71-906

PRENDERGAST, Francis X., 1925-
EXTENSION AND APPLICATIONS OF THE
SEDOV-BERDICHEVSKII VARIATIONAL PRINCIPLE.

Newark College of Engineering, D.Eng.Sc., 1970
Engineering, mechanics

University Microfilms, A XEROX Company, Ann Arbor, Michigan

EXTENSION AND APPLICATIONS OF THE SEDOV-BERDICHEVSKII VARIATIONAL PRINCIPLE

BY

FRANCIS X. PRENDERGAST

A DISSERTATION

PRESENTED IN PARTIAL FULFILLMENT OF

THE REQUIREMENTS FOR THE DEGREE

OF

DOCTOR OF ENGINEERING SCIENCE

AT

NEWARK COLLEGE OF ENGINEERING

Newark, New Jersey
1970

APPROVAL OF DISSERTATION

EXTENSION AND APPLICATIONS OF THE

SEDOV-BERDICHEVSKII VARIATIONAL PRINCIPLE

BY

FRANCIS X. PRENDERGAST

FOR

DEPARTMENT OF MECHANICAL ENGINEERING

NEWARK COLLEGE OF ENGINEERING

BY

FACULTY COMMITTEE

APPROVED: ________________________, CHAIRMAN

____

____

____

____

NEWARK, NEW JERSEY

1970

ABSTRACT

The Sedov-Berdichevskii variational principle is extended, and this extended principle is employed in the construction of a shell theory.

The extension of the principle is accomplished by the use of Lagrangian multipliers, and a redefinition of terms in order to maintain the original generality of the principle. The Euler equations of the extended variational principle provide, in addition to those obtained in the original principle, elastic and plastic kinematic relations, and, elastic and plastic constitutive relations. Also, the Euler equations of the original principle are obtained in a physically more meaningful form. Examples are given which show how this principle can be applied to various classical models whose formulation is well known.

A shell theory is derived from the extended variational principle by integrating the three-dimensional equations across the thickness of the shell. Tensor notation and the theory of surfaces in curvalinear coordinates are used. The derived theory is "exact" within the assumption that the shifted displacements and velocities vary linearly across the thickness of the shell.

A complete set of shell equations is derived, including momentum equations, equations involving internal degrees of freedom, entropy balance, constitutive relations, and some typical boundary conditions. An application is also given which shows how the derived equations reduce to the classical equations for a special case of an elastic shell.

ACKNOWLEDGMENT

The author is indebted to Dr. Harry Herman of the Mechanical Engineering Department of Newark College of Engineering for suggesting the topic of this dissertation, and for his invaluable assistance and guidance during its preparation.

The author also wishes to thank Mrs. Susan Elliot and Miss Linda Greenberg of the Bell Telephone Laboratories Library at Whippany, N. J. for their assistance in obtaining reference material, and Mrs. Nancy Campbell and Miss Leonora Jackson for the excellent typing in spite of the difficult notation.

Also, the author is indebted to his wife, Fran, for her patience, understanding, and support during the long years of preparation leading to this dissertation.

TABLE OF CONTENTS

NOMENCLATURE

Symbol		Description
A	-	Second rank tensor which represents internal degrees of freedom.
$A^{\alpha}{}_{\mu}$	-	Mixed components of tensor A.
a	-	Determinant of a_{ab}.
a_{β}	-	Surface base vector.
$a_{\alpha\beta}$	-	Surface metric tensor components.
B	-	Reciprocal of tensor A.
$B^{\mu}{}_{\alpha}$	-	Mixed components of tensor B.
B^{α}	-	Inertia effect.
$\vec{b}$	-	Burger's vector.
b_{ab}	-	Curvature tensor.
$\hat{C}^{\nu}{}_{\mu}$	-	Components of tensor A in Lagrangian frame.
C^{α}	-	Inertia effects.
c^{a}	-	External effects.
D	-	Time derivative operator.
$E^{\alpha\beta}$	-	Stress tensor components.
e_{ab}	-	Relative alternating tensor.
$\hat{e}^{(p)}_{\mu\nu}$	-	Components of plastic strain rate tensor.

F_{α}	-	Generalized force, body force.
F^{α}	-	Effective body force.
G	-	Metric tensor.
$\overset{o}{G}$	-	Initial metric tensor.
g	-	Determinant of $g_{\alpha\beta}$.
$\hat{g}$	-	Determinant of $\hat{g}_{\mu\nu}$.
g_{α}	-	Base vector in spacial frame.
$\hat{g}_{\mu}$	-	Base vector in Lagrangian frame.
$\overset{*}{g}_{\mu}$	-	Base vector in stress-free state.
$g_{\alpha\beta}$	-	Components of metric tensor in spacial frame.
$\overset{o}{g}_{\mu\nu}$	-	Components of initial metric tensor.
$\hat{g}_{\mu\nu}$	-	Components of metric tensor in Lagrangian frame.
H	-	Mean curvature.
h	-	Thickness of shell.
$h_{\alpha\beta}$	-	Stress tensor components.
I	-	Part of an integral equation.
J_{α}	-	Components of generalized velocity vector.
$\underset{\sim}{J}_{\alpha}$	-	Prescribed velocity on time boundary.
$J_{\alpha\beta}$	-	Components of generalized momentum density tensor.
$\underset{\sim}{J}_{\alpha\beta}$	-	Prescribed momentum density on time boundary.

K	-	Gaussian curvature.
$K_{\alpha\beta}{}^{\gamma}$	-	Potential function.
k	-	Yield surface parameter.
$L_{(P)}$	-	Tensors which characterize constant properties of the medium.
$L^{\alpha\beta\gamma}$	-	Resultant potential function.
ℓ^{α}	-	Surface stresses.
M^{ab}	-	Couple resultants.
$\tilde{M}^{ab}$	-	Couple resultant.
$\underset{\sim}{M}^{ab}$	-	Prescribed couple resultant on boundary.
$\mathcal{M}^{a}$	-	Effective body couples.
$\mathcal{M}^{3}$	-	Function of normal body forces.
m^{c}	-	Surface couples.
m^{3}	-	Function of normal surface forces.
N	-	Divergence of h at flux vector.
N^{ab}	-	Stress resultant.
$\tilde{N}^{ab}$	-	Stress resultant.
$\underset{\sim}{N}^{ab}$	-	Prescribed stress resultant on boundary.
n_{α}	-	Components of normal vector
$\hat{n}_{\mu}$	-	Components of normal vector.
$\bar{n}_{a}$	-	Shifted components of n_a

$P^{\alpha\beta}$ - Stress resultant.

p^{α} - External effects.

$p_{\alpha}{}^{\beta}$ - Components of stress tensor.

$\underset{\sim}{p}_{\alpha}{}^{\beta}$ - Prescribed stress on boundary.

Q^{α} - Stress resultant.

$\overline{Q}^{a}$ - Stress resultant.

$\tilde{Q}^{a}$ - Stress resultant.

$\underset{\sim}{Q}^{a}$ - Prescribed stress resultant on boundary.

$\tilde{\overline{Q}}^{a}$ - Stress resultant.

$\hat{Q}^{\mu\nu}$ - Generalized stress, yield surface stress

$\hat{Q}^{\mu\nu\lambda}$ - Generalized potential function.

$\mathcal{Q}^{\alpha}$ - Heat flux.

$\vec{q}$ - Heat flux vector.

$\hat{q}^{\mu}$ - Heat flux vector components.

$\hat{q}^{\mu\nu\lambda}$ - Components of potential function.

$\underset{\sim}{\hat{q}}^{\mu\nu\lambda}$ - Prescribed potential function on boundary.

$R^{\alpha\beta}$ - Couple resultant.

$\overline{R}_{abcd}$ - Covariant surface curvature tensor.

r - Position vector.

$\overline{r}$ - Surface position vector.

$\overset{*}{r}$ - Position vector in stress-free state.

S	-	Entropy
$\overline{S}$	-	Entropy
$\hat{S}^{\mu\nu}$	-	Components of dislocation density tensor.
$\hat{S}_{\mu\nu}{}^{\lambda}$	-	Components of dislocation density tensor.
T	-	A general tensor.
T^{a}	-	Couple resultant.
$\overline{T}^{a}$	-	Couple resultant.
$\tilde{T}^{a}$	-	Couple resultant.
$\tilde{\overline{T}}^{a}$	-	Couple resultant.
$\underset{\sim}{T}^{a}$	-	Prescribed couple resultant on boundary.
$\overline{T}^{ab}$	-	Shifted components of T.
t	-	Time
$t_{\alpha}{}^{\beta}$	-	Stress tensor components.
U	-	Internal energy.
v	-	Velocity vector.
v^{α}	-	Components of velocity vector.
$\overline{v}_{a}$	-	In-plane displacements of center surface.
W	-	Integral which represents parameters on the four dimensional boundary, Vt.
W^*	-	Prescribed functional for external actions and irreversible effects.
w	-	Normal displacement of center surface.

x_s - Parameter of dissipation function.

x^α - Spacial coordinates.

$\bar{x}_\alpha$ - Shifted components of x_α.

$x^\alpha{}_\mu$ - Displacement gradient.

α - Dislocation density tensor.

β_a - Rotation of normal fiber.

β_3 - Elongation of normal fiber.

$\Gamma_{\alpha\beta}{}^\gamma$ - Connection in spacial frame.

$\bar{\Gamma}_{ab}{}^c$ - Surface connection.

$\hat{\Gamma}_{\mu\nu}{}^\lambda$ - Connection in Lagrangian frame.

$\gamma_{\alpha\beta}$ - Part of strain tensor.

δ - Variational operator at constant t.

δ_1 - Total variational operator.

δ^a_b - Kronecher delta.

δ^{ab}_{cd} - Kronecher delta.

ε_{ab} - Fully antisymmetric tensor for two dimensions.

$\bar{\varepsilon}_{ab}$ - Fully antisymmetric surface tensor.

$\varepsilon^{(e)}_{\alpha\beta}$ - Components of elastic strain tensor.

$\varepsilon^{(p)}_{\alpha\beta}$ - Components of plastic strain tensor.

$\hat{\varepsilon}^{(e)}_{\mu\nu}$	-	Components of elastic strain tensor.
$\hat{\varepsilon}^{(p)}_{\mu\nu}$	-	Components of plastic strain tensor.
$\hat{\varepsilon}^{\omega\mu\nu}$	-	Fully antisymmetric tensor.
$\Theta_{\alpha\beta}$	-	Deformation tensor, internal degrees of freedom.
$\overline{\Theta}_{\alpha\beta}$	-	Shifted components of $\Theta_{\alpha\beta}$.
θ	-	Absolute temperature.
$\kappa_{\alpha\beta}$	-	Part of strain tensor
Λ	-	Lagrangian.
μ	-	Determinant of μ^a_c.
μ_α	-	Constant of proportionality.
μ^a_c	-	Shifter.
$\overline{\mu}^a_c$	-	Shifter, inverse of μ^a_c.
ξ^μ	-	Lagrangian coordinates.
$\xi^\mu{}_\alpha$	-	Displacement gradient.
π	-	Plastic deformation rate tensor.
$\overline{\pi}_{ab}$	-	Shifted components of π.
$\hat{\pi}_{\mu\nu}$	-	Components of π.
ρ	-	Density of the medium.
ρ_o	-	Density of median surface.

Σ - Area

σ - Area

σ_α - Dissipation function

$\sigma_\alpha{}^\beta$ - Components of stress tensor.

$\hat{\sigma}_\mu{}^{\nu\lambda}$ - Function of dislocation density tensor.

τ - Volume

$\tau_\alpha{}^\beta$ - Components of viscous stress tensor.

χ - Yield surface parameter.

$\hat{\psi}^{\mu\nu}$ - Plastic stress tensor.

$\hat{\Omega}_{\mu\nu}$ - Components of plastic vorticity tensor.

$\hat{\omega}_{\mu\nu}$ - Rotation tensor components

CHAPTER 1

INTRODUCTION

The purpose of this investigation is to extend the Sedov-Berdichevskii [1][1] variational principle, and to employ this extended principle in the construction of a shell theory. The Sedov-Berdichevskii variational principle provides a unifying basis for many of the previously developed continuum theories, such as elasticity, plasticity and dislocation theory. This principle thus interrelates the microscopic and macroscopic phenomena in elastic-plastic behavior, and, like all variational principles, it has the advantage of permitting a direct approach to the exact, or approximate, solution of problems.

The Sedov-Berdichevskii theory is formulated by the introduction of nine additional degrees of freedom for the material model; i.e., nine more degrees of freedom than in the classical theory of elasticity and three more degrees of freedom than in the ordinary theory of plasticity.

---

[1] Numbers in brackets designate references.

The new degrees of freedom are introduced by a second-order tensor, A, which together with its first-order covariant and time derivatives become additional arguments in the Lagrangian and serve as parameters in defining the state of the medium.

In a more recent paper, Sedov [15] discusses the possibility of introducing additional orders of derivatives of these variable parameters, and of the displacement gradients, in order to include new and even more complicated models. For this investigation, however, the first order of derivatives will be sufficient to provide the link between the theory of dislocations and plasticity.

Chapter 2 discusses the equations of the generalized model of continuous media. The definitions and equations used by Sedov and Berdichevskii in their development are presented and elaborated upon in order to present a complete basis for the subsequent discussion.

In Chapter 3, the Sedov-Berdichevskii variational principle is extended using the methods of Washizu [17]. The Euler equations of this extended variational principle, include, in addition to the Euler equations obtained from the Sedov-Berdichevskii principle in Chapter 2, elastic and plastic stress-strain relations as well as elastic and plastic kinematic relations.

The generalization of Chapter 3 is neither unique nor the most general possible. For instance, the variational principle could have been extended to include kinematic relations for total strain and constitutive relations for the dislocation tensors. However, the generalization was confined to the form given in Chapter 3 in order to permit insight into the physical meaning of many of the quantities introduced in the Sedov-Berdichevskii theory.

Chapter 3 also shows how the extended variational principle can be applied to some of the classical models such as an elastic body, an elastic-plastic body and an elastic body with couple stresses.

Chapter 4 reviews the definitions and equations of the geometry of a surface as a preliminary step to the derivation of a shell theory. The equations required in order to integrate terms involving second rank tensors are derived. In addition, similar equations involving third rank tensors are also derived. The equations in Chapter 4 provide the necessary geometric relationships which are used in the shell equation derivations of Chapter 5.

Chapter 5 applies the extended version of the Sedov-Berdichevskii variational principle in the derivation of a shell theory. The Euler equations of the variational principle are integrated to provide the equations of momentum, relations pertaining to the internal degrees of freedom, entropy balance, constitutive relations, and boundary conditions in terms of shell-type quantities such as, stress resultants, etc.

The only assumption made in this derivation is in the form of the shifted displacements and velocities. A linear variation in the normal direction is assumed and the extension of the normal fibers is included. The theory developed is "exact" within this approximation. The equations derived agree with classical shell theory when special cases are considered.

One of the consequences of this formulation is that the so-called "sixth equation" [12][1] of classical shell theory is obtained directly. This derivation is similar to Naghdi's in Ref. 12 except that Naghdi's interest in

---

[1] Naghdi did not consider the extension of the normal fibers in this work.

this paper is in the classical elastic effects only and therefore he sets the couple stress terms equal to zero before completing a shell theory including couple stresses.

CHAPTER 2

THE SEDOV-BERDICHEVSKII THEORY

In this chapter, the equations of the generalized model of continuous media are presented. The defining parameters of the model and the derivation of the generalized field equations by means of a variational principle are shown. The definitions and final equations are from the Sedov-Berdichevskii paper [1]. The derivation and intermediate steps were added by the author in order to clarify the results and present a complete theory under one cover. Also the applications of the equations in subsequent chapters do not always start from the final form.

Defining Parameters

The motion of a medium is defined with respect to an observer's frame in spacial coordinates x^{α}, time coordinate t, and basis g_{α}. The motion of the medium is also defined in a comoving frame with Lagrangian coordinates ξ^{μ}, time t, and basis $\hat{g}_{\mu}$.

The law of motion of the medium is determined by the equation

$$x^{\alpha} = x^{\alpha}(\xi^{\mu}, t). \tag{2.1}$$

In this chapter it is important to note that the lower Greek letters, α,β,γ, etc. represent spacial coordinates, while the middle letters, ξ,μ,ν, etc. represent Lagrangian coordinates. All indices range from 1 to 3. Also a repeated index will be used to indicate summation over this range.

The base vectors in the two coordinate systems are related by the following equations

$$\left.\begin{aligned} \hat{g}_\nu &= x^\alpha{}_\nu g_\alpha & \hat{g}^\mu &= \xi^\mu{}_\beta g^\beta \\ x^\alpha{}_\nu &= \frac{\partial x^\alpha}{\partial \xi^\nu} & \xi^\mu{}_\beta &= \frac{\partial \xi^\mu}{\partial x^\beta} . \end{aligned}\right\} \tag{2.2}$$

The contravariant and covariant base vectors are related by the components of the metric tensor,

$$\hat{g}_\nu = \hat{g}_{\nu\mu}\hat{g}^\mu \qquad g_\alpha = g_{\alpha\beta}g^\beta \tag{2.3}$$

where $\hat{g}_{\nu\mu}$ and $g_{\alpha\beta}$ are the covariant components of the metric tensor in the Lagrangian and spacial coordinate systems respectively. The metric components are related by

$$\hat{g}_{\nu\mu} = g_{\alpha\beta}x^\alpha{}_\nu x^\beta{}_\mu . \tag{2.4}$$

An invariant A can be expressed in terms of its components as

$$A = A^{\alpha}{}_{\mu} g_{\alpha} \hat{g}^{\mu} = \hat{C}^{\nu}{}_{\mu} \hat{g}_{\nu} \hat{g}^{\mu}. \tag{2.5}$$

The component representation, $A^{\alpha}{}_{\mu}$, behaves as a vector under transformation of either the spacial frame or the Lagrangian frame. The components, $\hat{C}^{\nu}{}_{\mu}$, behave as a second rank tensor under transformation of the Lagrangian frame. The quantities, $A^{\alpha}{}_{\mu}$, and, $\hat{C}^{\nu}{}_{\mu}$, are merely different ways of expressing the same tensor A. The metric tensor G can be expressed in this way as

$$G = g_{\alpha\beta} g^{\alpha} g^{\beta} = x^{\alpha}{}_{\mu} g_{\alpha} \hat{g}^{\mu} = \hat{g}_{\mu\nu} \hat{g}^{\mu} \hat{g}^{\nu}. \tag{2.6}$$

The components, $A^{\alpha}{}_{\mu}$, in this case become the displacement gradients, $x^{\alpha}{}_{\mu}$.

The covariant derivatives of A can be expressed in terms of the components, $A^{\alpha}{}_{\mu}$, as

$$A^{\beta}{}_{\mu|\alpha} = \frac{\partial A^{\beta}{}_{\mu}}{\partial x^{\alpha}} + \Gamma_{\alpha\gamma}{}^{\beta} A^{\gamma}{}_{\mu} - \hat{\Gamma}_{\mu\nu}{}^{\lambda} A^{\beta}{}_{\lambda} \xi^{\nu}{}_{\alpha} \tag{2.7}$$

or

$$A^{\beta}{}_{\mu|\nu} = \frac{\partial A^{\beta}{}_{\mu}}{\partial \xi^{\nu}} + \Gamma_{\alpha\gamma}{}^{\beta} A^{\gamma}{}_{\mu} x^{\alpha}{}_{\nu} - \hat{\Gamma}_{\mu\nu}{}^{\lambda} A^{\beta}{}_{\lambda} \qquad (2.8)$$

where $\Gamma_{\alpha\beta}{}^{\gamma}$ and $\hat{\Gamma}_{\mu\nu}{}^{\lambda}$ are defined by

$$\frac{\partial g_{\alpha}}{\partial x^{\beta}} = \Gamma_{\alpha\beta}{}^{\gamma} g_{\gamma} \quad \text{and} \quad \frac{\partial \hat{g}_{\mu}}{\partial \xi^{\nu}} = \hat{\Gamma}_{\mu\nu}{}^{\lambda} \hat{g}_{\lambda} \qquad (2.9)$$

With these definitions and the invariant relationships

$$\frac{\partial A}{\partial x^{\alpha}} g^{\alpha} = \frac{\partial A}{\partial \xi^{\mu}} \hat{g}^{\mu}, \qquad (2.10)$$

it can be shown that

$$A^{\beta}{}_{\mu|\alpha} g_{\beta} \hat{g}^{\mu} g^{\alpha} = A^{\beta}{}_{\mu|\nu} g_{\beta} \hat{g}^{\mu} \hat{g}^{\nu} \qquad (2.11)$$

or equivalently,

$$A^{\beta}{}_{\mu|\alpha} = A^{\beta}{}_{\mu|\nu} \xi^{\nu}{}_{\alpha}. \qquad (2.12)$$

The time derivative, DA, is defined by assuming that the Lagrangian coordinates and the Lagrangian basis $\hat{g}^{\mu}$ are constant with time. Thus the time derivative of A is

$$DA = \frac{d}{dt}\left(A^{\alpha}{}_{\mu} g_{\alpha} \hat{g}^{\mu}\right) = DA^{\alpha}{}_{\mu} g_{\alpha} \hat{g}^{\mu} \tag{2.13}$$

where

$$DA^{\alpha}{}_{\mu} = \frac{dA^{\alpha}{}_{\mu}}{dt} + \Gamma_{\beta\gamma}{}^{\alpha} A^{\beta}{}_{\mu} v^{\gamma} \tag{2.14}$$

and $v^{\gamma} = dx^{\gamma}/dt$ are the components of velocity of a point of the medium.

