraic system consisting of a set F and two operations defined on F which are single-valued functions of two variables, denoted by $a + b = c$ and $a \cdot b = c$, called addition and multiplication (not necessarily the addition and the multiplication of the arithmetic of ordinary numbers). The operations $+$ and \cdot satisfy axioms A.1 - A.5 (with the dummy operator o replaced first by $+$ and then by \cdot) and A.6:

- | | |
|-----------------------------|---|
| A.1 <u>Closure</u> | $\forall a, b \in F \rightarrow c = aob \in F,$ |
| A.2 <u>Associative law</u> | $\forall a, b, c \in F \rightarrow (aob)oc = ao(boc)$ |
| A.3 <u>Identity element</u> | $\forall a \in F \rightarrow \exists i \in F \exists ioa = aoi = a$ |
| A.4 <u>Inverse element</u> | $\forall a \in F \rightarrow \exists q \in F \exists qoa = aoq = i$ |
| A.5 <u>Commutative law</u> | $\forall a, b \in F \rightarrow aob = boa$ |
| A.6 <u>Distributive law</u> | $\forall a, b, c, e \in F \rightarrow$ |
| | $a \cdot (b+c) = a \cdot b + b \cdot c,$ |
| | $(b+c) \cdot a = b \cdot a + c \cdot a.$ |

If the multiplicative operation does not satisfy axioms A.3-A.5, the system is called a ring; if one of the operations (say $+$) satisfies A.1-A.4, it is called a group. If in addition to A.1-A.4, A.5 is also satisfied, the group is called and Abelian (or, commutative) group. Thus, a field is a commutative ring with a multiplicative inverse and a ring with respect to its additive operation is a commutative group (9).

to a finite number of values, it becomes advantageous to use fields that have only a finite number of elements.

In a finite field each function is equivalent to a polynomial. The number of elements cannot be selected arbitrarily; it must be of the form p^r , where p is a prime number and r is an integer. It can be shown that any two finite fields with the same number of elements are isomorphic (9), i.e., they have the same structure and differ only in the way the elements are named. Finite fields are called Galois fields (denoted by $GF(p^r)$), in honor of the French mathematician who first investigated their properties. Any function of n variables over $GF(p^r)$ can be represented by r functions in nr variables over $GF(p)$ (9).

In this chapter the field $GF(2)$ (also called the binary field, or mod 2 field) will be used (This applies to single-signed system). However, for sake of generality, some of the results will be given with respect to $GF(p^r)$.

The binary field $GF(2)$ has a very simple structure. It has two elements denoted by 0 and 1 and two binary operations denoted by (+) (called mod 2 addition)⁴ and

⁴The symbol + is also being used for addition in the usual sense. However, this will not cause any confusion because which field is used will be obvious.

(\cdot) (called binary multiplication). Operation rules for GF(2) are given in Figure 4.4

+	0	1
0	0	1
1	1	0

\cdot	0	1
0	0	0
1	0	1

+ exclusive OR
 \vee OR
 \cdot AND
 a' NOT a
 Basic conversion rules:
 $a' = 1 + a$
 $a \vee b = a + b + ab$

Figure 4.4 Operation rules for the Galois field GF(2).

In logic designer's language mod 2 addition (+) is also known as EXCLUSIVE-OR and the binary multiplication (\cdot) is known as AND (133). The INCLUSIVE-OR (or, simply OR) operation (\vee) is defined for two variables a and b such that $a \vee b = 1$ if and only if either a or b (or both) is 1. In logic design, it is more customary to use INCLUSIVE-OR, AND and NOT ($a' = 1$ if $a = 0$; $a' = 0$ if $a = 1$) operations (This may not be so). It is not difficult to see that

$$a' = 1 + a$$

and

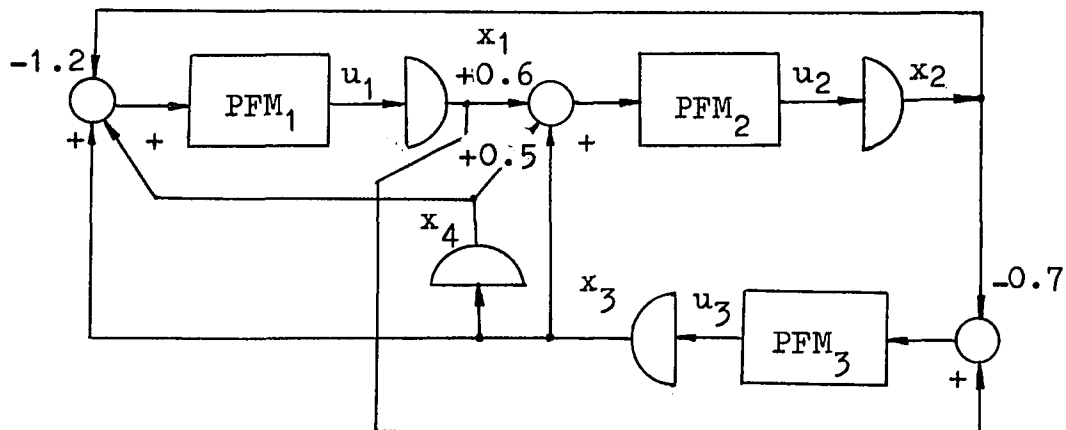
$$a \vee b = a + b + ab$$

To illustrate how the aforementioned linearization can be performed, the following example is given.

Example 4.2 Consider the neural network of Fig. 4.5a,



(a)



(b)

Figure 4.5 (a) Schematic representation of a neural network consisting of interconnections of two excitatory neurons (neuron 1 and neuron 3) and an inhibitory neuron (neuron 2).
 (b) Block diagram of a simplified model of the same neural network.

consisting of two excitatory neurons and an inhibitory neuron. A simplified model of this net is shown in Fig. 4.5b. Assuming the pulse repetition rate of the network to be sufficiently low, the effect of temporal summation can be neglected. Using the symbol 1 to denote the existence of a pulse and 0 to denote the absence of a pulse, the behavior of the net can be described by the equation

$$\underline{x}(k+1) = \mu \begin{bmatrix} 0 & -1.2 & 1 & 1 \\ 0.6 & 0 & 1 & 0.5 \\ 1 & -0.7 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \underline{x}(k)$$

The information contained in the above equation can also be displayed using the tables shown in Fig. 4.6.

$x_2(k)$	$x_3(k)$	$x_4(k)$	$x_1(k+1)$
0	0	0	0
0	0	1	1
0	1	0	1
0	1	1	1
1	0	0	0
1	0	1	0
1	1	0	0
1	1	1	1

$x_1(k)$	$x_2(k)$	$x_3(k+1)$
0	0	0
0	1	0
1	0	1
1	1	1

$x_1(k)$	$x_3(k)$	$x_4(k)$	$x_2(k+1)$
0	0	0	0
0	0	1	1
0	1	0	1
0	1	1	1
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	1

$x_3(k)$	$x_4(k+1)$
0	0
1	1

Figure 4.6 Truth tables of the system of Fig. 4.5b.

From these tables one can obtain:

$$x_1(k+1) = [1+x_2(k)][1+x_3(k)]x_4(k) \vee [1+x_2(k)][1+x_4(k)]x_2(k) \\ \vee [1+x_2(k)]x_3(k)x_4(k) \vee x_2(k)x_3(k)x_4(k) \quad (4.29a)$$

$$x_2(k+1) = x_1(k) + x_3(k) \vee x_4(k) \quad (4.29b)$$

and

$$x_3(k+1) = x_1(k) \quad (4.29c)$$

Also,

$$x_4(k+1) = x_3(k) \quad (4.29d)$$

Since $a \vee b = a + b + ab$, one can express the right hand side of equation (4.29) using only the operations (+) and (\cdot). To simplify notation, the k terms on the right hand side of the equations will be dropped. Hence, eq. (4.29a) becomes

$$x_1(k+1) = (1+x_2) \{ [(1+x_3)x_4 \vee (1+x_4)x_3] \vee x_3x_4 \} \vee x_2x_3x_4 \\ = (1+x_2) [(x_4+x_3x_4+x_3+x_3x_4+x_3x_4+x_3x_4+x_3x_4+x_3x_4) \\ \vee x_3x_4] \vee x_2x_3x_4 \\ = (1+x_2) [(x_3+x_4) \vee x_3x_4] \vee x_2x_3x_4 \\ = (1+x_2)(x_3+x_4+x_3x_4+x_3x_4+x_3x_4) \vee x_2x_3x_4 \\ = x_3+x_4+x_2x_3+x_2x_4+x_3x_4 \quad (4.30a)$$

Similarly, eq. (4.29b) becomes

$$x_2(k+1) = (x_1 \vee x_3) \vee x_4 = (x_1+x_3+x_1x_3)+x_4+(x_1x_4+x_3x_4+x_1x_3x_4) \\ = x_1+x_4+x_1x_3+x_1x_4+x_3x_4+x_1x_3x_4 \quad (4.30b)$$

The net of this example is nonlinear because of the presence of the product terms of the form $x_i x_j$ and $x_i x_j x_l$ ($i, j, l = 1, 2, 3; i \neq j \neq l$). In order to avoid such terms, simply define the following new variables

$$x_5(k) = x_1(k) x_2(k) \quad (4.31.1)$$

$$x_6(k) = x_1(k) x_3(k) \quad (4.31.2)$$

$$x_7(k) = x_1(k) x_4(k) \quad (4.31.3)$$

$$x_8(k) = x_2(k) x_3(k) \quad (4.31.4)$$

$$x_9(k) = x_2(k) x_4(k) \quad (4.31.5)$$

$$x_{10}(k) = x_3(k) x_4(k) \quad (4.31.6)$$

$$x_{11}(k) = x_1(k) x_2(k) x_3(k) \quad (4.31.7)$$

$$x_{12}(k) = x_1(k) x_2(k) x_4(k) \quad (4.31.8)$$

$$x_{13}(k) = x_1(k) x_3(k) x_4(k) \quad (4.31.9)$$

$$x_{14}(k) = x_2(k) x_3(k) x_4(k) \quad (4.31.10)$$

$$x_{15}(k) = x_1(k) x_2(k) x_3(k) x_4(k) \quad (4.31.11)$$

Substitution of eqs. (4.31) into eqs. (4.30a) and (4.30b) yields

$$x_1(k+1) = x_3(k) + x_4(k) + x_8(k) + x_9(k) + x_{10}(k) \quad (4.32.1)$$

and

$$\begin{aligned} x_2(k+1) &= x_1(k) + x_3(k) + x_4(k) + x_6(k) + x_7(k) \\ &\quad + x_{10}(k) + x_{13}(k) \end{aligned} \quad (4.32.2)$$

Also,

$$x_3(k+1) = x_1(k) \quad (\text{eq. (4.29c), repeated}) \quad (4.32.3)$$

and

$$x_4(k+1) = x_3(k) \quad (\text{eq. (4.29d), repeated}) \quad (4.32.4)$$

Since, from eq. (4.31.1), $x_5(k+1) = x_1(k+1) x_2(k+1)$, eqs. (4.32.1) and (4.32.2) give

$$\begin{aligned} x_5(k+1) &= x_3(k) + x_4(k) + x_2(k)x_3(k) + x_2(k)x_4(k) \\ &\quad + x_3(k)x_4(k) \end{aligned}$$

But, from eqs. (4.31.4), (4.31.5) and (4.31.6),

$$x_2(k)x_3(k) = x_8(k), \quad x_2(k)x_4(k) = x_9(k) \quad \text{and} \quad x_3(k)x_4(k) = x_{10}(k).$$

Thus, $x_5(k+1)$ can be written as

$$x_5(k+1) = x_3(k) + x_4(k) + x_8(k) + x_9(k) + x_{10}(k) \quad (4.32.5)$$

Similarly,

$$x_6(k) = x_6(k) + x_7(k) + x_{11}(k) + x_{12}(k) + x_{13}(k) \quad (4.32.6)$$

$$x_7(k+1) = x_3(k) + x_8(k) + x_{14}(k) \quad (4.32.7)$$

The rest of the equations can be obtained in the same manner; in matrix form they become

$$\underline{x}(k+1) = \begin{bmatrix} 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \underline{x}(k)$$

As discussed in Example 4.2, it is possible to use the truth table of the functions to be linearized first to express the function in terms of the logical operations (\vee , inclusive-OR), (\cdot , AND) and (a' , NOT a); then the function can be expressed in terms of the binary operations ($+$, mod 2 addition) and (\cdot , mod 2 multiplication). This transition was made possible through the transformations $a' = 1 + a$ and $a \vee b = a + b + ab$.

However, it is also possible to express any function in terms of mod 2 addition and multiplication of its variables, directly, without going to its description in terms of (\vee), (\cdot) and ($'$). The procedure is straightforward and suitable for implementation for programming purposes. This is described next.

The function $f(x_1, x_2, \dots, x_m)$ can be expressed in the following form

$$\begin{aligned}
 f(x_1, x_2, \dots, x_m) &= f_0 + \sum_{i=1}^m f_i x_i + \sum_{i=1}^m \sum_{j=i+1}^m f_{i+m} x_i x_j \\
 &+ \sum_{i=1}^m \sum_{j=i+1}^m \sum_{k=j+1}^m f_{i+m+\binom{m}{2}} x_i x_j x_k \\
 &+ \dots + f_{2^m - 1} x_1 x_2 \dots x_m \quad (4.33)
 \end{aligned}$$

where f_i ($i = 0, 1, 2, \dots, 2^m - 1$) are binary constants (0 or 1). In order to calculate f_0 set $x_1 = x_2 = \dots = x_m = 0$. Then, from eq. (4.33), obviously,

$$f_0 = f(0, 0, \dots, 0)$$

Now set $x_1 = 1$ and $x_2 = x_3 = \dots = x_m = 0$, then eq. (4.33) gives

$$f(1, 0, \dots, 0) = f_0 + f_1$$

Since f_0 is known f_1 can be calculated from

$$f_1 = f(1, 0, \dots, 0) + f_0$$

This procedure can easily be applied to obtain f_2, f_3, \dots, f_m . In order to calculate f_{m+1} , substitute $x_1 = x_2 = 1, x_3 = x_4 = \dots = x_m = 0$, then (4.31) yields

$$f_{m+1} = f(1, 1, 0, 0, \dots, 0) + f_0 + f_1 + f_2$$

Generalizing this idea, it is seen that the coefficient of a term of the form $\prod_{i=1}^{\nu} x_{\ell(i)}$ can be calculated from

$$f(x_1, \dots, x_m) \left| \begin{array}{l} x_{\ell(1)} = x_{\ell(2)} = \dots = x_{\ell(\nu)} = 1 \\ \text{(the remaining variables set to zero)} \end{array} \right.$$

$$+ f_0 + \sum_{i=1}^{\nu} f_{\ell(i)} + \sum \text{(coefficients of the terms}$$

of the form $x_{\ell(i)} x_{\ell(j)}$; $i, j = 1, \dots, \nu$; $i \neq j$) + \sum coefficients of the terms of the form $x_{\ell(i)} x_{\ell(j)} x_{\ell(k)}$; $i, j, k = 1, \dots, \nu$ $i \neq j \neq k$) +

(4.34)

Example 4.3 In this example the system considered in Example 4.1 will be linearized. For this purpose, let the symbol 1 denote the presence of a positive pulse and let the symbol 0 denote the presence of a negative pulse (This choice is, of course, arbitrary; one could also choose 0 for a negative pulse and 1 for a negative pulse). Information concerning state transitions is shown in Fig. 4.7.

$x_1(k)$	$x_2(k)$	$x_1(k+1)$	$x_2(k+1)$	$x_3(k+1)$
0	0	0	1	0
0	1	1	0	0
1	0	0	1	0
1	1	1	0	0

Figure 4.7 Truth table for the system of Example 4.1

Let $x_0(k) = 1$ and $x_3(k) = x_1(k)x_2(k)$. In this case, since there are only two variables, $x_1(k)$ and $x_2(k)$, any function $f[x_1(k), x_2(k)]$ can be written in the form:

$$f[x_1(k), x_2(k)] = f_0 + f_1 x_1(k) + f_2 x_2(k) + f_3 x_3(k)$$

where,

$$f_0 = f(0,0) ,$$

$$f_1 = f(0,0) + f(1,0),$$

$$f_2 = f(0,0) + f(0,1)$$

and

$$f_3 = f(0,0) + f(1,0) + f(0,1) + f(1,1)$$

Thus,

$$\begin{aligned}x_1(k+1) &= 0 + (0+0)x_1(k) + (0+1)x_2(k) + (0+0+1+1)x_3(k) \\ &= x_2(k),\end{aligned}$$

$$\begin{aligned}x_2(k+1) &= 1 + (1+1)x_1(k) + (1+0)x_2(k) + (1+1+0+0)x_3(k) \\ &= 1+x_2(k)\end{aligned}$$

and

$$x_3(k+1) = 0,$$

or in matrix form:

$$\underline{x}(k+1) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \underline{x}(k) = \underline{A} \underline{x}(k) \quad (4.35)$$

(GF(2))

Consider the characteristic polynomial of the matrix

\underline{A} ,

$$p(\lambda) = \begin{vmatrix} \lambda+1 & 0 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 1 & 0 & \lambda+1 & 0 \\ 0 & 0 & 0 & \lambda \end{vmatrix} = \lambda^2(\lambda+1)^2 \quad (\text{GF}(2)) \quad (4.36)$$

In Example 4.1 it was shown that the system had a cycle of period 2 and two transient states. The characteristic polynomial also contains this information; the term $(\lambda+1)^2$ shows that there is a cycle of period 2 and the term λ^2 shows that there are two transient states. This point will be elaborated during the rest of the chapter (see Examples 4.6 and 4.7).

