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ON THE CLASSIFICATION OF THE SOLUTION SPACE OF LINEAR ANISOTROPIC ELASTICITY

BY

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ON THE CLASSIFICATION OF THE

SOLUTION SPACE OF LINEAR ANISOTROPIC ELASTICITY

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FOR

DEPARTMENT OF MECHANICAL ENGINEERING

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ABSTRACT

A method is developed within the framework of Synge's function-space interpretation of problems in linear elasticity which allows the consideration of solutions to problems for anisotropic media in terms of the solutions of a corresponding isotropic problem and corresponding simpler anisotropic problems. The isotropic solution and simpler anisotropic solutions establish points in each of the statically and kinematically admissible spaces for the anisotropic problem. The established points are used to define sets of vectors in the statically and kinematically admissible spaces. The linear independence of these vectors can then be determined by a pair of criteria developed in this work. The independent vectors so established, when employed in Synge's hypercircle method, provide approximations to the solution of the anisotropic problem. Synge's expressions for the bounds on the approximations obtained are extended to allow more immediate usage. In addition the notion of the residual problem, which leads in some cases to an exact solution of the anisotropic problem, is developed.

ACKNOWLEDGMENTS

The author would like to take this opportunity to recognize those people who have helped make this dissertation a reality. Sincere gratitude is extended to the author's employer, Bell Telephone Laboratories, who have supported his entire graduate education and, specifically, provided him with the opportunity to devote full time study to this research for the past year under their Doctoral Support Program. In addition an acknowledgment is due to Miss Lynn Esposito of the mathematical typing service at the Whippany Labs for her diligence in preparing the typed manuscript.

In the technical aspects of this research the author is most grateful for the work of J. L. Synge which was responsible for the hypercircle theory, the basis for this investigation. In addition special recognition is due Dr. John Rausen, a member of the author's committee, whose careful scrutinization of the mathematics prevented a serious error. However, the most sincere appreciation is extended to the author's advisor, Dr. James L. Martin. His initial suggestion of the subject, subsequent guidance along the research path, and genuine concern for the author and his progress were indeed invaluable. His performance in this capacity was in the author's opinion exceptional.

Finally the author wishes to dedicate this dissertation to his wife, Mimi, and his daughter, Heather, who have made his efforts so much more meaningful.

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CHAPTER I

INTRODUCTION

A review of the literature reveals that a considerable amount of investigation has been done in the last thirty or forty years in the field of linear anisotropic elasticity. The approach taken by most investigators has been to extend the theories already developed for isotropic elasticity. Although many valuable results have been obtained by this method of analysis, it is somewhat surprising that very few of the contributions in this field have taken advantage of known solutions of corresponding problems with lesser degrees of anisotropy. As a result the manner in which the solution spaces of isotropic and anisotropic elasticity are interrelated is not completely understood.

Desired Objectives

The purpose of this research was to develop a method of systematically utilizing the solution space of isotropic elasticity to establish points in the solution space of anisotropic elasticity which then serve as points in solution space for problems with materials of more complex anisotropy. Such an analysis would provide the engineer with a better understanding of the relationships between isotropic and anisotropic solution space. The general plan of attack was to consider successive levels of material complexity, using the solution space of the previous level to establish points in the solution space of the next one, e.g., use isotropy to form initial estimates for transverse-isotropy and then use transverse-isotropy similarly with respect to orthotropy, etc.

Previous Investigations

As was indicated in the opening paragraph a considerable amount of material has been published concerning the field of anisotropic elasticity. Let us briefly consider the use of isotropic solution space in these works. The reader interested in the direct approach to the field by extension of isotropic solution theories is referred to the work by S. G. Lekhnitskii,[1] the most complete treatment discovered by the author.

One technique of utilizing the solutions of isotropic problems to obtain approximations of anisotropic problems

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[1] S. G. Lekhnitskii, *Theory of Elasticity of an Anisotropic Elastic Body*, trans. P. Fein (San Francisco: Holden-Day, Inc., 1963).

is a variation of the perturbation procedure often used in the approximate solution of nonlinear differential equations. Several methods of such a procedure, differing only in the definition of the perturbation parameter, were advanced by I. S. Sokolnikoff.[2] In this paper he considered, for example, the equation for Airy's stress function, U(x,y), in two-dimensional orthotropic elasticity written as

$$2 \frac{c_1}{c_3} \frac{\partial^4 U}{\partial x^4} + 2 \frac{\partial^4 U}{\partial x^2 \partial y^2} + 2 \frac{c_5}{c_3} \frac{\partial^4 U}{\partial y^4} = 0, \qquad (1.1)$$

where c_1, c_3, and c_5 are material constants. The perturbation parameters are introduced as

$$1 - \varepsilon_x \equiv 2 \frac{c_1}{c_3}, \qquad (1.2)$$
$$1 - \varepsilon_y \equiv 2 \frac{c_5}{c_3},$$

which results in the following form for (1.1):

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[2] I. S. Sokolnikoff, "Approximate Methods of Solution of Two-dimensional Problems in Anisotropic Elasticity," Proceedings of the Third Symposium in Applied Mathematics (New York: McGraw-Hill, 1950). pp. 1-11.

$$(1-\varepsilon_x)\frac{\partial^4 U}{\partial x^4} + 2\frac{\partial^4 U}{\partial x^2 \partial y^2} + (1-\varepsilon_y)\frac{\partial^4 U}{\partial y^4} = 0. \qquad (1.3)$$

Upon assuming that ε_x and ε_y are small, a solution of the form

$$U(x,y) = \sum_{i,j=0}^{\infty} U_{ij}(x,y)\varepsilon_x^i \varepsilon_y^j \qquad (1.4)$$

is proposed, and $U_{ij}(x,y)$ are determined so that $U(x,y)$ satisfies equation (1.1).

Sokolnikoff also obtains expansions based on two other definitions of the perturbation parameters in the same paper, and in a more recent publication Yih-O Tu[3] uses still another definition for the perturbation parameter, but the procedures are analogous so they need not be discussed further.

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[3] Yih-O Tu, "Perturbation Solution of Plane Stress Problems in Anisotropic Elasticity," SIAM Journal of Applied Mathematics, Vol. 16, No. 2, March 1968, pp. 374-386.

A somewhat different approach was taken by L. Goffi.[4] He showed that the equation for the stress function for a thin orthotropic plate could be reduced to the corresponding equation for an isotropic plate by assuming an approximate form of the shear modulus, G, in terms of the Young's moduli and the Poisson's coefficients, and a simple homographic deformation of one of the coordinates.

All of these proposed approximations possess common deficiencies. They do not admit to the possibility of an exact solution and require the degree of anisotropy of the material to be small. These restrictions were avoided in this investigation. Toward this end the basic approach taken by J. L. Martin was more appropriate.[5]

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[4] L. Goffi, "Proposed Approximate Solution for Problems of Plane Elasticity in Orthotropic Anisotropy," Giornale del Genio Civile, 10:601-611, 1962.

[5] J. L. Martin, "Preliminary Report on the Bounding of Anisotropic Elastic Problems from the Solutions of Corresponding Isotropic Elastic Problems Using the Method of the Hypercircle," (Mechanical Engineering Dept., Newark College of Engineering, Sept. 1967, Mimeographed). "Report No. 2," February, 1968. "Report No. 3," February 1969.

His work investigated the use of the corresponding isotropic solution in conjunction with the hypercircle method, a function-space approach developed by J. L. Synge and W. Prager which will be explained later, to obtain approximations to an anisotropic problem. The results of his work established the feasibility of such an approach and thereby paved the way for this investigation.

CHAPTER II

THE HYPERCIRCLE METHOD AND ITS APPLICATION

The method of the hypercircle is a technique for investigating boundary value problems in terms of function space. The method was developed by J. L. Synge and W. Prager in 1946 and first appeared in publication in October, 1947.[1] Another paper by Synge expanded the theory to include elasticity problems with body forces,[2] while still another paper presented the theory in regard to its applicability to boundary value problems of other fields.[3]

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[1] W. Prager and J. L. Synge, "Approximations in Elasticity Based on the Concept of Function Space," Quarterly of Applied Mathematics, 5, pp. 241-271 (1947).

[2] J. L. Synge, "The Method of the Hypercircle in Elasticity when Body Forces are Present," Quarterly of Applied Mathematics, 6, pp. 15-19 (1948).

[3] J. L. Synge, "The Method of the Hypercircle in Function-space for Boundary-value Problems," Proceedings of the Royal Society of London A, 191, pp. 447-67, 1947.

In the abstract to the latter article Synge gave the following statement which succinctly describes the method of the hypercircle:

> For certain boundary-value problems, the conditions to be satisfied are split into two parts, so that the solution of a given problem is the common solution of two relaxed problems. Solutions of the two relaxed problems are easy to obtain, and such solutions give information regarding the solution of the original problem. This information is interpreted by a function-space representation. If the scalar product in function-space is suitably defined, solutions of the relaxed problems locate the solution of the original problem on, or inside, a hypercircle in function-space. The approximation may be improved by introducing further solutions of the relaxed problems. If the center of the hypercircle is regarded as an approximate solution to the original problem, its error in a mean-square sense is immediately known.[4]

Synge has published a number of subsequent articles on various aspects of the method. These will not be reviewed since the material covered is all included in Synge's book.[5] It was from this work that the theory described and subsequently used in this research was derived.

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[4] *Ibid.*, p. 447.

[5] J. L. Synge, *The Hypercircle in Mathematical Physics* (Cambridge: Cambridge University Press, 1957).

Hypercircle Theory

The discussion of the hypercircle theory presented here considers only the case where the function space possesses a positive-definite metric since it will be shown that the definition of a metric appropriate to the function space for elasticity theory is indeed positive-definite. As stated previously the theory as outlined below follows the general approach given by Synge in his book except where specialization to elasticity theory lends simplification without detracting from the general treatment. A set of definitions of terms which are essential to the development will be given first, after which the theory will be outlined as briefly as possible while still maintaining a degree of continuity.

P-space. P-space is that domain of Euclidean 3-space with coordinates x_1, x_2, and x_3, upon which a problem of linear elasticity is defined.

F-space. The F-space defined on a given P-space consists of all real functions defined on the coordinates of the P-space such that these functions are piecewise continuous and square integrable.

F-vector. An F-vector, denoted by S, corresponds to a set of functions, $s_1, s_2, \ldots, s_M$, defined on P-space. These vectors possess the usual properties associated with elementary vector algebra with the exception that only an inner product, $S \cdot S'$, is defined in the case of the F-vector.

Linear subspace (L). L is a linear subspace of F-space if for the points X and Y in L, the points $aX + bY$ are also in L for all a and b satisfying $a + b = 1$.

Linear n-space (L_n). A linear n-space is a particular type of linear subspace defined as a space whose points X satisfy

$$X = A + \sum_{\rho=1}^{n} a_\rho T_\rho ,$$

where A is a fixed vector, T_ρ are n linearly independent fixed vectors, and a_ρ are n variable parameters.

Hyperplane of class n (H_n). A hyperplane of class n, also a particular type of linear subspace, is an infinite dimensional space whose points X satisfy

$$X \cdot S_\rho = b_\rho, \quad (\rho = 1,2,\ldots,n),$$

where S_ρ are n linearly independent fixed vectors and b_ρ are n fixed numbers.

Orthogonality of linear subspaces (L',L"). Two linear subspaces, L' and L", are orthogonal if and only if every vector T' lying in L' is orthogonal to every vector T" lying in L".

As indicated by the previous quote of Synge the hypercircle method is based upon splitting the boundary value problem into two parts. It is further required that the two solution spaces resulting must be orthogonal. Thus a boundary value problem can be stated equivalently as "it is required to find the intersection of two orthogonal linear subspaces of a function-space."[6] With this approach in mind Synge

[6] Ibid., p. 97.

developed several properties of linear orthogonal subspaces which will now be summarized.

Let L' and L" be two orthogonal, nonintersecting, linear subspaces with S' and S" denoting any points in each respective space.

<u>Vertices</u>. The vertices, V' and V", of L' and L" are the values of S' and S" for which $(S'-S'')^2$ is a minimum. The vertices have the properties:

1) the F-vector determined by (V'-V") is orthogonal to both L' and L"; and

2) the vertices are unique.

If L' and L" are finite dimensional then they are properly linear n-spaces. Let them be denoted by L_r' and L_s'' and let S_o' and S_o'' be the vectors corresponding to A in the definition of a linear n-space for L_r' and L_s'', respectively. Their closest approach is then

$$(V'-V'')^2 = \left(S_o'-S_o''\right)^2 - \left(a_\rho' T_\rho'\right)^2 - \left(a_\sigma'' T_\sigma''\right)^2, \qquad (2.1)$$

where the a_ρ' and a_σ'' are determined by

$$\sum_{\mu=1}^{r} a'_{\mu}T'_{\mu}\cdot T'_{\rho} + \left(S'_{o}-S''_{o}\right)\cdot T'_{\rho} = 0, \quad (\rho = 1,2,\ldots,r),$$

(2.2)

$$\sum_{\nu=1}^{s} a''_{\nu}T''_{\nu}\cdot T''_{\sigma} - \left(S'_{o}-S''_{o}\right)\cdot T''_{\sigma} = 0, \quad (\sigma = 1,2,\ldots,s).$$

Now let L' and L" be orthogonal and intersecting linear subspaces with S' and S" points in L' and L", respectively. Synge then proves the following properties:

1) L' and L" intersect at only one point, say S;

2) the vertices coincide, and V' = V" = S; and

3) the vectors (S-S') and (S-S") are orthogonal, hence

$$(S-S')\cdot(S-S'') = 0, \qquad (2.3)$$

or equivalently,

$$[S - \tfrac{1}{2}(S'+S'')]^2 = [\tfrac{1}{2}(S'-S'')]^2, \qquad (2.4)$$

which locates S on a hypersphere of radius $\frac{1}{2}|S'-S''|$ and center at $\frac{1}{2}(S'+S'')$.