The defining parameters of the medium can now be summarized as the following set of invariants.

$$\left.\begin{aligned} & v = v^{\alpha} g_{\alpha}, \\ & G = g_{\alpha\beta} g^{\alpha} g^{\beta} = x^{\alpha}{}_{\mu} g_{\alpha} \hat{g}^{\mu} = \hat{g}_{\mu\nu} \hat{g}^{\mu} \hat{g}^{\nu}, \\ & A = A^{\alpha}{}_{\mu} g_{\alpha} \hat{g}^{\mu} = \hat{C}^{\nu}{}_{\mu} \hat{g}_{\nu} \hat{g}^{\mu}, \\ & \frac{\partial A}{\partial \xi^{\mu}} \hat{g}^{\mu},\ DA,\ S,\ L_{(P)} \end{aligned}\right\} \tag{2.15}$$

where v is velocity; G, the metric tensor; S, entropy; and $L_{(P)}$, a set of tensors which characterize the physical and geometric properties of the medium. Included in the set, $L_{(P)}$, are the constants of the constitutive equations and an initial metric tensor, $\mathcal{G}$.

Physical Parameters

With the exception of A and its derivatives, the above set of parameters is the same as in classical elasticity. The physical meaning of the tensor, A, is defined as follows. If a particle with Lagrangian coordinates, ξ^{μ}, is removed from the body and released from external forces, the base vectors, $\hat{g}_{\nu}$, convert to a new state defined by $\overset{*}{g}_{\nu}$. This deformation is defined by the tensor, A.

$$\overset{*}{g}_{\mu} = \hat{C}^{\nu}{}_{\mu}\hat{g}_{\nu} = A^{\alpha}{}_{\mu}g_{\alpha} \tag{2.16}$$

Thus, the components, $\hat{C}^{\nu}{}_{\mu}$, of A describe the elastic deformation and the following physical parameters may be defined. The elastic strain tensor may be written as

$$\hat{\varepsilon}^{(e)}_{\mu\nu} = \frac{1}{2}\left(\hat{g}_{\mu\nu}-\overset{*}{g}_{\mu\nu}\right) \tag{2.17}$$

where $\overset{*}{g}_{\mu\nu}$ is the metric in the stress free state and is related to $g_{\alpha\beta}$ by

$$\overset{*}{g}_{\mu\nu} = g_{\alpha\beta}A^{\alpha}{}_{\mu}A^{\beta}{}_{\nu}. \tag{2.18}$$

The plastic strain tensor is

$$\hat{\varepsilon}^{(p)}_{\mu\nu} = \frac{1}{2}\left(\overset{*}{g}_{\mu\nu} - \overset{\circ}{g}_{\mu\nu}\right). \tag{2.19}$$

The plastic strain rate tensor is

$$\hat{e}^{(p)}_{\mu\nu} = \frac{d\hat{\varepsilon}^{(p)}_{\mu\nu}}{dt} = \frac{1}{2}\, g_{\alpha\beta}\left(A^{\alpha}{}_{\mu}DA^{\beta}{}_{\nu} + A^{\beta}{}_{\nu}DA^{\alpha}{}_{\mu}\right). \tag{2.20}$$

The elastic strain gradient tensor is

$$\hat{\varepsilon}^{(e)}_{\mu\nu|\lambda} = -\frac{1}{2}\, g_{\alpha\beta}\left(A^{\alpha}{}_{\mu}A^{\beta}{}_{\nu|\lambda} + A^{\beta}{}_{\nu}A^{\alpha}{}_{\mu|\lambda}\right) \tag{2.21}$$

The dislocation density tensor can be defined in terms of the tensor, A, by the following reasoning. Let a closed circuit, L, in the body be represented by

$$\oint dr = 0. \tag{2.22}$$

If, after the deformation defined by A, the vector, dr, becomes $d\overset{*}{r}$, the closed circuit, L, in this case is not necessarily zero, but is equal to the Burger's vector, $\vec{b}$.

$$\vec{b} = \oint d\overset{*}{r} = \oint \overset{*}{g}_{\mu} d\xi^{\mu} \tag{2.23}$$

With the application of equation (2.16), this becomes

$$\vec{b} = \oint A^{\alpha}{}_{\mu} g_{\alpha} d\xi^{\mu} \qquad (2.24)$$

By the use of Stoke's Theorem, the line integral can be converted to the surface integral and equation (2.24) becomes

$$\vec{b} = \iint_{\sigma} \hat{\varepsilon}^{\omega\mu\nu} A^{\alpha}{}_{\nu|\mu} \hat{n}_{\omega} g_{\alpha} d\sigma \qquad (2.25)$$

where $\hat{n}_{\omega}$ are the components of the normal vector, and where $\hat{\varepsilon}^{\omega\mu\nu}$ is the fully antisymmetric tensor, and is equal to $1/\hat{g}^{\frac{1}{2}}$ for even permutations of the indices. The dislocation density tensor is defined in terms of the limit of the Burger's vector over an area $d\sigma$ as $d\sigma$ approaches zero.

$$\frac{d\vec{b}}{d\sigma} = 2\hat{S}^{\omega\lambda} \hat{n}_{\omega} \overset{*}{g}_{\lambda} \qquad (2.26)$$

The dislocation density tensor can then be written as

$$\hat{S}^{\omega\lambda} = \frac{1}{2} \hat{\varepsilon}^{\omega\mu\nu} B^{\lambda}{}_{\alpha} A^{\alpha}{}_{\nu|\mu} \qquad (2.27)$$

where the derivatives of equation (2.25) and equation (2.16) were used, and $B^{\lambda}{}_{\alpha}$ are the components of the reciprocal of A. A third rank tensor can also be defined as

$$\hat{S}_{\mu\nu}{}^{\lambda} = B^{\lambda}{}_{\alpha}A^{\alpha}{}_{[\nu|\mu]}. \tag{2.28}$$

This will be used as an alternate form of the dislocation tensor.

Next the tensor, $\pi = \hat{\pi}_{\mu\nu}\hat{g}^{\mu}\hat{g}^{\nu}$, is defined as

$$\frac{d\overset{*}{g}_{\nu}}{dt} = \hat{\pi}_{\mu\nu}\overset{*}{g}^{\mu} = DA^{\alpha}{}_{\nu}g_{\alpha} \tag{2.29}$$

From this definition the components of π can be written as

$$\hat{\pi}_{\mu\nu} = B_{\mu\alpha}DA^{\alpha}{}_{\nu} \tag{2.30}$$

The angular velocity of plastic deformation is then the antisymmetric part of $\hat{\pi}_{\mu\nu}$ or

$$\hat{\Omega}_{\mu\nu} = \hat{\pi}_{[\mu\nu]} \tag{2.31}$$

while the symmetric part of $\hat{\pi}_{\mu\nu}$ is the plastic strain rate tensor in equation (2.20)

$$\hat{e}^{(p)}_{\mu\nu} = \hat{\pi}_{(\mu\nu)}. \qquad (2.32)$$

The parentheses are used to denote symmetry with respect to μ,ν.

A Variational Principle

The variational principle,

$$\delta_1 \int_V \int_{t_1}^{t_2} \Lambda d\tau dt + \delta_1 W + \delta_1 W^* = 0, \qquad (2.33)$$

is used to develop this theory. In this equation, Λ is the Lagrangian, $d\tau$ is a volume element, and can be written as

$$d\tau = \sqrt{\hat{g}}\, d\xi^1 d\xi^2 d\xi^3 = \sqrt{g}\, dx^1 dx^2 dx^3, \qquad (2.34)$$

where $\hat{g}$ is the determinant of $\hat{g}_{\mu\nu}$ and g is the determinant of $g_{\alpha\beta}$. In equation (2.33), V is an arbitrary volume, $\delta_1 W$ is an integral which represents the variations of the parameters on the boundary of the four dimensional domain Vt. $\delta_1 W^*$ is a prescribed functional which describes external actions and internal irreversible effects.

The variation, $\delta_1(\ )$, is defined as

$$\delta_1(\ ) = \delta(\ ) + D(\ )\delta t \qquad (2.35)$$

where $\delta(\ )$ is a variation taken at constant t, and δt is an infinitesimal shift of time.

The following equations define the variations of the system parameters and are used to carry out the indicated variation in the first term of equation (2.33). The Lagrangian is assumed to be a function of six quantities which are varied and of the physical and geometric relationship, $L_{(P)}$, which are held constant.

$$\Lambda = \Lambda\left(v, G, A, DA, S, \alpha, L_{(P)}\right) \qquad (2.36)$$

In this equation, α represents the invariant form of the dislocation tensor. It is assumed that the only dependency of the Lagrangian on the covariant derivative of A is through the dislocation tensor.

The variation of each of these parameters can be expressed in terms of the three independent variations, δx^α, $\delta A^\alpha{}_\mu$, δS as follows:

$$\left.\begin{aligned}
\delta v &= D\delta x^\alpha g_\alpha \\
\delta G &= x^\beta{}_\mu (\delta x^\alpha)_{|\beta} g_\alpha \hat{g}^\mu \\
\delta A &= \delta A^\alpha{}_\mu g_\alpha \hat{g}^\mu \\
\delta(DA) &= D(\delta A^\alpha{}_\mu) g_\alpha \hat{g}^\mu \\
\delta S &= \delta S \\
\delta\alpha &= \left[B^\lambda{}_\alpha\left(\delta A^\alpha{}_{[\nu}\right)_{|\mu]} - \hat{S}_{\mu\nu}{}^\omega B^\lambda{}_\alpha \delta A^\alpha{}_\omega\right] \hat{g}_\lambda \hat{g}^\mu \hat{g}^\nu
\end{aligned}\right\} \quad (2.37)$$

The last equation of (2.37) comes from the definition, (2.28), which yields

$$\delta S_{\mu\nu}{}^\lambda = \delta\left(B^\lambda{}_\alpha A^\alpha{}_{[\nu|\mu]}\right). \qquad (2.38)$$

This can be written as

$$\delta S_{\mu\nu}{}^{\lambda} = B^{\lambda}{}_{\alpha}\left(\delta A^{\alpha}{}_{[\nu}\right)_{|\mu]} + \delta B^{\lambda}{}_{\alpha} A^{\alpha}{}_{[\nu|\mu]}. \qquad (2.39)$$

In equation (2.39), $\delta B^{\lambda}{}_{\alpha}$ can be written as

$$\delta\left(B^{\lambda}{}_{\alpha} A^{\alpha}{}_{\lambda}\right) = B^{\lambda}{}_{\alpha}\delta A^{\alpha}{}_{\lambda} + \delta B^{\lambda}{}_{\alpha} A^{\alpha}{}_{\lambda} = 0, \qquad (2.40)$$

or

$$\delta B^{\lambda}{}_{\alpha} = -B^{\lambda}{}_{\alpha} B^{\omega}{}_{\beta}\delta A^{\beta}{}_{\omega}. \qquad (2.41)$$

Substituting equation (2.41) into (2.39) and making use of equation (2.28) yields

$$\delta S_{\mu\nu}{}^{\lambda} = B^{\lambda}{}_{\alpha}\left(\delta A^{\alpha}{}_{[\nu}\right)_{|\mu]} - \hat{S}_{\mu\nu}{}^{\omega} B^{\lambda}{}_{\alpha}\delta A^{\alpha}{}_{\omega} \qquad (2.42)$$

which is the last equation of the set, (2.37).

In addition to the variations of the defining parameters, it is also necessary to vary the volume element, $d\tau$. This can be put in terms of the independent variation, δx^{α}, by

$$\delta_1 d\tau = (\delta_1 x^{\alpha})_{|\alpha}\, d\tau \qquad (2.43)$$

The first term of equation (2.33) can be written as

$$\delta_1 \int_V \int_{t_1}^{t_2} \Lambda d\tau dt = \int_V \int_{t_1}^{t_2} \left(\delta\Lambda + \frac{d\Lambda}{dt}\,\delta t\right) d\tau dt + \int_V \int_{t_1}^{t_2} \Lambda \delta_1 (d\tau) dt \qquad (2.44)$$

The time-independent variation, $\delta\Lambda$, can be written as

$$\delta\Lambda = \frac{\partial\Lambda}{\partial A^{\alpha}{}_{\mu}}\,\delta A^{\alpha}{}_{\mu} + \frac{\partial\Lambda}{\partial DA^{\alpha}{}_{\mu}}\,\delta DA^{\alpha}{}_{\mu} + \frac{\partial\Lambda}{\partial S_{\mu\nu}{}^{\lambda}}\,\delta S_{\mu\nu}{}^{\lambda}$$

$$+ \frac{\partial\Lambda}{\partial x^{\alpha}{}_{\mu}}\,\delta x^{\alpha}{}_{\mu} + \frac{\partial\Lambda}{\partial v^{\alpha}}\,\delta v^{\alpha} + \frac{\partial\Lambda}{\partial S}\,\delta S. \qquad (2.45)$$

These variations can be put in terms of the independent variations using equations (2.37). They can also be manipulated to provide total variations, (δ_1), on the boundary. The first and last terms on the right side of equation (2.45) remain unchanged, while the other terms become

$$\frac{\partial\Lambda}{\partial DA^{\alpha}{}_{\mu}}\ \delta DA^{\alpha}{}_{\mu} = D\left[\frac{\partial\Lambda}{\partial DA^{\alpha}{}_{\mu}}\ \delta_1 A^{\alpha}{}_{\mu}\right] - D\left(\frac{\partial\Lambda}{\partial DA^{\alpha}{}_{\mu}}\right)\delta A^{\alpha}{}_{\mu}$$

$$- D\left(\frac{\partial\Lambda}{\partial DA^{\alpha}{}_{\mu}}\ DA^{\alpha}{}_{\mu}\right)\delta t, \qquad (2.46)$$

$$\frac{\partial\Lambda}{\partial S_{\mu\nu}{}^{\lambda}}\ \delta S_{\mu\nu}{}^{\lambda} = -\left(x^{\beta}{}_{\lambda}\hat{\sigma}_{\nu}{}^{\mu\lambda}B^{\nu}{}_{\alpha}\delta_1 A^{\alpha}{}_{\mu}\right)_{|\beta} + \left(\hat{\sigma}_{\lambda}{}^{\mu\nu}B^{\lambda}{}_{\alpha}\right)_{|\nu}\ \delta A^{\alpha}{}_{\mu}$$

$$+ \left(\hat{\sigma}^{\nu\mu\lambda}\pi_{\nu\mu}x^{\beta}{}_{\lambda}\right)_{|\beta}\ \delta t + \hat{\sigma}_{\lambda}{}^{\omega\nu}S_{\nu\omega}{}^{\mu}B^{\lambda}{}_{\alpha}\delta A^{\alpha}{}_{\mu},$$

$$(2.47)$$

where

$$\hat{\sigma}_{\nu}{}^{\mu\lambda} = \frac{\partial\Lambda}{\partial\hat{S}_{\mu\lambda}{}^{\nu}}\ .$$

Also,

$$\frac{\partial\Lambda}{\partial x^{\alpha}{}_{\mu}}\ \delta x^{\alpha}{}_{\mu} = \left(\frac{\partial\Lambda}{\partial x^{\alpha}{}_{\mu}}\ x^{\beta}{}_{\mu}\delta_1 x^{\alpha}\right)_{|\beta} - \left(\frac{\partial\Lambda}{\partial x^{\alpha}{}_{\mu}}\ x^{\beta}{}_{\mu}\right)_{|\beta}\ \delta x^{\alpha}$$

$$- \left(\frac{\partial\Lambda}{\partial x^{\alpha}{}_{\mu}}\ x^{\beta}{}_{\mu}Dx^{\alpha}\right)_{|\beta}\ \delta t \qquad (2.48)$$

$$\frac{\partial\Lambda}{\partial v^{\alpha}}\ \delta v^{\alpha} = D\left[\frac{\partial\Lambda}{\partial v^{\alpha}}\ \delta_1 x^{\alpha}\right] - D\ \frac{\partial\Lambda}{\partial v^{\alpha}}\ \delta x^{\alpha} - D\left[\frac{\partial\Lambda}{\partial v^{\alpha}}\ v^{\alpha}\right]\ \delta t.$$

$$(2.49)$$

A detailed derivation of (2.47) is given in Appendix A. The technique is similar for (2.46), (2.48), and (2.49).

The last term in equation (2.44) can be manipulated in a similar manner. It becomes

$$\int_V \int_{t_1}^{t_2} \Lambda\delta_1(d\tau)dt = \int_V \int_{t_1}^{t_2} \left[\left(\Lambda\delta_1 x^\alpha\right)_{|\alpha} - \Lambda_{|\alpha}\delta x^\alpha - \Lambda_{|\alpha} v^\alpha \delta t\right] d\tau dt. \tag{2.50}$$

The functional, $\delta_1 W^*$, is defined as

$$\delta_1 W^* = \int_V \int_{t_1}^{t_2} \Big\{ \rho\theta\delta S + F_\alpha \delta_1 x^\alpha - \tau_\alpha{}^\beta (\delta x^\alpha)_{|\beta} - \hat{Q}^{\mu\nu} B_{\mu\alpha} \delta A^\alpha{}_\nu$$

$$- \hat{Q}^{\mu\nu\lambda}\left(B_{\mu\alpha}\delta A^\alpha{}_\nu\right)_{|\lambda} + N\delta t \Big\} d\tau dt, \tag{2.51}$$

where θ is the absolute temperature, and $F_\alpha, \tau_\alpha{}^\beta, \hat{Q}^{\mu\nu}$, $\hat{Q}^{\mu\nu\lambda}$, N are generalized forces and stresses. By the usual manipulation, equation (2.51) can be rearranged as

$$\delta_1 W^* = \int_V \int_{t_1}^{t_2} \Big\{ \rho\theta\delta S + \left(F_\alpha + \tau_\alpha{}^\beta{}_{|\beta}\right)\delta x^\alpha \qquad \text{(Cont'd)}$$

$$+ \left(-\hat{Q}^{\mu\nu} + \hat{Q}^{\mu\nu\lambda}{}_{|\lambda}\right) B_{\mu\alpha}\delta A^{\alpha}{}_{\nu} +$$

$$+ \left[F_{\alpha}v^{\alpha} + N + \left(\tau_{\alpha}{}^{\beta}v^{\alpha} + \hat{Q}^{\mu\nu\lambda}\pi_{\mu\nu}x^{\alpha}{}_{\lambda}\right)_{|\beta}\right]\delta t\Big\} d\tau dt$$

$$- \int_{\Sigma}\int_{t_1}^{t_2} \left\{\tau_{\alpha}{}^{\beta}\delta_1 x^{\alpha} + \hat{Q}^{\mu\nu\lambda}x^{\beta}{}_{\lambda}B_{\mu\alpha}\delta_1 A^{\alpha}{}_{\nu}\right\} n_{\beta} d\sigma dt. \tag{2.52}$$

The variations on the boundary can be expressed in $\delta_1 W$ by

$$\delta_1 W = \int_{\Sigma}\int_{t_1}^{t_2}\left(\underset{\sim}{p}_{\alpha}{}^{\beta}\delta_1 x^{\alpha} + \hat{\underset{\sim}{q}}{}^{\mu\nu\lambda}x^{\beta}{}_{\lambda}B_{\mu\alpha}\delta_1 A^{\alpha}{}_{\nu}\right)n_{\beta} d\sigma dt$$

$$+ \int_{V} \rho\left[\underset{\sim}{J}_{\alpha}\delta_1 x^{\alpha} + \underset{\sim}{J}_{\alpha\beta}A^{\beta\mu}\delta_1 A^{\alpha}{}_{\mu}\right]_{t_1}^{t_2} d\tau \tag{2.53}$$

where ρ is the density of the medium and is a function of the Lagrangian coordinates only.

