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TRANSVERSE VIBRATION OF A CLASS OF ORTHOTROPIC PLATES

BY

NICHOLAS J. DE CAPUA

A DISSERTATION

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ABSTRACT

This study determines the eigenvalues, eigenvectors, and nodal patterns of a class of orthotropic plates whose geometry is governed by the equation

$$\left(\frac{x}{a}\right)^{\alpha} + \left(\frac{y}{b}\right)^{\beta} = 1,$$

where the parameters a, b, α, and β permit the plate geometry to vary over a range which includes the rhombus, circle, ellipse, square, and rectangle.

Variable thickness, inplane forces, and mixed or discontinuous boundary conditions are also considered. The following assumptions have beem employed:

i) plate is thin with respect to other dimensions,

ii) deflections are small,

iii) rotary inertia and shear are neglected.

The method of analysis employed is the Rayleigh-Ritz energy technique using xy-polynomials as the approximated deflection. Eigenvalues and eigenvectors were computed by the method of reductions, and the evaluation of double integrals was achieved by the numerical procedure of Gauss-Legendre quadratures.

The validity of the analysis was checked by comparison with known solutions for rectangular orthotropic plates, and isotropic plates with variable thickness, inplane forces, and mixed or discontinuous boundary conditions. It was found that the calculated frequencies and nodal patterns were in good agreement with existing data.

APPROVAL OF DISSERTATION

TRANSVERSE VIBRATION OF A CLASS OF

ORTHOTROPIC PLATES

BY

NICHOLAS J. DE CAPUA

FOR

DEPARTMENT OF MECHANICAL ENGINEERING

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BY

FACULTY COMMITTEE

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Newark, New Jersey

March 1971

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Finally, I wish to dedicate this work to my wife Lee, for her encouragement and understanding during the research, and my two children Nicky and Mary Beth.

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I. INTRODUCTION

The objective of this study is to develop a procedure for obtaining the fundamental and higher frequencies and mode shapes of a class of orthotropic plates. Variable thickness, inplane forces, and mixed or discontinuous boundary conditions are included in the analysis, whereas rotary inertia and shear are neglected.[1] It is also assumed that the plates are thin and that the amplitudes of vibration are small enough to ignore second order effects, i.e. the analysis is linear.

The class of plate geometries is governed by the equation

$$\left(\frac{x}{a}\right)^{\alpha} + \left(\frac{y}{b}\right)^{\beta} = 1. \qquad (1.1)$$

For a = b the rhombus, circle, and square result when $\alpha = \beta = 1$, 2 and 10, respectively. The square has slightly rounded corners and an area which is 1.427 percent smaller than the true square. For $a \neq b$ the diamond, ellipse, and rectangle result for $\alpha = \beta = 1$, 2 and 10. These configurations are indicated in Figure 1.

A solution for the frequencies and mode shapes is obtained by an approximate energy method, i.e. the Rayleigh-Ritz technique. The accuracy of the method depends to a great extent upon the set of functions that is chosen to

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[1] See section XI, Discussion of Assumptions, for details.

represent the plate deflection. In most previous investigations [1,9,26,48,67,72] this deflection shape is assumed to be the product of the normal mode shapes of a uniform transverse beam vibration. However, due to the generality of the present solution the use of normal mode shapes may not be feasible. An xy-polynomial is thus used as the approximated deflection shape. The polynomial approximation has been shown to give satisfactory results with the Rayleigh-Ritz technique [25,36,52].

II. HISTORICAL BACKGROUND

A. General

The vibration of thin isotropic plates has been studied for nearly 200 years [13], however the more general orthotropic, variable thickness plate with inplane forces has been examined in part for only about 30-40 years. A recent publication by Leissa [37] is an excellent document which gives a comprehensive set of available results for frequencies and mode shapes of plates through 1965. This document has hundreds of excellent references and summaries on the vibration of all kinds of plates.

B. Orthotropic Plates

The first major contribution to the vibration of orthotropic plates was done by Hearmon [27] in 1946. He used the Rayleigh method for estimating the fundamental frequency of rectangular plates and then attempted to corroborate his answers experimentally. In 1959 he [28] used Rayleigh's method again with characteristic beam functions as the deflection shape for the mixed boundary condition orthotropic rectangular plate. Other contributions were made by Sundara Raja Iyengar and Jagadish [60] who used an approximate Fourier series expansion method to obtain results similar to Hearmon's. Kanazawa and Kawai [33] solved various combinations of simply supported and clamped boundaries by superimposing edge moments on a simply supported plate. Mahalingam [39] used the Rayleigh-Ritz method with characteristic beam functions on rectangular plates with stiffeners. Kirk [34] also examined stiffened plates but used the simpler Rayleigh method for fundamental frequencies. Huffington and Hoppmann [31] obtained exact solutions for the rectangular plate with two opposite boundaries simply supported, and more recently Dickinson [20] used the sine series solution to rectangular plates with any combination of boundary conditions. Also, very recently J. E. Ashton [3] and [6] examined free-free plates by the Rayleigh-Ritz method and also performed an experimental investigation.

Very little work has been done on circular plates with rectangular orthotropy. In 1958 Hoppmann [30] did an experimental study on elliptical and circular plates with rectangular orthotropy and then tried unsuccessfully to corroborate the experimental data analytically.

Pandalai and Patel [51] examined circular plates with polar orthotropy for clamped and simply supported boundaries. Minkarah and Hoppmann [47] examined the same types of plates experimentally.

C. Variable Thickness Plates

The initial work on the vibration of variable thickness plates was done by Conway [14] in 1957. He obtained the exact solution for a circular plate with rigidity proportioned to r^m for various values of m. Barakat and Bauman [7] used a Ritz-Galerkin type of solution for a circular plate with parabolic thickness variation and Conway, Becker, and Dubil [15] solved the circular plate with linear thickness variation. Also, Harris [24] did an exact analysis with lenticular thickness variation.

In 1963 Plunkett [53] performed an experimental study of rectangular cantilever plates with linearly thickness variations. Appl and Byers [2] solved analytically the simply supported rectangular plate with linear thickness variation. They obtained upper and lower bounds on the fundamental frequency. Dawe [17] and [18] used a finite element approach and corroborated it with experimental data. Raju [54] also used finite element and experimental verification. In 1969 J. E. Ashton [4] and [5] used the Rayleigh-Ritz technique with characteristic beam functions for the assumed deflection to obtain the frequency and mode shapes of rectangular plates with clamped boundaries, and with two opposite boundaries clamped and two simply supported.

Maymon and Segal [41], in 1969, experimentally examined rhombic plates with diamond shaped cross sections.

D. Inplane Forces

Analysis of plates with inplane forces was first examined in 1933 by Bickley [10]. He used the Rayleigh method to obtain upper bounds and the Southwell method for lower bounds on the fundamental frequency of a clamped circular plate in hydrostatic tension. In 1943 Weinstein and Chien [69] used a variational technique to obtain lower bounds on the frequency of a clamped rectangular plate under hydrostatic tension. Upper bounds were obtained by the Rayleigh-Ritz technique with characteristic beam functions. The Rayleigh method was used by Herrmann [29] to obtain approximate fundamental frequencies of a rectangular plate with two opposite edges simply supported.

In 1962 Wah [64] determined the roots of the exact characteristic equation of a clamped circular plate and Martin [40] used a perturbation technique to obtain the same results.

E. Discontinuous Boundary Conditions

The fundamental frequencies of rectangular plates with discontinuous boundary conditions, i.e. a change in the boundary condition other than at a corner, were first obtained by Ota and Hamada [50] in 1958. They solved the problem by assuming a deflection function which satisfies the simply-supported boundary condition everywhere and applying distributed edge moments on the clamped portion. In 1963 Kurata and Okamura [35] did essentially the same thing.

Bartlett [8] used a variational approach to obtain upper and lower bounds for circular plates with discontinuous boundary conditions, and Noble [49] solved the same problem approximately.

F. Combined Conditions

The only study that could be found which included more than one effect, i.e. orthotropy and variable thickness, was by Salzman and Patel [57] in 1968. They used the method of Frobenius to obtain the frequency equation for a circular plate. However no data was presented.

III. RAYLEIGH-RITZ METHOD

The procedure of Rayleigh-Ritz [43,55,56,70] is an approximate energy method for determining the eigenvalues and eigenvectors of continuous-mass systems. It has been shown by Ritz that the eigenvalues determined by this method are upper bounds to the true eigenvalues [55]. The accuracy of the method increases as the number of trial functions increase, and is the most accurate for the lower modes. Two means of checking the accuracy of the results are comparison to known solutions and the satisfaction of the orthogonality condition for each mode.

The application of the Rayleigh-Ritz method to vibrating isotropic plates will be discussed in the remainder of this section. The kinetic and strain energies of a thin isotropic plate, neglecting rotary inertia and shear, and assuming small deflections, are given by [62]

$$V = \frac{D}{2} \iint_A \left[\left(\frac{\partial^2 w}{\partial x^2} \right)^2 + \left(\frac{\partial^2 w}{\partial y^2} \right)^2 + 2\nu \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} + 2(1-\nu) \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 \right] dxdy, \tag{3.1}$$

and

$$T = \tfrac{1}{2}\rho h \iint_A w^2 dxdy. \tag{3.2}$$

When the plate vibrates in a transverse normal mode the deflection can be written as

$$w(x,y,t) = W(x,y)\cos pt. \qquad (3.3)$$

Thus for an isotropic plate, vibrating harmonically with amplitude W(x,y), and natural frequency p, the maximum strain energy is,

$$V_{max} = \frac{D}{2} \int\!\!\int_A \left[\left(\frac{\partial^2 W}{\partial x^2}\right)^2 + \left(\frac{\partial^2 W}{\partial y^2}\right)^2 + 2\nu \frac{\partial^2 W}{\partial x^2}\frac{\partial^2 W}{\partial y^2} + 2(1-\nu)\left(\frac{\partial^2 W}{\partial x \partial y}\right)^2\right] dxdy, \qquad (3.4)$$

and the maximum kinetic energy is

$$T_{max} = \tfrac{1}{2}\rho h p^2 \int\!\!\int_A W^2 dxdy, \qquad (3.5)$$

where the integrations are to be taken over the domain of the plate.

For a conservative system the total energy must be a constant, so that

$$T_{max} = V_{max}. \qquad (3.6)$$

Thus from equations (3.4) and (3.5)

$$p^2 = \frac{V_{max}}{\frac{1}{2}\rho h \iint_A W^2 dxdy} \; . \qquad (3.7)$$

This expression is called Rayleigh's quotient. The Rayleigh-Ritz method consists of selecting a family of trial functions u_i, satisfying all[2] the boundary conditions of the problem, and constructing a linear combination

$$W_n = \sum_{i=1}^{n} A_i u_i , \qquad (3.8)$$

where the u_i are known functions of the spatial coordinates, linearly independent over the plate area and the A_i are unknown coefficients. Essentially, in doing that, one approximates an infinite degree-of-freedom system by an n-degrees-of-freedom system, so the constraints

$$A_{n+1} = A_{n+2} = \ldots = 0$$

are imposed on the system. Constraints have a tendency to raise the stiffness of the system, so the estimated frequency

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[2] This requirement will be reexamined shortly.

will be higher than the true frequency. By increasing the number of trial functions in the family, affects of constraints are reduced, resulting in an estimated frequency which is closer to the true frequency.

This trial family W_n is then substituted into Rayleigh's quotient. The Rayleigh-Ritz procedure is now employed. It states that the natural frequencies are determined by finding expressions for W_n that satisfy the boundary conditions and minimize Rayleigh's quotient with respect to each A_i. Thus

$$\frac{\partial V_{max}}{\partial A_i} - p^2 \frac{\partial}{\partial A_i} \left[\tfrac{1}{2}\rho h \iint_A W^2 dxdy \right] = 0. \qquad (3.9)$$

The partial derivatives in this expression are linear functions of the A_i, and hence, represent a set of n homogeneous equations in the A_i. Setting the determinant equal to zero gives the frequency equation, which has n real roots, $p_1, p_2, \ldots, p_n$.

It has been previously mentioned that the trial family of functions u_i must satisfy the boundary conditions. This, however, is a strict requirement which is sometimes difficult to obtain. It can be shown [43] that the chosen functions

u_i need satisfy only the "geometric" boundary conditions and it is not necessary to satisfy the "natural" boundary conditions. The "geometric" boundary conditions result purely from geometric compatibility, i.e. deflection and slope, while the "natural" boundary conditions are supplied by the moment or shear force balance. Such trial functions are said to be "admissible".

IV. ORTHOTROPIC PLATES

The maximum strain energy for an orthotropic plate vibrating harmonically is given by [38,63]

$$V_{max} = \frac{1}{2} \iint_A \left\{ D_x W_{xx}^2 + 2D_1 W_{xx} W_{yy} + D_y W_{yy}^2 + 4D_{xy} W_{xy}^2 \right\} dxdy, \tag{4.1}$$

where the flexural rigidities are given by

$$D_x = \frac{E_x h^3}{12(1-\nu_{xy}\nu_{yx})},$$

$$D_y = \frac{E_y h^3}{12(1-\nu_{xy}\nu_{yx})},$$

$$D_1 = \frac{\nu_{yx} E_x h^3}{12(1-\nu_{xy}\nu_{yx})},$$

$$D_{xy} = \frac{G_{xy} h^3}{12},$$

and rotary inertia and shear are neglected, and small deflections are assumed. Thus an orthotropic plate can be characterized by the constants E_x, E_y, ν_{xy}, ν_{yx} and G_{xy}, with $\nu_{xy}E_y = \nu_{yx}E_x$ because of the required symmetry of the stress-strain equations. Therefore, for an orthotropic plate, there are four independent elastic sonstants.

The maximum kinetic energy is

$$T_{max} = \frac{\rho h}{2} p^2 \iint_A W^2 dxdy. \qquad (4.2)$$

The functions chosen to represent the deflection W are given by

$$W(x,y) = F(x,y)\{A_1 + A_2 x + A_3 y + A_4 xy + \ldots\}, \qquad (4.3)$$

or in matrix notation

$$W = (\ldots A_i \ldots) \begin{pmatrix} \cdot \\ \cdot \\ \cdot \\ FG^i \\ \cdot \\ \cdot \\ \cdot \end{pmatrix} \equiv (A_i)(FG^i), \qquad (4.4)$$

where the A_i are the constants to be minimized, the G^i are the xy-polynomial functions, i.e.

$$G^1 = 1,$$

$$G^2 = x,$$

$$G^3 = y,$$

$$G^4 = xy,$$

etc.,

and F(x,y) are the boundary functions.

By the Rayleigh-Ritz method each FG^i must satisfy the "geometric" boundary conditions. This is achieved by a suitable choice of the function F(x,y) for simply supported, clamped, free, or mixed boundary conditions. A complete discussion of these boundary functions is given in Section VII.