This procedure can easily be implemented using a digital computer. For this purpose, it is convenient to assign an m -dimensional vector \hat{e}_j to each variable x_j of the form $x_j = x_{\ell(1)} x_{\ell(2)} \cdots x_{\ell(i)}$, such that the $\ell(1)$ th, $\ell(2)$ th, ..., $\ell(i)$ th elements of \hat{e}_j are 1 while the rest of the elements are zero. Let $\text{index}(\hat{e}_j)$ denote the number of nonzero elements of \hat{e}_j . It is not difficult to generate vectors \hat{e}_j with increasing index. To determine the coefficient of the term x_j in the expansion of a function $f(\underline{x}) = f(x_1, x_2, \dots, x_m)$, simply sum (mod 2) all the coefficients corresponding to each \hat{e}_i , where $\text{index}(\hat{e}_i) < \text{index}(\hat{e}_j)$ and \hat{e}_i has no nonzero element in positions corresponding to zeroes of \hat{e}_j ; and add $f(\hat{e}_j)$ (mod 2) to the result.

The listing of a Fortran program based on the above procedure is presented in Appendix E. A sample output of this program is given below.

ORIGINAL EQUATION

$$\underline{x}(k+1) = \mu \begin{bmatrix} 4.78 & 3.09 & -1.26 & -2.95 \\ -0.92 & 2.45 & 8.94 & 2.69 \\ 3.55 & 3.46 & 0.27 & 4.68 \\ 2.25 & 1.14 & -2.16 & -0.91 \end{bmatrix} \underline{x}(k)$$

TRANSFORMATIONS USED

$Y_1 = 1$	$Y_9 = X_1 X_4$
$Y_2 = X_1$	$Y_{10} = X_2 X_4$
$Y_3 = X_2$	$Y_{11} = X_3 X_4$
$Y_4 = X_3$	$Y_{12} = X_1 X_2 X_3$
$Y_5 = X_4$	$Y_{13} = X_1 X_2 X_4$
$Y_6 = X_1 X_2$	$Y_{14} = X_1 X_3 X_4$
$Y_7 = X_1 X_3$	$Y_{15} = X_2 X_3 X_4$
$Y_8 = X_2 X_3$	$Y_{16} = X_1 X_2 X_3 X_4$

LINEARIZED EQUATION

$$\underline{y}(k+1) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \underline{y}(k)$$

4.5 Determination of Cycles and Periods of the Linearized Net

Although the linearized net can have 2^{2^n} states only 2^n of these correspond to the original nonlinear net. The states belonging to the original system will be called natural states and the remaining states will be called artificial states. Note that the linearized net is such that a natural state is mapped always into a natural state, though an artificial state could be mapped into either an artificial state or into a natural state. From this fact it follows that a cycle can contain only one type of state, i.e., there cannot be any cycle containing both natural and artificial states. This point will be illustrated in Example 4.6.

To elucidate the concept of natural states, consider a nonlinear net with two variables x_1 and x_2 . Linearization of this net requires introduction of the variables $x_0 = 1$ and $x_3 = x_1 x_2$. Thus, the nonlinear net can assume

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

as its possible states, corresponding to $x_1 = x_2 = 0$, $x_1 = 1, x_2 = 0$; $x_1 = 0, x_2 = 1$ and $x_1 = x_2 = 1$, respectively. In the 4-dimensional vector space over the binary field, the vector $\underline{x} = (x_0, x_1, x_2, x_3)^T$ can also assume

the values

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

These states are the artificial states since they have no correspondence to the states of the nonlinear net.

In this section a summary of the methods for determining cycles and periods of a linear net will be presented. No distinction will be made between natural states and artificial states; this will be delayed until the next section. For generality, the results will be given with respect to the field $GF(p^r)$.

The system considered in this section is assumed to be describable by the following matrix equation:

$$\underline{x}(k+1) = \underline{A} \underline{x}(k) \quad (GF(p^r)) \quad (4.37)^5$$

where $\underline{x}(k)$ is an n -vector denoting the state of the system at time $t = kT$ and \underline{A} is an $n \times n$ matrix with elements from $GF(p^r)$. The operations in eq. (4.37) are performed with respect to the field $GF(p^r)$.

⁵The matrix \underline{A} should not be confused with the notation used in the previous sections.

Let X denote the $(p^r)^n$ -dimensional vector space (with respect to the field $GF(p^r)$) and let \underline{x}_i be any point in X . $\underline{A}\underline{x}_i, \underline{A}^2\underline{x}_i, \dots, \underline{A}^k\underline{x}_i$ also represent vectors from X . Since X is finite dimensional, there exists a $k \leq (p^r)^n$, such that the vector $\underline{A}^k\underline{x}_i$ is a linear combination of the previous terms, $\underline{x}_i, \underline{A}\underline{x}_i, \dots, \underline{A}^{k-1}\underline{x}_i$; i.e., there exist scalar constants c_0, c_1, \dots, c_{k-1} such that

$$\underline{A}^k\underline{x}_i + c_{k-1}\underline{A}^{k-1}\underline{x}_i + \dots + c_1\underline{A}\underline{x}_i + c_0\underline{x}_i = 0 \quad (4.38)$$

Defining by $f(\underline{A})$, the matrix polynomial

$$f(\underline{A}) = \underline{A}^k + c_{k-1}\underline{A}^{k-1} + \dots + c_1\underline{A} + c_0, \quad (4.39)$$

eq. (4.38) can be written in the following compact form:

$$f(\underline{A})\underline{x}_i = \underline{0} \quad (4.40)$$

There may be more than one polynomial of the form (4.39) satisfying eq. (4.40). The one with the lowest order is called the minimum polynomial of the vector \underline{x}_i (with respect to the matrix \underline{A}). There are polynomials for which (4.40) is true independent of the vector \underline{x}_i . The monic polynomial $m(\underline{A})$ of the lowest order satisfying $m(\underline{A}) = 0$ is said to be the minimum polynomial of \underline{A} . There is a close relationship between a matrix polynomial $f(\underline{A})$ and its regular polynomial $f(\lambda)$, obtained by replacing the matrix \underline{A} with a scalar λ . The minimum polynomial of \underline{A} , $m(\underline{A})$ is the least common multiple of all the minimum poly-

nomials of the vectors from X .

The Cayley-Hamilton theorem states that every square matrix satisfies its own characteristic equation, i.e., let $p(\lambda) = |\lambda I - \underline{A}|$, then $p(\underline{A}) = 0$. Thus, the minimum polynomial of \underline{A} , $m(\lambda)$ is a factor of the characteristic polynomial $p(\lambda)$.

A polynomial $f(\lambda)$ of degree N is called irreducible if no polynomial of degree less than N divides $f(\lambda)$ without a remainder⁶. The least positive integer k such that $f(\lambda)$ divides $\lambda^k - 1$ without remainder is called the exponent of $f(\lambda)$ [denoted by $xpo f(\lambda)$]. The exponent of the minimum polynomial of a matrix A is called exponent of A . It can be shown that any polynomial $f(\lambda)$ of degree N (over $GF(p^r)$) divides the polynomial $\lambda^{(p^r)^N - 1} - 1$ and that $f(\lambda)$ divides $\lambda^k - 1$ if and only if k is a factor of $(p^r)^N - 1$. Therefore, $xpo f(\lambda) \leq (p^r)^N - 1$ and $xpo f(\lambda) = \text{factor of } (p^r)^N - 1$. If $xpo f(\lambda) = (p^r)^N - 1$, $f(\lambda)$ is called primitive.

Let k_i ($i=1,2,\dots,\mu$) be the distinct periods that a net can exhibit and let v_i denote the number of cycles with period k_i . It is convenient to denote the cycle structure by

$$c_{k_1}^{v_1} c_{k_2}^{v_2} \dots c_{k_\mu}^{v_\mu} \quad (4.41)$$

⁶ See Peterson and Weldon (115, pp. 472-492) for a complete list of irreducible polynomial over $GF(2)$ of degree ≤ 36 .

In general, it is possible to partition the transition matrix A and determine the cycle structure from the cycle structure of the subsystems resulting from the partition. For this purpose, an understanding of some basic simpler structures is essential. In a linear net, the cycle structure is closely associated with the minimum polynomial of its transition matrix. If, for example, the minimum polynomial is both irreducible, primitive and is equal to the characteristic polynomial; except for the zero state, all the states form a single cycle. Another important case is when the characteristic polynomial is of the form $(n(\lambda))^{\ell}$ where $n(\lambda)$ is an irreducible polynomial. These cases are discussed next.

Case 1: $m(\lambda) = p(\lambda)$ is irreducible and primitive

In this case there will be exactly two cycles; one is the trivial cycle formed by the zero state $(0,0,\dots,0)^T$, which is present in every autonomous linear net. All the remaining states form the other cycle. There are no transient states. The period of the nontrivial cycle is $k_1 = (p^r)^n - 1$. Thus, the cycle structure is $C_1 C_{p^{rn}-1}$.

Example 4.4 Let $m(\lambda) = p(\lambda) = \lambda^4 + \lambda + 1$. This polynomial is primitive since it does not divide $\lambda^k - 1$ for $k < 2^4 - 1 = 15$; it corresponds to the matrix

$$\underline{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix} \quad (4.42)$$

This matrix is called the companion matrix of $m(\lambda)$.
 with $m(\lambda) = p(\lambda) = \lambda^4 + \lambda + 1$ has a trivial cycle formed
 by the zero state and a single nontrivial cycle formed by
 the remaining states, as shown in Fig. 4.8.

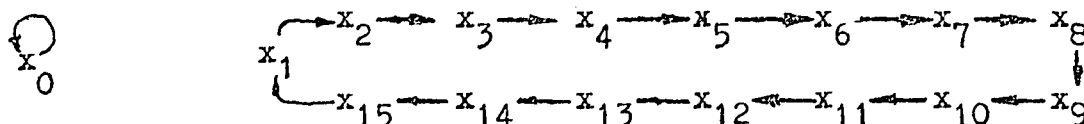


Figure 4.8 State transition diagram of an autonomous linear net with a primitive minimum polynomial ($m(\lambda) = p(\lambda) = \lambda^4 + \lambda + 1$).

Case 2: $m(\lambda) = p(\lambda)$ is irreducible

In this case, again there will not be any transient state⁷. Let k_1 be the smallest integer such that the minimum polynomial $m(\lambda)$ divides the polynomial $\lambda^{k_1} - 1$ without a remainder, i.e., let $k_1 = \text{xpo } m(\lambda)$. Then, for any nonzero state \underline{x} ,

$$[\underline{A}^{(p^r)^{n-1}} - \underline{I}] \underline{x} = m(\underline{A})[\underline{A}^{k_1} - \underline{I}] \underline{x} = \underline{0}$$

⁷The number of transient states is related to the factor λ^{n_t} of the characteristic polynomial. Thus, $n_t = 0$ if and only if $|\underline{A}| = 0$. This point will be discussed in Section 4.7.

or,

$$\underline{A}^{k_1} \underline{x} = \underline{x}$$

Therefore, there will be $\mathcal{V}_1 = \frac{p^{rn}-1}{k_1}$ nontrivial cycles of period k_1 . In symbolic form, the cycle structure is $C_1 C_{k_1}^{\frac{p^{rn}-1}{k_1}}$. Note that the period k_1 must be a factor of the integer $p^{rn}-1$. This is important since it restricts the allowable values of the period (See Table 4.1. For example, for $n = 10$, k_1 can only be 11 or 31, but for $n = 13$, it is 8191).

TABLE 4.1

Prime Factors of 2^n-1 .

<u>n</u>	<u>Prime Factors</u>	<u>n</u>	<u>Prime Factors</u>
3	7	11	23x89
4	3x5	12	3x3x5x7x13
5	31	13	8191
6	3x3x7	14	3x43x127
7	127	15	7x31x127
8	3x5x17	16	3x5x17x257
9	7x73	17	134071
10	3x11x31	18	3x3x3x7x19x73

In order to find the cycles, one can start with any nonzero state \underline{x} , and determine the k_1 states $\underline{A} \underline{x}$, $\underline{A}^2 \underline{x}$, $\underline{A}^3 \underline{x}$, ..., $\underline{A}^{k_1} \underline{x} = \underline{x}$. Selecting a state \underline{x} which does not belong to this cycle, another cycle can be obtained.

Repeating this process, all the cycles of the net can be determined.

Example 4.5 Let $m(\lambda) = p(\lambda) = \lambda^4 + \lambda^3 + \lambda^2 + \lambda + 1$. Note that $2^4 - 1 = 15 = 3 \times 5$. Therefore, this polynomial might divide $\lambda^5 + 1$. It is simple to see that it indeed divides $\lambda^5 + 1$. Thus, a net having this minimum polynomial will have $\frac{15}{5} = 3$ nontrivial cycles of period 5, as shown in Fig. 4.9.

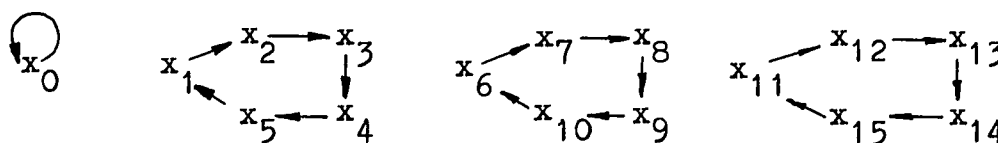


Figure 4.9 State transition diagram of an autonomous linear net with an irreducible minimum polynomial $[m(\lambda) = \lambda^4 + \lambda^3 + \lambda^2 + \lambda + 1]$.

Case 3: $m(\lambda) = p(\lambda) = [n(\lambda)]^\ell$, where $n(\lambda)$ is irreducible

Assume that the minimum polynomial is $m(\lambda) = p(\lambda) = [n(\lambda)]^\ell$, where $n(\lambda)$ is an irreducible polynomial of degree n_0 . In this case p^{rn_0-1} states will have $n(\lambda)$ as their minimum polynomial, $p^{2rn_0} - p^{rn_0}$ states $[n(\lambda)]^2$, ..., and the remaining $p^{\ell rn_0} - p^{(\ell-1)rn_0}$ states will be associated with the minimum polynomial $[n(\lambda)]^\ell$. The p^{rn_0-1} states associated with the minimum polynomial $n(\lambda)$ will form $v_1 = \frac{p^{rn_0}-1}{k_1}$ cycles of period $k_1 = \text{exp } n(\lambda)$, the

states corresponding to the minimum polynomial $[n(\lambda)]^2$ will form $\frac{p^{2rn_0-1}}{k_2}$ cycles of period $k_2 = \text{xpo}[n(\lambda)]^2 = p \cdot k_1$. In general, the states associated with the minimum polynomial $[n(\lambda)]^i$ ($i = 1, \dots, \ell$) will form $\nu_i = \frac{p^{irn_0 - p^{(i-1)rn_0}}}{k_i}$ cycles of period $k_i = \text{xpo}[n(\lambda)]^i = \nu p \cdot k_1$, where ν is the smallest integer such that $\nu p \geq i$. Therefore, the cycle structure will be

$$C_1 C_{k_1}^{\frac{rn_0}{p} - 1} C_{k_2}^{\frac{2rn_0}{p} - p} \dots C_k^{\frac{\ell rn_0}{p} - p} C_{k_\ell}^{(\ell-1)rn_0} \quad (4.43)$$

Because of its usefulness in computation of the cycle structures of more complex nets, the above results are summarized by the following theorem.

Theorem 4.3 Let the minimum polynomial of A , $m(\lambda)$ be of the form $m(\lambda) = [n(\lambda)]^\ell$ (over $\text{GF}(p^r)$) where, $n(\lambda)$ is an irreducible polynomial of degree n_0 . The cycle structure of the system described by eq. (4.35) will be given by (4.43), where $k_i = \text{xpo}[n(\lambda)]^i$ ($i = 1, \dots, \ell$). Furthermore, $k_i = \nu p \cdot k_1$, where ν is the smallest integer, such that $\nu p \geq i$.