The following list of definitions have been used to facilitate the writing of the complete variational principle.

$$\sigma_\alpha{}^\beta = - x^\beta{}_\mu \frac{\partial\Lambda}{\partial x^\alpha{}_\mu} - \Lambda\delta_\alpha{}^\beta \tag{2.54}$$

$$p_\alpha{}^\beta = \sigma_\alpha{}^\beta + \tau_\alpha{}^\beta \tag{2.55}$$

$$\hat{q}^{\mu\nu\lambda} = \hat{\sigma}^{\mu\nu\lambda} + \hat{Q}^{\mu\nu\lambda} \tag{2.56}$$

$$J_\alpha = \frac{\partial}{\partial v^\alpha}\left(\frac{\Lambda}{\rho}\right) \tag{2.57}$$

$$J_{\alpha\beta} = A_{\beta\mu} \frac{\partial}{\partial DA^\alpha{}_\mu}\left(\frac{\Lambda}{\rho}\right) \tag{2.58}$$

The complete variational equation is

$$\int_V \int_{t_1}^{t_2} \Bigg\{\left(- \rho D J_\alpha + p_\alpha{}^\beta{}_{|\beta} + F_\alpha\right)\delta x^\alpha$$

$$+ \left(\frac{\partial\Lambda}{\partial A^\alpha{}_\mu} + \left(\hat{\sigma}_\lambda{}^{\mu\nu} B^\lambda{}_\alpha\right)_{|\nu} + \hat{\sigma}_\lambda{}^{\omega\nu} S_{\nu\omega}{}^\mu B^\lambda{}_\alpha - \rho D\left(\frac{\partial\Lambda/\rho}{\partial DA^\alpha{}_\mu}\right)\right.$$

$$\left. - \hat{Q}^{\nu\mu} B_{\nu\alpha} + \hat{Q}^{\nu\mu\lambda}{}_{|\lambda} B_{\nu\alpha}\right)\delta A^\alpha{}_\mu$$

(Cont'd)

$$+ \left(\frac{\partial \Lambda}{\partial S} + \rho\theta\right)\delta S$$

$$+ \left(\rho \frac{d}{dt}\left(\frac{\Lambda}{\rho} - v^{\alpha}J_{\alpha} - A^{\beta\mu}J_{\alpha\beta}DA^{\alpha}{}_{\mu}\right) + \left(\hat{\sigma}^{\nu\mu\lambda}\hat{\pi}_{\nu\mu}x^{\beta}{}_{\lambda}\right.\right.$$

$$\left.\left. + p_{\alpha}{}^{\beta}v^{\alpha} + \hat{Q}^{\mu\nu\lambda}\hat{\pi}_{\mu\nu}x^{\alpha}{}_{\lambda}\right)_{|\beta} + N + F_{\alpha}v^{\alpha}\right)\delta t\Big\} d\tau dt$$

$$+ \int_{\Sigma}\int_{t_1}^{t_2}\Big\{\left(- p_{\alpha}{}^{\beta} + \underset{\sim}{p}_{\alpha}{}^{\beta}\right)n_{\beta}\delta_1 x^{\alpha}$$

$$+ \left(- \hat{q}^{\nu\mu\lambda} + \hat{\underset{\sim}{q}}{}^{\mu\nu\lambda}\right) x^{\beta}{}_{\lambda}B_{\nu\alpha}\delta_1 A^{\alpha}{}_{\mu}n_{\beta}\Big\} d\sigma dt$$

$$+ \int_{V}\Big\{\rho\left(-J_{\alpha}+\underset{\sim}{J}_{\alpha}\right)_{t_1}^{t_2}\delta_1 x^{\alpha} + \left(-J_{\alpha\beta}+\underset{\sim}{J}_{\alpha\beta}\right)_{t_1}^{t_2} A^{\beta\mu}\delta_1 A^{\alpha}{}_{\mu}\Big\} d\tau$$

$$= 0. \tag{2.59}$$

The four sets of Euler equations corresponding to the independent variations over the four-dimensional volume are

$$\delta x^{\alpha}: \quad -\rho DJ_{\alpha} + p_{\alpha}{}^{\beta}{}_{|\beta} + F_{\alpha} = 0 \tag{2.60}$$

$$\delta A^{\alpha}{}_{\mu}: \quad \frac{\partial\Lambda}{\partial A^{\alpha}{}_{\mu}} + \left(\hat{\sigma}_{\lambda}{}^{\mu\nu}B^{\lambda}{}_{\alpha}\right)_{|\nu} + \hat{\sigma}_{\lambda}{}^{\omega\nu}\hat{S}_{\nu\omega}{}^{\mu}B^{\lambda}{}_{\alpha}$$

$$- \rho D\left(\frac{\partial\Lambda/\rho}{\partial DA^{\alpha}{}_{\mu}}\right) - \left(\hat{Q}^{\nu\mu} - \hat{Q}^{\nu\mu\lambda}{}_{|\lambda}\right)B_{\nu\alpha} = 0 \qquad (2.61)$$

$$\delta S: \quad \frac{\partial\Lambda}{\partial S} + \rho\theta = 0 \qquad (2.62)$$

$$\delta t: \quad \rho\,\frac{d}{dt}\left(\frac{\Lambda}{\rho} - v^{\alpha}J_{\alpha} - A^{\beta\mu}J_{\alpha\beta}DA^{\alpha}{}_{\mu}\right)$$

$$+ \left(\hat{q}^{\nu\mu\lambda}\hat{\pi}_{\nu\mu}x^{\beta}{}_{\lambda} + p_{\alpha}{}^{\beta}v^{\alpha}\right)_{|\beta}$$

$$+ N + F_{\alpha}v^{\alpha} = 0 \qquad (2.63)$$

The boundary conditions for nonvanishing variations on the boundary are

$$\delta_1 x^{\alpha}: \quad p_{\alpha}{}^{\beta} = \underset{\sim}{p}_{\alpha}{}^{\beta} \qquad (2.64)$$

$$\delta_1 A^{\alpha}{}_{\mu}: \quad \hat{q}^{\nu\mu\lambda} = \underset{\sim}{\hat{q}}^{\nu\mu\lambda} \qquad (2.65)$$

on the surface, Σ, and

$$\delta_1 x^{\alpha}: \quad J_{\alpha} = \underset{\sim}{J}_{\alpha} \tag{2.66}$$

$$\delta_1 A^{\alpha}{}_{\mu}: \quad J_{\alpha\beta} = \underset{\sim}{J}_{\alpha\beta} \tag{2.67}$$

on the time boundary, or at t_1 and t_2.

Equations (2.60) are the momentum equations, equations (2.61) are the equations which specify the internal degrees of freedom due to the inclusion of components, $A^{\alpha}{}_{\mu}$, in terms of the determining parameters of the system. Equation (2.62) may be considered as a defining equation for temperature, while equation (2.63) is the energy equation.

If the Lagrangian of the system can be written as

$$\Lambda = \frac{1}{2} \rho v_{\alpha} v^{\alpha} - \rho U \tag{2.68}$$

or the difference between the kinetic and internal energies, the Euler equation for momentum can be written in the more familiar form,

$$p_{\alpha}{}^{\beta}{}_{|\beta} + F_{\alpha} = \rho D v_{\alpha} \, , \tag{2.69}$$

and the stress tensor can be written

$$p_\alpha{}^\beta = -x^\beta{}_\mu \frac{\partial}{\partial x^\alpha{}_\mu}\left(\frac{1}{2}\rho v_\alpha v^\alpha - \rho U\right) - \Lambda\delta_\alpha{}^\beta + \tau_\alpha{}^\beta \tag{2.70}$$

$$p_\alpha{}^\beta = -x^\beta{}_\mu \frac{1}{2} v_\alpha v^\alpha \frac{\partial\rho}{\partial x^\alpha{}_\mu} + x^\beta{}_\mu U \frac{\partial\rho}{\partial x^\alpha{}_\mu} + x^\beta{}_\mu \rho \frac{\partial U}{\partial x^\alpha{}_\mu}$$

$$-\Lambda\delta_\alpha{}^\beta + \tau_\alpha{}^\beta . \tag{2.71}$$

Using the identity,

$$\frac{\partial\rho}{\partial x^\alpha{}_\mu} = -\rho\xi^\mu{}_\alpha \tag{2.72}$$

equation (2.71) can be reduced to

$$p_\alpha{}^\beta = \rho x^\beta{}_\mu \frac{\partial U}{\partial x^\alpha{}_\mu} + \tau_\alpha{}^\beta . \tag{2.73}$$

Equations (2.61) can be reduced to more physically meaningful equations by multiplying both sides by $A_{\beta\mu}$

$$A_{\beta\mu} \frac{\partial\Lambda}{\partial A^\alpha{}_\mu} + A_{\beta\mu}\left(\hat{\sigma}_\lambda{}^{\mu\nu} B^\lambda{}_\alpha\right)_{|\nu} + A_{\beta\mu}\hat{\sigma}_\lambda{}^{\omega\nu}\hat{S}_{\nu\omega}{}^\mu B^\lambda{}_\alpha$$

$$- \rho A_{\beta\mu} D\left(\frac{\partial\Lambda/\rho}{\partial DA^\alpha{}_\mu}\right) - A_{\beta\mu}B_{\nu\alpha}\left(\hat{Q}^{\nu\mu} - \hat{Q}^{\nu\mu\lambda}{}_{|\lambda}\right) = 0. \tag{2.74}$$

With the aid of equations (2.58) and (2.28), this can be written as

$$\rho DJ_{\alpha\beta} = \frac{\partial\Lambda}{\partial DA^{\alpha}_{\mu}} DA_{\beta\mu} + \frac{\partial\Lambda}{\partial A^{\alpha}_{\mu}} A_{\beta\mu} - \hat{Q}^{\mu\nu}A_{\alpha\nu}A_{\beta\mu}$$

$$+ \left(A_{\beta\mu}A_{\alpha\nu}\hat{Q}^{\nu\mu\lambda}x^{\gamma}{}_{\lambda}\right)\Big|_{\gamma} - \hat{Q}^{\nu\mu\lambda}\left(A_{\beta\mu}A_{\alpha\nu}\right)\Big|_{\lambda}$$

$$+ \left(A_{\beta\mu}A_{\alpha\lambda}\hat{\sigma}^{\lambda\mu\nu}x^{\gamma}{}_{\nu}\right)\Big|_{\gamma}, \qquad (2.75)$$

where the identity

$$\hat{\sigma}_{\lambda}{}^{\mu\nu}B^{\lambda}{}_{\alpha}A_{\beta\mu|\nu} = A_{\beta\mu}B^{\lambda}{}_{\alpha}\hat{\sigma}_{\lambda}{}^{\omega\nu}\hat{S}_{\nu\omega}{}^{\mu} \qquad (2.76)$$

was used in the simplification.

With the definitions

$$K_{\alpha\beta}{}^{\gamma} = \hat{q}^{\mu\nu\lambda}A_{\alpha\nu}A_{\beta\mu}x^{\gamma}{}_{\lambda} \qquad (2.77)$$

and

$$\rho h_{\alpha\beta} = \frac{\partial\Lambda}{\partial A^{\alpha}_{\mu}} A_{\beta\mu} + \frac{\partial\Lambda}{\partial(DA^{\alpha}_{\mu})} DA_{\beta\mu}$$

$$- \hat{Q}^{\mu\nu}A_{\alpha\mu}A_{\beta\nu} - \hat{Q}^{\mu\nu\lambda}(A_{\alpha\mu}A_{\beta\nu})|_{\lambda}, \qquad (2.78)$$

equations (2.75) can be written as

$$\rho DJ_{\alpha\beta} = K_{\alpha\beta}{}^{\gamma}{}_{|\gamma} + \rho h_{\alpha\beta}. \tag{2.79}$$

In this form the antisymmetric part of (2.79) can be considered as an internal angular momentum balance where $J_{[\alpha\beta]}$ are the components of the moment of momentum tensor, $K_{[\alpha\beta]}{}^{\gamma}$ are the components of the torque stress tensor, and $h_{[\alpha\beta]}$ are components of a stress-type tensor and applied body couples.

The energy equation (2.63) can be converted to an entropy balance equation by the following manipulations. Expanding $d\Lambda/dt$ in terms of the arguments of Λ, equation (2.63) can be written

$$\frac{\partial\Lambda}{\partial S}\frac{dS}{dt} + \frac{\partial\Lambda}{\partial A^{\alpha}{}_{\mu}} DA^{\alpha}{}_{\mu} + \frac{\partial\Lambda}{\partial DA^{\alpha}{}_{\mu}} DDA^{\alpha}{}_{\mu} + \sigma_{\nu}{}^{\mu\lambda} DS_{\mu\lambda}{}^{\nu}$$

$$+ \frac{\partial\Lambda}{\partial x^{\alpha}{}_{\mu}} Dx^{\alpha}{}_{\mu} + \frac{\partial\Lambda}{\partial v^{\alpha}} Dv^{\alpha} - 2\rho v_{\alpha} Dv^{\alpha}$$

$$- D\left(A^{\beta\mu} J_{\alpha\beta} DA^{\alpha}{}_{\mu}\right) + \left(\hat{q}^{\nu\mu\lambda}\hat{\pi}_{\nu\mu}\right)_{|\lambda} + p_{\alpha}{}^{\beta}{}_{|\beta} v^{\alpha} + p_{\alpha}{}^{\beta} v^{\alpha}{}_{|\beta}$$

$$+ N + F_{\alpha} v^{\alpha} = 0. \tag{2.80}$$

Using Euler equations (2.60) and (2.62), this can be simplified to

$$\rho\theta \frac{dS}{dt} = N + \frac{\partial\Lambda}{\partial A^{\alpha}{}_{\mu}} DA^{\alpha}{}_{\mu} + \rho D\left(\frac{\partial\Lambda/\rho}{\partial DA^{\alpha}{}_{\mu}} A^{\beta}{}_{\mu}B^{\nu}{}_{\beta}DA^{\alpha}{}_{\nu}\right)$$

$$- \rho D \frac{\partial\Lambda/\rho}{\partial DA^{\alpha}{}_{\mu}} DA^{\alpha}{}_{\mu} + \hat{\sigma}^{\nu\mu\lambda}\hat{\pi}_{\nu\lambda|\mu} + \frac{\partial\Lambda}{\partial x^{\alpha}{}_{\mu}} x^{\beta}{}_{\mu}v^{\alpha}{}_{|\beta}$$

$$- \rho D\left(A^{\beta\mu}DA^{\alpha}{}_{\mu}J_{\alpha\beta}\right) + \left(\hat{q}^{\nu\mu\lambda}\hat{\pi}_{\nu\mu}\right)_{|\lambda} + p_{\alpha}{}^{\beta}v^{\alpha}{}_{|\beta} \tag{2.81}$$

With the use of equations (2.54), (2.55), (2.56), (2.57), (2.58), (2.61) and the continuity equation

$$v^{\alpha}{}_{|\alpha} = 0, \tag{2.82}$$

equation (2.81) can be reduced to

$$\rho\theta \frac{dS}{dt} = N + \hat{Q}^{\mu\nu}\hat{\pi}_{\mu\nu} + \hat{Q}^{\mu\nu\lambda}\hat{\pi}_{\mu\nu|\lambda} + \hat{\tau}^{\mu\nu}\hat{v}_{\mu|\nu} \tag{2.83}$$

This is now in the form of an entropy balance. If only the internal entropy production is considered, it can be written as

$$\sigma = \sigma_1 + \sigma_2 + \sigma_3 \tag{2.84}$$

$$\left.\begin{aligned} \sigma_1 &= -\frac{\hat{q}^{\mu}}{\theta}\,\theta_{|\mu} \\ \sigma_2 &= \hat{Q}^{\mu\nu}\hat{\pi}_{\mu\nu} + \hat{Q}^{\mu\nu\lambda}\hat{\pi}_{\mu\nu|\lambda} \\ \sigma_3 &= \hat{\tau}^{\mu\nu}\hat{v}_{\mu|\nu} \end{aligned}\right\} \tag{2.85}$$

where σ_1 represents the irreversible effects due to heat conduction, σ_2 is due to plastic deformations and the motion of dislocations, and σ_3 is due to viscous dissipation.

In accordance with the general theory of irreversible processes, it is assumed that generalized forces are related to the fluxes through the equations

$$\left.\begin{aligned} -\theta^{-1}\hat{q}^{\mu} &= \mu_1 \frac{\partial\sigma_1}{\partial\theta_{|\mu}} \\ \hat{Q}^{\mu\nu} &= \mu_2 \frac{\partial\sigma_2}{\partial\hat{\pi}_{\mu\nu}} \\ \hat{Q}^{\mu\nu\lambda} &= \mu_2 \frac{\partial\sigma_2}{\partial\hat{\pi}_{\mu\nu|\lambda}} \end{aligned}\right\} \tag{2.86}$$

(Cont'd)

$$\hat{\tau}^{\mu\nu} = \mu_3 \left.\frac{\partial \sigma_3}{\partial \hat{v}_{\mu|\nu}}\right) ,$$

where σ_1, σ_2, and σ_3 are functions of their respective fluxes and also of x_S, constant or variable parameters, and where μ_1, μ_2, and μ_3 are constants of proportionality.

In the theory of plasticity, it is the assumed form of σ_2 and x_S which determine the yield surface and strain hardening characteristics of the medium.

Summary of Variational Equations

In summarizing the equations derived in this chapter, it is assumed that the Lagrangian can be written in the form

$$\Lambda = \frac{1}{2} \rho v^2 - \rho U. \qquad [2.68][1]$$

The Euler equations are presented in the simplified, or physically meaningful, form.

The complete variational equation is

---

[1] Brackets indicate a repeated equation number.

$$\int_V \int_{t_1}^{t_2} \Bigg\{ \left[-\rho D v_\alpha + p_\alpha{}^\beta{}_{|\beta} + F_\alpha \right] \delta x^\alpha$$

$$+ \Big[-\rho D J_{\alpha\beta} + \left(\hat{q}^{\mu\nu\lambda} A_{\alpha\nu} A_{\beta\mu} x^\gamma{}_\lambda \right)_{|\lambda}$$

$$+ \frac{\partial \Lambda}{\partial A^\alpha{}_\mu} A_{\beta\mu} + \frac{\partial \Lambda}{\partial (DA^\alpha{}_\mu)} DA_{\beta\mu} - \hat{Q}^{\mu\nu} A_{\alpha\mu} A_{\beta\nu}$$

$$- \hat{Q}^{\mu\nu\lambda} \left(A_{\alpha\mu} A_{\beta\nu} \right)_{|\lambda} \Big] A^{\beta\xi} \delta A^\alpha{}_\xi + \left[\frac{\partial \Lambda}{\partial S} + \rho\theta \right] \delta S$$

$$+ \Big[-\rho\theta \frac{dS}{dt} - \theta \frac{\hat{q}^\mu}{\theta}{}_{|\mu} + \frac{\hat{q}^\mu}{\theta} \theta_{|\mu} + \hat{Q}^{\mu\nu} \hat{\pi}_{\mu\nu}$$

$$+ \hat{Q}^{\mu\nu\lambda} \hat{\pi}_{\mu\nu|\lambda} + \hat{\tau}^{\mu\nu} \hat{v}_{\mu|\nu} \Big] \delta t \Bigg\} d\tau dt$$

$$+ \int_\Sigma \int_{t_1}^{t_2} \Bigg\{ \left[-p_\alpha{}^\beta + \underset{\sim}{p}_\alpha{}^\beta \right] n_\beta \delta_1 x^\alpha$$

$$+ \left[-\hat{q}^{\mu\nu\lambda} + \underset{\sim}{\hat{q}}^{\mu\nu\lambda} \right] x^\beta{}_\lambda B_{\nu\alpha} \delta_1 \Lambda^\alpha{}_\mu n_\beta \Bigg\} d\sigma dt$$

$$+ \int_V \Bigg\{ \rho \left[-J_\alpha + \underset{\sim}{J}_\alpha \right]_{t_1}^{t_2} \delta_1 x^\alpha + \left[-J_{\alpha\beta} + \underset{\sim}{J}_{\alpha\beta} \right]_{t_1}^{t_2} A^{\beta\mu} \delta_1 A^\alpha{}_\mu \Bigg\} d\tau$$

$$= 0. \qquad (2.87)$$

The Euler equations corresponding to the independent variations in the above equation are the coefficients in the square brackets. Obviously, the entropy balance was obtained by making use of the Euler equations corresponding to δx^{α}, $\delta A^{\alpha}{}_{\mu}$ and δS in the volume, Vt.

CHAPTER 3

EXTENDED VARIATIONAL PRINCIPLE

In this chapter, the Sedov-Berdichevski variational principle, discussed in Chapter 2, is extended in the manner of Washizu [17]. To do this it is necessary to make certain assumptions concerning the form of the Lagrangian. First, it is assumed that the Lagrangian can be written

$$\Lambda = \frac{1}{2}\rho v^2 - \rho U \qquad [2.68]$$

as in the latter part of Chapter 2. Also it is assumed that the arguments of U include

$$U = U\left(\hat{\varepsilon}^{(e)}_{\mu\nu}, \hat{\varepsilon}^{(p)}_{\mu\nu}, x^{\alpha}{}_{\mu}, \hat{S}^{\mu\nu\lambda}, DA^{\alpha}{}_{\mu}, S, A^{\alpha}{}_{\mu}\right) \qquad (3.1)$$

where the dependency on components, $A^{\alpha}{}_{\mu}$ and $x^{\alpha}{}_{\mu}$ is exclusive of their appearance in the elastic and plastic strains. This allows for the introduction of Lagrangian multipliers in connection with the kinematic relationship while still maintaining the generality of the original model.