Recall from Section III that the eigenvalues are determined from

$$\frac{\partial V_{max}}{\partial A_i} - p^2 \frac{\partial}{\partial A_i}\left[\tfrac{1}{2}\rho h \iint_A W^2 dxdy\right] = 0, \qquad (4.5)$$

where for an orthotropic plate from equation (4.1)

$$\frac{\partial V_{max}}{\partial A_i} = \frac{\partial}{\partial A_i}\frac{1}{2} \iint_A \left[D_x W_{xx}^2 + 2D_1 W_{xx} W_{yy} + D_y W_{yy}^2 + 4D_{xy} W_{xy}^2\right] dxdy, \qquad (4.6)$$

or

$$\frac{\partial V_{max}}{\partial A_i} = \frac{1}{2} \iint_A \left[2D_x W_{xx} \frac{\partial W_{xx}}{\partial A_i} + 2D_1 W_{xx} \frac{\partial W_{yy}}{\partial A_i} + 2D_1 W_{yy} \frac{\partial W_{xx}}{\partial A_i} + 2D_y W_{yy} \frac{\partial W_{yy}}{\partial A_i} + 8D_{xy} W_{xy} \frac{\partial W_{xy}}{\partial A_i}\right] dxdy. \qquad (4.7)$$

Examining the derivatives $\frac{\partial W_{xx}}{\partial A_i}$, $\frac{\partial W_{yy}}{\partial A_i}$ and $\frac{\partial W_{xy}}{\partial A_i}$ in detail, consider first $\frac{\partial W_{xx}}{\partial A_i}$. From equation (4.4)

$$\frac{\partial W_{xx}}{\partial A_i} = \frac{\partial}{\partial A_i} \left\{ (A_i)(FG^i)_{xx} \right\}$$

$$= \frac{\partial}{\partial A_i} \left\{ (\ldots A_i \ldots) \begin{pmatrix} \vdots \\ (FG^i)_{xx} \\ \vdots \end{pmatrix} \right\}$$

$$= (0,0,0,\ldots,1,0,0,\ldots) \begin{pmatrix} \vdots \\ (FG^i)_{xx} \\ \vdots \end{pmatrix}$$

Thus

$$\frac{\partial W_{xx}}{\partial A_i} = (FG^i)_{xx}. \tag{4.8}$$

Similarly,

$$\frac{\partial W_{yy}}{\partial A_i} = (FG^i)_{yy}, \tag{4.9}$$

and

$$\frac{\partial W_{xy}}{\partial A_i} = (FG^i)_{xy}. \tag{4.10}$$

Equation (4.7) can therefore be written

$$\frac{\partial V_{max}}{\partial A_i} = \frac{1}{2} \iint_A \Big[2D_x A_j (FG^j)_{xx} (FG^i)_{xx} + 2D_1 A_j \left[(FG^j)_{xx} (FG^i)_{yy} + (FG^j)_{yy} (FG^i)_{xx} \right] + 2D_y A_j (FG^j)_{yy} (FG^i)_{yy} + 8D_{xy} A_j (FG^j)_{xy} (FG^i)_{xy} \Big] dxdy. \tag{4.11}$$

Also,

$$\frac{\partial}{\partial A_i} \left[\tfrac{1}{2} \rho h \iint_A W^2 dxdy \right] = \rho h \iint_A A_j (FG^j)(FG^i) dxdy. \tag{4.12}$$

Equation (4.5) can now be written as

$$[C_{ij}]\{A_i\} - \frac{p^2 \rho h}{D_y} [B_{ij}]\{A_i\} = 0, \tag{4.13}$$

where

$$C_{ij} = \iint_A \left\{ \frac{D_x}{D_y} (FG^j)_{xx}(FG^i)_{xx} + \frac{D_1}{D_y} \left[(FG^j)_{xx}(FG^i)_{yy} \right.\right.$$

$$\left. + (FG^j)_{yy}(FG^i)_{xx} \right] + (FG^j)_{yy}(FG^i)_{yy}$$

$$\left. + 4 \frac{D_{xy}}{D_y} (FG^j)_{xy}(FG^i)_{xy} \right\} dxdy, \qquad (4.14)$$

and

$$B_{ij} = \iint_A (FG^j)(FG^i) dxdy. \qquad (4.15)$$

$[C_{ij}]$ and $[B_{ij}]$ are square, symmetric matrices with real number elements and $\{A_i\}$ is the column matrix defining the eigenvectors of the specific natural mode.

Consider now, the specific geometric boundary of the plate defined by

$$1 - \left(\frac{x}{a}\right)^\alpha - \left(\frac{y}{b}\right)^\beta = 0, \qquad (4.16)$$

and introduce the normalized variables

$$X = \frac{x}{a}, \quad Y = \frac{y}{a}, \quad \text{with} \quad P = \left(\frac{a}{b}\right)^\beta.$$

Thus equation (4.16) becomes

$$1 - X^{\alpha} - PY^{\beta} = 0, \tag{4.17}$$

and (4.13) becomes

$$[C_{ij}]\{A_i\} - \frac{p^2 \rho h a^4}{D_y} [B_{ij}]\{A_i\} = 0, \tag{4.18}$$

with

$$C_{ij} = \int_0^1 \int_0^{R(1-X^{\alpha})^{1/\beta}} \Bigg\{ \frac{D_x}{D_y} (FG^j)_{XX}(FG^i)_{XX} + \frac{D_1}{D_y} \Big[(FG^j)_{XX}(FG^i)_{YY} + (FG^j)_{YY}(FG^i)_{XX} \Big] + (FG^j)_{YY}(FG^i)_{YY} + 4\frac{D_{xy}}{D_y} (FG^j)_{XY}(FG^i)_{XY} \Bigg\} dYdX, \tag{4.19}$$

and

$$B_{ij} = \int_0^1 \int_0^{R(1-X^{\alpha})^{1/\beta}} (FG^j)(FG^i)dYdX, \tag{4.20}$$

where R is the aspect ratio, i.e.

$$R = b/a.$$

The integrations in (4.19) and (4.20) can be carried out over the first quadrant of the plate area, as indicated by the integration limits, only when the boundary conditions are the same for all four quadrants. For mixed or discontinuous boundary conditions the integration must be over the entire area.

By defining

$$\omega^2 = \frac{p^2 \rho h a^4}{D_y} , \qquad (4.21)$$

(4.18) becomes

$$[C_{ij}]\{A_i\} - \omega^2 [B_{ij}]\{A_i\} = 0. \qquad (4.22)$$

Thus the fundamental and higher frequencies can be determined from the eigenvalues of (4.22). First, this equation must be reduced to standard eigenvalue form, i.e.

$$[D_{ij}]\{\psi_i\} = \lambda\{\psi_i\}. \qquad (4.23)$$

Both $[C_{ij}]$ and $[B_{ij}]$ are positive definite so that the conversion can be accomplished by the series of matrix operations outlined in Bishop, Gladwell, and Michaelson [11]. The eigenvalues and eigenvectors of equation (4.23) can now be obtained through a numerical technique.

The specific eigenvalue evaluation method is discussed in Section IX - Computational Techniques.

Results for specific values of α, β and elastic properties are given in Section X - Results.

V. VARIABLE THICKNESS ISOTROPIC PLATES

The inclusion of variable thickness into the solution of the free vibration of isotropic plates means that the thickness and flexural rigidity are now functions of the x and y coordinates. Thus

$$h = h(x,y),$$
$$D = D(x,y) = \frac{Eh^3(x,y)}{12(1-\nu^2)} . \qquad (5.1)$$

The maximum strain and kinetic energy are therefore

$$V_{max} = \iint_A \frac{D(x,y)}{2} \left\{(\nabla^2 W)^2 - (1-\nu)\left[W_{xx}W_{yy}-W_{xy}^2\right]\right\}dxdy, \qquad (5.2)$$

where

$$\nabla^2 W = \frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2} ,$$

and

$$T_{max} = \frac{\rho p^2}{2} \iint_A h(x,y)W^2 dxdy. \qquad (5.3)$$

As before, rotary inertia and shear are neglected and small deflections are assumed.

Define

$$h(x,y) = \bar{h}H(x,y),$$

and

$$D(x,y) = \bar{D}H^3(x,y),$$

where $\bar{h}$ and $\bar{D}$ are constants such that

$$\bar{D} = \frac{E\bar{h}^3}{12(1-\nu^2)} .$$

Thus equations (5.2) and (5.3) can be written as

$$V_{max} = \frac{\bar{D}}{2} \iint_A H^3(x,y)\left\{(\nabla^2 W)^2 + 2(1-\nu)\left[W_{xx}W_{yy} - W_{xy}^2\right]\right\}dxdy, \tag{5.4}$$

and

$$T_{max} = \frac{\rho p^2\bar{h}}{2} \iint_A H(x,y)W^2 dxdy. \tag{5.5}$$

Carrying out the Rayleigh-Ritz procedure as in the previous section yields

$$[C_{ij}]\{A_i\} - \omega^2[B_{ij}]\{A_i\} = 0, \tag{5.6}$$

where

$$\omega^2 = \frac{\rho \bar{h} p^2 a^4}{\bar{D}} \tag{5.7}$$

$$C_{ij} = \int_0^1 \int_0^{R(1-X^\alpha)^{1/\beta}} H^3(X,Y)\Big\{\nabla^2(FG^j)\nabla^2(FG^i) - (1-\nu)\Big[(FG^j)_{XX}(FG^i)_{YY} + (FG^i)_{XX}(FG^j)_{YY} - 2(FG^j)_{XY}(FG^i)_{XY}\Big]\Big\}dYdX, \tag{5.8}$$

and

$$B_{ij} = \int_0^1 \int_0^{R(1-X^\alpha)^{1/\beta}} H(X,Y)(FG^j)(FG^i)dYdX. \tag{5.9}$$

Here, as in the previous section, the integrations are carried out over one-quarter of the area. This will yield correct results only for boundary conditions, geometries and thickness variations that are symmetric about the X and Y axes. Otherwise the integrations must be carried out over the entire area.

Specific results are given in Section X - Results.

VI. ISOTROPIC PLATES WITH INPLANE FORCES

In this section the effects of forces acting in the plane of the undeformed middle surface of the plate are considered. The inplane force intensities N_x, N_y, and N_{xy} are assumed to be constants. This assumption can be realized in one of the following two ways:

(1) The boundary of plate provides no fixity in the plane of the plate.

(2) The deflection is sufficiently small relative to the initial tension or compression in the plate so that the inplane forces are not significantly affected.

The normal forces N_x and N_y are positive if the plate is in tension, the shear force N_{xy} is positive according to the accepted convention of elasticity. See Figure 2.

The maximum strain and kinetic energies for isotropic plates with inplane force intensities N_x, N_y, and N_{xy} are [69]

$$V_{max} = \frac{D}{2} \iint_A \left\{ (\nabla^2 W)^2 - (1-\nu)\left[W_{xx}W_{yy} - W_{xy}^2\right]\right\} dxdy$$

$$+ \frac{1}{2} \iint_A \left\{ N_x W_x^2 + N_y W_y^2 + 2N_{xy}W_xW_y \right\} dxdy, \tag{6.1}$$

and

$$T_{max} = \frac{\rho p^2 h}{2} \iint_A W^2 dxdy. \tag{6.2}$$

Once again, assuming a deflecting function given by

$$W = (\ldots A_i \ldots) \begin{pmatrix} \vdots \\ FG^i \\ \vdots \end{pmatrix}, \tag{6.3}$$

and performing the Ritz minimization

$$\frac{\partial V_{max}}{\partial A_i} - p^2 \frac{\partial}{\partial A_i} \left\{ \frac{\rho h}{2} \iint_A W^2 dxdy \right\}, \tag{6.4}$$

yields

$$[C_{ij}]\{A_i\} - \omega^2 [B_{ij}]\{A_i\} = 0, \tag{6.5}$$

with

$$\begin{aligned} C_{ij} = \int_0^1 \int^{R(1-X^\alpha)^{1/\beta}} \Big\{ & \nabla^2(FG^j)\nabla^2(FG^i) - (1-\nu)\Big[(FG^j)_{XX}(FG^i)_{YY} \\ & + (FG^i)_{XX}(FG^j)_{YY} - 2(FG^j)_{XY}(FG^i)_{XY}\Big] \\ & + \frac{N_x a^2}{D} (FG^j)_X (FG^i)_X + \frac{N_y a^2}{D} (FG^j)_Y (FG^i)_Y \\ & + \frac{N_{xy} a^2}{D} \Big[(FG^j)_X (FG^i)_Y + (FG^j)_Y (FG^i)_X\Big] \Big\} dYdX, \end{aligned} \tag{6.6}$$

$$B_{ij} = \int_0^1 \int_0^{R(1-X^{\alpha})^{1/\beta}} (FG^j)(FG^i)\,dY\,dX, \qquad (6.7)$$

and

$$\omega^2 = \frac{\rho h p^2 a^4}{D} \,. \qquad (6.8)$$

Specific results are given in Section X - Results.

VII. BOUNDARY CONDITIONS

A. General

As stated in Section III when using a Rayleigh-Ritz procedure each trial function must be "admissible". This means it must satisfy the "geometric" boundary conditions, i.e. the conditions on deflection and slope, and need not satisfy the "natural" boundary conditions, i.e. second and third derivatives or combinations of them. However, in order to have a rapid convergence it is desirable to have the trial function satisfy both the "geometric" and "natural" boundary conditions.

Recall the trial functions

$$W = (A_i)(FG^i) \tag{7.1}$$

so that each (FG^i) must satisfy the boundary conditions.

The first and second derivatives of W are

$$W_X = (A_i)\left(FG^i_X + F_X G^i\right),$$

and (7.2)

$$W_{XX} = (A_i)\left(FG^i_{XX} + 2F_X G^i_X + F_{XX} G^i\right) .$$

B. Clamped Boundaries

The boundary conditions for a clamped plate are

$$W = 0,$$

and

$$\frac{\partial W}{\partial n} = 0,$$

for deflection and normal slope, and

$$M_X \neq 0, \quad M_Y \neq 0, \quad M_{XY} \neq 0,$$

for bending moments. These can be achieved by having

$$W = W_X = W_Y = 0,$$

and

$$W_{XX} \neq 0, \quad W_{YY} \neq 0, \quad W_{XY} \neq 0.$$

Therefore from equation (7.2) the boundary function F must satisfy

$$F = F_X = F_Y = 0,$$

and

$$F_{XX} \neq 0, \quad F_{YY} \neq 0, \quad F_{XY} \neq 0,$$

on the boundary.

a. <u>α and β Even.</u> When α and β are even the boundary of the plate can be described by the equation

$$1 - X^{\alpha} - PY^{\beta} = 0, \tag{7.3}$$

for all four quadrants. Thus the boundary function which satisfies both the "geometric" and "natural" boundary conditions is

$$F = (1 - X^{\alpha} - PY^{\beta})^{2}. \tag{7.4}$$

b. <u>α and β Odd</u>. For odd values of α and β equation (7.3) does not describe the plate boundary for all four quadrants. A different equation must be used for each quadrant as follows:

$$\begin{aligned} &\text{First quadrant:} && 1 - X^{\alpha} - PY^{\beta} = 0, \\ &\text{Second quadrant:} && 1 + X^{\alpha} - PY^{\beta} = 0, \\ &\text{Third quadrant:} && 1 + X^{\alpha} + PY^{\beta} = 0, \\ &\text{Fourth quadrant:} && 1 - X^{\alpha} + PY^{\beta} = 0. \end{aligned} \tag{7.5}$$

Thus the boundary function for the clamped condition with α and β odd becomes

$$F = (1 - X^{\alpha} - PY^{\beta})^2 (1 + X^{\alpha} - PY^{\beta})^2 (1 + X^{\alpha} + PY^{\beta})^2 (1 - X^{\alpha} + PY^{\beta})^2. \tag{7.6}$$

This satisfies both the "geometric" and "natural" boundary conditions.

C. <u>Simple Supports</u>

For the simply supported plate the deflection and moments vanish at the boundary, i.e.

$$W = 0 \quad \text{and} \quad M_n = 0,$$

and the slope is non-zero, i.e.

$$\frac{\partial W}{\partial n} \neq 0 \ .$$

These can be achieved by having

$$W = 0, \quad W_X \neq 0, \quad W_Y \neq 0,$$

and

$$W_{XX} = W_{YY} = W_{XY} = 0,$$

and from equation (7.2) this indicates

$$F = 0,$$

$$F_X \neq 0, \quad F_Y \neq 0,$$

$$F_{XX} = F_{YY} = F_{XY} = 0 \ .$$

a. α and β Even. As previously mentioned equation (7.3) can be used to describe the plate shape for all four quadrants. Hence the following function can be written to satisfy both the "geometric" and "natural" boundary conditions:

$$F = (1 - X^{\alpha} - PY^{\beta})X^{\frac{1-\alpha}{2}} \, Y^{\frac{1-\beta}{2}} \ . \tag{7.7}$$

However, this function is not well-behaved over the whole domain of integration, i.e. for $\alpha > 1$ and $\beta > 1$ the function F ceases to be defined for zero values of X and Y. The following function:

$$F = 1 - X^{\alpha} - PY^{\beta}, \tag{7.8}$$

satisfies the "geometric" condition and not the "natural" condition, but is "admissable" and does give rapid convergence and satisfactory results.

b. α and β Odd. For odd values of α and β equation (7.3) must be replaced with equations (7.5) and the "admissible" boundary function for simple supports becomes,

$$F = (1 - X^{\alpha} - PY^{\beta})(1 + X^{\alpha} - PY^{\beta})(1 + X^{\alpha} + PY^{\beta})(1 - X^{\alpha} + PY^{\beta}). \tag{7.9}$$

D. Free Boundary

The conditions on a free boundary are that bending moments and shearing forces both vanish. This can be achieved by $W \neq 0$, $W_X \neq 0$, $W_Y \neq 0$, for deflection and shopes, $W_{XX} = W_{YY} = W_{XY} = 0$, for bending moments, and $W_{XXX} = W_{XYY} = W_{YXX} = W_{YYY} = 0$ for shears. The boundary function

$$F = X + Y + 1, \tag{7.10}$$

for both even and odd values of α and β, satisfies only the geometric conditions, but is "admissable", and gives satisfactory results.