Example 4.6 Let $m(\lambda) = (\lambda+1)^2$. In this case $n_0 = 1$ and $\ell = 2$. Therefore, the period of the first nontrivial cycle corresponding to the minimum polynomial $(\lambda+1)$ is $k_1 = 1$; there will be only one such cycle since $\nu_1 = \frac{2-1}{1} = 1$. corresponding the the minimum polynomial $(\lambda+1)^2$, there

will be cycles of period $k_2 = 2k_1 = 2$. The number of cycles of period 2 is $\nu_2 = \frac{2^2 - 2}{2} = 1$. The state transition diagram is shown in Fig. 4.10.

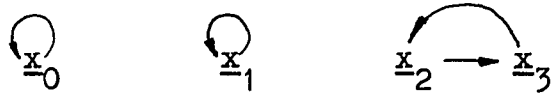


Figure 4.10 State transition diagram of an autonomous linear net with $m(\lambda) = p(\lambda) = (\lambda+1)^2$.

Now, consider again the PFM system treated in Example 4.1. In Example 4.3 the characteristic polynomial of its equivalent linear net was found to be $p(\lambda) = \lambda^2(\lambda+1)^2$. The minimum polynomial is $\lambda(\lambda+1)^2$. The term λ simply indicates that there are transient states but the length of any transient chain is ≤ 1 (i.e., every transient state maps into a cyclic state). The polynomial $(\lambda+1)^2$ was just shown to be associated with two nontrivial cycles of periods 1 and 2. There is also the trivial cycle formed by the zero state. In this particular case, the trivial cycle and the cycle with period 1 are formed by artificial states. Thus, one can easily predict a cycle of period 2. The state diagram of the linearized network is shown in Fig. 4.11.

$$\text{Let } a = \prod_{i=1}^{\mu_a} C_{k_a}(i) \quad \text{and} \quad b = \prod_{j=1}^{\mu_b} C_{k_b}(i) \quad (4.44)$$

$$\underline{x}_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \underline{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \underline{x}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \underline{x}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\underline{x}_4 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \underline{x}_5 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad \underline{x}_6 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \quad \underline{x}_7 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

$$\underline{x}_8 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \quad \underline{x}_9 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \underline{x}_{10} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \underline{x}_{11} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\underline{x}_{12} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad \underline{x}_{13} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \quad \underline{x}_{14} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \quad \underline{x}_{15} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

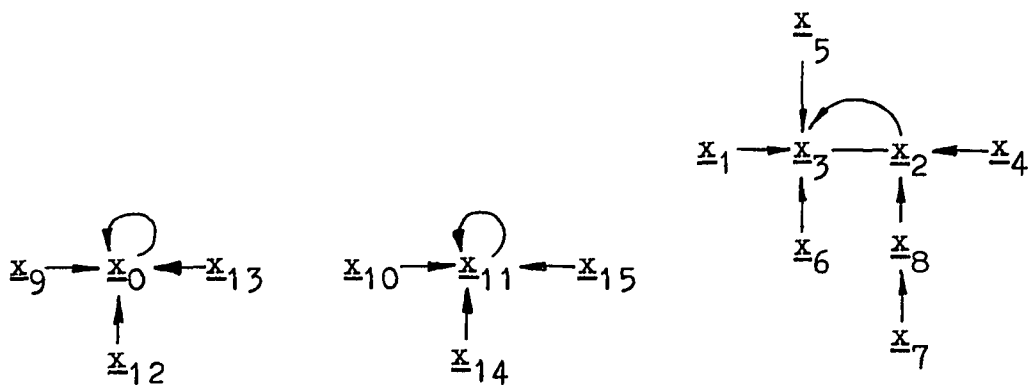


Figure 4.11 State diagram of the equivalent linear net of the PFM system of Example 4.1. Only the states \underline{x}_1 - \underline{x}_4 correspond to the original system; \underline{x}_0 and \underline{x}_5 - \underline{x}_{15} are artificial states.

be two cycle structures and let $a \otimes b$ denote the cycle structure

$$C = \prod_{i=1}^{r_a} \prod_{j=1}^{r_b} C_{\text{LCM}[k_a(i), k_b(j)]}^{v_a(i) v_b(j) \text{GCD}[k_a(i), k_b(i)]} \quad (4.45)$$

The following theorem relates the cycle structure of an autonomous linear net with transition matrix \underline{A} of the form

$$\underline{A} = \underline{A}_1 \oplus \underline{A}_2 = \begin{bmatrix} \underline{A}_1 & \\ & \underline{A}_2 \end{bmatrix} \quad (4.46)^7$$

to the cycle structure a_1, a_2 of its subnetworks with transition matrices A_1 and A_2 , respectively.

Theorem 4.4 [Harrison (62)] The cycle structure of an autonomous linear net whose transition matrix \underline{A} is of the form $\underline{A} = \underline{A}_1 \oplus \underline{A}_2$ is $a = a_1 \otimes a_2$, where \underline{A}_1 and \underline{A}_2 are nonsingular matrices and a_1 and a_2 are the cycle structures of the subnetworks associated with the transition matrices \underline{A}_1 and \underline{A}_2 , respectively.

Proof: Let \underline{x}_1 and \underline{x}_2 denote the states of the net corresponding to the submatrices \underline{A}_1 and \underline{A}_2 , respectively. The period k of any cycle will be the smallest integer such that

⁷ $\underline{A}_1 \oplus \underline{A}_2$ is called the direct sum of the matrices \underline{A}_1 and \underline{A}_2 .

$$\underline{A}^k \underline{x} = \begin{bmatrix} \underline{A}_1^k & \\ & \underline{A}_2^k \end{bmatrix} \begin{bmatrix} \underline{x}_1 \\ \dots \\ \underline{x}_2 \end{bmatrix} = \begin{bmatrix} \underline{x}_1 \\ \dots \\ \underline{x}_2 \end{bmatrix}$$

Let k_1 and k_2 be the smallest integers satisfying $\underline{A}_1^{k_1} \underline{x}_1 = \underline{x}_1$ and $\underline{A}_2^{k_2} \underline{x}_2 = \underline{x}_2$, respectively; then it is $k = \text{LCM}(k_1, k_2)$. If \underline{A}_1 has ν_1 cycles of period k_1 and \underline{A}_2 has ν_2 cycles of period k_2 , then, there will be $\nu_1 \nu_2 \text{GDC}(k_1, k_2)$ cycles of period k_2 .

(QED)

Theorem 4.4 is useful for computing the cycle structures of complex nets. Consider, for example, a transition matrix \underline{A} of the form $\underline{A} = \underline{A}_1 \oplus \underline{A}_2 \oplus \dots \oplus \underline{A}_N$. The cycle structure will be $a = a_1 \boxtimes a_2 \boxtimes \dots \boxtimes a_N$, where a_1, a_2, \dots, a_N are the cycle structure corresponding to the subnetworks with transition matrices $\underline{A}_1, \underline{A}_2, \dots, \underline{A}_N$, respectively. Therefore, the cycle structure of any net can be determined from Theorem 4.3 and Theorem 4.4. For this purpose it is useful to transform the transition matrix \underline{A} into its classical canonical form (13), in which case the polynomials $m_1(\lambda) = |\lambda \underline{I} - \underline{A}_1|$, \dots , $m_N(\lambda) = |\lambda \underline{I} - \underline{A}_N|$ are irreducible polynomials.

4.6 State Diagram and The Transition Matrix

As illustrated in Example 4.1 (Fig. 4.3), the cycle structure of a net can conveniently be represented using

a directed graph in which the vertices represent the possible states of the net. A transition from state \underline{x}_i to state \underline{x}_j is indicated by a directed edge connecting the corresponding vertices. These graphs are called state transition diagrams. If the dimension of the net is large, this graphical presentation loses its convenience. In this case, the same information can be better presented in matrix form.

Corresponding to a state transition diagram, the transition matrix Q is defined by

$$q_{ij} = \begin{cases} 1 & \text{if there is a transition from state } \underline{x}_j \text{ to } \underline{x}_i \\ 0, & \text{otherwise} \end{cases} \quad (4.47)$$

In network theory this matrix is known as the incidence matrix (79). Since there is only one transition from a state into another state, the transition matrix contains exactly one 1 in each column. Let at $t = kT$ the net be at state \underline{x}_i . Let $\underline{y}(k)$ be a 2^n -dimensional column vector containing a 1 at its i th row as its only nonzero element. The behavior of the net can be described by

$$\underline{y}(k+1) = Q \underline{y}(k) \quad (4.48)$$

Since a 1 in the i th column of $\underline{y}(k)$ corresponds to the state \underline{x}_i , it is

$$\underline{x}(k) = [\underline{x}_1; \underline{x}_2; \cdots; \underline{x}_{2^n}] \underline{y}(k) \quad (4.49)$$

Let

$$\underline{P} = [\underline{x}_1 | \underline{x}_2 | \dots | \underline{x}_{2n}] \quad (4.50)$$

\underline{P}^{-1} exists since there is a 1-1 relationship between each \underline{x}_i and \underline{y}_i . Let

$$\underline{x}(k+1) = \underline{A} \underline{x}(k) \quad (4.51)$$

be the linearized equations of the systems considered in Section 4.2. Then, from (4.51) and (4.49), it follows that

$$\underline{Q} = \underline{P}^{-1} \underline{A} \underline{P}, \quad \text{or} \quad \underline{A} = \underline{P} \underline{Q} \underline{P}^{-1} \quad (4.52)^6$$

Consider the characteristic polynomial of the transition matrix \underline{Q} , $p_Q(\lambda) = |\lambda \underline{I} - \underline{Q}|$. All the elements in any row of the matrix $|\lambda \underline{I} - \underline{Q}|$ corresponding to first states are zero, except for the diagonal element which is λ . Thus, the determinant $|\lambda \underline{I} - \underline{Q}|$ can be expanded in terms of the rows corresponding to the first states. Each of these rows will contribute a factor λ to the characteristic polynomial. If a state can be reached only from the eliminated states, there will again be a λ in the diagonal position of the row corresponding to that

⁶This transformation is called a similarity transformation. A very important property of this transformation is the invariance of the characteristic values. Note that

$$p_A(\lambda) = |\lambda \underline{I} - \underline{A}| = |\lambda \underline{I} - \underline{PQP}^{-1}| = |\underline{P}(\lambda \underline{I} - \underline{Q})\underline{P}^{-1}| = |\lambda \underline{I} - \underline{Q}| = p_Q(\lambda).$$

state as the only element. Expanding with respect to these rows and repeating the procedure, it is seen that the characteristic polynomial will have λ^{n_t} as a factor, where n_t is the number of transient (natural) states.

Now, consider a first state \underline{x} . Let ℓ_t be the number of transient states generated by \underline{x} and let k_c be the period of the cycle that \underline{x} enters after ℓ_t transitions (see Fig. 4.12). It follows that

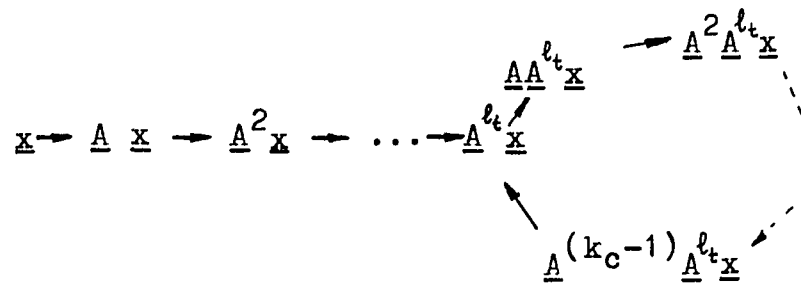


Figure 4.12 Illustration of state transitions.

$$\underline{A}^{\ell_t} [\underline{A}^{k_c} + \underline{I}] \underline{x} = \underline{0} \quad (4.53)$$

Therefore, $\lambda^{\ell_t} (\lambda^{k_c} + 1)$ is the minimum polynomial of \underline{x} .

Since the minimum polynomial of the transition matrix \underline{A} is the least common multiple of the minimum polynomials of the vectors \underline{x} , it will be in the form

$$\lambda^{\ell_t} (\lambda^{k_1} + 1) (\lambda^{k_2} + 1) \dots (\lambda^{k_v} + 1) \quad (4.54)$$

where ℓ_t is the length of the longest transient chain

and k_i ($i = 1, \dots, \nu$) are distinct cycle lengths. Expansion of the minimum polynomial into the form given by (4.54) is not always unique since, in general

$$(a+b)^{p^r} = a^{p^r} + b^{p^r} \quad (4.55)$$

Example 4.8 Consider again the system treated in Example 4.1 (see also Example 4.6). Let

$$\underline{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \underline{x}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \underline{x}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \underline{x}_4 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Then, from (4.50)

$$\underline{P} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The state transition diagram is shown in Fig. 4.13.



Figure 4.13 State transition diagram.

From the state transition diagram, the transition matrix is easily obtained as

$$\underline{y}(k+1) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \underline{y}(k)$$

Note that

$$\underline{x}(k+1) = \underline{PQP}^{-1} \underline{x}(k) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \underline{x}(k) \quad (4.56)$$

(4.56) is the same as (4.35). The characteristic polynomial is $p(\lambda) = \lambda^2(\lambda^2+1)$ and the minimum polynomial is $m(\lambda) = \lambda(\lambda^2+1)$. The factor λ^2 of the characteristic polynomial indicates two transient states. The factor λ of the minimum polynomial shows that these transient states are first states and the factor (λ^2+1) designates a cycle of period 2.

4.7 Conclusions

In this chapter, to gain further insight to periodic behavior of CRPFM systems, a time-discretized approximation was considered. This approximation reduced the system to one containing unit delays and threshold elements. However, except for oscillations having very short periods, it was not possible to obtain analytical results directly from the resulting nonlinear equations.

Since the output of a modulator assumes only a finite number of states (e.g., at a given time, the output of a CRPFM either contains an impulse or not), it was found to be advantageous to consider the system equations with respect to a finite field [GF(2) or GF(3)]. By introdu-

cing extra variables, the system equations were "linearized" (with respect to a finite field) using Fukunaga's method (133) for nonlinear switching nets.

After the linearization, the characteristic equation of the system can be used to obtain information about periodic behavior. For example, a factor λ^{n_t} in the characteristic polynomial means that there are n_t transient states, a factor λ^{l_t} in the minimum polynomial means that the length of the longest transient chain is l_t and factors of the form $(\lambda^{k_i} + 1)$ in the minimum polynomial mean that there are cycles of period(s) k_i ($i = 1, 2, \dots$).

In this chapter the main concentration was given to a system consisting of interconnections of CRPFM's and ideal delays, for which the aforementioned analysis is particularly suitable. For the double-signed system, it was assumed that there are no impulse cancellations. This condition can be relaxed by using the field GF(3).

CHAPTER 5

CONCLUSIONS AND TOPICS FOR FUTURE RESEARCH5.1 Summary of The Results

In this dissertation the dynamic behavior of complete-reset pulse frequency modulation (CRPFM) systems are considered. In Chapter 1, a review of previous work on PFM systems is presented. Also, a brief discussion is given on the neuron and the relation of CRPFM to neural modeling. Chapters 2, 3 and 4 discuss the results of this work which encompasses two basic aspects, namely, stability and oscillatory behavior. The results are summarized at the end of these chapters.

In Chapter 2, two different approaches are presented for global finite-pulse stability (GFPS) (Def. 2.1, p. 56), the first is an improved Lyapunov-like method which is also applicable to more general type of PFM systems (e.g., a PFM system with a nonlinear continuous part); however, its difficulty of application increases with the order of the system. The second approach is a direct approach involving the application of inequalities to the system equations; it is easy to use and, at the same time, provides bounds on the number of pulses emitted by each modulator. Such number is not only indicative of

the energy spent by the corresponding modulator (e.g., in a spacecraft control system which employ controlling jets) but also represents a measure of the degree of stability. A comparison is presented between these stability criteria and previous stability conditions available for special classes of CRPFM systems (e.g., systems with integral PFM or relaxation PFM); in representative examples, the direct GFPS criterion yields comparable (or better) stability regions (of system parameters) with respect to the other criteria.

In Chapter 3, oscillatory motion is considered. A matrix relationship is presented for IPFM systems with time-invariant LP's, which relates the period of oscillation to the net number of pulses emitted by each modulator over that period. This relation shows that, though pure periodic motion is possible in single-modulator systems, in multi-modulator systems, it can exist only in the "ideal" case when all the components of a certain vector of system parameters are rational numbers. Practically, however, the observed motion may "look like" periodic motion, at least over some observation interval. Thus, it can be considered "periodic" within a certain (measurement and/or observation) accuracy. This consideration lead to the definition of the concept of " ϵ_e -nearly periodic motion" and to the development of

expressions for bounds on the deviation of the true motion from periodic motion.