Derivation of Extended Principle

The elastic strain tensor, equation (2.17) can be written, with the aid of equations (2.4) and (2.18), as

$$\hat{\varepsilon}^{(e)}_{\mu\nu} = \frac{1}{2}\left(x_{\alpha\mu}x^{\alpha}{}_{\nu} - A_{\alpha\mu}A^{\alpha}{}_{\mu}\right). \tag{3.2}$$

Similarly, the plastic strain tensor becomes

$$\hat{\varepsilon}^{(p)}_{\mu\nu} = \frac{1}{2}\left(A_{\alpha\mu}A^{\alpha}{}_{\nu} - \overset{\circ}{g}_{\mu\nu}\right). \tag{3.3}$$

With these assumptions the variation of the Lagrangian (considering only $\delta x^{\alpha}{}_{\mu}$ and $\delta A^{\alpha}{}_{\mu}$ temporarily) is

$$\delta\Lambda = \frac{\partial\Lambda}{\partial x^{\alpha}{}_{\mu}}\,\delta x^{\alpha}{}_{\mu} + \frac{\partial\Lambda}{\partial A^{\alpha}{}_{\mu}}\,\delta A^{\alpha}{}_{\mu} + \ldots$$

$$= \left(\frac{1}{2}\,v^2 - U\right)\frac{\partial\rho}{\partial x^{\alpha}{}_{\mu}}\,\delta x^{\alpha}{}_{\mu} - \rho\,\frac{\partial U}{\partial x^{\alpha}{}_{\mu}}\,\delta x^{\alpha}{}_{\mu}$$

$$- \rho\,\frac{\partial U}{\partial A^{\alpha}{}_{\mu}}\,\delta A^{\alpha}{}_{\mu} + \ldots$$

$$= -\Lambda\xi^{\mu}{}_{\alpha}\delta x^{\alpha}{}_{\mu} - \rho\,\frac{\partial U}{\partial\hat{\varepsilon}^{(e)}_{\mu\nu}}\,x_{\alpha\mu}\delta x^{\alpha}{}_{\nu}$$

$$+ \rho \frac{\partial U}{\partial \hat{\varepsilon}^{(e)}_{\mu\nu}} A_{\alpha\mu} \delta A^{\alpha}{}_{\nu} - \rho \frac{\partial U}{\partial \hat{\varepsilon}^{(p)}_{\mu\nu}} A_{\alpha\mu} \delta A^{\alpha}{}_{\nu} + \frac{\partial U}{\partial A^{\alpha}{}_{\mu}} \delta A^{\alpha}{}_{\mu}$$

$$+ \frac{\partial U}{\partial x^{\alpha}{}_{\mu}} \delta x^{\alpha}{}_{\mu} + \ldots$$

$$= - \Lambda \delta x^{\alpha}|\alpha - \rho \frac{\partial U}{\partial \hat{\varepsilon}^{(e)}_{\mu\nu}} \delta \hat{\varepsilon}^{(e)}_{\mu\nu} - \rho \frac{\partial U}{\partial \hat{\varepsilon}^{(p)}_{\mu\nu}} \delta \hat{\varepsilon}^{(p)}_{\mu\nu}$$

$$+ \frac{\partial U}{\partial A^{\alpha}{}_{\mu}} \delta A^{\alpha}{}_{\mu} + \frac{\partial U}{\partial x^{\alpha}{}_{\mu}} \delta x^{\alpha}{}_{\mu} + \ldots \qquad (3.4)$$

where the meaning of $A^{\alpha}{}_{\mu}$ and $x^{\alpha}{}_{\mu}$ in the final form is as in the stated assumptions.

With these variations substituted for those in equation (2.45), the variational equation can be written as

$$\delta_1 \int_V \int_{t_1}^{t_2} \Lambda d\tau dt + \delta_1 W + \delta_1 W^*$$

$$+ \delta_1 \int_V \int_{t_1}^{t_2} \hat{\sigma}^{\mu\nu} \left(\hat{\varepsilon}^{(e)}_{\mu\nu} - \frac{1}{2} x_{\alpha\mu} x^{\alpha}{}_{\mu} + \frac{1}{2} A_{\alpha\mu} A^{\alpha}{}_{\mu} \right) d\tau dt$$

$$+ \delta_1 \int_V \int_{t_1}^{t_2} \hat{\psi}^{\mu\nu} \left(\hat{\varepsilon}^{(p)}_{\mu\nu} - \frac{1}{2} A_{\alpha\mu} A^{\alpha}{}_{\nu} + \overset{\circ}{g}_{\mu\nu} \right) d\tau dt = 0 \qquad (3.5)$$

where $\hat{\sigma}^{\mu\nu}$ and $\hat{\psi}^{\mu\nu}$ are Lagrangian multipliers and are subject to variation.

The additional integrals of equation (3.4) can be expanded as

$$\delta_1 \int_V \int_{t_1}^{t_2} \hat{\sigma}^{\mu\nu}\left(\hat{\varepsilon}^{(e)}_{\mu\nu} - \frac{1}{2} x_{\alpha\mu}x^{\alpha}{}_{\mu} + \frac{1}{2} A_{\alpha\mu}A^{\alpha}{}_{\nu}\right) d\tau dt$$

$$= \int_V \int_{t_1}^{t_2} \left\{ \delta\hat{\sigma}^{\mu\nu}\left(\hat{\varepsilon}^{(e)}_{\mu\nu} - \frac{1}{2} x_{\alpha\mu}x^{\alpha}{}_{\nu} + \frac{1}{2} A_{\alpha\mu}A^{\alpha}{}_{\mu}\right)\right.$$

$$+ \hat{\sigma}^{\mu\nu}\delta\hat{\varepsilon}^{(e)}_{\mu\nu} - \hat{\sigma}^{\mu\nu}x_{\alpha\mu}\delta x^{\alpha}{}_{\nu} + \hat{\sigma}^{\mu\nu}A_{\alpha\mu}\delta A^{\alpha}{}_{\nu}$$

$$\left. + \frac{d}{dt}\left[\hat{\sigma}^{\mu\nu}\left(\hat{\varepsilon}^{(e)}_{\mu\nu} - \frac{1}{2} x_{\alpha\mu}x^{\alpha}{}_{\nu} + \frac{1}{2} A_{\alpha\mu}A^{\alpha}{}_{\nu}\right)\right]\delta t \right\} d\tau dt. \tag{3.6}$$

The term, $-\hat{\sigma}^{\mu\nu}x_{\alpha\mu}\delta x^{\alpha}{}_{\nu}$, can be further expanded by

$$-\hat{\sigma}^{\mu\nu}x_{\alpha\mu}\delta x^{\alpha}{}_{\nu} = -\hat{\sigma}^{\mu\nu}x_{\alpha\mu}x^{\beta}{}_{\nu}\delta x^{\alpha}{}_{|\beta}$$

$$= \sigma_{\alpha}{}^{\beta}{}_{|\beta}\,\delta x^{\alpha} + \left(\sigma_{\alpha}{}^{\beta}v^{\alpha}\right)_{|\beta}\delta t$$

$$- \left(\sigma_{\alpha}{}^{\beta}\delta_1 x^{\alpha}\right)_{|\beta} \tag{3.7}$$

where the identity

$$\sigma_{\alpha}{}^{\beta} = \hat{\sigma}^{\mu\nu} x_{\alpha\mu} x^{\beta}{}_{\nu} \tag{3.8}$$

has been used. Similarly the last integral of equation (3.5) is expanded to

$$\delta_1 \int_V \int_{t_1}^{t_2} \hat{\psi}^{\mu\nu}\left(\hat{\varepsilon}^{(p)}_{\mu\nu} - \frac{1}{2} A_{\alpha\mu} A^{\alpha}{}_{\nu} + \overset{\circ}{g}_{\mu\nu}\right) d\tau dt$$

$$= \int_V \int_{t_1}^{t_2} \left\{ \delta\hat{\psi}^{\mu\nu}\left(\hat{\varepsilon}^{(p)}_{\mu\nu} - \frac{1}{2} A_{\alpha\mu} A^{\alpha}{}_{\nu} + \overset{\circ}{g}^{\mu\nu}\right)\right.$$

$$+ \hat{\psi}^{\mu\nu} \delta \hat{\varepsilon}^{(p)}_{\mu\nu} - \hat{\psi}^{\mu\nu} A_{\alpha\mu} \delta A^{\alpha}{}_{\nu}$$

$$\left. + \frac{d}{dt}\left[\hat{\psi}^{\mu\nu}\left(\hat{\varepsilon}^{(p)}_{\mu\nu} - \frac{1}{2} A_{\alpha\mu} A^{\alpha}{}_{\nu} + \overset{\circ}{g}_{\mu\nu}\right)\right] \delta t \right\} d\tau dt \tag{3.9}$$

The complete variational equation can then be written by collecting terms with the same independent variation, i.e., δx^{α}, $\delta A^{\alpha}{}_{\mu}$, δt, δS, $\delta\hat{\varepsilon}^{(e)}_{\mu\nu}$, $\delta\hat{\varepsilon}^{(p)}_{\mu\nu}$, $\delta\hat{\sigma}^{\mu\nu}$, $\delta\hat{\psi}^{\mu\nu}$, on the volume, Vt, and $\delta_1 x^{\alpha}$, $\delta_1 A^{\alpha}{}_{\mu}$ on the boundary. With the substitution of equations (2.46), (2.47), (2.48), (2.49), (2.50), (2.52), (2.53), (3.6) and (3.9) into equation (3.5), and the use of the defining equations (2.55), (2.56), (2.57), (2.58), it becomes

$$\int_V \int_{t_1}^{t_2} \Bigg\{ \Bigg[-\rho D v_\alpha + \left(p_\alpha{}^\beta + \rho \frac{\partial U}{\partial x^\alpha{}_\mu} x^\beta{}_\mu \right)_{|\beta} + F_\alpha \Bigg] \delta x^\alpha$$

$$+ \Bigg[\left(\hat{\sigma}^{\nu\mu} - \hat{\psi}^{\nu\mu} \right) A_{\alpha\nu} + \frac{\partial \Lambda}{\partial A^\alpha{}_\mu} + \left(\hat{\sigma}_\lambda{}^{\mu\nu} B^\lambda{}_\alpha \right)_{|\nu} + \hat{\sigma}_\lambda{}^{\omega\nu} \hat{S}_{\nu\omega}{}^\mu B^\lambda{}_\alpha$$

$$-\rho D \frac{\partial \Lambda / \rho}{\partial D A^\alpha{}_\mu} - \left(\hat{Q}^{\nu\mu} - \hat{Q}^{\nu\mu\lambda}{}_{|\lambda} \right) A_{\alpha\nu} \Bigg] \delta A^\alpha{}_\mu$$

$$+ \left[\frac{\partial \Lambda}{\partial S} + \rho\theta \right] \delta S$$

$$+ \Bigg[\rho \frac{d}{dt} \Bigg(-\frac{1}{2} v^\alpha v_\alpha - U - A^{\beta\mu} J_{\alpha\beta} D A^\alpha{}_\mu + \hat{\sigma}^{\mu\nu} \Big(\hat{\varepsilon}^{(e)}_{\mu\nu}$$

$$- \frac{1}{2} x_{\alpha\mu} x^\alpha{}_\nu + \frac{1}{2} A_{\alpha\mu} A^\alpha{}_\nu \Big) + \hat{\psi}^{\mu\nu} \Big(\hat{\varepsilon}^{(p)}_{\mu\nu}$$

$$- \frac{1}{2} A_{\alpha\mu} A^\alpha{}_\nu + \overset{\circ}{g}_{\mu\nu} \Big) \Bigg) + \Big(\hat{q}^{\nu\mu\lambda} \hat{\pi}_{\nu\mu} x^\beta{}_\lambda + p_\alpha{}^\beta v^\alpha$$

$$+ \rho \frac{\partial U}{\partial x^\alpha{}_\mu} x^\beta{}_\mu v^\alpha \Big)_{|\beta} + N + F_\alpha v^\alpha \Bigg] \delta t$$

$$+ \left[\hat{\varepsilon}^{(e)}_{\mu\nu} - \frac{1}{2} x_{\alpha\mu} x^\alpha{}_\nu + \frac{1}{2} A_{\alpha\mu} A^\alpha{}_\nu \right] \delta \hat{\sigma}^{\mu\nu}$$

$$+ \left[\hat{\varepsilon}^{(p)}_{\mu\nu} - \frac{1}{2} A_{\alpha\mu} A^\alpha{}_\nu + \frac{1}{2} \overset{\circ}{g}_{\mu\nu} \right] \delta \hat{\psi}^{\mu\nu}$$

$$+ \left[-\rho \frac{\partial U}{\partial \hat{\varepsilon}^{(e)}_{\mu\nu}} + \hat{\sigma}^{\mu\nu} \right] \delta \hat{\varepsilon}^{(e)}_{\mu\nu}$$

$$+ \left[- \rho \frac{\partial U}{\partial \hat{\varepsilon}^{(p)}_{\mu\nu}} + \hat{\psi}^{\mu\nu} \right] \delta \hat{\varepsilon}^{(p)}_{\mu\nu} \Bigg\} d\tau dt$$

$$+ \int_{\Sigma} \int_{t_1}^{t_2} \Bigg\{ \left[- p_{\alpha}{}^{\beta} - \rho \frac{\partial}{\partial x^{\alpha}{}_{\mu}} x^{\beta}{}_{\mu} + \underset{\sim}{\hat{p}}{}^{\mu\nu\lambda} \right] n_{\beta} \delta_1 x^{\alpha}$$

$$+ \left[-\hat{q}^{\mu\nu\lambda} + \underset{\sim}{\hat{q}}{}^{\mu\nu\lambda} \right] x^{\beta}{}_{\lambda} A_{\alpha\nu} \delta_1 A^{\alpha}{}_{\mu} n_{\beta} \Bigg\} d\sigma dt$$

$$+ \int_{V} \Bigg\{ \rho \left[-v_{\alpha} + \underset{\sim}{J}_{\alpha} \right]_{t_1}^{t_2} \delta_1 x^{\alpha}$$

$$+ \rho \left[-J_{\alpha\beta} + \underset{\sim}{J}_{\alpha\beta} \right]_{t_1}^{t_2} A^{\beta\mu} \delta_1 A^{\alpha}{}_{\mu} \Bigg\} d\tau = 0. \qquad (3.10)$$

By the technique of separating the dependency of the Lagrangian into the elastic and plastic strain tensor components from the more general functions $x^{\alpha}{}_{\mu}$ and $A^{\alpha}{}_{\mu}$, the variational principle has been extended to contain the elastic and plastic constitutive relations and the elastic and plastic kinematic relations, in addition to the equations previously derived in Chapter 2. This has been done with no lessening of the generalities of the Sedov-Berdichevskii physical model except for the assumption implied by equation (2.68).

Discussion of Extended Principle

The Euler equations of the extended principle, equation (3.10) are:

Momentum.

$$\delta x^{\alpha}: \quad \left(p_{\alpha}{}^{\beta} + \rho \frac{\partial U}{\partial x^{\alpha}{}_{\mu}} x^{\beta}{}_{\mu}\right)_{|\beta} + F_{\alpha} = \rho D v_{\alpha} \tag{3.11}$$

Internal Degrees of Freedom.

$$\begin{aligned} \delta A^{\alpha}{}_{\mu}: \quad & \left(\hat{\sigma}^{\nu\mu} - \hat{\psi}^{\nu\mu} - \hat{Q}^{\nu\mu} + \hat{Q}^{\nu\mu\lambda}{}_{|\lambda}\right) A_{\alpha\nu} \\ & + \frac{\partial \Lambda}{\partial A^{\alpha}{}_{\mu}} + \left(\hat{\sigma}_{\lambda}{}^{\mu\nu} B^{\lambda}{}_{\alpha}\right)_{|\nu} + \hat{\sigma}_{\lambda}{}^{\omega\nu} \hat{S}_{\nu\omega}{}^{\mu} B^{\lambda}{}_{\alpha} \\ & - \rho D \frac{\partial \Lambda/\rho}{\partial D A^{\alpha}{}_{\mu}} = 0 \end{aligned} \tag{3.12}$$

Temperature-Entropy.

$$\delta S: \quad \frac{\partial \Lambda}{\partial S} + \rho\theta = 0 \tag{3.13}$$

Energy.

$$\delta t: \quad \rho \frac{d}{dt}\left(-\frac{1}{2} v^{\alpha} v_{\alpha} - U - A^{\beta\mu} J_{\alpha\beta} DA^{\alpha}{}_{\mu}\right)$$

$$+\left(\hat{q}^{\nu\mu\lambda}\hat{\pi}_{\nu\mu} x^{\beta}{}_{\lambda} + p_{\alpha}{}^{\beta} v^{\alpha} + \rho x^{\beta}{}_{\mu} \frac{\partial U}{\partial x^{\alpha}{}_{\mu}} v^{\alpha}\right)_{|\beta}$$

$$+ N + F_{\alpha} v^{\alpha} = 0 \tag{3.14}$$

Elastic Strain-Displacement

$$\delta\hat{\sigma}^{\mu\nu}: \quad \hat{\varepsilon}^{(e)}_{\mu\nu} = \frac{1}{2}\left(x_{\alpha\mu} x^{\alpha}{}_{\nu} - A_{\alpha\mu} A^{\alpha}{}_{\nu}\right) \tag{3.15}$$

Plastic Strain-Displacement

$$\delta\hat{\psi}^{\mu\nu}: \quad \hat{\varepsilon}^{(p)}_{\mu\nu} = \frac{1}{2}\left(A_{\alpha\mu} A^{\alpha}{}_{\nu} - \mathring{g}_{\mu\nu}\right) \tag{3.16}$$

Elastic Stress-Strain. (symmetric)

$$\delta\hat{\varepsilon}^{(e)}_{\mu\nu}: \quad \hat{\sigma}^{\mu\nu} = \rho \frac{\partial U}{\partial \hat{\varepsilon}^{(e)}_{\mu\nu}} \tag{3.17}$$

Plastic Stress-Strain.

$$\delta\hat{\varepsilon}^{(p)}_{\mu\nu}: \quad \hat{\psi}^{\mu\nu} = \rho \frac{\partial U}{\partial \hat{\varepsilon}^{(p)}_{\mu\nu}} \tag{3.18}$$

Boundary Conditions-Space

$$\delta_1 x^\alpha: \quad \underset{\sim}{p}_\alpha{}^\beta = p_\alpha{}^\beta + \rho \frac{\partial U}{\partial x^\alpha{}_\mu} x^\beta{}_\mu \tag{3.19}$$

$$\delta_1 A^\alpha{}_\mu: \quad \underset{\sim}{\hat{q}}^{\mu\nu\lambda} = \hat{q}^{\mu\nu\lambda} \tag{3.20}$$

Boundary Condition-Time (t_1&t_2)

$$\left.\begin{aligned} &\delta_1 x^\alpha: \quad \underset{\sim}{J}_\alpha = v_\alpha \\ &\delta_1 A^\alpha{}_\mu: \quad \underset{\sim}{J}_{\alpha\beta} = J_{\alpha\beta} \end{aligned}\right\} \tag{3.21}$$

The energy equation (3.14) has been simplified by the use of equations (3.15) and (3.16), and the Lagrangian multipliers are defined physically by equations (3.17) and (3.18).

It is interesting to note the differences in the form between the above equations and the corresponding equations derived in Chapter 2. The momentum equations, (3.11), for instance, show explicitly that other quantities besides the usual symmetric stress tensor may appear. One such quantity is, of course, the anti-symmetric stress tensor which arises when couple stresses are present. This case will be discussed below.

The internal degrees of freedom equations, (3.12), now show clearly how the stresses, defined by $\hat{\sigma}^{\nu\mu}$ and $\hat{\psi}^{\mu\nu}$, are coupled with the plastic and dislocation tensors. A further explanation of this will also be given in the next section. Equation (3.12) can be put in a form similar to that derived in Chapter 2, by multiplying both sides by $A_{\beta\mu}$. Doing this and manipulating the results as was done above, gives the relationship

$$\rho DJ_{\alpha\beta} = \frac{\partial\Lambda}{\partial DA^{\alpha}_{\ \mu}} DA_{\beta\mu} + \frac{\partial\Lambda}{\partial A^{\alpha}_{\ \mu}} A_{\beta\mu}$$

$$+ \left(\hat{\sigma}^{\nu\mu} - \hat{\psi}^{\nu\mu} - \hat{Q}^{\nu\mu}\right) A_{\alpha\nu} A_{\beta\mu}$$

$$+ \left(A_{\beta\mu} A_{\alpha\nu} \hat{Q}^{\nu\mu\lambda} x^{\gamma}_{\ \lambda}\right)_{|\gamma} - \hat{Q}^{\nu\mu\lambda} \left(A_{\beta\mu} A_{\alpha\nu}\right)_{|\lambda}$$

$$+ \left(A_{\beta\mu} A_{\alpha\lambda} \hat{\sigma}^{\lambda\mu\nu} x^{\gamma}_{\ \nu}\right)_{|\gamma}. \qquad (3.22)$$

This equation can be reduced to

$$\rho DJ_{\alpha\beta} = K_{\alpha\beta}{}^{\gamma}{}_{|\gamma} + \rho h_{\alpha\beta} \qquad (3.23)$$

where

$$K_{\alpha\beta}{}^{\gamma} = \hat{q}^{\mu\nu\lambda} A_{\alpha\nu} A_{\beta\mu} x^{\gamma}_{\ \lambda} \qquad (3.24)$$

and

$$\rho h_{\alpha\beta} = \frac{\partial \Lambda}{\partial A^{\alpha}_{\ \mu}} A_{\beta\mu} + \frac{\partial \Lambda}{\partial (DA^{\alpha}_{\ \mu})} DA_{\beta\mu} + \left(\hat{\sigma}^{\nu\mu} - \hat{\psi}^{\nu\mu} - \hat{Q}^{\nu\mu}\right) A_{\dot{\alpha}\nu} A_{\beta\mu}$$

$$- \hat{Q}^{\mu\nu\lambda} \left(A_{\alpha\mu} A_{\beta\nu}\right)_{|\lambda} . \tag{3.25}$$

The energy equation can also be rearranged in the same manner as in Chapter 2 to yield the entropy balance equation

$$\rho\theta \frac{dS}{dt} = \theta \left(\frac{\hat{q}^{\mu}}{\theta}\right)_{|\mu} + \frac{\hat{q}^{\mu}}{\theta} \theta_{|\mu} + \hat{Q}^{\mu\nu} \hat{\pi}_{\mu\nu} + \hat{Q}^{\mu\nu\lambda} \pi_{\mu\nu|\lambda} + \hat{\tau}^{\mu\nu} \hat{v}_{\mu|\nu} . \tag{3.26}$$

This, of course, is exactly the same as in Chapter 2.