Now the essence of the hypercircle method is to obtain information about S, the intersection of L' and L", in terms of the vertices of the linear n-spaces, L_r' and L_s'', which are subsets of L' and L" respectively. The value lies in the fact that while a method for determining S which would in effect require a complete description of L' and L" may not be available, it may be relatively simple to obtain a few points, S_r' and S_s'', in each of L' and L" upon which to define L_r' and L_s''.[7]

To outline the construction suppose that r+1 points, $S_0', S_1', \ldots, S_r'$, are obtained in L' and s+1 points, $S_0'', S_1'', \ldots, S_s''$, are obtained in L". These points then determine r vectors, T_ρ', in L' and s vectors, T_σ'', in L" as

$$T_\rho' \equiv S_\rho' - S_0' \quad (\rho = 1,2,\ldots,r),$$
$$T_\sigma'' \equiv S_\sigma'' - S_0'' \quad (\sigma = 1,2,\ldots,s). \qquad (2.5)$$

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[7] *Ibid.*, pp. 98-107.

Then with these vectors linear n-spaces are defined as

$$L_r': \quad X = S_o' + \sum_{\rho=1}^{r} a_\rho' T_\rho'$$

and (2.6)

$$L_s'': \quad X = S_o'' + \sum_{\sigma=1}^{s} a_\sigma'' T_\sigma'',$$

which lie in L' and L" respectively.

The following orthogonalities result with respect to L', L", L_r', and L_s'':

$$(S-V')\cdot(S-V'') = 0;$$

$$(S-V')\cdot T_\sigma'' = 0, \quad \sigma = (1,2,\ldots,s); \text{ and} \quad (2.7)$$

$$(S-V'')\cdot T_\rho' = 0, \quad \rho = (1,2,\ldots,r),$$

where V' and V" are the vertices of L_r' and L_s'' and S is the intersection of L' and L".

The first of equations (2.7) is recognized by recalling the form of (2.3) as requiring S to lie on a hypersphere. The second and third relations of (2.7) combined are seen to place S on a hyperplane of class r + s. Thus S must lie on the intersection of a hypersphere and a hyperplane of class r + s, which is a hypercircle of class r + s. Figure 1 gives a geometric interpretation of the construction.[8]

Once the hypercircle has been constructed it is important to know how closely the solution can be approximated. A simple but important bound on S is the fact that if the radius of the hypersphere is small, then S can be approximated closely by the center of the hypersphere. A somewhat more complicated expression for bounds on S results from the fact that S lies on a hypercircle. These bounds are

$$\left| S^2 - C^2 - R^2 \right| \leqq 2R \left| C_o' \right| , \qquad (2.8)$$

where

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[8] *Ibid.*, pp. 98-109.

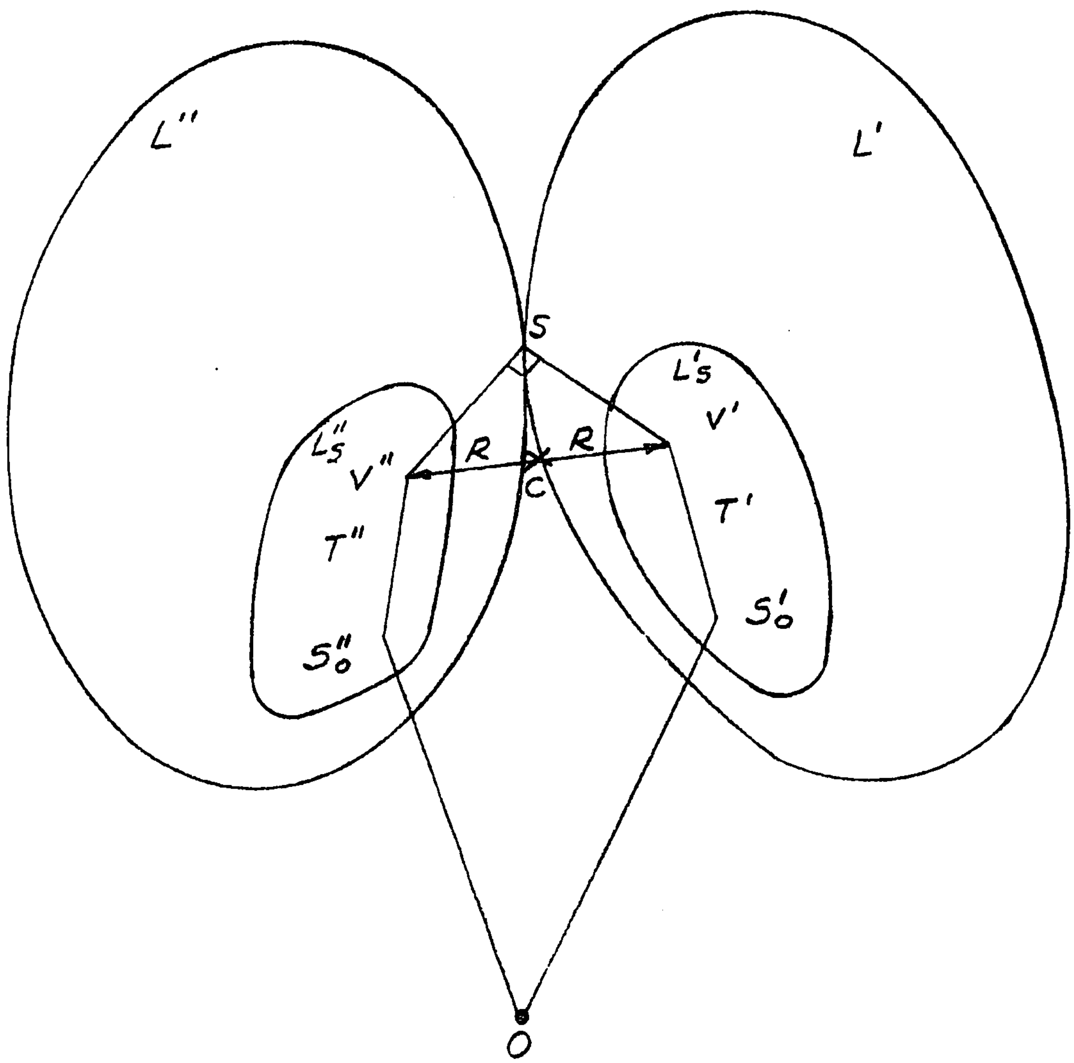

FIGURE 1

GEOMETRIC DIAGRAM FOR THE HYPERCIRCLE

$$C = \tfrac{1}{2}(V'+V''),$$

$$R = \tfrac{1}{2}|V' - V''|,$$

$$C^2 + R^2 = \tfrac{1}{2}(V'^2+V''^2),$$

and

$$C_o'^2 = C^2 - \sum_{\rho=1}^{r} \left(C\cdot I_\rho'\right)^2 - \sum_{\sigma=1}^{s} \left(C\cdot I_\sigma''\right)^2,$$

I_ρ' and I_σ'' resulting from the orthonormalization of T_ρ' and T_σ''.

The above provide only mean-square bounds on S, i.e., they bound the strain energy associated with the correct solution. Bounds can also be obtained for the value of S·G, where G is any F-vector. These bounds are important for they will provide point-bounds if G is chosen to be the Green's tensor for the medium. The bounds on S·G are given by

$$|S\cdot G - C\cdot G| \leqq R|G_o|, \tag{2.9}$$

where

$$G_o^2 = G^2 - \left(\sum_{\rho=1}^{r} C_\rho' T_\rho'\right)^2 - \left(\sum_{\sigma=1}^{s} C_\sigma'' T_\sigma''\right)^2,$$

with C_ρ' and C_σ'' being obtained from

$$\sum_{\mu=1}^{r} C_\mu' T_\mu' \cdot T_\rho' - G \cdot T_\rho' = 0 \qquad (\rho = 1,2,\ldots,r),$$

$$\sum_{\nu=1}^{s} C_\nu'' T_\nu'' \cdot T_\sigma'' - G \cdot T_\sigma'' = 0 \qquad (\sigma = 1,2,\ldots,s).[9]$$

Hypercircle Applied to Linear Elasticity

The previous discussion has set down the major features of the hypercircle method as developed by Synge which will be used in this research. Synge also shows that the general problem of linear elasticity can be treated by the hypercircle method.[10] The

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[9] *Ibid.*, pp. 110-113.

[10] *Ibid.*, pp. 330-339.

following discussion relates how Synge divides the elastic problem and defines the F-vectors and scalar products required.

To begin a discussion of the elastic equilibrium problem one must first set down the defining rules, the field equations of linear elasticity and the boundary conditions appropriate to a mixed problem.

Equilibrium equations. The equations of equilibrium are

$$\tau_{ij,j} + F_i = 0 \quad \text{in} \quad V, \qquad (i,j = 1,2,3), \qquad (2.10)$$

where τ_{ij} are the components of a symmetric stress tensor, F_i are the components of the body force per unit volume, and V is the volume of the body.

Generalized Hooke's law. The stress components are related to the strain components by

$$\tau_{ij} = \alpha_{ijk\ell} e_{k\ell}$$

and (2.11)

$$e_{ij} = C_{ijk\ell} \tau_{k\ell},$$

where e_{ij} are the components of a symmetric strain tensor and $\alpha_{ijk\ell}$ and $C_{ijk\ell}$ are the components of the elastic moduli and compliance tensors, respectively. The symmetry of τ_{ij} and e_{ij} place symmetry requirements on $\alpha_{ijk\ell}$ and $C_{ijk\ell}$ given by

$$\alpha_{ijk\ell} = \alpha_{k\ell ij} = \alpha_{jik\ell}$$

and (2.12)

$$C_{ijk\ell} = C_{k\ell ij} = C_{jik\ell}.$$

<u>Strain-displacement relations</u>. The components of e_{ij} are related to the displacement components, u_i, by

$$e_{ij} = \tfrac{1}{2}(u_{i,j}+u_{j,i}). \qquad (2.13)$$

<u>Displacement boundary conditions</u>. The displacements are specified by

$$u_i = u_i^* \quad \text{on} \quad B_1, \qquad (2.14)$$

where B_1 is part of the boundary, B.

Stress boundary conditions. The stresses must satisfy

$$\tau_{ij}n_j = \overline{X}_i \text{ on } B_2, \qquad (2.15)$$

where B_2 is the remainder of the boundary and n_j are the components of the unit normal vector on B_2.

Compatibility equations. The relations necessary to insure single-valued displacements when equations (2.13) are integrated are given by

$$e_{ij,k\ell} + e_{k\ell,ij} - e_{ik,j\ell} - e_{j\ell,ik} = 0. \qquad (2.16)$$

[11]

These equations are not part of the field equations since they are only necessary if the formulation of the problem is made in terms of the stresses.

Synge defines the function space so that any point or vector in it corresponds to any state of stress in V,

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[11] I. S. Sokolnikoff, Mathematical Theory of Elasticity (New York: McGraw-Hill, 1956), p. 25.

$$S \leftrightarrow \tau_{ij}.$$

One may alternately employ strain vectors (2.17)

$$S \leftrightarrow e_{ij},$$

to establish F-vectors. The only requirements on τ_{ij} is that it is symmetric.

The scalar product is defined by

$$S \cdot S' = \int_V \tau_{ij} e'_{ij} dV, \qquad (2.18)$$

which is shown to be commutative by the symmetry properties (2.12). This definition gives the metric as

$$S^2 = \int_V \tau_{ij} e_{ij} dV = \int_V \alpha_{ijk\ell} e_{k\ell} e_{ij} dV. \qquad (2.19)$$

In linear elasticity thermodynamic restrictions require $\alpha_{ijk\ell}e_{k\ell}e_{ij}$ to be a positive-definite quadratic form. Therefore the metric defined by (2.19) is positive-definite.

The linear subspaces L' and L" are defined by

$$L': \quad S' \leftrightarrow \tau'_{ij}, \quad \tau'_{ij,j} + F_i = 0 \text{ in } V, \ (\tau_{ij}n_j)_{B_2} = \bar{X}_i,$$

$$L'': \quad S'' \leftrightarrow \tau''_{ij}, \quad C\left(\tau''_{ij}\right) = 0, \quad \left(u''_i\right)_{B_1} = u^*_i, \tag{2.20}$$

where $C\left(\tau''_{ij}\right)$ is shorthand for the compatibility relations (2.16) written in terms of stress components. Note that vectors T' and T" lying in L' and L" then satisfy

$$L': \quad T' \leftrightarrow \tau'_{ij}, \quad \tau'_{ij,j} = 0 \text{ in } V, \ (\tau_{ij}n_j)_{B_2} = 0,$$

$$L'': \quad T'' \leftrightarrow \tau''_{ij}, \quad C\left(\tau''_{ij}\right) = 0, \ \left(u''_i\right)_{B_1} = 0. \tag{2.21}$$

These linear subspaces are easily shown to be orthogonal by considering

$$T'\cdot T'' = \int_V \tau'_{ij} e''_{ij}\, dV$$

$$= \int_V \tau'_{ij} u''_{i,j}\, dV$$

$$= \int_{B_1+B_2} u''_i \tau'_{ij} n_j\, dB - \int_V u''_i \tau'_{ij,j}\, dV$$

$$= 0 \qquad (2.22)$$

and the definition of orthogonal linear subspaces. Hence the solution of the problem of elastic equilibrium consists then of finding the intersection of the two orthogonal linear subspaces, L' and L", defined by (2.20).[12]

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[12] J. L. Synge, _The Hypercircle in Mathematical Physics_, _op. cit._, pp. 336-339.