Oscillatory behavior is further studied, in Chapter 4, on a time discretized approximation of the CRPFM system. This approximation reduces the system to one containing unit delays and threshold elements. However, it is still difficult to obtain analytical results directly from the resulting (nonlinear) equations, though information concerning short cycles have been obtained. Since the output of a modulator assumes only a finite number of states (e.g., at a given time, the output of a CRPFM either contains a pulse or not), it is advantageous to consider the system equations with respect to a finite field $[GF(p^r)]$. By introducing extra variables, the system equations are "linearized" (with respect to a finite field) using Fukunaga's method for nonlinear switching nets. After the linearization, the characteristic equation of the system is used to obtain information about periodic behavior in terms of possible frequencies of oscillation.

5.2 Suggestions For Future Research

Below, several problems arising from the present work are stated.

1) The applicability of the sufficient conditions for global finite-pulse stability (GFPS), developed in Section 2.3, can be extended and improved by transforming the system. This has been illustrated in examples (Examples 2.2 and 2.4). However, there remains the development of general rules for this transformation as well as methods for optimization of such transformations to yield maximum parameter regions sufficient for stability.

2) The GFPS condition presented in Section 2.3 include Condition 3 on p. 65, which allows the TF to be nonlinear, but imposes certain restriction of the nonlinearities. It would be desirable to relax these restrictions to allow the TF to contain such nonlinearities as dead-zones and/or hysteresis.

3) A Lyapunov-like theorem for GFPS is presented in Section 2.2, which requires less restrictive conditions than previous Lyapunov-like methods used for these systems. This theorem was applied to individual examples. However, the application of this approach to a CRPFM system in its general form could possibly yield new stability criteria directly in terms of system parameters.

4) The definition of near periodicity introduced in Section 3.2 may be modified by associating a linear functional with the modulator input vector $\underline{e}(t)$ (e.g.,

some integral). This is especially meaningful for cases where $\underline{e}(t)$ has discontinuities.

5) The problem considered in Section 3.4 can be reversed to that of the determination of the period for given accuracy ϵ_e and observation interval $(0, a]$ such that the motion is ϵ_e -n.p. Iterative methods could be used to attack this difficult problem, which may not always have a solution (e.g., the motion may not be periodic, in which case the iteration will not converge).

6) In Section 4.4, the system equations describing a CRPFM system with ideal delays are linearized by introduction of extra variables. In certain cases it might be possible to minimize the number of variables necessary. This point needs further research. Another interesting problem is the determination of an optimal reverse transformation, with which a switching network can be transformed into a threshold type network, such that the number of threshold devices are minimized.

7) Demodulation of a PFM signal is usually accomplished by passing the signal through a linear (lowpass) filter. It might be possible to obtain a better performance (in terms of signal-to-noise ratio) from a filter (of the same or smaller order) where certain states are reset upon arrival of a signal impulse. Determination

of the optimal or suboptimal demodulation filter is (at present) an unsolved problem of considerable practical importance.

8) Investigation of the dynamic behavior of randomly connected large-scale CRPFM systems from the macroscopic point of view might lead to certain physiological results. This investigation can be carried out by defining certain macroscopic variables (e.g., a sum formed by the TF outputs) and using the law of large numbers to find the mean values of these variables.

9) The scope of this work has been limited to complete-reset PFM; it would be desired to consider Partial Reset PFM (see pp. 20-21) which has not been studied previously (except for the special case of output-reset PFM in which only a single state is reset).

APPENDIX A

PROOF OF THEOREM 2.2

In this appendix, matrix inequality (2.14), which provides bounds on the number of impulses emitted by each modulator, will be derived.

Using eq. (1.24b) in eq. (1.24a), applying Conditions 2 and 3 of Section 2.3 and (1.24c), the following inequality is obtained:

$$S_i \leq |z_i(t_{i,K_i}(t))| = \left| \int_{t_{i,K_i}(t)-1}^{t_{i,K_i}(t)} f_i[r_i(\tau), y_i(\tau), t_{i,K_i}(t), \tau] d\tau \right|$$

$$\int_{t_{i,K_i}(t)-1}^{t_{i,K_i}(t)} \left\{ \alpha_i |r_i(\tau)| + \beta_i [|y_{0i}(\tau)| + \sum_{j=1}^m \sum_{k=1}^{K_j(t)} |M_j g'_{ij}(\tau - t_{j,k})|] \right\} d\tau$$

Summing the above inequalities for all intervals

$$[t_{i,0}, t_{i,1}], [t_{i,1}, t_{i,2}], \dots, [t_{i,K_i}(t)-1, t_{i,K_i}(t)]$$

assuming $t_{i,0} = 0$ and using the inequality

$$\left| \int [f_1(t) + f_2(t)] dt \right| \leq \int |f_1(t)| dt + \int |f_2(t)| dt$$

yields

$$S_{iK_i}(t) \leq \int_0^{t_{i,K_i}(t)} \left[\alpha_i |r_i(\tau)| + \beta_i |y_{0i}(\tau)| + \beta_i \sum_{j=1}^m \sum_{k=1}^{K_j(t)} |M_j g'_{ij}(\tau - t_{j,k})| \right] d\tau$$

(A.1)

Recognizing that $t_{i,K_i(t)} < t < t_{i,K_i(t)+1}$, the upper limit of the integrals in inequality (A.1) can be extended from $t_{i,K_i(t)}$ to t . Considering also that

$$\sum_{k=1}^{K_j(t)} \int_0^t |M_j g'_{ij}(\tau - t_{j,k})| d\tau \leq K_j(t) \int_0^t |M_j g'_{ij}(\tau)| d\tau,$$

and dividing both sides of inequality (A.1) by S_i yields:

$$K_i(t) \leq v_i(t) + \sum_{i=1}^m K_j(t) h'_{ij}(t) \quad (\text{A.2})$$

where $v_i(t)$ and $h'_{ij}(t)$ are defined in eqs. (2.11) and (2.13), respectively. In vector form, inequality (A.2) becomes

$$\underline{k}(t) \leq \underline{v}(t) + \underline{H}'(t) \underline{k}(t)$$

from which inequality (2.14) is obtained.

APPENDIX BPROOF OF THEOREM 3.3

Consider the input vector $\underline{e}(t)$ to the modulator block in the interval $t \in ([i-1]T, iT]$.

$$\|\underline{e}(t+T) - \underline{e}(t)\| \leq \|\underline{r}(t+T) - \underline{r}(t)\| + \|\underline{y}(t+T) - \underline{y}(t)\| \quad (\text{B.1})$$

From (3.8a), it follows that $\|\underline{r}(t+T) - \underline{r}(t)\| \leq \epsilon_r$. Therefore,

$$\|\underline{e}(t+T) - \underline{e}(t)\| \leq \epsilon_r + \|\underline{y}(t+T) - \underline{y}(t)\| \quad (\text{B.2})$$

(3.8b) yields

$$\|\underline{y}_1(t+T) - \underline{y}_0(t)\| \leq \epsilon_0 \quad (\text{B.3})$$

From (3.9) and (B.3), it follows that

$$\|\underline{\xi}_1(t+T) - \underline{\xi}_0(t)\| \leq \epsilon_0 \sigma \quad (\text{B.4})$$

Therefore, using (3.7b),

$$\begin{aligned} \|\underline{y}_2(t+T) - \underline{y}_1(t)\| &= \|\underline{y}_1(t+T) + \underline{\xi}_1(t+T) - \underline{y}_0(t) - \underline{\xi}_0(t)\| \\ &\leq \|\underline{y}_1(t+T) - \underline{y}_0(t)\| + \|\underline{\xi}_1(t+T) - \underline{\xi}_0(t)\| \\ &\leq \epsilon_0(1+\sigma) \end{aligned} \quad (\text{B.5})$$

During the interval $t \in (0, T]$, it is $\underline{y}_2(t+T) = \underline{y}(t+T)$ and $\underline{y}_1(t) = \underline{y}(t)$. Therefore,

$$\|\underline{y}(t+T) - \underline{y}(t)\| \leq \epsilon_0(1+\sigma), \quad t \in (0, T] \quad (\text{B.6})$$

Because of (3.9), inequality (B.5) implies

$$\|\underline{z}_2(t+T) - \underline{z}_1(t)\| \leq \epsilon_0(1+\sigma)\sigma \quad (\text{B.7})$$

Repeating the steps in (B.5) and using (B.5) and (B.7) yields:

$$\begin{aligned} \|\underline{y}_3(t+T) - \underline{y}_2(t)\| &\leq \|\underline{y}_2(t+T) - \underline{y}_1(t)\| + \|\underline{z}_2(t+T) - \underline{z}_1(t)\| \\ &\leq \epsilon_0(1+\sigma)^2 \end{aligned} \quad (\text{B.8})$$

Recognizing that during the interval $t \in (T, 2T]$, $\underline{y}_3(t+T) = \underline{y}(t+T)$, and $\underline{y}_2(t) = \underline{y}(t)$, inequality (B.8) gives

$$\|\underline{y}(t+T) - \underline{y}(t)\| \leq (1+\sigma)^2 \epsilon_0, \quad t \in (T, 2T] \quad (\text{B.9})$$

Repeating the previous steps for each consecutive interval yields

$$\|\underline{y}(t+T) - \underline{y}(t)\| \leq (1+\sigma)^n \epsilon_0, \quad t \in ([n-1]T, nT] \quad (\text{B.10})$$

Therefore, from (B.2) and (B.10),

$$\begin{aligned} \|\underline{e}(t+T) - \underline{e}(t)\| &\leq \epsilon_r + (1+\sigma)^n \epsilon_0, \quad t \in ([n-1]T, nT] \quad (\text{B.11}) \\ &(n = 1, 2, \dots) \end{aligned}$$

In the given observation interval $t \in (0, a]$, one can select an $\epsilon_e = \epsilon_r + K\epsilon_0$, such that

$$\|\underline{e}(t+T) - \underline{e}(t)\| \leq \epsilon_e, \quad t \in (0, a]$$

where, $K = (1+\sigma)^{\lceil a/T \rceil + 1}$. Clearly, with this value of ϵ_e Def. 3.1 is satisfied. Therefore, the CRPFM system of Fig. 3.1 is ϵ_e -n.p in the observation interval $(0, a]$.

APPENDIX CPROOF OF THEOREM 3.4

Let v_j denote the number of pulses emitted by the j th modulator in the interval $(0, T]$, and let

$$v = \sum_{j=1}^m v_j \quad (C.1)$$

Then, from (1.24a), (3.6) and (3.7b), it follows that

$$\underline{\xi}_0(t) = \sum_{j=1}^m M_{l_j} b_j \underline{g}_{l_j}(t-t_j) = \sum_{i=1}^m \sum_{j=1}^i M_i b_{ij} \underline{g}_i(t-t_{ij}) \quad (C.2)$$

The output vector is given by

$$\underline{y}(t) = \underline{y}_0(t) + \sum_{k=0}^{n-1} \underline{\xi}_k(t), \quad t \in (0, T] \quad (C.3)$$

Consider the integral of the input vector of the modulator block, $\underline{e}(t)$ over a period; it is

$$\int_{(n-1)T}^{nT} \underline{e}(t) dt = \underline{S} \underline{q} \quad (C.4a)$$

where, \underline{S} is the $m \times m$ diagonal matrix

$$\underline{S} = \text{diag}[S_i] \quad (C.4b)$$

and \underline{q} is an m -dimensional column vector as defined by (3.35c).

Now, let

$$\underline{y}'(t) \triangleq \sum_{k=1}^{n-1} \underline{z}_k(t) - \underline{z}_0(t-kT) \quad (\text{C.5})$$

and,

$$\underline{v}_n \triangleq \int_{(n-1)T}^{nT} \underline{r}(t) dt - \int_0^T \underline{r}(t) dt \quad (\text{C.6})$$

Also, let

$$\underline{z}(t) \triangleq \int \underline{z}_0(t) dt, \quad (\text{C.7})$$

and,

$$\underline{v}_n \triangleq \int_{(n-1)T}^{nT} [\underline{y}'(t) + \underline{y}_0(t)] dt + \underline{z}(nT) + \underline{v}_n \quad (\text{C.8})$$

Noting that $\underline{e}(t) = \underline{r}(t) + \underline{y}(t)$ and that

$$\int_{(n-1)T}^{nT} \sum_{n=0}^{n-1} \underline{z}_0(t-nT) dt = \underline{z}(nT) - \underline{z}(0),$$

and using (C.5)-(C.8) in (C.3) yields

$$\underline{r}_0 T - \underline{z}(0) + \underline{v}_n = \underline{s} \underline{q} \quad (\text{C.9})$$

where,

$$\underline{r}_0 \triangleq \int_0^T \underline{r}(t) dt.$$

Now, consider the term \underline{v}_n . In Appendix B it was shown that

$$\|\underline{z}_{n+1}(t+T) - \underline{z}_n(t)\| \leq (1+\sigma)^n \sigma \epsilon_0 \text{ [generalization of (B.7)]}$$

Therefore,

$$\begin{aligned}
\|\underline{x}_n(t) - \underline{x}_0(t-nT)\| &\leq \|\underline{x}_n(t) - \underline{x}_{n-1}(t-T)\| + \|\underline{x}_{n-1}(t-T) - \underline{x}_{n-2}(t-2T)\| \\
&\quad + \dots + \|\underline{x}_1(t-(n-1)T) - \underline{x}_0(t-nT)\| \\
&= \epsilon_0 (1+\sigma)^{n-1} \tag{C.10}
\end{aligned}$$

Hence, from (C.5) and (C.10), one obtains

$$\|\underline{y}'(t)\| \leq \epsilon_0 [(1+\sigma)^n/\sigma - n], \quad t \in (0, T]. \tag{C.11}$$

Since, $\|\underline{x}(t+T) - \underline{x}(t)\| \leq \epsilon_r$, (C.6) yields

$$\begin{aligned}
\|\underline{y}_n\| &= \left\| \int_0^T \{\underline{x}[t+(n-1)T] - \underline{x}(t)\} dt \right\| \leq \int_0^T \{\|\underline{x}[t+(n-1)T] \\
&\quad - \underline{x}[t+(n-2)T]\| + \dots + \|\underline{x}(t+T) - \underline{x}(t)\|\} dt \leq \frac{n-1}{T} \epsilon_r \tag{C.12}
\end{aligned}$$

Equations (C.1), (C.7) and the hypothesis of the theorem give

$$\|\underline{z}(t)\| = \left\| \int \left[\sum_{i=1}^m \sum_{j=1}^{j_j} M_i g_i(t-t_{ij}) \right] dt \right\| \leq \frac{B_g}{a_g} e^{-a_g(t-T)} \tag{C.13}$$

From the hypothesis of Theorem 3.4, it also follows that

$$\left\| \int_{(n-1)T}^{nT} \underline{y}_0(t) dt \right\| \leq \frac{B_0}{a_0} \left[e^{-a_0(n-1)T} - e^{-a_0 nT} \right] \tag{C.14}$$

Substitution of (C.10)-(C.14) into (C.8) yield

$$\|\underline{y}_n\| \leq \left[\epsilon_0 \frac{(1+\sigma)^n}{\sigma} + n\epsilon_r \right] T + \frac{B_0}{a_0} e^{-a_0(n-1)T} + \frac{B_g}{a_g} e^{-a_g(n-1)T} \tag{C.15}$$

Let

$$\underline{h}_j(t) = \int M_j \underline{g}_j(t) dt, \quad (j = 1, \dots, m).$$

Then, (C.1) and (C.7) yield:

$$\underline{z}(0) = \sum_{j=1}^m \sum_{k=1}^{j_j} b_{jk} \underline{h}_j(0) \quad (C.16)$$

Let

$$q_j = \sum_{k=1}^{j_j} b_{jk},$$

i.e., the number of positive pulses less the number of negative pulses, emitted by the ith modulator in the interval $(0, T]$. With this substitution (C.16) becomes

$$\underline{z}(0) = \sum_{j=1}^m q_j \underline{h}_j(0). \quad (C.17)$$

Substituting (C. 17) into (C.9), using (3.35a)-(3.35d) and matrix notation, one finally obtains the relation

$$\underline{P} \underline{q} = \underline{T} \underline{r}_0 + \underline{\psi} \quad (C.18)$$

APPENDIX D

COMPUTER PROGRAM FOR THE CALCULATION
OF THE RESPONSE OF CRPFM SYSTEMS

Method

This program calculates the response of the CRPFM system of Fig. 3.1 with finite-dimensional, time-invariant LP and TF's. It is assumed that the combined equations of the TF's are given in the form

$$\dot{\underline{x}}_1(t) = \underline{A}_1 \underline{x}_1(t) + \underline{B}_1 \underline{e}(t) , \quad (D.1)$$

$$\underline{z}(t) = \underline{C}_1 \underline{x}_1(t). \quad (D.2)$$

Also, it is assumed that the LP is described by the equations

$$\dot{\underline{x}}_2(t) = \underline{A}_2 \underline{x}_2(t) + \underline{B}_2 \underline{u}(t) \quad (D.3)$$

$$\underline{y}(t) = \underline{C}_2 \underline{x}_2(t) + \underline{D} \underline{u}(t) \quad (D.4)$$

Description of all the parameters used in the program are given in the listing (presented at the end of the Appendix).