Application of the Variational Principle to Classical Models

The best way to show the physical significance of the extended Euler equations is by giving examples of some well-known classical models. The procedure for applying the principle to specific models is as follows:

1. Assume the proper form for the Lagrangian.
2. Eliminate the generalized forces or stresses which do not apply to the particular model.
3. Assume a form for σ if the system is a dissipative one.

Elastic Body. For an elastic body, the Lagrangian takes the form

$$\Lambda = \frac{1}{2}\rho v^2 - \rho U\left(\hat{\varepsilon}_{\mu\nu}, S, L_p\right) \tag{3.27}$$

The only remaining terms in $\delta_1 W^*$ are those involving θ, F_α, and N, or

$$\delta W^* = \int_V \int_{t_1}^{t_2} \left[\rho\theta dS + F_\alpha \delta_1 x^\alpha - \mathrm{div}\ \vec{q}\right] d\tau dt. \tag{3.28}$$

Also, the tensor components, $A^\alpha{}_\mu$, become equal to $x^\alpha{}_\mu$ and the elastic strain becomes

$$\hat{\varepsilon}^{(e)}_{\mu\nu} = \frac{1}{2}\left(x_{\alpha\mu} x^\alpha{}_\nu - \overset{\circ}{g}_{\mu\nu}\right) \tag{3.29}$$

The field equations and boundary conditions are:

$$\text{Momentum:} \quad \sigma_\alpha{}^\beta{}_{|\beta} + F_\alpha = \rho D v_\alpha \tag{3.30}$$

$$\text{Temperature:} \quad \frac{\partial\Lambda}{\partial S} + \rho\theta = 0 \tag{3.31}$$

$$\text{Entropy:} \quad \rho\theta \frac{dS}{dt} = -\mathrm{div}\ \vec{q} \tag{3.32}$$

Strain-Displacement: $\hat{\varepsilon}^{(e)}_{\mu\nu} = \frac{1}{2}\left(\hat{g}_{\mu\nu} - \overset{\circ}{g}_{\mu\nu}\right)$ (3.33)

Constitutive Equation: $\hat{\sigma}^{\mu\nu} = \rho \dfrac{\partial U}{\partial \hat{\varepsilon}^{(e)}_{\mu\nu}}$ (3.34)

Boundary Conditions: $\sigma_{\alpha}{}^{\beta} = \underset{\sim}{p}_{\alpha}{}^{\beta}$ on Σ (3.35)

$$\underset{\sim}{J}_{\alpha} = v_{\alpha} \quad \text{at} \quad t_1, t_2 \qquad (3.36)$$

Also the relation between the stress tensor in Eulerian coordinates and in Lagrangián coordinates is

$$\sigma_{\alpha}{}^{\beta} = \hat{\sigma}^{\mu\nu} x_{\alpha\mu} x^{\beta}{}_{\nu} \qquad (3.37)$$

The equation for the internal degrees of freedom is not applicable for this model since the Lagrangian is not a function of $A^{\alpha}{}_{\mu}$, i.e., there is no variation $\delta A^{\alpha}{}_{\mu}$.

Elastic-Plastic Body. The Lagrangian for this model takes the form

$$\Lambda = \frac{1}{2}\rho v^2 - \rho U\left(\hat{\varepsilon}^{(e)}_{\mu\nu}, \hat{\varepsilon}^{(p)}_{\mu\nu}, S, L_p\right) \qquad (3.38)$$

where the dependency of the internal energy function on the plastic strain tensor implies a material "with a memory". The functional $\delta_1 W^*$ becomes

$$\delta_1 W^* = \int_V \int_{t_1}^{t_2} \left[\rho\theta\delta S + F_\alpha \delta_1 x^\alpha - \hat{Q}^{\mu\nu} B_{\alpha\mu} \delta A^\alpha{}_\nu - \text{div}\,\vec{q}\right] d\tau dt. \tag{3.39}$$

Under these assumptions the field equations become as follows:

Momentum: $\sigma_\alpha{}^\beta{}_{|\beta} + F_\alpha = \rho D v_\alpha$ (3.40)

Internal Degrees of Freedom: $\hat{Q}^{\mu\nu} = \hat{\sigma}^{\mu\nu} - \hat{\psi}^{\mu\nu}$ (3.41)

Temperature: $\frac{\partial \Lambda}{\partial S} + \rho\theta = 0$ (3.42)

Entropy: $\rho\theta \frac{dS}{dt} = -\text{div}\,\vec{q} + \hat{Q}^{\mu\nu}\hat{\pi}_{\mu\nu}$ (3.43)

Kinematic Equations:

$$\hat{\varepsilon}^{(e)}_{\mu\nu} = \frac{1}{2}\left(x_{\alpha\mu} x^\alpha{}_\nu - A_{\alpha\mu} A^\alpha{}_\nu\right) \tag{3.44}$$

$$\hat{\varepsilon}^{(p)}_{\mu\nu} = \frac{1}{2}\left(A_{\alpha\mu} A^\alpha{}_\nu - \mathring{g}_{\mu\nu}\right) \tag{3.45}$$

Constitutive Relations:

$$\hat{\sigma}^{\mu\nu} = \rho \frac{\partial U}{\partial \hat{\varepsilon}^{(e)}_{\mu\nu}} \tag{3.46}$$

$$\hat{\psi}^{\mu\nu} = \rho \frac{\partial U}{\partial \hat{\varepsilon}_{\mu\nu}^{(p)}} \qquad (3.47)$$

The boundary conditions are the same as for the elastic body.

In the above field equations there is a dissipative term in the entropy balance equation (3.43) which involves $\hat{Q}^{\mu\nu}$. This term can be written as in Chapter 2, equation (2.86),

$$\hat{Q}^{\mu\nu} = \mu_2 \frac{\partial \sigma_2}{\partial \hat{e}_{\mu\nu}^{p}} \, . \qquad (3.48)$$

The form of σ_2 can be chosen to give relationships for the yield surface as given in various theories of plasticity. For instance, if

$$\sigma_2 = k \sqrt{e_{\mu\nu}^{(p)} \, e^{\mu\nu(p)}} \, , \qquad (3.49)$$

the yield surface is

$$\hat{Q}_{\mu\nu}\hat{Q}^{\mu\nu} = k^2. \qquad (3.50)$$

In this equation k can either be a constant or a function such as

$$k = k(\chi) \tag{3.51}$$

where

$$\chi = \sqrt{\hat{\varepsilon}^{(p)}_{\mu\nu} \varepsilon^{\mu\nu(p)}} \tag{3.52}$$

Thus, the yield surface in terms of the stress tensor becomes

$$\left(\hat{\sigma}^{\mu\nu} - \hat{\psi}^{\mu\nu}\right)\left(\hat{\sigma}_{\mu\nu} - \hat{\psi}_{\mu\nu}\right) = k^2 \tag{3.53}$$

where equation (3.41) has been used. This example, shows the meaning of the internal degrees of freedom equation and how the stress tensors are coupled with the dissipative terms of plasticity.

Elastic Body with Couple Stresses. To derive the equations for this model, it is necessary to extend the original meaning of the Sedov-Berdichevskii model to include independent rotations among the varied quantities in the Lagrangian. This is done by defining the components of the rotation tensor as

$$\hat{\omega}_{\mu\nu} = \frac{1}{2} A_{\alpha[\mu} A^{\alpha}{}_{\nu]} \tag{3.54}$$

The independent variation of $\hat{\omega}_{\mu\nu}$ can now be assured by using the equations in which the variation of $A^{\alpha}{}_{\mu}$ appears. Since this model is elastic, the last step in this derivation is to let

$$A^{\alpha}{}_{\mu} \rightarrow \chi^{\alpha}{}_{\mu}.$$

The physical meaning of the terms will then be obvious.

The Lagrangian in this case takes the form

$$\Lambda = \frac{1}{2}\rho v^2 - \rho U\left(\hat{\varepsilon}_{\mu\nu}, \hat{\omega}_{\mu\nu}, \hat{\omega}_{\mu\nu|\lambda}, S, L_p\right) \qquad (3.55)$$

and the functional $\delta_1 W^*$ becomes

$$\delta_1 W^* = \int_V \int_{t_1}^{t_2} \left\{\rho\theta dS + F_\alpha \delta_1 x^\alpha - \hat{Q}^{\mu\nu\lambda}\left(B_{\mu\nu}\delta A^{\alpha}{}_{\nu}\right)_{|\lambda} + N\delta t\right\} d\tau dt. \qquad (3.56)$$

Since this model is a little more complicated than the previous one, the steps beginning with the general equations are shown. The momentum equations are

$$\left(p_\alpha{}^\beta + \rho \frac{\partial U}{\partial x^\alpha{}_\mu} x^\beta{}_\mu\right)_{|\beta} + F_\alpha = \rho D v_\alpha. \qquad [3.11]$$

But

$$p_\alpha{}^\beta = \sigma_\alpha{}^\beta \qquad (3.57)$$

and

$$\rho \frac{\partial U}{\partial x^\alpha{}_\mu} = \rho \frac{\partial U}{\partial \hat{\omega}_{\lambda\nu}} \frac{\partial \omega_{\lambda\nu}}{\partial x^\alpha{}_\mu}$$

$$= \rho \frac{\partial U}{\partial \hat{\omega}_{\mu\nu}} x_{\alpha\nu} \qquad (3.58)$$

where the identity $x^\alpha{}_\mu = A^\alpha{}_\mu$ was used. If the anti-symmetric stress tensor is defined as

$$\hat{\sigma}^{[\mu\nu]} = \rho \frac{\partial U}{\partial \hat{\omega}_{\mu\nu}} , \qquad (3.59)$$

the momentum equations can be written

$$\bar{\sigma}_\alpha{}^\beta{}_{|\beta} + F_\alpha = \rho D v_\alpha \qquad (3.60)$$

where

$$\bar{\sigma}_\alpha{}^\beta = \sigma_\alpha{}^\beta + \hat{\sigma}^{[\mu\nu]} x_{\alpha\mu} x^\beta{}_\nu \qquad (3.61)$$

The most convenient form of the internal degrees of freedom equation for this case is equation (3.22). It can be written as

$$\rho D \frac{\partial \Lambda}{\partial DA^{\alpha}_{\ \mu}} A_{\beta\mu} = \frac{\partial \Lambda}{\partial A^{\alpha}_{\ \mu}} A_{\beta\mu} + \left(A_{\beta\mu} A_{\alpha\nu} \hat{Q}^{\nu\mu\lambda} x^{\gamma}_{\ \lambda}\right)_{|\gamma}$$

$$- \hat{Q}^{\nu\mu\lambda}\left(A_{\beta\mu} A_{\alpha\nu}\right)_{|\lambda} + \left(A_{\beta\mu} A_{\alpha\lambda} \hat{\sigma}^{\lambda\mu\nu} x_{\nu}\right)_{|\gamma}. \tag{3.62}$$

The first term is zero since the Lagrangian is not a function of $DA^{\alpha}_{\ \mu}$, the third term on the right is zero since $x_{\beta\mu|\lambda}$ is zero, and the remaining terms become

$$\frac{\partial \Lambda}{\partial A^{\alpha}_{\ \mu}} A_{\beta\mu} = -\rho \frac{\partial U}{\partial \hat{\omega}_{\mu\nu}} x_{\alpha\nu} x_{\beta\mu}, \tag{3.63}$$

$$\left(A_{\beta\mu} A_{\alpha\nu} \hat{Q}^{\nu\mu\lambda} x^{\gamma}_{\ \lambda}\right)_{|\gamma} = Q^{\alpha\beta\gamma}{}_{|\gamma}, \tag{3.64}$$

$$\left(A_{\beta\mu} A_{\alpha\lambda} \hat{\sigma}^{\lambda\mu\nu} x^{\gamma}_{\ \nu}\right)_{|\gamma} = \sigma^{\alpha\beta\gamma}{}_{|\gamma}. \tag{3.65}$$

The internal degrees of freedom equations which now become the angular momentum equations are

$$\sigma^{[\alpha\beta]} + \sigma^{\alpha\beta\gamma}{}_{|\gamma} + \bar{Q}^{\alpha\beta} = 0, \tag{3.66}$$

where $\bar{Q}^{\alpha\beta} = Q^{\alpha\beta\gamma}{}_{|\gamma}$ and represent body couples and $\sigma^{\alpha\beta\gamma}$ is the couple stress tensor.

Since this model is elastic, there should be no dissipation involving the quantity $\hat{Q}^{\mu\nu}$. A check of the entropy balance equation (3.26) shows a term

$$\sigma_2 = \hat{Q}^{\mu\nu\lambda}\hat{\pi}_{\mu\nu|\lambda} \tag{3.67}$$

If $\hat{\pi}_{\mu\nu}$ is put in terms of the tensor component $A^{\alpha}{}_{\mu}$, equation (3.67) becomes

$$\sigma_2 = \hat{Q}^{\mu\nu\lambda}\left(A_{\alpha\mu}DA^{\alpha}{}_{\nu}\right)_{|\lambda} \tag{3.68}$$

Since $A^{\alpha}{}_{\nu} = x^{\alpha}{}_{\nu}$ and the covariant derivative of $x^{\alpha}{}_{\nu}$ is zero, the entropy production due to the couple stresses is zero as expected.

The remaining field equations and the boundary conditions for this model merely involve substitution in the general equation and will not be presented here. This model is derived from basic principles in the classical way by Eringen [2].

CHAPTER 4

GEOMETRY OF A SURFACE

In order to apply the Sedov-Berdichevskii variational principle to the theory of shells, it is necessary to review, briefly, the equations of the geometry of a surface. This chapter will be devoted to this task.

The references for this material are Green and Zerna [4], McConnell [7], and Naghdi [10]. The notation follows closely that used by Naghdi [10] with the exception that the roles of the Latin and Greek indices are reversed. In this chapter, and in Chapter 5, a Greek index has a range from 1 to 3, while a Latin index ranges from 1 to 2. This will make the surface geometry equations compatible (in notation) with the equations of Chapters 2 and 3.

Coordinate System

A coordinate system in three dimensional space is chosen in the form

$$r\left(x^{\alpha}\right) = \bar{r}\left(x^{a}\right) + x^{3}a_{3}\left(x^{a}\right) \qquad (4.1)$$

with the restriction

$$r_{,a}\cdot a_3 = 0 \quad \text{and} \quad a_3\cdot a_3 = 1 \ . \tag{4.2}$$

This defines a system of coordinates in space where $\bar{r}$ is the position vector to a point on a surface, and a_3 is a unit vector normal to the surface. It is obvious from equation (4.1) that

$$r\left(x^a,0\right) = \bar{r}\left(x^a\right) \ . \tag{4.3}$$

The surface base vector is defined as

$$a_a = \bar{r}_{,a} = g_a\left(x^b,0\right) \tag{4.4}$$

which leads to

$$a_a\cdot a_3 = 0 \tag{4.5}$$

and by differentiating the last equation in (4.2) to

$$a_{3,a}\cdot a_3 = 0 \ . \tag{4.6}$$

Some of the relationships which result from these definitions are:

$$\left.\begin{aligned} g_a &= a_a + x^3 a_{3,a} \\ g_3 &= a_3 \\ g_a \cdot g_3 &= g_{a3} = 0 \\ g_3 \cdot g_3 &= a_3 \cdot a_3 = g_{33} = 1 \end{aligned}\right\} \quad (4.7)$$

where $g_{\alpha\beta}$ is the metric in the three dimensional space.

First Fundamental Form

The first fundamental form is defined as the square of a line element on the surface, $x_3 = 0$, and is given by

$$d\bar{r} \cdot d\bar{r} = a_{ab} dx^a dx^b \ . \quad (4.8)$$

The quantities, a_{ab}, are the components of the surface metric tensor.

A two dimensional ε-system can be defined by

$$\varepsilon_{ab} = \varepsilon_{ab3} \quad \text{and} \quad e_{ab} = e_{ab3} \quad (4.9)$$

A surface ε-system can then be defined as

$$\bar{\varepsilon}_{ab} = a^{\frac{1}{2}} e_{ab} \tag{4.10}$$

which is analogous to the space equivalent

$$\varepsilon_{ab} = g^{\frac{1}{2}} e_{ab} \tag{4.11}$$

where a and g are the determinants of a_{ab} and $g_{\alpha\beta}$ respectively.

From equations (4.10) and (4.11) and the definitions thus far, the following relationships can be written:

$$\left.\begin{aligned} a^{ab} a_{bc} &= \delta^a_c \\ \varepsilon_{ab} &= \left(\frac{g}{a}\right)^{\frac{1}{2}} \bar{\varepsilon}_{ab} \\ \varepsilon^{ab} &= \left(\frac{a}{g}\right)^{\frac{1}{2}} \bar{\varepsilon}^{ab} \end{aligned}\right\} \tag{4.12}$$

In the above equations, e_{ab} is the relative alternating tensor defined by

$$\left.\begin{aligned} e_{11} = e_{22} = e^{11} = e^{22} = 0 \\ e_{12} = e^{12} = -e_{21} = -e^{21} = 1. \end{aligned}\right\} \quad (4.13)$$

Surface Tensors

Also, in the above equations $\bar{\varepsilon}_{ab}$ and $\bar{\varepsilon}^{ab}$ are absolute surface tensor components. The meaning of surface tensor components can be made clear by an example. Consider a tensor T and write it in terms of its components with respect to both the g_α basis and the a_a, a_3 basis.

$$T = T^\alpha_\beta g_\alpha g^\beta = \bar{T}^\alpha_\beta a_\alpha a^\beta \quad (4.14)$$

The components, T^α_β, are the usual spacial components of a tensor defined throughout the metric space. The components $\bar{T}^\alpha_\beta$ are the shifted components of the tensor T and are invariants with respect to a surface transformation. Although $\bar{T}^\alpha_\beta$ is called a surface tensor it may also be a function of the coordinate x^3. From now on, unless defined otherwise, a bar over the tensor components implies shifted components in the sense of equation (4.14).

The surface Kronecher delta is defined by

$$\bar{\varepsilon}^{ab}\bar{\varepsilon}_{\ell m} = \delta^{ab}_{\ell m} \qquad (4.15)$$

from which the following set of equations may be derived:

$$\left.\begin{aligned} \delta^{ab}_{\ell b} &= \delta^{a}_{\ell} \\ \delta^{ab}_{ab} &= 2 \\ \bar{T}^{ab} - \bar{T}^{ba} &= \delta^{ab}_{\ell m}\bar{T}^{\ell m} \\ \delta^{ab}_{\ell m}\bar{T}^{\ell}_{a} &= \delta^{b}_{m}\bar{T}^{a}_{a} - \bar{T}^{b}_{m} \end{aligned}\right\} \qquad (4.16)$$

The following relations between surface base vectors can be derived in a manner similar to their three dimensional counterparts.

$$\left.\begin{aligned} a_a \times a_3 &= \bar{\varepsilon}_{ba} a^b \\ a_a \times a_b &= \bar{\varepsilon}_{ab} a^3 \\ a^a \times a^b &= \bar{\varepsilon}^{ab} a_3 \end{aligned}\right\} \qquad (4.17)$$

Second Fundamental Form

The second fundamental form of the surface is defined by

$$-d\bar{r}\cdot da_3 = b_{ab}dx^a dx^b \tag{4.18}$$

from which

$$b_{ab} = -a_a\cdot a_{3,b} = a_3\cdot a_{a,b} \ . \tag{4.19}$$

Two invariants of the tensor b_{ab} can be formed. The mean curvature is defined as

$$H = \frac{1}{2}\, a^{ab}b_{ab} = \frac{1}{2}\, b^a_a \ . \tag{4.20}$$

Note that a^{ab} in this equation is used to raise the index of a surface tensor. It plays the same role with respect to surface tensors that $g^{\alpha\beta}$ does with respect to space tensors.

The second invariant is called the Gaussian curvature, and is

$$K = \left|b^a_b\right| = \frac{1}{2}\, \delta^{ab}_{\ell m} b^\ell_a b^m_b \tag{4.21}$$

Covariant Differentiation

A double vertical line will be used to distinguish the surface covariant derivative from the space derivative. It is defined as

$$\bar{T}^{a}{}_{b||c} = \bar{T}^{a}{}_{b,c} + \bar{\Gamma}_{c\ell}{}^{a}\bar{T}^{\ell}{}_{b} - \bar{\Gamma}_{bc}{}^{\ell}\bar{T}^{a}{}_{\ell} \tag{4.22}$$

where $\bar{\Gamma}_{bc}{}^{a}$ is the surface Christoffel symbol defined by

$$\bar{\Gamma}_{bc}{}^{a} = a^{a}\cdot a_{b,c} \ . \tag{4.23}$$

The surface Christoffel symbols are also equal to their space equivalents evaluated at $x_3 = 0$.