CHAPTER III

THE CORRESPONDING ISOTROPIC SOLUTION

The previous chapter has described the hypercircle method and in addition has established that the linear theory of elasticity satisfies the requirements for its use. However, as yet, nothing has been said concerning a method of obtaining the required set of points in each of L' and L". Of course one method which Synge seems to have used for most of his examples, is to seek by trial and error a few functions which satisfy the required conditions thus obtaining a few points with which to build the hypercircle.[1] It is obvious that a systematic approach would be preferable, but just what would that approach be? Perhaps something similar to Southwell's relaxation method would be appropriate.

The Initial Points, S_o' and S_o''

The first step in developing a systematic method of generating points in L' and L" is of course obtaining the initial point in each space. In addition these two

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[1] See for example Synge, Hypercircle in Math. Phy., pp. 339-343.

points must become the basis for the construction of more points in L' and L". It will be shown in this chapter how these two initial points can be obtained from the solution of a corresponding isotropic problem. The following chapter will develop the complete estimation procedure originating from these initial points.

Kinematic and static admissibility of the initial isotropic solution. Consider a problem in linear anisotropic elasticity defined by the set of equations given in Chapter II. Of these the equations of equilibrium (2.18) and the stress boundary conditions (2.23) are referred to as the static requirements of the problem. Similarly the displacement boundary conditions (2.22) and the equations of compatibility (2.24) are called the kinematic requirements of the problem. These equations are

Static requirements,

$$\tau_{ij,j} + F_i = 0, \qquad (3.1)$$

$$(\tau_{ij}n_j)_{B_2} = \overline{X}_i; \qquad (3.2)$$

Kinematic requirements,

$$u_i = u_i^* \quad \text{on} \quad B_1, \tag{3.3}$$

$$C(e) = 0, \tag{3.4}$$

where $C(e)$ is a shorthand notation for the left-hand side of the compatibility relations (2.24), and B_1 and B_2 are portions of the boundary B such that $B_1 + B_2 = B$.

Recall the definitions of the two subspaces L' and L'',

$$L': \quad S' \leftrightarrow \tau'_{ij}, \quad \tau'_{ij,j} + F_i = 0, \quad \left(\tau'_{ij} n_j\right)_{B_2} = \overline{X}_i,$$

(3.5)

$$L'': \quad S'' \leftrightarrow \tau''_{ij}, \quad C(e'') = 0, \quad \left(u''_i\right)_{B_1} = u_i^*.$$

Note that under these definitions of L' and L'', the stress vectors τ'_{ij} and τ''_{ij} satisfy the static and kinematic requirements, respectively. Thus τ'_{ij} and τ''_{ij} will be designated as statically and kinematically admissible solutions, respectively, for the anisotropic problem.

It should also be observed that in neither the static nor the kinematic requirements is any specification made concerning the constitution of the medium. Thus the vectors τ'_{ij} and τ''_{ij} for an anisotropic medium can be derived from the known solution of a corresponding problem for any type of material. Now, of course, an isotropic material is the logical type of medium to exploit in this respect since solutions in isotropic elasticity are frequently available.

The following theorem is an immediate consequence of the previous discussion.

Theorem 1. Given any linear anisotropic elasticity problem, the stress vector and the strain vector obtained from the solution of an isotropic problem of the same geometry, body forces, and boundary conditions are, respectively, statically and kinematically admissible solutions for the given anisotropic problem.

The preferred isotropic solution. The preceding theorem establishes potential initial points in L' and L" for the hypercircle method applied to anisotropic problems. In fact more than one point is established in L' and L" by the isotropic solution, since in general the isotropic material constants appear in the expressions for the component of the stress and strain vectors.[2] Nothing has as yet been required of these constants so they will be allowed to take on any real values, thus establishing infinite subsets of L' and L", denoted by L_I' and L_I''.

Recall from the hypercircle discussion that with only one point in each of L' and L", call them S_o' and S_o'', an approximation to the solution is the center of the hypersphere determined by S_o' and S_o'', which is $\frac{1}{2}\left(S_o'+S_o''\right)$. This point will be called an isotropic approximation. Theorem 2, stated below, then follows from the preceding discussion.

Theorem 2. Given any problem of linear anisotropic elasticity there exist an infinite number of isotropic approximations.

Of course the next question now naturally arises. Which one of the infinite number of isotropic approximations is the best or most desirable? In order to

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[2] Either the stress or the strain may not be a function of the isotropic material constants. But since one of them must contain these constants the generality of the discussion that follows is not affected.

answer this question the manner in which the approximation is to be used must first be established.

A method preferred by Synge is the minimization of the square of the radius of the hypersphere,

$$R^2 = \tfrac{1}{2}\left(S_o' - S_o''\right)^2. \qquad (3.6)$$

In the present situation the natural minimization is with respect to the isotropic material constants, ν and E. While this would yield the best mean-square approximation to S, it is frequently, as will be seen shortly, not the most facile technique. Such an approach was taken by Martin[3] in his initial investigative reports, and following him this investigator examined the idea further. The major drawback was found to be in the solution of the two equations which result from differentiating (3.6) with respect to ν and E and equating the resulting expressions with zero. While their solution was obtainable, it involved considerable

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[3] Martin, op. cit.

difficulty.[4] Therefore, because of these difficulties and because the initial isotropic approximation was only the first step of a procedure for developing better and better approximations, a different approach was developed.

Before explaining the method chosen for selecting ν and E consider the constitutive laws for both anisotropic and isotropic materials written in matrix form.

Anisotropic

$$\begin{Bmatrix} e_{11} \\ e_{22} \\ e_{33} \\ e_{12} \\ e_{13} \\ e_{23} \end{Bmatrix} = \begin{Bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{12} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{13} & C_{23} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{14} & C_{24} & C_{34} & C_{44} & C_{45} & C_{46} \\ C_{15} & C_{25} & C_{35} & C_{45} & C_{55} & C_{56} \\ C_{16} & C_{26} & C_{36} & C_{46} & C_{56} & C_{66} \end{Bmatrix} \begin{Bmatrix} \tau_{11} \\ \tau_{22} \\ \tau_{33} \\ \tau_{12} \\ \tau_{13} \\ \tau_{23} \end{Bmatrix} \qquad (3.7)$$

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[4] For an example of the difficulties associated with a specific problem see Appendix A.

Isotropic

$$\begin{Bmatrix} e_{11} \\ e_{22} \\ e_{33} \\ e_{12} \\ e_{13} \\ e_{23} \end{Bmatrix} = \begin{bmatrix} \frac{1}{E} & -\frac{\nu}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & \frac{1}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & -\frac{\nu}{E} & \frac{1}{E} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{2(1+\nu)}{E} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{2(1+\nu)}{E} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{2(1-\nu)}{E} \end{bmatrix} \begin{Bmatrix} \tau_{11} \\ \tau_{22} \\ \tau_{33} \\ \tau_{12} \\ \tau_{13} \\ \tau_{23} \end{Bmatrix} \tag{3.8}$$

Upon examining the above it is apparent that no matter how ν and E are chosen the quality of the approximation will be limited. However, the necessity is a starting point. Recall the definition of the points S_o' and S_o''. If the stress components for the isotropic solution are used in (3.7) and (3.8), the statically admissible strains are then obtained by (3.7) and of course the kinematically admissible strains by (3.8). It is then proposed that what will be called the preferred isotropic solution have values of ν and E so that, individually, maximum functional matching of the kinematically admissible and statically admissible strain components is achieved. Proceeding in this manner each problem must be considered separately, since the form of the isotropic stress vector will influence how ν and E will be chosen. Recalling that the original plan was to estimate only the next level of anisotropy by the preceding level, the constitutive relation (3.7) will generally be transversally-isotropic if its approximation is sought in terms of isotropic solutions. This, of course, makes the prospects of a better quality of approximation much brighter.

The motivation for selecting ν and E in this manner is two-fold. First it is a direct procedure. Also some of the anisotropic constants are introduced into the isotropic strains in the same manner in which they are introduced into the statically admissible strains, thus giving rise to a bit of optimism relative to obtaining an exact solution by the procedure to be described in the subsequent material.

Once the values of ν and E have been determined the initial portion of the approximation procedure is complete. An initial hypersphere of known radius has been established for the problem. Thus the following theorem is obvious.

<u>Theorem 3</u>. The preferred isotropic solution establishes an initial hypersphere of known radius for the anisotropic problem.

Note! Before proceeding to obtain additional points in L' and L" the strains derived from the preferred isotropic stresses by the anisotropic constitutive relations should be checked in the compatibility equations. If compatibility is satisfied then check the displacements for boundary condition satisfaction. Under certain conditions the preferred isotropic stresses can also be the anisotropic stresses. An example of this situation occurs in the problem solved in Appendix C, part I. Examination of the final solution reveals that the stresses are exactly those of the preferred isotropic solution.

CHAPTER IV

RESIDUAL PROBLEM PROCEDURE

Theorem 3 has established the existence of a hypersphere for the anisotropic problem, the initial step required in the hypercircle method. Two sets of linearly independent vectors must now be generated. One set lying within L' is a set of homogeneous statically admissible vectors, while the other set lies within L" and contains homogeneous kinematically admissible vectors. The term homogeneous is attached to indicate that the vectors satisfy the homogeneous static and kinematic conditions ($F_i = \overline{X}_i = u_i^* = 0$ in (2.10), (2.14), and (2.15)), respectively. These sets of vectors are generated by the fundamental process of this work as solutions of a series of residual problems.

The required sets of homogeneous vectors in L' and L" could alternately be generated by trial and error or by a second and more systematic method which is that of pyramid F-vectors and is described by Synge.[1] However,

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[1] Synge, Hypercircle in Math. Phy., op. cit., pp. 168-213.

neither of these approaches exploits either known or generated solutions of corresponding problems with simpler material characteristics as a basis for the two required sets of vectors.

The residual problem procedure which will now be developed has in fact dual applicability. The process hopefully will reduce the complexity of a given anisotropic problem so that an exact solution can be obtained. Additionally the required sets of vectors can be generated by combining the results of the residual problem process with a linear independence criterion.

Residual Problem

The basic feature of the procedure to be developed is the residual problem.

Definition. Given any anisotropic problem a residual problem is defined by the set of equations obtained for τ_{K_o}, by letting $\tau = \tau_o'' + \tau_{K_o}$ or for τ_{S_o} by letting $\tau = \tau_o' + \tau_{S_o}$ in the defining equations for the anisotropic problem.

Let τ denote the stress vector which solves the anisotropic problem. Now consider the following definitions:

$$\tau_{S_o} \equiv \tau - \tau_o', \tag{4.1}$$

$$\tau_{K_o} \equiv \tau - \tau_o'', \tag{4.2}$$

where τ_o' and τ_o'' are respectively the statically and kinematically admissible stresses derived from the preferred initial approximation as previously described. It should be noted that the definitions have the effect of transferring the origin of function space to L' in the case of τ_{S_o} and to L" in the case of τ_{K_o}.

The stress vector τ_{S_o} denotes the solution of the residual problem which results from subtracting the statically admissible solution τ_o' from the anisotropic problem. Since τ_o' satisfies equilibrium and the stress boundary conditions, τ_{S_o} must satisfy the homogeneous equilibrium equations and zero resultants on the stress boundary. However, τ_o' does not satisfy the kinematic conditions in general, so τ_{S_o} must satisfy modified compatibility relations and displacement boundary conditions.

Similarly the stress vector τ_{K_o} denotes the solution of the residual problem resulting from the subtraction of

τ''_o from the original problem. Since τ''_o satisfies compatibility and the displacement boundary conditions, τ_{K_o} must be the solution of a problem with zero displacement boundary conditions and the usual compatibility relations. In addition the residual problem is characterized by modified equilibrium equations and stress boundary conditions.

The residual problem chosen depends on the nature of the original problem. For instance if the anisotropic problem has only displacement boundaries it might be advantageous to use (4.2), while stress boundaries would point toward the use of (4.1). The complete set of equations for τ_{S_o} and τ_{K_o} are given in Appendix B. In subsequent discussions concerning the residual problem it is assumed that definition (4.2) applies. Thus the theory will be based on transferring the origin to L", however, this is only done as a matter of convenience and does not restrict the generality of the theory.

For a particular problem the investigation begins by using isotropic solutions to generate initial approximations and a residual problem for a transversally-isotropic medium. If the resulting residual problem or further residual problems generated by linearly independent solutions of any residual problem can be solved exactly, the transversally-isotropic solution will be available and can be used on problems of greater material complexity for forming preferred initial approximations. Such a solution would also prove valuable in the process of obtaining the desired linearly independent sets of vectors as will become apparent in the discussions to follow.

Homogeneous Vectors

Now that the residual problem has been defined its role in the development of the sets of homogeneous vectors will be established. The following theorem provides another necessary step in obtaining these sets.

Theorem 4. An isotropic solution of the residual problem for a corresponding transversally-isotropic problem can be obtained.

In order to prove the theorem it is only necessary to state that the equations which define the residual problem are complete and the existence of a solution is assured. The fact that τ_{K_o} exists then establishes that the corresponding isotropic solution τ'_{K_o} must also exist, since the isotropic solution corresponds merely to a special case of the anisotropic solution.

