INFORMATION TO USERS

This dissertation was produced from a microfilm copy of the original document. While the most advanced technological means to photograph and reproduce this document have been used, the quality is heavily dependent upon the quality of the original submitted.

The following explanation of techniques is provided to help you understand markings or patterns which may appear on this reproduction.

1. The sign or "target" for pages apparently lacking from the document photographed is "Missing Page(s)". If it was possible to obtain the missing page(s) or section, they are spliced into the film along with adjacent pages. This may have necessitated cutting thru an image and duplicating adjacent pages to insure you complete continuity.

2. When an image on the film is obliterated with a large round black mark, it is an indication that the photographer suspected that the copy may have moved during exposure and thus cause a blurred image. You will find a good image of the page in the adjacent frame.

3. When a map, drawing or chart, etc., was part of the material being photographed the photographer followed a definite method in "sectioning" the material. It is customary to begin photoing at the upper left hand corner of a large sheet and to continue photoing from left to right in equal sections with a small overlap. If necessary, sectioning is continued again – beginning below the first row and continuing on until complete.

4. The majority of users indicate that the textual content is of greatest value, however, a somewhat higher quality reproduction could be made from "photographs" if essential to the understanding of the dissertation. Silver prints of "photographs" may be ordered at additional charge by writing the Order Department, giving the catalog number, title, author and specific pages you wish reproduced.

73-9593

BATIWALLA, Bomi Homi, 1945-
VIBRATION AND BUCKLING OF ELASTIC PLATES
WITH SHEAR AND ROTATORY INERTIA.

Newark College of Engineering, D.Eng.Sc.,
1972
Engineering, general

University Microfilms, A XEROX Company, Ann Arbor, Michigan

VIBRATION AND BUCKLING OF ELASTIC PLATES WITH SHEAR AND ROTATORY INERTIA

BY

BOMI HOMI BATIWALLA

A DISSERTATION

PRESENTED IN PARTIAL FULFILLMENT OF

THE REQUIREMENTS FOR THE DEGREE

OF

DOCTOR OF ENGINEERING SCIENCE

AT

NEWARK COLLEGE OF ENGINEERING

Newark, New Jersey

1972

ABSTRACT

The effects of shear, rotatory inertia and inplane forces on the transverse vibration of thin plates are studied. In addition, the effect of shear on the buckling of thin plates is examined. A general differential equation of motion is derived for an isotropic thin plate subjected to normal and inplane forces with the consideration of shear and rotatory inertia. The method of internal constraints and Hamilton's principle are utilized.

The resulting fourth order differential equation is solved for simply supported plates of various shapes by employing a finite difference technique. The shapes examined are a square, a circle, a circular annulus, and an elliptic annulus. The differential equation is written in its finite difference form and finally as a matrix. The value of the matrix is determined using the lower and upper decomposition method. The first few natural frequencies and the critical buckling loads are obtained using an iterative

technique.

The numerical results for the several shapes examined show that the inclusion of shear, rotatory inertia, and inplane forces result in substantially lower natural frequencies. The inclusion of shear effect in the buckling analysis also results in significantly lowering the critical buckling load.

As a check on the numerical technique employed in the study, natural frequencies and critical buckling loads neglecting the effects of shear and rotatory inertia were also determined. Excellent agreement between these numerical results and analytical data obtained from classical theories is obtained.

APPROVAL OF DISSERTATION

VIBRATION AND BUCKLING OF ELASTIC PLATES
WITH SHEAR AND ROTATORY INERTIA

BY

BOMI HOMI BATIWALLA

FOR

DEPARTMENT OF MECHANICAL ENGINEERING
NEWARK COLLEGE OF ENGINEERING

BY

FACULTY COMMITTEE

APPROVED: ______________________, Chairman

____

____

____

____

Newark, New Jersey

May 1972

ACKNOWLEDGEMENTS

The author wishes to express his sincere appreciation to his advisor, Dr. Benedict C. Sun, for his constant guidance, encouragement and time during the author's doctoral research at Newark College of Engineering.

The valuable advice on the numerical technique by Dr. Phyllis Fox were very important in completing this study. Her time and effort are very much appreciated.

The comments of Dr. Harry Herman, Dr. James L. Martin and Dr. Charles E. Wilson of the Mechanical Engineering Department and Dr. William L. Haberman, Chairman of the Mechanical Engineering Department were very helpful in bringing this study to its conclusion. Their advice and time are gratefully acknowledged.

Finally, the author wishes to express his appreciation to the Foundation for Advancement of Graduate Study in Engineering at Newark College of Engineering and the Graduate Division for their financial support and encouragement. The author also wishes to thank the staff of the Computer

Science Department for the use of their computer facilities.

DEDICATED

To My Pauline

TABLE OF CONTENTS

TABLE OF CONTENTS (cont'd)

TABLE OF CONTENTS (cont'd)

LIST OF FIGURES

LIST OF FIGURES (cont'd)

LIST OF FIGURES (cont'd)

LIST OF FIGURES (cont'd)

LIST OF TABLES

LIST OF TABLES (cont'd)

LIST OF TABLES (cont'd)

I. INTRODUCTION

In the classical theory for the vibration of thin plates, the natural frequencies are obtained without consideration of the effects of shear and rotatory inertia. In recent years a few researchers have formulated the vibration problem which includes these effects. No numerical results, however, were presented. The shear effects were also disregarded in the buckling analysis. It is known that these effects lower the natural frequencies and the buckling loads of the plate because of increased inertia and flexibility.

The objective of this study is to investigate the effects of shear, rotatory inertia and inplane forces on the flexural vibration and the effect of shear on the buckling of elastic isotropic plates. It is known that these effects are very important for vibration and stability problems for "relatively thick" thin plates. The design of nuclear pressure vessels frequently calls for an accurate analysis of this type. In this study the simply supported square, circular, annular circular and annular elliptical plates are considered.

In Section II of this thesis a brief historical background for the vibration and stability of plates is given.

In Section III a general differential equation of motion considering the effects of shear and rotatory inertia and subjected to normal and in-plane forces, is derived by the method of internal constraints and using Hamilton's principle. This method assumes that the elastic displacements must comply with the special equations of constraints. It is assumed that the plates are thin and that the amplitudes of vibrations and the deflection of the middle surface are small enough to ignore second order effects. The differential equation can be reduced to a number of special equations for plate problems in statics and dynamics. These are shown in Section IV.

In Section V a finite difference technique is employed for computing the natural frequencies and the critical buckling loads. The fourth order differential equation is first reduced to a second order equation, then written in its finite difference form and finally in the form of a matrix. The correct values of the natural frequency or the critical buckling load would make the value of their respective determinants to be zero. The determinant is eveluated using the lower and upper decomposition method which is briefly explained in Appendix A.

In Section VI fundamental frequencies for a square plate for various thickness-to-side length ratio and for a circular plate for various thickness-to-diameter ratio are computed with and without the effects of shear and rotatory inertia. The first four natural frequencies are also evaluated for a circular plate to see in what degree these effects influence the higher natural frequencies. Critical buckling loads for circular and annular circular plates for various thickness-to-diameter ratios are calculated considering the effects of shear. This is computed for both circular and annular circular plates so that the shear effects to buckling on both simply and multiply connected regions may be examined. The effects of tensile and compressive inplane forces, acting on both inner and outer edges of an annular circular and annular elliptical plate, on their natural frequencies are studied with and without the effects of shear and rotatory inertia. Discussion to these results are also given in Section VI.

Conclusions are drawn in Section VII and recommendations are outlined in Section VIII.

II HISTORICAL BACKGROUND

In the earlier period of investigation of transverse vibration of elastic bodies the effects of shear and rotatory inertia were not considered. In 1848 D. Bernoulli and Euler were among the first to present a "classical theory" for the transverse vibration of elastic bars by neglecting these effects. In 1889 an approximate method, due to Lord Rayleigh (49) which took into account the effect of rotatory inertia did not improve the results very much.

In 1921 S. Timoshenko (60) showed the importance of shear in the transverse vibration of bars, and also showed that the effects of shear and rotatory inertia previously disregarded by other authors were equally important. It is well-known that both these effects decrease the computed frequencies because of increased inertia and flexibility of the system.

In the case of flexural vibration of plates, there is no agreement among results following application of the "classical theory", by Lagrange and the theories by Lord Rayleigh (50) in 1889 and by H. Lamb (29) in 1917. In the classical two dimensional theory used by Lagrange it was assumed that the velocity of straight crested waves is inversely proportional to the wave-length. This assumption is good for wave-lengths which are large in comparison with the thickness of the plate but does not hold well for waves of small length or for higher natural frequencies. Therefore Lagrange's theory gives good results only for fundamental frequency of thin plate.

For static deflection of thin elastic plates, the classical theory used by G. Kirchhoff (20) in 1850 neglects the effects of shear deformation and the effects of normal stress. Results obtained from Kirchhoff's theory are applicable only in certain cases.

In the middle of the twentieth century E. Reissner (52), (53), (54), H. Hencky (22), L. Bolle (6), M. Schafer (56) and A. Kromm (27), (28) presented new theories for the deflection of thin plates taking into account the effect of shear. These authors used hypotheses concerning the stress distribution over the plate cross-section in order to obtain the equation of equilibrium. These theories are referred to as "engineering theories" in order to distinguish them from other theories .

In 1945 Reissner (52) obtained the equation of equilibrium using the hypotheses that the stress due to the bending moment varies linearly, whereas those due to the shear vary according to a parabolic law across the section of the plate. At the same time he included the effects due to the normal stress which was previously disregarded by G. Kirchhoff.

L. Bolle (6) in his important work in 1947, independently from Reissner, obtained the same equilibrium equations and boundary conditions on the deformation of thin plates by taking into account the effects of shear. The equations of equilbrium by Reissner taking into account the transverse deformation of the plate due to shear were also derived by Green (19) in 1949 using Castigliano's Theorem. Schafer (56) in 1952 and Kromm (28) in 1953, also studied the effect of shear on the static deflection of plates.

A first presentation of a consistent theory for dynamic behaviour of plates including the effects of shear deformation and rotatory inertia was made by Uflyand (61) in 1948. However in 1951 Mindlin (38) unquestionably made the most profound contribution to this subject. His paper showed how a more comprehensive two-dimensional theory of bars may be deduced directly from the three-dimensional equations of elasticity. He also suggested a formula for the value of the constant 'k' which took into account the non-linear distribution of shear stress across the cross section of the plate.

In 1961 Volterra (66) included the effects of shear and rotatory inertia in the vibration study of elastic bars and plates by the "method of internal constraints", which assumes that the elastic displacements must comply with certain equations of constraint. Lee (30) in 1963 studied the effects of shear and rotatory inertia on the vibration of a wedge by a generalized minimum principle; a step-by-step iteration method is generalized to apply to a coupled simultaneous differential equation in order to obtain an approximate solution for the flexural vibration frequencies of a wedge. In 1966 Callahan (7), (8) included the effects of shear and rotatory inertia by assuming certain functions which took into account these effects and formulated the vibration problem in the form of an infinite determinant. No numerical work was done by these above-mentioned authors.

The elastic stability of plates has been treated by several researchers neglecting the effects of shear. Saint Venant in 1883 was among the first to derive a differential equation for the stability of a plate. In the past two decades several researchers such as Dean (12), Conway and Leissa (10), Mansfield (34), Robinson (55), Timoshenko (58), Yamaki (71) and several others studied the plate stability problem under several loading conditions for various shaped plates. In 1970 Brand and Uthgenannt (62) studied the stability of orthotropic annular circular plates under uniform compressive forces applied at both edges for several boundary conditions, and obtained critical buckling loads by solving the equilibrium equation using a finite difference technique. The only known numerical solution for the vibration with the influence of inplane forces for a simply supported circular plate was obtained by Wah (68) in 1962. The work done by all the above researchers was without the consideration of shear.

No numerical work is available on the effects of shear and rotatory inertia on the natural frequency of transverse vibration of plates or the effects of shear on the critical buckling load of plates. In this study a general differential equation of motion is derived for a plate subjected to normal and inplane forces and considering the effects of shear and rotatory inertia. Natural frequencies and critical buckling loads are computed including these effects and compared with values obtained by using the classical theory.

III <u>DERIVATION OF THE DIFFERENTIAL EQUATION</u>

A. <u>METHOD OF INTERNAL CONSTRAINTS</u>.

In deriving the equation of motion for a thin elastic plate by the classical Lagrange theory, the effects of shear and rotatory inertia were neglected. We take these effects into considerations by the "Method of Internal Constraints" (66). This method assumes that the elastic displacements must comply with special equations of constraints.

We assume that the components of the elastic displacements $\bar{U}$, $\bar{V}$, $\bar{W}$, in the x, y, and z, directions may be developed in a Taylor series in z with the coefficients being functions of the variables x, y and t.

$$\begin{aligned}
\bar{U}(x,y,z,t) &= U_0(x,y,t) + z\,U_1(x,y,t) + \tfrac{1}{2} z^2 U_2(x,y,t) + \ldots\ldots \\
\bar{V}(x,y,z,t) &= V_0(x,y,t) + z\,V_1(x,y,t) + \tfrac{1}{2} z^2 V_2(x,y,t) + \ldots\ldots \qquad (3.1) \\
\bar{W}(x,y,z,t) &= W_0(x,y,t) + z\,W_1(x,y,t) + \tfrac{1}{2} z^2 W_2(x,y,t) + \ldots\ldots
\end{aligned}$$

In discussing flexural vibrations and buckling we make the following assumptions concerning the development of the series:

a. Terms of higher order than the second order in z are neglected.

b. The particles of the plate which were originally in the xy-plane move only in the z-direction.

$$\bar{U}(x,y,0,t) = \bar{V}(x,y,0,t) = 0$$

c. Finally every plane originally perpendicular to the x and y axes respectively remain plane.

$$\bar{U}(x,y,z,t) = -\bar{U}(x,y,-z,t)$$

$$\bar{V}(x,y,z,t) = -\bar{V}(x,y,-z,t)$$

With the above assumptions equations (3.1) reduce to the following:

$$\bar{U} = z\, U_1(x,y,t)$$

$$\bar{V} = z\, V_1(x,y,t)$$

$$\bar{W} = W_o(x,y,t) + z\, W_1(x,y,t) + \tfrac{1}{2} z^2 W_2(x,y,t) \qquad (3.2)$$

To determine the coefficients $W_1(x,y,t)$ and $W_2(x,y,t)$ we satisfy the requirement that:

$$\sigma_z = q_o(x,y,t) \quad \text{at} \quad z = +\frac{H}{2}$$

$$\sigma_z = 0 \quad \text{at} \quad z = -\frac{H}{2}$$

The expression for the normal stress σ_z is:

$$\sigma_z = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \frac{\partial \bar{W}}{\partial z} + \frac{\nu E}{(1+\nu)(1-2\nu)} \left(\frac{\partial \bar{U}}{\partial x} + \frac{\partial \bar{V}}{\partial y}\right) \quad (3.3)$$

Differentiating equation (3.2) and substituting in the above equation we have:

$$\sigma_z = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} (W_1 + z\, W_2) + \frac{\nu E}{(1+\nu)(1-2\nu)} z \left(\frac{\partial U_1}{\partial x} + \frac{\partial V_1}{\partial y}\right) \quad (3.4)$$

Using the above conditions that $\sigma_z = q_o(x,y,t)$ at $z = \frac{H}{2}$ and $\sigma_z = 0$ at $z = -\frac{H}{2}$ we have the following:

$$q_o(x,y,t) = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} (W_1 + \frac{H}{2} W_2) + \frac{\nu EH}{2(1+\nu)(1-2\nu)} \left(\frac{\partial U_1}{\partial x} + \frac{\partial V_1}{\partial y}\right)$$

$$0 = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} (W_1 - \frac{H}{2} W_2) - \frac{\nu EH}{2(1+\nu)(1-2\nu)} \left(\frac{\partial U_1}{\partial x} + \frac{\partial V_1}{\partial y}\right)$$

$$(3.5)$$

Solving the above equation (3.5) simultaneously, the coefficients $W_1(x,y,t)$ and $W_2(x,y,t)$ are:

$$W_1(x,y,t) = \frac{(1+\nu)(1-2\nu)}{2E(1-\nu)} q_o(x,y,t) \tag{3.6}$$

$$W_2(x,y,t) = -\frac{\nu}{1-\nu}\left(\frac{\partial U_1}{\partial x} + \frac{\partial V_1}{\partial y}\right) + \frac{(1+\nu)(1-2\nu)}{E(1-\nu)H} q_o$$

B. STRAIN-DISPLACEMENT RELATIONS

The following are the expressions for the strain displacement relations:

$$\varepsilon_x = \frac{\partial \bar{U}}{\partial x} \qquad \gamma_{xy} = \frac{\partial \bar{U}}{\partial y} + \frac{\partial \bar{V}}{\partial x}$$

$$\varepsilon_y = \frac{\partial \bar{V}}{\partial y} \qquad \gamma_{yz} = \frac{\partial \bar{V}}{\partial z} + \frac{\partial \bar{W}}{\partial y} \qquad (3.7)$$

$$\varepsilon_z = \frac{\partial \bar{W}}{\partial z} \qquad \gamma_{zx} = \frac{\partial \bar{W}}{\partial x} + \frac{\partial \bar{U}}{\partial z}$$

Differentiating the expressions for $\bar{U}$, $\bar{V}$ and $\bar{W}$ in equation (3.2) and neglecting the terms containing z^2 in comparison with the linear terms, the equations (3.7) reduce to:

$$\varepsilon_x = z \frac{\partial U_1}{\partial x} \qquad \varepsilon_y = z \frac{\partial V_1}{\partial y}$$

$$\gamma_{xy} = z \left(\frac{\partial U_1}{\partial y} + \frac{\partial V_1}{\partial x}\right)$$

$$\varepsilon_z = \frac{(1+\nu)}{2E(1-\nu)} q_o - z \frac{\nu}{1-\nu} (\frac{\partial U_1}{\partial x} + \frac{\partial V_1}{\partial y}) + z \frac{(1+\nu)(1-2\nu)}{2E(1-\nu)} H q_o$$

$$\gamma_{yz} = V_1 + \frac{\partial W_o}{\partial y} + z \frac{(1+\nu)(1-2\nu)}{2E(1-\nu)} \frac{\partial q_o}{\partial y}$$

$$\gamma_{zx} = U_1 + \frac{\partial W_o}{\partial x} + z \frac{(1+\nu)(1-2\nu)}{2E(1-\nu)} \frac{\partial q_o}{\partial x} \qquad (3.8)$$

C. STRESS-STRAIN RELATIONS

The following are the stress-strain relations:

$$\sigma_x = 2 G \varepsilon_x + \lambda (\varepsilon_x + \varepsilon_y + \varepsilon_z)$$

$$\sigma_y = 2 G \varepsilon_y + \lambda (\varepsilon_x + \varepsilon_y + \varepsilon_z)$$

$$\sigma_z = 2 G \varepsilon_z + \lambda (\varepsilon_x + \varepsilon_y + \varepsilon_z)$$

$$\tau_{xy} = G \gamma_{xy} \qquad (3.9)$$

$$\tau_{yz} = k G \gamma_{yz}$$

$$\tau_{zx} = k G \gamma_{zx}$$

The factor 'k' is introduced in order to take into account the non-linear distribution of shear stresses across the cross section of the plate. It has the same significance as the Timoshenko shear coefficient. The value $\frac{\pi^2}{12}$ for k^2 for $\nu = 0.3$ is suggested by Mindlin (30), and is unity if shear effects are neglected.

Mindlin obtained the value of k from an equation where the wave velocity of the wave length is given in the form of a trancendental equation. The value of k obtained from the equation was tested with that obtained from the known exact solution for straight crested flexural waves.