Eqs. (D.1)-(D.4) can be rearranged into the following form:

$$\dot{\underline{x}} = \underline{A} \underline{x} + \underline{B} \underline{u} + \underline{F} \underline{r} \quad (D.5)$$

where,

$$\underline{x} = \begin{bmatrix} \underline{x}_1 \\ \underline{x}_2 \end{bmatrix}, \quad \underline{A} = \begin{bmatrix} \underline{A}_1 & \underline{B}_1 \underline{C}_2 \\ \underline{0} & \underline{A}_2 \end{bmatrix}, \quad (D.5b)$$

$$\underline{B} = \begin{bmatrix} \underline{B}_1 \underline{D}_2 \\ \underline{B}_2 \end{bmatrix}, \quad \text{and} \quad \underline{F} = \begin{bmatrix} \underline{B}_1 \\ \underline{Q} \end{bmatrix}$$

The solution of (D.5) between the firing instances are

$$\underline{x}(t) = e^{\underline{A}(t-t_k)} \underline{x}(t_k^+) + \int_{t_k^+}^t e^{\underline{A}(t-\tau)} \underline{F} \underline{r}(\tau) d\tau \quad (D.6)$$

Let t^* be an instant between the firing times t_k and t_{k+1} . If $\underline{r}(t)$ is approximately constant from time t^* up to time $t^* + \Delta t$, then (D.6) yields

$$\underline{x}(t^* + \Delta t) = e^{\underline{A}\Delta t} \underline{x}(t^*) + \underline{Q} \underline{r}(t^*) \quad (D.7)$$

where

$$\underline{Q} \triangleq \int_0^{\Delta t} e^{\underline{A}\tau} \underline{F} d\tau \quad (D.8)$$

If at $t = t_{k+1}$, the ℓ_{k+1} th modulator emits an impulse of polarity b_{k+1} and strength $M \ell_{k+1}$, then the state at $t = t_{k+1}^+$ is given by

$$\underline{x}(t_{k+1}^+) = \underline{x}(t_{k+1}^-) + b_{k+1} M \ell_{k+1} \underline{B}_{k+1} \quad (D.9)$$

Immediately upon emission of the impulse the state of the l_{k+1} th TF is also reset to zero.

The program is based on eqs. (D.7) and (D.9). The system output, $\underline{y}(t)$ and the output vector of the TF's, $\underline{z}(t)$ are evaluated using eqs. (D.4) and (D.2) at time instances $\Delta t, 2\Delta t, 3\Delta t, \dots$. Each time the outputs of the TF's are compared with their threshold values and impulse emissions are decided in accordance with eq. (1.24c). The details of the program are explained by comment cards in the listing.

Input Data

Card	Quantities	Format
1	M, N1, N2, NE, NSS, MLOT, T(1), TIMAX, D10 (See the program listing for a description of the parameters).	(6I3, 3E15.8)
2	Matrix \underline{A}_1 (row by row)	(8F10.4)
	Matrix \underline{B}_1	(8F10.4)
	Matrix \underline{C}_1	(8F10.4)
	S_i, M_i ($i = 1, \dots, m$)	(8F10.4)
	Matrix \underline{A}_2	(8F10.4)
	Matrix \underline{B}_2	(8F10.4)
	Matrix \underline{C}_2	(8F10.4)
	Matrix \underline{D}	(8F10.4)
	$\underline{x}(0)$ (initial state)	(8F10.4)

Output

1. Input data: $M, N_1, N_2, NE, NS, T(1), TIMAX, D10,$
Matrices $\underline{A}_1, \underline{B}_1, \underline{C}_1, \underline{A}_2, \underline{B}_2, \underline{C}_2, \underline{D}; S_i, M_i$ ($i = 1, 2, \dots, m$); $\underline{x}(0)$ (initial state); ...
2. The augmented fundamental matrix \underline{A} , Associated states to be zeroed immediately after each firing, the matrix \underline{B} , the matrices $e^{\underline{A}T}$ and $\underline{Q}(T)$ (T is the main discretization interval)
3. A plot of the first five states if $MLOT = 0$.
4. t_k, b_k, l_k (pulse emission instant, polarity of the pulse and the number of modulator firing)
5. $\underline{x}(t_k^+)$ (the state immediately after an impulse emission).

C	PROGRAM	PFM2	1	
C	PURPOSE	PFM2	2	
C		PFM2	3	
C		PFM2	4	
C		PFM2	5	
C		PFM2	6	
C		PFM2	7	
C		PFM2	8	
C		PFM2	9	
C		PFM2	10	
C		PFM2	11	
C		PFM2	12	
C		PFM2	13	
C		PFM2	14	
C		PFM2	15	
C		PFM2	16	
C		PFM2	17	
C		PFM2	18	
C		PFM2	19	
C		PFM2	20	
C		PFM2	21	
C		PFM2	22	
C		PFM2	23	
C		PFM2	24	
C		PFM2	25	
C		PFM2	26	
C		PFM2	27	
C		PFM2	28	
C		PFM2	29	
C		PFM2	30	
C		PFM2	31	
C		PFM2	32	
C		PFM2	33	
C		PFM2	34	
C		PFM2	35	

TO CALCULATE THE RESPONSE OF A SYSTEM CONSISTING OF
 INTERCONNECTIONS OF PULSE-FREQUENCY MODULATORS AND LINEAR
 TIME-INVARIANT SUBSYSTEMS.

DESCRIPTION OF PARAMETERS

INPUT

M -- NUMBER OF MODULATORS.
 N1 -- DIMENSION OF THE MODULATOR BLOCK.
 N2 -- DIMENSION OF THE LINEAR PART.
 NE -- NUMBER OF INPUTS.
 NSS -- NUMBER OF SINGLE-SIGNED MODULATORS IN THE SYSTEM.
 (MODULATORS 1-NSS ARE SINGLE-SIGNED, REMAINING
 MODULATORS ARE DOUBLE-SIGNED-NSSKM).
 MLNT -- AN INPUT PARAMETER WHICH WHEN SET TO ZERO, THE
 PROGRAM ALSO GENERATES A PLOT OF THE STATES.
 T -- MAIN DISCRETIZATION INTERVAL.
 TIMAX -- SOLUTION INTERVAL.
 DIO -- DESIRED ACCURACY PERCENTAGE.
 A -- AUGMENTED FUNDAMENTAL MATRIX (CONTAINS THE FUNDAMENTAL
 MATRICES, A1 AND A2 OF THE MODULATORS AND THE LINEAR
 PART, RESPECTIVELY).
 MODULATOR BLOCK AND THE LINEAR PART, RESPECTIVELY).
 B1 -- INPUT COUPLING MATRIX OF THE MODULATOR BLOCK.
 C1 -- STATE COUPLING MATRIX OF THE MODULATOR BLOCK.
 S -- VECTOR OF THRESHOLD VALUES.
 AM -- VECTOR CONTAINING THE IMPULSE STRENGTHS.
 B -- AUGMENTED INPUT COUPLING MATRIX (CONTAINS B1*02 AND
 B2).
 C2 -- STATE COUPLING MATRIX OF THE LINEAR PART.
 D2 -- INPUT COUPLING MATRIX OF THE LINEAR PART.
 X -- VECTOR OF INITIAL STATES, LATER, THE AUGMENTED STATE

C C VECTOR. 36 PFM2

C C INPUT VECTOR. 37 PFM2

C C 38 PFM2

C C 39 PFM2

C C 40 PFM2

C C 41 AN AUXILIARY STORAGE MATRIX EVALUATED AT TIME=TPFM2

C C 42 AUGMENTED STATE TRANSITION MATRIX EVALUATED AT TIME=TPFM2

C C 43 T/2, T/J, ..., T/2**10. PFM2

C C 44 VECTOR OF IMPULSE STRENGTHS (SIGNED). PFM2

C C 45 VECTOR IDENTIFYING THE STATES TO BE ZEROED AFTER EACH PFM2

C C 46 IMPULSE EMISSION. PFM2

C C 47 NUMBER OF IMPULSES EMITTED. PFM2

C C 48 VECTOR FORMED BY THE MODULATOR'S FIRING AT THE SAME TIME PFM2

C C 49 INTEGRAL OF THE MATRIX EXP(AI*T)*B1, FROM T=0 TO T=T. PFM2

C C 50 INTEGRAL OF THE MATRIX EXP(AI*T)*B1, FROM T=0 TO T=T1. PFM2

C C 51 OUTPUT VECTOR OF THE MODULATOR TIMING FILTERS. PFM2

C C 52 MATRIX USED TO STORE DATA TO BE PLOTTED. PFM2

C C 53 PFM2

C C 54 PFM2

C C 55 PFM2

C C 56 PFM2

C C 57 PFM2

C C 58 PFM2

C C 59 PFM2

C C 60 PFM2

C C 61 PFM2

C C 62 PFM2

C C 63 PFM2

C C 64 PFM2

C C 65 PFM2

C C 66 PFM2

C C 67 PFM2

C C 68 PFM2

C C 69 PFM2

C C 70 PFM2

INTERNAL

DUMMY - AN AUXILIARY STORAGE MATRIX.

EAT - AUGMENTED STATE TRANSITION MATRIX EVALUATED AT TIME=TPFM2

EPS - VECTOR OF IMPULSE STRENGTHS (SIGNED).

IZFRDX - VECTOR IDENTIFYING THE STATES TO BE ZEROED AFTER EACH IMPULSE EMISSION.

JJ - NUMBER OF IMPULSES EMITTED.

L - VECTOR FORMED BY THE MODULATOR'S FIRING AT THE SAME TIME

Q - INTEGRAL OF THE MATRIX EXP(AI*T)*B1, FROM T=0 TO T=T.

Q1 - INTEGRAL OF THE MATRIX EXP(AI*T)*B1, FROM T=0 TO T=T1.

Z - OUTPUT VECTOR OF THE MODULATOR TIMING FILTERS.

ZZ - MATRIX USED TO STORE DATA TO BE PLOTTED.

DECISION

INDEX - AN INTERNAL PARAMETER WHICH IS SET TO 1 IF ALL THE OUTPUTS OF THE TIMING FILTERS ARE SMALLER THAN SMINUS, PFM2 TO 3 IF ONE OUTPUT EXCEEDS SPLUS AND TO 2, OTHERWISE. PFM2

INDEX1 - MAXIMUM NUMBER OF BISECTIONS OF THE MAIN INTERVAL PFM2

INDEX2 - CURRENT MAXIMUM NUMBER OF BISECTIONS OF THE MAIN INTERVAL. PFM2

INDEX3 - CURRENT NUMBER OF BISECTIONS USED. PFM2

IPLOT - NUMBER OF STATE VARIABLES TO BE PLOTTED (FIRST 5 STATES ARE PLOTTED IF MLOT=0, ONLY N STATES IF NK5). PFM2

KPLOT - NUMBER OF POINTS TO BE PLOTTED. WHEN KPLOT REACHES 101 PFM2 PLOT IS GENERATED. PFM2

REMARKS

THE PROCEDURE TERMINATES IF

(1) MORE THAN (INDEX1) BISECTIONS OF THE MAIN INCREMENT ARE NECESSARY TO GET SATISFACTORY ACCURACY.

```

C          (2) SOLUTION INTERVAL IS EXCEEDED.
C
C          SUBROUTINES AND FUNCTION SUBROUTINES REQUIRED.
C          SURROUTINES EXPAT AND PLOT. IF THE INPUT VECTOR IS NOT
C          CONSTANT, A FUNCTION SUBPROGRAM IS NECESSARY.
C
C          PFM2 71
C          PFM2 72
C          PFM2 73
C          PFM2 74
C          PFM2 75
C          PFM2 76
C          PFM2 77
C          PFM2 78
C          PFM2 79
C          PFM2 80
C          PFM2 81
C          PFM2 82
C          PFM2 83
C          PFM2 84
C          PFM2 85
C          PFM2 86
C          PFM2 87
C          PFM2 88
C          PFM2 89
C          PFM2 90
C          PFM2 91
C          PFM2 92
C          PFM2 93
C          PFM2 94
C          PFM2 95
C          PFM2 96
C          PFM2 97
C          PFM2 98
C          PFM2 99
C          PFM2 100
C          PFM2 101
C          PFM2 102
C          PFM2 103
C          PFM2 104
C          PFM2 105

```

(2) SOLUTION INTERVAL IS EXCEEDED.
 SUBROUTINES AND FUNCTION SUBROUTINES REQUIRED.
 SURROUTINES EXPAT AND PLOT. IF THE INPUT VECTOR IS NOT
 CONSTANT, A FUNCTION SUBPROGRAM IS NECESSARY.

```

*****
DIMENSION C1(4,4),C2(4,4),D2(4,4),S(4),AM(4),R(4),ZZ(5,101),B(8,8)
1,X(8),Z(4),SPLUS(4),SMINUS(4),IZEROX(4,4),IZMAX(4),L(4),EPS(4),T(4)
20)
COMMON N,NJ,NE,A(8,8),B1(4,4),EAT(40,8,8),Q(40,4,4),DUMMY(8,8)
*****
READ AND PRINT THE INPUT PARAMETERS.
INDEX1=40
READ 101,M,N1,N2,NE,NSS,MLOT,T(1),TIMAX,D10
101 FORMAT(6I3,3E15.8)
PRINT 102,M,N1,NE,N2,T(1),TIMAX,D10
102 FORMAT(IH,T11,'NUMBER OF MODULATORS',I,I3,T51,'ORDER OF THE MODP',I3,
//,'ORDER OF THE P',I3,
//,'T11,'NUMBER OF INPUTS',I3,
//,'T11,'ORDER OF THE P',I3,
//,'T11,'DISCRETIZATION INTERVAL',E15.8,T51,'TIP',I3,
//,'T11,'DESIRED ACCURACY PERCENTAGE',E15.8)
PRINT 123, INDEX1
123 FORMAT(IH,/,T11,'MAXIMUM NUMBER OF BISECTIONS OF THE MAIN INTERV',I3,
//)
N=N1+N2
IF(NSS)131,131,132
132 PRINT 134,NSS
134 FORMAT(IH,/,T11,'NUMBER OF SINGLE-SIGNED MODULATORS =',I3)
131 CONTINUE

```



```

106 PRINT 106
    FORMAT(1H ,//,T30,'MATRIX C1',/)
    PRINT 3200,((C1(I,J),J=1,N1),I=1,M)
    PRINT 107
107 FORMAT(1H ,//,T30,'THE FUNDAMENTAL MATRIX OF THE LINEAR PLANT',/)
    PRINT 3201,((A(I,J),J=N11,N),I=N11,N)
3201 FORMAT((10X,4(E15.8,4X)))
    PRINT 108
108 FORMAT(1H ,//, T30,'THE MATRIX B2',)
    PRINT 3200,((B(I,J),J=1,M),I=N11,N)
    PRINT 109
109 FORMAT(1H ,//,T30,'THE MATRIX C2',)
    PRINT 3201,((C2(I,J),J=1,N2),I=1,NE)
    PRINT 110
110 FORMAT(1H ,//,T30,'THE MATRIX D2',)
    PRINT 3200,((D2(I,J),J=1,M),I=1,NE)
    PRINT 111,(I,S(I),AM(I),I=1,M)
111 FORMAT(1H ,//,15X,'THRESHOLD',9X,'PULSE STRENGTH',/,(5X,I3,5X,E15.8,
    18,5X,E15.8))
    PRINT 112,(I,X(I),I=1,N)
112 FORMAT(1H ,//,15X,'INITIAL STATE X(0)',/,(10X,I3,5X,E15.8))
C
C
C
    CONSTRUCT THE A MATRIX
    DO 1 I=1,N1
    DO 1 J=1,N2
    JN1=J+N1
    D=0.0
    DO 2 K=1,NF
    2 D=D+B1(I,K)*C2(K,J)
    A(I,JN1)=D
    1 A(JN1,I)=0.0
    PRINT 113,((A(I,J),J=1,N),I=1,N)
113 FORMAT(1H ,//,T30,'THE AUGMENTED FUNDAMENTAL MATRIX A',/,(10X,4(E15.8,4X))
    PFM2 141
    PFM2 142
    PFM2 143
    PFM2 144
    PFM2 145
    PFM2 146
    PFM2 147
    PFM2 148
    PFM2 149
    PFM2 150
    PFM2 151
    PFM2 152
    PFM2 153
    PFM2 154
    PFM2 155
    PFM2 156
    PFM2 157
    PFM2 158
    PFM2 159
    PFM2 160
    PFM2 161
    PFM2 162
    PFM2 163
    PFM2 164
    PFM2 165
    PFM2 166
    PFM2 167
    PFM2 168
    PFM2 169
    PFM2 170
    PFM2 171
    PFM2 172
    PFM2 173
    PFM2 174
    PFM2 175