The plan is now evident. A new residual problem can be generated by subtracting the kinematically admissible stresses τ''_{K_o}. The isotropic solution is obtained and the procedure repeated. The following theorem establishes that the isotropic solution of each new residual problem provides one point in each of L' and L".

Theorem 5. The initial isotropic approximation to each new residual problem for a corresponding transversally-isotropic problem establishes new statically and kinematically admissible solutions of the original anisotropic problem.

Proof. The preferred isotropic solution of the (n+1)st residual problem is τ'_{K_n}, and the equations which it satisfies are given below.

Equilibrium Equations

$$\tau'_{K_n ij,j} + \tau''_{oij,j} + \sum_{m=0}^{n-1} \tau''_{K_m ij,j} + F_i = 0 \qquad (4.3)$$

Stress Boundary Conditions

$$\left[\tau'_{K_n ij} + \tau''_{oij} + \sum_{m=0}^{n-1} \tau''_{K_m ij}\right] n_j = \overline{X}_i \quad \text{on} \quad B_T \qquad (4.4)$$

Displacement Boundary Conditions

$$u''_{K_n ij} = 0 \quad \text{on} \quad B_u \qquad (4.5)$$

The compatibility, stress-strain, and strain displacement relations are as given previously.

From the above equations it is obvious that the stress vector

$$\tau'_{(n+1)} \equiv \tau'_{K_n} + \tau''_o + \sum_{m=0}^{n-1} \tau''_{K_m} \qquad (4.6)$$

satisfies the equilibrium equations and stress boundary conditions for the original anisotropic problem, thus it determines a point in L'. Likewise the strain,

$$e''_{(n+1)} \equiv e''_{K_n} + \sum_{m=0}^{n-1} e''_{K_m} + e''_o = C^{I(n+1)}\tau'_{(n+1)}, \quad (4.7)$$

satisfies compatibility, and the displacement,

$$u''_{(n+1)} \equiv u''_{K_n} + \sum_{m=0}^{n-1} u''_{K_n} + u''_o \quad (4.8)$$

derived from the corresponding strain, satisfies the original displacement boundary conditions. Thus the stress

$$\tau''_{(n+1)} \equiv \tau''_{K_n} + \sum_{m=0}^{n-1} \tau''_{K_m} + \tau''_o = \alpha e''_{(n+1)}, \quad (4.9)$$

defines a point in L".

A closer look at the equations reveals that the vectors (4.6), (4.7), (4.8), and (4.9) are the same vectors as those obtained in the initial isotropic approximation except for a different set of material constants. It is then obvious that unless τ'_o is not a function of the isotropic material constants in which case the problem would be statically determinant, the vector given by (4.6) is a different point in L' if the isotropic constants are different. Similarly the point in L" is different from τ''_o.

Linear independence. The two sets of homogeneous vectors will now be defined.

Definition. The set of homogeneous statically admissible vectors {T'} contains vectors,

$$T'_n \equiv \tau'_n - \tau'_{(n-1)}, \quad n > 0. \tag{4.10}$$

Definition. The set of homogeneous kinematically admissible vectors {T"} has members,

$$T''_n \equiv \tau''_n - \tau''_{(n-1)}, \quad n > 0. \tag{4.11}$$

Recalling the previous discussion the members of {T'} and {T"} as defined above are simply the differences of two isotropic solutions of the initial problem. Thus their linear independence or dependence is a basic property of the form of the isotropic solution. While an infinite number of distinct points can always be obtained in this manner in each of L' and L", unless of course the problem is statically determinant, the linear dependence of the vectors defined by these points is open to question. In general the property of linear independence will be a function of the values chosen for the successive sets of isotropic constants.

In some problems it may be completely apparent how to choose the isotropic material constants to insure linear independence, but in general this will not be the case. It is therefore necessary to establish criteria for choosing the isotropic constants which will be sufficient to guarantee linear independence of at least some finite set of τ_n' and τ_n''.

Criterion for {T"}. Assume that T_n'' is the first linearly dependent vector in {T"}. Thus

$$T_n'' = \sum_{m=1}^{n-1} \bar{p}_m T_m'', \tag{4.12}$$

for some set of constants, $\bar{p}_m$, not all zero. This can also be written

$$E_n'' = \sum_{m=1}^{n-1} p_m E_m'' \tag{4.13}$$

where $E''_n = e''_n - e''_{(n-1)} = CT''_n$, since C is an invertible linear transformation and therefore preserves linear independence.[2]

Consider the vector

$$T'_n = \tau'_n - \tau'_{(n-1)} = \alpha^{I_n} e''_n - \alpha^{I(n-1)} e''_{(n-1)}. \qquad (4.14)$$

Using (4.3) and (4.4) along with (4.6), the following relations on (4.14) are obtained:

$$\left.\begin{aligned} &\left[\alpha^{I_n}_{ijk\ell}\, e''_{nk\ell} - \alpha^{I(n-1)}_{ijk\ell}\, e''_{(n-1)k\ell}\right]_{,j} = 0, \\ &\text{and} \\ &\left[\alpha^{I_n}_{ijk}\, e''_{nk\ell} - \alpha^{I(n-1)}_{ijk\ell}\, e_{(n-1)k\ell}\right] n_j = 0 \quad \text{on} \quad B_T. \end{aligned}\right\} \qquad (4.15)$$

By adding and subtracting $\alpha^{I_n} e''_{(n-1)}$ and using the assumed linear dependence (4.13), the above relations become

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[2] Paul R. Halmos, Finite-Dimensional Vector Spaces (Princeton, N. J., D. Van Nostrand Co., Inc., (1958)), p. 64.

$$\left. \begin{aligned} &\left[\alpha^{I_n}_{ijk\ell}\sum_{m=1}^{n-1} p_m E''_{mk\ell} + \left(\alpha^{I_n}_{ijk\ell} - \alpha^{I_{(n-1)}}_{ijk\ell}\right) e''_{(n-1)k\ell}\right]_{,j} = 0, \\ \text{and}& \\ &\left[\alpha^{I_n}_{ijk\ell}\sum_{m=1}^{n-1} p_m E''_{mk\ell} + \left(\alpha^{I_n}_{ijk\ell} - \alpha^{I_{(n-1)}}_{ijk\ell}\right) e''_{(n-1)k\ell}\right] n_j = 0 \quad \text{on} \quad B_T. \end{aligned} \right\} \tag{4.16}$$

Letting the bracketed term in (4.16) be denoted by a''_{ij}, consider

$$\int_B a''_{ij} n_j U''_{ri} dB = \int_V a''_{ij,j} U''_{ri} dV + \int_V a''_{ij} U''_{ri,j} \, dV, \tag{4.17}$$

where $U''_r = u''_r - u''_{(r-1)}$ is the displacement vector corresponding to E''_r and is therefore zero on B_u. Hence (4.17) becomes

$$\int_V a''_{ij} E''_{rij} \, dV = 0, \tag{4.20}$$

for all $r = 1,2,\ldots,(n-1)$.

Now (4.20) represents a set of (n-1) equations for the (n-1) constants, p_m, which can be written as

$$g_{mr} p_m = b_r, \quad m,r = 1,2,\ldots,(n-1), \tag{4.21}$$

where

$$g_{mr} = \int_V \alpha^{I_n}_{ijk\ell} E''_{mk\ell} E''_{rij} \, dV,$$

and

$$b_r = \int_V \left(\alpha^{I(n-1)}_{ijk\ell} - \alpha^{I_n}_{ijk\ell} \right) e''_{(n-1)k\ell} E''_{rij} \, dV.$$

So (4.21) has a nontrivial solution (which implies the linear dependence of E''_n) unless $b_r = 0$ for all r and $|g_{mr}| \neq 0$. Now b_r and g_{mr} are functions of the isotropic material constants, so if the constants can be chosen such that $b_r = 0$ and $|g_{mr}| \neq 0$, $p_m = 0$ for all m are the solutions of (4.21), and so (4.13) implies that $E''_n = 0$. If in fact $E''_n \neq 0$, the assumed linear dependence leads to a contradiction. Thus E''_n is also linearly independent.

Criterion for {T'}. Assume that T_n' is the first linearly dependent vector in {T'}. Thus

$$T_n' = \sum_{m=1}^{n-1} q_m T_m', \qquad (4.22)$$

for some set of constants, q_m, not all zero. Consider

$$\int_B T_{rij}' n_j U_{ni}'' \, dB = \int_V T_{rij,j}' U_{ni}'' \, dV + \int_V T_{rij}' U_{ni,j}'' \, dV,$$

$$r = 1,2,\ldots,n-1, \qquad (4.23)$$

where $U_n'' = u_n'' - u_{(n-1)}'' = 0$ on B_u. Thus (4.23) becomes

$$\int_V T_{rij}' U_{ni,j}'' \, dV = \int_V T_{rij}' E_{nij}'' \, dV = 0. \qquad (4.24)$$

Now

$$E_n'' = e_n'' - e_{(n-1)}'' = C^{I_n} \tau_n' - C^{I(n-1)} \tau_{(n-1)}',$$

which can be written

$$E_n'' = C^{I_n} T_n' + \left(C^{I_n} - C^{I(n-1)}\right)\tau'_{(n-1)}. \qquad (4.25)$$

Thus using the linear dependence assumption (4.22) and (4.25) in (4.24) yields

$$h_{mr} q_m = d_r, \quad m,r = 1,2,\ldots,(n-1), \qquad (4.26)$$

where

$$h_{mr} = \int_V C^{I_n}_{ijk\ell}\, T'_{mk\ell} T'_{rij}\, dV,$$

and

$$d_r = \int_V \left(C^{I(n-1)}_{ijk\ell} - C^{I_n}_{ijk\ell}\right)\tau'_{(n-1)k\ell} T''_{rij}\, dV.$$

Hence (4.26) is exactly analogous to the criterion (4.21). In order to insure linear independence of T_n', d_r must be zero for all r and $|h_{mr}| \neq 0$.

If at anytime either of the above criterion cannot be satisfied, the set of vectors cannot be expanded by simply adding more isotropic solution points. It then becomes necessary to consider the use of transversally-isotropic solution points, if they are available, in the same way as the isotropic solutions were used to obtain additional linearly independent vectors.

CHAPTER V

COMPLETING THE PROCESS

The residual problem technique and the associated theorems and criteria have provided some important results concerning the characterization of anisotropic solution space beginning with points in isotropic solution space. While the residual problem procedure was born from the geometric interpretation provided by the hypercircle method, the process can be employed independently as a type of relaxation technique to obtain some exact solutions of anisotropic problems. Moreover, if the exact solution cannot be obtained, the procedure provides sets of linearly independent vectors for use in the hypercircle method, the extent of these sets being determined by the independence criteria.

Point Bounds

The approximations obtained by the hypercircle method used with the bounds given in Chapter II provide only mean-square estimates of the quality of the solution. However, Synge advanced a method, developed by Prager, for obtaining bounds at a point for the solution.[1] This method utilizes the Green's tensor appropriate to the type of material being considered and also requires that the solution has been located on a hypercircle.

The expression for the bounds on the displacement components at a point results from consideration of the following equation developed by Synge:

$$u_p(x') = \int_V u_i^{(p)} F_i \, dV - \int_{B_1} u_i \tau_{ij}^{(p)} n_j \, dB + \int_{B_2} u_i^{(p)} \tau_{ij} n_j \, dB$$

$$+ \int_{B_1} u_i^{(p)} \tau_{ij} n_j \, dB - \int_{B_2} u_i \tau_{ij}^{(p)} n_j \, dB, \qquad (5.1)$$

where u_i and τ_{ij} denote the solution of the anisotropic problem, $u_i^{(p)}$ and $\tau_{ij}^{(p)}$ are the fundamental Green's

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[1] Synge, Hypercircle in Math. Phy., op. cit., pp. 350-53.

solution, F_i are the body forces, and u_i and $\tau_{ij}n_j$ are specified on B_1 and B_2, respectively. The coordinates of the point are x' and the coordinates of the body, x, have their origin at x'. Thus $u_i^{(p)}$ and $\tau_{ij}^{(p)}$ are functions of x and x' while u_i and τ_{ij} are simply functions of x. The subscripts i and j denote components of x and p denotes components of x'.

Now as Synge points out the first three integrals can be evaluated since they contain only known functions. However, the last two integrals cannot be evaluated since $\tau_{ij}n_j$ is not known on B_1 and u_i is not known on B_2.

Synge proposes a scheme of handling these last two integrals by considering auxiliary functions $u_i^{'(p)}$ and $\tau_{ij}^{''(p)}$ with the following properties:

$$u_i^{'(p)} = u_i^{(p)} \quad \text{on} \quad B_1 \quad \text{and}$$

$$\int_{B_2} u_i^{'(p)} \tau_{ij} n_j \, dB = 0; \tag{5.2}$$

$$\tau_{ij,j}^{''(p)} = 0 \quad \text{in} \quad V,$$

$$\tau_{ij}^{''(p)} n_j = \tau_{ij}^{(p)} n_j \quad \text{on} \quad B_2, \quad \text{and} \tag{5.3}$$

$$\int_{B_1} u_i \tau_{ij}^{''(p)} n_j \, dB = 0.$$

With these definitions he shows that $u_p(x')$ can be written as

$$u_p(x') - \int_V u_i^{(p)} F_i dV + \int_{B_1} u_i \tau_{ij}^{(p)} n_j dB - \int_{B_2} u_i^{(p)} \tau_{ij} n_j dB$$

$$- \int_V u_i'^{(p)} F_i dV = S\cdot(S'^{(p)} - S''^{(p)}), \qquad (5.4)$$

where

$$S \leftrightarrow \tau_{ij}, S'^{(p)} \leftrightarrow \tau_{ij}'^{(p)}, \quad \text{and} \quad S''^{(p)} \leftrightarrow \tau_{ij}''^{(p)}.$$

The scalar product on the right is bounded by the relations given in Chapter II by equations (2.9), so $u_p(x')$ is bounded.