D. STRAIN ENERGY

The expression for strain energy including the energy of plate compression and transverse shear:

$$U_E = \tfrac{1}{2} \int_V \Big[(\sigma_x \varepsilon_x + \sigma_y \varepsilon_y + \sigma_z \varepsilon_z)\, dx\, dy\, dz + (\tau_{xy} \gamma_{xy} + \tau_{yz} \gamma_{yz} + \tau_{zx} \gamma_{zx})\, dx\, dy\, dz \Big] \qquad (3.10)$$

Using equations (3.9) evaluate the following:

$$\sigma_x \varepsilon_x = (2 G \varepsilon_x + \lambda e) \varepsilon_x = 2 G \varepsilon_x 2 + \lambda e \varepsilon_x$$

$$\sigma_y \varepsilon_y = (2 G \varepsilon_y + \lambda e) \varepsilon_y = 2 G \varepsilon_y 2 + \lambda e \varepsilon_y \qquad (3.11)$$

$$\sigma_z \varepsilon_z = (2 G \varepsilon_z + \lambda e) \varepsilon_z = 2 G \varepsilon_z 2 + \lambda e \varepsilon_z$$

$$\tau_{xy} \gamma_{xy} = (G \gamma_{xy}) \gamma_{xy} = G \gamma_{xy} 2$$

$$\tau_{yz} \gamma_{yz} = (k G \gamma_{yz}) \gamma_{yz} = k G \gamma_{yz} 2 \qquad (3.12)$$

$$\tau_{zx} \gamma_{zx} = (k G \gamma_{zx}) \gamma_{zx} = k G \gamma_{zx} 2$$

Where $e = \varepsilon_x + \varepsilon_y + \varepsilon_z$

Adding the equations of (3.11) we have:

$$(\sigma_x \varepsilon_x + \sigma_y \varepsilon_y + \sigma_z \varepsilon_z) = \lambda e (\varepsilon_x + \varepsilon_y + \varepsilon_z) + 2 G (\varepsilon_x^2 + \varepsilon_y^2 + \varepsilon_z^2) \qquad (3.13)$$

Similarly, adding the equations of (3.12) we have:

$$(\tau_{xy} \gamma_{xy} + \tau_{yz} \gamma_{yz} + \tau_{zx} \gamma_{zx}) = G (\gamma_{xy}^2 + k \gamma_{yz}^2 + k \gamma_{zx}^2) \quad (3.14)$$

Substituting equations (3.13) and (3.14) into equation (3.10) we have the following expression for the strain energy:

$$U_E = \tfrac{1}{2} \int_V [\lambda e(\varepsilon_x + \varepsilon_y + \varepsilon_z) + 2 G (\varepsilon_x^2 + \varepsilon_y^2 + \varepsilon_z^2) + G (\gamma_{xy}^2 + k \gamma_{yz}^2 + k \gamma_{zx}^2)] \, dx \, dy \, dz \quad (3.15)$$

Substituting the strain-displacement relations (3.8) into the above equation (3.15) for strain-energy yields:

$$
\begin{aligned}
U_E = \tfrac{1}{2} \int_V & \Big[\lambda \{ z \frac{\partial U_1}{\partial x} + z \frac{\partial V_1}{\partial y} - \frac{\nu}{(1-\nu)} z \left(\frac{\partial U_1}{\partial x} + \frac{\partial V_1}{\partial y} \right) \\
& + \frac{(1+\nu)(1-2\nu)}{E(1-\nu)} (\tfrac{1}{2} + z/H)\, q_o \}^2 + 2\, G\, \{ z^2 \left(\frac{\partial U_1}{\partial x} \right)^2 + z^2 \left(\frac{\partial V_1}{\partial y} \right)^2 \\
& + \frac{\nu^2}{(1-\nu)^2} z^2 \left(\frac{\partial U_1}{\partial x} + \frac{\partial V_1}{\partial y} \right)^2 + \frac{(1+\nu)^2 (1-2\nu)^2}{E^2 (1-\nu)^2} (\tfrac{1}{2} + z/H)^2\, q_o^2 \\
& - \frac{2\,(1+\nu)(1-2\nu)}{E(1-\nu)^2} z (\tfrac{1}{2} + z/H) \left(\frac{\partial U_1}{\partial x} + \frac{\partial V_1}{\partial y} \right) q_o \} + \{ G\, z^2 \left(\frac{\partial U_1}{\partial y} + \frac{\partial V_1}{\partial x} \right) \\
& + k \left(V_1 + \frac{\partial W_o}{\partial y} + z \frac{(1+\nu)(1-2\nu)}{2E(1-\nu)} \frac{\partial q_o}{\partial y} \right)^2 + k \left(U_1 + \frac{\partial W_o}{\partial x} + z \frac{(1+\nu)(1-2\nu)}{2E(1-\nu)} \right. \\
& \left. \left(\frac{\partial q_o}{\partial x} \right)^2 \} \right] dx\, dy\, dz. \qquad (3.16)
\end{aligned}
$$

Simplifying and integrating equation (3.16) with respect to z we obtain:

$$U_E = \frac{1}{2} \int_A \Big[\frac{EI}{1-\nu^2} \left(\frac{\partial U_1}{\partial x}\right) + \frac{EI}{1-\nu^2} \left(\frac{\partial V_1}{\partial y}\right)$$

$$+ \frac{2\nu EI}{1-\nu^2} \left(\frac{\partial U_1}{\partial x} \frac{\partial V_1}{\partial y}\right) + \frac{E}{2(1+\nu)} \Big\{ I\left(\frac{\partial U_1}{\partial y}\right)^2 + I\left(\frac{\partial V_1}{\partial x}\right)^2$$

$$+ 2\ I\left(\frac{\partial U_1}{\partial y} \frac{\partial V_1}{\partial x}\right) + K\ H\ V_1{}^2 + 2\ k\ H\ V_1 \frac{\partial W_o}{\partial y} + k\ H\ U_1{}^2$$

$$+ 2\ k\ U_1 \frac{\partial W_o}{\partial x} + k\ H\ \left(\frac{\partial W_o}{\partial y}\right)^2 + k\ H\ \left(\frac{\partial W_o}{\partial x}\right)^2$$

$$+ \frac{(2G+\lambda)(1+\nu)^2(1-2\nu)^2}{E^2(1-\nu)^2} \frac{H}{3} q_o{}^2 + \frac{G\ K\ I(1+\nu)^2(1-2\nu)^2}{4E^2(1-\nu)^2}$$

$$\Big\{\left(\frac{\partial q_o}{\partial x}\right)^2 + \left(\frac{\partial q_o}{\partial y}\right)^2\Big\}\Big]\ dx\ dy. \qquad (3.17)$$

where $I = \int_{-H/2}^{+H/2} z^2\ dz = \frac{H^3}{12}$

E. KINETIC ENERGY

The expression for the kinetic energy is:

$$K_E = \frac{\rho}{2} \int_V \left[\left(\frac{\partial \bar{U}}{\partial t}\right)^2 + \left(\frac{\partial \bar{V}}{\partial t}\right)^2 + \left(\frac{\partial \bar{W}}{\partial t}\right)^2 \right] dx\, dy\, dz \qquad (3.18)$$

The expressions for the particle velocities are obtained by differentiating equation (3.2) with respect to time and neglecting the terms containing z^2 in comparison with the linear terms. Thus we have:

$$\frac{\partial \bar{U}}{\partial t} = z \frac{\partial \bar{U}_1}{\partial t} \quad ; \quad \frac{\partial \bar{V}}{\partial t} = z \frac{\partial \bar{V}_1}{\partial t}$$

$$\frac{\partial \bar{W}}{\partial t} = \frac{\partial W_o}{\partial t} + z \frac{(1+\nu)(1-2\nu)}{2E(1-\nu)} \frac{\partial q_o}{\partial t} \qquad (3.19)$$

Substituting equation (3.19) into equation (3.18) we have the following expression for kinetic energy:

$$K_E = \frac{\rho}{2} \int_V \left[\left(z \frac{\partial U_1}{\partial t} \right)^2 + \left(z \frac{\partial V_1}{\partial t} \right)^2 + \left(\frac{\partial W_o}{\partial t} \right)^2 \right.$$

$$\left. + \; z \frac{(1+\nu)(1-2\nu)}{2E(1-\nu)} \frac{\partial q_o}{\partial t})^2 \right] dx\ dy\ dz$$

Integrating the above expressions for kinetic energy with respect to z we obtain:

$$K_E = \frac{\rho}{2} \int_A \left[I \left(\frac{\partial U_1}{\partial t} \right)^2 + I \left(\frac{\partial V_1}{\partial t} \right)^2 + H \left(\frac{\partial W_o}{\partial t} \right)^2 \right.$$

$$\left. + \; \frac{(1+\nu)^2(1-2\nu)^2}{4E^2(1-\nu)^2} \; I \left(\frac{\partial q_o}{\partial t} \right)^2 \right] dx\ dy \qquad (3.20)$$

F. WORK DONE BY EXTERNAL FORCES

Let the intensity of the normally distributed force be $q_o(x,y,t)$ and the magnitude of the in-plane forces acting in the middle plane of the plate per unit length be N_x, N_y, and N_{xy} as shown in Fig. 1.

Projecting these forces on the x and y axes and assuming that there are no body forces or tangential forces acting in these directions on the faces of the plate, we obtain the following equations of equilibrium in the x and y directions respectively:

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0$$
$$\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} = 0 \qquad (3.21)$$

In considering the projection of the forces shown in Fig. 1., on the z-axis, we must take into account the bending of the plate and the resulting small angles between the forces N_x and N_y that act on the opposite sides of the element. As a result of this bending the projection of the normal forces N_x on the z-axis is:

$$- \quad N_x \, dy \frac{\partial W}{\partial x} \quad + \quad (N_x + \frac{\partial N_x}{\partial x} dx)(\frac{\partial W}{\partial x} + \frac{\partial^2 W}{\partial x^2} dx) \; dy \qquad (3.22)$$

After simplification, if the small quantities of higher than the second order are neglected, this projection becomes:

$$N_x \frac{\partial^2 W}{\partial x^2} dx \; dy \quad + \quad \frac{\partial N_x}{\partial x} \frac{\partial W}{\partial x} dx \; dy \qquad (3.23)$$

Similarly, the projection of the inplane forces N_y on the z-axis is:

$$N_y \frac{\partial^2 W}{\partial y^2} dx \; dy \quad + \quad \frac{\partial N_y}{\partial y} \frac{\partial W}{\partial y} dx \; dy \qquad (3.24)$$

Considering the projection of the shearing forces N_{xy} on the z-axis, we observe that the slope of the deflection surface in the y-direction on the two opposite sides of the element is $\frac{\partial W}{\partial y}$ and $\frac{\partial W}{\partial y} + \frac{\partial^2 W}{\partial x \partial y} dx$. Hence the jection of the shearing forces on the z-axis is equal to:

$$N_{xy} \frac{\partial^2 W}{\partial x \partial y} dx\ dy + \frac{\partial N_{xy}}{\partial x} \frac{\partial W}{\partial y} dx\ dy \qquad (3.25)$$

An analogous expression can be obtained for the projection of the shearing forces $N_{yx} = N_{xy}$ on the z-axis. The final expression for the projection of all the shearing forces on the z-axis can be written as:

$$2\ N_{xy} \frac{\partial^2 W}{\partial x \partial y} dx\ dy + \frac{\partial N_{xy}}{\partial x} \frac{\partial W}{\partial y} dx\ dy + \frac{\partial N_{xy}}{\partial y} \frac{\partial W}{\partial x} dx\ dy \qquad (3.26)$$

Adding expressions (3.23), (3.24), and (3.26) to the load q_o dx dy acting on the element and using equation (3.21) yields:

$$q_o + N_x \frac{\partial^2 W}{\partial x^2} + N_y \frac{\partial^2 W}{\partial y^2} + 2\ N_{xy} \frac{\partial^2 W}{\partial x \partial y} \qquad (3.27)$$

In our case we have uniform boundary force, that is, $N_{xy} = 0$ and $N_x = N_y = N$. Therefore equation (3.27) reduces to:

$$q_o + N\nabla^2 W \tag{3.28}$$

The virtual work done by the external force $q_o(x,y,t)$ and the inplane force N in a virtual displacement δW is:

$$\delta W_E = \int_A (q_o + N\nabla^2 W) \ \delta W \ dx \ dy \tag{3.29}$$

G. HAMILTON'S PRINCIPLE AND THE DIFFERENTIAL EQUATION:

Hamilton's principle is:

$$\delta \int_{t_o}^{t_1} (U_E - K_E)\ dt = \int_{t_o}^{t_1} \delta\ W_E\ dt \tag{3.30}$$

which can also be written as:

$$\delta \int_{t_o}^{t_1} (K_E - U_E + W_E)\ dt = 0 \tag{3.31}$$

Substituting the expressions (3.16), (3.20) and (3.29) for the kinetic energy, strain energy and the work done by the external forces respectively into the above equation (3.31) yields:

$$\delta \int_{t_o}^{t_1} \{\frac{\rho}{2} \int_A [I(\frac{\partial U_1}{\partial t})^2 + I(\frac{\partial V_1}{\partial t})^2 + H(\frac{\partial W_o}{\partial t})^2$$

$$+ \frac{(1+\nu)^2(1-2\nu)^2}{4E^2(1-\nu)^2}\ I(\frac{\partial q_o}{\partial t})^2]\ dx\ dy - \frac{1}{2} \int_A [\frac{EI}{1-\nu^2}\ (\frac{\partial U_1}{\partial x})^2$$

$$+ \frac{EI}{1-\nu^2}\left(\frac{\partial V_1}{\partial y}\right)^2 + \frac{2\nu EI}{1-\nu^2}\left(\frac{\partial U_1}{\partial x}\frac{\partial V_1}{\partial y}\right) + \frac{E}{2(1+\nu)} I\left(\frac{\partial U_1}{\partial y}\right)^2$$

$$+ I\left(\frac{\partial V_1}{\partial x}\right)^2 + 2\, I\left(\frac{\partial U_1}{\partial y}\frac{\partial V_1}{\partial x}\right) + k\, H\, V_1^{\,2} + 2\, k\, H\, V_1 \frac{\partial W_o}{\partial y}$$

$$+ k\, H\, U_1^2 + 2\, k\, H\, U_1 \frac{\partial W_o}{\partial x} + k\, H\left(\frac{\partial W_o}{\partial y}\right)^2 + k\, H\left(\frac{\partial W_o}{\partial x}\right)^2\}$$

$$+ \frac{(2G+\lambda)(1+\nu)^2(1-2\nu)^2}{E^2(1-\nu)^2}\frac{H}{3} q_o^2 + \frac{G\, k\, I(1+\nu)^2(1-2\nu)^2}{4E^2(1-\nu)^2}\{\left(\frac{\partial q_o}{\partial x}\right)^2$$

$$+ \left(\frac{\partial q_o}{\partial y}\right)^2\}]\ dx\ dy\}\ dt + \int_{t_o}^{t_1}\int_A (q_o + N\ \nabla^2 W)\ \delta W\ dx\ dy\ dt = 0 \qquad (3.32)$$

Performing the variation and grouping the δU_1, δV_1 and δW_1 terms separately and setting each of them equal to zero we have:

<u>δU_1 terms</u>:

$$D \frac{\partial^2 U_1}{\partial x^2} + \frac{(1+\nu)}{2} D \frac{\partial^2 V_1}{\partial x \partial y} + \frac{1-\nu}{2} D \frac{\partial^2 U_1}{\partial y^2}$$

$$- \frac{E\,H\,k}{2(1+\nu)} \frac{\partial W_o}{\partial x} - \frac{E\,H\,k}{2(1+\nu)} U_1 - \frac{\rho H^3}{12} \frac{\partial^2 U_1}{\partial t^2} = 0 \tag{3.33}$$

δV_1 terms:

$$D \frac{\partial^2 V_1}{\partial y^2} + \frac{1+\nu}{2} D \frac{\partial^2 U_1}{\partial x \partial y} + \frac{1-\nu}{2} D \frac{\partial^2 V_1}{\partial x^2}$$

$$- \frac{E\,H\,k}{2(1+\nu)} \frac{\partial W_o}{\partial y} - \frac{E\,H\,k}{2(1+\nu)} V_1 - \frac{\rho H^3}{12} \frac{\partial^2 V_1}{\partial t^2} = 0 \tag{3.34}$$

δW_o terms:

$$\frac{E\,H\,k}{2(1+\nu)} \nabla^2 W_o + \frac{E\,H\,k}{2(1+\nu)} \left(\frac{\partial U_1}{\partial x} + \frac{\partial V_1}{\partial y}\right)$$

$$- \rho H \frac{\partial^2 W_o}{\partial t^2} + q_o + N \nabla^2 W_o = 0 \tag{3.35}$$

Therefore we have 3 equations in terms of U_1, V_1 and W_o. These are:

$$D \frac{\partial^2 U_1}{\partial x^2} + \frac{1+\nu}{2} D \frac{\partial^2 V_1}{\partial x \partial y} + \frac{1-\nu}{2} D \frac{\partial^2 U_1}{\partial y^2}$$

$$- \frac{E\ H\ k}{2(1+\nu)} \frac{\partial W_o}{\partial x} - \frac{E\ H\ k}{2(1+\nu)} U_1 = \frac{\rho H^3}{12} \frac{\partial^2 U_1}{\partial t^2} \quad . \tag{3.36}$$

$$D \frac{\partial^2 V_1}{\partial y^2} + \frac{1+\nu}{2} \frac{\partial^2 U_1}{\partial x \partial y} + \frac{1-\nu}{2} D \frac{\partial^2 V_1}{\partial x^2}$$

$$- \frac{E\ H\ k}{2(1+\nu)} \frac{\partial W_o}{\partial y} - \frac{E\ H\ k}{2(1+\nu)} V_1 = \frac{\rho H^3}{12} \frac{\partial^2 V_1}{\partial t^2} \quad . \tag{3.37}$$

$$\frac{E\ H\ k}{2(1+\nu)} \nabla^2 W_o + \frac{E\ H\ k}{2(1+\nu)} \left(\frac{\partial U_1}{\partial x} + \frac{\partial V_1}{\partial y}\right)$$

$$= \rho H \frac{\partial^2 W_o}{\partial t^2} - q_o - N \nabla^2 W_o \quad . \tag{3.38}$$

Differentiating equation (3.36) with respect to x and equation (3.37) with respect to y and adding them together results in:

$$\left(D \nabla^2 - \frac{E H k}{2(1+\nu)} - \frac{\rho H^3}{12} \frac{\partial^2}{\partial t^2}\right) \left(\frac{\partial U_1}{\partial x} + \frac{\partial V_1}{\partial y}\right)$$

$$= \frac{E H k}{2(1+\nu)} \nabla^2 W_o \qquad (3.39)$$

Eliminating the term $\left(\frac{\partial U_1}{\partial x} + \frac{\partial V_1}{\partial y}\right)$ between the above equation and equation (3.38) we obtain:

$$\left(\nabla^2 - \frac{2(1+\nu)}{E k} \frac{\partial^2}{\partial t^2}\right) \left(D \nabla^2 - \frac{\rho H^3}{12} \frac{\partial^2}{\partial t^2}\right) W_o$$

$$+ \rho H \frac{\partial^2 W_o}{\partial t^2} = \left(1 - \frac{H^2}{6(1-\nu)k} \nabla^2 + \frac{\rho H^2(1+\nu)}{6Ek} \frac{\partial^2}{\partial t^2}\right)$$

$$(q_o + N \nabla^2 W_o) \qquad (3.40)$$

The above equation (3.40) is the general differential equation of motion of an elastic plate under normal and inplane forces, including the effects of shear and rotatory inertia.

Neglecting the terms due to shear and rotatory inertia in equation (3.40) we obtain the classical equation:

$$D \nabla^4 W_o + \rho H \frac{\partial^2 W_o}{\partial t^2} = q_o + N \nabla^2 W_o \tag{3.41}$$

IV. EQUATIONS DEDUCIBLE FROM THE GENERAL EQUATION:

1. Neglecting shear and rotatory inertia:

a). Static plate equation with a uniform load:

$$D\nabla^4 W_o = q_o\ (x,y) \tag{4.1}$$

b). Forced plate vibration:

$$D\nabla^4 W_o + \rho H \frac{\partial^2 W_o}{\partial t^2} = q_o\ (x,y,t) \tag{4.2}$$

c). Buckling problem:

$$D\nabla^4 W_o + N\nabla^2 W_o = 0 \tag{4.3}$$

d). Vibration with inplane forces:

$$D\nabla^4 W_o + N\nabla^2 W_o + \rho H \frac{\partial^2 W_o}{\partial t^2} = 0 \tag{4.4}$$

Cases a) and b) have been studied by several researchers such as Boidine (3), Leissa (31), Molachlan (36), McNitt (37), Reid (51) and Yu (73) in rectangular, polar, and elliptical coordinates for various shaped plates and for several boundary conditions.

Case c) has been studied by Bradley (4), Conway (11), Dean (12), Herrmann (23), Mansfield

(34), Yamaki (71) and several other researchers for various loading and boundary conditions.

Case d) has been studied by Kaul (25), Lurie (33), Wah (68), and several other investigators for various shaped plates with different boundary and loading conditions.

2. Including the effects of shear and rotatory inertia:

a). Static plate equation with a uniform load:

$$D\nabla^4 W_o = \left(1 - \frac{H^2}{6(1-\nu)k}\nabla^2\right) q_o\,(x,y) \qquad (4.5)$$

b). Forced plate vibration:

$$\left(\nabla^2 - \frac{2\rho(1+\nu)}{E\ k}\frac{\partial^2}{\partial t^2}\right)\left(D\ \nabla^2 - \frac{\rho H^3}{12}\frac{\partial^2}{\partial t^2}\right) W_o$$

$$+ \rho H\frac{\partial^2 W_o}{\partial t^2} = \left(1 - \frac{H^2}{6(1-\nu)k}\nabla^2 + \frac{\rho H^2(1+\nu)}{6Ek}\frac{\partial^2}{\partial t^2}\right) q_o \qquad (4.6)$$

c). Buckling problem:

$$D\nabla^4 W_o = \left(1 - \frac{H^2}{6(1-\nu)k}\nabla^2\right)(-N\ \nabla^2 W_o) \qquad (4.7)$$

d). Vibration with inplane forces:

$$\left(\nabla^2 - \frac{2\rho(1+\nu)}{Ek}\frac{\partial^2}{\partial t^2}\right)\left(D\ \nabla^2 - \frac{\rho H^3}{12}\frac{\partial^2}{\partial t^2}\right) W_o$$

$$+ \rho H\frac{\partial^2 W_o}{\partial t^2} = \left(1 - \frac{H^2}{6(1-\nu)k}\nabla^2 + \frac{\rho H^2(1+\nu)}{6Ek}\frac{\partial^2}{\partial t^2}\right) N\ \nabla^2 W_o \qquad (4.8)$$

Case a) is identical to the equation obtained by Reissner (54) and studied by Green (19). Case b) has been studied by Callahan (8), Fettis (14) and Lee (30) who developed certain functions which took into account the effect of shear and rotatory inertia. No numerical work is available.

To the best knowledge of this author cases c) and d) have not been previously derived. Equations (4.7) and (4.8) are derived in this study and are solved for simply supported square, circular, annular circular and annular elliptical plates using the finite difference method.

V. NUMERICAL PROCEDURE

A. Reduction of the fourth order differential equation to a second order:

Equation (4.8) is:

$$\left(\frac{E\,H^3}{12(1-\nu^2)} + \frac{H^2 N}{6(1-\nu)k}\right) \nabla^4 W_o = \left(\frac{\rho H^3}{12} + \frac{\rho H^3}{6k(1-\nu)} + \frac{\rho H^2(1+\nu)N}{6Ek}\right) \frac{\partial^2}{\partial t^2} \nabla^2 W_o + N\,\nabla^2 W_o - \rho H \frac{\partial W_o}{\partial t^2} - \frac{2\,(1+\nu)}{Ek} \frac{\rho H^3}{12} \frac{\partial^4 W}{\partial t^4}\,. \tag{5.1}$$

Define:

$$\begin{aligned}
\bar{A} &= \frac{\dfrac{\rho H^3}{12} + \dfrac{\rho H^3}{6k(1-\nu)} + \dfrac{\rho H^2(1+\nu)N}{6Ek}}{\dfrac{EH^3}{12(1-\nu^2)} + \dfrac{H^2N}{6(1-\nu)k}} \\
\bar{B} &= \frac{N}{\dfrac{EH^3}{12(1-\nu^2)} + \dfrac{H^2N}{6(1-\nu)k}} \\
\bar{C} &= \frac{\rho H}{\dfrac{EH^3}{12(1-\nu^2)} + \dfrac{H^2N}{6(1-\nu)k}} \\
\bar{D} &= \frac{\dfrac{\rho H^3(1+\nu)}{6Ek}}{\dfrac{EH^3}{12(1-\nu^2)} + \dfrac{H^2N}{6(1-\nu)k}}
\end{aligned} \tag{5.2}$$

Substituting expressions (5.2) into equation (5.1) we have:

$$\nabla^4 W_o = \bar{A} \frac{\partial^2}{\partial t^2} \nabla^2 W_o + \bar{B} \nabla^2 W_o - \bar{C} \frac{\partial W_o}{\partial t^2} - \bar{D} \frac{\partial^4 W_o}{\partial t^4} \qquad (5.3)$$

Let $W_o(x,y,t) = W(x,y) \, Cos(pt)$ (5.4)

where 'p' is the frequency in radians per second.