Some of the equations derived from these definitions are:

$$\left.\begin{aligned}
a_{ab||c} &= a^{ab}{}_{||c} = \bar{\varepsilon}_{ab||c} = \bar{\varepsilon}^{ab}{}_{||c} = 0 \\
a_{a||b} &= \bar{\Gamma}_{ab}{}^{3}a_{3} = b_{ab}a_{3} \\
a^{3}{}_{||c} &= a^{3}{}_{,c} = -b_{ca}a^{a} \\
a_{3||c} &= a_{3,c} = -b^{a}_{c}a_{a}
\end{aligned}\right\} \tag{4.24}$$

Third Fundamental Form

The last set of equations, (4.24), is known as Weingarten's formula. This can be used to define a third fundamental form

$$da_3 \cdot da_3 = b^{\ell}_{a} b_{\ell b} dx^a dx^b \qquad (4.25)$$

which can be used to obtain the following relationships:

$$\left.\begin{aligned} & b^a_a b^b_r - b^b_a b^a_r = 0 \\ & K\delta^b_r = 2Hb^b_r - b^b_a b^a_r \\ & b_{ac} b_{bd} - b_{ab} b_{cd} = \overline{R}_{dabc} \\ & b_{ab||c} - b_{ac||b} = 0 \end{aligned}\right\} \qquad (4.26)$$

where $\overline{R}_{dabc}$ is the covariant surface curvature tensor. Since the space is two dimensional $\overline{R}_{dabc}$ has only one independent component,

$$\overline{R}_{1212} = aK. \qquad (4.27)$$

The Space-Surface Shifter

The derivative of equation (4.1) with respect to x^α gives

$$\left.\begin{aligned} g_a &= a_a - x^3 b_a^c a_c \\ g_3 &= a_3 \, . \end{aligned}\right\} \qquad (4.28)$$

If the definition

$$\mu_a^c = \delta_a^c - x^3 b_a^c \qquad (4.29)$$

is employed, (4.28) can be written

$$g_a = \mu_a^c a_c \, . \qquad (4.30)$$

Note that μ_c^a plays the role of a shifter between the surface base vector and the space base vector. If μ_c^a has a unique inverse, then the contravariant base vectors may be related by

$$g^a = \bar{\mu}_c^a a^c \, . \qquad (4.31)$$

Naghdi [10] has shown that the inverse of μ^a_c exists, and is unique if

$$|x_3| < |R_{min}| \quad . \qquad (4.32)$$

In words, this says that the half thickness of the shell must be less than the minimum radius of curvature. This is obviously a weak restriction in view of the usual engineering application of shell theory.

Thus with the existence of the above shifter assured, the relation between the tensor components in equation (4.14) becomes

$$\left.\begin{aligned} T^a_b &= \bar{\mu}^a_n \mu^\ell_b \bar{T}^n_\ell \\ T^3_b &= \mu^\ell_b \bar{T}^3_\ell \\ T^a_3 &= \bar{\mu}^a_n \bar{T}^n_3 \\ T^3_3 &= \bar{T}^3_3 \end{aligned}\right\} \qquad (4.33)$$

Some additional equations involving μ^a_b are:

$$\left.\begin{aligned} \mu &= \left|\mu^a_b\right| = \left(\frac{g}{a}\right)^{\frac{1}{2}} \\ \mu &= 1 - 2x^3 H + \left(x^3\right)^2 K \\ \varepsilon_{ab} &= \mu \bar{\varepsilon} \\ \mu^b \bar{\mu}^a_b &= \delta^a_c \\ \mu &= \frac{1}{2}\,\delta^{am}_{\ell n}\mu^{\ell}_a \mu^n_m \end{aligned}\right\} \qquad (4.34)$$

<u>Space-Surface Relationships</u>

With the above definitions and equations, the space tensor components and their derivatives can be put in terms of surface tensors. This is done by writing out the space covariant derivative and using equation (4.33), and the following values for the space Christoffel symbols:

$$\left.\begin{aligned}
\Gamma_{ab}{}^{c} &= \bar{\Gamma}_{ab}{}^{c} + \bar{\mu}^{c}_{n}\mu^{n}_{a||b} \\
\Gamma_{b3}{}^{a} &= -\bar{\mu}^{a}_{\ell} b^{\ell}_{b} \\
\Gamma_{ab}{}^{3} &= \mu^{n}_{a} b_{nb} \\
\Gamma_{a3}{}^{3} &= \Gamma_{33}{}^{a} = \Gamma_{33}{}^{3} = 0
\end{aligned}\right\} \quad (4.35)$$

The most important results of this manipulation are expressed in the following set of equations:

$$T_{a|b} = \mu^{n}_{a}\left(\bar{T}_{n||b} - b_{nb}\bar{T}^{3}\right) \quad (4.36)$$

$$T^{a}{}_{|b} = \bar{\mu}^{a}_{n}\left(\bar{T}^{n}{}_{||b} - b^{n}_{b}\bar{T}^{3}\right) \quad (4.37)$$

$$T_{a|3} = \mu^{n}_{a}\bar{T}_{n,3} \quad (4.38)$$

$$T^{a}{}_{|3} = \bar{\mu}^{a}_{n}\bar{T}^{n}{}_{,3} \quad (4.39)$$

$$T_{3|a} = \bar{T}_{3,a} + b^{\ell}_{a}\bar{T}_{\ell} \quad (4.40)$$

$$T^{3}{}_{|a} = \bar{T}^{3}{}_{,a} + b_{a\ell}\bar{T}^{\ell} \quad (4.41)$$

$$T^{3}{}_{|3} = T_{3|3} = T_{3,3} = \bar{T}_{3,3} = \bar{T}^{3}{}_{,3} \quad (4.42)$$

$$T_{ab|c} = \mu^n_a \mu^\ell_b \left[\overline{T}_{n\ell||c} - b_{\ell c}\overline{T}_{n3} - b_{nc}\overline{T}_{3\ell} \right] \tag{4.43}$$

$$T_{a3|c} = \mu^n_a \left[\overline{T}_{n3|c} + b^\ell_c \overline{T}_{n\ell} - b_{nc}\overline{T}_{33} \right] \tag{4.44}$$

$$T_{3a|c} = \mu^n_a \left[\overline{T}_{3n||c} + b^\ell_c \overline{T}_{\ell n} - b_{nc}\overline{T}_{33} \right] \tag{4.45}$$

$$T_{33|c} = \overline{T}_{33,c} + b^a_c \overline{T}_{3a} + b^a_c \overline{T}_{a3} \tag{4.46}$$

$$T_{ab|3} = \mu^m_a \mu^n_b \overline{T}_{mn,3} \tag{4.47}$$

$$T_{a3|3} = \mu^n_a \overline{T}_{m3,3} \tag{4.48}$$

$$T_{3a|3} = \mu^n_a \overline{T}_{3a,3} \tag{4.49}$$

$$T_{33|3} = \overline{T}_{33,3} \tag{4.50}$$

In addition to these equations for the covariant derivatives of first and second rank tensors, some special relationships will be used in the shell theory of Chapter 5. These are

$$\mu_{|\ell} = \mu \bar{\mu}^c_a \mu^a_{\ell||c} \tag{4.51}$$

$$\mu T^{a}|_{a} = \left(\mu T^{a}\right)_{||a} - \mu\bar{\mu}^{a}_{n} b^{n}_{a} T^{3} \qquad (4.52)$$

$$\mu T^{3}|_{3} = \left(\mu T^{3}\right)_{|3} + \mu\bar{\mu}^{a}_{n} b^{n}_{a} T^{3} \qquad (4.53)$$

$$\mu\mu^{c}_{a} T^{ab}|_{b} = \left(\mu\mu^{c}_{n} T^{nb}\right)_{||b} - \mu\mu^{c}_{a}\bar{\mu}^{b}_{\ell} b^{\ell}_{b} T^{a3} - \mu b^{c}_{b} T^{3b} \qquad (4.54)$$

$$\mu T^{3a}|_{a} = \left(\mu T^{3a}\right)_{||a} + \mu\mu^{c}_{a} b_{cb} T^{ab} - \mu\bar{\mu}^{a}_{c} b^{c}_{a} T^{33} \qquad (4.55)$$

$$\mu^{c}_{a} T^{a3}|_{3} = \left(\mu^{c}_{a} T^{a3}\right)_{,3} . \qquad (4.56)$$

$$\mu_{,3} = \mu\bar{\mu}^{c}_{a}\mu^{a}_{c,3} = -\mu\bar{\mu}^{c}_{a} b^{a}_{c} \qquad (4.57)$$

$$\mu\mu^{c}_{a} T^{a3}|_{3} = \left(\mu\mu^{c}_{a} T^{a3}\right)_{,3} + \mu\mu^{c}_{a}\bar{\mu}^{d}_{b} b^{b}_{d} T^{a3} \qquad (4.58)$$

$$\mu T^{33}|_{3} = \left(\mu T^{33}\right)_{,3} + \mu\bar{\mu}^{d}_{b} b^{b}_{d} T^{33} \qquad (4.59)$$

Most of these equations have been given in Naghdi [10]. The few that are not given can be derived in a few simple steps.

Besides this group of equations it is necessary to derive similar relationships for a third rank tensor $T^{\alpha\beta c}|_{c}$ which will be used in Chapter 5. Consider the covariant derivative of T^{abc} with respect to x^{c}.

$$T^{abc}{}_{|c} = T^{abc}{}_{,c} + \Gamma_{dc}{}^{c}T^{abd} + \Gamma_{dc}{}^{b}T^{adc} + \Gamma_{dc}{}^{a}T^{dbc} + \Gamma_{3c}{}^{c}T^{ab3} + \Gamma_{3c}{}^{b}T^{a3c} + \Gamma_{3c}{}^{a}T^{3bc} \tag{4.60}$$

With the help of (4.35) and (4.22), this can be written as

$$T^{abc}{}_{|c} = T^{abc}{}_{||c} + \bar{\mu}^{c}_{n}\mu^{n}_{d||c}T^{abd} + \bar{\mu}^{b}_{n}\mu^{n}_{d||c}T^{adc} + \bar{\mu}^{a}_{n}\mu^{n}_{d||c}T^{dbc} - \bar{\mu}^{c}_{e}b^{e}_{c}T^{ab3} - \bar{\mu}^{b}_{e}b^{e}_{c}T^{a3c} - \bar{\mu}^{a}_{e}b^{e}_{c}T^{3bc} \tag{4.61}$$

Next, both sides are multiplied by $\mu\mu^{d}_{a}\mu^{i}_{b}$ and become

$$\begin{aligned}\mu\mu^{\ell}_{a}\mu^{i}_{b}T^{abc}{}_{|c} = {} & \left(\mu\mu^{\ell}_{a}\mu^{i}_{b}T^{abc}\right)_{||c} + \mu\mu^{\ell}_{a}\mu^{i}_{b}\bar{\mu}^{c}_{n}\mu^{n}_{d||c}T^{abd} \\ & + \mu\mu^{\ell}_{a}\mu^{i}_{b}\bar{\mu}^{b}_{n}\mu^{n}_{d||c}T^{adc} + \mu\mu^{\ell}_{a}\mu^{i}_{b}\bar{\mu}^{a}_{n}\mu^{n}_{d||c}T^{dbc} \\ & - \mu\mu^{\ell}_{a}\mu^{i}_{b}\bar{\mu}^{c}_{e}b^{e}_{c}T^{ab3} - \mu\mu^{\ell}_{a}\mu^{i}_{b}\bar{\mu}^{b}_{e}b^{e}_{c}T^{a3c} \\ & - \mu\mu^{\ell}_{a}\mu^{i}_{b}\bar{\mu}^{a}_{e}b^{e}_{c}T^{3bc} - \mu_{||c}\mu^{\ell}_{a}\mu^{i}_{b}T^{abc} \\ & - \mu\mu^{\ell}_{a||c}\mu^{i}_{b}T^{abc} - \mu\mu^{\ell}_{a}\mu^{i}_{b||c}T^{abc}\end{aligned} \tag{4.62}$$

This can be simplified with the aid of (4.51) to

$$\mu\mu^{\ell}_{a}\mu^{i}_{b}T^{abc}|_{c} = \left(\mu\mu^{\ell}_{a}\mu^{i}_{b}T^{abc}\right)_{||c} - \mu\mu^{\ell}_{a}\mu^{i}_{b}\bar{\mu}^{c}_{e}b^{e}_{c}T^{abc}$$

$$+ \mu\mu^{\ell}_{a}b^{i}_{c}T^{a3c} - \mu\mu^{i}_{b}b^{\ell}_{c}T^{3bc} \qquad (4.63)$$

By a similar procedure, the following equations can be derived:

$$\mu\mu^{\ell}_{a}\mu^{i}_{b}T^{ab3}|_{3} = \left(\mu\mu^{\ell}_{a}\mu^{i}_{b}T^{ab3}\right)_{,3} + \mu\mu^{\ell}_{a}\mu^{i}_{b}\bar{\mu}^{c}_{e}b^{e}_{c}T^{ab3} \qquad (4.64)$$

$$\mu\mu^{\ell}_{a}T^{a3c}|_{c} = \left(\mu\mu^{\ell}_{a}T^{a3c}\right)_{||c} + \mu\mu^{\ell}_{a}\mu^{m}_{n}b_{mc}T^{anc}$$

$$- \mu\mu^{\ell}_{a}\bar{\mu}^{c}_{m}b^{m}_{c}T^{a33} - \mu b^{\ell}_{c}T^{33c} \qquad (4.65)$$

$$\mu\mu^{\ell}_{a}T^{a33}|_{3} = \left(\mu\mu^{\ell}_{a}T^{a33}\right)_{,3} + \mu\mu^{\ell}_{a}\bar{\mu}^{c}_{m}b^{m}_{c}T^{a33} \qquad (4.66)$$

$$\mu\mu^{c}_{b}T^{3ba}|_{a} = \left(\mu\mu^{c}_{b}T^{3ba}\right)_{||a} + \mu\mu^{c}_{b}\mu^{m}_{n}b_{ma}T^{bna}$$

$$- \mu\mu^{c}_{b}\bar{\mu}^{n}_{m}b^{m}_{n}T^{3b3} - \mu b^{c}_{a}T^{33a} \qquad (4.67)$$

$$\mu\mu^{c}_{b}T^{3b3}|_{3} = \left(\mu\mu^{c}_{b}T^{3b3}\right)_{,3} + \mu\mu^{c}_{b}\bar{\mu}^{n}_{m}b^{m}_{n}T^{3b3} \qquad (4.68)$$

$$\mu T^{33a}{}_{|a} = \left(\mu T^{33a}\right)_{||a} + \mu\mu^{m}_{n} b_{ma} T^{n3a}$$

$$+ \mu\mu^{m}_{n} b_{ma} T^{3na} - \mu\bar{\mu}^{a}_{\ell} b^{\ell}_{a} T^{333} \qquad (4.69)$$

$$\mu T^{333}{}_{|3} = \left(\mu T^{333}\right)_{,3} + \mu\bar{\mu}^{a}_{\ell} b^{\ell}_{a} T^{333} \qquad (4.70)$$

CHAPTER 5

APPLICATION TO SHELL THEORY

In this chapter the extended Sedov-Berdichevskii variational principle is used to derive a set of shell equations. The total generality of the Sedov-Berdichevskii theory is maintained where possible, and special cases are shown to illustrate agreement with the usual shell theory derivations.

Momentum Equations

The momentum equations in three dimensions are obtained from the integral

$$I_1 = \int_V \int_{t_1}^{t_2} \left[-\rho D v_\alpha + t_\alpha{}^\beta{}_{|\beta} + F_\alpha \right] \delta x^\alpha d\tau \; dt \qquad (5.1)$$

in equation (3.10), where the definition

$$t_\alpha{}^\beta = \rho \frac{\partial}{\partial x^\alpha{}_\mu} x^\beta{}_\mu + p_\alpha{}^\beta \qquad (5.2)$$

has been used.

The variations, δx_α, can be written in terms of shifted components by applying equations (4.30), (4.28).

$$\left.\begin{aligned} \delta x_a &= \mu_a^c \delta \bar{x}_c \\ \delta x_3 &= \delta \bar{x}_3 \end{aligned}\right\} \qquad (5.3)$$

If $\bar{x}_a$ and $\bar{x}_3$ are assumed to have a linear variation in the x_3 direction, they can be written

$$\left.\begin{aligned} \bar{x}_a &= \bar{v}_a + x^3 \beta_a \\ \bar{x}_3 &= w + x^3 \beta_3 \end{aligned}\right\} \qquad (5.4)$$

where $\bar{v}_a$, β_a, β_3, w are functions only of x_1, x_2. The physical meaning of these quantities can be deduced from equations (5.4), i.e. the motion of a point on the middle surface of the shell ($x^3 = 0$) is described by $\bar{v}_a$, w. A rotation of the normal before deformation is defined by β_a and the elongation of the originally normal fibers is denoted by β_3. Thus, the only physical assumption implied by the above assumption is that straight normal fibers before deformation remain straight, but not necessarily normal or unextended.

Putting (5.4) into (5.3) yields

$$\left.\begin{aligned} \delta x_a &= \mu_a^c \delta \bar{v}_c + \mu_a^c x^3 \delta \beta_c \\ \delta x_3 &= \delta w + x^3 \delta \beta_3 \ . \end{aligned}\right\} \qquad (5.5)$$

From equation (2.34) and the relation

$$\mu(a)^{1/2} = (g)^{1/2} \ , \qquad (5.6)$$

which is one of equations (4.34), the volume element, $d\tau$, can be written

$$d\tau = \mu(a)^{1/2}\, dx^1 dx^2 dx^3 \qquad (5.7)$$

With (5.5) and (5.7) substituted in (5.1) and the remaining components expressed as subtensors, (5.1) becomes

$$I_1 = \int_v \int_{t_1}^{t_2} \Bigg\{ \Big[-\rho\mu\mu_a^c Dv^a + \mu\mu_a^c \left(t^{ab}{}_{|b} + t^{a3}{}_{|3} \right) + \mu\mu_a^c F^a \Big] \delta\bar{v}_c$$

$$+ \Big[-\mu\rho Dv^3 + \mu \left(t^{3b}{}_{|b} + t^{33}{}_{|3} \right) + \mu F^3 \Big] \delta w$$

$$+ \Big[-\rho\mu\mu_a^c x^3 Dv^a + \mu\mu_a^c x^3 \left(t^{ab}{}_{|b} + t^{a3}{}_{|3} \right)$$

$$+ \mu\mu_a^c x^3 F^a \Big] \delta\beta_c$$

$$+ \Big[-\mu x^3 \rho Dv^3 + \mu x^3 \left(t^{3b}{}_{|b} + t^{33}{}_{|3} \right)$$

$$+ \mu x^3 F^3 \Big] \delta\beta_3 \Bigg\} (a)^{\frac{1}{2}} \, dx^1 dx^2 dx^3 dt \qquad (5.8)$$

From equation (4.54), the second term becomes

$$\mu\mu_a^c t^{ab}{}_{|b} = \left(\mu\mu_n^c t^{nb} \right)_{||b} - \mu\mu_a^c \bar{\mu}_\ell^b b_b^\ell t^{a3} - \mu b_b^c t^{3b} \qquad (5.9)$$

Applying (4.58) to the next term yields

$$\mu\mu_a^c t^{a3}{}_{|3} = \left(\mu\mu_a^c t^{a3} \right)_{,3} + \mu\mu_a^c \bar{\mu}_\ell^b b_b^\ell t^{a3} \ . \qquad (5.10)$$

Similarly, with the use of (4.55) and (4.59), the next two terms with covariant derivatives become

$$\mu t^{3b}{}_{|b} = \left(\mu t^{3a}\right)_{||a} + \mu\mu^{c}_{a}b_{cb}t^{ab} - \mu\bar{\mu}^{a}_{c}b^{c}_{a}t^{33} \qquad (5.11)$$

$$\mu t^{33}{}_{|3} = \left(\mu t^{33}\right)_{,3} + \mu\bar{\mu}^{a}_{c}b^{c}_{a}t^{33} \,. \qquad (5.12)$$

The remaining terms of this type are obtained by multiplying the above equations by x^3.

$$\mu\mu^{c}_{a}x^{3}t^{ab}{}_{|b} = \left(\mu\mu^{c}_{n}x^{3}t^{nb}\right)_{||b} - \mu\mu^{c}_{a}\bar{\mu}^{b}_{\ell}b^{\ell}_{b}x^{3}t^{a3} - \mu b^{c}_{b}x^{3}t^{3b} \qquad (5.13)$$

$$\mu\mu^{c}_{a}x^{3}t^{a3}{}_{|3} = \left(\mu\mu^{c}_{a}x^{3}t^{a3}\right)_{,3} - \mu\mu^{c}_{a}t^{a3} + \mu\mu^{c}_{a}\bar{\mu}^{b}_{\ell}b^{\ell}_{b}x^{3}t^{a3} \qquad (5.14)$$

$$\mu x^{3}t^{3b}{}_{|b} = \left(\mu x^{3}t^{3a}\right)_{||a} + \mu\mu^{c}_{a}b_{cb}x^{3}t^{ab} - \mu\bar{\mu}^{a}_{c}b^{c}_{a}x^{3}t^{33} \qquad (5.15)$$

$$\mu x^{3}t^{33}{}_{|3} = \left(\mu x^{3}t^{33}\right)_{,3} - \mu t^{33} + \mu\bar{\mu}^{a}_{c}b^{c}_{a}t^{33} \qquad (5.16)$$