Now an extension of Synge's expression yields a most important result which is advanced by the following theorem.

<u>Theorem 7</u>. The residual problem solution space generates a constant which when added to the function space spanned by the integrals,

$$\int_{B_1} u_i \tau_{ij}^{(p)} n_j dB$$

and

$$\int_{B_2} u_i^{(p)} \tau_{ij} n_j dB,$$

completely spans the solution space of the given anisotropic problem.

Proof. Returning to equations (5.2) and (5.3) a variation is proposed to the definitions of $u_i^{'(p)}$ and $\tau_{ij}^{''(p)}$. Consider

$$u_i^{'(p)} = u_i^{(p)} + a^{(p)} v_i \quad \text{and}$$

$$\tau_{ij}^{''(p)} = \tau_{ij}^{(p)} + b^{(p)} R_{ij}, \tag{5.5}$$

where $a^{(p)}$ and $b^{(p)}$ are functions of the coordinates x'. Then the requirements on $u_i^{'(p)}$ and $\tau_{ij}^{''(p)}$ result in the following conditions to be satisfied by v_i and R_{ij}:

$$v_i = 0 \quad \text{on} \quad B_1 \quad \text{and}$$

$$a^{(p)}\int_{B_2} v_i \tau_{ij} n_j dB = - \int_{B_2} u_i^{(p)} \tau_{ij} n_j dB; \tag{5.6}$$

$$R_{ij,j} = 0 \quad \text{in} \quad V,$$

$$R_{ij} n_j = 0 \quad \text{on} \quad B_2, \quad \text{and} \tag{5.7}$$

$$b^{(p)}\int_{B_1} u_i R_{ij} n_j dB = - \int_{B_1} u_i \tau_{ij}^{(p)} n_j dB.$$

The functions $a^{(p)}$ and $b^{(p)}$ can be determined by the above after v_i and R_{ij} are chosen.

By examining (5.6) and (5.7) it can be seen that the displacements v_i can be obtained from the homogeneous kinematically admissible vectors, T_n''. Likewise the homogeneous statically admissible vectors, T_n', provide appropriate candidates for R_{ij}.

Now consider the integrals from (5.1) which were to be altered. The first integral can be written as

$$\int_{B_1} u_i^{(p)} \tau_{ij} n_j dB = \int_{B_1} u_i'^{(p)} \tau_{ij} n_j dB + \int_{B_2} u_i'^{(p)} \tau_{ij} n_j dB$$

$$= \int_B u_i'^{(p)} \tau_{ij} n_j dB$$

$$= \int_B \left(u_i^{(p)} + a^{(p)} v_i \right) \tau_{ij} n_j dB, \qquad (5.8)$$

by using (5.2) and (5.5). Similarly the second integral becomes

$$\int_{B_2} u_i \tau_{ij}^{(p)} n_j dB = \int_{B_1} u_i \tau_{ij}''^{(p)} n_j dB + \int_{B_2} u_i \tau_{ij}''^{(p)} n_j dB$$

$$= \int_B u_i \tau_{ij}''^{(p)} n_j dB$$

$$= \int_B u_i \left(\tau_{ij}^{(p)} + b^{(p)} R_{ij} \right) n_j dB, \qquad (5.9)$$

by (5.3) and (5.5).

The integrals can now be converted into volume integrals by Green's theorem. Thus (5.8) and (5.9) become

$$\int_B \left(u_i^{(p)}+a^{(p)}v_i\right)\tau_{ij}n_j dB = \int_V \left[\left(u_i^{(p)}+a^{(p)}v_i\right)\tau_{ij}\right]_{,j} dV, \tag{5.10}$$

and

$$\int_B u_i\left(\tau_{ij}^{(p)}+b^{(p)}R_{ij}\right)n_j dB = \int_V \left[u_i\left(\tau_{ij}^{(p)}+b^{(p)}R_{ij}\right)\right]_{,j} dV. \tag{5.11}$$

Performing the indicated differentiation yields

$$\int_V \left[\left(u_i^{(p)}+a^{(p)}v_i\right)\tau_{ij}\right]_{,j} dV = \int_V \left(u_{i,j}^{(p)}+a^{(p)}v_{i,j}\right)\tau_{ij}\, dV$$

$$+ \int_V \left(u_i^{(p)}+a^{(p)}v_i\right)\tau_{ij,j}\, dV, \tag{5.12}$$

and

$$\int_V \left[u_i\left(\tau_{ij}^{(p)}+b^{(p)}R_{ij}\right)\right]_{,j} dV = \int_V u_{i,j}\left(\tau_{ij}^{(p)}+b^{(p)}R_{ij}\right)dV$$

$$+ \int_V u_i\left(\tau_{ij,j}^{(p)}+b^{(p)}R_{ij,j}\right)dV. \tag{5.13}$$

By using the strain displacement relations and the indicated correspondences for v_i and R_{ij} the first integral of each of (5.12) and (5.13) becomes

$$\int_V \left(u_{i,j}^{(p)}+a^{(p)}v_{i,j}\right)\tau_{ij}\ dV = \int_V \left(e_{ij}^{(p)}+a^{(p)}e''_{nij}\right)\tau_{ij}\ dV, \tag{5.14}$$

and

$$\int_V u_{i,j}\left(\tau_{ij}^{(p)}+b^{(p)}R_{ij}\right)dV = \int_V e_{ij}\left(\tau_{ij}^{(p)}+b^{(p)}\tau'_{nij}\right)dV, \tag{5.15}$$

where both are in the form of the scalar product. Making the associations $S \leftrightarrow \tau_{ij}$, $S' \leftrightarrow \tau'_{nij}$, $S'' \leftrightarrow \tau''_{nij}$, and $S^{(p)} \leftrightarrow \tau_{ij}^{(p)}$, the expressions (5.12) and (5.13) can be written as

$$\int_V \left[\left(u_i^{(p)} + a^{(p)}v_i\right)\tau_{ij}\right]_{,j} dV = S^{(p)}\cdot S + a^{(p)}S''\cdot S$$

$$+ \int_V \left(u_i^{(p)}+a^{(p)}v_i\right)\tau_{ij,j}\ dV,$$

(5.16)

and

$$\int_V \left[u_i\left(\tau_{ij}^{(p)}+b^{(p)}R_{ij}\right)\right]_{,j} dV = S\cdot S^{(p)} + b^{(p)}S\cdot S'$$

$$+ \int_V u_i\left(\tau_{ij,j}^{(p)}+b^{(p)}R_{ij,j}\right)dV.$$

(5.17)

Now using the fact that $\tau_{ij,j} = - F_i$, $\tau_{ij,j}^{(p)} = 0$, and $R_{ij,j} = 0$ in (5.16) and (5.17) and using the relationships (5.8) through (5.11) in equation (5.1), $u_p(x')$ becomes

$$u_p(x') = \int_V u_i^{(p)} F_i \, dV - \int_{B_1} u_i \tau_{ij}^{(p)} n_j \, dB + \int_{B_2} u_i^{(p)} \tau_{ij} n_j \, dB$$

$$+ S^{(p)} \cdot S + a^{(p)} S'' \cdot S - \int_V u_i^{(p)} F_i \, dV$$

$$- a^{(p)} \int_V v_i F_i dV - S \cdot S^{(p)} - b^{(p)} S \cdot S'. \qquad (5.18)$$

Adding like terms gives

$$u_p(x') = - \int_{B_1} u_i \tau_{ij}^{(p)} n_j \, dB + \int_{B_2} u_i^{(p)} \tau_{ij} n_j \, dB$$

$$- a^{(p)} \int_V v_i F_i \, dV + S \cdot \left(a^{(p)} S'' - b^{(p)} S' \right), \qquad (5.19)$$

which is similar to Synge's result (5.4) except that v_i, $a^{(p)} S''$, and $b^{(p)} S'$ are explicitly known from the residual problem procedure.

The importance of (5.19) is that the functions represented by the terms on the right-hand side of the equation form a function space which spans the solution space for the displacement of the anisotropic problem.

Now Synge also obtains an expression for $e_{pq}(x')$ the strain components at a point, which is analogous to (5.1).[2]

$$e_{pq}(x') = - \int_V u_i^{(pq)} F_i \, dV + \int_{B_1} u_i \tau_{ij}^{(pq)} n_j \, dB$$

$$- \int_{B_2} u_i^{(pq)} \tau_{ij} n_j \, dB + \int_{B_2} u_i \tau_{ij}^{(pq)} n_j \, dB$$

$$- \int_{B_1} u_i^{(pq)} \tau_{ij} n_j \, dB, \qquad (5.20)$$

where

$$u_i^{(pq)} = - \tfrac{1}{2}\left(\frac{\partial u_i^{(p)}}{\partial x_q'} + \frac{\partial u_i^{(q)}}{\partial x_p'}\right) = \tfrac{1}{2}\left(\frac{\partial u_i^{(p)}}{\partial x_q} + \frac{\partial u_i^{(q)}}{\partial x_p}\right) = u_i^{(qp)},$$

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[2] Ibid., pp. 352-53.

and

$$e_{ij}^{(pq)} = e_{ij}^{(qp)} = \tfrac{1}{2}\left(\frac{\partial e_{ij}^{(p)}}{\partial x_q} + \frac{\partial e_{ij}^{(q)}}{\partial x_p}\right) = -\tfrac{1}{2}\left(\frac{\partial e_{ij}^{(p)}}{\partial x_q'} + \frac{\partial e_{ij}^{(q)}}{\partial x_p'}\right).$$

Similar to the procedure for the displacement equation, let $u_i'^{(pq)}$ and $\tau_{ij}''^{(pq)}$ be such that

$$u_i'^{(pq)} = u_i^{(pq)} \quad \text{on} \quad B_1,$$

$$\int_{B_2} u_i'^{(pq)} \tau_{ij} n_j \, dB = 0; \tag{5.21}$$

$$\tau_{ij,j}''^{(pq)} = 0 \quad \text{in} \quad V, \tag{5.22}$$

$$\tau_{ij}''^{(pq)} n_j = \tau_{ij}^{(pq)} n_j \quad \text{on} \quad B_2, \quad \text{and} \tag{5.22}$$

$$\int_{B_1} u_i \tau_{ij}''^{(pq)} n_j \, dB = 0.$$

And again let

$$u_i'^{(pq)} = u_i^{(pq)} + \ell^{(pq)} v_i$$

and (5.23)

$$\tau_{ij}''^{(pq)} = \tau_{ij}^{(pq)} + m^{(pq)} R_{ij}.$$

It follows that $\ell^{(pq)}$ and $m^{(pq)}$ are determined from

$$\ell^{(pq)} \int_{B_2} v_i \tau_{ij} n_j \, dB = - \int_{B_2} u_i^{(pq)} \tau_{ij} n_j \, dB$$

and (5.24)

$$m^{(pq)} \int_{B_1} u_i R_{ij} n_j \, dB = - \int_{B_1} u_i \tau_{ij}^{(pq)} n_j \, dB.$$

Exactly the same procedure as that for $u_p(x')$ yields

$$e_{pq}(x') = \int_{B_1} u_i \tau_{ij}^{(pq)} n_j \, dB - \int_{B_2} u_i^{(pq)} \tau_{ij} n_j \, dB$$

$$+ \ell^{(pq)} \int_V v_i F_i \, dV - S \cdot (\ell^{(pq)} S'' - m^{(pq)} S'), \tag{5.25}$$

where the correspondences

$$S'' \leftrightarrow \tau''_{nij},$$

$$S' \leftrightarrow \tau'_{nij}, \quad \text{and}$$

$$S \leftrightarrow \tau_{ij},$$

have been used, and v_i are kinematically admissible displacement components derived from τ''_{nij}. An expression for $\tau_{pq}(x')$ follows directly from (5.25) by using Hooke's law.

Again (5.25) is important since it indicates that the functions on the right completely span the solution space for the strains (and stresses) of the anisotropic problem. Furthermore these functions can be derived from the residual problem function space and the function space spanned by Green's fundamental solution.

Based on the previous discussion the following theorem can be stated.

<u>Theorem 8</u>. The displacements $u_p(x')$, strains $e_{pq}(x')$, and stresses $\tau_{pq}(x')$, at a point can be explicitly bounded in terms of functions which completely span the function space of $u_p(x')$, $e_{pq}(x')$, and $\tau_{pq}(x')$, respectively.

The proof of theorem 8 is immediate from the discussion following equations (5.19) and (5.25).

Accomplishments and Prospective Research

The research reported upon herein has provided a method of obtaining approximations to the solutions of anisotropic problems. The technique involves the use of Synge's hypercircle method in conjunction with the residual problem technique which has been developed in the present investigation. The residual problem technique is initiated by the solution of an isotropic problem which corresponds to the anisotropic problem. In addition the residual problem technique has been shown to yield exact solutions of the anisotropic problem in some cases. Finally Synge's expression for point bounds has been extended to facilitate its usage.

A few areas for prospective research are apparent. The expressions for point bounds developed by Synge and extended by this investigation involve, the Green's fundamental solution. Since this has only been determined in terms of simple functions for isotropy and transverse-isotropy, an investigation of approximating the integrals which involve the Green's fundamental solution is suggested. Of course an alternative approach is to completely reconsider the question of point bounds by some other technique. A suggestion for the latter might be the paper by Stallybrass.[4]

A final proposal for study would be to consider approximation of the Green's fundamental solution for anisotropic bodies in terms of that for isotropic bodies or even transversally-isotropic bodies since the latter has been determined explicitly by Kröner.[5]

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[4] M. P. Stallybrass, "On a Pointwise Variational Principle of Elasticity and Mathematical Physics," Stanford Research Institute, Menlo Park, Calif., AFCRL-63-784, November, 1963.