Differentiating equation (5.4) with respect to time and substituting into equation (5.3) yields:

$$\nabla^4 W = - \bar{A} \, p^2 \nabla^2 W + \bar{B} \, \nabla^2 W + \bar{C} p^2 W - \bar{D} \, p^4 W. \qquad (5.5)$$

The above equation can be written as:

$$\nabla^4 W = -(\bar{A} p^2 - \bar{B}) \, \nabla^2 W - (\bar{D} \, p^4 - \bar{C} \, p^2) W \qquad (5.6)$$

Introduce a function M defined as:

$$M = - \bar{a}^2 \left(\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2}\right),$$ $\bar{a}$ is the grid size,

which can be written as:

$$M = - \bar{a}^2 \, \nabla^2 W \qquad (5.7)$$

Differentiating equation (5.7) twice with respect to x and y respectively and adding yields:

$$\nabla^2 M = - \bar{a}^2 \, \nabla^4 W \qquad (5.8)$$

Substituting equations (5.7) and (5.8) into equation (5.6) we have:

$$-\frac{1}{\bar{a}^2}\,\nabla^2 M \;=\; (\bar{A}\;p^2 - \bar{B})\,\frac{M}{\bar{a}^2} \;-\; (\bar{D}\;p^4 - \bar{C}\;p^2)\;W \qquad (5.9)$$

Multiplying the above equation (5.9) by $-\bar{a}^4$ yields:

$$\bar{a}^2\;\nabla^2 M \;=\; -\,\bar{a}^2(\bar{A}\;p^2 - \bar{B})\;M \;+\; \bar{a}^4(\bar{D}\;p^4 - \bar{C}\;p^2)W \qquad (5.10)$$

Define:

$$\begin{aligned} R &= \bar{a}^2\;\bar{A} \\ S &= \bar{a}^2\;\bar{B} \\ T &= \bar{a}^2\;\bar{C} \\ U &= \bar{a}^2\;\bar{D} \end{aligned} \qquad (5.11)$$

Substituting the expressions (5.11) into equation (5.10) yields:

$$-\bar{a}^2\;\nabla^2 M \;=\; (Rp^2 - S)M \;-\; (Up^4 - Tp^2)W \qquad (5.12)$$

Thus instead of a fourth order differential equation we have two second order differential equations (5.7) and (5.12).

Recall that for a simply supported boundary condition $W = 0$ and $\frac{\partial^2 W}{\partial x^2} = \frac{\partial^2 W}{\partial y^2} = 0$. These conditions are suitable for the above numerical technique since we can set:

$$W_{ij} = 0$$

$$\text{and } M_{ij} = 0$$

for the mesh points on the boundaries.

B. FINITE DIFFERENCE METHOD:

The Laplacian operator ∇^2 is written in its finite difference form as:

$$\bar{a}^2 \nabla^2 W_{ij} = (W_{i+1,j} + W_{i-1,j} + W_{i,j+1} + W_{i,j-1} - 4 W_{i,j}) \tag{5.13}$$

Note that there are five points involved in the above finite difference equation; points to the right, left, above and below the central point (x_i , y_i). This finite difference representation for the Laplacian operator has an error of $0(\bar{a}^2)$, provided that W is sufficiently smooth.

It is convenient to represent the above equation (5.13) pictorially, where the linear combination of W's is represented graphically as:

$$\nabla^2 W_{ij} = \frac{1}{a^2} \begin{bmatrix} & 1 & \\ 1 & -4 & 1 \\ & 1 & \end{bmatrix} W_{ij} \qquad (5.14)$$

Writing equation (5.7) in pictorial form:

$$\begin{bmatrix} & 1 & \\ 1 & -4 & 1 \\ & 1 & \end{bmatrix} W_{ij} = M_{ij} \qquad (5.15)$$

This can also be written as:

$$\begin{bmatrix} & -1 & \\ -1 & 4 & -1 \\ & -1 & \end{bmatrix} W_{ij} = M_{ij} \qquad (5.16)$$

Writing equation (5.12) in pictorial form:

$$\begin{bmatrix} & -1 & \\ -1 & 4 & -1 \\ & -1 & \end{bmatrix} M_{ij} = (Rp^2-S)M_{ij} - (Up^4-Tp^2)\, W_{ij} \qquad (5.17)$$

Eliminating M_{ij} from the above equation by using equation (5.16):

$$\begin{bmatrix} & -1 & \\ -1 & 4 & -1 \\ & -1 & \end{bmatrix} \begin{bmatrix} & -1 & \\ -1 & 4 & -1 \\ & -1 & \end{bmatrix} W_{ij} = (Rp^2-S)$$

$$\begin{bmatrix} & -1 & \\ -1 & 4 & -1 \\ & -1 & \end{bmatrix} W_{ij} - (Up^4-Tp^2)\; W_{ij} \qquad (5.18)$$

The pictorial representation for the Laplacian operator when written for any particular grid becomes an augmented coefficients matrix, the W_{ij}'s and the M_{ij}'s become column vectors. These are shown for various shaped grids in Section VI - Results and Discussion. Defining the pictorial operator for any general grid as matrix $[\alpha]$, we can write the equation (5.18) in terms of a matrix $[\alpha]$ and column vectors:

$$[\alpha]\ [\alpha]\ \{W_{ij}\} = (Rp^2 - S)\ [\alpha]\ \{W_{ij}\} - (Up^4 - Tp^2)\ \{W_{ij}\} \qquad (5.19)$$

Define: $[\beta] = [\alpha]\ [\alpha]$ (5.20)

Then equation (5.19) becomes:

$$[\beta]\ \{W_{ij}\} = (Rp^2 - S)\ [\alpha]\ \{W_{ij}\} - (Up^4 - Tp^2)\ \{W_{ij}\} \qquad (5.21)$$

The above equation is used to study the vibration and buckling problem, in which the effects of shear, rotatory inertia and inplane forces are included in the constants R, S, T and U.

C. VIBRATION PROBLEM:

In order to evaluate the natural frequencies we rearrange equation (5.21) and define a matrix $[P]$ as follows:

$$[P] = [\beta] - (Rp^2 - S)\,[\alpha] - (Up^4 - Tp^2)\,[I] \qquad (5.22)$$

Thus the vibration problem with inplane forces including the effects of shear and rotatory inertia for any simply supported plate reduces to:

$$[P]\,\{W_{ij}\} = 0$$

The determinant of matrix $[P]$ is evaluated to obtain the natural frequencies of the plate. The matrix $[P]$ has the natural frequency of the plate 'p' as the only unknown and the correct value of 'p' makes the determinant of matrix $[P]$ equal to zero. The order of the matrix $[P]$ depends on the number of mesh points taken for a particular grid as shown in Section VI.

D. BUCKLING PROBLEM:

For evaluating the critical buckling load we set the constants R, T, and U in equation (5.12) to be equal to zero. Thus equation (5.21) reduces to:

$$[\beta] \{W_{ij}\} = -S [\alpha] \{W_{ij}\} \qquad (5.24)$$

Rearrange the above equation and define:

$$[B] = [\beta] + S [\alpha] \qquad (5.25)$$

Then the buckling problem including the effects of shear for a simply supported plate reduces to:

$$[B] \{W_{ij}\} = 0 \qquad (5.26)$$

The determinant of matrix [B] is evaluated to determine the critical buckling load of a plate. The matrix [B] has the critical buckling load as the unknown and the correct value makes the determinant of matrix [B] equal to zero.

E. NUMERICAL EXAMPLE : ANNULAR ELLIPTICAL PLATE:

We shall now illustrate with an example how the finite difference technique is applied and how the matrices $[\alpha]$, $[P]$ and $[B]$ are formulated for an annular elliptical plate subjected to inplane forces and simply supported on its inner and outer edges.

It is evident from symmetry that the calculations need be extended over an area of one-fourth the plate only, as shown in Fig. 7. This area of the plate is divided into a number of square mesh of mesh size $\bar{a} = a/6$. This yields 28 mesh points for computation. With reference to Fig. 7 we write the difference equations at all grid points not on the boundaries for which M and W are different from zero. At the remaining nodes on the boundaries M and W are zero from the boundary conditions.

The difference equations for equation (5.16) for different mesh points are written as:

$n = 2$:

$$\frac{4}{a_1} W_2 - \frac{2}{1 + a_1} W_3 - \frac{4}{1 + a_1} W_6 = M_2$$

$n = 3$:

$$- W_2 + 4 W_3 - 2 W_7 = M_3$$

$n = 6$:

$$- \frac{2}{1 + a_2} W_2 + \frac{4}{a_2} W_6 - \frac{2}{1 + a_2} W_7 - \frac{2}{1 + a_2} W_{12} = M_6$$

$n = 7$:

$$- \frac{2}{1 + a_3} W_3 - \frac{2}{1 + a_3} W_6 + \frac{4}{a_3} W_7 - \frac{2}{1 + a_3} W_{13} = M_7$$

$n = 11$:

$$+ \frac{4}{a_4 \; a_5} W_{11} - \frac{4}{(1 + a_4)(1 + a_5)} W_{12} - \frac{4}{(1 + a_4)(1 + a_5)} W_{17}$$

$$= M_{11}$$

$n = 12$:

$$- W_6 - W_{11} + 4 W_{12} - W_{13} - W_{18} = M_{12}$$

$n = 13$:

$$- \frac{2}{1 + a_6} W_7 - \frac{2}{1 + a_6} W_{12} + \frac{4}{a_6} W_{13} - \frac{2}{1 + a_6} W_{19} = M_{13}$$

$n = 16:$

$$+ \frac{4}{a_7} W_{16} - \frac{4}{1 + a_7} W_{17} - \frac{2}{1 + a_7} W_{22} = M_{16}$$

$n = 17:$

$$- W_{11} - W_{16} + 4 W_{17} - W_{18} - W_{23} = M_{17}$$

$n = 18:$

$$- W_{12} - W_{17} + 4 W_{18} - W_{19} - W_{24} = M_{18}$$

$n = 19:$

$$- \frac{4}{(1+a_8)(1+a_9)} W_{13} - \frac{4}{(1+a_8)(1+a_9)} W_{18} + \frac{4}{a_8 a_9} W_{19} = M_{19}$$

$n = 22:$

$$- W_{16} + 4 W_{22} - 2 W_{23} = M_{22}$$

$n = 23:$

$$- \frac{2}{1+a_{10}} W_{17} - \frac{2}{1+a_{10}} W_{22} + \frac{4}{a_{10}} W_{23} - \frac{2}{1+a_{10}} W_{24} = M_{23}$$

$n = 24:$

$$- \frac{4}{(1+a_{11})(1+a_{12})} W_{18} - \frac{4}{(1+a_{11})(1+a_{12})} W_{23} + \frac{4}{a_{11}a_{12}} W_{24} = M_{24}$$

(5.27)

The above finite difference equations (5.27) can be written in a matrix form as:

$$[\text{COEFFICIENT MATRIX}]\,\{W_n\} = \{M_n\} \qquad (5.28)$$

This coefficient matrix is previously defined as matrix $[\alpha]$, and W_n and M_n are column vectors. The coefficient matrix is shown as:

Grid Size: $\bar{a} = 1.00$

The fractions of the grid size are:

$a_1 = 0.250$	$a_7 = 0.810$
$a_2 = 0.450$	$a_8 = 0.200$
$a_3 = 0.925$	$a_9 = 0.350$
$a_4 = 0.200$	$a_{10} = 0.850$
$a_5 = 0.300$	$a_{11} = 0.425$
$a_6 = 0.675$	$a_{12} = 0.325$

$4/a_1$	$-2/(1+a_1)$	$-4/(1+a_1)$	0	0	0	0	0	0	0	0	0	0	0
- 1	4	0	- 2	0	0	0	0	0	0	0	0	0	0
$-2/(1+a_2)$	0	$4/a_2$	$-2/(1+a_2)$	0	$-2/(1+a_2)$	0	0	0	0	0	0	0	0
0	$-2/(1+a_3)$	$-2/(1+a_3)$	$4/a_3$	0	0	$-2/(1+a_3)$	0	0	0	0	0	0	0
0	0	0	0	$4/a_4a_5$	$-4/(1+a_4)(1+a_5)$	0	0	$-4/((1+a_4)(1+a_5))$	0	0	0	0	0
0	0	- 1	0	- 1	4	- 1	0	0	- 1	0	0	0	0
0	0	0	$-2/(1+a_6)$	0	$-2/(1+a_6)$	$4/a_6$	0	0	0	$-2/(1+a_6)$	0	0	0
0	0	0	0	0	0	0	$4/a_7$	$-4/(1+a_7)$	0	0	$-2/(1+a_7)$	0	0
0	0	0	0	- 1	0	0	- 1	4	- 1	0	0	- 1	0
0	0	0	0	0	- 1	0	0	- 1	4	- 1	0	0	- 1
0	0	0	0	0	0	$-4/((1+a_8)(1+a_9))$	0	0	$-4/((1+a_8)(1+a_9))$	$4/a_8a_9$	0	0	0
0	0	0	0	0	0	0	- 1	0	0	0	4	- 2	0
0	0	0	0	0	0	0	0	$-2/(1+a_{10})$	0	0	$-2/(1+a_{10})$	$4/a_{10}$	$-2/(1+a_{10})$
0	0	0	0	0	0	0	0	0	$-4/((1+a_{11})(1+a_{12}))$	0	0	$-4/((1+a_{11})(1+a_{12}))$	$4/a_{11}a_{12}$

The Coefficient Matrix [α] for the Annular Elliptical Plate

Once matrix $[\alpha]$ is obtained, the matrices $[\beta]$, $[P]$ and $[B]$ are easily obtained from equations (5.20), (5.23) and (5.25) respectively.

In order to compute the natural frequency 'p' or the critical buckling load 'Ncr'; the value of 'p' or 'Ncr' is considered correct if it makes the determinant of its respective matrix to be zero. These values are approximated by using the following technique. First the matrices $[P]$ and $[B]$ are written as a product of their respective upper and lower triangular matrices. It is known from the lower and upper decomposition method (Appendix A) that the value of the determinant is the trace of the upper triangular matrix. The trace of the respective upper triangular matrices contain 'p' or 'Ncr'. Different values of 'p' or 'Ncr' are tried so that the correct value will make their respective determinant equal to zero. The iterative procedure used for this is the interpolation or the false position method (57).

Matrices $[P]$ and $[B]$ are evaluated by the lower and upper decomposition method (Appendix A) to determine the natural frequencies and the critical buckling load of the plate respectively. The computer program used to determine the matrices $[\beta]$, $[P]$ and $[B]$ and to evaluate the frequency 'p' and the critical buckling load 'Ncr' is given in the Appendix C. The results are given in Section VI D.

The numerical technique has been shown to give excellent results. The accuracy of the results depend on the grid size. Several mesh points were chosen to obtain good results. The values of the natural frequencies and critical buckling loads obtained using 40 to 50 mesh points compared very well (0 - 0.1%) with those obtained using 20 to 30 mesh points. Results obtained using the above numerical technique neglecting the effects of shear and rotatory inertia compared very well (0 - 0.5%) with those obtained by previous researchers.

VI. RESULTS AND DISCUSSIONS

A. Square Plate:

Using the technique discussed in Section V, the fundamental frequency for a simply supported square plate is determined using equation (5.23). The normal and inplane forces are not included in the computation. It is evident from symmetry that the grid need be extended only over an area of one-eighth of the plate; twenty mesh points are taken as shown in Fig. 2. The finite difference equations for these mesh points and the coefficient matrix $[\alpha]$ are given in Appendix B.1.

Natural frequencies for four different thickness-to-side ratios are computed to study the effects of shear and rotatory inertia. These effects decrease the natural frequency by 3.90 to .92 percent for plate thickness-to-side ratio of 0.1 to .025 respectively. The results are given in Table 1 and are also plotted in Fig. 8. It can be seen that the effects of shear and rotatory inertia are fairly significant for a relatively thick thin plate.

The results obtained by neglecting shear and rotatory inertia are in good agreement with data obtained by Conway and Leissa (10), Vet (63), Young (72) and others with classical theory.

B. Circular Plate:

Fundamental frequencies and critical buckling loads are studied for a simply supported circular plate. Due to the symmetry of the plate, only one eighth of the plate is used to construct the grid as shown in Fig. 3. The finite difference equations and their coefficient matrix are given in Appendix B.2. Higher natural frequencies and their respective mode shapes are also computed by extending the grid over half the plate as shown in Fig. 14. The difference equations for the 39 interrior mesh points and the method for determining the mode shapes are given in Appendix B.2.

Vibration:

Table 2. lists the fundamental frequencies for various thickness - to - diameter ratios. The percentage frequency decrease due to the effects of shear and rotatory inertia, is from 4.15% to 1.48% as the thickness to diameter ratio decreases from 0.1 to 0.025. The fundamental frequencies for these thickness - to - diameter ratios are plotted in Fig. 9.

Table 3. lists the first four natural frequencies and their respective mode shapes. The mode shapes are computed to check the accuracy of the computed frequencies. The nodal pattern for the third natural mode is plotted in Fig. 15. The percentage frequency decrease for the first four natural frequencies due to the effect of shear and rotatory inertia are 4.21, 4.98, 5.89 and 7.36 percent respectively, for a thickness to diameter ratio of 0.1. λp in Table 3. is defined as:

$$\lambda_p = pa^2\sqrt{\rho/D}$$

One notes that for fairly thick plates the effect of shear and rotatory inertia included in the computation give significant corrections to the classical frequencies. The correction is equally significant to higher natural frequencies.

Buckling:

Considering the effect of shear, the critical buckling loads for different thickness to diameter ratios are calculated from equation (5.26).

The critical buckling load reduces from 5.71 to 0.39 percent for thickness-to-diameter ratio of 0.1 to 0.025 respectively. The results are given in Table 4 and plotted on curves shown in Fig. 10 in which the quantity λ_N is the critical buckling load parameter defined as:

$$\lambda_N = Ncr\ a^2 \sqrt{\rho/D}$$

It is again observed that the effect of shear reduces the critical buckling load significantly for a relatively thick thin plate.

The values of the natural frequencies and the critical buckling load computed without the effects of shear and rotatory inertia, and shear respectively agree well with data obtained by Boidine (3), Conway (11), Dean (12), Yamaki (71) and others using the classical theory.

C. Annular Circular Plate:

For an annular circular plate shown in Fig. 4, the effects of shear on the critical buckling load for various thickness - to - diameter ratios is studied. The interrelationship between the tensile and compressive inplane forces on the natural frequency including the effects of shear and rotatory inertia is also examined. Both inner and outer edges are simply supported and subjected to the same inplane force intensity. Due to symmetry one eighth of the plate is used to construct the grid. The grid and mesh points are shown in Fig. 5. Twenty one mesh points are adopted to obtain their finite difference equations and the coefficient matrix $[\alpha]$. These are given in Appendix B.3.

Critical buckling loads for four different thickness-to-diameter ratios are evaluated using equation (5.27) including the effect of shear. The classical critical buckling load are also obtained by neglecting the effect of shear from the above equation. It is seen that the critical buckling load decreases from the classical case by 28.05% to 2.41% for a thickness-to-the-outside diameter ratio of 0.1 to 0.25 respectively. The results are tabulated in Table 5 and plotted in Fig. 11. Thus we see that the effects of shear are very significant on the buckling load as the thin plate thickness increases.

Vibration with Inplane Forces:

Next the influence of inplane forces on the natural frequencies is investigated. Fundamental frequencies for a range of tensile inplane forces and compressive inplane forces (not greater than the critical buckling load) are determined. These frequencies are computed from equation (5.2.3) including the effects of shear and rotatory inertia. The results are compared in Table 6 and shown in Fig. 12. The quantity ϕ_C in Table 6 is defined as:

$$\phi_C = \frac{Ncr\ a^2}{27.25\ D}$$

It is observed that the effects of shear and rotatory inertia are more important when the plate is under compressive inplane forces and become less significant as the tensile inplane forces get larger.

The values of the natural frequencies and the critical buckling loads evaluated using the classical theory agree well with the data obtained by Mansfield (34), Raju (48), Wah (68), Yamaki (71) and others.

D. Annular Elliptical Plate:

An annular elliptical plate was used as an illustration of the numerical technique discussed in Section - V.E. Fundamental frequencies for different values of tensile and compressive inplane forces, and the critical buckling load for a thickness-to-the-outer major axis of 0.1 were computed using equation (5.23). This equation includes the effects of shear and rotatory inertia. The results are compared in Table 7 and plotted on a curve in Fig. 13. The quantity ϕ_E is a multiple of the critical bcukling load defined as:

$$\phi_E = \frac{Ncr\ a^2}{27.4\ D}$$

It is again observed that the effects of shear and rotatory inertia are more important when the plate is subjected to compressive inplane forces.

To the best knowledge of the author no data using the classical theory is available in the existing literature concerning the critical buckling load or the natural frequency of an annular elliptical plate.

VII. CONCLUSION

From the numerical results obtained in this study for various shaped plates it is concluded that the effects of shear and rotatory inertia in vibration analysis gives lower values for the fundamental frequencies as compared to the results for the classical theory. Furthermore the influence of shear and rotatory inertia become more significant for higher modes of vibration. These effects become more significant as the plate thickness is increased and are very significant for the annular plates.

Inclusion of shear effect in the buckling study also shows that the critical buckling load is smaller than predicted by the classical non-shear case. Again, the shear effect becomes significant as the plate thickness is increased, particularly for annular plates.

It is also shown in the study that for annular plates the effects of shear and rotatory inertia are very important when the plates are subjected to large compressive inplane forces on both inner and outer edges and become less significant for large tensile inplane forces.