```

```

PFM2 176
PFM2 177
PFM2 178
PFM2 179
PFM2 180
PFM2 181
PFM2 182
PFM2 183
PFM2 184
PFM2 185
PFM2 186
PFM2 187
PFM2 188
PFM2 189
PFM2 190
PFM2 191
PFM2 192
PFM2 193
PFM2 194
PFM2 195
PFM2 196
PFM2 197
PFM2 198
PFM2 199
PFM2 200
PFM2 201
PFM2 202
PFM2 203
PFM2 204
PFM2 205
PFM2 206
PFM2 207
PFM2 208
PFM2 209
PFM2 210

PRINT 1141
CALCULATE THE NUMBER OF STATE VARIABLES TO BE ZEROED IMMEDIATELY
AFTER THE ITH MODULATOR FIRES(FROM THE NONZERO ELEMENTS OF C1).
DO 5 I=1,M
  I1=0
  DO 6 J=1,N1
    IF(C1(I,J))7,6,7
  7 I1=I1+1
  IZEROX(I,I1)=J
6 CONTINUE
PRINT 114,I,(IZEROX(I,J),J=1,I1)
  IZMAX(I)=I1
5 CONTINUE
1141 FORMAT(IH,/,/,T10,MODULATOR,I,T30,ASSOCIATED STATES TO BE ZEROED
  1 IMMEDIATELY AFTER EACH FIRING,/)
  114 FORMAT(IH 10X,I3,20X,10I6)

C
C CONSTRUCT THE MATRIX S.
DO 200 I=1,N1
DO 200 J=1,M
  D=0.0
  DO 201 K=1,NE
    DO 201 I,K) #D2(K,J)
  201 D=D+B1(I,K)*D2(K,J)
  200 B(I,J)=D
  115 PRINT 115,((B(I,J),J=1,M),I=1,M)
  115 FORMAT(IH,/,/,T30,THE MATRIX B',/,/(10X,2(E15.8,4X)))
DO 9 I=1,M
  SPLUS(I)=S(I)*(1.+D10)
  9 SMINUS(I)=S(I)*(1.-D10)
C
C *****
C TT=T(1)
C CALL EXPAT(TT,1)

```



```

      PRINT 3000, ((EAT(1,I,J),J=1,N),I=1,N)
      FORMAT(IH,/,/,20X,EXP(A*T),/,/(4(2X,E15.8)))
      PRINT 3002, ((Q(1,I,J),J=1,NE),I=1,N1)
      FORMAT(IH,/,/,20X,Q(T),/,/(2(2X,E15.8)))
C
C *****
C INITIALIZE VARIABLES.
      60 CONTINUE
      PRINT 1593
      1593 FORMAT(IH1,////,5X,INO, OF PULSES EMITTED',9X,TIME',11X,INO, OFPFM2 221
      1 MODULATOR FIRING',10X, PULSE STRENGTH',/) PFM2 222
      1592 FORMAT(IH1,////) PFM2 223
      JJ=1 PFM2 224
      TIME=0.0 PFM2 225
      KPLOT=0 PFM2 226
C PFM2 227
C TIME1 DENOTES THE TIME USED IN THE ITERATION PROCESS. PFM2 228
      10 TIME1=0.0 PFM2 229
      INDEX2=1 PFM2 230
      INDEX3=1 PFM2 231
      11 TIME=TIME+T(1) PFM2 232
      KPLOT=KPLOT+1 PFM2 233
      IF(TIME-TI*MAX)12,12,13 PFM2 234
      12 CONTINUE PFM2 235
      DO 602 I=1,N PFM2 236
      DUMMY(I,1)=0.0 PFM2 237
      DO 602 J=1,N PFM2 238
      602 DUMMY(I,1)=DUMMY(I,1)+EAT(INDEX3,I,J)*X(J) PFM2 239
C PFM2 240
      CALL INPUT(NE,TIME,R) PFM2 241
C PFM2 242
      DO 603 I=1,N1 PFM2 243
      DUMMY(I,2)=0.0 PFM2 244
      DO 603 J=1,NE PFM2 245

```



```

521 EPS(K)=-AM(I)
522 GO TO 501
524 PRINT 524
524 FORMAT(IH ,10X, 'ERROR IN THRESHD')
523 GO TO 13
523 EPS(K)=AM(I)
501 CONTINUE
501 IF(I-N)530,531,531
531 CONTINUE
C
C *****
C END OF THRESHOLD COMPARISONS.
DO 507 I=1,N
507 X(I)=DUMMY(I,I)
569 IF(INDEX3-1)572,571,573
572 PRINT 575
575 FORMAT(IH ,10X, 'ERROR, INDEX3 BECOMES NEGATIVE')
GO TO 13
C
C *****
C ARRANGE DATA FOR THE SUBROUTINE PLOT, IF PLOT IS DESIRED.
571 GO TO (21,22,23),INDEX
21 IF(MLOT)11,880,11
C PLOT THE OUTPUT.
680 IF(KPLOT-101)1111,1111,24
1111 DO 1112 I=1,IPLOT
1112 ZZ(I,KPLOT)=X(I)
GO TO 11
24 KPLOT=0
PRINT 1592
PRINT 1973,(X(I),I=1,N)
1973 FORMAT(IH1,////////,10X,'X=',6(E15.8,5X))
C CALL PLOT(ZZ,IPLOT,101)
PRINT 1593
GO TO 11
PFM2 281
PFM2 282
PFM2 283
PFM2 284
PFM2 285
PFM2 286
PFM2 287
PFM2 288
PFM2 289
PFM2 290
PFM2 291
PFM2 292
PFM2 293
PFM2 294
PFM2 295
PFM2 296
PFM2 297
PFM2 298
PFM2 299
PFM2 300
PFM2 301
PFM2 302
PFM2 303
PFM2 304
PFM2 305
PFM2 306
PFM2 307
PFM2 308
PFM2 309
PFM2 310
PFM2 311
PFM2 312
PFM2 313
PFM2 314
PFM2 315

```



```
C
C
C      GO TO 10
C      SATISFACTORY ACCURACY CANNOT BE ACHIEVED WITH THE MAIN
C      DISCRETIZATION INTERVAL, BISECT IT.
C      23 TIME = TIME-T(1)
C      30 INDEX3=INDEX3+1
C      580 IF(INDEX2-INDEX3)580,581,581
C      580 INDEX2=INDEX2+1
C      582 IF(INDEX1-INDEX2)582,583,583
C      584 PRINT 584
C      584 FORMAT(1H ,10X,1SATISFACTORY ACCURACY CANNOT BE ACHIEVED AFTER INDP
C      1EX1 ITERATIONS,1)
C      GO TO 13
C      583 CONTINUE
C      TT=T(INDEX3)
C      CALL EXPAT(TT,INDEX3)
C      581 TIME1=TIME1+T(INDEX3)
C      GO TO 12
C      573 GO TO (30,22,81),INDEX
C      81 TIME1=TIME1-T(INDEX3)
C      GO TO 30
C      13 CONTINUE
C      STOP
C      END
PFM2 351
PFM2 352
PFM2 353
PFM2 354
PFM2 355
PFM2 356
PFM2 357
PFM2 358
PFM2 359
PFM2 360
INDP PFM2 361
PFM2 362
PFM2 363
PFM2 364
PFM2 365
PFM2 366
PFM2 367
PFM2 368
PFM2 369
PFM2 370
PFM2 371
PFM2 372
PFM2 373
PFM2 374
```

```

1 EAT
2 EAT
3 EAT
4 EAT
5 EAT
6 EAT
7 EAT
8 EAT
9 EAT
10 EAT
11 EAT
12 EAT
13 EAT
14 EAT
15 EAT
16 EAT
17 EAT
18 EAT
19 EAT
20 EAT
21 EAT
22 EAT
23 EAT
24 EAT
25 EAT
26 EAT
27 EAT
28 EAT
29 LFEAT
30 EAT
31 EAT
32 EAT
33 EAT
34 EAT
35 EAT

*****
SUBROUTINE EXPAT
*****
PURPOSE
  TO COMPUTE THE STATE TRANSITION MATRIX AND ITS INTEGRAL.
*****
DESCRIPTION OF PARAMETERS
  T - DISCRETIZATION INTERVAL.
  L - NUMBER OF TIMES SUBROUTINE EXPAT IS CALLED BY THE
      MAIN PROGRAM, AT EACH CALL EXPAT GENERATES THE
      MATRICES EAT(L)=EXP(A*T(L)) AND Q(L)=INTEGRAL OF
      EXP(A1*T)*B1 FROM 0 TO T(L).
  N - DIMENSION OF THE AUGMENTED FUNDAMENTAL MATRIX.
  N1 - DIMENSION OF THE FUNDAMENTAL MATRIX A1, OF THE
      MODULATORS.
  NE - NUMBER OF INPUTS TO THE PFN SYSTEM.
  A - AUGMENTED FUNDAMENTAL MATRIX.
  B1 - INPUT COUPLING MATRIX OF THE MODULATORS.
  EAT - STATE TRANSITION MATRIX EXP(A*T).
  Q - INTEGRAL OF THE MATRIX EXP(A1*T)*B1, FROM 0 TO T.
  Q1 - INTEGRAL OF THE MATRIX EXP(A1*T)*B1, FROM 0 TO T(L).
*****
REMARKS
  THE DISCRETIZATION INTERVAL, T MUST BE SMALLER THAN
  2/MAX(SUM ABS(A(I,J))).
  ACCURACY CHECK IS NOT PROVIDED. HOWEVER, THE PARAMETER LFEAT
  CAN BE SET TO LARGER VALUES (10-200) FOR GREATER ACCURACY.
*****
DIMENSION Q(8,8)
*****

```


APPENDIX E

COMPUTER PROGRAM FOR LINEARIZING A NONLINEAR NET

This program linearizes eq. (4.24) (for the single-signed system) by increasing the number of variables of the system. The procedure is described in Section 4.4 (pp. 166-168). A subroutine for calculation of the coefficients of the characteristic polynomial of the linearized system is also included.

Input data

Card	Quantities	Format
1	+Xbb+1bbb1bbb2bbb3bbb4bbb5bbb6bbb7bbb8 bbb9bb10bb11bb12bb13bb14bb15bb16bb17bb18 (b denotes blank space)	(20A4)
2	N (order of the system)	(I4)
3	R(1),...,R(n) (input vector), Connection matrix of the system <u>D</u> .	(8F10.5)

Output (see pp. 170-171)

1. Input data: The connection matrix D,
2. Fundamental matrix of the linearized system A.
3. Characteristic polynomial of A.


```

1  LNET
2  LNET
3  LNET
4  LNET
5  LNET
6  LNET
7  LNET
8  LNET
9  LNET
10 LNET
11 LNET
12 LNET
13 LNET
14 LNET
15 LNET
16 LNET
17 LNET
18 LNET
19 LNET
20 LNET
21 LNET
22 LNET
23 LNET
24 LNET
25 LNET
26 LNET
27 LNET
28 LNET
29 LNET
30 LNET
31 LNET
32 LNET
33 LNET
34 LNET
35 LNET

PROGRAM LNET
TO LINEARIZE A NONLINEAR NET CONSISTING OF THE INTERCONNECTIONS OF PULSE FREQUENCY MODULATORS AND IDEAL DELAYS BY INCREASING THE NUMBER OF VARIABLES OF THE SYSTEM.

DESCRIPTION OF PARAMETERS
INPUT
N      = NUMBER OF MODULATORS (DIMENSION OF THE NET).
A      = CONNECTION MATRIX OF THE NET.
R      = INPUT VECTOR (MUST BE SPECIFIED).
COMMON JB,IP

DIMENSION JB(16,16),IP(16),A(4,4),JY(16,4),INDEX(16),R(4),IDUMMY(4),
1),IY(16),NZERO(4),NO1(4),NY(16),LINE1(16),LINE2(16),NUMBER(16),NX
2(16),JX(16,5)
READ 511,JPLUSX,JPLUS1,(NUMBER(I),I=1,16)
511  FORMAT(20A4)
READ 1511,N
1511  FORMAT(I4)
READ 1512,(R(I),I=1,N)
READ 1512((A(I,J),J=1,N),I=1,N)
1512  FORMAT(8F10.5)
N1=N-1
N2=2**N
N22=N+2
NHALF=N/2
N2HALF=N2/2
N11=N2-1

GENERATE Y(I,J).
IK=1
DO 101 I=1,N
JK=2*IK
DO 102 J=1,N2,JK

```



```

108 DO I=1,N2
109   I1=0
110   I1=I1+1
111   IF(I1-N2)114,114,115
112   J=0
113   J=J+1
114   IF(J-N)111,111,118
115   IF(JX(I,J)-JY(I1,J))112,113,112
116   PRINT 116
117   FORMAT(1H ,10X,'ERROR AT STATEMENT NO. 111.')
```

```

118   GO TO 500
119   NX(I)=NY(I1)
120   CONTINUE
121   DO 130 J=1,N2
122   JB(1,J)=1
123   DO 131 I=1,N
124   DO 131 J=1,N2
125   I1=I+1
126   IYJ=IY(J)
127   JB(I1,J)=JY(IY,J,I)
128   DO 316 I1=022,I2
129   IX2=IY(I1)
130   NOZMAX=0
131   NOIMAX=0
132   DO 303 J=1,N
133   JF(JY(IX2,J))301,301,302
134   NOZMAX=NOZMAX+1
135   NOZERO(NOZMAX)=J
136   GO TO 303
137   NOIMAX=NOIMAX+1
138   NOI(NOIMAX)=J
139   CONTINUE
140   DO 326 J1=1,N2
141   JB(I1,J1)=1
142   IYJ=IY(J1)

```

```

LNET 71
LNET 72
LNET 73
LNET 74
LNET 75
LNET 76
LNET 77
LNET 78
LNET 79
LNET 80
LNET 81
LNET 82
LNET 83
LNET 84
LNET 85
LNET 86
LNET 87
LNET 88
LNET 89
LNET 90
LNET 91
LNET 92
LNET 93
LNET 94
LNET 95
LNET 96
LNET 97
LNET 98
LNET 99
LNET 100
LNET 101
LNET 102
LNET 103
LNET 104
LNET 105

```

```

318 DO 310 J=1,NO1*MAX
      JJ=NO1(J)
310 JB(I1,J1)=JB(I1,J1)*JX(IYJ,JJ)
326 CONTINUE
316 CONTINUE
      DO 140 J1=1,N2
          IX2=IY(J1)
          NOZMAX=0
          NO1MAX=0
          DO 141 J=1,N
              IF(JY(IX2,J))142,142,143
          142 NOZMAX=NOZMAX+1
              NOZERD(NOZMAX)=J
          GO TO 141
          143 NO1MAX=NO1MAX+1
              NO1(NO1MAX)=J
          141 CONTINUE
          J11=J1-1
          DO 144 I1=1,N2
              IF(J11)315,315,309
          309 DO 306 J=1,J11
                  IX1=IY(J)
                  K=0
          305 K=K+1
                  IF(K=NOZMAX)313,313,304
          313 NZ=NOZERO(K)
                  IF(JY(IX1,NZ))306,305,306
          304 JB(I1,J1)=JB(I1,J1)+JB(I1,J)
                  IF(JB(I1,J1)-1)306,306,307
          307 JB(I1,J1)=0
          306 CONTINUE
          315 CONTINUE
          144 CONTINUE
          140 CONTINUE
          DO 2221 I=1,N

```

```

LNET 106
LNET 107
LNET 108
LNET 109
LNET 110
LNET 111
LNET 112
LNET 113
LNET 114
LNET 115
LNET 116
LNET 117
LNET 118
LNET 119
LNET 120
LNET 121
LNET 122
LNET 123
LNET 124
LNET 125
LNET 126
LNET 127
LNET 128
LNET 129
LNET 130
LNET 131
LNET 132
LNET 133
LNET 134
LNET 135
LNET 136
LNET 137
LNET 138
LNET 139
LNET 140