Upon substitution of equations (5.9) through (5.16) into (5.8), the integral becomes

$$
\begin{aligned}
I_1 = \int_v \int_{t_1}^{t_2} \Bigg\{ & \Big[-\rho\mu\mu_a^c Dv^a + \left(\mu\mu_n^c t^{nb}\right)_{||b} + \left(\mu\mu_a^c t^{a3}\right)_{,3} \\
& \qquad - \mu b_b^c t^{3b} + \mu\mu_a^c F^a \Big] \delta\bar{v}_c \\
& + \Big[-\rho\mu Dv^3 + \left(\mu t^{3a}\right)_{||a} + \left(\mu t^{33}\right)_{,3} \\
& \qquad + \mu\mu_a^c b_{cb} t^{ab} + \mu F^3 \Big] \delta w \\
& + \Big[-\rho\mu\mu_a^c x^3 Dv^a + \left(\mu\mu_n^c x^3 t^{nb}\right)_{||b} + \left(\mu\mu_a^c x^3 t^{a3}\right)_{,3} \\
& \qquad - \mu b_b^c x^3 t^{3b} - \mu\mu_a^c t^{a3} + \mu\mu_a^c x^3 F^a \Big] \delta\beta_c \\
& + \Big[-\rho\mu x^3 Dv^3 + \left(\mu x^3 t^{3a}\right)_{||a} + \left(\mu x^3 t^{33}\right)_{,3} \\
& \qquad + \mu\mu_a^c b_{cb} x^3 t^{ab} - \mu t^{33} \\
& \qquad + \mu x^3 F^3 \Big] \delta\beta_3 \Bigg\} (a)^{\frac{1}{2}}\, dx^3 dx^2 dx^1 dt \qquad (5.17)
\end{aligned}
$$

Next, each term in (5.17) is integrated with respect to x^3, and the resulting stress resultants and couples are defined by the following:

$$\left.\begin{aligned}
N^{ab} &= \int_{-h/2}^{h/2} \mu\mu_c^b t^{ca} dx^3 & M^{ab} &= \int_{-h/2}^{h/2} \mu\mu_c^b x^3 t^{ca} dx^3 \\
Q^a &= \int_{-h/2}^{h/2} \mu t^{3a} dx^3 & T^a &= \int_{-h/2}^{h/2} \mu x^3 t^{3a} dx^3 \\
Q^3 &= \int_{-h/2}^{h/2} \mu t^{33} dx^3 & & \\
\overline{Q}^a &= \int_{-h/2}^{h/2} \mu t^{a3} dx^3 & \overline{T}^a &= \int_{-h/2}^{h/2} \mu x^3 t^{a3} dx^3 \\
\ell^c &= \left[\mu\mu_a^c t^{a3}\right]_{-h/2}^{h/2} & m^c &= \left[\mu\mu_a^c x^3 t^{a3}\right]_{-h/2}^{h/2} \\
\ell^3 &= \left[\mu t^{33}\right]_{-h/2}^{h/2} & m^3 &= \left[\mu x^3 t^{33}\right]_{-h/2}^{h/2} \\
F^c &= \int_{-h/2}^{h/2} \mu\mu_a^c F^a dx^3 & M^c &= \int_{-h/2}^{h/2} \mu\mu_a^c x^3 F^a dx^3
\end{aligned}\right\} \quad (5.18)$$

$$\mathcal{F}^3 = \int_{-h/2}^{h/2} \mu F^3 dx^3 \qquad \mathcal{M}^3 = \int_{-h/2}^{h/2} \mu x^3 F^3 dx^3$$

$$\mathcal{B}^c = \int_{-h/2}^{h/2} \rho\mu\mu_a^c Dv^a dx^3 \qquad \mathcal{C}^c = \int_{-h/2}^{h/2} \rho\mu\mu_a^c x^3 Dv^a dx^3$$

$$\mathcal{B}^3 = \int_{-h/2}^{h/2} \rho\mu Dv^a dx^3 \qquad \mathcal{C}^3 = \int_{-h/2}^{h/2} \rho\mu x^3 Dv^3 dx^3$$

$$p^c = \ell^c + \mathcal{F}^c - \mathcal{B}^c \qquad c^c = m^c + \mathcal{M}^c - \mathcal{C}^c$$

$$p^3 = \ell^3 + \mathcal{F}^3 - \mathcal{B}^3 \qquad c^3 = m^3 + \mathcal{M}^3 - \mathcal{C}^3$$

Note that the asymmetry property of t^{a3} has made it necessary to define the quantities, $\overline{Q}^a$ and $\overline{T}^a$. In the classical theory of elastic shells, these quantities, of course, equal Q^a and T^a respectively. In fact, if the assumption is made that the fibers originally normal to the surface do not elongate, then T^a does not enter the momentum equations. It is also worthy of note at this point that N^{ab} and M^{ab} are asymmetric due to the asymmetry of t^{ab} in addition to that caused by μ_a^c. In other words, in the general case, where t^{ab} is not necessarily symmetric, N^{ab} and M^{ab} are non-symmetric even for the cases of thin shells where μ_a^c is replaced by the Kronecker delta.

With the definitions (5.18), equation (5.17) becomes

$$I_1 = \int_\Sigma \int_{t_1}^{t_2} \Big\{ \left[N^{bc}{}_{\|b} - b_b^c Q^b + p^c \right] \delta \bar{v}_c + \left[Q^a{}_{\|a} + b_{cb} N^{bc} + p^3 \right] \delta w + \left[M^{bc}{}_{\|b} - b_b^c T^b + b_b^c \bar{T}^b - \bar{Q}^c + c^c \right] \delta \beta_c + \left[T^b{}_{\|b} + b_{cb} M^{bc} - Q^3 + c^3 \right] \delta \beta_3 \Big\} (a)^{1/2}\, dx^2 dx^1 dt \tag{5.19}$$

Since the variations of the displacements in (5.19) are independent and arbitrary even when I_1 is considered part of (3.10), the momentum equations can be written as

$$\delta \bar{v}_c: \quad N^{bc}{}_{\|b} - b_b^c Q^b + p^c = 0 \tag{5.20}$$

$$\delta w: \quad Q^{a}{}_{||a} + b_{cb}N^{bc} + p^{3} = 0 \tag{5.21}$$

$$\delta \beta_{c}: \quad M^{bc}{}_{||b} - b^{c}_{b}T^{b} + b^{c}_{b}\bar{T}^{b} - \bar{Q}^{c} + c^{c} = 0 \tag{5.22}$$

$$\delta \beta_{3}: \quad T^{b}{}_{||b} + b_{cb}M^{bc} - Q^{3} + c^{3} = 0 \tag{5.23}$$

If the stress tensor, $t^{\alpha\beta}$, is symmetric, these equations reduce to

$$N^{bc}{}_{||b} - b^{c}_{b}Q^{b} + p^{c} = 0 \tag{5.24}$$

$$Q^{a}{}_{||a} + b_{cb}N^{bc} + p^{3} = 0 \tag{5.25}$$

$$M^{bc}{}_{||b} - Q^{c} + c^{c} = 0 \tag{5.26}$$

$$T^{b}{}_{||b} + b_{cb}M^{bc} - Q^{3} + c^{3} = 0 \tag{5.27}$$

These are in agreement wtth the usual results of elastic shell theory in which elongation of the normal fibers is permitted [5].

Internal Degrees of Freedom

The internal degrees of freedom are expressed through the Euler equation associated with $\delta A^{\alpha}{}_{\mu}$ in the general variational equation (3.10). Another form of this equation is shown as equation (3.23). This is a more convenient starting point to illustrate the derivation of the shell equations.

Consider the equation

$$I_2 = \int_V \int_{t_1}^{t_2} \left[E^{\alpha\beta} + K^{\alpha\beta\gamma}{}_{|\gamma} \right] \delta\Theta_{\alpha\beta} d\tau dt \qquad (5.28)$$

where $E^{\alpha\beta} = \rho h^{\alpha\beta} - \rho D J^{\alpha\beta}$ and where $J^{\alpha\beta}$, $K^{\alpha\beta\gamma}$ and $h^{\alpha\beta}$ are the contravariant forms of equations (2.58), (3.24) and (3.25), and

$$\delta\Theta_{\alpha\beta} = B^{\mu}{}_{\alpha} \delta A_{\beta\mu} \ . \qquad (5.29)$$

At this point it is necessary to know how $\Theta_{\alpha\beta}$ varies across the thickness of the shell so that the integration may be carried out. It is assumed for simplicity that $\overline{\Theta}_{\alpha\beta}$, the shifted components of $\Theta_{\alpha\beta}$, are independent of x^3. This is sufficient for this demonstration and furthermore, agrees with Naghdi [12] for the case where $\Theta_{\alpha\beta}$ represents the rotation tensor as in a body with couple stresses.

The shifted components of $\delta\Theta_{\alpha\beta}$ are:

$$\left.\begin{aligned} \delta\Theta_{ab} &= \mu_a^c\mu_b^d\delta\overline{\Theta}_{cd} \\ \delta\Theta_{a3} &= \mu_a^c\delta\overline{\Theta}_{c3} \\ \delta\Theta_{3a} &= \mu_a^c\delta\overline{\Theta}_{3c} \\ \delta\Theta_{33} &= \delta\overline{\Theta}_{33} \ . \end{aligned}\right\} \qquad (5.30)$$

The equations (5.30), (5.7) can be substituted in (5.28) to give

$$I_2 = \int_v \int_{t_1}^{t_2} \Big\{ \Big[\mu\mu_a^c E^{ad} - \mu\mu_a^c x^3 b_b^d E^{ab} + \mu\mu_a^c \mu_e^d K^{aeb}\big|_b$$

$$+ \mu\mu_a^c \mu_e^d K^{ae3}\big|_3 \Big] \delta\overline{\Theta}_{cd}$$

$$+ \Big[\mu E^{c3} - \mu x^3 b_a^c E^{a3} + \mu\mu_a^c K^{a3b}\big|_b$$

$$+ \mu\mu_a^c K^{a33}\big|_3 \Big] \delta\overline{\Theta}_{c3}$$

$$+ \Big[\mu E^{3c} - \mu x^3 b_a^c E^{3a} + \mu\mu_b^c K^{3ba}\big|_a$$

$$+ \mu\mu_b^c K^{3b3}\big|_3 \Big] \delta\overline{\Theta}_{3c}$$

$$+ \Big[\mu E^{33} + \mu K^{33a}\big|_a$$

$$+ \mu K^{333}\big|_3 \Big] \delta\overline{\Theta}_{33} \Big\} (a)^{\frac{1}{2}}\, dx^2 dx^1 dt \quad (5.31)$$

where (4.29) has been used to make the form of the terms involving the second rank tensor $E^{\alpha\beta}$ agree with the definitions of the stress and couple resultants defined previously. The following definitions can now be used:

$$
\left.
\begin{aligned}
P^{ab} &= \int_{-h/2}^{h/2} \mu\mu_c^b E^{ca} dx^3 & R^{ab} &= \int_{-h/2}^{h/2} \mu\mu_c^b x^3 E^{ca} dx^3 \\
P^{c3} &= \int_{-h/2}^{h/2} \mu E^{c3} dx^3 & R^{c3} &= \int_{-h/2}^{h/2} \mu x^3 E^{c3} dx^3 \\
P^{3c} &= \int_{-h/2}^{h/2} \mu E^{3c} dx^3 & R^{3c} &= \int_{-h/2}^{h/2} \mu x^3 E^{3c} dx^3 \\
P^{33} &= \int_{-h/2}^{h/2} \mu E^{33} dx^3 & &
\end{aligned}
\right\} \quad (5.32)
$$

With the substitution of (5.32) and the use of (4.63) through (4.70), equation (5.31) becomes

$$I_2 = \int_v \int_{t_1}^{t_2} \Bigg\{ \Big[P^{dc} - b^d_b R^{bc} + \left(\mu\mu^c_a \mu^d_e K^{aeb}\right)_{||b} + \left(\mu\mu^c_a \mu^d_b K^{ab3}\right)_{,3}$$

$$- \mu\mu^c_a b^d_b K^{a3b} - \mu\mu^d_b b^c_a K^{3ba} \Big] \delta\overline{\Theta}_{cd}$$

$$+ \Big[P^{c3} - b^c_a R^{a3} + \left(\mu\mu^c_a K^{a3b}\right)_{||b} + \left(\mu\mu^c_a K^{a33}\right)_{,3}$$

$$+ \mu\mu^c_a \mu^d_b b_{de} K^{abe} - \mu b^c_a K^{33a} \Big] \delta\overline{\Theta}_{c3}$$

$$+ \Big[P^{3c} - b^c_a R^{3a} + \left(\mu\mu^c_a K^{3ab}\right)_{||b} + \left(\mu\mu^c_a K^{a33}\right)_{,3}$$

$$+ \mu\mu^c_a \mu^d_b b_{de} K^{abe} - \mu b^c_a T^{33a} \Big] \delta\overline{\Theta}_{3c}$$

$$+ \Big[P^{33} + \left(\mu K^{33a}\right)_{||a} + \left(\mu K^{333}\right)_{,3} + \mu\mu^a_b b_{ac} K^{b3c}$$

$$+ \mu\mu^a_b b_{ac} K^{3bc} \Big] \delta\overline{\Theta}_{33} \Bigg\} (a)^{\frac{1}{2}} \, dx^2 dx^1 dt \qquad (5.33)$$

The following definitions can now be applied to the terms involving $K^{\alpha\beta\gamma}$:

$$\left.\begin{aligned}
L^{abc} &= \int_{-h/2}^{h/2} \mu\mu^c_e\mu^b_d K^{eda} dx^3 \\
L^{3bc} &= \left[\mu\mu^c_e\mu^b_d K^{ed3}\right]_{-h/2}^{h/2} \\
L^{a3c} &= \int_{-h/2}^{h/2} \mu\mu^c_d K^{d3a} dx^3 \\
L^{ab3} &= \int_{-h/2}^{h/2} \mu\mu^b_d K^{3da} dx^3 \\
L^{33c} &= \left[\mu\mu^c_a K^{a33}\right]_{-h/2}^{h/2} \\
L^{a33} &= \int_{-h/2}^{h/2} \mu K^{33a} dx^3 \\
L^{333} &= \left[\mu K^{333}\right]_{-h/2}^{h/2}
\end{aligned}\right\} \qquad (5.34)$$

With the substitution of (5.34) into (5.33), the final generalized form of the integral involving the internal degrees of freedom in shell theory notation becomes

$$I_2 = \int_\Sigma \int_{t_1}^{t_2} \Bigg\{ \Big[P^{dc} - b^d_b R^{bc} + L^{bdc}{}_{\|b} + L^{3dc} - b^d_b L^{b3c} - b^c_a L^{ad3} \Big] \delta\overline{\Theta}_{cd}$$

$$+ \Big[P^{c3} - b^c_a R^{a3} + L^{b3c}{}_{\|b} + L^{33c} + b_{de} L^{edc} - b^c_a L^{a33} \Big] \delta\overline{\Theta}_{c3}$$

$$+ \Big[P^{3c} - b^c_a R^{3a} + L^{bc3}{}_{\|b} + L^{33c} + b_{de} L^{edc} - b^c_a L^{a33} \Big] \delta\overline{\Theta}_{3c}$$

$$+ \Big[P^{33} + L^{a33}{}_{\|a} + L^{333} + b_{ac} L^{c3a} + b_{ac} L^{ca3} \Big] \delta\overline{\Theta}_{33} \Bigg\} (a)^{\frac{1}{2}} \, dx^2 dx^1 dt \quad (5.35)$$

Since each of these variations is considered independent even when I_2 is part of (3.10), the equations pertaining to the internal degrees of freedom can be written:

$$\delta\overline{\Theta}_{cd}\colon \quad P^{dc} - b^d_b R^{bc} + L^{bdc}\|_b + L^{3dc} - b^d_b L^{b3c} - b^c_a L^{ad3} = 0 \qquad (5.36)$$

$$\delta\overline{\Theta}_{c3}\colon \quad P^{c3} - b^c_a R^{a3} + L^{b3c}\|_b + L^{33c} + b_{de} L^{edc} - b^c_a L^{a33} = 0 \qquad (5.37)$$

$$\delta\overline{\theta}_{3c}\colon \quad P^{3c} - b^c_a R^{3a} + L^{bc3}\|_b + L^{33c} + b_{de} L^{edc} - b^c_a L^{a33} = 0 \qquad (5.38)$$

$$\delta\overline{\theta}_{33}\colon \quad P^{33} + L^{a33}\|_a + L^{333} + b_{ac} L^{c3a} + b_{ac} L^{ca3} = 0 \qquad (5.39)$$

Obviously, equations (5.37) and (5.38) will not be independent if the variation is either symmetric or antisymmetric. The example of the elastic body with couple stresses will show the effect of antisymmetry on these equations in a subsequent section.

Entropy Balance

Instead of trying to integrate the entire energy equation as it appears in the variational principle, it was noted that the energy equation could be transformed into the entropy balance which is a much simpler form. The pertinent integral is given by

$$I_3 = \int_V \int_{t_1}^{t_2} \left[-\rho\theta \frac{ds}{dt} - q^a{}_{|a} + Q^{\alpha\beta}\pi_{\alpha\beta} + Q^{\alpha\beta\gamma}\pi_{\alpha\beta|\gamma} \right] \delta t \, d\tau \, dt \tag{5.40}$$

where the viscous dissipation term has been dropped.

Entropy term. The first term in brackets can be written as

$$I_{(s)} = \int_{-h/2}^{h/2} \mu\rho\theta \frac{ds}{dt} \, dx^3 = \rho_0\theta \int_{-h/2}^{h/2} \mu \frac{\rho}{\rho_0} \frac{ds}{dt} \, dx^3 \tag{5.41}$$

where ρ_0 is the density at the center surface, $x^3 = 0$, and it is assumed that the temperature is uniform across the thickness. With the definition,

$$\bar{S} = \int_{-h/2}^{h/2} \mu\rho s \, dx^3, \tag{5.42}$$

the entropy flow term can be written as

$$I_{(s)} = \rho_0\theta \frac{d\bar{S}}{dt} . \tag{5.43}$$

Heat Flow. Consider next the integral

$$I_{(q)} = \int_{-h/2}^{h/2} \mu\left(q^a{}_{|a} + q^3{}_{|3}\right)dx^3 \ . \tag{5.44}$$

This can be rewritten, using (4.52) and (4.53), as

$$I_{(q)} = \int_{-h/2}^{h/2} \left[\left(\mu q^a\right)_{\|a} + \left(\mu q^3\right)_{,3}\right]dx^3 \tag{5.45}$$

With the definitions

$$\left.\begin{aligned} \mathcal{Q}^a &= \int_{-h/2}^{h/2} \mu q^a dx^3 \ , \\ &\text{and} \\ \mathcal{Q}^3 &= \Big[\mu q^3\Big]_{-h/2}^{h/2} \ , \end{aligned}\right\} \tag{5.46}$$

$I_{(q)}$ can be written as

$$I_{(q)} = \mathcal{Q}^a{}_{\|a} + \mathcal{Q}^3 \tag{5.47}$$

where $Q^{a}_{||a}$ represents an average value over the shell thickness, and Q^{3} is the heat flow through the top and bottom surfaces.

Plastic Dissipation. Before integrating this term with respect to x^3, it is necessary to determine how the plastic flow tensor, $\pi_{\alpha\beta}$, varies with respect to x^3. It has already been sssumed that the shifted displacements are linear in x^3, and that the shifted components, $\bar{\Theta}_{\alpha\beta}$ are constant. It can easily be shown that the linear distribution in shifted displacements implies a quadratic strain distribution of the form

$$\left.\begin{aligned} \varepsilon_{ab} &= \varepsilon^{(0)}_{ab} + x^3\varepsilon^{(1)}_{ab} + \left(x^3\right)^2\varepsilon^{(2)}_{ab} \ . \\ \varepsilon_{a3} &= \varepsilon^{(0)}_{a3} + x^3\varepsilon^{(1)}_{a3} \\ \varepsilon_{33} &= \varepsilon^{(0)}_{33} \end{aligned}\right\} \quad (5.48)$$

These relationships have been shown in Naghdi [10], and Habip and Ebcioglu [6]. Since $\pi_{\alpha\beta}$ is the sum of a plastic strain rate and vorticity, the assumption of a quadratic distribution in x^3 would be compatible with previous assumptions.