In general, it is concluded that the effects of shear and rotatory inertia on the natural frequencies and the effect of shear on the buckling load previously disregarded are very pronounced for sufficiently thick plates, particularly in the case of annular plates.

VIII. RECOMMENDATION

Equation of Motion:

The general differential equation of motion derived in this study is for thin isotropic elastic plates. The same method of analysis could be followed to obtain an equation of motion for orthotropic elastic plates. Further work could be carried out by following the work of Mossakowski (42), Pandalai (46) and Uthgennant and Brand (62).

Boundary Conditions:

The numerical method employed in this study is applicable to simply supported boundaries because only in this case does the fourth order partial differential equation reduce to a second order as shown in Section - V. It is felt that numerical techniques by Bramble (5) and Ehrlich (20) for solving biharmonic equations may be useful for other boundary conditions.

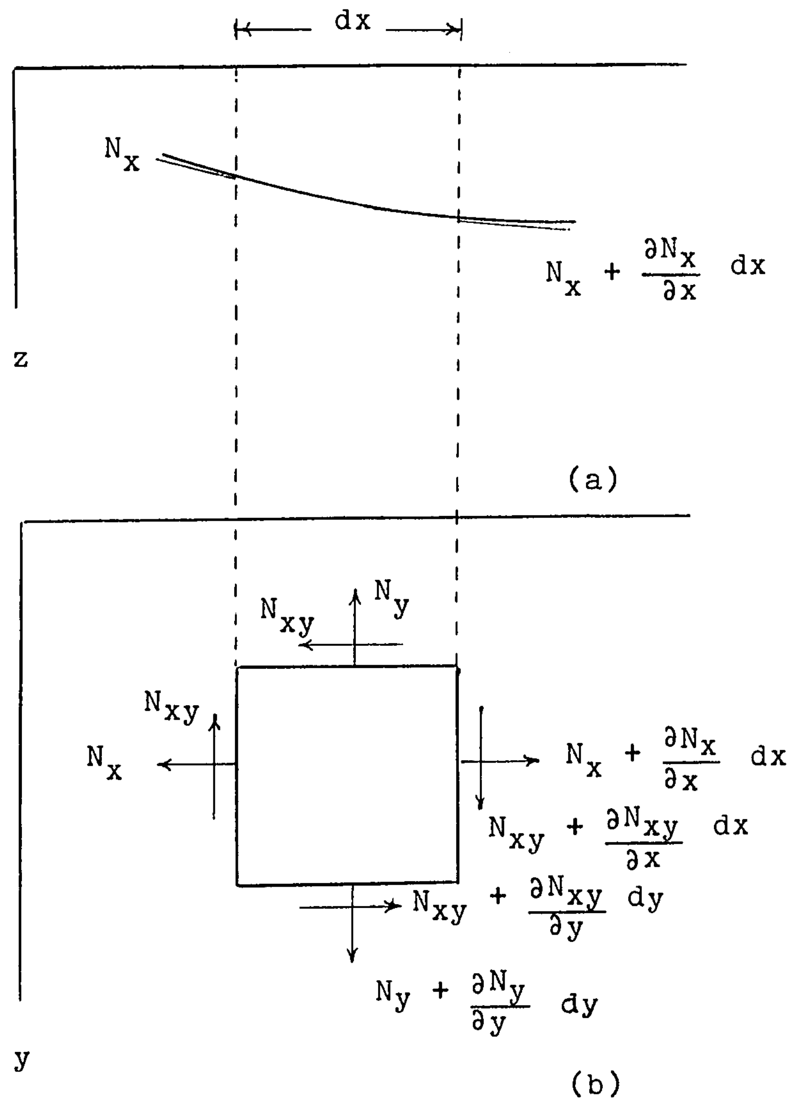


Figure 1. a) Deflected Surface due to Inplane Forces.

1. b) Equilibrium of a Small Element Subjected to Inplane Forces.

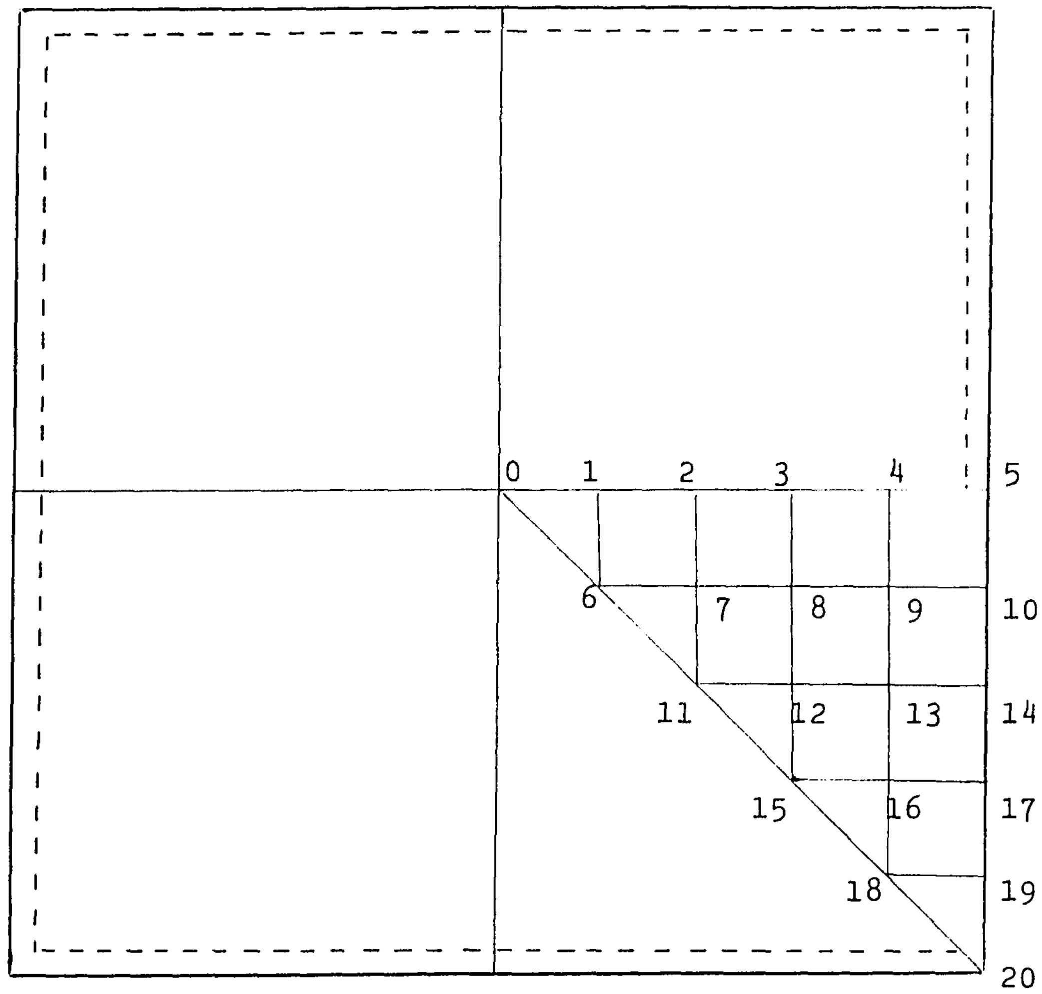


Figure 2 Simply Supported Square Plate with its Grid (2a = 10").

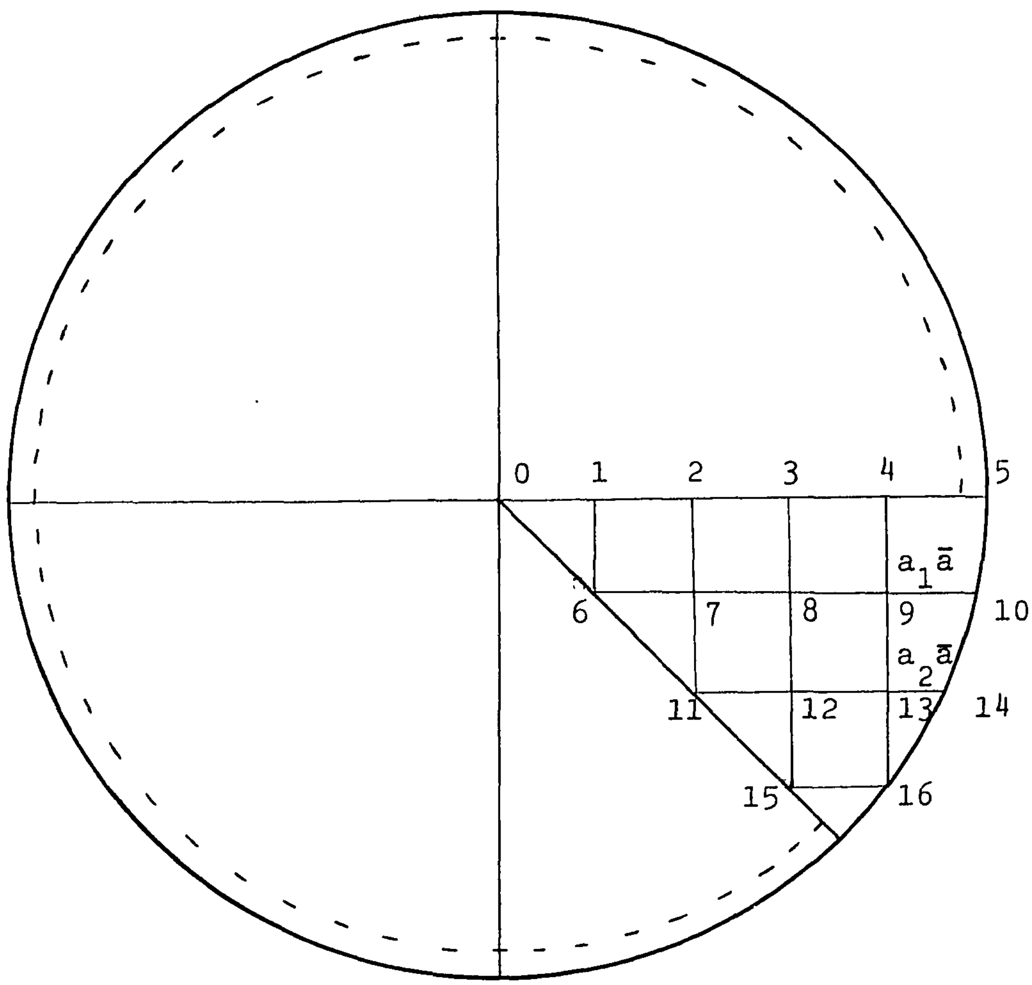


Figure 3 Simply Supported Circular Plate with its Grid (2a = 10").

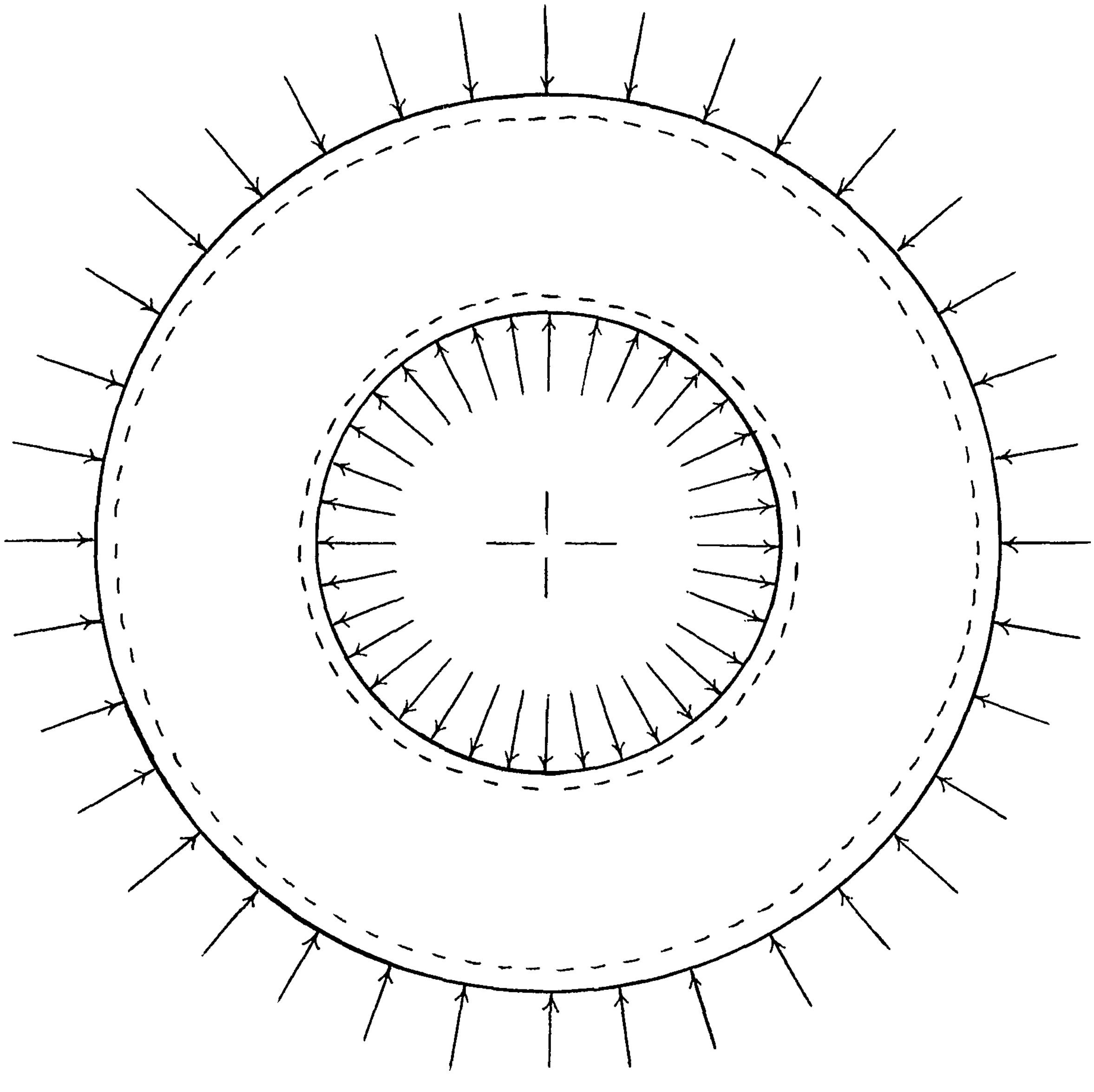


Figure. 4 Simply Supported Annular Circular Plate (a_i/a_o = 0.5, $2a_o$ = 10") under Hydrostatic Compression.

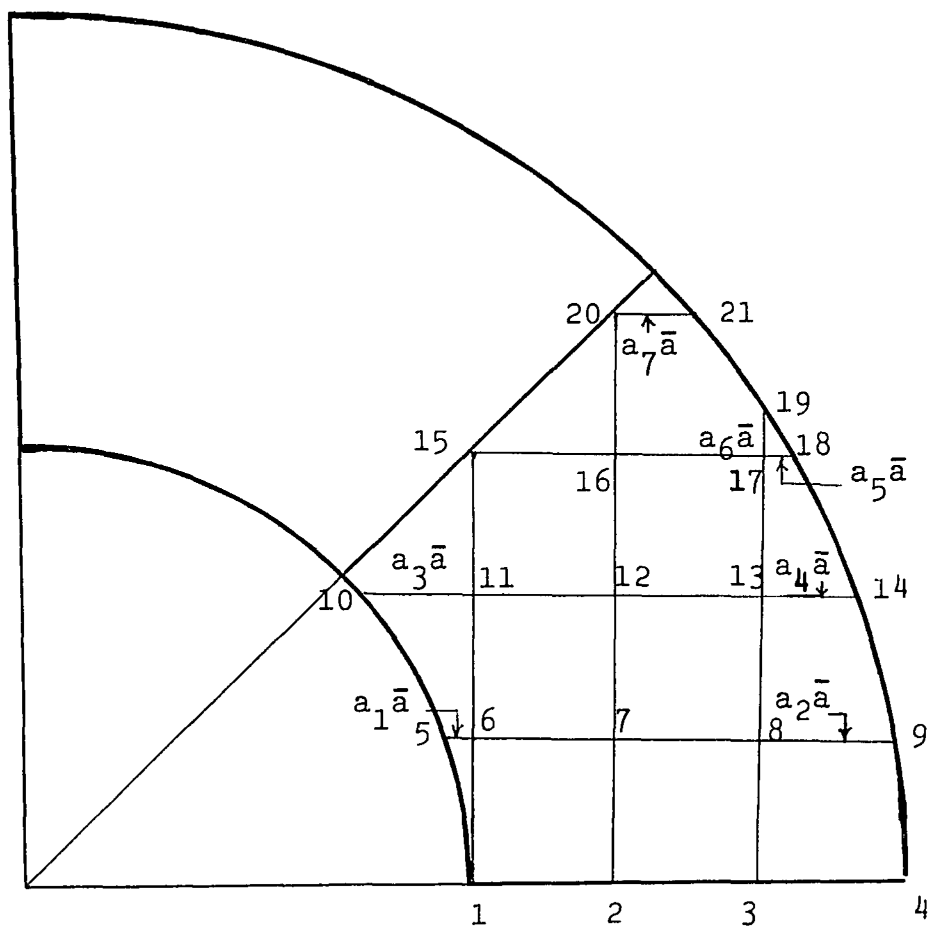


Figure 5 Grid for an Annular Circular Plate Shown in Figure 4.

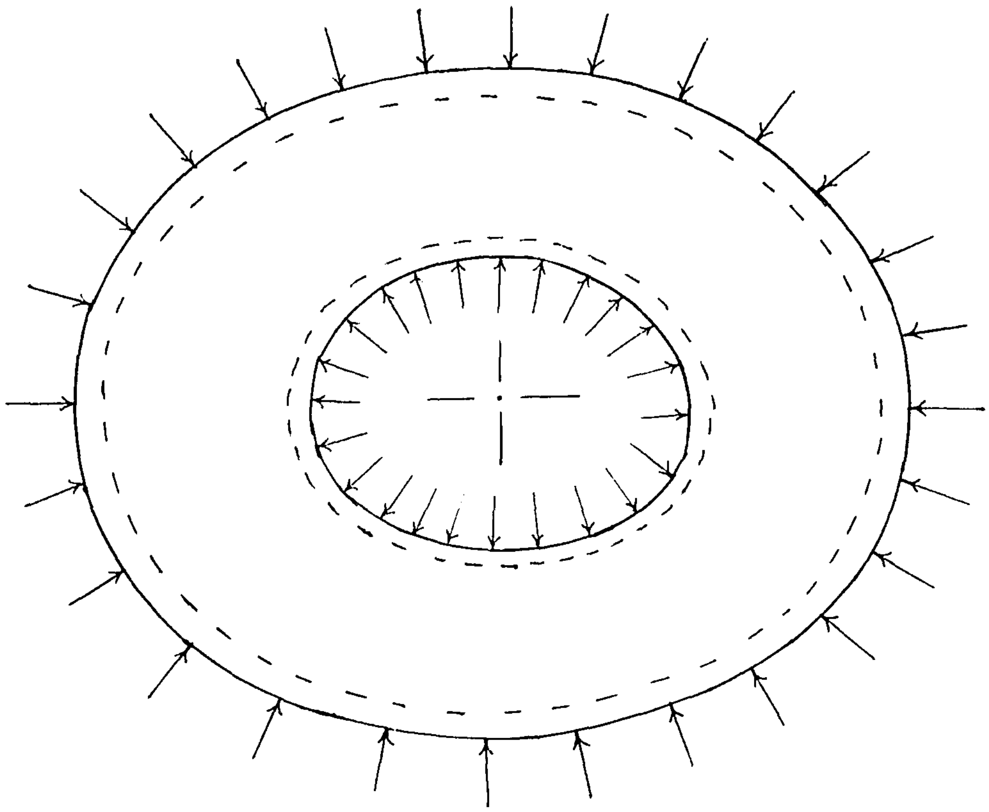


Figure 6 Simply Supported Annular Elliptical Plate ($H/2a_o = 0.1$, $b_o/a_o = 0.4$, $b_i/b_o = 0.5$) under Hydrostatic Compression.

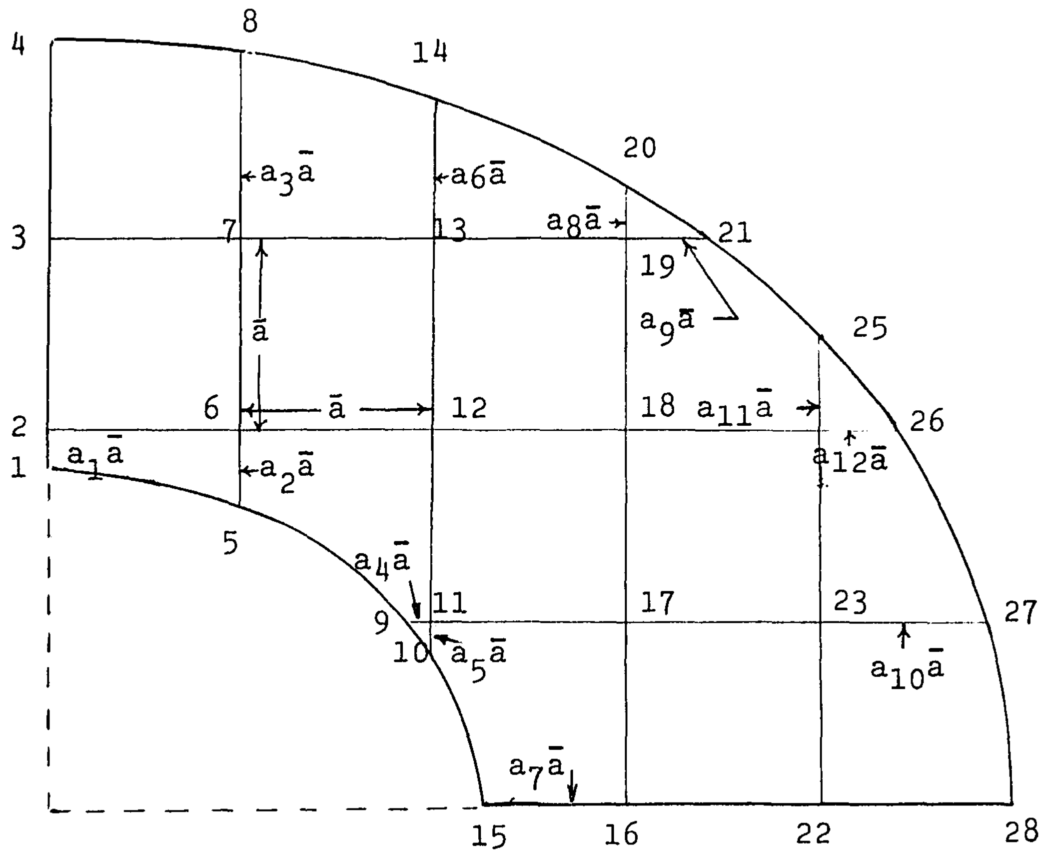


Figure 7 Grid for an Annular Elliptical Plate Shown in Figure 6 .