[5] E. Kroner, "Das Fundamentalintegral der anisotropen elastischen Differentialgleichungen," Zeitschrift fur Physik, Bd. 136, S. 402-410 (1953).

APPENDIX A

MINIMIZATION OF INITIAL HYPERSPHERE RADIUS

An example is given here of a straightforward minimization of the radius of the initial hypersphere with respect to the isotropic material constants, ν and E. The example given is the initial hypersphere for the bending of a transversally isotropic circular beam by a transverse force, the solution of which is given in Appendix C. It is presented here to demonstrate the difficulties associated with such a procedure.

The radius of the initial hypersphere is

$$R^2 = \tfrac{1}{4}\left(S_o'-S_o''\right)^2$$

$$= \tfrac{1}{4}\int_V \left(\tau_{oij}'-\tau_{oij}''\right)\left(e_{oij}'-e_{oij}''\right) dV. \qquad (A.1)$$

Recall that

$$e''_{oij} = - \frac{\nu_o}{E_o} \delta_{ij}\tau'_{oKK} + \frac{(1+\nu_o)}{E_o} \tau'_{oij},$$

$$e'_{oij} = C_{ijk\ell}\tau'_{ok\ell},$$

and (A.2)

$$\tau''_{oij} = \alpha_{ijk\ell}e''_{ok\ell}.$$

So

$$R^2 = \tfrac{1}{4}\int_V \left[\tau'_{oij} - \alpha_{ijk\ell}\left(- \frac{\nu_o}{E_o} \delta_{k\ell}\tau'_{oKK} + \frac{(1+\nu_o)}{E_o} \tau'_{o\kappa\ell}\right)\right]$$

$$\left[C_{ijrs}\tau'_{ors} - \left(- \frac{\nu_o}{E_o} \delta_{ij}\tau'_{oKK} + \frac{(1+\nu_o)}{E_o} \tau'_{oij}\right)\right] dV$$

(A.3)

In the problem under consideration $\tau'_{o11} = \tau'_{o22} = \tau'_{o12} = 0$. Using this and the form of $C_{ijk\ell}$, for transverse isotropy, R^2 can be written as

$$R^2 = \tfrac{1}{4}\left\{\left[C_{33} - \frac{2}{E_o} + \frac{1}{E_o^2}\left(2[\alpha_{11}+\alpha_{12}]\nu_o^2 - 4\alpha_{13}\nu_o + \alpha_{33}\right)\right]\int_V \tau'^2_{o33}\, dV\right.$$

$$+ 2\left[C_{55} + \frac{1}{E_o^2}(1+\nu_o)^2\,\alpha_{55} - \frac{2}{E_o}(1+\nu_o)\right]\int_V$$

$$\left[\tau'^2_{o13} + \tau'^2_{023}\right] dV. \qquad (A.4)$$

The isotropic solutions are

$$\tau'_{o33} = -\frac{4W}{\pi a^4}(\ell - z)x,$$

$$\tau'_{o13} = \frac{3 + 2\nu_o}{1 + \nu_o}\,\frac{W}{2\pi a^2}\left(1 - \frac{x^2}{a^2} - \frac{1 - 2\nu_o}{3 + 2\nu_o}\,\frac{y^2}{a^2}\right), \quad (A.5)$$

$$\tau'_{o23} = -\frac{1 + 2\nu_o}{1 + \nu_o}\,\frac{W}{\pi a^4}\,xy.$$

The required integrals are

$$\int_V \tau'^2_{o33}\, dV = \frac{4}{3}\,\frac{W^2 \ell^3}{\pi a^4} ,$$

$$\int_V \tau'^2_{o13}\, dV = \frac{28\nu_o^2 + 52\nu_o + 27}{24(1+\nu_o)^2}\,\frac{W^2 \ell}{\pi a^2} , \qquad (A.6)$$

$$\int_V \tau'^2_{o23}\, dV = \frac{(1+2\nu_o)^2}{24(1+\nu_o)^2}\,\frac{W^2 \ell}{\pi a^2} .$$

Let $n_o = 1 + \nu_o$ and $K_o = \frac{1}{E_o}$. The expression for R^2 becomes

$$R^2 = \frac{1}{12}\,\frac{W^2 \ell}{\pi a^2} \left\{ 4\,\frac{\ell^2}{a^2} \left[C_{33} - 2K_o + K_o^2 \left(2\alpha'' n_o^2 - 4\alpha' n_o + \bar{\alpha} \right) \right] + \frac{8n_o^2 - 2n_o + 1}{n_o^2} \left[C_{55} - 2K_o n_o + K_o^2 n_o^2 \alpha_{55} \right] \right\} , \qquad (A.7)$$

where

$$\alpha'' = \alpha_{11} + \alpha_{12}$$

$$\alpha' = \alpha_{11} + \alpha_{12} + \alpha_{13}$$

$$\bar{\alpha} = 2(\alpha_{11}+\alpha_{12}+2\alpha_{13}) + \alpha_{33}.$$

Now taking $\frac{\partial R^2}{\partial n_o}$ and equating the result to zero gives

$$8 \frac{\ell^2}{a^2} K_o^2 \left[(n_o-1)(\alpha_{11}+\alpha_{12}) - \alpha_{13}\right] n_o^3 \cdot$$

$$+ (n_o-1)\left(C_{55}-2K_o n_o+K_o^2 n_o^2 \alpha_{55}\right)$$

$$+ \left(8n_o^2-2n_o+1\right)\left(K_o^2 n_o^2 \alpha_{55}-K_o n_o\right) = 0. \quad \text{(A.8)}$$

Similarly for $\frac{\partial R^2}{\partial K_o} = 0$,

$$16 \frac{\ell^2}{a^2} \left[2K_o\left(2[\alpha_{11}+\alpha_{12}][n_o-1]^2 - 4\alpha_{13}[n_o-1] + \alpha_{33}\right) - 2\right] n_o^2$$

$$+ 8\left(8n_o^2-2n_o+1\right)\left(K_o n_o^2 \alpha_{55}-n_o\right) = 0 . \quad \text{(A.9)}$$

Let $K_o n_o = u_o$, (A.8) and (A.9) become, respectively,

$$u^2\left\{n_o^2\left[8\,\frac{\ell^2}{a^2}\,(\alpha_{11}+\alpha_{12}) + 8\alpha_{55}\right] - n_o\left[8\,\frac{\ell^2}{a^2}\,(\alpha_{11}+\alpha_{12}+\alpha_{13}) + \alpha_{55}\right]\right\} - u\left[8n_o^2 - 1\right] + C_{55}(n_o-1) = 0, \qquad \text{(A.10)}$$

and

$$u\left[4\,\frac{\ell^2}{a^2}\,n_o\left(2[\alpha_{11} + \alpha_{12}][n_o-1]^2 - 4\alpha_{13}[n_o-1] + \alpha_{33}\right) + n_o\alpha_{55}\left(8n_o^2-2n_o+1\right)\right] - 4\,\frac{\ell^2}{a^2}\,n_o^2 - n_o\left(8n_o^2-2n_o+1\right) = 0. \qquad \text{(A.11)}$$

These two equations must be solved for u_o and n_o, once the α's and $\frac{\ell}{a}$ ratio are specified.

APPENDIX B

RESIDUAL PROBLEM EQUATIONS

The complete set of equations for both types of residual problem approaches are given below. The first set of equations pertain to the residual problem with solution $\tau_{K_n ij}$, which results from the subtraction of the kinematically admissible stress vector for the preceding problem. The second set of equations are for the residual problem with solution $\tau_{S_n ij}$ resulting from the removal of the statically admissible stress vector for the preceding problem.

Residual Problem Based in L"

Equilibrium Equations

$$\tau_{K_n ij,j} + \sum_{m=0}^{n-1} \tau''_{K_m ij,j} + \tau''_{oij,j} + X_i = 0 \qquad \text{(B.1)}$$

Stress Boundary Conditions

$$\left[\tau_{K_n ij} + \sum_{m=0}^{n-1} \tau''_{K_m ij} + \tau''_{oij}\right] n_j = \overline{X}_i \quad \text{on} \quad B_2 \qquad \text{(B.2)}$$

Displacement Boundary Conditions

$$u_{K_n i} = 0 \quad \text{on} \quad B_1 \tag{B.3}$$

Compatibility Equations

$$C\left(e_{K_n}\right) = 0 \tag{B.4}$$

Strain Displacement Relations

$$e_{K_n ij} = \tfrac{1}{2}\left(u_{K_n i,j} + u_{K_n j,i}\right) \tag{B.5}$$

Constitutive Relations

$$e_{K_n ij} = C_{ijk\ell}\tau_{K_n k\ell} \tag{B.6}$$

Residual Problem Based in L'

Equilibrium Equations

$$\tau_{S_n ij,j} = 0 \tag{B.7}$$

Stress Boundary Conditions

$$\tau_{S_n ij} n_j = 0 \quad \text{on} \quad B_2 \tag{B.8}$$

Displacement Boundary Conditions

$$u_{S_n i} + \sum_{m=0}^{n-1} u'_{S_m i} + u'_{oi} = u_i^* \quad \text{on} \quad B_1 \tag{B.9}$$

Compatibility Equations

$$C\left(e_{S_n}\right) + \sum_{m=0}^{n-1} C\left(e'_{S_m}\right) + C\left(e'_o\right) = 0 \tag{B.10}$$

Strain Displacement Relations

$$e_{S_n ij} = \tfrac{1}{2}\left(u_{S_n i,j} + u_{S_n j,i}\right) \tag{B.11}$$

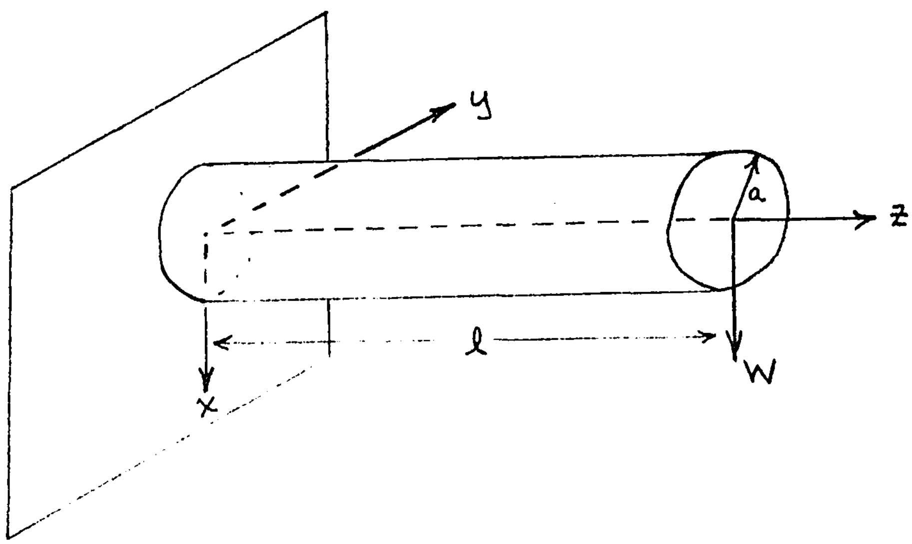
Constitutive Relations

$$e_{S_n ij} = C_{ijk\ell} \tau_{S_n ij} \tag{B.12}$$

APPENDIX C

FLEXURE OF A CANTILEVER BY A TERMINAL LOAD

The solution for the flexure of a circular cylindrical cantilever beam by a transverse load on the free end is obtained for transverse-isotropy and orthotropy as an example of how the residual problem technique can produce exact solutions. The problem is shown in the figure below.



Transverse Isotropy

For the transversally isotropic beam the plane of isotropy is assumed to be perpendicular to the z-axis. The equations for this particular problem are given by equations (C.1) through (C.6).