E. Mixed or Discontinuous Boundary Conditions

The mixed or discontinuous boundary conditions discussed herein are for combinations of clamped and simple supports for each of the four quadrants, e.g. see Figure 3 (for $\alpha = \beta = 2$). As discussed in previous sections, for values of α and β even, equation (7.3) can be used to describe the plate shape for all four quadrants. On the other hand, for odd values of α and β the plate shape is described by a different equation for each quadrant. It is this characteristic of describing the plate shape that makes the mixed or discontinuous boundary conditions easier to develop for odd values of α and β than for even values.

a. <u>α and β Odd</u>. The "admissable" boundary function F satisfying the mixed boundary conditions C - SS - SS - SS for the four consecutive quadrants can be written as

$$F = (1 - X^{\alpha} - PY^{\beta})^2 (1 + X^{\alpha} - PY^{\beta})(1 + X^{\alpha} + PY^{\beta})(1 - X^{\alpha} + PY^{\beta}). \tag{7.11}$$

By squaring the first term of this product, the clamped conditions,

$$F = 0, \quad F_x = 0, \quad F_y = 0,$$

are satisfied on the boundary of the first quadrant and the simply supported conditions,

$$F = 0, \quad F_x \neq 0, \quad F_y \neq 0,$$

are satisfied over the remainder of the boundaries. Similarly, for C - C - SS - SS the boundary function becomes

$$F = (1 - X^{\alpha} - PY^{\beta})^2 (1 + X^{\alpha} - PY^{\beta})^2 (1 + X^{\alpha} + PY^{\beta})(1 - X^{\alpha} + PY^{\beta}), \tag{7.12}$$

and for C - C - C - SS it is

$$F = (1 - X^{\alpha} - PY^{\beta})^2 (1 + X^{\alpha} - PY^{\beta})^2 (1 + X^{\alpha} + PY^{\beta})^2 (1 - X^{\alpha} + PY^{\beta}), \tag{7.13}$$

and finally for C - SS - C - SS one obtains

$$F = (1 - X^{\alpha} - PY^{\beta})^2 (1 + X^{\alpha} - PY^{\beta})(1 + X^{\alpha} + PY^{\beta})^2 (1 - X^{\alpha} + PY^{\beta}). \tag{7.14}$$

b. <u>α and β Even</u>. As mentioned above, the fact that only one equation, (7.3), is necessary to describe the plate shape for even values of α and β, makes it more difficult to develop a boundary function which satisfies the mixed or discontinuous boundary conditions. This occurs because the boundary function F is not written as the product of four functions which satisfy each of the four quadrants of the plate. One method of artificially writing the boundary function in this manner is to approximate the shape of the plate for each quadrant with a different polynomial which includes both even and odd powers of X. For example for $\alpha = \beta = 2$ and $b/a = 1$ (the circle), one can generate four different polynomials which approximate, very closely, the plate shape for each quadrant. The following fifth order polynomials were found which achieve this:

$$\begin{aligned}
&\text{First quadrant:} && Y_1 = -11.48X^5 + 24.72X^4 - 18.91X^3 \\
& && \quad + 5.365X^2 - .636X + 1.01 - Y, \\
&\text{Second quadrant:} && Y_2 = 11.48X^5 + 24.72X^4 + 18.91X^3 \\
& && \quad + 5.365X^2 + .636X + 1.01 - Y, \\
&\text{Third quadrant:} && Y_3 = -11.48X^5 - 24.72X^4 - 18.91X^3 \\
& && \quad - 5.365X^2 - .636X - 1.01 - Y, \\
&\text{Fourth quadrant:} && Y_4 = 11.48X^5 - 24.72X^4 + 18.91X^3 \\
& && \quad - 5.36X^2 + .636X - 1.01 - Y.
\end{aligned} \tag{7.15}$$

These are indicated in Figure 4. Thus the "admissible" boundary function F satisfying the mixed or discontinuous condition C - SS - SS - SS for the circular plate, $\alpha = \beta = 2$ and $b/a = 1$, is

$$F = Y_1^2 \cdot Y_2 \cdot Y_3 \cdot Y_4, \tag{7.16}$$

and for C - C - SS - SS,

$$F = Y_1^2 \cdot Y_2^2 \cdot Y_3 \cdot Y_4, \tag{7.17}$$

and C - C - C - SS,

$$F = Y_1^2 \cdot Y_2^2 \cdot Y_3^2 \cdot Y_4, \tag{7.18}$$

and finally C - SS - C - SS,

$$F = Y_1^2 \cdot Y_2 \cdot Y_3^2 \cdot Y_4. \tag{7.19}$$

Similarly, polynomial expressions can be found for elliptical shapes or for any shape with even values of α and β, and the mixed or discontinuous boundary function can be constructed.

For $\alpha = \beta = 10$ and $b/a = 1$ the square results and this can be more easily represented with straight line segments. Thus the boundary functions for the mixed boundary condition square is as follows:

C - SS - SS - SS: $F = (X-1)^2(Y-1)(X+1)(Y+1),$ (7.20)

C - C - SS - SS: $F = (X-1)^2(Y-1)^2(X+1)(Y+1),$ (7.21)

C - C - C - SS: $F = (X-1)^2(Y-1)^2(X+1)^2(Y+1),$ (7.22)

C - SS - C - SS: $F = (X-1)^2(Y-1)(X+1)^2(Y+1).$ (7.23)

Similar expressions can be written for rectangular shapes also.

VIII. ORTHOTROPIC PLATES OF VARIABLE THICKNESS WITH INPLANE FORCES

In this section the previous discussions are generalized to include the effects of:

i) orthotropy,

ii) variable thickness,

iii) inplane forces,

iv) mixed or discontinuous boundary conditions,

in one eigenvalue problem so that all or some of the conditions can be considered simultaneously.

The general relationship for the maximum strain energy of an orthotropic, variable thickness thin plate with inplane forces is

$$V_{max} = \frac{1}{2} \iint_A \Big\{ D_x W_{xx}^2 + 2D_1 W_{xx} W_{yy} + D_y W_{yy}^2 + 4D_{xy} W_{xy}^2 + N_x W_x^2 + N_y W_y^2 + 2N_{xy} W_x W_y \Big\} dxdy, \tag{8.1}$$

where

$$D_x = \frac{E_x h^3(x,y)}{12(1-\nu_{yx}\nu_{xy})}, \qquad D_y = \frac{E_y h^3(x,y)}{12(1-\nu_{yx}\nu_{xy})},$$

$$D_1 = \nu_{yx} D_x, \qquad D_{xy} = \frac{G_{xy} h^3(x,y)}{12},$$

and N_x, N_y, and N_{xy} are inplane force intensities. The maximum kinetic energy is

$$T_{max} = \frac{\rho p^2}{2} \iint_A h(x,y) W^2 dx dy. \tag{8.2}$$

Assuming a deflection function of the form

$$W = (\ldots A_i \ldots) \begin{pmatrix} \vdots \\ FG^i \\ \vdots \end{pmatrix},$$

and carrying out the Ritz minimization yields

$$[C_{ij}]\{A_i\} - \omega^2 [B_{ij}]\{A_i\} = 0, \tag{8.3}$$

where,

Case A. for even values of α and β,

$$C_{ij} = \int_{-1}^{1} \int_{-R(1-X^\alpha)^{1/\beta}}^{R(1-X^\alpha)^{1/\beta}} \left(\left\{ \frac{\overline{D}_x}{\overline{D}_y} \left(F\ G^j\right)_{XX} \left(F\ G^i\right)_{XX} + \frac{\overline{D}_1}{\overline{D}_y} \left[\left(F\ G^j\right)_{XX} \left(F\ G^i\right)_{YY} + \left(F\ G^j\right)_{YY} \left(F\ G^i\right)_{XX} \right] + \left(F\ G^j\right)_{YY} \left(F\ G^i\right)_{YY} + 4 \frac{\overline{D}_{xy}}{\overline{D}_y} \left(F\ G^j\right)_{XY} \left(F\ G^i\right)_{XY} \right\} H^3(X,Y) \right), \tag{8.4}$$

$$B_{ij} = \int_{-1}^{1} \int_{-R(1-X^{\alpha})^{1/\beta}}^{R(1-X^{\alpha})^{1/\beta}} H(X,Y)\left(F\ G^{j}\right)\left(F\ G^{i}\right) dY dX, \quad (8.5)$$

where

$$h(x,y) = \bar{h}H(x,y),$$

$$\bar{D}_x = \frac{E_x \bar{h}^3}{12(1-\nu_{yx}\nu_{xy})}, \qquad \bar{D}_y = \frac{E_y \bar{h}^3}{12(1-\nu_{yx}\nu_{xy})},$$

$$\bar{D}_1 = \nu_{yx}\bar{D}_x, \qquad \bar{D}_{xy} = \frac{G_{xy}\bar{h}^3}{12},$$

$$\omega^2 = \frac{\rho\bar{h}p^2a^4}{\bar{D}_y},$$

$$\bar{h} = \text{constant}.$$

Case B. for odd values of α and β,

$$
\begin{aligned}
C_{ij} = \int_0^1 \int_0^{R(1-X^{\alpha})^{1/\beta}} \Bigg(\Bigg\{ & \frac{\bar{D}_x}{\bar{D}_y} \left(F\ G^j\right)_{XX} \left(F\ G^i\right)_{XX} + \frac{\bar{D}_1}{\bar{D}_y} \Big[\left(F\ G^j\right)_{XX} \left(F\ G^i\right)_{YY} \\
& + \left(F\ G^j\right)_{YY} \left(F\ G^i\right)_{XX} \Big] + \left(F\ G^j\right)_{YY} \left(F\ G^i\right)_{YY} \\
& + 4\, \frac{\bar{D}_{xy}}{\bar{D}_y} \left(F\ G^j\right)_{XY} \left(F\ G^i\right)_{XY} \Bigg\} H^3(X,Y) \\
& + \frac{N_x a^2}{\bar{D}_y} \left(F\ G^i\right)_X \left(F\ G^j\right)_X + \frac{N_y a^2}{\bar{D}_y} \left(F\ G^i\right)_Y \left(F\ G^j\right)_Y \\
& + \frac{N_{xy} a^2}{\bar{D}_y} \Big[\left(F\ G^j\right)_X \left(F\ G^i\right)_Y + \left(F\ G^j\right)_Y \left(F\ G^i\right)_X \Big] \Bigg) dY dX \\
+ \int_{-1}^{0} \int_0^{R(1+X^{\alpha})^{1/\beta}} \Bigg(\Bigg\{ & \frac{\bar{D}_x}{\bar{D}_y} \left(F\ G^j\right)_{XX} \left(F\ G^i\right)_{XX} + \frac{\bar{D}_1}{\bar{D}_y} \Big[\left(F\ G^j\right)_{XX} \left(F\ G^i\right)_{YY} \\
& + \left(F\ G^j\right)_{YY} \left(F\ G^i\right)_{XX} \Big] + \left(F\ G^j\right)_{YY} \left(F\ G^i\right)_{YY} \\
& + 4\, \frac{\bar{D}_{xy}}{\bar{D}_y} \left(F\ G^j\right)_{XY} \left(F\ G^i\right)_{XY} \Bigg\} H^3(X,Y) \\
& + \frac{N_x a^2}{\bar{D}_y} \left(F\ G^i\right)_X \left(F\ G^j\right)_X + \frac{N_y a^2}{\bar{D}_y} \left(F\ G^j\right)_Y \left(F\ G^i\right)_Y
\end{aligned}
\qquad (8.6)
$$

$$+ \frac{N_{xy}a^2}{\bar{D}_y} \left[\left(F\, G^j\right)_X\left(F\, G^i\right)_Y + \left(F\, G^j\right)_Y\left(F\, G^i\right)_X\right]\Bigg) dY dX$$

$$+ \int_{-1}^{0} \int_{R(-1-X^\alpha)^{1/\beta}}^{0} \Bigg(\Bigg\{\frac{\bar{D}_x}{\bar{D}_y} \left(F\, G^j\right)_{XX}\left(F\, G^i\right)_{XX} + \frac{\bar{D}_1}{\bar{D}_y}\Bigg[\left(F\, G^j\right)_{XX}\left(F\, G^i\right)_{YY}$$

$$+ \left(F\, G^j\right)_{YY}\left(F\, G^i\right)_{XX}\Bigg] + \left(F\, G^j\right)_{YY}\left(F\, G^i\right)_{YY}$$

$$+ 4\,\frac{\bar{D}_{xy}}{\bar{D}_y} \left(F\, G^i\right)_{XY}\left(F\, G^j\right)_{XY}\Bigg\} H^3(X,Y)$$

$$+ \frac{N_x a^2}{\bar{D}_y} \left(F\, G^i\right)_X\left(F\, G^j\right)_X + \frac{N_y a^2}{\bar{D}_y} \left(F\, G^j\right)_Y\left(F\, G^i\right)_Y$$

$$+ \frac{N_{xy}a^2}{\bar{D}_y} \left[\left(F\, G^j\right)_X\left(F\, G^i\right)_Y + \left(F\, G^j\right)_Y\left(F\, G^i\right)_X\right]\Bigg) dY dX$$

$$+ \int_{0}^{1} \int_{R(-1+X^\alpha)^{1/\beta}}^{0} \Bigg(\Bigg\{\frac{\bar{D}_x}{\bar{D}_y} \left(F\, G^j\right)_{XX}\left(F\, G^i\right)_{XX} + \frac{\bar{D}_1}{\bar{D}_y}\Bigg[\left(F\, G^j\right)_{XX}\left(F\, G^i\right)_{YY}$$

$$+ \left(F\, G^j\right)_{YY}\left(F\, G^i\right)_{XX}\Bigg] + \left(F\, G^j\right)_{YY}\left(F\, G^i\right)_{YY}$$

$$+ 4\,\frac{\bar{D}_{xy}}{\bar{D}_y} \left(F\, G^j\right)_{XY}\left(F\, G^i\right)_{XY}\Bigg\} H^3(X,Y) \qquad \text{(8.6) (cont'd)}$$

$$+ \frac{N_x a^2}{\bar{D}_y}\left(F\ G^i\right)_X\left(F\ G^j\right)_X + \frac{N_y a^2}{\bar{D}_y}\left(F\ G^j\right)_Y\left(F\ G^i\right)_Y$$

$$+ \frac{N_{xy} a^2}{\bar{D}_y}\left[\left(F\ G^j\right)_X\left(F\ G^i\right)_Y + \left(F\ G^j\right)_Y\left(F\ G^i\right)_X\right]\Bigg) dYdX, \quad \begin{matrix}(8.6)\\ (\text{cont'd})\end{matrix}$$

and

$$B_{ij} = \int_0^1 \int_0^{R(1-X^\alpha)^{1/\beta}} H(X,Y)\left(F\ G^j\right)\left(F\ G^i\right) dYdX$$

$$+ \int_{-1}^0 \int_0^{R(1+X^\alpha)^{1/\beta}} H(X,Y)\left(F\ G^j\right)\left(F\ G^i\right) dYdX$$

$$+ \int_{-1}^0 \int_{R(-1-X^\alpha)^{1/\beta}}^0 H(X,Y)\left(F\ G^j\right)\left(F\ G^i\right) dYdX$$

$$+ \int_0^1 \int_{R(-1+X^\alpha)^{1/\beta}}^0 H(X,Y)\left(F\ G^j\right)\left(F\ G^i\right) dYdX \ . \quad (8.7)$$

IX. COMPUTATIONAL TECHNIQUE

In order to solve for the eigenvalues and eigenvectors of

$$[C_{ij}]\{A_i\} - \omega^2[B_{ij}]\{A_i\} = 0, \qquad (9.1)$$

each of the elements C_{ij} and B_{ij} must be determined first. As derived in previous sections these elements are double integrals of xy-polynomials whose order depends on the values of α and β and the boundary function F. Gaussian quadrature [12] integration technique is employed for this double integration. The Gaussian quadrature rule of order n yields exact results whenever the integrand is a polynomial of degree $\leq 2n-1$.

The rule of order n on interval [-1,1] is given as

$$\int_{-1}^{1} f(x)dx = \sum_{k=1}^{n} w_k f(x_k), \qquad (9.2)$$

where the abscissas x_k ($k = 1,2,\ldots,n$) are the n zeros of the Legendre polynomials of order n, i.e. $P_n(x_k) = 0$, and the weights w_k are given by

$$w_k = \frac{2\left(1-x_k^2\right)}{[nP_{n-1}(x_k)]^2} . \qquad (9.3)$$

The weights and abscissas for Gaussian quadrature rules of orders 2 through 64 are given by Stroud and Secrest [59].