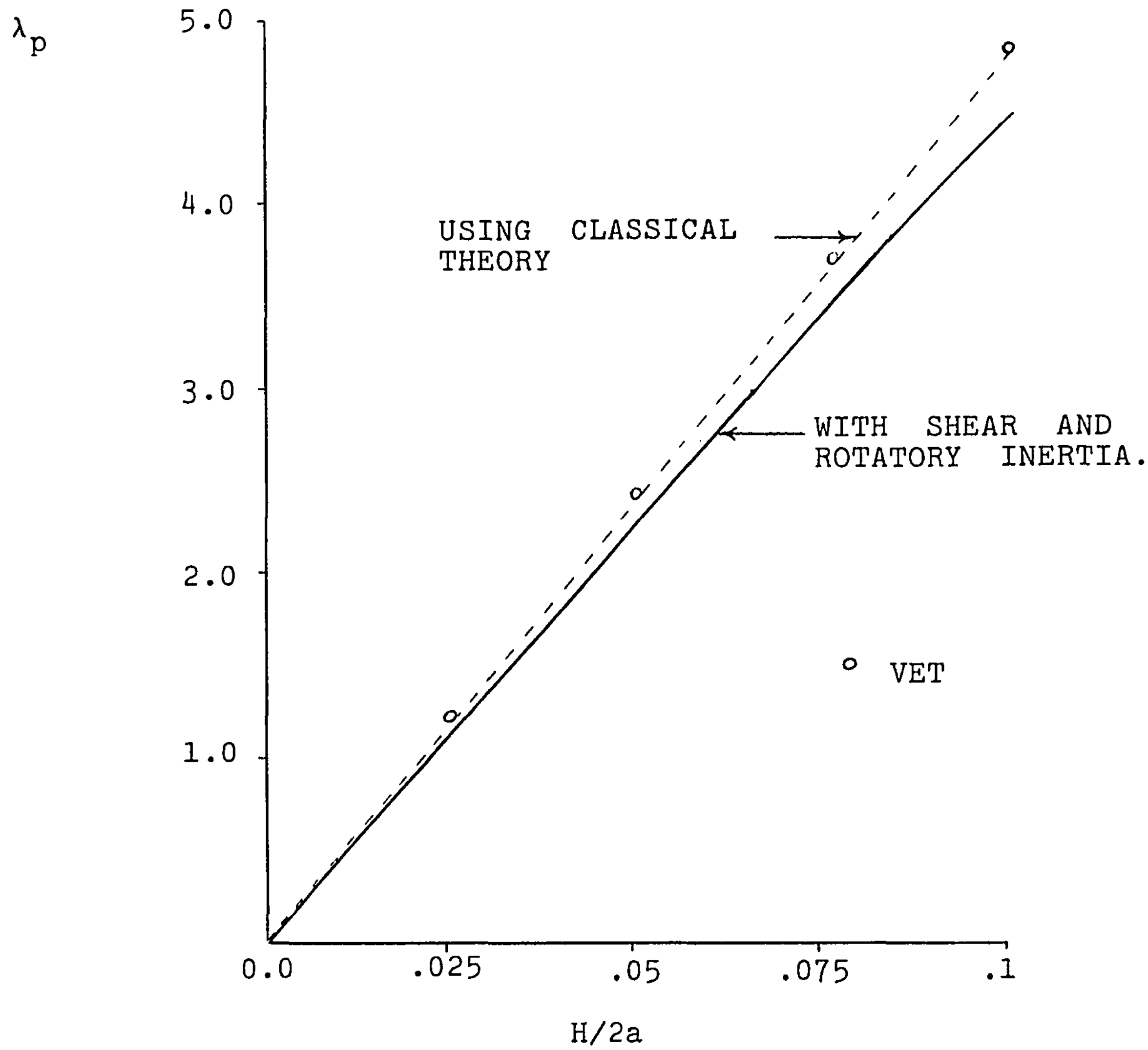


Figure 8 Fundamental Frequency Parameters for a Simply Supported Square Plate (2a = 10").

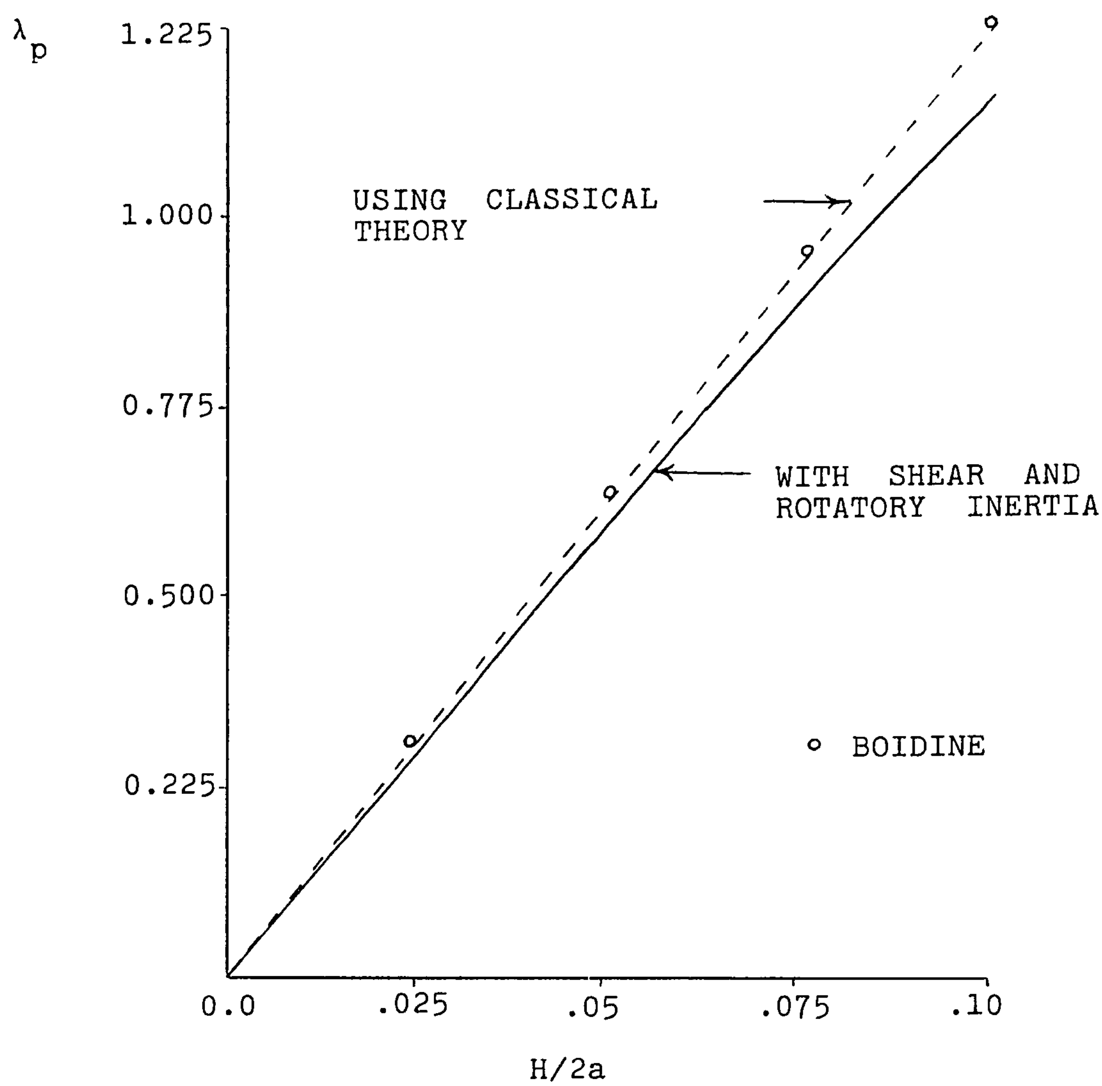


Figure 9 Fundamental Frequency Parameters of a Simply Supported Circular Plate (2a = 10").

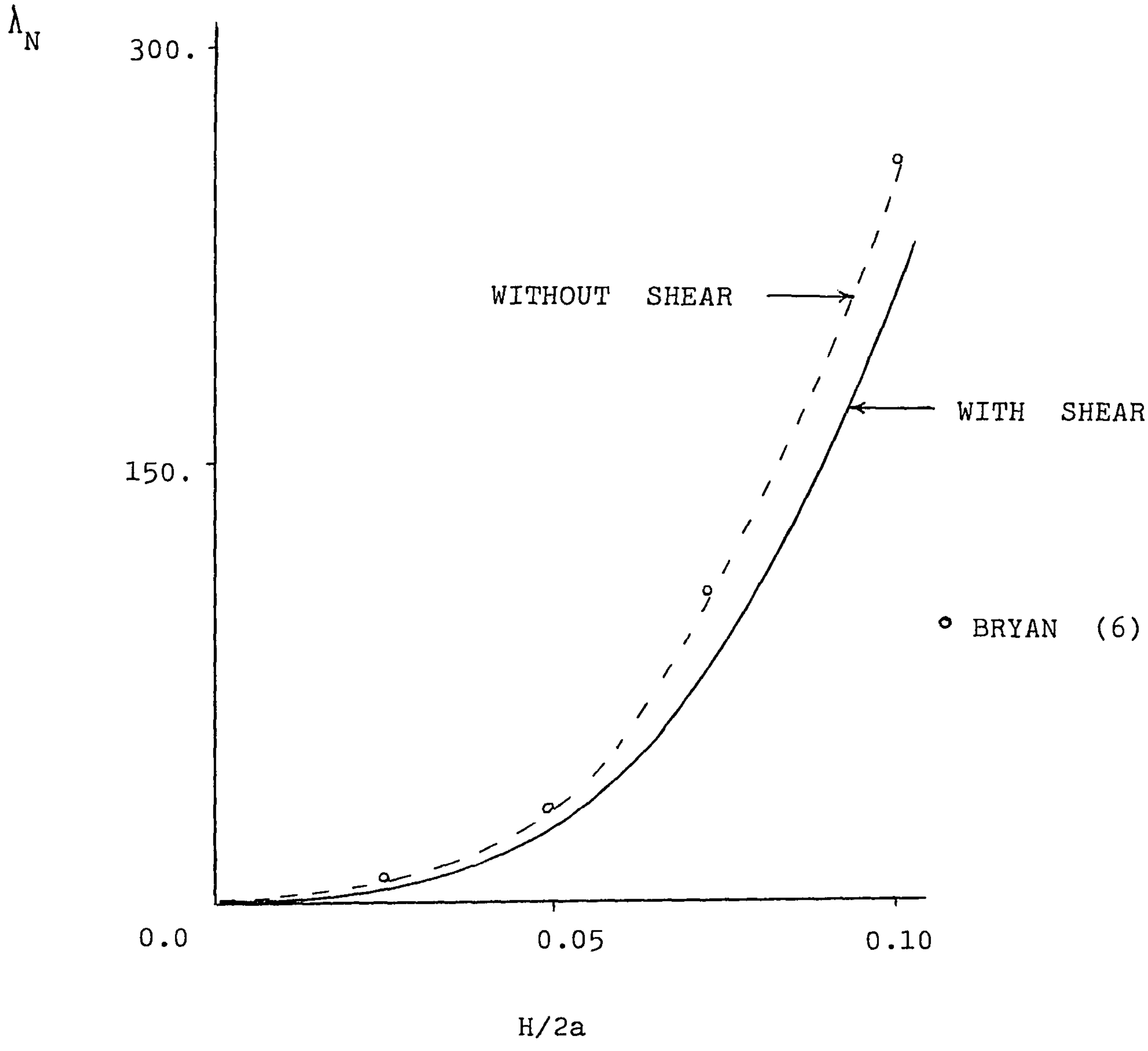


Figure 10 Critical Buckling Load Parameter for a Simply Supported Circular Plate. (2a = 10").

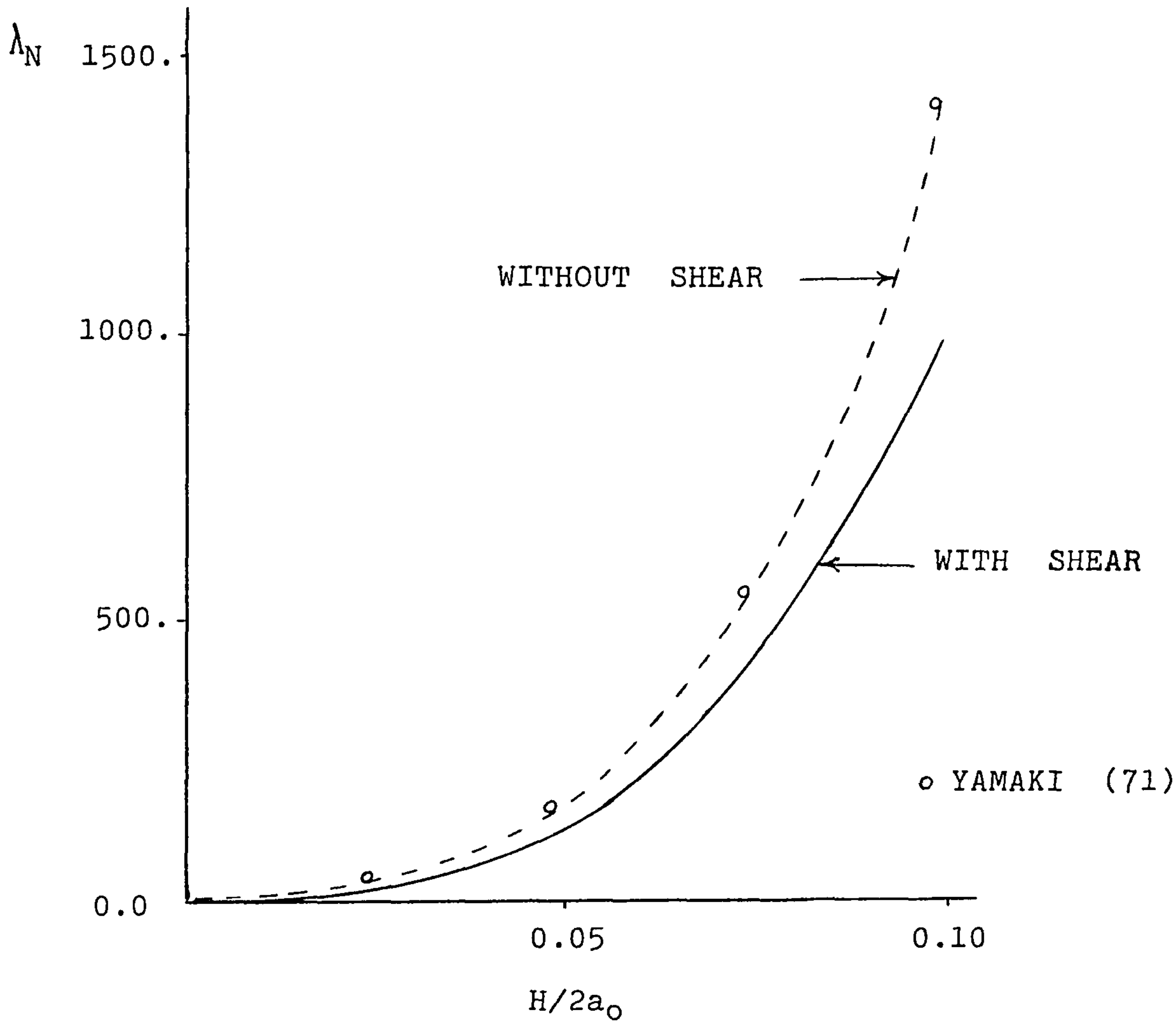


Figure 11 Critical Buckling Load Parameter for a Simply Supported Annular Circular Plate ($a_i/a_o = 0.5$, $2a_o = 10''$).

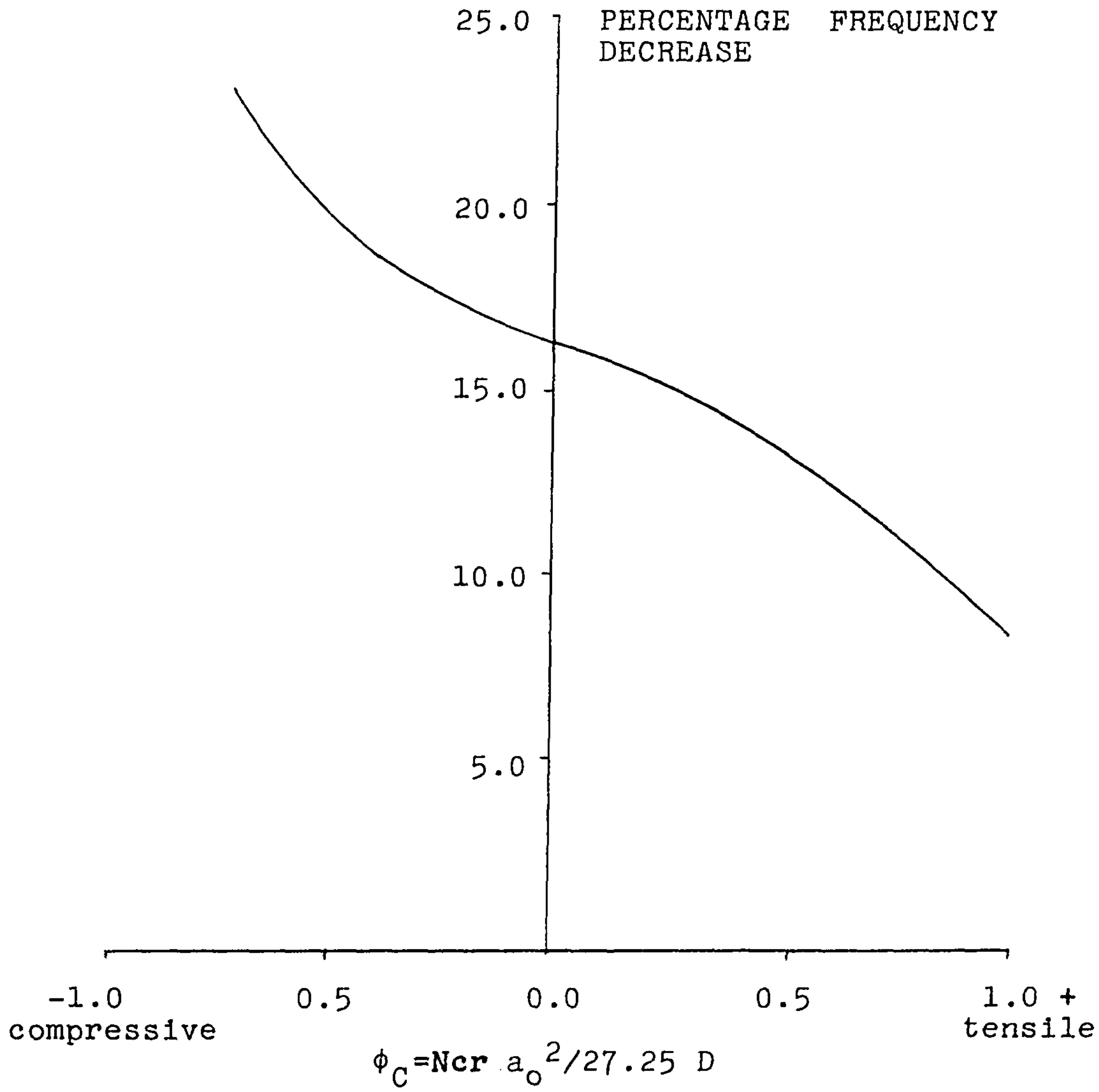
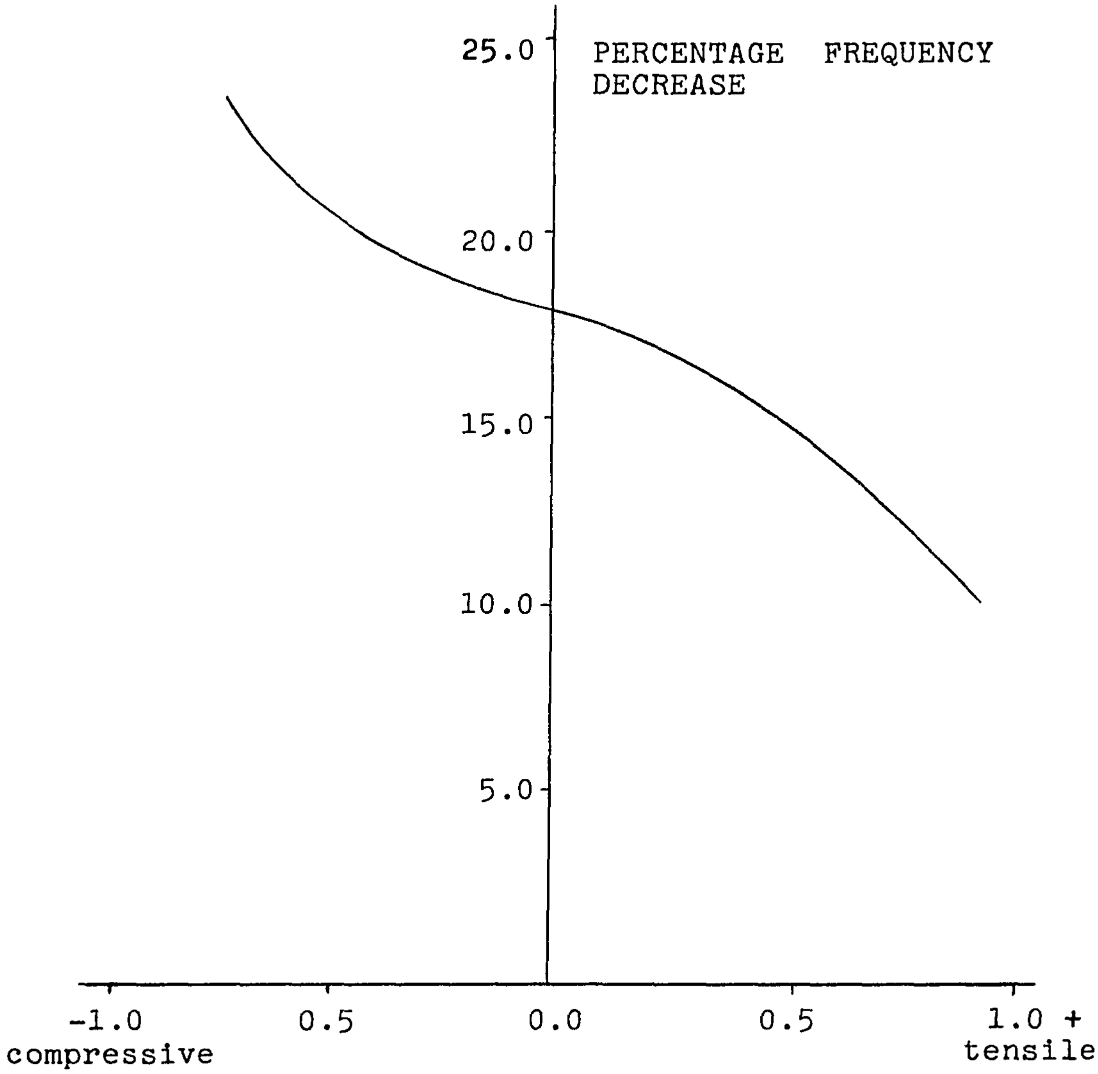


Figure 12 Percentage Frequency Decrease for a Simply Supported Annular Circular Plate Subjected to Inplane Forces N (a_1/a_o = 0.5, $H/2a_o$ = 0.1, $2a_o$ = 10").