Equilibrium equations

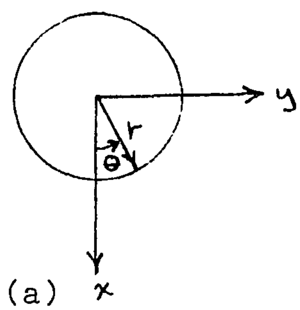
$$\frac{\partial \tau_{11}}{\partial x} + \frac{\partial \tau_{12}}{\partial y} + \frac{\partial \tau_{13}}{\partial z} = 0 \qquad \text{(a)}$$

$$\frac{\partial \tau_{12}}{\partial x} + \frac{\partial \tau_{22}}{\partial y} + \frac{\partial \tau_{33}}{\partial z} = 0 \qquad \text{(b)} \qquad \text{(C.1)}$$

$$\frac{\partial \tau_{13}}{\partial x} + \frac{\partial \tau_{23}}{\partial y} + \frac{\partial \tau_{33}}{\partial z} = 0 \qquad \text{(c)}$$

Stress boundary conditions

Lateral surface ($n_3 = 0$)



$$\tau_{11} \cos\theta + \tau_{12} \sin\theta = 0 \qquad \text{(a)}$$

$$\tau_{12} \cos\theta + \tau_{22} \sin\theta = 0 \qquad \text{(b)}$$

$$\tau_{13} \cos\theta + \tau_{23} \sin\theta = 0 \qquad \text{(c)}$$

Face $(z = \ell)$

$$\int_A \tau_{13}\, dA = W \qquad \text{(d)}$$

$$\int_A \tau_{23}\, dA = 0 \qquad \text{(e)}$$

(C.2)

$$\int_A \tau_{33}\, dA = 0 \qquad \text{(f)}$$

$$\int_A (\tau_{13} y - \tau_{23} x)\, dA = 0 \qquad \text{(g)}$$

Face $(z = 0)$

$$\int_A \tau_{33} x\, dA = W\ell \qquad \text{(h)}$$

Displacement boundary conditions

At the origin $(x,y,z) = (0,0,0)$

$$u = v = w = 0 \qquad \text{(a)}$$

(C.3)

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 \qquad \text{(b)}$$

Strain-displacement relations

$$e_{11} = \frac{\partial u}{\partial x} \quad \text{(a)} \qquad e_{12} = \tfrac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \quad \text{(e)}$$

$$e_{22} = \frac{\partial v}{\partial y} \quad \text{(b)} \qquad e_{13} = \tfrac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \quad \text{(f)} \qquad \text{(C.4)}$$

$$e_{33} = \frac{\partial w}{\partial z} \quad \text{(c)} \qquad e_{23} = \tfrac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \quad \text{(g)}$$

Constitutive relations

$$\begin{Bmatrix} e_{11} \\ e_{22} \\ e_{33} \\ e_{12} \\ e_{13} \\ e_{23} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{13} & 0 & 0 & 0 \\ C_{13} & C_{13} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(C_{11}-C_{12}) & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{55} \end{bmatrix} \begin{Bmatrix} \tau_{11} \\ \tau_{22} \\ \tau_{33} \\ \tau_{12} \\ \tau_{13} \\ \tau_{23} \end{Bmatrix}$$

(C.5)

Compatibility conditions

$$\frac{\partial^2 e_{11}}{\partial y^2} + \frac{\partial^2 e_{22}}{\partial x^2} = \frac{\partial^2 e_{12}}{\partial x \partial y} \qquad \text{(a)}$$

$$\frac{\partial^2 e_{22}}{\partial z^2} + \frac{\partial^2 e_{33}}{\partial y^2} = \frac{\partial^2 e_{23}}{\partial y \partial z} \qquad \text{(b)}$$

$$\frac{\partial^2 e_{33}}{\partial x^2} + \frac{\partial^2 e_{11}}{\partial z^2} = \frac{\partial^2 e_{13}}{\partial x \partial z} \qquad \text{(c)}$$

(C.6)

$$2\,\frac{\partial^2 e_{11}}{\partial y \partial z} = \frac{\partial}{\partial x}\left(-\frac{\partial e_{23}}{\partial x} + \frac{\partial e_{13}}{\partial y} + \frac{\partial e_{12}}{\partial z}\right) \qquad \text{(d)}$$

$$2\,\frac{\partial^2 e_{22}}{\partial x \partial z} = \frac{\partial}{\partial y}\left(\frac{\partial e_{23}}{\partial x} - \frac{\partial e_{13}}{\partial y} + \frac{\partial e_{12}}{\partial z}\right) \qquad \text{(e)}$$

$$2\,\frac{\partial^2 e_{33}}{\partial x \partial y} = \frac{\partial}{\partial z}\left(\frac{\partial e_{23}}{\partial x} + \frac{\partial e_{13}}{\partial y} - \frac{\partial e_{12}}{\partial z}\right) \qquad \text{(f)}$$

The isotropic solution for this problem is given by Sokolnikoff[1] as

$$\tau'_{o11} = \tau'_{o22} = \tau'_{o12} = 0,$$

$$\tau'_{o33} = -\frac{4W}{\pi a^4}(\ell - z)x,$$

$$\tau'_{o13} = \frac{3 + 2\nu_o}{1 + \nu_o}\frac{W}{2\pi a^2}\left(1 - \frac{x^2}{a^2} - \frac{1 - 2\nu_o}{3 + 2\nu_o}\frac{y^2}{a^2}\right), \qquad (C.7)$$

$$\tau'_{o23} = -\frac{1 + 2\nu_o}{1 + \nu_o}\frac{W}{\pi a^4}xy.$$

From τ'_{oij} the isotropic strain components are

$$e''_{o11} = -\frac{\nu_o}{E_o}\tau'_{o33},$$

$$e''_{o22} = -\frac{\nu_o}{E_o}\tau'_{o33},$$

$$e''_{o33} = \frac{1}{E_o}\tau'_{o33}, \qquad \text{(C.8) Cont'd}$$

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[1] Sokolnikoff, op. cit., p. 213.

$$e''_{o12} = 0,$$

$$e''_{o13} = \frac{2(1+\nu_o)}{E_o} \tau'_{o13}, \qquad (C.8)$$

$$e''_{o23} = \frac{2(1+\nu_o)}{E_o} \tau'_{o23}.$$

Now by using the constitutive relations (C.5), the components of $\bar{\gamma}_{oij}$ are

$$\bar{\gamma}_{o11} = \left(C_{13} + \frac{\nu_o}{E_o}\right) \tau'_{o33},$$

$$\bar{\gamma}_{o22} = \left(C_{13} + \frac{\nu_o}{E_o}\right) \tau'_{o33},$$

$$\bar{\gamma}_{o33} = \left(C_{33} - \frac{1}{E_o}\right) \tau'_{o33},$$

$$\bar{\gamma}_{o12} = 0, \qquad (C.9)$$

Cont'd

$$\bar{\gamma}_{o13} = \left(C_{55} - \frac{2(1+\nu_o)}{E_o}\right)\tau'_{o13},$$

$$\bar{\gamma}_{o23} = \left(C_{55} - \frac{2(1+\nu_o)}{E_o}\right)\tau'_{o23}. \qquad (C.9)$$

Choosing

$$\frac{\nu_o}{E_o} = -C_{13} \quad \text{and} \quad \frac{2(1+\nu_o)}{E_o} = C_{55}$$

yields

$$\nu_o = -\frac{2C_{13}}{C_{55} + 2C_{13}} \quad \text{and} \quad E_o = \frac{2}{C_{55} + 2C_{13}} . \qquad (C.10)$$

Thus the preferred isotropic solution becomes

$$\tau'_{o11} = \tau'_{o22} = \tau'_{o12} = 0,$$

$$\tau'_{o33} = -\frac{4W}{\pi a^4}(\ell - z)x,$$

$$\tau'_{o13} = \frac{3C_{55} + 2C_{13}}{C_{55}} \frac{W}{2\pi a^2}\left(1 - \frac{x^2}{a^2} - \frac{C_{55} + 6C_{13}}{3C_{55} + 2C_{13}} \frac{y^2}{a^2}\right), \qquad (c.11)$$

Cont'd

$$\tau'_{o23} = -\frac{C_{55} - 2C_{13}}{C_{55}} \frac{W}{\pi a^4} xy. \qquad (C.11)$$

Using the transversally-isotropic tensor of the moduli, the inverse of (C.5), the kinematically admissible stress component, τ''_{oij}, are

$$\tau''_{o11} = \left[\alpha_{11}C_{13} + \alpha_{12}C_{13} + \alpha_{13}\frac{C_{55} + 2C_{13}}{2}\right]\tau'_{o33} = R_{o13}\tau'_{o33},$$

$$\tau''_{o22} = R_{o13}\tau'_{o33},$$

$$\tau''_{o33} = \left[2\alpha_{13}C_{13} + \alpha_{33}\frac{C_{55} + 2C_{13}}{2}\right]\tau'_{o33} = R_{o33}\tau'_{o33}, \qquad (C.12)$$

$$\tau''_{o12} = 0,$$

$$\tau''_{o13} = \tau'_{o13},$$

$$\tau''_{o23} = \tau'_{o23}.$$

The residual problem is now generated by subtracting τ''_{oij}, and the resulting equations for $\tau_{K_o ij}$ are given below.

Equilibrium equations

$$\frac{\partial \tau_{K_o 11}}{\partial x} + \frac{\partial \tau_{K_o 12}}{\partial y} + \frac{\partial \tau_{K_o 13}}{\partial z} = - \frac{\partial \tau''_{o11}}{\partial x} \qquad \text{(a)}$$

$$\frac{\partial \tau_{K_o 12}}{\partial x} + \frac{\partial \tau_{K_o 22}}{\partial y} + \frac{\partial \tau_{K_o 23}}{\partial z} = 0 \qquad \text{(b)}$$

$$\frac{\partial \tau_{K_o 13}}{\partial x} + \frac{\partial \tau_{K_o 23}}{\partial y} + \frac{\partial \tau_{K_o 33}}{\partial z} = [1 - R_{o33}] \frac{\partial \tau'_{o33}}{\partial z} \qquad \text{(c)}$$

(C.13)

In obtaining these recall that

$$\frac{\partial \tau''_{o13}}{\partial z} = \frac{\partial \tau''_{o23}}{\partial z} = \frac{\partial \tau''_{o22}}{\partial y} = 0,$$

and that

$$\frac{\partial \tau''_{o13}}{\partial x} + \frac{\partial \tau''_{o23}}{\partial y} + \frac{\partial \tau''_{o33}}{\partial z} = \frac{\partial \tau'_{o13}}{\partial x} + \frac{\partial \tau'_{o23}}{\partial y} + R_{o33} \frac{\partial \tau'_{o33}}{\partial z}$$

$$= \frac{\partial \tau'_{o13}}{\partial x} + \frac{\partial \tau'_{o23}}{\partial y} + \frac{\partial \tau'_{o33}}{\partial z} + [R_{o33} - 1] \frac{\partial \tau'_{o33}}{\partial z}$$

$$= [R_{o33} - 1] \frac{\partial \tau'_{o33}}{\partial z} .$$

Stress boundary conditions

Lateral surface ($n_3 = 0$)

$$\tau_{K_o 11} \cos\theta + \tau_{K_o 12} \sin\theta = -\tau''_{o11} \cos\theta \qquad \text{(a)}$$

$$\tau_{K_o 12} \cos\theta + \tau_{K_o 22} \sin\theta = -\tau''_{o22} \sin\theta \qquad \text{(b)}$$

$$\tau_{K_o 13} \cos\theta + \tau_{K_o 23} \sin\theta = 0 \qquad \text{(c)}$$

Face ($z = \ell$)

$$\int_A \tau_{K_o 13}\, dA = 0 \qquad \text{(d)} \quad \text{(c.14)}$$

Cont'd

$$\int_A \tau_{K_o 23} \, dA = 0 \qquad \text{(e)}$$

$$\int_A \tau_{K_o 33} \, dA = 0 \qquad \text{(f)} \qquad \text{(C.14)}$$

$$\int_A \left(\tau_{K_o 13} y - \tau_{K_o 23} x \right) dA = 0 \qquad \text{(g)}$$

Face (z = 0)

$$\int_A \tau_{K_o 33} x \, dA = [1 - R_{o33}] W\ell \qquad \text{(h)}$$

The displacement boundary conditions, strain-displacement relations, constitutive relations, and compatibility conditions are the same as given in (C.3), (C.4), (C.5), and (C.6).

Now a statically admissible solution of the residual problem is

$$\tau'_{K_o 11} = \tau'_{K_o 22} = -R_{o13}\tau'_{o33},$$

$$\tau'_{K_o 33} = (1-R_{o33})\tau'_{o33}, \qquad (C.15)$$

$$\tau'_{K_o 12} = \tau'_{K_o 13} = \tau'_{K_o 23} = 0.$$

As indicated in the residual problem procedure the solution should be checked to see if it satisfies the kinematic relations for the anisotropic problem. The strain components, $e'_{K_o ij}$, computed by (C.5) are

$$e'_{K_o 11} = -\left[C_{11}R_{o13} + C_{12}R_{o13} + C_{13}(R_{o33}-1)\right]\tau'_{o33},$$

$$e'_{K_o 22} = e'_{K_o 11},$$

$$e'_{K_o 33} = -\left[2C_{13}R_{o13} + C_{33}(R_{o33}-1)\right]\tau'_{o33}, \qquad (C.16)$$

$$e'_{K_o 12} = e'_{K_o 13} = e'_{K_o 23} = 0.$$

Now from successive use of (C.5) and its inverse the following identities are obtained:

$$(C_{11}+C_{12})\alpha_{13} + C_{13}\alpha_{33} = 0; \quad \text{and}$$

$$2\alpha_{13}C_{13} + \alpha_{33}C_{33} = 1. \tag{C.17}$$

Thus, referring to (C.12) R_{o13} and R_{o33} can be written as

$$R_{o13} = \alpha_{13}\left(\frac{C_{55} + 2C_{13}}{2} - C_{33}\right)$$

and (C.18)

$$R_{o33} = 1 + \alpha_{33}\left(\frac{C_{55} + 2C_{13}}{2} - C_{33}\right).$$

Using (C.18) in (C.16) gives

$$e'_{K_o11} = - \left[(C_{11}+C_{12})\alpha_{13} + C_{13}C_{33}\right]\left[\frac{C_{55} + 2C_{13}}{2} - C_{33}\right]\tau'_{o33}$$

$$= 0,$$

and (C.19)

$$e'_{K_o33} = -\left[2C_{13}\alpha_{13} + C_{33}\alpha_{33}\right]\left[\frac{C_{55} + 2C_{13}}{2} - C_{33}\right]\tau'_{o33}$$

$$= \left(C_{33} - \frac{C_{55}+2C_{13}}{2}\right)\tau'_{o33}.$$

Hence the components of e'_{K_oij} are

$$e'_{K_o11} = e'_{K_o22} = e'_{K_o12} = e'_{K_o13} = e'_{K_o23} = 0$$

and (C.20)

$$e'_{K_o33} = \left(C_{33} - \frac{C_{55}+2C_{13}}{2}\right)\tau'_{o33}.$$

$$= R_{K_o33}\tau'_{o33}.$$

Checking e'_{K_oij} in the compatibility conditions (C.6) gives

$$\frac{\partial^2 e'_{K_o 33}}{\partial y^2} = 0,$$

$$\frac{\partial^2 e'_{K_o 33}}{\partial x^2} = 0, \qquad (C.21)$$

$$2\,\frac{\partial^2 e'_{K_o 33}}{\partial x \partial y} = 0,$$

which are all satisfied.