In general, if the integral over the interval r,s is required, a simple transformation may reduce the interval r,s to [-1,1], i.e.

$$\int_r^s f(x)dx = \tfrac{1}{2}(s-r) \int_{-1}^{1} f[\tfrac{1}{2}(s-r)x + \tfrac{1}{2}(s+r)]dx$$

$$= \tfrac{1}{2}(s-r) \sum_{i=1}^{n} w_i f[\tfrac{1}{2}(s-r)x_i + \tfrac{1}{2}(s+r)]. \quad (9.4)$$

If the abscissas and the weights are symmetric about origin, this becomes

$$\int_r^s f(x)dx = \tfrac{1}{2}(s-r) \sum_{i=1}^{n/2} w_i \{f[\tfrac{1}{2}(s-r)x_i + \tfrac{1}{2}(s+r)]$$

$$+ f[-\tfrac{1}{2}(s-r)x_i + \tfrac{1}{2}(s+r)]\}. \quad (9.5)$$

The double integral can now be written as [71]

$$\phi \equiv \int_r^s g(x) \int_{c(x)}^{d(x)} f(x,y)\,dy\,dx$$

$$= \tfrac{1}{2}(s-r) \sum_{i=1}^{n/2} w_i \left\{ g\left[\frac{s-r}{2} x_i + \frac{s+r}{2}\right] \int_{c\left(\frac{s-r}{2} x_i + \frac{s+r}{2}\right)}^{d\left(\frac{s-r}{2} x_i + \frac{s+r}{2}\right)} f\left(\frac{s-r}{2} x_i + \frac{s+r}{2}, y\right) dy \right.$$

$$\left. + g\left[- \frac{s-r}{2} x_i + \frac{s+r}{2}\right] \int_{c\left(- \frac{s-r}{2} x_i + \frac{s+r}{2}\right)}^{d\left(- \frac{s-r}{2} x_i + \frac{s+r}{2}\right)} f\left(- \frac{s-r}{2} x_i + \frac{s+r}{2}, y\right) dy \right\}. \tag{9.6}$$

Let

$$u_i \equiv \frac{s-r}{2} x_i + \frac{s+r}{2} ,$$

and

$$v_i \equiv - \frac{s-r}{2} x_i + \frac{s+r}{2} ,$$

then

$$\phi = \tfrac{1}{2}(s-r) \sum_{i=1}^{n/2} w_i \left\{ g(u_i) \int_{c(u_i)}^{d(u_i)} f(u_i,y)\,dy \right.$$

$$\left. + g(v_i) \int_{c(v_i)}^{d(v_i)} f(v_i,y)\,dy \right\}, \tag{9.7}$$

and finally,

$$\phi = \tfrac{1}{2}(s-r) \sum_{i=1}^{n/2} w_i \Bigg\{ g(u_i)\, \frac{d(u_i) - c(u_i)}{2} \Bigg[\sum_{j=1}^{n/2} w_j \Bigg\{ f\Bigg(u_i, \frac{d(u_i) - c(u_i)}{2}\, y_j + \frac{d(u_i) + c(u_i)}{2}\Bigg) + f\Bigg(u_i, - \frac{d(u_i) - c(u_i)}{2}\, y_j + \frac{d(u_i) + c(u_i)}{2}\Bigg)\Bigg\}\Bigg]$$

$$+ g(v_i)\, \frac{d(v_i) - c(v_i)}{2} \Bigg[\sum_{j=1}^{n/2} w_j \Bigg\{ f\Bigg(v_i, \frac{d(v_i) - c(v_i)}{2}\, y_j + \frac{d(v_i) + c(v_i)}{2}\Bigg) + f\Bigg(v_i, - \frac{d(v_i) - c(v_i)}{2}\, y_j + \frac{d(v_i) + c(v_i)}{2}\Bigg)\Bigg\}\Bigg]\Bigg\}. \tag{9.8}$$

Equation (9.8) can be programmed on a computer to obtain the exact values of the double integrals. Thus all the elements C_{ij} and B_{ij} of equation (9.1) can be determined for particular values of α, β, a/b, and the boundary function F.

Equation (9.1) is then converted into the standard eigenvalue form

$$[D_{ij}]\{\psi_i\} = \lambda\{\psi_i\} \qquad (9.9)$$

as discussed in Section IV. The eigenvalues and eigenvectors of equation (9.9) can now be obtained through various techniques. The method of reduction [21] is employed in this investigation. This method is relatively efficient and is based on successive reductions of the matrix $[D_{ij}]$.

Equation (9.9) yields the eigenvalues directly, however the eigenvectors of the problem may be obtained by transforming $\{\psi_i\}$ back to $\{A_i\}$ in equation (9.1).

Based upon these techniques a general computer program was developed which determines the eigenvalues and eigenvectors for the general class of plates whose geometry is given by

$$\left(\frac{x}{a}\right)^{\alpha} + \left(\frac{y}{b}\right)^{\beta} = 1$$

for orthotropic plates with variable thickness, inplane forces, and mixed or discontinuous boundary conditions.

The eigenvalues λ, give directly, the natural frequencies ω of the problem since

$$\omega = 1/\lambda$$

and the corresponding eigenvectors $\{A_i\}$ can be substituted into the equation

$$(A_i)(FG^i) = 0$$

to give the nodal patterns.

X. RESULTS

A. General

Recall from Section IV that the assumed deflection shape is the product of the boundary function F, and the XY polynomial G^i. The boundary function employed is dependent upon the particular boundary conditions and the plate geometry while the polynomials G^i depend on the plate vibration modes that are being examined. Since the fundamental mode is the lowest symmetrical mode, the following polynomial expression is used in obtaining it

$$(A_i)(G^i) = A_1 + A_2Y^2 + A_3X^2 + A_4X^2Y^2 + A_5Y^4 + \dots \tag{10.1}$$

The higher modes of vibration can be computed by using separately the symmetric and anti-symmetric XY-polynomials. For doubly anti-symmetric modes, there is an odd powered XY-polynomial given as

$$(A_i)(G^i) = A_1XY + A_2XY^3 + A_3X^3Y + A_4X^3Y^3 + A_5XY^5 + \dots \tag{10.2}$$

For the anti-symmetric modes there are two groups of XY-polynomials. They are

$$(A_i)(G^i) = A_1X + A_2X^3 + A_3XY^2 + A_4X^3Y^2 + A_5X^5 + \dots \tag{10.3}$$

and

$$(A_i)(G^i) = A_1Y + A_2Y^3 + A_3X^2Y + A_4X^2Y^3 + A_5Y^5 + \dots \quad (10.4)$$

The computation time is approximately proportional to the square of the number of polynomial terms. Thus a number of terms must be chosen so that it will yield accurate results with a reasonable amount of computation time.

It has been found that a 21-term polynomial gives excellent accuracy for the first six modes and yet consumes a relatively small amount of computer time per run, e.g. about 2 minutes per case on a GE 635 computer.

The 21-term polynomials can be written in a compact notation by just noting the powers of the X and Y terms respectively. Thus the four 21-term polynomials given by equations (10.1), (10.2), (10.3) and (10.4) can be abbreviated respectively as

$$\begin{array}{llllll} (0,0) & (0,2) & (0,4) & (0,6) & (0,8) & (0,10) \\ (2,0) & (2,2) & (2,4) & (2,6) & (2,8) & \\ (4,0) & (4,2) & (4,4) & (4,6) & & \\ (6,0) & (6,2) & (6,4) & & & \\ (8,0) & (8,2) & & & & \\ (10,0) & & & & & \end{array} \quad (10.5)$$

(1,1) (1,3) (1,5) (1,7) (1,9) (1,11)
(3,1) (3,3) (3,5) (3,7) (3,9)
(5,1) (5,3) (5,5) (5,7) (10.6)
(7,1) (7,3) (7,5)
(9,1) (9,3)
(11,1)

(1,0) (3,0) (5,0) (7,0) (9,0) (11,0)
(1,2) (3,2) (5,2) (7,2) (9,2)
(1,4) (3,4) (5,4) (7,4) (10.7)
(1,6) (3,6) (5,6)
(1,8) (3,8)
(1,10)

and

(0,1) (0,3) (0,5) (0,7) (0,9) (0,11)
(2,1) (2,3) (2,5) (2,7) (2,9)
(4,1) (4,3) (4,5) (4,7) (10.8)
(6,1) (6,3) (6,5)
(8,1) (8,3)
(10,1)

B. Orthotropic Plates

1. Fundamental frequencies. For orthotropic plates the fundamental frequencies, $\omega = p\Big/\sqrt{D_y/\rho h a^4}$, are presented for the following plate configurations:

a) $\alpha = \beta = 1$, $R = 1.0$ (rhombus)

b) $\alpha = \beta = 1$, $R = .5$ (diamond)

c) $\alpha = \beta = 2$, $R = 1.0$ (circle)

d) $\alpha = \beta = 2$, $R = .5$ (ellipse)

e) $\alpha = \beta = 10$, $R = 1.0$ (square)

f) $\alpha = \beta = 10$, $R = .5$ (rectangle).

Each of these shapes has been investigated with the following 9 cases of elastic properties

	D_{xy}/D_y	D_x/D_y
i)	1/3	1/3
ii)	1/3	1/2
iii)	1/3	1
iv)	1/2	1/3
v)	1/2	1/2
vi)	1/2	1
vii)	1	1/3
viii)	1	1/2
ix)	1	1

and $\nu_{xy} = 1/3$.

The "a" in the frequency equation is the side length for a square plate; the length in the x-direction for a rectangular plate; the diameter for a circular plate; the x-direction diameter for an elliptical plate; the diagonal for a rhomboidal plate; and the x-direction diagonal for a diamond plate.

Tables 1a through 1f show the fundamental frequencies for orthotropic plates, and Tables 1g through 1ℓ are for simply supported plates. Since cases e) and f) are the only orthotropic plate configurations which can be compared to the existing literature known to the author these will represent test cases. The comparison of this data with existing literatures is indicated below for $D_{xy}/D_y = 1/3$, $D_x/D_y = 1/3$, $\nu_{xy} = 1/3$.

$\alpha = \beta$	R	Shape	B.C.	Present Study	Literature
10	1.0	Square	Clamped	30.91	30.98 Hearmon [28]
10	.5	Rectangle	Clamped	96.42	96.87 Hearmon [28]
10	1.0	Square	SS	17.61	18.02 Hearmon [28]
10	.5	Rectangle	SS	48.15	48.68 Hearmon [28]

Although the present analysis gives slightly smaller frequencies than Hearmon's it must be pointed out that both studies were analyzed using approximate energy techniques which give upper bounds to the frequency. Thus since the frequencies determined herein are respectively smaller than Hearmon's it can be concluded that the present analysis gives results which are closer to the true values.

As a result of the scarcity of orthotropic data for plate shapes other than the square and rectangle, the validity of the results for these configurations was checked by examining the isotropic plate as a special case, i.e. $D_{xy}/D_y = 1/3$,

$D_x/D_y = 1$, $\nu_{xy} = 1/3$. The comparisons with existing data yield excellent agreement as indicated:

$\alpha = \beta$	R	Shape	B.C.	Present Study	Literature
1	1.0	Rhombus	Clamped	71.96	71.98 Young [72]
1	1.0	Rhombus	SS	39.48	39.48 Hearmon [26]
1	.5	Diamond	Clamped	169.26	170[3] Conway, et al. [16]
1	.5	Diamond	SS	91.82	92[3] Conway, et al. [16]
2	1.0	Circle	Clamped	40.86	40.87 McLeod, et al. [42]
2	1.0	Circle	SS	19.93	19.92 McLeod, et al. [42]
2	.5	Ellipse	Clamped	109.50	110.0 Shibaoka [58]
2	.5	Ellipse	SS	53.03	53.3[3] Leissa [36]

2. Higher frequencies. The accuracy of each frequency can be ascertained through examination of the orthogonality relation for any two eigenvectors. Another indication of good accuracy is the clarity and consistency of the nodal patterns.

For the elliptical plate, case d), the higher modes are examined and data is presented for the following four cases of elastic properties:

	D_{xy}/D_y	D_x/D_y
i)	1/3	1
ii)	1/3	1/3
iii)	1	1
iv)	1	1/3

[3] Interpolated from data in reference indicated.

Tables 2 through 5 show the frequencies and mode shapes for the clamped orthotropic elliptic plate. Also indicated are plots of the nodal lines for each mode [22].

C. Variable Thickness Plates

As a check on the validity of the variable thickness solution the clamped isotropic thin circular plate with axisymmetric parabolic thickness variation was analyzed by the present method and the fundamental frequency was compared to the results given by Barakat and Baumann [7], who used the Ritz-Galerkin method. They present data for a thickness variation given by

$$h = \bar{h}(1+\alpha' r^2)$$

for different values of α'. The comparison of their fundamental frequencies, $\omega = p/\sqrt{D/\rho\bar{h}a^4}$, with the present solution is as follows:

$\alpha = \beta$	R	Shape	α'	B.C.	Present Study	Literature
2	1.0	Circular	.5	Clamped	54.79	55.34 [7]
2	1.0	Circular	.9	Clamped	66.41	66.79 [7]

These plate configurations are indicated in Figure 5. As an example of the variable thickness orthotropic plate, data is presented on the first six eigenvalues, eigenvectors and nodal patterns for the clamped elliptic plate of linearly varying thickness in the x-direction. The configuration is indicated in Figure 6. Elastic properties for this case are $D_{xy}/D_y = 1/3$, $D_x/D_y = 1/3$, $\nu_{xy} = 1/3$. Results are shown in Table 6.

D. Plates With Inplane Forces

The validity of the analysis of plates with inplane forces is checked by comparisons with existing solutions on the isotropic circular plate and square plate in hydrostatic tension. These comparisons of the fundamental frequencies, $\omega = p/\sqrt{D/\rho h a^4}$, are as follows:

$\alpha = \beta$	R	Shape	Inplane Force Ta^2/D	B.C.	Present Study	Literature
2	1.0	Circular	100	Clamped	109.93	109.92 Bickley [10]
10	1.0	Square	400	SS	91.01	91.02 Herrmann [29]
10	1.0	Square	400	Clamped	101.13	101[4] Weinstein, et al. [69]

In this study a representative orthotropic case with inplane forces is presented in detail. The chosen configuration is elliptical, ($\alpha = \beta = 2$, $R = .5$) with elastic properties $\nu_{xy} = 1/3$, $D_{xy}/D_y = 1/3$, and $D_x/D_y = 1/3$, with hydrostatic compression $Ta^2/D_y = -10$. This configuration is demonstrated in Figure 7 and the first six frequencies, mode shapes and nodal patterns are shown in Table 7.

[4] Interpolated from data in the reference indicated.

E. Mixed and Discontinuous Boundary Conditions

1. Choice of polynomials. As a result of the unsymmetrical boundary conditions it was found that the polynomials represented by equations (10.5), (10.6), (10.7) and (10.8) did not yield the most accurate results. However, by constructing a polynomial which included terms from all four of these polynomials, i.e.

$$(A_i)(G^i) = A_1 + A_2X + A_3Y + A_4XY + A_5X^2Y + A_6XY^2 + A_7X^2Y^2 + \ldots, \qquad (10.9)$$

extremely accurate results were obtained by using 21 terms as in the previous cases.

2. Comparison to literature. The validity of the mixed boundary condition case was checked by examining the isotropic rhomboid ($\alpha = \beta = 1$, $R = 1.0$), which is actually a square with side length $a/\sqrt{2}$, and comparing the results with known solutions. The four combinations of boundary conditions were studied and the fundamental frequencies, $\omega = p/\sqrt{D/pha^4}$, compared to the data in the literature. These cases are indicated in Figure 8 and the comparisons with known solutions are as follows:

B.C.	Present Study	Literature
C-C-C-SS	63.60	63.66 Kanazawa & Kawai [33]
C-C-SS-SS	54.42	54.20 Kanazawa & Kawai [33]
C-SS-SS-SS	47.29	47.29 Iguchi [32]
C-SS-C-SS	57.90	57.89 Hamada [23]

The frequencies and nodal patterns of the first four modes are presented and compared to the literature for the two cases C-SS-SS-SS and C-SS-C-SS. These are shown in Figures 9 and 10. As can be seen from these figures the results are excellent.

XI. DISCUSSION OF ASSUMPTIONS

A. Rotary Inertia and Shear

The inclusion of rotary inertia and shear in an analysis serves to decrease the computed frequencies because of increased inertia and flexibility of the system. The justification for ignoring these effects in a plate vibration analysis is a function of the ratio of thickness to plate dimension, h/a, and the flexural frequencies considered in the study. Considering the ratio of plate flexural frequency to the thickness shear frequency $\bar{\omega}$ of a plate having infinite dimensions in the x and y directions, i.e. $\bar{\omega} = \pi\sqrt{G/\rho h}$, one obtains

$$\frac{p}{\bar{\omega}} = \left(\frac{h}{2a}\right)^2 \frac{\omega}{\pi^2} \sqrt{\frac{8}{3(1-\nu)}} \simeq \left(\frac{h}{2a}\right)^2 \frac{\omega}{5} \; . \qquad (11.1)$$

As a result of work done by Mindlin [44,45,46] the following observations may be made

a) for $p/\bar{\omega} > 1$ all computed frequencies are completely wrong if rotary inertia and shear are neglected,

b) for $p/\bar{\omega} << 1$ classical plate theory gives accurate results.