```

```

      GO TO (272,273,274,275),N1
272 PRINT 282,A(I,1),A(I,2)
      GO TO 221
273 PRINT 283,A(I,1),A(I,2),A(I,3)
      GO TO 221
274 PRINT 284,A(I,1),A(I,2),A(I,3),A(I,4)
      GO TO 221
275 PRINT 285,A(I,1),A(I,2),A(I,3),A(I,4),A(I,5)
282 FURMAT(1H,20X,2(F5,2,3X))
283 FURMAT(1H,20X,3(F5,2,3X))
284 FURMAT(1H,20X,4(F5,2,3X))
285 FURMAT(1H,20X,5(F5,2,3X))
221 IF(I-NHALF)2221,286,2221
286 PRINT 287
287 FURMAT(1H,8X,1X(K+1)= 1,40X,1X(K))
2221 CONTINUE
      C CHECK THE PROGRAM.
      PRINT 276
276 FURMAT(1H,//////////)
      PRINT 30
30 FURMAT(1H,12X,11,5X,1NY,3X,1NX,1 INDEX,17X,1JY(I,J)),13X,
1JX(I,J),1,1)
      DO 10 I=1,N2
      PRINT 20,J,JY(I),NY(I),INDEX(I),JY(I,1),JY(I,2),JY(I,3),JY(I,4),
20 FURMAT(1H,10X,4(I3,3X),10X,4I3,10X,4I3)
      10 CONTINUE
      PRINT 223
223 FURMAT(1H,////////,10X,1TRANSFORMATIONS USED,1,1,20X,1Y = 1,1,21X)
1,11,1,1)
      DO 230 I=2,N2
      JY2=IY(I)
      INDEX1=INDEX(IY2)
      JJ=0
      DO 231 J=1,N

```

```

LNET 141
LNET 142
LNET 143
LNET 144
LNET 145
LNET 146
LNET 147
LNET 148
LNET 149
LNET 150
LNET 151
LNET 152
LNET 153
LNET 154
LNET 155
LNET 156
LNET 157
LNET 158
LNET 159
LNET 160
LNET 161
LNET 162
LNET 163
LNET 164
LNET 165
LNET 166
LNET 167
LNET 168
LNET 169
LNET 170
LNET 171
LNET 172
LNET 173
LNET 174
LNET 175

```

```

232 IF(JY(IY2,J))231,231,232
    JJ=JJ+1
    IDUMMY(JJ)=J
231 CONTINUE
241 GO TO (241,242,243,244),INDEX
241 PRINT 251,I,IDUMMY(1)
251 FORMAT(1H ,19X,1Y = X',/,21X,12,3X,11,/,)
    GO TO 230
242 PRINT 252,I,IDUMMY(1),IDUMMY(2)
252 FORMAT(1H ,19X,1Y = X X',/,21X,12,3X,11,2X,11,/,)
    GO TO 230
243 PRINT 253,I,IDUMMY(1),IDUMMY(2),IDUMMY(3)
253 FORMAT(1H ,19X,1Y = X X X',/,21X,12,3X,11,2X,11,2X,11,/,)
    GO TO 230
244 PRINT 254,I,IDUMMY(1),IDUMMY(2),IDUMMY(3),IDUMMY(4)
254 FORMAT(1H ,19X,1Y = X X X X',/,21X,12,3X,11,2X,11,2X,11,2X,11,2X,11,/,)
230 CONTINUE
    PRINT 260
260 FORMAT(1H ,////,10X,'LINEARIZED EQUATION',/)
    DO 2261 I=1,NZ
    GO TO (292,293,294,295),N1
292 PRINT 262,JB(I,1),JB(I,2),JB(I,3),JB(I,4)
    GO TO 261
293 PRINT 263,JB(I,1),JB(I,2),JB(I,3),JB(I,4),JB(I,5),JB(I,6),JB(I,7)
    IJB(I,8)
    GO TO 261
294 PRINT 264,JB(I,1),JB(I,2),JB(I,3),JB(I,4),JB(I,5),JB(I,6),JB(I,7)
    IJB(I,8),JB(I,9),JB(I,10),JB(I,11),JB(I,12),JB(I,13),JB(I,14),JB(I,15),JB(I,16)
    IJB(I,17),JB(I,18),JB(I,19),JB(I,20),JB(I,21),JB(I,22)
    PRINT 265,JB(I,23),JB(I,24),JB(I,25),JB(I,26),JB(I,27),JB(I,28),JB(I,29)
    GO TO 261
295 PRINT 265,JB(I,1),JB(I,2),JB(I,3),JB(I,4),JB(I,5),JB(I,6),JB(I,7)
    IJB(I,8),JB(I,9),JB(I,10),JB(I,11),JB(I,12),JB(I,13),JB(I,14),JB(I,15),JB(I,16)
    IJB(I,17),JB(I,18),JB(I,19),JB(I,20),JB(I,21),JB(I,22)
    PRINT 266,JB(I,23),JB(I,24),JB(I,25),JB(I,26),JB(I,27),JB(I,28),JB(I,29)

```

```

LNFT 176
LNFT 177
LNFT 178
LNFT 179
LNFT 180
LNFT 181
LNFT 182
LNFT 183
LNFT 184
LNFT 185
LNFT 186
LNFT 187
LNFT 188
LNFT 189
LNFT 190
LNFT 191
LNFT 192
LNFT 193
LNFT 194
LNFT 195
LNFT 196
LNFT 197
LNFT 198
LNFT 199
LNFT 200
LNFT 201
LNFT 202
LNFT 203
LNFT 204
LNFT 205
LNFT 206
LNFT 207
LNFT 208
LNFT 209
LNFT 210

```

```

1(I,29),JB(I,30),JB(I,31),JB(I,32)
261 IF(I-N2HALF)2261,296,2261
296 PRINT 297
297 FORMAT(1H+,11X,'Y(K+1) = ',53X,'Y(K)')
2261 CONTINUE
262 FORMAT(1H ,20X,4I3)
263 FORMAT(1H ,20X,S13)
264 FORMAT(1H ,20X,16I3)
265 FORMAT(1H ,20X,22I3)
266 FORMAT(1H+,85X,10I3)
CALL CHAPOL(N2,JB,IP)
DO 362 I=1,32
LINE1(I)=JPLANK
LINE2(I)=JPLANK
362 LINE2(I)=NUMBER(N2)
L=1
DO 350 J=1,N11
IF(IP(I))351,350,351
351 LINE1(L)=JPLUSX
IF(I-N11)353,354,353
353 LINE2(L+1)=NUMBER(N2-I)
354 L=L+1
350 CONTINUE
IF(IP(N2))355,356,355
355 LINE1(L)=JPLUS1
356 PRINT 360,LINE2
360 FORMAT(1H ,///,10X,'CHARACTERISTIC POLYNOMIAL',/,27X,16A4)
PRINT 361,LINE1
361 FORMAT(1H ,20X,'P(X) = X ',20A4)
500 CONTINUE
STOP
END
LNFT 211
LNFT 212
LNFT 213
LNFT 214
LNFT 215
LNFT 216
LNFT 217
LNFT 218
LNFT 219
LNFT 220
LNFT 221
LNFT 222
LNFT 223
LNFT 224
LNFT 225
LNFT 226
LNFT 227
LNFT 228
LNFT 229
LNFT 230
LNFT 231
LNFT 232
LNFT 233
LNFT 234
LNFT 235
LNFT 236
LNFT 237
LNFT 238
LNFT 239
LNFT 240
LNFT 241
LNFT 242

```

```

1 SUBROUTINE CHAPOL(M,IA,IP)
2 SUPROUTINE FOR CALCULATING THE COEFFICIENTS OF THE CHARACTERISTIC
3 COMMON IA,IP
4 POLYNOMIAL P(X)=X**M+P(1)*X**(N-1)+...+P(N-1)*X+P(N), OF A MATRIX
5 DIMENSION IA(16,16),IP(16),TR(16),AN(16,16),C(16)
6 INITIALIZE IAN (MATRIX A MULTIPLIED BY ITSELF N TIMES).
7 DO 111 I=1,M
8 DO 111 J=1,N
9 IAN(I,J)=0.0
10 DO 1 I=1,M
11 AN(I,I)=1.0
12 DO 2 N=1,M
13 CALCULATE IAN.
14 TR(N)=0.0
15 DO 3 J=1,M
16 DO 33 I=1,M
17 C(I)=AN(I,J)
18 DO 3 I=1,M
19 D=0.0
20 DO 5 K=1,M
21 D=D+IA(I,K)*C(K)
22 AN(I,J)=D
23 CALCULATE THE TRACE OF A MULTIPLIED BY ITSELF N TIMES.
24 DO 6 I=1,M
25 TR(N)=TR(N)+AN(I,I)
26 CONTINUE
27 NOW CALCULATE THE COEFFICIENTS OF THE CHARACTERISTIC POLYNOMIAL
28 USING RUCHER'S FORMULA.
29 DO 25 H=1,M
30 IP(N)=TR(N)
31 N1=N-1
32 IF(N-1)12,25,12
33 DO 8 I=1,N1
34 IP(N)=IP(N)+IP(I)*TR(N-I)
35 IP(N)=-IP(N)/N

```



```

DO 21 N=1,M
  IF(IP(N))20,21,22
20 IP(N)=-IP(N)
22 O=IP(N)
  IP2=D/2.
  IF(IP(N)-IP2*2)24,24,23
24 IP(N)=O
  GO TO 21
23 IP(N)=1
21 CONTINUE
105 PRINT 105,(IP(I),I=1,M)
  FORMAT(1H ,///,1H ,10X,1CHAPOL P(X) = 1,16I3)
  RETURN
  END

```

```

CPDL 36
CPDL 37
CPDL 38
CPDL 39
CPDL 40
CPDL 41
CPDL 42
CPDL 43
CPDL 44
CPDL 45
CPDL 46
CPDL 47
CPDL 48
CPDL 49

```

REFERENCES

1. Abate, J.E., "Linear and Adaptive Delta Modulation," Ph.D. Dissertation, Newark College of Engineering, Newark, N.J., 1967.
2. Accardi, L., "Rank and Reverbrations in Neural Networks," Kybernetik, vol 8, no. 4, 1971, pp. 123-127.
3. Amerio, L. and G. Prouse, Almost-Periodic Functions and Functional Equations, Van Nostrand Reinhold Comp., 1971.
4. Anninos, P.A., B. Beek, T.J. Csermely, E.M. Harth and G. Pertile, "Dynamics of Neural Structures," J. Theoret. Biol., vol. 26, 1970, pp. 121-148.
5. Barnett, S. and C. Storey, Matrix Methods in Stability Theory, New York, Barnes and Noble, Inc., 1970.
6. Bayly, E.J., "Spectral Analysis of Pulse Frequency Modulation in the Nervous Systems," IEEE Trans. BME, vol. 15, October 1968, pp. 257-265.
7. Bellman, R., Introduction to Matrix Analysis, New York, McGraw-Hill, 1960.
8. Besicovitch, A. S., Almost Periodic Functions, Dover Publications, Inc., 1954.
9. Birkoff, G. and T. C. Barteo, Modern Applied Algebra, New York, McGraw-Hill Book Co., 1970.
10. Blanchard, J. G., "The Optimization of Pulse Frequency Control Systems," Ph.D. Dissertation, Univ. of Calif California, Berkeley, 1966.
11. Bombi, F. and D. Ciscato, "A Modified IPFM in Control Systems," IEEE Trans. AC, vol. AC-12, no. 6, December 1967, pp. 784-785.
12. Bombi, F. and D. Ciscato, "Noise Effects in IPFM Systems," IFAC Pulse Symposium, April 1968, Budapest, Hungary
13. Booth, T.L., Sequential Machines and Automata Theory, New York, John Wiley and Sons, Inc., 1967, pp. 257-321.

14. Broughton, M. B., "The Transition Matrix Analysis of Linear Asynchronous Discrete/Continuous Systems," IFAC Pulse Symposium, April 1968, Budapest, Hungary.
15. Broughton, M. B., "Parameter Identification in Plant Adaptive Pulse Frequency Modulated Control Systems," Int. Electronic Conf., Toronto, October, 1969.
16. Broughton, M. B., "Plant Adaptive Pulse Frequency Modulated Feedback Control Systems," Ph. D. Dissertation, Queen's Univ., Kingston, Ontario, Canada, September 1971.
17. Broughton, M. B., "Plant Adaptive Pulse Frequency Modulated Control Systems," Proc. 5th Hawaii Int. Conference on System Sciences, Honolulu, Hawaii, 11-13 January 1972, pp. 392-394.
18. Caianiello, E. R., "Outline of a Theory of Thought-Process and Thinking Machines," J. Theor. Biol., vol. 1, 1961, pp. 204-235.
19. Caianiello, E. R., A. De Luca and L. M. Ricciardi, "Reverbrations and Control of Neural Networks," Kybernetik, vol. 4, no. 1, 1967, pp. 10-18.
20. Caianiello, E. R., A. De Luca and L. M. Ricciardi, "Neural Networks: Reverbrations, Constants of Motion, General Behavior," in Neural Networks, Ed. E. R. Caianiello, Springer Verlag, 1969.
21. Cattermol, K. W., Principles of Pulse Code Modulation, Illiffe Books Ltd., London, 1969.
22. Chekhovoi, Yu. N., "Stability on the Whole of Pulse Frequency Modulated Control Systems," Avtomatika i Telemekhanika, no. 6, June 1968, pp. 79-87.
23. Chekhovoi, Yu. N., "Limiting Boundedness of Sampled-Data Control Systems with Pulse Frequency Modulation," Avtomatika i Telemekhanika, no. 9, September 1970, pp. 44-51.
24. Chekhovoi, Yu. N., "Stability in The Large of a Pulse Frequency System of The Second Kind in The Simplest Critical Case," Avtomatika i Telemekhanika, no. 4, April 1971, pp. 178-182.
25. Chekhovoi, Yu. V., "Pulse Frequency Systems of Second

- Kind and Their Stability," Avtomatika i Telemekhanika, no. 11, November 1972, pp. 72-83.
26. Clark, J.P.C., "An Analysis of Pulse Frequency Modulation Systems," Ph. D. Dissertation, University of Washington, 1965.
 27. Clark, J. P. C. and E. Noges, "Stability of a Class of Pulse Frequency Modulation Closed Loop Control Systems," 1966 IEEE Int. Conv. Rec., vol. 14, p. 179.
 28. Clark, R. N., "Analysis of Oscillations in Pulse Modulated Satellite Attitude Control Systems," Ph. D. Dissertation, Stanford Univ., 1969.
 29. Clark, R. N. and G. P. Franklin, "Limit Cycle Operation in Pulse Modulated Systems," 1969 JACC, Boulder, Colorado.
 30. Clark, R. N., "Limit Cycle Oscillation in a Satellite Attitude Control System," Automatica, vol. 6, 1970, pp. 801-807.
 31. Datta, K. B., "Stability of Combined Integral Pulse Frequency and Pulse Width Modulated Systems," Int. J. Control, vol. 13, no. 6, June 1971, pp. 1161-1169.
 32. Datta, K. b., "Analysis of Combined Integral Pulse and Pulse Width Modulated Control System," Int. J. Control, vol. 14, no. 5, November 1971, pp. 801-809.
 33. De Luca, A., "On Some Dynamical Properties of Linear and Affine Networks," Kybernetik, vol, 8, no, 4, 1971, pp. 123-127.
 34. Derzhavin, O. M., "Block Diagrams for Pulse Time Modulators of Type 1," Automation and Remote Control, no. 4, April 1967, pp. 74-79.
 35. Dymkov, V. I., "Absolute Stability of Pulse Frequency Systems," Automation and Remote Control, no. 10, October 1967, pp. 109-114.
 36. Dymkov, V. I., "Periodic States in Pulse Frequency Systems," Automation and Remote Control, no. 11, November 1967, pp. 101-108.