The displacements $u'_{K_o i}$ are obtained by using (C.20) in (C.4) and integrating which gives

$$u'_{K_o} = R_{K_o 33}\left[\frac{2W}{\pi a^4}\left(\ell - \frac{z}{3}\right)z^2\right],$$

$$v'_{K_o} = 0, \qquad (C.22)$$

$$w'_{K_o} = -\,R_{K_o 33}\left[\frac{4W}{\pi a^4}\left(\ell - \frac{z}{2}\right)xz\right],$$

and these displacements satisfy (C.3).

Therefore $\tau'_{K_o ij}$ is the solution of the residual transversally-isotropic problem, and the complete solution of the problem, τ_{ij}, is

$$\tau_{ij} = \tau''_{oij} + \tau'_{K_o ij},$$

so

$$\tau_{11} = \tau_{22} = \tau_{12} = 0,$$

$$\tau_{33} = -\frac{4W}{\pi a^4}(\ell - z)x,$$

$$\tau_{13} = \frac{3C_{55} + 2C_{13}}{C_{55}} \frac{W}{2\pi a^2}\left(1 - \frac{x^2}{a^2} - \frac{C_{55}+6C_{13}}{3C_{55}+2C_{13}} \frac{y^2}{a^2}\right), \tag{C.23}$$

$$\tau_{23} = -\frac{C_{55} - 2C_{13}}{C_{55}} \frac{W}{\pi a^4} xy.$$

Orthotropy

In the orthotropic beam the three axes of symmetry are aligned with the coordinate axes. The set of equations which apply are identical to those for transverse isotropy except for the constitutive relations.

Constitutive relations

$$\begin{Bmatrix} e_{11} \\ e_{22} \\ e_{33} \\ e_{12} \\ e_{13} \\ e_{23} \end{Bmatrix} = \begin{Bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{Bmatrix} \begin{Bmatrix} \tau_{11} \\ \tau_{22} \\ \tau_{33} \\ \tau_{12} \\ \tau_{13} \\ \tau_{23} \end{Bmatrix} \tag{C.24}$$

As indicated earlier the initial solution used here will be that just determined for a transversally-isotropic beam, which will be denoted by τ'_{Tij}.

Using (C.23) and (C.5) the kinematically admissible strain components, e''_{Tij}, are

$$e''_{T11} = C_{13}\tau'_{T33},$$

$$e''_{T22} = C_{13}\tau'_{T33},$$

$$e''_{T33} = C_{33}\tau'_{T33},$$

$$e''_{T12} = 0, \tag{C.25}$$

Cont'd

$$e''_{T13} = C_{55}\tau'_{T13},$$
$$e''_{T23} = C_{55}\tau'_{T23}. \qquad (C.25)$$

Now using (C.25) in the inverse of (C.24) gives the kinematically admissible stress, τ''_{Tij}.

$$\tau''_{T11} = (\alpha_{11}C_{13}+\alpha_{12}C_{13}+\alpha_{13}C_{33})\tau'_{T33} = R_{T11}\tau'_{T33}$$

$$\tau''_{T22} = (\alpha_{12}C_{13}+\alpha_{22}C_{13}+\alpha_{23}C_{33})\tau'_{T33} = R_{T22}\tau'_{T33}$$

$$\tau''_{T33} = (\alpha_{13}C_{13}+\alpha_{23}C_{13}+\alpha_{33}C_{33})\tau'_{T33} = R_{T33}\tau'_{T33}$$

$$\tau''_{T12} = 0 \qquad (C.26)$$

$$\tau''_{T13} = \tau'_{T13}$$

$$\tau''_{T23} = \frac{C_{55}}{C_{66}}\,\tau'_{T23}.$$

From successive use of (C.24) and its inverse the following identities are obtained:

$$\alpha_{11}C_{13} + \alpha_{12}C_{33} + \alpha_{13}C_{33} = 0;$$
$$\alpha_{12}C_{13} + \alpha_{22}C_{23} + \alpha_{23}C_{33} = 0; \qquad (C.27)$$

and

$$\alpha_{13}C_{13} + \alpha_{23}C_{23} + \alpha_{33}C_{33} = 1.$$

The use of these identities gives equivalent expressions for the R_T's,

$$R_{T11} = \alpha_{12}(C_{13}-C_{23}),$$
$$R_{T22} = \alpha_{22}(C_{13}-C_{23}), \qquad (C.28)$$
$$R_{T33} = 1 + \alpha_{23}(C_{13}-C_{23}).$$

Thus upon forming the residual problem by subtracting τ''_{Tij}, the equations resulting are given below. Again only equilibrium and stress boundary conditions are given as the others remain the same.

Equilibrium equations

$$\frac{\partial \tau_{K_o11}}{\partial x} + \frac{\partial \tau_{K_o12}}{\partial y} + \frac{\partial \tau_{K_o13}}{\partial z} = - \frac{\partial \tau''_{T11}}{\partial x} \quad \text{(a)}$$

$$\frac{\partial \tau_{K_o12}}{\partial x} + \frac{\partial \tau_{K_o22}}{\partial y} + \frac{\partial \tau_{K_o23}}{\partial z} = 0 \quad \text{(b)}$$

$$\frac{\partial \tau_{K_o13}}{\partial x} + \frac{\partial \tau_{K_o23}}{\partial y} + \frac{\partial \tau_{K_o33}}{\partial z} = - \frac{\partial \tau''_{T13}}{\partial x} - \frac{\partial \tau''_{T23}}{\partial y} - \frac{\partial \tau''_{T33}}{\partial z} \quad \text{(c)}$$

(C.29)

Stress boundary conditions

Lateral surface ($n_3 = 0$)

$$\tau_{K_o11} \cos\theta + \tau_{K_o12} \sin\theta = - \tau''_{T11} \cos\theta \quad \text{(a)}$$

$$\tau_{K_o12} \cos\theta + \tau_{K_o22} \sin\theta = - \tau''_{T22} \sin\theta \quad \text{(b)}$$

$$\tau_{K_o13} \cos\theta + \tau_{K_o23} \sin\theta = - \left[\tau''_{T13} \cos\theta + \tau''_{T23} \sin\theta\right] \quad \text{(c)}$$

Face $(z = \ell)$

$$\int_A \tau_{K_o 13} \, dA = 0 \qquad \text{(d)}$$

$$\int_A \tau_{K_o 23} \, dA = 0 \qquad \text{(e)}$$

$$\int_A \tau_{K_o 33} \, dA = 0 \qquad \text{(f)}$$

$$\text{(C.30)}$$

$$\int_A \left(\tau_{K_o 13} y - \tau_{K_o 23} x\right) dA = 0 \qquad \text{(g)}$$

Face $(z = 0)$

$$\int_A \tau_{K_o 33} x \, dA = (1 - R_{T33}) W\ell . \qquad \text{(h)}$$

Now the equilibrium equations (a) and (c) are rewritten after performing the differentiation of the terms on the right hand sides as

$$\frac{\partial \tau_{K_o11}}{\partial x} + \frac{\partial \tau_{K_o12}}{\partial y} + \frac{\partial \tau_{K_o13}}{\partial z} = \alpha_{12}(C_{13}-C_{23})\ \frac{4W}{\pi a^4}\ (\ell-z) \qquad \text{(a)}$$

$$\frac{\partial \tau_{K_o13}}{\partial x} + \frac{\partial \tau_{K_o23}}{\partial y} + \frac{\partial \tau_{K_o33}}{\partial z} = L\ \frac{W}{\pi a^4}\ x, \qquad \text{(c)}$$

where

$$L = \left\{ \frac{C_{55}(3C_{66}+C_{55}) + 2C_{13}(C_{66}-C_{55}) - 4C_{55}C_{66}R_{T33}}{C_{55}C_{66}} \right\}.$$

A potential solution, τ_{K_oij}, of the following form is assumed:

$$\tau_{K_o11} = \alpha_{12}(C_{23}-C_{13})\tau'_{T33};$$

$$\tau_{K_o22} = \alpha_{22}(C_{23}-C_{13})\tau'_{T33};$$

$$\tau_{K_o33} = \alpha_{23}(C_{23}-C_{13})\tau'_{T33};$$

$$\tau_{K_o12} = 0; \qquad \text{(C.31)}$$

Cont'd

$$\tau_{K_o13} = M_1 \frac{W}{\pi a^2}\left[1 - \frac{x^2}{a^2} - M_2 \frac{y^2}{a^2}\right]; \quad \text{and}$$

$$\tau_{K_o23} = M_3 \frac{W}{\pi a^4} xy,$$

where M_1, M_2, and M_3 will be determined by the stress boundary conditions and equilibrium equation (c), displacement boundary conditions, and compatibility relations.

The boundary condition (C.30d) yields that $M_2 = 3$, and equilibrium equation (c) gives

$$- 2M_1 + M_3 + 4(1-R_{T33}) = L,$$

or

$$M_3 - 2M_1 = \frac{C_{55}(3C_{66}+C_{55}) + 2C_{13}(C_{66}-C_{55}) - 4C_{55}C_{66}}{C_{55}C_{66}}. \quad (C.32)$$

All the other static conditions are satisfied so another condition can be imposed since one of M_1 or M_3 remains arbitrary.

Computing the orthotropic strains by (C.24) gives

$$e_{K_o11} = (C_{11}\alpha_{12}+C_{12}\alpha_{22}+C_{13}\alpha_{23})(C_{23}-C_{13})\tau'_{T33},$$

$$e_{K_o22} = (C_{12}\alpha_{12}+C_{22}\alpha_{22}+C_{23}\alpha_{23})(C_{23}-C_{13})\tau'_{T33},$$

$$e_{K_o33} = (C_{13}\alpha_{12}+C_{23}\alpha_{22}+C_{33}\alpha_{23})(C_{23}-C_{13})\tau'_{T33},$$

$$e_{K_o12} = 0, \qquad (C.33)$$

$$e_{K_o13} = C_{55}\ M_1\ \frac{W}{\pi a^2}\left[1 - \frac{x^2}{a^2} - 3\ \frac{y^2}{a^2}\right],$$

$$e_{K_o23} = C_{66}\ M_3\ \frac{W}{\pi a^4}\ xy.$$

Identities analogous to (C.27) are

$$C_{11}\alpha_{12} + C_{12}\alpha_{22} + C_{13}\alpha_{23} = 0,$$

$$C_{13}\alpha_{12} + C_{23}\alpha_{22} + C_{33}\alpha_{23} = 0, \quad \text{and} \qquad (C.34)$$

$$C_{12}\alpha_{12} + C_{22}\alpha_{22} + C_{23}\alpha_{23} = 1.$$

Hence

$$e_{K_o11} = e_{K_o33} = e_{K_o12} = 0,$$

and (C.35)

$$e_{K_o22} = (C_{23}-C_{13})\tau_{T33}.$$

The only unsatisfied compatibility condition is (e).

$$8(C_{23}-C_{13}) = C_{66}M_3 + 6C_{55}\, M_1 \qquad \text{(C.36)}$$

Solving (C.32) and (C.36) simultaneously for M_1 and M_3 gives

$$M_1 = \frac{8C_{55}C_{23} - 2C_{13}(3C_{55}+C_{66}) - C_{55}(C_{55}-C_{66})}{2C_{55}(3C_{55}+C_{66})}$$

and

$$M_3 = \frac{3C_{55}(C_{55}-C_{66}) - 2C_{13}(3C_{55}+C_{66}) + 8C_{66}C_{23}}{C_{66}(3C_{55}+C_{66})} .$$

Integrating the strain-displacement relations gives

$$u_{K_o} = -\frac{\partial W_o}{\partial x} z + 2C_{55} M_1 \frac{W}{\pi a^2}\left[1 - \frac{x^2}{a^2} - 3\frac{y^2}{a^2}\right] z + U_o(x,y),$$

$$v_{K_o} = -\frac{\partial W_o}{\partial y} z + 2C_{66} M_3 \frac{W}{\pi a^4} xyz + V_o(x,y), \qquad (C.38)$$

$$w_{K_o} = W_o(x,y),$$

which obviously will satisfy the displacement boundary conditions if U_o, V_o, and W_o are chosen properly.

Thus $\tau_{K_o ij}$ as given by (C.31) is the solution to the residual problem. Hence $\tau_{ij} = \tau''_{Tij} + \tau'_{K_o ij}$, and

$$\tau_{11} = \tau_{22} = \tau_{12} = 0,$$

$$\tau_{33} = -\frac{4W}{\pi a^4}(\ell - z)x,$$

$$(C.39)$$

$$\tau_{13} = \frac{2C_{55} + C_{66} + 2C_{23}}{3C_{55} + C_{66}} \frac{2W}{\pi a^2}\left[1 - \frac{x^2}{a^2} - \frac{C_{66} + 6C_{23}}{2C_{55} + C_{66} + 2C_{23}} \frac{y^2}{a^2}\right],$$

$$\tau_{23} = -\frac{(C_{55} - 2C_{23})}{3C_{55} + C_{66}} \frac{4W}{\pi a^4} xy.$$

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