As an indication of when the assumption of ignoring rotary inertia and shear is justified consider the following cases of plate dimensions and frequencies computed.

i) $h/a < 1/10$ and $\omega < 500 \Rightarrow p/\bar{\omega} < .25$,

ii) $h/a < 1/10$ and $\omega < 100 \Rightarrow p/\overline{\omega} < .05$,

iii) $h/a < 1/20$ and $\omega < 500 \Rightarrow p/\overline{\omega} < .0625$.

Thus, ignoring the effects of rotary inertia and shear is more justifiable for cases ii) and iii) than for i).

B. Small Deflections

The assumption of small deflections in plate theory means that the extension of the middle surface, which is a non-linear second order effect, is ignored. The assumption is justified in plate vibration studies if the deflections are small compared to the plate thickness, e.g. the error in fundamental frequency [65,66] is less than 2% if the deflection $\delta < h/5$ and less than 1% for $\delta < h/10$.

XII. CONCLUSIONS AND RECOMMENDATIONS

The frequencies and nodal patterns have been determined for a class of orthotropic thin plates with the considerations of variable thickness, inplane forces, and mixed or discontinuous boundary conditions. Rotary inertia and shear are neglected and small deflections are assumed. This class of plates includes the rhombus, circle, ellipse, square and rectangle as special cases.

The method of analysis employed was the Rayleigh-Ritz energy procedure using 21 terms of XY-polynomials as the approximate deflection function. This technique has shown to give excellent results when compared to known solutions for plates with various conditions and properties. These comparisons with existing data have indicated that the technique is excellent and could be extended to many other plate configurations which have not yet been investigated.

Results which have not appeared in previous literature have been presented for the following configurations:

	Material Properties	Plate Shapes	Plate Thickness	Boundary Conditions	Inplane Forces	Data Presented
A)	Orthotropic (9 cases)	Rhombus, Diamond, Circle, Ellipse, Square, Rectangle	Uniform	Clamped, SS	None	Fundamental frequencies
B)	Orthotropic (4 cases)	Ellipse	Uniform	Clamped	None	Six frequencies and nodal patterns
C)	Orthotropic (1 case)	Ellipse	Linear variation in x-direction	Clamped	None	Six frequencies and nodal patterns
D)	Orthotropic (1 case)	Ellipse	Uniform	Clamped	Hydrostatic compression	Six frequencies and nodal patterns

An attempt to extend the Rayleigh-Ritz procedure to large amplitude vibrations breaks down because of the non-linearities involved in this problem. However, the simpler Rayleigh principle [61], which uses only the boundary function F as the approximated deflection function, can be applied to obtain the fundamental frequencies approximately [19].

The determination of natural frequencies of transverse vibration of plates has been studied for many years and by numerous methods. It has been shown through the literature that only a handful of plate problems can be solved exactly. The others can only be solved by an approximate method such as the one shown herein. Although the technique is an approximate one, the use of computers yields extremely accurate results.

Eigenvalues and eigenvectors are of fundamental importance in characterizing the dynamic behavior of a plate. Based upon the results of this investigation the dynamic response of the plate to specific inputs may be studied.

Other possible areas of extension of this work could be to geometries which are not symmetric with respect to the origin and to multiply connected domains, i.e. plates with holes.

The entire discussion and analysis presented herein has been concerned with plates. However, a large area for future vibration problems seems possible with the application of the Rayleigh-Ritz technique to shell vibrations [67]. Instead of the basic plate equation used herein, i.e.

$$\left(\frac{x}{a}\right)^{\alpha} + \left(\frac{y}{b}\right)^{\beta} = 1,$$

the shell would be defined by

$$\left(\frac{x}{a}\right)^{\alpha} + \left(\frac{y}{b}\right)^{\beta} + \left(\frac{z}{c}\right)^{\gamma} = 1,$$

where variations in a, b, c, α, β, and γ would yield a number of different configurations.

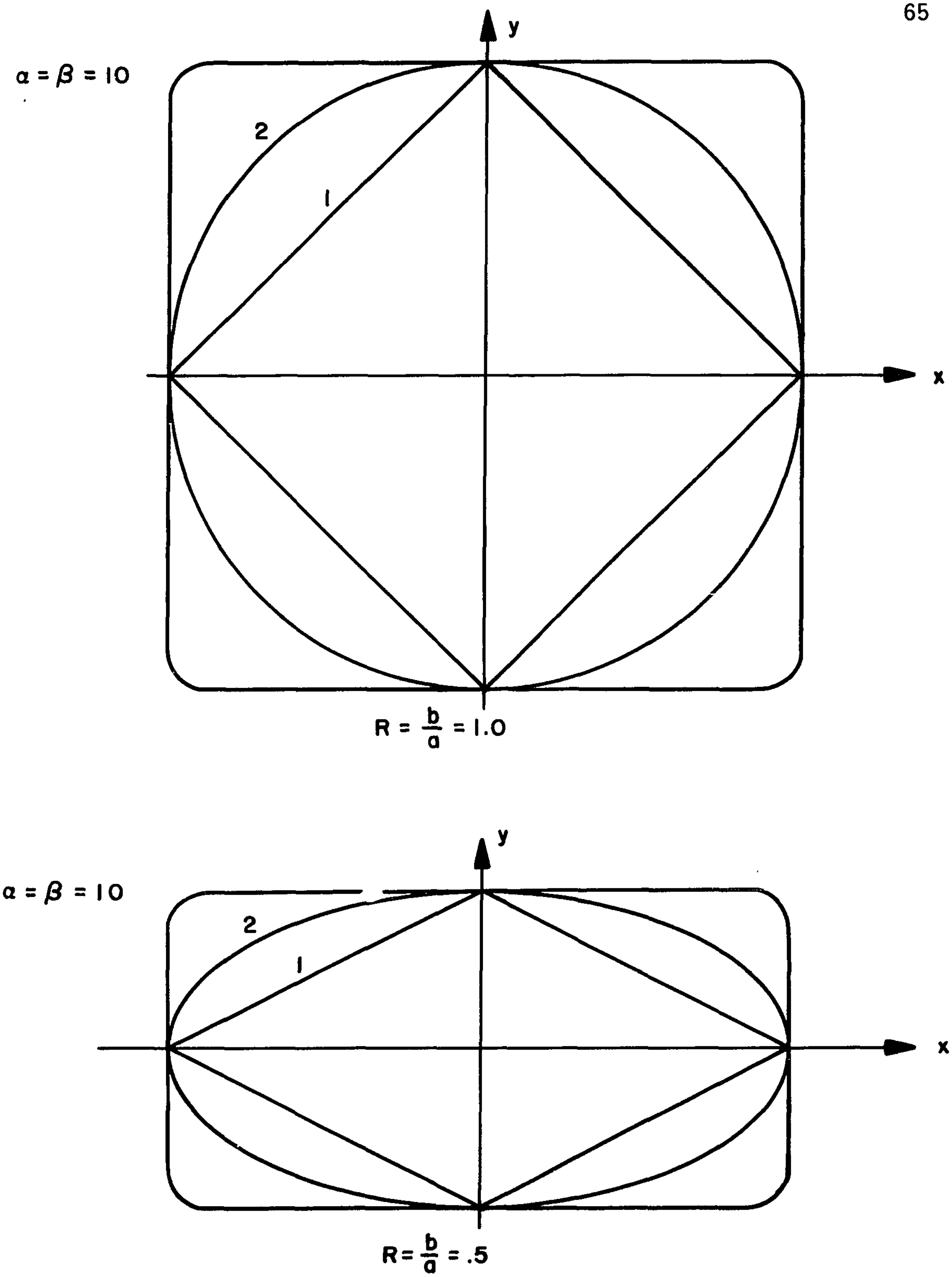


FIGURE 1 PLATE GEOMETRIES

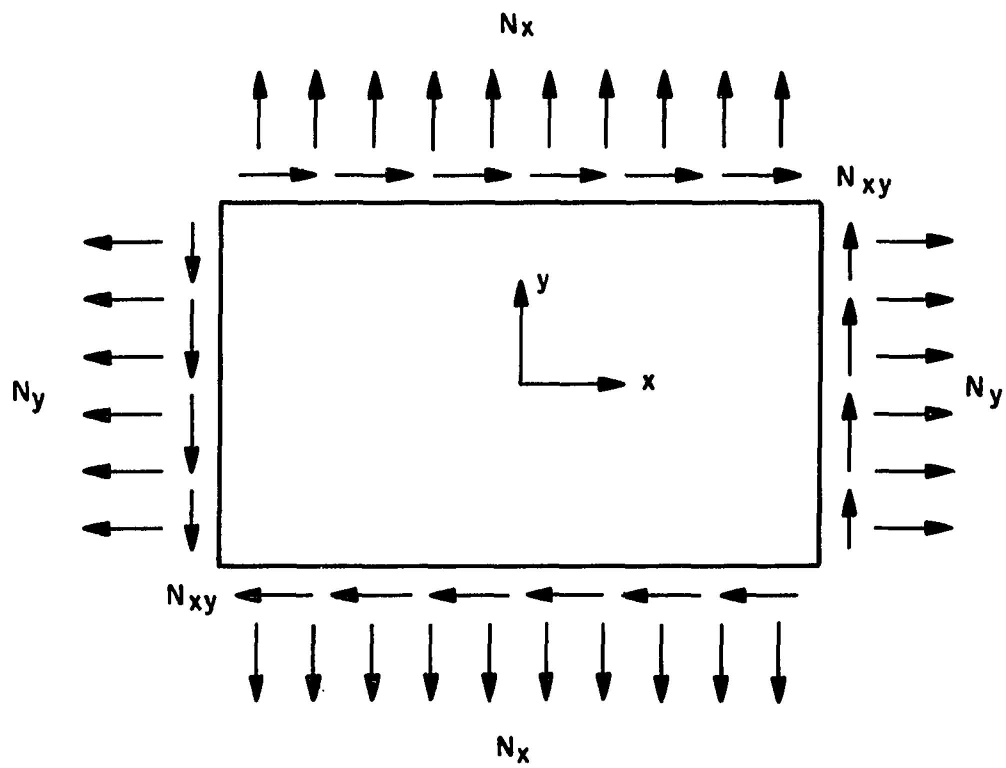


FIGURE 2 SIGN CONVENTION FOR INPLANE FORCE INTENSITIES

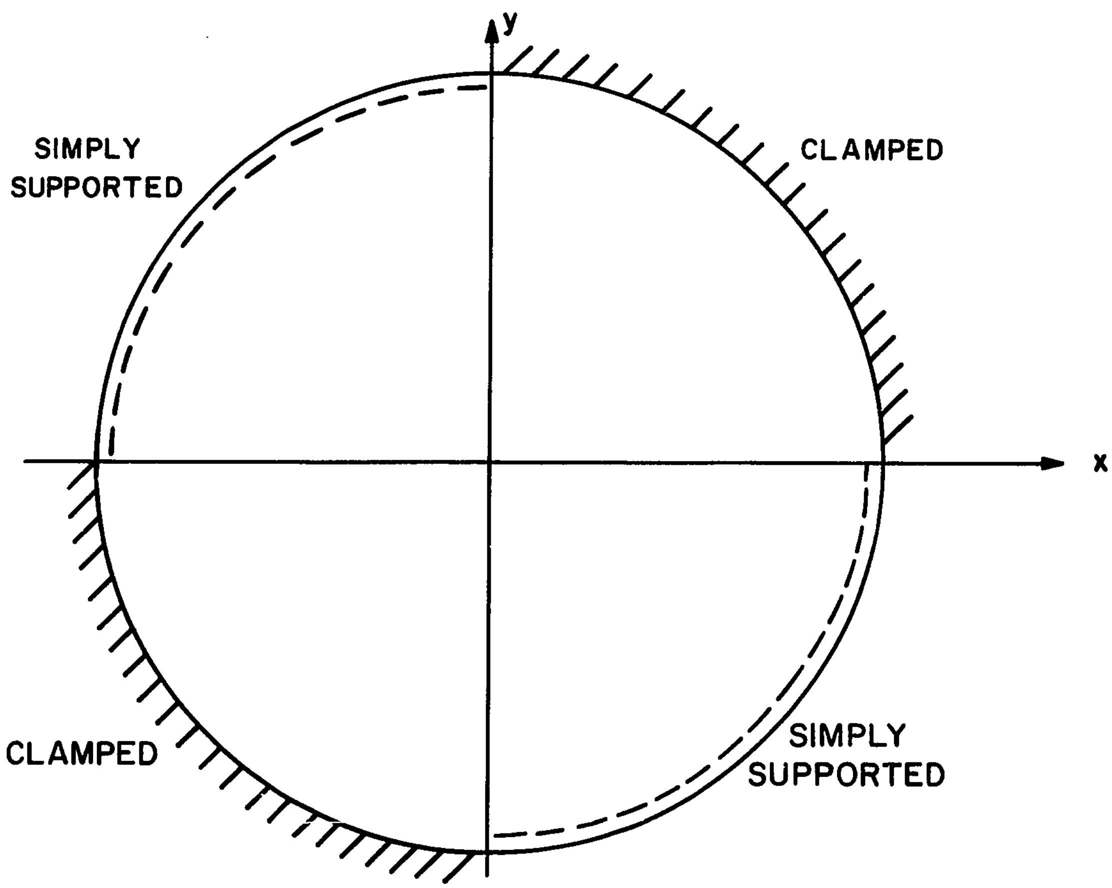
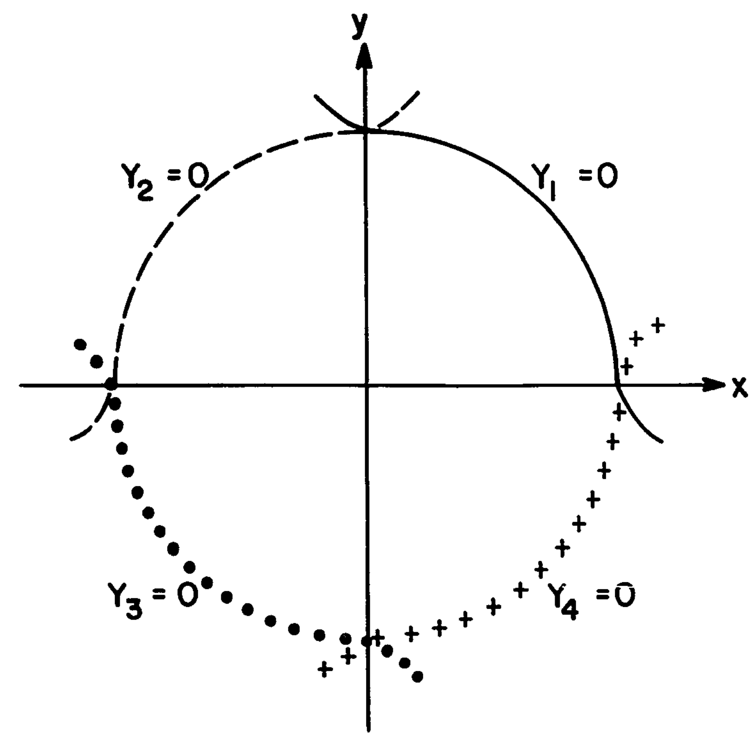


FIGURE 3 DIFFERENT BOUNDARY CONDITIONS
FOR EACH QUADRANT



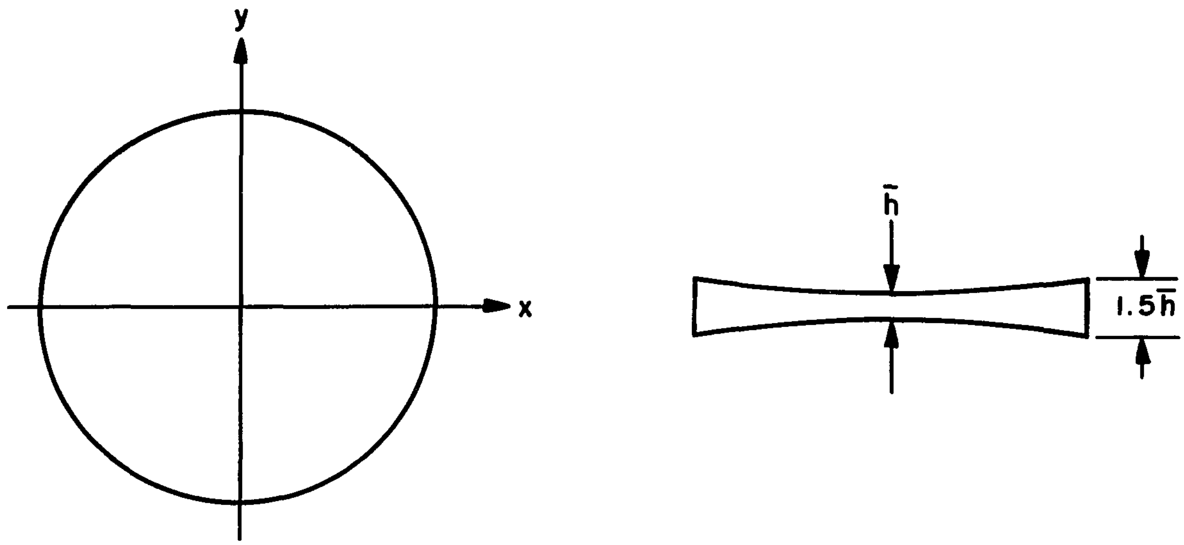
———— $Y_1 = -11.48X^5 + 24.72X^4 - 18.91X^3 + 5.365X^2 - .636X + 1.01 - y$