37. Eldden, C.M. and A. J. Ley, "A Digital Transfer Function Analyser Based on Pulse Rate Techniques," IFAC Pulse Symposium, April 1968, Budapest, Hungary.
38. Farrenkopf, R. L., A. E. Sabroff and P. C. Wheeler, "Integral Pulse Frequency On-Off Attitude Control," Guidance and Control-II, vol. 13 of Progress in Astronautics and Aeronautics, Langford and Mundo Editions, New York, N.Y., Academic Press, 1964.
39. Fink, A. M., "Almost Periodic Functions Invented for Specific Purposes," SIAM Review, vol 14, no. 4,
40. Gantmacher, F. R., Application of the Theory of Matrices, New York, Interscience Publishers, Inc., 1959,
41. Gelig, A. Kh., "Stabilization of Nonlinear Systems with Pulse Frequency Modulation," Automation and Remote Control, no. 6., June 1967, pp. 75-82.
42. Gelig, A. Kh., "Absolute Stability of Nonlinear Pulse Systems and Width and Time Modulation," Avtomatika i Telemekhanika, vol 29, no. 7, July 1968, pp. 33-43.
43. Gelig, A. Kh., "Stability of Sampled-Data Systems with Pulse Frequency Modulation of The 2nd Type," Avtomatika i Telemekhanika, no. 11, November 1971, pp. 53- 60.
44. Gelig, A. Kh., "Stability of Multidimensional Asynchronous Sampled-Data System with Frequency Modulation of The Second Kind," Avtomatika i Telemekhanika, no. 3, March 1972, pp. 52-59.
45. Gelig, A. Kh., "Ultimate Boundedness and Stability of a Class of Pulse-Frequency Systems," Avtomatika i Telemekhanika, no. 8, August, 1972, pp. 51-58.
46. Gestri, G., "Pulse Frequency Modulation in Neural Systems," Biophysical Journal, vol. 11, January 1971, pp. 98-109.
47. Golomb, S. W., Shift Register Sequences, San Fransisco, Holden-Day, Inc., 1967.
48. Guy, W. J., "Oscillations in Pulse Frequency Modulated Control Systems," Ph. D. Dissertation, Newark College of Engineering, Newark, N.J., 1970.

49. Guy, W. J. Jr, M. Kurland and A. U. Meyer, "Initial Condition Zones for IPFM System Stability," Proc. Eight Annual Allerton Conference on Circuit and System Theory, Monticello, Ill., 7-9 October 1970 p. 164-165.
50. Guyton, A. C., Textbook of Medical Physiology, Philadelphia, W. B. Saunders Company, 1971 (Fourth Edition).
51. Gülçür, H. Ö. and A. U. Meyer, "Finite Pulse Stability of Interconnected Systems with Complete-Reset Pulse Frequency Modulators," Tenth Annual Allerton Conference on Circuit and System Theory, 4-6 October 1972, Monticello, Illinois.
52. Gülçür, H. Ö. and A. U. Meyer, "Finite Pulse Stability of Interconnected Systems with Complete-Reset Pulse Frequency Modulators," IEEE Trans. AC, vol. AC-18, no. 4, August 1973, pp. 387-392.
53. Gülçür, H. Ö. and A. U. Meyer, "Comparison of Stability Criteria for Interconnected Systems with Pulse Frequency Modulation," To be published.
54. Hahn, W., Stability of Motion, Springer-verlag New York Inc., 1967, p. 264.
55. Harmon, L. D. and E. R. Lewis, "Neural Modeling," Physiological Reviews, vol. 46, July 1966, pp. 513-591.
56. Harmuth, H. F., Transmission of Information by Orthogonal Functions, Springer-Verlag, Berlin, 1970.
57. Harrison, M. A., Lectures on Linear Sequential Machines, New York, Academic Press, 1969.
58. Harterbrot, H., "Differential Equations for a Pulse Frequency Modulator of the Second Kind and Numerical Solution," Soviet Automatic Control, vol 14, March-April, 1969.
59. Hodgkin, A. L. and A. F. Huxley, "A Quantitative Description of Membrane Current and its Application to Conduction and Excitation in Nerve," J. Physiol. vol. 117, pp. 500-544.

60. Hutchinson, C. E. and Y. T. Chon, "The Effect of Pulse Frequency Modulation on Noise," IEEE Trans. AC, vo. AC-12, October, 1967, p. 621.
61. Hutchinson, C. E., Y. J. Chon and R. A. Leuchs, "An Analysis of the Effect of Integral and Sigma Pulse Frequency Modulation on White Noise," Information and Control, vol. 13, September 1968, pp. 173-185.
62. Hynes, R. J., "The Minimum-Time Regulator Problem for Pulse Frequency Modulated Control Systems," Ph. D. Dissertation, Univ. of Washington, 1969.
63. Ivey, K. A., "A-C Carrier Control Systems, New York, John Wiley and Sons, Inc., 1964.
64. Jones, R. W., C.C. Li, A. U. Meyer and R. B. Pinter, "Pulse Modulation in Physiological Systems, Phenomenological Aspects," IRE Trans. Biomedical Electronics, vol. BME-8, no. 1, January 1961, pp. 59-67.
65. Jury, E. I. and J. Blanchard, "A Nonlinear Discrete System Equivalence of Integral Pulse Frequency Modulated Systems," Joint Automatic Control Conference, June 1967, Philadelphia, Pa., pp. 680-695.
66. Jury, E. I. and E. P. F. Kan, "Stability of a Dual Mode (Pulse Width and Pulse Frequency) Modulated Feedback System," Report, Dept. of E.E. and Comp. Sciences, Electronics Research Lab., Univ. of California, Berkeley, 1968.
67. Jury, E. I. and Y. Z. Tsypkin, "Theory of Discrete Automatic Control Systems (Review)," Automation and Remote Control, vol. 31, June 1970, pp. 915-936.
68. Kan, E. P. F. and E. I. Jury, "On Popov Criteria for SPFM Systems," Int. J. Control, vol. 13, no. 6, 1971.
69. Kan, P. F. E., "L₁ and L_p Stability of an IPFM System," Int. J. Control, vol. 14, no. 1, 1971, pp. 27-31.
70. Kauffman, S. A., "Metabolic Stability and Epigenesis in Randomly Constructed Genetic Nets," J. Theor. Biol., vol. 22, 1969, pp. 437-467.
71. King-Smith, E. A. and J. R. Cumpston, "Periodic Cycles in IPFM Systems," IFAC Pulse Symposium, April 1968, Budapest, Hungary.

72. King-Smith, E. A. and J. R. Cumpston, "The Stability of IFM Systems," Int. J. Control, vol. 7, no.4, 1968, pp. 301-316.
73. Koenig, H. E., Y. Tokad, and H. K. Kesevan, Analysis of Discrete Physical Systems, New York, McGraw-Hill Book Co., 1967.
74. Kohavi, Z., Switching and Finite Automata Theory, New York, McGraw-Hill Book Co., 1970.
75. Kuntsevich, V. M. and Yu N. Chekhovoi, "Investigating Stability of Sampled Data Control Systems with Pulse Frequency Modulation Using Lyapunov's Direct Method," Automation and Remote Control, no. 2, February 1967, pp. 80-92
76. Kuntsevich, V. M. and Yu. N. Chekhovoi, "Stability of Control Systems with Double (Pulse Frequency and Pulse Width) Modulation," Automation and Remote Control, no. 7, July 1967, pp. 54-63.
77. Kuntsevich, V. M., "Investigating the Stability of a Multivariable Control System with Pulse-Frequency Modulation Using the Direct Lyapunov Method," Avtomatika i Telemekhanika, vol 29, no. 7, September 1968, pp. 71-83.
78. Kuntsevich, V. M. and Yu. N. Chekhovoi, "Fundamentals of Nonlinear Control Systems with Pulse Frequency and Pulse Width Modulation," Automatica, (IFAC), vol. 7, January 1971, pp. 73-81.
79. Kuntsevich, M. and Yu. N. Chekhovoi, "Asymptotic Stability on the Whole of One Class of Pulse Frequency Systems of The Second Type," Avtomatika i Telemekhanika, no. 3, March, 1971, pp. 178-182.
80. Kuntsevich, V. M., "Global Asymptotic Stability of Two Classes of Control Systems with Pulse Duration and Puls4 Frequency Modulations," Automation and Remote Control, vol. 33, no. 7, pp. 1140-32, July 1972.
81. Lancaster, P., Theory of Matrices, New York, Acad. Press, 1960.

82. Landahl, H. D. and R. Runge, "Outline of a Matrix Calculus for Neural Nets," Bull. Math. Biophys., vo. 8, 1946, pp. 75-81,
83. La Salle, J. and S. Lefschetz, Stability by Lyapunov's Direct Method, New York, Acad. Press, 1961.
84. Lee, H. C., "Integral Pulse Frequency Modulation with Technological and Biological Applications," Ph.D. Dissertation, McGill University, Montreal, Quebec, July 1969.
85. Lee, H. C. and J. H. Milsum, "Statistical Analysis of Multiunit Multipath Neural Communication," Mathematical Biosciences, no. 11, pp. 181-202, 1971.
86. Leibovic, K. N., Nervous System Theory, New York, Acad. Press, 1972.
87. Lermentov, M. and E. Noges, "Geometrical Aspects of Admissible Regions in Fixed Time Pulse Frequency Modulated Control Systems," IEEE Trans. AC, February 1972, pp. 135-137.
88. Lermentov, M., "A Gradient Method for Optimization of PFM Systems," (Tech. Note), IEEE Trans. AC, Vol. AC-18, no. 2, April 1973, pp. 177-178.
89. Li, C. C., "Integral Pulse Frequency Modulated Control Systems," Ph. D. Dissertation, Northwestern Univ., Evanston, Illinois, June 1961.
90. Li, C. C. and R. W. Jones, "Integral Pulse Frequency Modulation Control Systems," Proc. 1963, IFAC Congress, Basle, Switzerland, September 1963.
91. Maeda, H., "A New Stability Criterion of Relay Systems and Its Application to Pulse Frequency Modulated Systems," Proc. 3rd Hawaii Int. Conf., Honolulu Hawaii, 14-16 January, 1970, pp. 1040-1043.
92. McCulloch, W. S. and W. H. Pitts, "A Logical Calculus of The Ideas Immanent in Nervous Activity," Bull. Math. Biophys., vol. 5, 1943, pp. 115-133.
93. Meyer, A. U., "Pulse Frequency Modulation and Its Effects in Feedback Control Systems," Ph. D. Dissertation, Northwestern Univ., Evanston, Ill., August 1961.

94. Meyer, A. U., "Discussion of 'Analysis of a New Class of Pulse Modulation Feedback Systems'," IEEE Trans. AC, vol. 10, April 1965, pp. 211-214.
95. Monopoli, R. V. and D. K. Wylie, "A New Model for Neural Pulse Frequency Modulation," Third Princeton Conference on Information Sciences and Systems, March 1969, Princeton, N.J.
96. Murphy, G. and K. L. West, "The Use of Pulse Frequency Modulation for Adaptive Control," Proc. NEC, vol. 18, October 1962, pp. 271-277.
97. Nightingale, J., M. and G. A. Richards, "Error Analysis in Binary Rate Modulation Systems," IFAC Pulse Symposium, April 1968, Budapest, Hungary.
98. Onyshko, S., "Optimized Pulse Frequency Modulation," Ph. D. Dissertation, Univ. Washington, 1966.
99. Onyshko, S. and S. Roges, "Pulse Frequency Modulation and Dynamic Programming," IEEE Trans. AC, vol. 14, October 1969, pp. 558-561.
100. Pai, M. A. and B. S. Varadarajan, "Stability of Feedback Systems with Pulse Frequency Modulation," 13th Midwest Symposium on Circuit Theory, May 1970.
101. Pavlidis, T. and E. I. Jury, "Analysis of a New Class of Pulse Frequency Modulation Feedback Systems," IEEE Trans. AC, vol. 10, January 1965, pp. 35-43.
102. Pavlidis, T., "A New Model for Simple Neural Nets and Its Application in The Design of a Neural Oscillator," Bull. Math. Biophys., vol. 27, 1965, pp. 215-229.
103. Pavlidis, T., "Stability of a Class of Discontinuous Dynamical Systems," Information and Control, vol. 9, 1966, pp. 298-322.
104. Pavlidis, T., "Stability of Systems Described by Differential Equations Containing Impulses," IEEE Trans. AC, vol. 12, February 1967, pp. 80-92.
105. Pavlidis, T., "Optimal Control of Pulse Frequency Modulation Systems," IEEE Trans. AC, vol. 11, October 1966, pp. 676-684.
106. Pavlidis, T., "Design of Neural Nets with Intermittent Response and Certain Other Relevant Studies," Bull. Math. Biophys., March 1966, pp. 51-74.

107. Peterson, W. W. and E. J. Weldon, Jr., Error Correcting Codes, Second Edition, Cambridge, Massachusetts, The MIT Press, 1972.
108. Pshenichikov, A. M., "Choice of Frequency Range For Industrial Pulse Frequency Telemetry System Devices,"
109. Rochelle, R. W., "Pulse Frequency Modulation," IRE Trans. Space Elecs. and Telemetry, vol. SET-8, no. 2, June 1962, pp. 107-112.
110. Ross, A. E., "Theoretical Study of Pulse Frequency Modulation," IRE Proc., vol. 37, no. 11, November 1949, pp. 1277-1286.
111. Ruch, T. C., H. D. Patton, J. W. Woodbury and A. L. Towe, Neurophysiology, Philadelphia, W. B. Saunders Co., 1965 (Second Edition).
112. Salikov, L. M. and L. B. Solobev, "Designing Systems with Pulse Frequency Modulation," Automation and Remote Control, no. 10, October 1970, pp. 182-185.
113. Schaefer, R. A., "New Pulse Modulation Method," Electronics, vol. 35, October 12, 1962, pp. 50-53.
114. Schaefer, R. A., "Pulse Ratio Modulation for Attitude Control," Space Aeronautics, May 1963, pp. 90-93.
115. Schwartz, M., W. R. Bennett and S. Stein, Communication Systems and Techniques, McGraw-Hill Book Comp. 1966.
116. Shortle, G. E., Jr. and F. J. Alexandro, Jr., "Stability of Class of Pulse Frequency Modulated Control Systems," 1966 JACC, Boulder, Colo.
117. Sigorskii, V. P., L. S. Sitkinov and L. L. Utyakov, "Multistable Pulse Frequency Elements," Avtomatika i Telemekhanika, vol 27, no. 2, pp. 76-81, February 1966.
118. Skoczowski, S., "New Type of Discrete Control Systems with Pulse Frequency Modulation," Pomiary Automat Kontrola (Poland), vol. 13, no. 12, December 1967, pp. 537-540. (in Polish).
119. Skoczowski, S. and L. Trasiejski, "The Control Performance of Pulse Frequency Modulated Feedback Systems and Possibilities of Its Improvement," Arch. Autom. & Telemekh., (Poland), vol. 16, no. 3, 1971, pp. 327-338.

120. Skoczowski, S. and M. Tarasiejski, "Performance Comparison of Pulse Frequency Modulated Control Systems by Digital and Analogue Simulation," Messen Steuern Regeln (Germany), vol 15, no. 2, pp. 71-74, Feb. 1 1972.
121. Skoog, R. A. and G. L. Blankenship, "Generalized Pulse Modulated Feedback Systems; Norms, Gains, Lipschitz Constants and Stability," IEEE Trans. Ac, vol. 15, June 1970, pp. 300-315.
122. Stoep, D. R. and F. J. Alexandro, "Bounds on The Optimal Performance of Pulse Controlled Linear Systems," 1967 JACC, Philadelphia, Pa., 1967.
123. Stoep, D. R., "Synthesis of Optimal Pulse Controlled Control Systems," "Ph. D. Dissertation, Univ. of Washington, 1967.
124. Taylor, R. G., "A Neural Model Using Positive Logic With Adjustable Delay," Design Project, Newark College of Engineering, Newark, N.J. 1972.
125. Taylor, J. G., "Spontaneous Behaviour in Neural Networks," J. Theor. Biol., vol. 36, 1972, pp. 513-528.
126. Varadarajan, M. S., "Stability of Feedback Systems with pulse Frequency Modulation," Int. J. Control, vol. 13, no. 2, 1971, pp. 265-273.
127. Varadarajan, M. S. and Pai, "Periodic Oscillations in IPFM Feedback Systems," int. J. Control, vol. 14, no. 6, 1971, pp. 1175-1181.
128. Wylie, B. K. and R. V. Monopoli, "Stability Analysis of a Class of Pulse Frequency Modulated Error Sampled Control Systems," Proc. of the 13th Midwest Symposium on Circuit Theory, May 7-8, 1970.
129. Willems, J. C., The Analysis of Feedback Systems, Cambridge, Massachusetts, The MIT Press, (Research Monograph No. 62), 1971.
130. Zames, G., "On the Input-Output Stability of Time-Varying Nonlinear Feedback Systems, Part 1: Conditions Derived From the Concept of Loop-Gain, Conicity and Positivity" IEEE Trans. AC, vol. AC-11, pp. 228-238. April 1966. (Also Part II, vol. 11, pp. 465-476, July 1966).

131. DeFonseca, S. J. L. and W. S. McCulloch, "Synthesis and Linearization of Nonlinear Feedback Shift Registers," Quart. Progr. Rep. No. 86, Research Laboratory of Electronics, M.I.T., 1967, pp. 355-366.
132. Fukunaga, K., "A Theory of Nonlinear Autonomous Sequential Nets Using z Transforms," IEEE Trans. EC, vol. 13, 1964, pp. 310-312.
133. Peatman, J. B., The Design of Digital Systems, New York, McGraw-Hill Book Company, 1972.