$\phi_E = Ncr\ a_o^2/27.4\ D$

Figure 13 Percentage Frequency Decrease for a Simply Supported Annular Elliptical Plate Subjected to Inplane Forces N ($H/2a_o$=0.1, b_o/a_o = 0.4, a_i/a_o = 0.5, b_i/b_o = 0.5, $2a_o$ = 10").

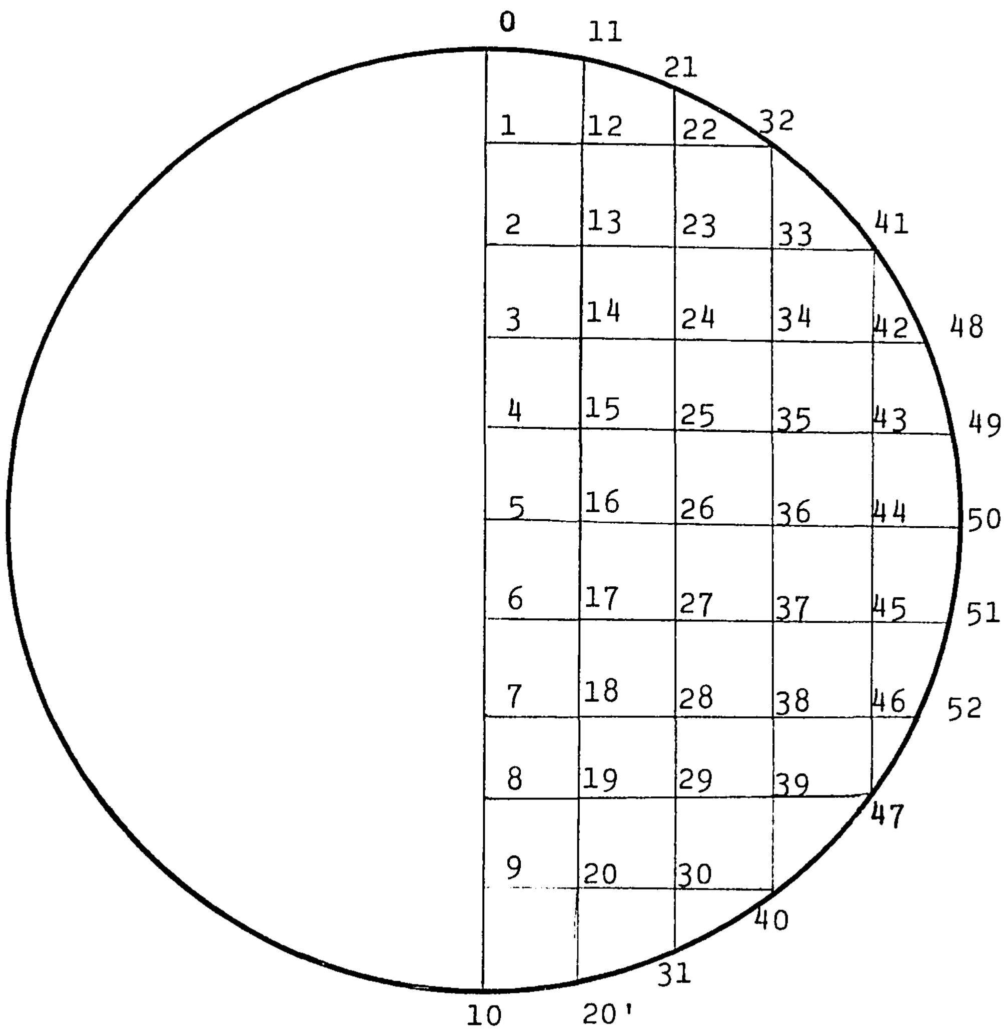


Figure 14 Simply Supported Circular Plate with its Grid Extended Over Half the Plate (2a = 10").

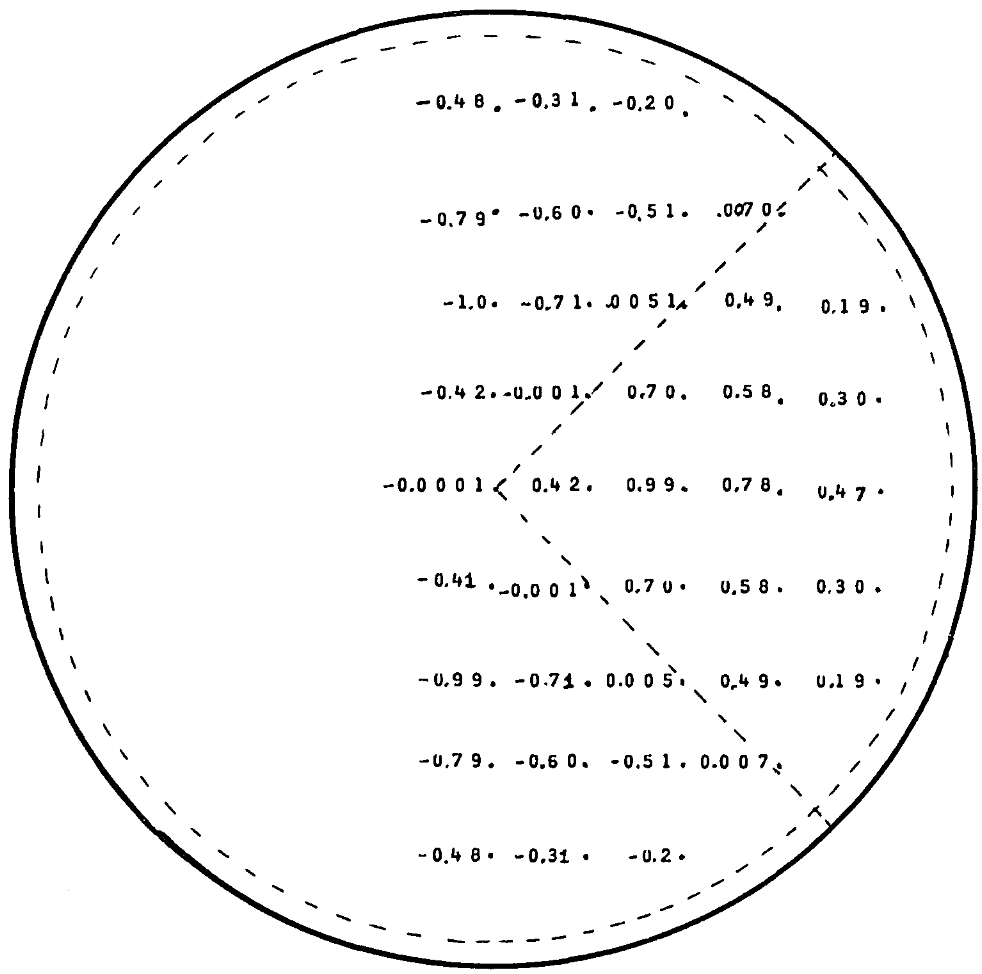


Figure 15 Mode Shape for the Third Natural Mode.

$\frac{H}{2a}$	λ_p Without Shear and Rotatory Inertia.	λ_p With Shear and Rotatory Inertia.	% Decrease in Frequency due to Shear and Rotatory Inertia.
0.025	1.210	1.199	0.92
0.050	2.420	2.380	1.45
0.075	3.620	3.520	2.48
0.100	4.840	4.650	3.90

Table 1 Fundamental Frequency Parameters for a Simply Supported Square Plate with and without the Effects of Shear and Rotatory Inertia (2a = 10").

$\frac{H}{2a}$	λ_p Without Shear and Rotatory Inertia.	λ_p With Shear and Rotatory Inertia.	% Decrease in Frequency due to Shear and Rotatory Inertia.
0.025	0.3062	0.3010	1.48
0.050	0.6125	0.5980	2.26
0.075	0.9180	0.8850	3.08
0.100	1.225	1.172	4.15

Table 2 Fundamental Frequency Parameters for a Simply Supported Circular Plate with and without the Effects of Shear and Rotatory Inertia (2a = 10").

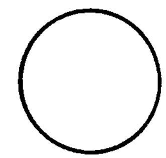
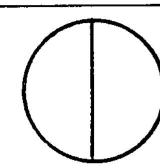
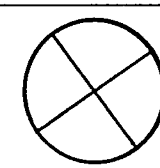
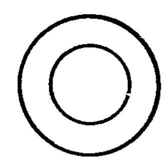
* Frequencies	λ_p Without Shear and Rotatory Inertia	λ_p With Shear and Rotatory Inertia	% Decrease in frequency due to Shear and Rotatory Inertia	Mode Shapes
Fundamental	4.98	4.715	4.21	
Second	13.962	13.210	4.98	
Third	25.721	24.206	5.89	
Fourth	29.833	27.643	7.36	

Table 3 Values of the First Four Natural Frequencies for a Simply Supported Circular Plate (H/2a = 0.1, 2a = 10").

* (In ascending order of magnitude)

$\frac{H}{2a}$	λ_N Without Shear	λ_N With Shear	% Decrease in the Critical Buckling Load due to Shear.
0.025	2.064	2.071	0.39
0.050	23.080	23.450	1.48
0.075	94.08	97.26	3.23
0.100	246.12	261.48	5.71

Table 4 Critical Buckling Load Parameters for a Simply Supported Circular Plate with and without the Effects of Shear (2a = 10").

$\frac{H}{2a}$	λ_N Without Shear.	λ_N With Shear	% Decrease in the Critical Buckling Load due to Shear.
0.025	13.10	13.44	2.41
0.050	138.80	152.02	9.05
0.075	517.12	631.23	18.60
0.100	1235.15	1720.45	28.03

Table 5 Critical Buckling Load Parameters for an Annular Circular Plate with and without the Effects of Shear ($a_i/a_o = 0.5$, $2a_o = 10''$).

ϕ_C	λ_p Without Shear and Rotatory Inertia.	λ_p With Shear and Rotatory Inertia.	% Decrease in Frequency due to Shear and Rotatory Inertia.
+1.00	53.60	48.82	8.52
+0.75	50.01	45.80	10.08
+0.50	46.82	37.62	13.23
+0.25	43.72	36.56	15.12
0.0	40.04	33.60	16.06
-0.25	37.51	30.40	17.71
-0.50	33.63	27.50	19.02
-0.75	28.11	21.41	23.26
-1.00 -1.392	20.80 0.0	0.0 -	28.05% Decrease in the Critical Buckling Load.

Table 6 Frequency Parameters for a Simply Supported Annular Circular Plate ($a_i/a_o = 0.5$, $H/2a_o = 0.1$, $2a = 10''$) Subjected to Inplane Forces.

ϕ_E	λ_p Without Shear and Rotatory Inertia.	λ_p With Shear and Rotatory Inertia.	% Decrease in Frequency due tp Shear and Rotatory Inertia.
+1.00	50.04	45.20	10.30
+0.75	47.60	41.23	11.28
+0.50	44.50	37.80	15.30
+0.25	41.80	34.61	17.21
+0.0	38.50	31.74	18.05
-0.25	34.70	27.92	19.80
-0.50	30.82	22.40	20.91
-0.75	26.23	19.75	24.80
-1.00	18.62	0.0	28.2% Decrease in the Critical Buckling Load.
-1.398	0.0	-	

Table 7 Frequency Parameters for a Simply Supported Annular Elliptical Plate Subjected to Inplane Forces N ($H/2a_o = 0.1$. $b_o/a_o = 0.4$, $a_i/a_o = 0.5$, $b_i/b_o = 0.5$, $2a_o = 10''$).

APPENDIX A

A. LOWER AND UPPER DECOMPOSITION METHOD

Definition: A lower triangular matrix is a square matrix $[C] = C_{ij}$ such that $C_{ij} = 0$ for $i < j$. Similarly, if $C_{ij} = 0$ for $i > j$, then $[C]$ is a upper triangular matrix (41).

L U Theorem:

Given a square matrix A of order n, let $[A_k]$ denote the principal minor matrix made from the first k rows and columns. Assume that det $[A_k]$ is not equal to zero for $k = 1,2, \ldots\ldots\ldots, n-1$. Then there exists a unique lower triangular matrix $L = [m_{ij}]$, with $m_{11} = m_{22} = \ldots\ldots = m_{nn} = 1$, and a unique upper triangular matrix $U = [u_{ij}]$ so that $L\,U = A$. Moreover,

$$\text{Det } [A] = u_{11}\, u_{22}\, u_{33} \ldots\ldots\ldots u_{nn}.$$

The above technique is used to evaluate the Matrices [P] and [B] of equations (5.23) and (5.26) respectively.

APPENDIX - B.1

RECTANGULAR PLATE

The pictorial equation (5.16) is written in finite difference form for a grid of a rectangular plate shown in Fig. 2.

The difference equations are:

$$4\,W_o - 4\,W_1 = M_o$$

$$-W_o + 4\,W_1 - W_2 - 2\,W_6 = M_1$$

$$-W_1 + 4\,W_2 - W_3 - 2\,W_7 = M_2$$

$$-W_2 + 4\,W_3 - W_4 - 2\,W_8 = M_3$$

$$-W_3 + 4\,W_4 - W_5 - 2\,W_9 = M_4$$

$$-2\,W_1 + 4\,W_6 - 2\,W_7 = M_6$$

$$-W_2 - W_6 + 4\,W_7 - W_8 - W_{11} = M_7$$

$$-W_3 - W_7 + 4\,W_8 - W_9 - W_{12} = M_8$$

$$-W_4 - W_8 + 4\,W_9 - W_{10} - W_{13} = M_9$$

$$-2\,W_7 + 4\,W_{11} - 2\,W_{12} = M_{11}$$

$$-W_8 - W_{11} + 4\,W_{12} - W_{13} - W_{15} = M_{12}$$

$$- W_9 - W_{12} + 4 W_{13} - W_{14} - W_{16} = M_{13}$$

$$- 2 W_{12} + 4 W_{15} - 2 W_{16} = M_{15}$$

$$- W_{13} - W_{15} + 4 W_{16} - W_{17} - W_{18} = M_{16}$$

$$- 2 W_{16} + 4 W_{18} - 2 W_{19} = M_{18}$$

Grid size : $\bar{a} = 1.00$

4	- 4	0	0	0	0	0	0	0	0	0	0	0	0	0
- 1	4	- 1	0	0	- 2	0	0	0	0	0	0	0	0	0
0	- 1	4	- 1	0	0	- 2	0	0	0	0	0	0	0	0
0	0	- 1	4	- 1	0	0	- 2	0	0	0	0	0	0	0
0	0	0	- 1	4	0	0	0	- 2	0	0	0	0	0	0
0	- 2	0	0	0	4	- 2	0	0	0	0	0	0	0	0
0	0	- 1	0	0	- 1	4	- 1	0	- 1	0	0	0	0	0
0	0	0	- 1	0	0	- 1	4	- 1	0	- 1	0	0	0	0
0	0	0	0	- 1	0	0	- 1	4	0	0	- 1	0	0	0
0	0	0	0	0	0	- 2	0	0	4	- 2	0	0	0	0
0	0	0	0	0	0	0	- 1	0	- 1	4	- 1	- 1	0	0
0	0	0	0	0	0	0	0	- 1	0	- 1	4	0	- 1	0
0	0	0	0	0	0	0	0	0	0	- 2	0	4	- 2	0
0	0	0	0	0	0	0	0	0	0	0	- 1	- 1	4	- 1
0	0	0	0	0	0	0	0	0	0	0	0	0	- 2	4

The Coefficient Matrix [α] for the Square Plate

APPENDIX B.2

CIRCULAR PLATE

The pictorial equation (5.16) is written in finite difference form for a grid of a circular plate shown in Fig. 3. The difference equations are:

$$4\, W_o - 4\, W_1 = M_o$$

$$- W_o + 4\, W_1 - W_2 - 2\, W_6 = M_1$$

$$- W_1 + 4\, W_2 - W_3 - 2\, W_7 = M_2$$

$$- W_2 + 4\, W_3 - W_4 - 2\, W_8 = M_3$$

$$- W_3 + 4\, W_4 - 2\, W_9 = M_4$$

$$- 2\, W_1 + 4\, W_6 - 2\, W_7 = M_6$$

$$- W_2 - W_6 + 4\, W_7 - W_8 - W_{11} = M_7$$

$$- W_3 - W_7 + 4\, W_8 - W_9 - W_{12} = M_8$$

$$- \frac{2}{1 + a_1} W_4 - \frac{2}{1 + a_1} W_8 + \frac{4}{a_1} W_9 - \frac{2}{1 + a_1} W_{13} = M_9$$

$$- 2\, W_7 + 4\, W_{11} - 2\, W_{12} = M_{11}$$

$$- W_8 - W_{11} + 4\, W_{12} - W_{13} - W_{15} = M_{12}$$

$$- \frac{2}{1 + a_2} W_9 - \frac{2}{1 + a_2} W_{12} + \frac{4}{a_2} W_{13} = M_{13}$$

$$- 2\ W_{12} + 4\ W_{15} = M_{15}$$

Grid size : $\bar{a} = 1.00$

The fractions of the grid size are :

$$a_1 = 0.916$$

$$a_2 = 0.625$$

4	- 4	0	0	0	0	0	0	0	0	0	0	0
- 1	4	- 1	0	0	- 2	0	0	0	0	0	0	0
0	- 1	4	- 1	0	0	- 2	0	0	0	0	0	0
0	0	- 1	4	- 1	0	0	- 2	0	0	0	0	0
0	0	0	- 1	4	0	0	0	- 2	0	0	0	0
0	- 2	0	0	0	4	- 2	0	0	0	0	0	0
0	0	- 1	0	0	- 1	4	- 1	0	- 1	0	0	0
0	0	0	- 1	0	0	- 1	4	- 1	0	- 1	0	0
0	0	0	0	$-2/(1+a_1)$	0	0	$-2/(1+a_1)$	$4/a_1$	0	0	$-2/(1+a_1)$	0
0	0	0	0	0	0	- 2	0	0	4	- 2	0	0
0	0	0	0	0	0	0	- 1	0	- 1	4	- 1	- 1
0	0	0	0	0	0	0	0	$-2/(1+a_2)$	0	$-2/(1+a_2)$	$4/a_2$	0
0	0	0	0	0	0	0	0	0	0	- 2	0	4

The Coefficient Matrix [α] for the Circular Plate

The pictorial equation (5.16) is written in finite difference form for a grid of a circular plate shown in Fig. 14. The difference equations are:

$$4\,W_1 - W_2 - 2\,W_{12} = M_1$$

$$-W_1 + 4\,W_2 - W_3 - 2\,W_{13} = M_2$$

$$-W_2 + 4\,W_3 - W_4 - 2\,W_{14} = M_3$$

$$-W_3 + 4\,W_4 - W_5 - 2\,W_{15} = M_4$$

$$-W_4 + 4\,W_5 - W_6 - 2\,W_{16} = M_5$$

$$-W_5 + 4\,W_6 - W_7 - 2\,W_{17} = M_6$$

$$-W_6 + 4\,W_7 - W_8 - 2\,W_{18} = M_7$$

$$-W_7 + 4\,W_8 - W_9 - 2\,W_{19} = M_8$$

$$-W_8 + 4\,W_9 - 2\,W_{20} = M_9$$

$$-\frac{2}{1+a_1}W_1 + \frac{4}{a_1}W_{12} - \frac{2}{1+a_1}W_{13} - \frac{2}{1+a_1}W_{22} = M_{12}$$

$$- W_2 - W_{12} + 4\,W_{13} - W_{14} - W_{23} = M_{13}$$

$$- W_3 - W_{13} + 4\,W_{14} - W_{15} - W_{24} = M_{14}$$

$$- W_4 - W_{14} + 4\,W_{15} - W_{16} - W_{25} = M_{15}$$

$$- W_5 - W_{15} + 4\,W_{16} - W_{17} - W_{26} = M_{16}$$

$$- W_6 - W_{16} + 4\,W_{17} - W_{18} - W_{27} = M_{17}$$

$$- W_7 - W_{17} + 4\,W_{18} - W_{19} - W_{28} = M_{18}$$

$$- W_8 - W_{18} + 4\,W_{19} - W_{20} - W_{29} = M_{19}$$

$$- \frac{2}{1+a_1} W_9 - \frac{2}{1+a_1} W_{19} + \frac{4}{a_1} W_{20} - \frac{2}{1+a_1} W_{30} = M_{20}$$

$$- \frac{2}{1+a_2} W_{12} + \frac{4}{a_2} W_{22} - \frac{2}{1+a_2} W_{23} = M_{22}$$

$$- W_{13} - W_{22} + 4\,W_{23} - W_{24} - W_{33} = M_{23}$$

$$- W_{14} - W_{23} + 4\,W_{24} - W_{25} - W_{34} = M_{24}$$

$$- W_{15} - W_{24} + 4\,W_{25} - W_{26} - W_{35} = M_{25}$$

$$- W_{16} - W_{25} + 4 W_{26} - W_{27} - W_{36} = M_{26}$$

$$- W_{17} - W_{26} + 4 W_{27} - W_{28} - W_{37} = M_{27}$$

$$- W_{18} - W_{27} + 4 W_{28} - W_{29} - W_{38} = M_{28}$$

$$- W_{19} - W_{28} + 4 W_{29} - W_{30} - W_{39} = M_{29}$$

$$- \frac{2}{1+a_2} W_{20} - \frac{2}{1+a_2} W_{29} + \frac{4}{a_2} W_{30} = M_{30}$$

$$- W_{23} + 4 W_{33} - W_{34} = M_{33}$$

$$- W_{24} - W_{33} + 4 W_{34} - W_{35} - W_{42} = M_{34}$$

$$- W_{25} - W_{34} + 4 W_{35} - W_{36} - W_{43} = M_{35}$$

$$- W_{26} - W_{35} + 4 W_{36} - W_{37} - W_{44} = M_{36}$$

$$- W_{27} - W_{36} + 4 W_{37} - W_{38} - W_{45} = M_{37}$$

$$- W_{28} - W_{37} + 4 W_{38} - W_{39} - W_{46} = M_{38}$$

$$- W_{29} - W_{38} + 4 W_{39} = M_{39}$$

$$- \frac{2}{1+a_2} W_{34} + \frac{4}{a_2} W_{42} - \frac{2}{1+a_2} W_{43} = M_{42}$$

$$- \frac{2}{1+a_1} W_{35} - \frac{2}{1+a_1} W_{42} + \frac{4}{a_1} W_{43} - \frac{2}{1+a_1} W_{44} = M_{43}$$

$$- W_{36} - W_{43} + 4 W_{44} - W_{45} = M_{44}$$

$$- \frac{2}{1+a_1} W_{37} - \frac{2}{1+a_1} W_{44} + \frac{4}{a_1} W_{45} - \frac{2}{1+a_1} W_{66} = M_{45}$$

$$- \frac{2}{1+a_2} W_{38} - \frac{2}{1+a_2} W_{45} + \frac{4}{a_2} W_{46} = M_{46}$$

From the above difference equations the coefficient matrix $[\alpha]$ is constructed and finally the matrix $[P]$ of order 39 is obtained. The fundamental and the three consecutive frequencies are computed. With knowledge of nodal patterns from the classical case it is noted that when using half the plate for computation, care should be taken to assign proper signs to the difference equations. The respective mode shapes are obtained by substituting the value of 'p' into the matrix [P] and computing the eigen vectors using the computer program given in Appendix C.