— — — $Y_2 = +11.48X^5 + 24.72X^4 + 18.91X^3 + 5.365X^2 + .636X + 1.01 - y$

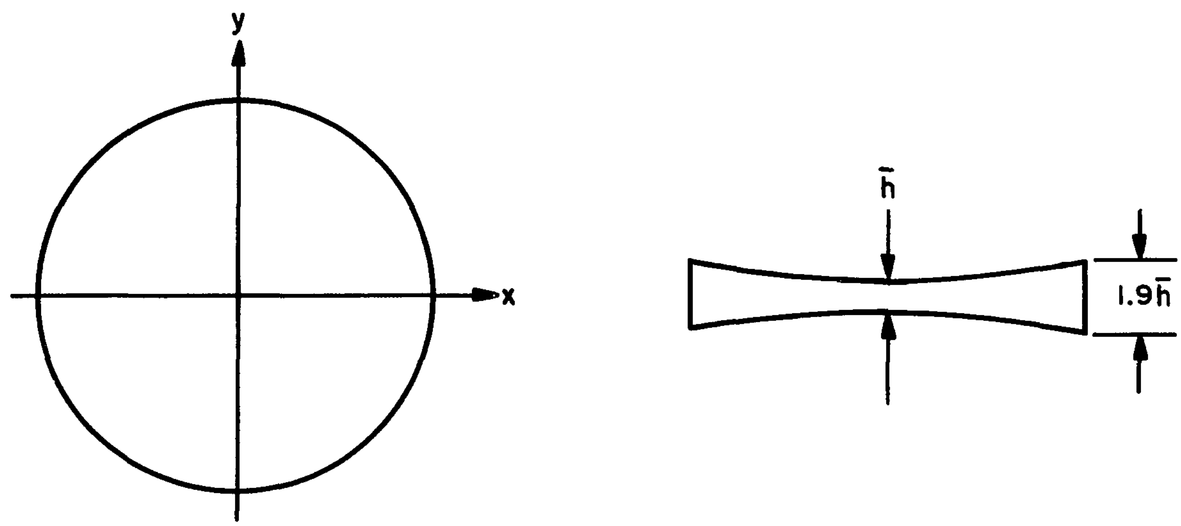
• • • • • $Y_3 = -11.48X^5 - 24.72X^4 - 18.91X^3 - 5.365X^2 - .636X - 1.01 - y$

+ + + + + $Y_4 = 11.48X^5 - 24.72X^4 + 18.91X^3 - 5.365X^2 + .636X - 1.01 - y$

FIGURE 4 MIXED BOUNDARY CONDITIONS FOR α ANB β EVEN

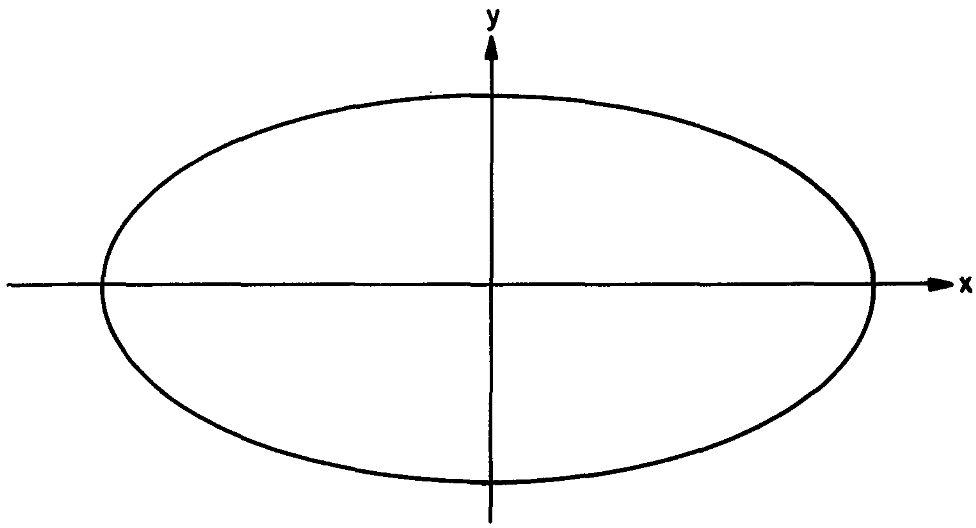


$\alpha = \beta = 2, \; R = 1.0$ $\qquad$ $\alpha^{1} = .5$

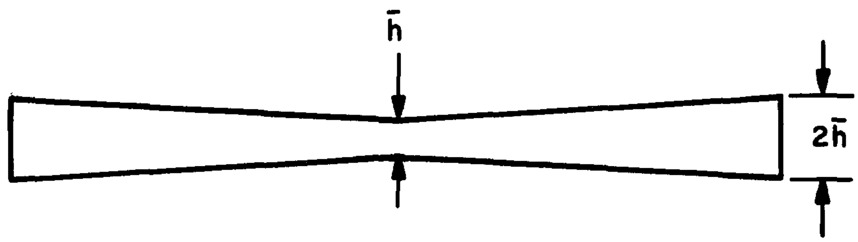


$\alpha = \beta = 2, \; R = 1.0$ $\qquad$ $\alpha^{1} = .9$

FIGURE 5. VARIABLE THICKNESS CHECK CASE

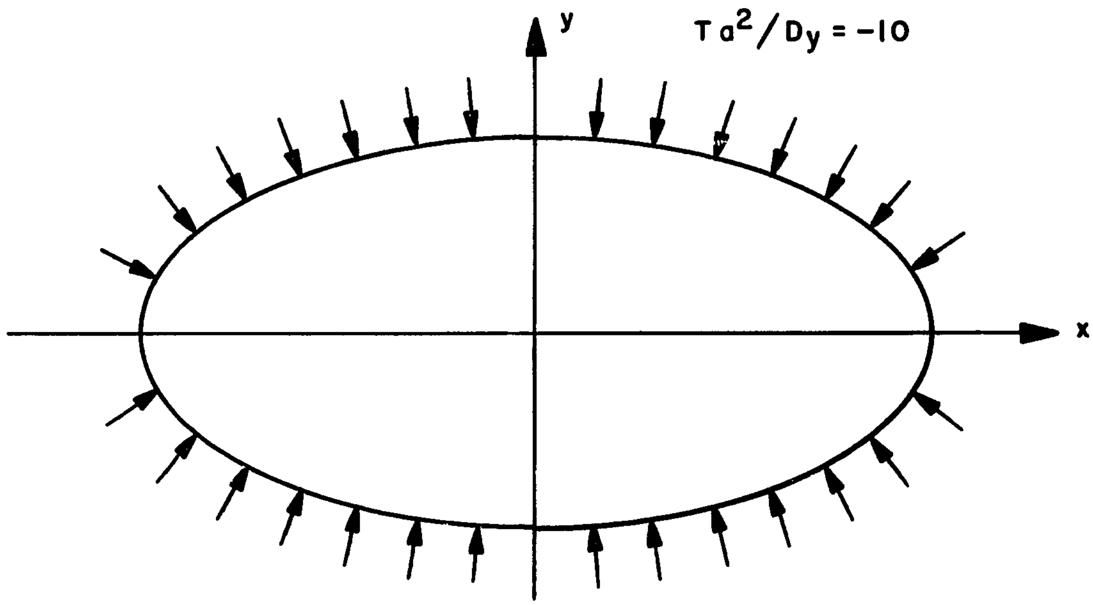


$\alpha = \beta = 2$, $R = .5$



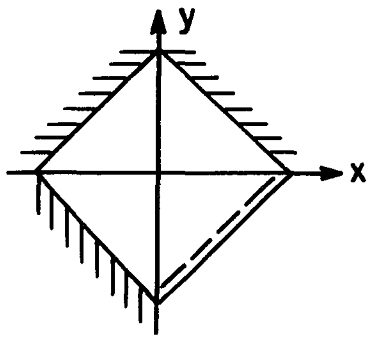
$\gamma_{xy} = 1/3$, $D_{xy}/D_y = 1/3$, $D_x/D_y = 1/3$

FIGURE 6. CLAMPED ORTHOTROPIC ELLIPTIC PLATE WITH LINEARLY VARYING THICKNESS



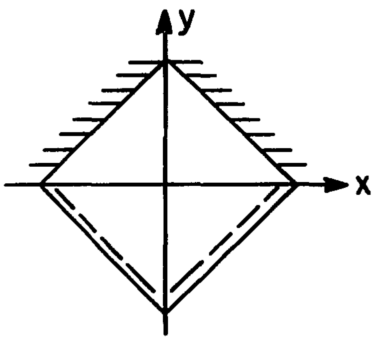
$\alpha = \beta = 2$, R=.5

FIGURE 7. CLAMPED ORTHOTROPIC ELLIPSE UNDER HYDROSTATIC COMPRESSION



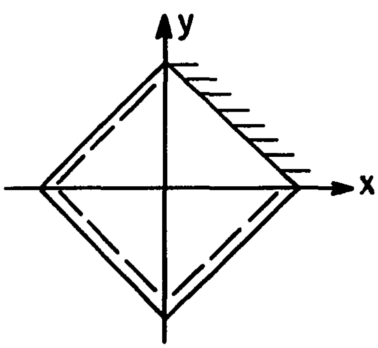
C–C–C–SS

PRESENT SOLUTION	63.60
KANAZAWA & KAWAI [33]	63.66



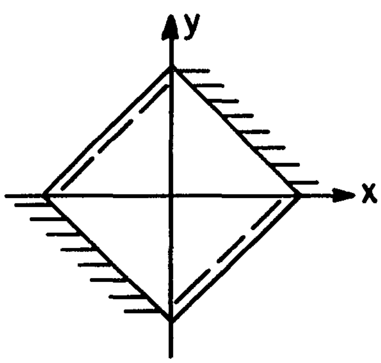
C–C–SS–SS

PRESENT SOLUTION	54.42
KANAZAWA & KAWAI [33]	54.20



C–SS–SS–SS

PRESENT SOLUTION	47.29
IGUCHI [32]	47.29

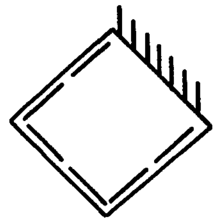


C–SS–C–SS

PRESENT SOLUTION	57.90
HAMADA [23]	57.89

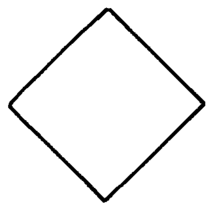
FIGURE 8 MIXED BOUNDARY CONDITION CHECK CASES FOR $\alpha=\beta=1$, R = 1.0

SHAPE AND BC



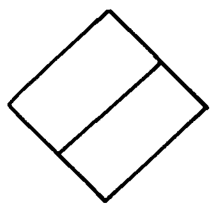
$\alpha = \beta = 1$, $R = 1.0$
C – SS – SS – SS

MODE 1



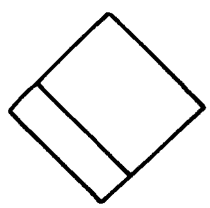
PRESENT SOLUTION 47.29
LITERATURE 47.29 [32]

MODE 2



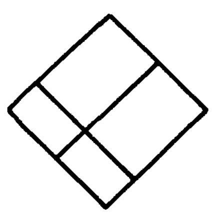
PRESENT SOLUTION 103.32
LITERATURE 103.35 [32]

MODE 3



PRESENT SOLUTION 117.30
LITERATURE 117.28 [32]

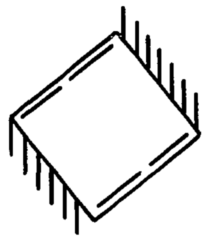
MODE 4



PRESENT SOLUTION 174.24
LITERATURE 172.23 [32]

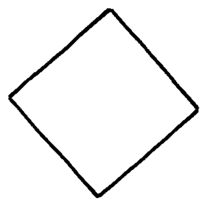
FIGURE 9 FREQUENCIES AND NODAL PATTERNS OF FIRST FOUR MODES: C–SS–SS–SS

SHAPE AND BC



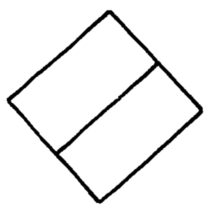
$\alpha = \beta = 1$, $R = 1.0$
SS – C – SS – C

MODE 1



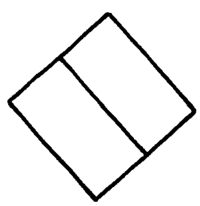
PRESENT SOLUTION 57.90
LITERATURE 57.89 [23]

MODE 2



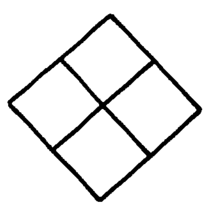
PRESENT SOLUTION 109.49
LITERATURE 109.49 [23]

MODE 3



PRESENT SOLUTION 138.66
LITERATURE 138.64 [23]

MODE 4



PRESENT SOLUTION 189.40
LITERATURE 189.17 [23]

FIGURE 10 FREQUENCIES AND NODAL PATTERNS OF FIRST FOUR MODES: C–SS–C–SS

Table 1 **Fundamental Frequencies of Orthotropic Plates**

$$\omega = \frac{P}{\sqrt{\frac{D_y}{\rho h a^4}}}$$

$\gamma_{xy} = 1/3$

D_x/D_y \ D_{xy}/D_y	1/3	1/2	1
1/3	61.36	65.15	75.22
1/2	64.52	68.12	77.79
1	71.96	75.22	84.11

Table 1a Clamped Rhomboid, $\alpha=\beta=1$, R=1.0

$\gamma_{xy} = 1/3$

D_x/D_y \ D_{xy}/D_y	1/3	1/2	1
1/3	157.94	165.05	183.97
1/2	161.30	168.10	186.39
1	169.26	175.49	192.53

Table 1b Clamped Rhomboid, $\alpha=\beta=1$, R=.5

$\gamma_{xy} = 1/3$

D_x/D_y \ D_{xy}/D_y	1/3	1/2	1
1/3	35.33	37.25	42.49
1/2	36.80	38.65	43.72
1	40.86	42.53	47.18

Table 1c Clamped Circular, $\alpha=\beta=2$, R=1.0

Table 1 (continued)

$$\omega = \frac{P}{\sqrt{\frac{D_y}{\rho h a^4}}}$$

$\gamma_{xy} = 1/3$

D_x/D_y \ D_{xy}/D_y	1/3	1/2	1
1/3	106.92	109.74	117.52
1/2	107.61	110.38	118.05
1	109.50	112.14	119.58

Table 1d Clamped Ellipse, $\alpha=\beta=2$, R=.5

$\gamma_{xy} = 1/3$

D_x/D_y \ D_{xy}/D_y	1/3	1/2	1
1/3	30.91 *	32.36	36.26
1/2	32.28	33.69	37.70
1	35.99	37.29	40.85

Table 1e Clamped Square, $\alpha=\beta=10$, R=1.0

$\gamma_{xy} = 1/3$

D_x/D_y \ D_{xy}/D_y	1/3	1/2	1
1/3	96.42 *	98.24	103.39
1/2	96.93	98.77	103.97
1	98.34	100.21	105.48

Table 1f Clamped Rectangle $\alpha=\beta=10$, R=.5

*Check Cases

Table 1 (continued)

$$\omega = \frac{P}{\sqrt{\frac{D_y}{\rho h a^4}}}$$

$\gamma_{xy} = 1/3$

D_x/D_y \ D_{xy}/D_y	1/3	1/2	1
1/3	30.79	30.84	30.95
1/2	33.53	33.55	33.60
1	39.48	39.48	39.48

Table 1g Simply Supported Rhomboid, $\alpha=\beta=1$, R=1.0

$\gamma_{xy} = 1/3$

D_x/D_y \ D_{xy}/D_y	1/3	1/2	1
1/3	82.46	83.49	85.69
1/2	85.20	86.11	88.08
1	91.82	92.50	94.00

Table 1h Simply Supported Rhomboid, $\alpha=\beta=1$, R=.5

$\gamma_{xy} = 1/3$

D_x/D_y \ D_{xy}/D_y	1/3	1/2	1
1/3	17.22	17.42	17.82
1/2	17.95	18.14	18.53
1	19.93	20.11	20.49

Table 1i Simply Supported Circular, $\alpha=\beta=2$, R=1.0

Table 1 (continued)

$$\omega = \frac{p}{\sqrt{\frac{D_y}{\rho h a^4}}}$$

$\nu_{xy} = 1/3$

D_x/D_y \ D_{xy}/D_y	1/3	1/2	1
1/3	51.62	52.35	53.76
1/2	52.02	52.70	54.03
1	53.08	53.63	54.80

Table 1j Simply Supported Ellipse, $\alpha=\beta=2, R=.5$

$\nu_{xy} = 1/3$

D_x/D_y \ D_{xy}/D_y	1/3	1/2	1
1/3	17.61*	19.11	22.89
1/2	18.07	19.56	23.31
1	19.39	20.78	24.43

Table 1k Simply Supported Square, $\alpha=\beta=10$, $R=1.0$

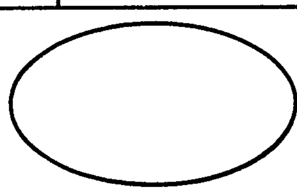
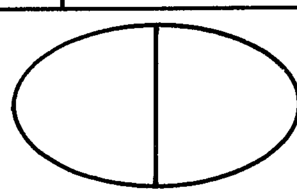
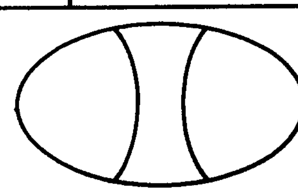
$\nu_{xy} = 1/3$

D_x/D_y \ D_{xy}/D_y	1/3	1/2	1
1/3	48.15*	50.42	56.45
1/2	48.33	50.60	56.66
1	48.84	51.11	57.20

Table 1l Simply Supported Rectangle $\alpha=\beta=10$, $R=.5$

* Check Cases

Table 2 First Six Eigenvalues with Eigenvectors and Nodal Patterns for Clamped Orthotropic Elliptic Plate ($\alpha = \beta = 2$, R=.5), $\gamma_{xy}=1/3$, $D_{xy}/D_y=1/3$, $D_x/D_y=1/3$

106.92		145.73		192.92	
* A_{00}	.6193	A_{10}	.5227	A_{00}	-.0223
A_{20}	-1.0000	A_{12}	-.2064	A_{20}	.4778
A_{40}	.7446	A_{14}	.3201	A_{40}	-1.0000
A_{60}	-.3401	A_{16}	-.0111	A_{60}	.9820
A_{80}	.1116	A_{18}	.0349	A_{80}	-.5295
$A_{10,0}$	-.0248	$A_{1,10}$	.0007	$A_{10,0}$	.1296
A_{02}	-.3127	A_{30}	-1.0000	A_{02}	-.0257
A_{22}	.3576	A_{32}	.3148	A_{22}	-.0877
A_{42}	-.1974	A_{34}	-.4717	A_{42}	.2364
A_{62}	.0481	A_{36}	-.0194	A_{62}	-.2268
A_{82}	.0143	A_{38}	-.0510	A_{82}	.0892
A_{04}	.2642	A_{50}	.8889	A_{04}	-.0020
A_{24}	-.3243	A_{52}	-.2254	A_{24}	.3698
A_{44}	.1911	A_{54}	.3333	A_{44}	-.5732
A_{64}	-.0839	A_{56}	.0538	A_{64}	.2943
A_{06}	-.0392	A_{70}	-.4870	A_{06}	-.0279
A_{26}	.0174	A_{72}	.0666	A_{26}	.0779
A_{46}	.0166	A_{74}	-.1511	A_{46}	-.0830
A_{08}	.0162	A_{90}	.1828	A_{08}	-.0005
A_{28}	-.0199	A_{92}	.0148	A_{28}	.0577
$A_{0,10}$	-.0024	$A_{11,0}$	-.0402	$A_{0,10}$	-.0036
					

* The subscripts indicate the powers of the xy-polynomial associated with the coefficient (see equations 10.5, 10.6, 10.7, 10.8)

Table 2 (continued)

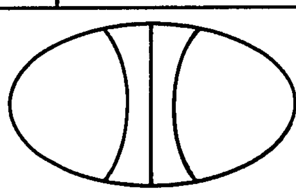
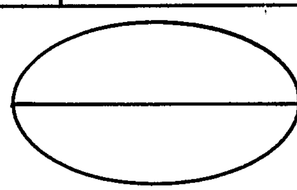
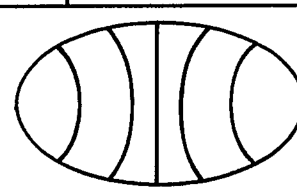
249.19		277.53		282.60	
A_{10}	-.0441	A_{01}	.2186	A_{10}	.1435
A_{12}	-.0515	A_{21}	-.5953	A_{12}	.0052
A_{14}	-.0218	A_{41}	.7747	A_{14}	.0057
A_{16}	-.0824	A_{61}	-.6217	A_{16}	.0211
A_{18}	-.0151	A_{81}	.3133	A_{18}	.0063
$A_{1,10}$	-.0164	$A_{10,1}$	-.0772	$A_{1,10}$	.0258
A_{30}	.4322	A_{03}	-.3794	A_{30}	-.0430
A_{32}	.0199	A_{23}	.8951	A_{32}	-.1065
A_{34}	.3961	A_{43}	1.0000	A_{34}	-.1536
A_{36}	.2190	A_{63}	.6502	A_{36}	-.1499
A_{38}	.1043	A_{83}	-.2024	A_{38}	-.2379
A_{50}	-.9404	A_{05}	.4590	A_{50}	.3196
A_{52}	.1275	A_{25}	-.9722	A_{52}	.5541
A_{54}	-.6309	A_{45}	.9291	A_{54}	.4153
A_{56}	-.1860	A_{65}	-.4016	A_{56}	.5568
A_{70}	1.0000	A_{07}	-.2597	A_{70}	-.8846
A_{72}	-.1809	A_{27}	.4546	A_{72}	-.8472
A_{74}	.3298	A_{47}	-.2775	A_{74}	-.4280
A_{90}	-.5753	A_{09}	.1280	A_{90}	1.0000
A_{92}	.0803	A_{29}	-.1767	A_{92}	.4138
$A_{11,0}$	.1454	$A_{0,11}$	-.0300	$A_{11,0}$	-.4002
					

Table 3 First Six Eigenvalues with Eigenvectors and Nodal Patterns for Clamped Orthotropic Elliptic Plate ($\alpha=\beta=2$, R=.5), $\nu_{xy}=1/3$, $D_{xy}/D_y=1/3$, $D_x/D_y=1$

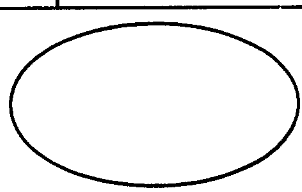
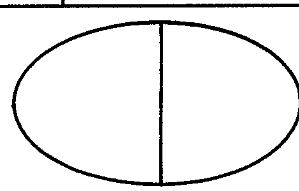
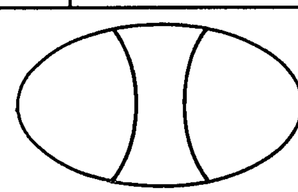
109.50		157.97		223.87	
A_{00}	.8712	A_{10}	.7555	A_{00}	-.0353
A_{20}	-1.0000	A_{12}	-.4930	A_{20}	.6632
A_{40}	.5022	A_{14}	.5384	A_{40}	-1.0000
A_{60}	-.1679	A_{16}	-.0970	A_{60}	.7097
A_{80}	.0687	A_{18}	.0603	A_{80}	-.2761
$A_{10,0}$	-.0292	$A_{1,10}$	-.0032	$A_{10,0}$	.0474
A_{02}	-.4995	A_{30}	-1.0000	A_{02}	-.0370
A_{22}	.5725	A_{32}	.7238	A_{22}	-.4541
A_{42}	-.2547	A_{34}	-.5765	A_{42}	.8870
A_{62}	-.0290	A_{36}	.0583	A_{62}	-.6634
A_{82}	.1103	A_{38}	-.0910	A_{82}	.2377
A_{04}	.3915	A_{50}	.6163	A_{04}	-.0199
A_{24}	-.3473	A_{52}	-.3703	A_{24}	.6194
A_{44}	.2104	A_{54}	.4763	A_{44}	-.7032
A_{64}	-.2105	A_{56}	.1986	A_{64}	.2356
A_{06}	-.0715	A_{70}	-.2638	A_{06}	-.0603
A_{26}	.0474	A_{72}	-.0639	A_{26}	-.0724
A_{46}	.0669	A_{74}	-.4382	A_{46}	.1691
A_{08}	.0245	A_{90}	.1215	A_{08}	.0138
A_{28}	-.0275	A_{92}	.1750	A_{28}	.0762
$A_{0,10}$	-.0040	$A_{11,0}$	-.0454	$A_{0,10}$	-.0096
					

Table 3 (continued)

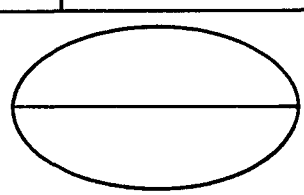
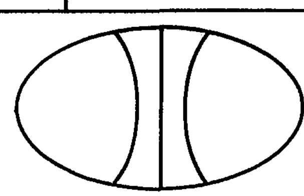
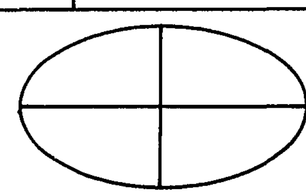
279.46		307.95		352.14	
A_{01}	.2520	A_{10}	.7555	A_{11}	.1837
A_{21}	-.5771	A_{12}	-.4930	A_{31}	-.4555
A_{41}	.5900	A_{14}	.5384	A_{51}	.5131
A_{61}	-.3591	A_{16}	-.0970	A_{71}	-.3457
A_{81}	.1384	A_{18}	.0603	A_{91}	.1446
$A_{10,1}$	-.0275	$A_{1,10}$	-.0032	$A_{11,1}$	-.0299
A_{03}	-.4476	A_{30}	-1.0000	A_{13}	-.3535
A_{23}	.9479	A_{32}	.7239	A_{33}	.8388
A_{43}	-.9023	A_{34}	-.5765	A_{53}	-.8885
A_{63}	.4862	A_{36}	.0583	A_{73}	.5213
A_{83}	-.1284	A_{38}	-.0910	A_{93}	-.1434
A_{05}	.5420	A_{50}	.6163	A_{15}	.4924
A_{25}	-1.0000	A_{52}	-.3703	A_{35}	-1.0000
A_{45}	.7980	A_{54}	.4763	A_{55}	.8700
A_{65}	-.2892	A_{56}	.1986	A_{75}	-.3279
A_{07}	-.3139	A_{70}	-.2638	A_{17}	-.3095
A_{27}	.5168	A_{72}	-.0639	A_{37}	.5693
A_{47}	-.2975	A_{74}	-.4382	A_{57}	-.3423
A_{09}	.1538	A_{90}	.1215	A_{19}	.1792
A_{29}	-.1870	A_{92}	.1750	A_{39}	-.2237
$A_{0,11}$	-.0372	$A_{11,0}$	-.0454	$A_{1,11}$	-.0410
					

Table 4 First Six Eigenvalues with Eigenvectors and Nodal Patterns for Clamped Orthotropic Elliptic Plates ($\alpha=\beta=2$, R=.5), $\nu_{xy}=1/3$, $D_{xy}/D_y=1$, $D_x/D_y=1/3$

117.52		175.47		238.73	
A_{00}	.9023	A_{10}	.5942	A_{00}	-.0226
A_{20}	-1.0000	A_{12}	-.1522	A_{20}	.4713
A_{40}	.6235	A_{14}	.4908	A_{40}	-.9775
A_{60}	-.2592	A_{16}	.0631	A_{60}	1.0000
A_{80}	.0979	A_{18}	.0978	A_{80}	-.5695
$A_{10,0}$	-.0299	$A_{1,10}$	.0290	$A_{10,0}$	.1470
A_{02}	-.4558	A_{30}	-1.0000	A_{02}	-.0681
A_{22}	.2688	A_{32}	.0888	A_{22}	.1019
A_{42}	-.0525	A_{34}	-.6197	A_{42}	-.1441
A_{62}	-.0441	A_{36}	-.1787	A_{62}	.1229
A_{82}	.0616	A_{38}	-.1927	A_{82}	-.4382
A_{04}	.4459	A_{50}	.8671	A_{04}	.0053
A_{24}	-.3631	A_{52}	.0172	A_{24}	.5366
A_{44}	.1948	A_{54}	.4221	A_{44}	-.8114
A_{64}	-.1501	A_{56}	.2375	A_{64}	.4356
A_{06}	-.0595	A_{70}	-.4853	A_{06}	-.0942
A_{26}	-.0186	A_{72}	-.0694	A_{26}	.3488
A_{46}	.1304	A_{74}	-.2198	A_{46}	-.4026
A_{08}	.0339	A_{90}	.1944	A_{08}	-.0126
A_{28}	-.0690	A_{92}	.0574	A_{28}	.2326
$A_{0,10}$	-.0012	$A_{11,0}$	-.0461	$A_{0,10}$	-.0236

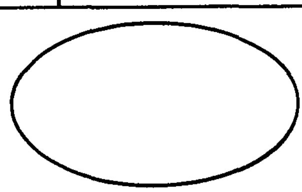
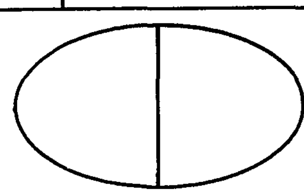
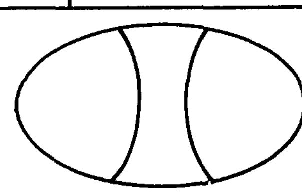




Table 4 (continued)

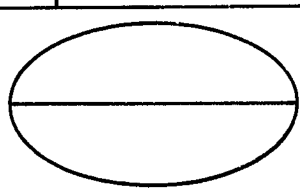
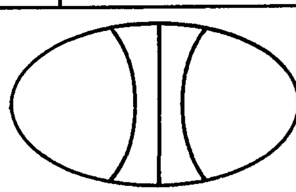
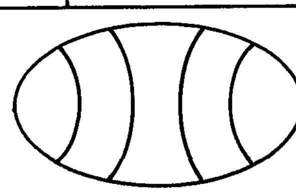
296.94		309.16		387.61	
A_{01}	.3636	A_{10}	-.0388	A_{00}	-.0015
A_{21}	-.5895	A_{12}	-.0991	A_{20}	.0727
A_{41}	.5245	A_{14}	-.0535	A_{40}	-.4770
A_{61}	-.3075	A_{16}	-.2142	A_{60}	.9878
A_{81}	.1235	A_{18}	-.1120	A_{80}	-.9092
$A_{10,1}$	-.0262	$A_{1,10}$	-.1058	$A_{10,0}$	.3214
A_{03}	-.6624	A_{30}	.3861	A_{02}	-.0017
A_{23}	.9172	A_{32}	.2498	A_{22}	.1612
A_{43}	-.6372	A_{34}	.5897	A_{42}	-.4526
A_{63}	.2869	A_{36}	.6732	A_{62}	.5445
A_{83}	-.0689	A_{38}	.4811	A_{82}	-.2624
A_{05}	.8291	A_{50}	-.8788	A_{04}	-.0219
A_{25}	-1.0000	A_{52}	-.3324	A_{24}	.1972
A_{45}	.6091	A_{54}	-.9284	A_{44}	-.8309
A_{65}	-.1899	A_{56}	-.6747	A_{64}	.8214
A_{07}	-.5021	A_{70}	1.0000	A_{06}	.3189
A_{27}	.4959	A_{72}	.2524	A_{26}	.4628
A_{47}	-.1848	A_{74}	.5175	A_{46}	-1.0000
A_{09}	.2621	A_{90}	-.6159	A_{08}	-.0432
A_{29}	-.2089	A_{92}	-.0864	A_{28}	.3918
$A_{0,11}$	-.0650	$A_{11,0}$	.1650	$A_{0,10}$	-.0168
					

Table 5 First Six Eigenvalues with Eigenvectors and Nodal Patterns for Clamped Orthotropic Ellipic Plates ($\alpha=\beta=2$, R=.5), $\nu_{xy}=1/3$, $D_{xy}/D_y=1$, $D_x/D_y=1$