APPENDIX B.3

ANNULAR CIRCULAR PLATE

The pictorial equation (5.16) is written in finite difference form for a grid of an annular circular plate shown in Fig. 5.

The difference equations are:

$$4\,W_2 - W_3 - 2W_7 = M_2$$

$$-W_2 + 4\,W_3 - 2\,W_8 = M_3$$

$$\frac{4}{a_1}\,W_6 - \frac{2}{1 + a_1}\,W_7 - \frac{2}{1 + a_1}\,W_{11} = M_6$$

$$-W_2 - W_6 + 4\,W_7 - W_8 - W_{12} = M_7$$

$$-\frac{2}{1 + a_2}\,W_3 - \frac{2}{1 + a_2}\,W_3 + \frac{4}{a_2}\,W_7 - \frac{2}{1 + a_2}\,W_{13} = M_8$$

$$-\frac{2}{1 + a_3}\,W_6 - \frac{4}{a_3}\,W_{11} - \frac{2}{1 + a_3}\,W_{12} - \frac{2}{a + a_3}\,W_{15} = M_{11}$$

$$-W_7 - W_{11} + 4\,W_{12} - W_{13} - W_{16} = M_{12}$$

$$-\frac{2}{1 + a_4}\,W_8 - \frac{2}{1 + a_4}\,W_{12} + \frac{4}{a_4}\,W_3 - \frac{2}{1 + a_4}\,W_7 = M_{13}$$

$$-2\,W_{11} + 4\,W_{15} - 2\,W_{16} = M_{15}$$

Grid size : $\bar{a} = 0.833$

The fraction of the grid size are:

$a_1 = 0.150$, $a_2 = 0.900$, $a_3 = 0.750$, $a_4 = 0.70$

4	- 1	0	- 2	0	0	0	0	0	0	0	0
- 1	4	0	0	- 2	0	0	0	0	0	0	0
0	0	$4/a_1$	$-2/(1+a_1)$	0	$-2/(1+a_1)$	0	0	0	0	0	0
- 1	0	- 1	4	- 1	0	- 1	0	0	0	0	0
0	$-2/(1+a_2)$	0	$-2/(1+a_2)$	$4/a_2$	0	0	$-2/(1+a_2)$	0	0	0	0
0	0	$-2/(1+a_3)$	0	0	$4/a_3$	$-2/(1+a_3)$	0	$-2/(1+a_3)$	0	0	0
0	0	0	- 1	0	- 1	4	- 1	0	- 1	0	0
0	0	0	0	$-2/(1+a_4)$	0	$-2/(1+a_4)$	$4/a_4$	0	0	$-2/(1+a_4)$	0
0	0	0	0	0	- 2	0	0	4	- 2	0	0
0	0	0	0	0	0	- 1	0	- 1	4	- 1	- 1
0	0	0	0	0	0	0	$-4/(1+a_5)(1+a_6)$	0	$-4/(1+a_5)(1+a_6)$	$4/a_5a_6$	0
0	0	0	0	0	0	0	0	0	$-8/(1+a_7)^2$	0	$4/a_7^2$

The Coefficient Matrix $[\alpha]$ for the Annular Circular Plate

APPENDIX C

COMPUTER PROGRAM

Vibration Problem: Annular Elliptical Plate

```
      PROGRAM FREQ
C     SHEAR AND ROTATORY INERTIA
      COMMON M,N,R,S,T,U,D(19,19),E(19,19)
      DIMENSION P(2), A(20,20),B(20,20),C(20,20),F(20,20)
      READ(5,1) M,N
      READ(5,2) DEN,THI,RAD,PRAT,EOD,SCOF,PFORCE
      READ(5,3) G,G1,G2,G3,G4,G5,G6,G7,G8,G9,G10,G11,G12
      DEN1 = EOD*THI**3/(12.*(1.-PRAT**2))
      DEN2 = THI**2*PFORCE/(6.*(1.-PRAT)*SCOF))
      DENOM =DEN1 + DEN2
      R1 = DEN*THI**3/12
      R2 = DEN*THI**3/(6.*SCOF*(1.-PRAT))
      R3 = DEN*THI**2*(1. + PRAT)/(6.*EOD*SCOF)
      R = G**2*(R1 + R2 + R3)/DENOM
      S = G**2*PFORCE/DENOM
      T = G**4*DEN*THI/DENOM
      U1 = DEN**2*THI**3*(1.+ PRAT)/(6.*EOD*SCOF)
      U = G**4*U1/DENOM
      DO 148 II = 1,N
```

COMPUTER PROGRAM (cont'd)

```
      DO 148 JJ = 1,N
      D(II,JJ) + 0.0
  148 CONTINUE
C     THE ELEMENTS OF THE MATRIX (α) OF EQUATION
C     (5.16) ARE READ IN AS:
      D(1,1) = 4.0/G1
      D(1,2) = -2.0/(1.0 + G1)
      D(1,3) = -4.0/(1.0 + G1)
      D(2,1) = -1.0
      D(2,2) = 4.0
      D(2,4) = -2.0
      D(3,1) = -2.0/(1.0 + G2)
      D(3,3) = 4.0/G2
      D(3,4) = -2.0/(1.0 + G2)
      D(3,6) = -2.0/(1.0 + G2)
      D(4,2) = -2.0/(1.0 + G3)
      D(4,3) = -2.0/(1.0 + G3)
      D(4,4) = 4.0/G3
      D(4,7) = -2.0/(1.0 + G3)
      D(5,5) = 4.0/G4*G5
      D(5,6) = -4.0/((1.0 + G4)(1.0 + G5))
      D(5,9) = -4.0/((1.0 + G4)(1.0 + G5))
```

COMPUTER PROGRAM (cont'd)

```
D(6,3) = -1.0
D(6,5) = -1.0
D(6,6) = 4.0
D(6,7)= -1.0
D(6,10) = -1.0
D(7,4) = -2.0/(1.0 + G6)
D(7,6) = -2.0/(1.0 + G6)
D(7,7) = 4.0/G6
D(7,11) = -2.0/(1.0 + G6)
D(8,8) = 4.0/G7
D(8,9) = -4.0/(1.0 + G7)
D(8,12)= -2.0/(1.0 + G7)
D(9,5) = -1.0
D(9,8) = -1.0
D(9,9) = 4.0
D(9,10) = 1.0
D(9,13) = -1.0
D(10,6) = -1.0
D(10,9) = -1.0
D(10,10) = 4.0
D(1-,11) = -1.0
D(10,14) = -1.0
D(11,7) = -4.0/((1.0 + G8)(1.0 + G9))
```

COMPUTER PROGRAM (cont'd)

```
      D(11,10) = -4.0/((1.0 + G8)(1.0 + G9))
      D11,11) = 4.0/G8*G9
      D(12,8) = -1.0
      D(12,12) = 4.0
      D(12,13) = -2.0
      D(13,9) = -2.0/(1.0 + G10)
      D13,12) = -2.0/(1.0 + G10)
      D(13,13) = 4.0/G10
      D(13,14) = -2.0/(1.0 + G10)
      D(14,10) = -4.0/((1.0 + G11)(1.0 + G12))
      D(14,13) = -4.0/((1.0 + G11)(1.0 + G12))
      D(14,14) = 4.0/(G11*G12)
      DO  173  K = 1,N
      DO  169  I = 1,N
      SUM = 0.0
      DO 154 J = 1,N
  154 SUM = D(K,J)*D(J,I) + SUM
  169 E(K,I) = SUM
C     E(K,I)  IS  MATRIX  (β)  DEFINED  BY  EQUATION
C     (5.20).
  173 CONTINUE
      DO 26 IK = 70000, 100000, 10000
      DI = IK
```

COMPUTER PROGRAM (contd'd)

```
      H = 250.0
      EI = DI + 10000.0
      ERROR = 0.01
      AI = DI -H
      BI = EI - H
      YL = DET(AI)
      WRITE(6.100)YL
      XL = A1
      XR = A1 + H
      YR = DET(XR)
      WRITE(6,200)YR
      L = 1
6     IF(YR)14,7,14
7     P(L) = XR
      WRITE(6,12)XR
      STOP
14    IF(YR*YL)22,15,17
15    P(L) = XL
      WRITE(6,12)XL
      STOP
17    WRITE(6,34)YL,YR
      WRITE(6,25)XL,XR
```

COMPUTER PROGRAM (cont'd)

```
      XL = XR
      YL = YR
      XR = XR + H
      IF(XR - B1)20,20,18
18    L = L -1
      WRITE(6,36)DI,EI
      GO TO 26
20    YR = DET(XR)
      GO TO 6
22    WRITE(6,34)YL,YR
      DELTA = ABS(YL)*(XR - XL)/(ABS(YL) + ABS(YR))
      WRITE(6,77)DELTA
      IF(ABS(XR - XI) - ERROR)23,23,50
50    IF(DELTA - ERROR)23,23,24
23    P(L) = XI
      WRITE(6,12)XI
      STOP
24    YI = DET(XI)
      IF(YI*YR)32,23,29
29    XR = XI
      YR = YI
      GO TO 22
32    XL = XI
```

COMPUTER PROGRAM (cont'd)

```
      YL = YI
      GO TO 22
26    CONTINUE
1     FORMAT(2I5)
2     FORMAT(E10.4,3F5.2,E10.4,F5.3,F12.1)
3     FORMAT(13F6.4)
12    FORMAT(10X,17HTHE FREQUENCY  1S=,E17.9)
34    FORMAT(10X,3HYL=,E15.7,10X,3HYR=,E15.7)
25    FORMAT(10X,3HXL=,E17,9,10X,3HXR=,E17.9)
36    FORMAT(5X, 'NO ROOTS BETWEEN''E17.9,'AND', E17.9)
77    FORMAT(10X,6HDELTA=,E15.7)
100   FORMAT(10X,3HYL=,E15.7)
200   FORMAT(10X,3HYR=,E15.7)
      STOP
      END

      FUNCTION DET(P)
C     THE FOLLOWING SUBROUTING DETERMINES THE MATRIX(P)
C     USING EQUATION (5.22) AND THEN CALCULATES THE
C     VALUE OF THE DETERMINANT OF THE MATRIX(P) BY THE
C     LOWER AND UPPER DECOMPOSITION METHOD.
```

COMPUTER PROGRAM (cont'd)

```
      COMMON M,N,R,S,T,U,D(19,19),E(19,19)
      DIMENSION A(20,20),B(20,20),C(20,20),F(20,20)
      DO 115 KK = 1,N
      DO 112 II = 1,N
      F(KK,II) = 0.0
      IF(KK.EQ.II) F(KK,II)=1.0
  112 A(KK,II)=E(KK,II)-D(KK,II)*(R*P**2-S)+F(KK,II)*(U*P**4-T*P**2)
C     MATRIX A(KK,II)  IS  MATRIX(P)  DEFINED  BY  EQUATION  (5.22)
  115 CONTINUE
      N1 = N - 1
      DO 89 L = 1,N1
      DO 33 I = 1,N
      DO 33 J = 1,N
      B(I,J) = 0.0
      IF(I.EQ.J)  B(I,J) = 1.0
  33  CONTINUE
      K = L + 1
      DO 44 I = K,N
  44  B(I,L) = -A(I,L)/A(L.L)
      DO 73 K = 1,N
      DO 69 I + 1,N
      SUM = 0.0
      DO 54 J I,N
```

COMPUTER PROGRAM (cont'd)

```
54  SUM = SUM + B(K,J)*A(J,I)
69  C(K,I) = SUM
73  CONTINUE
    DO 84 I =1,N
    DO 84 J =1,N
84  A(I,J) = C(I,J)
89  CONTINUE
    DET = 1.0
    DO 132 I =1,N
132 DET = DET*A(I,I)
    RETURN
    END

C       WHEN THE NATURAL FREQUENCY IS TO BE DETER-
C       MINED WITHOUT THE EFFECTS OF SHEAR AND
C       ROTATORY INERTIA THAT IS BY USING THE CLAS-
C       SICAL THEORY THE FOLLOWING CHANGES ARE MADE
C       IN THE ABOVE PROGRAM:
        DEN2 = 0.0
        DENOM = DEN1
        R1 = 0.0
        R2 = 0.0
```

COMPUTER PROGRAM (cont'd)

```
      R3 = 0.0
      R = 0.0
      U1 = 0.0
      U = 0.0
```

Buckling Problem: Annular Elliptical Plate

```
      PROGRAM LOAD
C     SHEAR AND ROTATORY INERTIA
      COMMON M,N,G,D(20,20),E(20,20),DEN,THI,RAD,PRAT,SCOF,EOD
      DIMENSION PFORCE(2),A(20,20),B(20,20),C(20,20)
      READ(5,1)M,N
      READ(5,2)DEN,THI,RAD,PRAT,EOD,SCOF
      READ(5,3)G,G1,G2, G3,G4,G5,G6,G7,G8,G9,G10,G11,G12
      DO 148 II = 1,N
      DO 148 JJ = 1,N
      D(II,JJ) = 0.0
  148 CONTINUE
C     THE ELEMENTS OF THE MATRIX (α) DEFINED BY
C     EQUATION (5.16) ARE READ IN AS:
      D(1,1) = 4.0/G1
      D(1,2) = -2.0/(1.0 + G1)
```

COMPUTER PROGRAM (cont'd)

```
    D(1,3) = -4.0/(1.0 + G1)
    D(2,1) = -1.0
    D(2,2) = 4.0
    D(2,4) = -2.0
    D(3,3) = 4.0/G2
    D(3,4) = -2.0/(1.0 + G2)
    .....
    .....
    .....
    .....
    .....
    D(13,13) = 4.0/G10
    D(13,14) = -2.0/(1.0 + G10)
    D(14,10) = -4.0/((1.0 + G11)(1.0 + G12))
    D(14,13) = -4.0/((1.0 + G11)(1.0 + G12))
    D(14,14) = 4.0/(G11*G12)
    DO 173 K = 1,N
    DO 169 I = 1,N
    SUM = 0.0
    DO 154 J = 1,N
154 SUM = D(K,J)*D(J,I) + SUM
```

COMPUTER PROGRAM (cont'd)

```
    169 E(K,I) = SUM
    173 CONTINUE
C       E(K,I)  IS  MATRIX(β)  DEFINED  BY  EQUATION  (5.20).
        DO 26 IK = 70000, 100000, 10000
        DI = IK
        H = 250.0
        EI = DI + 10000.0
        ERROR = 0.01
        AI = DI - H
        BI = EI - H
        YL = DET(AI)
        WRITE(6,100) YL
        XL = A1
        XR = A1 + H
        YR = DET(XR)
        WRITE(6,200)YR
        L = 1
    6   IF(YR)14,7,14
    7   PFORCE(L) = XR
        WRITE(6,12)XR
        STOP
    14  IF(YR*YL)22,15,17
```

COMPUTER PROGRAM (cont'd)

```
15  PFORCE(L) = XL
    WRITE(6,12)XL
    STOP
17  WRITE(6,34)YL,YR
    WRITE(6,25)XL,XR
    XL = XR
    YL = YR
    XR = XR + H
    IF(XR - B1)20,20,18
18  L = L - 1
    WRITE(6,36)DI,EI
    GO TO 26
20  YR = DET(XR)
    GO TO 6
22  WRITE(6,34)YL,YR
    DELTA = ABS(YL)*(XR - XL)/(ABS(YL) + ABS(YR))
    WRITE(6,77)DELTA
    XI = XL + DELTA
    IF(ABS(XR - XI) - ERROR)23,23,50
50  IF(DELTA - ERROR)23,23,24
23  PFORCE(L) = XI
    WRITE(6,12)XI
    STOP
```

COMPUTER PROGRAM (cont'd)

```
24   YI = DET(XI)
     IF(YI*YR)32,23,29
29   XR = XI
     YR = YI
     GO TO 22
32   XL = XI
     YL = YI
     GO TO 22
26   CONTINUE
1    FORMAT(215)
2    FORMAT(E10.4,3F5.2,E10.4,F5.3,F12.1)
3    FORMAT(13F6.4)
12   FORMAT(10X,21HTHE BUCKLING LOAD IS=,E17.9)
34   FORMAT(10X,3HYL=,E15.7,10X,3HYR=,E15.7)
25   FORMAT(10X,3HXL=,E17,9,10X,3HXR=,E17.9)
36   FORMAT(5X, 'NO ROOTS BETWEEN''E17.9,'AND',E17.9)
77   FORMAT(10X,6HDELTA=,E15.7)
100  FORMAT(10X,3HYL=,E15.7)
200  FORMAT(10X,3HYR=,E15.7)
     STOP
     END
```

COMPUTER PROGRAM (cont'd)

```
      FUNCTION  DET(PFORCE)
C     THE  FOLLOWING  SUBROUTINE  DETERMINES  THE  MATRIX(B)
C     DEFINED  BY  EQUATION  (5.25)  AND  THEN  CALCULATES
C     THE  VALUE  OF  THE  DETERMINANT  OF  THE  MATRIX(B)
C     BY  THE  LOWER  AND  UPPER  DECOMPOSITION  METHOD.
      COMMON M.N.G.D(20,20),E(20,20),DEN,THI,RAD,SCOF,PRAT,EOD
      DIMENSION A(20,20),B(20,20),C(20,20)
      DEN1 = EOD*THI**3/(12.0*(1.0 - PRAT**2))
      DEN2 = THI**2*PFORCE/(6.0*(1.0 - PRAT)*SCOF)
      DENOM = DEN1 - DEN2
      S = G**2*PFORCE/DENOM
      DO 115 KK = 1,N
      DO 112 II = 1,N
  112 A(KK,II) = E(KK,II) - D(KK,II)*S
C     MATRIX  A(KK,II)  IS  MATRIX(B)  DEFINED  BY
C     EQUATION  (5.26).
  115 CONTINUE
      N1 = N - 1
      DO 89 L = 1,N1
      DO 33 I = 1,N
      DO 33 J = 1,N
      B(I,J) = 0.0
      IF(I.EQ.J)  B(I,J) = 1.0
```

COMPUTER PROGRAM (cont'd)

```
33  CONTINUE
    K = L + 1
    DO 44 I = K,N
44  B(I,L) = -A(I,L)/A(L,L)
    DO 73 K = 1,N
    DO 69 I = 1,N
    SUM = 0.0
    DO 54 J 1,N
54  SUM = SUM + B(K,J)*A(J,I)
69  C(K,I) = SUM
73  CONTINUE
    DO 84 I = 1,N
    DO 84 J = 1,N
84  A(I,J) = C(I,J)
89  CONTINUE
    DET = 1.0
    DO 132 I = 1,N
132 DET = DET*A(I,I)
    RETURN
    END
```

COMPUTER PROGRAM (contd)

```
C        WHEN THE CRITICAL BUCKLING LOAD IS TO BE
C        DETERMINED WITHOUT THE EFFECTS OF SHEAR
C        AND ROTATORY INERTIA THAT IS BY USING THE
C        CLASSICAL THEORY THE FOLLOWING CHANGES ARE
C        MADE IN THE ABOVE PROGRAM:

         DEN2 = 0.0
         DENOM = DEN1
```

MODE SHAPES:

To solve the system [A] {X} = {B}

```
      SUBROUTINE  LINEEQ(N,NN,A,B,X,DIGITS)
      DIMENSION A(NN,NN),B(NN),X(NN)
      DIMENSION  UL(30,30),IPS(30),SCALES(30),R(30),DX(30)
      NO = 30
C    N  = THE  NUMBER  OF  EQUATIONS  TO  BE  SOLVE
C    NN   THE  DIMENSION  NUMBER  OF  A, B, X IN  THE  MAIN  PROGRAM
C    NO = THE  DIMENSION  NUMBER  OF  THE  WORK  SPACES  UL,IPS,SCALES,
C         R,DX
      CALL  DECOMP(N,NN,A,UL,SCALES,IPS,NO)
      CALL  SOLVE(N,NN,UL,B,X,IPS,NO)
      CALL  IMPROV(N,NN,A,UL,B,X,IPS,R,DX,DIGITS,NO)
      RETURN
      END

      SUBROUTINE  DECOMP(N,NN,A,UL,SCALES,IPS,NO)
C         SUBROUTINE  FOR  SOLVING  A  LINEAR  SYSTEM
C         FROM  COMPUTER  SOLUTION  OF  LINEAR  ALGEBRAIC  SYSTEMS
C         BY  FORSYTHE  AND  MOLER
C         PRENTICE  HALL,  1967,  PAGE  68-70
      DIMENSION  A(NN,NN)
```

```
      DIMENSION  UL(NO,NO),SCALES(NO),IPS(NO)
C     INITIALIZE IPS,UL  AND  SCALES
      DO 5 I=1,N
      IPS(I)=I
      ROWNRM=0.0
      DO 2 J=1,N
      UL(I,J)=A(I,J)
      IF(ROWNRM=ABS(UL(I,J)))1,2,2
 1    ROWNRM=ABS(UL(I,J))
 2    CONTINUE
      IF(ROWNRM) 3,4,3
 3    SCALES(I) = 1.0/ROWNRM
      GO TO 5
 4    CALL SING(I)
      SCALES(I) = 0.0
 5    CONTINUE
C     GAUSSIAN ELIMINATION WITH PARTIAL PIVOTING
      NM1 = N-1
      DO 17 K=1,NM1
      BIG = 0.0
      DO 11 I=K,N
      IP=IPS(I)
      SIZE=ABS(UL(IP,K))*SCALES(IP)
      IF(SIZE-BIG)11,11,10
```

```
10        BIG=SIZE
          IDXPIV=I
11        CONTINUE
          IF(BIG) 13,12,13
12        CALL SING(2)
          GO TO 17
13        IF(IDXPIV = K) 14,15,14
14        J=IPS(K)
          IPS(K)=IPS(IDXPIV)
          IPS(IDXPIV) = J
15        KP=IPS(K)
          PIVOT=UL(KP,K)
          KP1=K+1
          DO 16 I=KP1,N
          IP=IPS(I)
          EM = -UL(IP,K)/PIVOT
          UL(IP,K) = -EM
          DO 16 J=KP1,N
          UL(IP,J) = UL(IP,J)+EM*UL(KP,J)
C         INNER LOOP,USE MACHINE LANGUAGE CODING IF
C         COMPILER DOES NOT PRODUCE EFFICIENT CODE.
16        CONTINUE
17        CONTINUE
          KP=IPS(N)
```

```
      IF(UL(KP,N))19,18,19
18    CALL SING(2)
19    RETURN
      END

      SUBROUTINE SOLVE (N,NN,UL,B,X,IPS,NO)
      DIMENSION B(NN),X(NN)
      DIMENSION UL(NO,NO),IPS(NO)
      NP1=N+1
      IP=IPS(1)
      X(1)=B(IP)
      DO 2 1=2,N
      IP=IPS(I)
      IM1=I-1
      SUM = 0.0
      DO 1 J=1,IM1
 1    SUM=SUM+UL(IP,J)*X(J)
 2    X(I)=B(IP)-SUM
      IP=IPS(N)
      X(N)=X(N)/UL(IP,N)
      DO 4 IBACK=2,N
      I=NP1-IBACK
C     I GOES FROM (N-1),...,1
      IP=IPS(I)
```

```
      IP1=I+1
      SUM = 0.0
      DO 3 J=IP1,N
 3    SUM=SUM+UL(IP,J)*X(J)
 4    X(I)=(X(I)=SUM)/UL(IP,I)
      RETURN
      END

      SUBROUTINE SING(IWHY)
11    FORMAT(54H MATRIX WITH ZERO ROW IN DECOMPOSE.)
12    FORMAT(54H SINGULAR MATRIX IN DECOMPOSE.  ZERO DIVIDE
      IN SOLVE.)
13    FORMAT (54H NO CONVERGENCE IN IMPRUV.  MATRIX IS NEARLY
      SINGULAR.)
      GO TO (1,2,3),IWHY
 1    PRINT 11
      GO TO 10
 2    PRINT 12
      GO TO 10
 3    PRINT 13
10    RETURN
      END

      SUBROUTINE IMPRUV(N,NN,A,UL,B,X,IPS,R,DX,DIGITS,NO)
```

```
      DIMENSION A(NN,NN),X(NN)
      DIMENSION UL(NO,NO),IPS(NO),R(NO),DX(NO
C     USES ABS(),AMAX1(),ALOG10()
      DOUBLE PRECISION SUM
      EPS = 1.0E=8
      ITMAX = 16
C     *** EPS AND ITMAX ARE MACHINE DEPENDENT. ***
      XNORM = 0,0
      DO 1 I=1,N
 1    XNORM=AMAX1(XNORM,ABS(X(I)))
      IF (XNORM)3,2,3
 2    DIGITS = -ALOGIO(EPS)
      GO TO 10
 3    DO 9 ITER=1,ITMAX
      DO 5 I=1,N
      SUM = 0.0
      DO 4 J=1,N
 4    SUM=SUM+A(I,J)*DBLE(X(J))
      SUM=B(I)-SUM
 5    R(I)=SUM
C     *** IT IS ESSENTIAL THAT A(I,J)*X(J) YIELD A DOUBLE
C     PRECISION RESULT AND THAT THE ABOVE + AND - BE
C     DOUBLE PRECISION.***
      CALL SOLVE (N,NN,UL,R,DX,IPS,NO)
      DXNORM = 0.0
```

```
      DO 6 I=1,N
      T=X(I)
      X(I)=X(I)+DX(I)
      DXNORM=AMAX1(DXNORM,ABS(X(I)-T))
 6    CONTINUE
      IF(ITER-1)8,7,8
 7    DIGITS=-ALOG10(AMAX1(DXNORM/XNORM,EPS))
 8    IF (DXNORM-EPS*XNORM)10,10,9
 9    CONTINUE
C     ITERATION DID NOT CONVERGE
      CALL SING(3)
10    RETURN
      END
```

NOMENCLATURE

W_E	work done by external forces, lb - in
K_E	kinetic energy, lb - in
$W(x,y)$	normal mode deflection amplitude, inch
p	natural frequency, rad/sec
Λ	$= p/10^4$
n	mesh points
$\bar{a}$	grid size
a_i	fraction of grid size $(i = 1,2, \ldots.)$
i,j	indices
q_o	normal force intensity, lb/in^2
N_x, N_y, N_{xy}	inplane force intensity, lb/in
λ_p	$= pa^2\sqrt{\rho/D}$ natural frequency parameter
λN	$= Ncr\ a^2\sqrt{\rho/D}$ critical buckling load parameter
ϕ_C	$= \dfrac{Ncr\ a_o^2}{27.25\ D}$ multiple of the critical buckling load for an annular circular plate

NOMENCLATURE (cont'd)

ϕ_E	$= \frac{Ncr\ a_o^2}{27.4D}$ multiple of the critical buckling load for an annular elliptical plate	
x,y,z	rectangular coordinates	
t	time, seconds	
a,b	plate dimensions, inches	
a_o, b_o	outer major and minor axes respectively of an annular elliptical plate	
a_i, b_i	inner major and minor axes respectively of an annular elliptical plate	
ρ	mass density per unit area of the plate, 0.00073 lbs sec^2/inch^2 for steel	
E	modulus of elasticity of an isotropic plate, 30 x 10^6 lbs/inch^2 for steel	
ν	poisson's ratio, 0.3 for steel	
G	$= E/2(1+\nu)$ modulus of rigidity of an isotropic plate	
H	plate thickness	
λ	$= \frac{\nu E}{(1+\nu)(1-2\nu)}$ Lame's constant	

NOMENCLATURE (cont'd)

k^2	=	$\Pi^2/12$ shear stress factor
$\bar{U},\bar{V},\bar{W}$		plate displacements in the x, y, and z directions respectively
U_E		Strain energy, lb - in
D	=	$\frac{EH^3}{12(1-\nu^2)}$ Flexural rigidity lb - in
$\nabla^2 W$	=	$\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2}$
$\nabla^4 W$	=	$\frac{\partial^4 W}{\partial x^4} + 2\frac{\partial^4 W}{\partial x^2 \partial y^2} + \frac{\partial^4 W}{\partial y^2}$
$[I]$		unitary matrix
Ncr		critical buckling load, lb/in
I	=	$\int_{-H/2}^{H/2} z^2\, dz$

REFERENCES

1. Anderson, R. A., "Flexural Vibration in Beams according to Timoshenko's Theory", Journal of Applied Mechanics, No. 20, 1953, pp 504.

2. Armenekas, A. and Hermann, G., "Vibrations and Stability of Plates under initial Stress", Paper No. 2500, J. Eng. Mech. Division, Proc. American Soc. Civil Engg., E.M. 3, Vol. 86, June 1960, pp 65-74.

3. Boidine, R. Y., "The Fundamental Frequencies of Thin Flat Circular Plates Simply Supported Along a Circle of Arbitrary Radius", A.S.M.E., Paper No. APMW - 10, J. APPL. Mech., Vol. 26, Dec. 1959, pp 666-668.

4. Bradley, W. A., "Stability of Equilateral Triangular Plates", Proc. American Soc. of Civil Engg., Vol. 89, E.M. 1 (J. Engg. Mech. Div.), Part 1, Feb. 1963, pp 37-56.

REFERENCES (cont'd)

5. Bramble, J. H., "A Second Order Finite Difference Analog of the First Biharmonic Boundary Value Problem", Numerische Math., Vol. 9, No. 3, 1966, pp 236-249.

6. Bolle, L., "Contribution au Probleme Lineaire de Flexion d'un Plaque Elastique", Tech. de la Suisse Romande, Nos. 21, 22, Oct. 1947.

7. Callahan, W. R., "On The Flexural Vibrations of Circular and Elliptical Plates", Quart. Appl. Math. Vol. 13, No. 4 Jan. 1956, pp 371-380.

8. Callahan, W. R., "Flexural Vibrations of Elliptical Plates when Transverse Shear and Rotatory Inertia are Considered", J. Accoust. Soc. Am., Vol. 36, No. 5, May 1966, pp 823-829.

9. Callegari, A., "Membrane Buckling Problem", Comm. Pure and Appl. Math., Vol XXIV, No. 4 July 1971 , pp 449-528.

REFERENCES (cont'd)

10. Conway, H. D., and Leissa, A. W., "A Method of Investigating certain Eigen Value of the Buckling and Vibration of Plates.", Journal of Applied Mechanics, Vol. 81, 1959.

11. Conway, H. D., "The Bending and Buckling and Flexural Vibration of Simply Supported Plates", J. Appl. Mech., Vol. 2 June 1961, pp 288-291.

12. Dean, W. R., "The Elastic Stability of an Annular Plate", Proceedings of the Royal Society of London, England, Series A, Vol. 106, 1924.

13. Ehlrich, L. W., "Solving the Biharmonic Equation as Coupled Finite Difference Equations", Journal on Numerical Analysis Vol. 8, No. 2, June 1971, pp 278-287.

14. Fettis, H. E., "Effects of Rotatory Inertia on Higher Modes of Vibration", Journal Aero. Sciences, Vol. 16, 1949, pp 445.

REFERENCES (cont'd)

15. Flugge, W. and Gerdeen, J. C., "Collapse of a Simply Supported Circular Plate Under a Uniform Load", International Journal of Solids and Structures, 3, 4 July 1967, pp 667-689.

16. Forsythe, G. E., and Moler, C. B., Computer Solution of Linear Algebraic Systems, Englewood Cliffs, N.J. Prentice - Hall, 1967.

17. Frederick, D. and Chang, T. S., Continuum Mechanics, Allyn and Bacon, Inc., June 1969.

18. Gerald, C., Applied Numerical Analysis, Reading, Mass., Addison - Wesley, 1970.

19. Green, A. E., "On Reissner's Theory of Bending of Elastic Plates", Quarterly of Applied Math., Vol. 7, No. 2, 1949, pp 223.

20. Hadjidimos, A., "The Numerical Solution of a Model Problem of a Biharmonic Equation By Using the Alternating Direction Implicit Methods", Num. Math., Vol. 17, No. 4, 1971, pp 301-317.

REFERENCES (cont'd)

21. Hans, L. S., "The Buckling and Deflection of Isocoles - Triangular Plates", A.S.M.E. Trans. 82 E. Journal of Applied Mechanics, 1, Mar. 1960, pp 207-208.

22. Hencky, H., "Uber die Berucksitigung der Schulwerzenung in ebenem Platten", Ingenieur - Archiv, XVI Band, 1947, pp 72.

23. Herrmann, G., "Influence of Large Amplitudes on Flexural Motions of Elastic Plates", NACA. TN D - 3598, 1965.

24. Hildebrand, F. B., Advanced Calculus for Applications, Prentice - Hall, Inc., 1962.

25. Kaul, R. K., and Tewari, S. G., "On the Bounds of Eigen Value of a Clamped Plate in Tension", Journal of Appl. Mech., Vol. 25, No. 1 Mar. 1958, pp 52-56.

26 Kirchhoff, G., "Uber das Gleichgewicht und die Bewegung einer Elastichen Scheibe", Zeitchrift fur Reine und Angewandte Mathematik, Vol. 40, 1850, pp 57.

REFERENCES (cont'd)

27. Kromm, A., "Verallgemeinerte Theorie der Plattenstatik", Ingenieur Archiv, XXI Band, 1953, pp 266.

28. Kromm, A., "Uber die Randquerkrafte bei Gestutzten Platten", Zeitschrift fur Angewandte Math. und Mechanik, Band 35, Heft 6, 1955, pp 231.

29. Lamb, H., "On Waves in an Elastic Plate", Proceedings Royal Soc. of London, Vol. 93, Series A, 1917, pp 114.

30. Lee, H. C., "A Generalized Minimum Principle and its Application to the Vibration of a Wedge with Rotatory Inertia and Shear", Journal of Applied Mechanics, Trans. Am. Soc. Engg., Jun 1963, pp 176-180.

31. Leissa, A. W., "Vibration of a Simply Supported Elliptical Plate", J. Sound and Vibration, 1967.

32. Leissa, A. W., "Vibration of Plates", NASA SP - 160, 1969.

REFERENCES (cont'd)

33. Lurie, H., "Lateral Vibrations as Related to Structural Stability", Journal of Applied Mech., Vol. 19, No. 2, June 1952, pp 195-204.

34. Mansfield, E. H., "On the Buckling of an Annular Plate", Quart. J. Mech. Appl. Math., 13, 1, Feb 1960, pp 16-23.

35. Martin, C. J., "Vibrations of a Circular Elastic Plate Under Uniform Tension". Proc. 4th U.S. Natl. Congr. Appl. Mech., 1962, pp 227-284.

36. Molachlan, N. V., "Vibrational Problems in Elliptical Co-ordinates", Quart. Appl. Math., Vol. 5, No. 3, 1947, pp 289-297.

37. Mc Nitt, R. P., "Free Vibration of a Clamped Elliptical Plate", J. Aerospace Sci., Vol. 29 No. 9 Sept. 1962, pp 1124-1125.

38. Mindlin, R. D. "Influence of Rotatory Inertia and Shear on Flexural Motions of Isotropic Elastic Plates", Journal of Applied Mechanics, Vol. 18, Mar. 1951, pp 31-38.

REFERENCES (cont'd)

39. Mindlin, R. D., "Thickness - Shear and Flexural Vibrations of a Circular Disk", J. Appl. Phy., Vol. 25, No. 10, Oct. 1954, pp 1329-1332.

40. Mindlin, R. D., Schacknow, A., and Deresiewicz, H., "Flexural Vibrations of a Rectangular Plate", J. Appl. Mech., Vol. 23, No. 23, Sept. 1956, pp 430-436.

41. Mitchell, J., "Computational Methods in Partial Differential Equation", John Wiley & Sons, New York, New York, 1969.

42. Mossakowski, J., "Buckling of Circular Plates with Cylinderical Orthotropy", Archiwum Mech. Stosowanej, Vol. 12, 1960, pp 583-596.

43. Noble, B., "The Vibration and Buckling of a Circular Plate Clamped on Part of its Boundary and Simply Supported on the Remainder", Proc. 9th Midwest Conf. Sol. Fluid Mech., Aug 1965.

44. Nowacki, W., "Dynamics of Elastic Systems", John Wiley & Sons., Inc., 1963.

REFERENCES (cont'd)

45. Pagano, N., and Chou, P. C., Theory of Elasticity, Van Nostrand Co., Inc., 1967.

46. Pandalai, K. A., and Patel, S. A., "Buckling of Orthotropic Circular Plates", Journal of the Royal Aeronautical Society, Vol. 69, April 1965, pp 279-280.

47. Pochhammer, L., and Chree., "Uber die Fortplanzungsgechwindigkeiten Kleiner Schwingungen in einem Unbegrenzten Isotropen Kreiszylinder", Zeitschrift fur die Reine und Angewandte Mathematik, Crelle, Vol. 81, 1876, pp 324.

48. Raju, P. N., "Vibrations of Annular Plates", J. Aeron. Soc. India, Vol. 14, No. 2, May 1962, pp 37-52.

49. Lord Rayleigh, Theory of Sound, Dover Publications, New York, 1945.

50. Lord Rayleigh, "On the Free Vibrations of an Infinite Elastic Plate of Homogenious Isotropic Elastic Matter", Proc. Royal Soc. of London, Vol. 10, 1889, pp 225.

REFERENCES (cont'd)

51. Reid, W. P., "Free Vibration of a Circular Plate", J. Soc. Ind. Appl. Math., Vol. 10, No. 4, Dec. 1963, pp 668-674.

52. Reissner, E., "The Effects of Shear Deformation on the Bending of Elastic Plates", Journal of Applied Mechanics, Vol. 12, 1945, pp 68-70.

53. Reissner, E., "On the Bending of Elastic Plates", Quarterly of Applied Mechanics, Vol. 5, 1947, pp 55.

54. Reissner, E., "On Axi-symmetrical Vibrations of a Circular Plate of Uniform Thickness, Including the Effects of Shear and Rotatory Inertia", Journal Accoust. Soc. Am., Vol. 26, No. 2, Mar. 1954, pp 252.

55. Robinson, N. I., "Buckling of Parobolic and Semi-Elliptical Plates", A I A A Journal, 6, 7, June 1969, pp 1204 (Technical Notes).

56. Schafer, M., "Uber eine Verfeineeung der Klassichen Theorie dunner Schwachgebougener Platten", Zeitschrift fur Angewandte Mathematik und Mechanik, Band 32, Heft 6, 1952, pp 161.

57. Southworth and Deleeuw, "Digital Computations and Numerical Methods", McGraw - Hill Book Co., 1965.

58. Timoshenko and Gere, "Theory of Elastic Stability", McGraw - Hill Book Co., 1961, Second Edition.

59. Timoshenko and Woinowsky-Krieger, "Theory of Plates and Shells", McGraw - Hill Book Co., 1959, Second Edition.

60. Timoshenko, S., "On the Correction for Shear of the Differential Equation for Transverse Vibration of Prismatic Bar", Philosophical Magazine, Vol. 41, Series 6, 1921, pp 742.

61. Uflyand, Y. S., "The Propagation of Waves in the Transverse Vibrations of Bars and Plates", Prikl. Mat. Mech., Vol. 12, 1948, pp 287-300.

62. Uthgenannt, E. B., and Brand, R. S., "Buckling of Orthotropic Annular Plates", A I A A Journal, Vol. 8, No. 11, Nov. 1970.

REFERENCES (cont'd)

63. Vet, Maarten., "Natural Frequencies of Thin Rectangular Plates", Mach. Design, Vol. 37, No. 13, June 1965, pp 183-185.

64. Vijay Kumar, K. and Joga Rao, C. V., "Buckling of Polar Orthotropic Annular Plates", Journal of the Engg. Mech. Division, Proceedings of the American Society if Civil Engg., 97, E.M. 3, June 1971, pp 701-710.

65. Volterra, E., "Method of Internal Constraints and it Application", Trans. of American Soc. Civil Engg., Vol. 128, Part I 1963, pp 509-513.

66. Volterra, E., "Effect of Shear Deformation on the Benging of Rectangular Plates", Trans. ASME., Sept. 1960, pp 594.

67. Volterra, E., and Zachmanoglou Dynamics of Vibrations, Columbus, Ohio, C. E. Merril Books, 1965.

68. Wah, Thein., "Vibrations of Circular Plates", Journal A.S.A., Vol. 34, No. 3, Mar. 1962.

REFERENCES (cont'd)

69. White, R. N., and Cottingham, W. S., "Stability of Plates under Partial Edge Loadings", Proc. Am. Soc. Civil Engg. 88 E.M.S. (J. Engg. Mech. Div.) Oct. 1962, pp 67-68.

70. Woinowski - Kreiger, S., "Buckling Stability of Circular Plates with Cylinderical Aeolotropy", Ingenieur Archiv, Vol. 26, 1958, pp 129-131.

71. Yamaki, N., "Buckling of a Thin Annular Plate Under Compression", Journal of Applied Mech., Vol. 25, June 1958, pp 267-273.

72. Young, D., "Vibration of Rectangular Plates by the Ritz Method", Journal of Applied Mech., Vol. 17, No. 4, Dec. 1950, pp 448-453.

73. Yu, Y. Y., "Flexural Vibrations of Elastic Sandwich Plates", J. Aeron. Sciences, April 1960, pp 273-282.

VITA

Bomi Homi Batiwalla was born . He received his Bachelor of Mechanical Engineering from Bangalore University, India in 1968. After receiving his degree he registered at the Graduate School of Stevens Institute of Technology, Hoboken, New Jersey.

In 1969 he was awarded the degree of Master of Mechanical Engineering. He then registered as a doctoral student and a Research Assistant under Dr. Benedict C. Sun at Newark College of Engineering until his graduation in May 1972.