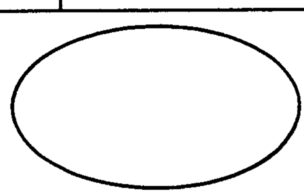
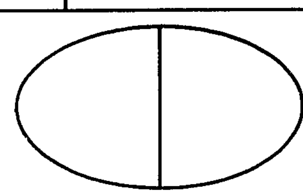
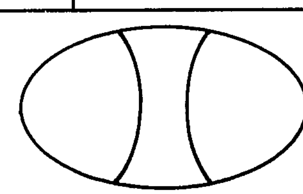
119.58		185.37		264.37	
A_{00}	1.0000	A_{10}	.7703	A_{00}	-.0326
A_{20}	-.8973	A_{12}	-.3823	A_{20}	.6252
A_{40}	.4319	A_{14}	.6429	A_{40}	-1.0000
A_{60}	-.1535	A_{16}	-.0431	A_{60}	.7779
A_{80}	.0860	A_{18}	.1031	A_{80}	-.3368
$A_{10,0}$	-.0422	$A_{1,10}$	.6396	$A_{10,0}$	.0668
A_{02}	-.5651	A_{30}	-1.0000	A_{02}	-.1061
A_{22}	.4057	A_{32}	.4429	A_{22}	-.1335
A_{42}	-.1060	A_{34}	-.5864	A_{42}	.3762
A_{62}	-.1020	A_{36}	-.0410	A_{62}	-.3153
A_{82}	-.1564	A_{38}	-.3295	A_{82}	.1277
A_{04}	.5043	A_{50}	.6547	A_{04}	.0687
A_{24}	-.3172	A_{52}	-.1821	A_{24}	.6118
A_{44}	.2010	A_{54}	.4278	A_{44}	-.0673
A_{64}	-.3313	A_{56}	.4871	A_{64}	.2197
A_{06}	-.8726	A_{70}	-.2985	A_{06}	-.1634
A_{26}	.0115	A_{72}	-.1015	A_{26}	.1884
A_{46}	.3009	A_{74}	-.4524	A_{46}	-.0261
A_{08}	.0372	A_{90}	.1386	A_{08}	.0253
A_{28}	-.1249	A_{92}	.1630	A_{28}	.1398
$A_{0,10}$	.0001	$A_{11,0}$	-.0496	$A_{0,10}$	-.0341
					

Table 5 (continued)

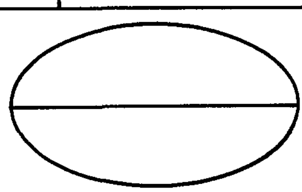
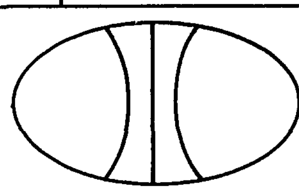
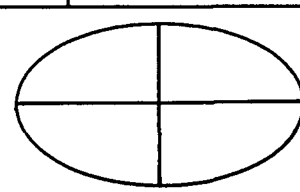
298.23		358.90		406.47	
A_{01}	.3821	A_{10}	-.0628	A_{11}	.2191
A_{21}	.5674	A_{12}	-.1747	A_{31}	-.4348
A_{41}	.4427	A_{14}	.0693	A_{51}	.4341
A_{61}	-.2193	A_{16}	-.4044	A_{71}	-.2721
A_{81}	.0734	A_{18}	.0046	A_{91}	.1101
$A_{10,1}$	-.0132	$A_{1,10}$	-.1409	$A_{11,1}$	-.0225
A_{03}	-.7070	A_{30}	.5554	A_{13}	-.4384
A_{23}	.9413	A_{32}	.0655	A_{33}	.7779
A_{43}	-.6083	A_{34}	.8416	A_{53}	-.6622
A_{63}	.2498	A_{36}	.5736	A_{73}	.3389
A_{83}	-.0557	A_{38}	.2345	A_{93}	-.0864
A_{05}	.8949	A_{50}	-1.0000	A_{15}	.6737
A_{25}	-1.0000	A_{52}	.2528	A_{35}	-1.0000
A_{45}	.5521	A_{54}	-.7285	A_{55}	.7006
A_{65}	-.1514	A_{56}	-.2170	A_{75}	-.2280
A_{07}	-.5443	A_{70}	.8975	A_{17}	-.4482
A_{27}	.5348	A_{72}	-.3177	A_{37}	.5778
A_{47}	-.2032	A_{74}	.2830	A_{57}	-2640
A_{09}	.2819	A_{90}	-.4441	A_{19}	.3024
A_{29}	-.2072	A_{92}	.1399	A_{39}	-.2769
$A_{0,11}$	-.7192	$A_{11,0}$	.0984	$A_{1,11}$	-.0698
					

Table 6 First Six Eigenvalues with Eigenvectors and Nodal Patterns for Clamped Orthotropic Elliptic Plate with Linearly Varying Thickness, $\nu_{xy}=1/3$, $D_{xy}/D_y=1/3$, $D_x/D_y=1/3$

134.26		197.65		271.89	
A_{00}	.0316	A_{10}	.0401	A_{00}	.0040
A_{20}	-.1569	A_{12}	-.0118	A_{20}	-.1202
A_{40}	.5022	A_{14}	.0233	A_{40}	.5012
A_{60}	-.9830	A_{16}	-.0005	A_{60}	-1.0000
A_{80}	1.0000	A_{18}	-.0009	A_{80}	.9891
$A_{10,0}$	-.3959	$A_{1,10}$	-.0019	$A_{10,0}$	-.3816
A_{02}	-.0109	A_{30}	-.2033	A_{02}	.0084
A_{22}	.0351	A_{32}	.0651	A_{22}	-.0416
A_{42}	-.1071	A_{34}	-.0740	A_{42}	.1358
A_{62}	.1927	A_{36}	.0369	A_{62}	-.2171
A_{82}	-.1263	A_{38}	.0256	A_{82}	.1276
A_{04}	.0148	A_{50}	.5860	A_{04}	.0004
A_{24}	-.0744	A_{52}	-.2128	A_{24}	-.0957
A_{44}	.1390	A_{54}	.0510	A_{44}	.3172
A_{64}	-.0753	A_{56}	-.1111	A_{64}	-.3005
A_{06}	.0004	A_{70}	-1.0000	A_{06}	.0107
A_{26}	-.0047	A_{72}	.3318	A_{26}	-.0945
A_{46}	-.0156	A_{74}	.0355	A_{46}	.1858
A_{08}	.0012	A_{90}	.8959	A_{08}	.0020
A_{28}	.0001	A_{92}	-.1878	A_{28}	-.0323
$A_{0,10}$	-.0001	$A_{11,0}$	-.3199	$A_{0,10}$	.0020

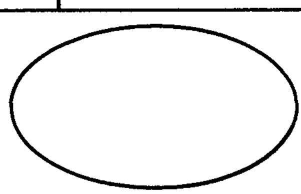
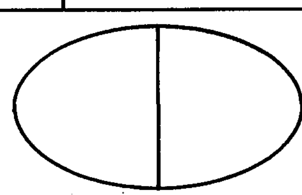
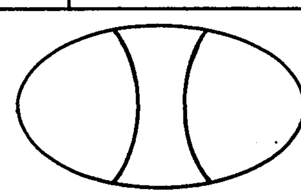
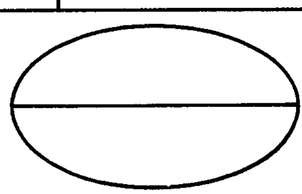
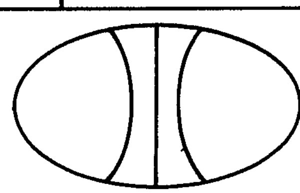




Table 6 (continued)

330.51		358.38		442.20	
A_{01}	.0123	A_{10}	-.0106	A_{11}	.0191
A_{21}	-.0951	A_{12}	-.0186	A_{31}	-.1414
A_{41}	.3943	A_{14}	-.0062	A_{51}	.5118
A_{61}	-.8943	A_{16}	-.0303	A_{71}	-1.0000
A_{81}	1.0000	A_{18}	-.0098	A_{91}	.9828
$A_{10,1}$	-.4273	$A_{1,10}$	-.0063	$A_{11,1}$	-.3775
A_{03}	-.0194	A_{30}	.1471	A_{13}	-.0295
A_{23}	.1233	A_{32}	.0974	A_{33}	.1926
A_{43}	-.4310	A_{34}	.1203	A_{53}	-.5756
A_{63}	.6889	A_{36}	.1697	A_{73}	.8100
A_{83}	-.4010	A_{38}	.0537	A_{93}	-.4230
A_{05}	.0246	A_{50}	-.5509	A_{15}	.0422
A_{25}	-.1689	A_{52}	-.2497	A_{35}	-.2516
A_{45}	.4745	A_{54}	-.3175	A_{55}	.5746
A_{65}	-.4500	A_{56}	-.2354	A_{75}	-.4479
A_{07}	-.0110	A_{70}	1.0000	A_{17}	-.0157
A_{27}	.0400	A_{72}	.3178	A_{37}	.0568
A_{47}	-.0314	A_{74}	.2504	A_{57}	-.0551
A_{09}	.0066	A_{90}	-.9010	A_{19}	.0126
A_{29}	-.0293	A_{92}	-.1554	A_{39}	-.0412
$A_{0,11}$	-.0007	$A_{11,0}$	.3194	$A_{1,11}$	-.0001





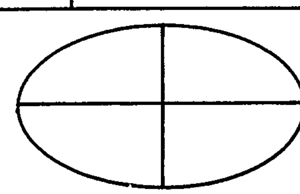


Table 7 First Six Eigenvalues with Eigenvectors and Nodal Patterns for Clamped Orthotropic Elliptic Plate ($\alpha=\beta=2$, R=.5) with Inplane Forces $Ta^2/D_y = -10$, $\nu_{xy}=1/3$, $D_{xy}/D_y=1/3$, $D_x/D_y=1/3$

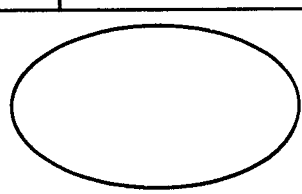
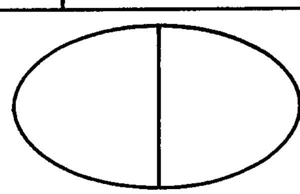
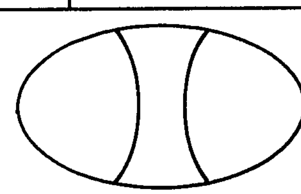
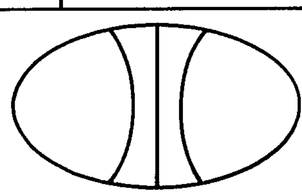
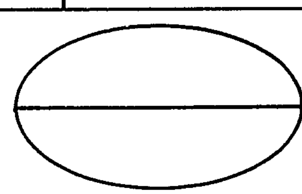
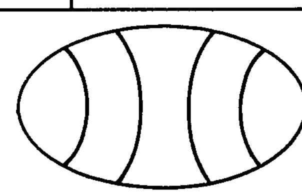
92.66		127.93		172.37	
A_{00}	.5080	A_{10}	.4590	A_{00}	-.0184
A_{20}	-1.0000	A_{12}	-.2260	A_{20}	.4181
A_{40}	.9049	A_{14}	.2687	A_{40}	-.9473
A_{60}	-.4967	A_{16}	-.0315	A_{60}	1.0000
A_{80}	.1802	A_{18}	.0259	A_{80}	-.5738
$A_{10,0}$	-.0368	$A_{1,10}$	-.0014	$A_{10,0}$	.1473
A_{02}	-.3524	A_{30}	-.9967	A_{02}	-.0209
A_{22}	.4411	A_{32}	.3410	A_{22}	-.0913
A_{42}	-.2668	A_{34}	-.4564	A_{42}	.2167
A_{62}	.0913	A_{36}	.0023	A_{62}	-.2021
A_{82}	-.0067	A_{38}	-.0405	A_{82}	.0786
A_{04}	.2366	A_{50}	1.0000	A_{04}	.0017
A_{24}	-.3407	A_{52}	-.2515	A_{24}	.2975
A_{44}	.2343	A_{54}	.3560	A_{44}	-.5148
A_{64}	-.0932	A_{56}	.0367	A_{64}	.2857
A_{06}	-.0572	A_{70}	-.6104	A_{06}	-.0211
A_{26}	.0408	A_{72}	-.0954	A_{26}	.0465
A_{46}	.0019	A_{74}	-.1523	A_{46}	-.0604
A_{08}	.0152	A_{90}	.2420	A_{08}	.0015
A_{28}	-.0180	A_{92}	-.0037	A_{28}	.0366
$A_{0,10}$	-.0029	$A_{11,0}$	-.0513	$A_{0,10}$	-.0021
					

Table 7 (continued)

226.46		261.39		290.66	
A_{10}	.0386	A_{01}	.1962	A_{00}	-.0015
A_{12}	.0444	A_{21}	-.5553	A_{20}	.0755
A_{14}	.0149	A_{41}	.7534	A_{40}	-.4930
A_{16}	.0652	A_{61}	-.6307	A_{60}	1.0000
A_{18}	.0078	A_{81}	.3298	A_{80}	-.8948
$A_{1,10}$	.0110	$A_{10,1}$	-.0835	$A_{10,0}$	.3083
A_{30}	-.3935	A_{03}	-.3601	A_{02}	-.0028
A_{32}	-.0174	A_{23}	.8699	A_{22}	.0904
A_{34}	-.3358	A_{43}	-1.0000	A_{42}	-.1482
A_{36}	-.1737	A_{63}	.6688	A_{62}	.0685
A_{38}	-.0725	A_{83}	-.2130	A_{82}	-.0020
A_{50}	.8998	A_{05}	.4279	A_{04}	-.0050
A_{52}	-.1005	A_{25}	-.9348	A_{24}	.0602
A_{54}	.5757	A_{45}	.9230	A_{44}	-.4079
A_{56}	.1575	A_{65}	-.4096	A_{64}	.4238
A_{70}	-1.0000	A_{07}	-.2573	A_{06}	-.0028
A_{72}	.1467	A_{27}	.4621	A_{26}	.1643
A_{74}	-.3158	A_{47}	-.2900	A_{46}	-.3006
A_{90}	.5967	A_{09}	.1224	A_{08}	-.0063
A_{92}	-.0661	A_{29}	-.1698	A_{28}	.0395
$A_{11,0}$	-.1549	$A_{0,11}$	-.0313	$A_{0,10}$	-.0003
					

Nomenclature

x, y	rectangular coordinates
a, b	plate dimensions
α, β	exponents of x and y in plate geometry
V	strain energy
T	kinetic energy
$D = Eh^3/12(1-\nu^2)$	flexural rigidity of isotropic plate
h	plate thickness
E	modulus of elasticity of isotropic plate
ν	Poisson's Ration of isotropic plate
$w(x,y,t)$	plate deflection
ρ	plate density
p	natural frequency, rad/sec
$W(x,y)$	normal mode deflection amplitude
V_{max}	maximum strain energy
T_{max}	maximum kinetic energy
$\nabla^2 W$	$\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2}$
W_{xx}, W_{XX}	$\frac{\partial^2 W}{\partial x^2}, \frac{\partial^2 W}{\partial X^2}$
W_{yy}, W_{YY}	$\frac{\partial^2 W}{\partial y^2}, \frac{\partial^2 W}{\partial Y^2}$
W_{xy}, W_{XY}	$\frac{\partial^2 W}{\partial x \partial y}, \frac{\partial^2 W}{\partial X \partial Y}$
u_i	trial family of functions
A_i	coefficients of trial family of functions
i, j, k	indices

Nomenclature (Cont'd)

$$D_x = \frac{E_x h^3}{12(1-\nu_{xy}\nu_{yx})}$$

$$D_y = \frac{E_y h^3}{12(1-\nu_{xy}\nu_{yx})}$$

$$D_1 = \nu_{yx} D_x$$

$$D_{xy} = G_{xy} h^3/12$$

rigidity constants for orthotropic plates

E_x modulus in x-direction

E_y modulus in y-direction

G_{xy} rigidity modulus for shear stresses

ν_{xy}, ν_{yx} Poisson's ratios

$F(x,y), F_1, F_2, F_3, F_4$ boundary functions

$G(x,y)$ xy polynomials

$[C_{ij}], [B_{ij}], [D_{ij}]$ square symmetric matrices

$\{A_i\}, \{\psi\}$ column matrices

$X = x/a$, $Y = y/a$ transformed coordinates

$P = (a/b)^\beta$, $R = b/a$ aspect ratio constants

ω^2 dimensionless frequency

$h(x,y)$ variable plate thickness

$D(x,y)$ flexural rigidity with variable thickness

$\bar{h}$ thickness amplitude

Nomenclature (Cont'd)

$H(x,y)$	plate thickness function
$\overline{D}$	rigidity amplitude
N_x, N_y, N_{xy}	inplane force intensities
w_k	Gaussian quadrature weights
x_k	zeros of Legendre polynomials
$P_n(x_k)$	Legendre polynomials
$\overline{\omega}$	thickness shear frequency

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