Therefore, it is assumed that

$$\left.\begin{aligned}
\pi_{ab} &= \mu_a^c\left(\bar{\pi}_{cb}^{(0)} + x^3\bar{\pi}_{c|3}^{(1)}\right) \\
\pi_{a3} &= \left(\bar{\pi}_{c3}^{(0)} + x^3\bar{\pi}_{c3}^{(1)}\right) \\
\pi_{3a} &= \left(\bar{\pi}_{3c}^{(0)} + x^3\bar{\pi}_{3c}^{(1)}\right) \\
\pi_{33} &= \bar{\pi}_{33} \ .
\end{aligned}\right\} \qquad (5.49)$$

The integral form of the dissipation term then becomes

$$\begin{aligned}
I_{(P)} = \int_{-h/2}^{h/2} \Big[& \mu\mu_a^\ell Q^{am}\bar{\pi}_{\ell m}^{(0)} + x^3\mu\mu_a^\ell Q^{am}\bar{\pi}_{\ell m}^{(1)} \\
& + \mu Q^{\ell 3}\bar{\pi}_{\ell 3}^{(0)} + x^3\mu Q^{\ell 3}\bar{\pi}_{\ell 3}^{(1)} + \mu Q^{3\ell}\bar{\pi}_{3\ell}^{(0)} \\
& + x^3\mu Q^{3\ell}\bar{\pi}_{3\ell}^{(1)} + \mu Q^{33}\bar{\pi}_{33}\Big] dx^3 \qquad (5.50)
\end{aligned}$$

The following definitions can be used to simplify (5.50). The notation is chosen to show that when $Q^{\alpha\beta}$ is equated to the stress tensor $p^{\alpha\beta}$, the associated stress and couple resultants also agree.

$$\left.\begin{aligned}
\tilde{N}^{m\ell} &= \int_{-h/2}^{h/2} \mu\mu^{\ell}_{a}Q^{am}dx^3 \qquad \tilde{M}^{m\ell} = \int_{-h/2}^{h/2} \mu\mu^{\ell}_{a}x^3Q^{am}dx^3 \\
\tilde{Q}^{a} &= \int_{-h/2}^{h/2} \mu Q^{3a}dx^3 \qquad \tilde{T}^{a} = \int_{-h/2}^{h/2} \mu x^3Q^{3a}dx^3 \\
\tilde{\tilde{Q}}^{a} &= \int_{-h/2}^{h/2} \mu Q^{a3}dx^3 \qquad \tilde{\tilde{T}}^{a} = \int_{-h/2}^{h/2} \mu x^3Q^{a3}dx^3 \\
\tilde{Q}^{3} &= \int_{-h/2}^{h/2} \mu Q^{33}dx^3 \ .
\end{aligned}\right\} \quad (5.51)$$

$I_{(P)}$ is, then,

$$I_{(P)} = \tilde{N}^{m\ell}\bar{\pi}^{(0)}_{\ell m} + \tilde{M}^{m\ell}\bar{\pi}^{(1)}_{\ell m} + \tilde{Q}^{a}\bar{\pi}^{(0)}_{a3} + \tilde{T}^{a}\bar{\pi}^{(1)}_{a3}$$

$$+ \tilde{\tilde{Q}}^{a}\bar{\pi}^{(0)}_{3a} + \tilde{\tilde{T}}^{a}\bar{\pi}^{(1)}_{3a} + \tilde{Q}^{3}\bar{\pi}_{33} \quad (5.52)$$

Dislocation Term. The assumption for the quadratic form of π_{ab} is again used to help integrate this term, but in this case the form of the shifted components is different.

$$\left.\begin{aligned} \pi_{ab} &= \mu^{\ell}_{a}\mu^{m}_{b}\bar{\pi}_{\ell m} \\ \pi_{a3} &= \mu^{\ell}_{a}\bar{\pi}_{\ell 3} \\ \pi_{3a} &= \mu^{\ell}_{a}\bar{\pi}_{3\ell} \\ \pi_{33} &= \bar{\pi}_{33} \end{aligned}\right\} \qquad (5.53)$$

This assumption is necessary so that the defining equations of resultant quantities agree in form with those of the internal degrees of freedom section.

Equations (4.43) through (4.50) are used to shift the components, $\pi_{\alpha\beta|\gamma}$, and the definitions (5.34) are extended to $Q^{\alpha\beta\gamma}$ by denoting them by $\tilde{L}^{abc}$, etc. With these substitutions, the dissipation term becomes

$$\begin{aligned} I_{(D)} = {} & \tilde{L}^{cba}\left(\bar{\pi}_{ab||c} - b_{ac}\bar{\pi}_{b3} - b_{ab}\bar{\pi}_{3c}\right) \\ & + \tilde{L}^{c3a}\left(\bar{\pi}_{a3||c} + b^{\ell}_{c}\bar{\pi}_{a\ell} - b_{ac}\bar{\pi}_{33}\right) \\ & + \tilde{L}^{ca3}\left(\bar{\pi}_{3a||c} + b^{\ell}_{c}\bar{\pi}_{\ell a} - b_{ac}\bar{\pi}_{33}\right) \end{aligned} \qquad (5.54)$$

Entropy Balance in Shell Notation. If all of the above terms are collected, and δt in equation (5.40) is considered arbitrary, the entropy balance becomes

$$\rho_0 \theta \frac{d\bar{S}}{dt} = -Q^a{}_{\|a} - Q^3 + \tilde{L}^{cba}\bar{\pi}_{ab\|c} + \tilde{L}^{c3a}\bar{\pi}_{a3\|c}$$

$$+ \tilde{L}^{ca3}\bar{\pi}_{3a\|c} + \left(b^m_c\tilde{L}^{c3\ell} + b^\ell_c\tilde{L}^{cm3}\right)\bar{\pi}_{\ell m}$$

$$- b_{bc}\tilde{L}^{cab}\bar{\pi}_{a3} - b_{bc}\tilde{L}^{abc}\bar{\pi}_{3a}$$

$$+ \left(-b_{ac}\tilde{L}^{c3a} - b_{ac}\tilde{L}^{ca3}\right)\bar{\pi}_{33} + \tilde{N}^{m\ell}\bar{\pi}^{(0)}_{\ell m}$$

$$+ \tilde{M}^{m\ell}\bar{\pi}^{(1)}_{\ell m} + \tilde{Q}^a\bar{\pi}^{(0)}_{a3} + \tilde{T}^a\bar{\pi}^{(1)}_{a3} + \tilde{\tilde{Q}}^a\bar{\pi}^{(0)}_{3a}$$

$$+ \tilde{\tilde{T}}^a\bar{\pi}^{(1)}_{3a} + \tilde{Q}^{33}\bar{\pi}_{33} \qquad (5.55)$$

In (5.55), $\bar{\pi}_{\ell m}$ and $\bar{\pi}^{(0)}_{\ell m}$, $\bar{\pi}^{(1)}_{\ell m}$ are not independent quantities, but merely different ways of expressing π_{ab}, the space components. This procedure interrelates explicitly plastic deformation and dislocations.

<u>Constitutive Equations</u>

In (3.10), the integral pertaining to the stress-strain equations is

$$I_4 = \int_{-h/2}^{h/2} \left[\left(\sigma^{\alpha\beta} - \rho \frac{\partial U}{\partial \varepsilon_{\alpha\beta}^{(e)}} \right) \delta \varepsilon_{\alpha\beta}^{(e)} + \left(\psi^{\alpha\beta} - \rho \frac{\partial U}{\partial \varepsilon_{\alpha\beta}^{(p)}} \right) \delta \varepsilon_{\alpha\beta}^{(p)} \right] d\tau dt \qquad (5.56)$$

where all of the tensor components have been converted to Eulerian form.

In keeping with the assumptions of the previous section, it is now assumed that

$$\left.\begin{aligned} \varepsilon_{ab}^{(e)} &= \mu_a^{\ell}\left(\gamma_{\ell b}^{(e)} + x^3 \kappa_{\ell b}^{(e)}\right) \\ \varepsilon_{a3}^{(e)} &= \gamma_{a3}^{(e)} + x^3 \kappa_{a3}^{(e)} \\ \varepsilon_{33}^{(e)} &= \gamma_{33}^{(e)} \end{aligned}\right\} \qquad (5.57)$$

and

$$\left.\begin{aligned}\varepsilon_{ab}^{(p)} &= \mu_a^\ell\left(\gamma_{\ell b}^{(p)} + x^3\kappa_{\ell b}^{(p)}\right)\\ \varepsilon_{a3}^{(p)} &= \gamma_{a3}^{(p)} + x^3\kappa_{a3}^{(p)}\\ \varepsilon_{33}^{(p)} &= \gamma_{33}^{(p)} .\end{aligned}\right\} \quad (5.58)$$

With substitutions similar to those previously made, equation (5.56) becomes

$$\begin{aligned}I_4 = \int_v\int_{t_1}^{t_2}\Bigg[&\left(\mu\mu_a^\ell\sigma^{ab} - \rho\mu\mu_a^\ell \frac{\partial U}{\partial\varepsilon_{ab}^{(e)}}\right)\delta\gamma_{\ell b}^{(e)}\\ &+ \left(x^3\mu\mu_a^\ell\sigma^{ab} - x^3\rho\mu\mu_a^\ell \frac{\partial U}{\partial\varepsilon_{ab}^{(e)}}\right)\delta\kappa_{\ell b}^{(e)}\\ &+ \left(\mu\sigma^{a3} - \rho\mu \frac{\partial U}{\partial\varepsilon_{a3}^{(e)}}\right)\delta\gamma_{a3}^{(e)}\\ &+ \left(\mu x^3\sigma^{a3} - \rho\mu \frac{\partial U}{\partial\varepsilon_{a3}^{(e)}}\right)\delta\kappa_{a3}^{(e)}\\ &+ \left(\mu\sigma^{33} - \rho\mu \frac{\partial U}{\partial\varepsilon_{33}^{(e)}}\right)\delta\gamma_{33}^{(e)} ,\\ &+ \text{...similar plastic terms...}\Bigg](a)^{1/2}dx^3dx^2dx^1dt\end{aligned} \quad (5.59)$$

Noting that

$$\frac{\partial U}{\partial \gamma_{\ell b}^{(e)}} = \mu_a^\ell \frac{\partial U}{\partial \varepsilon_{ab}^{(e)}} , \tag{5.60}$$

and defining

$$\overline{U} = \int_{-h/2}^{h/2} \rho\mu U dx^3 , \tag{5.61}$$

yields

$$I_4 = \int_\Sigma \int_{t_1}^{t_2} \Bigg[\left(N^{b\ell} - \frac{\partial \overline{U}}{\partial \gamma_{\ell b}^{(e)}} \right) \delta\gamma_{\ell b}^{(e)} + \left(M^{b\ell} - \frac{\partial \overline{U}}{\partial \kappa_{\ell b}^{(e)}} \right) \delta\kappa_{\ell b}^{(e)}$$

$$+ \left(Q^a - \frac{\partial \overline{U}}{\partial \gamma_{a3}^{(e)}} \right) \delta\gamma_{a3}^{(e)} + \left(T^a - \frac{\partial \overline{U}}{\partial \kappa_{a3}^{(e)}} \right) \delta\kappa_{a3}^{(e)}$$

$$+ \left(Q^3 - \frac{\partial \overline{U}}{\partial \gamma_{33}^{(e)}} \right) \delta\gamma_{33}^{(e)}$$

$$+ \text{ similar plastic terms} \ldots \Bigg] (a)^{1/2} \, dx^2 dx^1 dt \tag{5.62}$$

The Euler equations of (5.62) give the constitutive relations between stress and couple resultants and the strain measures. Under the assumption of small displacements they agree with those in Naghdi [9]. In general, however, they are not exactly the same as Habip's [5]. The difference is in the method of defining the measure of strain. Habip [5] uses three independent terms for ε_{ab},

$$\varepsilon_{ab} = \gamma_{ab}^{(0)} + x^3\gamma_{ab}^{(1)} + \left(x^3\right)^2\gamma_{ab}^{(2)} \qquad (5.63)$$

whereas in this work only two are independent as in equations (5.57).

<u>Boundary Conditions</u>

Integration of the boundary terms of (3.10) is straightforward and presents no new methods or techniques. As an example of the results for the stress and couple resultants at the boundary, integration yields

$$\left.\begin{aligned} N^{ab}\bar{n}_b &= \underset{\sim}{N}^{ab}\bar{n}_b \\ M^{ab}\bar{n}_b &= \underset{\sim}{M}^{ab}\bar{n}_b \\ Q^{a}\bar{n}_a &= \underset{\sim}{Q}^{a}\bar{n}_a \\ T^{a}\bar{n}_a &= \underset{\sim}{T}^{a}\bar{n}_a \end{aligned}\right\} \qquad (5.64)$$

where $\bar{n}_a$ are the shifted components of n_a, the unit normal to the boundary surface.

Summary of Shell Equations

The field equations of shell theory derived in this chapter can be summarized in the following:

Momentum

$$N^{bc}{}_{||b} - b^c_b Q^b + p^c = 0 \qquad [5.20]$$

$$Q^a{}_{||a} + b_{cb}N^{bc} + p^3 = 0 \qquad [5.21]$$

$$M^{bc}{}_{||b} - b^c_b T^b + b^c_b \bar{T}^b - \bar{Q}^c + c^c = 0 \qquad [5.22]$$

$$T^b{}_{||b} + b_{cb}M^{bc} - Q^3 + c^3 = 0 \qquad [5.23]$$

Symbols are defined in equation (5.18).

Internal Degrees of Freedom

$$P^{dc} - b^{d}_{b}R^{bc} + L^{bdc}{}_{||b} + L^{3dc} - b^{d}_{b}L^{b3c} - b^{c}_{a}L^{ad3} = 0 \qquad [5.36]$$

$$P^{c3} - b^{c}_{a}R^{a3} + L^{b3c}{}_{||b} + L^{33c} + b_{de}L^{edc} - b^{c}_{a}L^{a33} = 0 \qquad [5.37]$$

$$P^{3c} - b^{c}_{a}R^{3a} + L^{bc3}{}_{||b} + L^{33c} + b_{de}L^{edc} - b^{c}_{a}L^{a33} = 0 \qquad [5.38]$$

$$P^{33} + L^{a33}{}_{||a} + L^{333} + b_{ac}L^{c3a} + b_{ac}L^{ca3} = 0 \qquad [5.39]$$

Symbols are defined in (5.32) and (5.34).

Entropy Balance

$$\begin{aligned}\rho_0\theta\,\frac{d\bar{S}}{dt} = {} & -Q^{a}{}_{||a} - Q^{3} + \tilde{L}^{cba}\bar{\pi}_{ab||c} + \tilde{L}^{c3a}\bar{\pi}_{a3||c} \\ & + \tilde{L}^{ca3}\bar{\pi}_{3a||c} + \left(b^{m}_{c}\tilde{L}^{c3\ell} + b^{\ell}_{c}\tilde{L}^{cm3}\right)\bar{\pi}_{\ell m} \\ & - b_{bc}\tilde{L}^{cab}\bar{\pi}_{a3} - b_{bc}\tilde{L}^{abc}\bar{\pi}_{3a} \\ & - \left(b_{ac}\tilde{L}^{c3a} + b_{ac}\tilde{L}^{ca3}\right)\bar{\pi}_{33} + \tilde{N}^{m\ell}\bar{\pi}^{(0)}_{\ell m} + \tilde{M}^{m\ell}\bar{\pi}^{(1)}_{\ell m} \\ & + \tilde{Q}^{a}\bar{\pi}^{(0)}_{a3} + \tilde{T}^{a}\bar{\pi}^{(1)}_{a3} + \tilde{\tilde{Q}}^{a}\bar{\pi}^{(0)}_{3a} + \tilde{\tilde{T}}^{a}\bar{\pi}^{(1)}_{3a} + \tilde{Q}^{33}\bar{\pi}_{33}\end{aligned} \qquad [5.55]$$

Symbols are defined by (5.49), (5.51) and (5.34), where $\tilde{L}^{abc}$ denotes that $Q^{\alpha\beta\gamma}$ is used in the integrand instead of $K^{\alpha\beta\gamma}$.

Constitutive Equations

$$\left.\begin{aligned} N^{b\ell} &= \frac{\partial \bar{U}}{\partial \gamma^{(e)}_{\ell b}} & P^{b\ell} &= \frac{\partial \bar{U}}{\partial \gamma^{(p)}_{\ell b}} \\ M^{b\ell} &= \frac{\partial \bar{U}}{\partial \kappa^{(e)}_{\ell b}} & R^{b\ell} &= \frac{\partial \bar{U}}{\partial \kappa^{(p)}_{\ell b}} \\ Q^{a} &= \frac{\partial \bar{U}}{\partial \gamma^{(e)}_{a3}} & P^{c3} &= \frac{\partial \bar{U}}{\partial \gamma^{(p)}_{c3}} \\ T^{a} &= \frac{\partial \bar{U}}{\partial \kappa^{(e)}_{a3}} & R^{c3} &= \frac{\partial \bar{U}}{\partial \kappa^{(p)}_{c3}} \\ Q^{3} &= \frac{\partial \bar{U}}{\partial \gamma^{(e)}_{33}} & P^{33} &= \frac{\partial \bar{U}}{\partial \gamma^{(p)}_{33}} \end{aligned}\right\} \qquad (5.65)$$

The symbols in this case are defined by (5.18) with $t^{\alpha\beta} = \sigma^{\alpha\beta}$, the symmetric stress tensor, and by (5.32) with $E^{\alpha\beta} = \psi^{\alpha\beta}$, a symmetric plastic stress tensor.

An Application

Assume as in Chapter 3, that the rotation tensor can be written as

$$\hat{\omega}_{\mu\nu} = \frac{1}{2} A_{\alpha[\mu} A^{\alpha}{}_{\nu]} \ . \qquad [3.54]$$

Also assume that the Lagrangian is a function of $\hat{\varepsilon}_{\mu\nu}$, $\hat{\omega}_{\mu\nu}$, s, and L_p only, and that there are no terms in $\delta_1 W^*$ other than those for the simple elastic case as in (3.28). It is further assumed that the body is elastic and therefore $A^{\alpha}{}_{\mu} \to X^{\alpha}{}_{\mu}$.

These assumptions, together with (3.59), reduce equation (5.35) to

$$I_2 = \int_V \int_{t_1}^{t_2} \left\{ \left(P^{dc} - b^d_b R^{bc}\right) \delta\bar{\omega}_{cd} + \left(P^{c3} - b^c_a R^{a3}\right) \delta\bar{\omega}_{c3} + \left(P^{3c} - b^c_a R^{3a}\right) \delta\bar{\omega}_{3c} \right\} a^{\frac{1}{2}} \, dx^2 dx^1 dt \qquad (5.66)$$

The Euler equations of (5.66) are

$$\bar{\varepsilon}_{cd}\left(P^{dc} - b^d_b R^{bc}\right) = 0 \qquad (5.67)$$

$$P^{c3} - b^c_a R^{a3} = P^{3c} - b^c_a R^{3a} \qquad (5.68)$$

By the above assumptions, P^{dc} and R^{bc} are exactly equal to N^{dc} and M^{bc} and thus equation (5.67) is the so-called sixth equation of momentum of the classical theory. Naghdi [10] has derived this equation in a similar manner. Equation (5.68) serves to reduce

$$M^{bc}{}_{||b} - b^c_b T^b + b^c_b \bar{T}^b - \bar{Q}^c + C^c = 0 \qquad [5.22]$$

to the classical equivalent

$$M^{bc}{}_{||b} - Q^c + C^c = 0 \ . \qquad [5.26]$$

Though these equations are immediately derivable from the symmetry property of the stress tensor, their derivation in this manner serves to illustrate the role of the internal degrees of freedom equations even in the simplest model.

<u>Summary</u>

This paper extends the Sedov-Berdichevskii variational principle [1] and employs this extended principle in the construction of a shell theory.

The extension of the principle is accomplished by the use of Lagrangian multipliers, and a redefinition of terms in order to maintain the original generality of the principle. The Euler equations of the extended variational principle provide, in addition to those obtained in the original principle, elastic and plastic kinematic relations, and, elastic and plastic comtitutive relations. Also, the original Euler equations are expressed in a more physically meaningful form. Examples are given which show how this principle can be applied to various classical models whose formulation is well known.

The extended principle is also used to derive a shell theory. This is done by integrating the three-dimensional equations across the thickness of the shell. The derived theory is "exact" within the assumptions concerning the variation of displacements and velocities across the thickness of the shell. A linear form is assumed for the shifted displacements and velocities.

A complete set of equations is derived including momentum equations, equations involving internal degrees of freedom, an entropy balance, constitutive relations, and some typical boundary conditions. An application is also given which shows how the derived equations reduce to the classical equations for a specified case of an elastic shell.

APPENDIX A

DERIVATION OF TYPICAL VARIATIONAL TERM

Consider the term

$$\frac{\partial \Lambda}{\partial \hat{S}_{\mu\nu}{}^{\lambda}} \, \delta\left(\hat{S}_{\mu\nu}{}^{\lambda}\right).$$

It is required to express this term in terms of the independent variations, $\delta A^{\alpha}{}_{\mu}$, and δt for the volume, and $\delta_1 A^{\alpha}{}_{\mu}$ for the boundary.

By using (2.42) and defining

$$\hat{\sigma}^{\mu\nu}{}_{\lambda} = \frac{\partial \Lambda}{\partial \hat{S}_{\mu\nu}{}^{\lambda}} \qquad \text{(A-1)}$$

the above term can be written as

$$\hat{\sigma}_{\lambda}{}^{\mu\nu}\delta\left(\hat{S}_{\mu\nu}{}^{\lambda}\right) = \hat{\sigma}_{\lambda}{}^{\mu\nu}B^{\lambda}{}_{\alpha}\delta\left(A^{\alpha}{}_{[\nu|\mu]}\right) - \hat{\sigma}_{\lambda}{}^{\mu\nu}S_{\mu\nu}{}^{\omega}B^{\lambda}{}_{\alpha}\delta A^{\alpha}{}_{\omega} \qquad \text{(A-2)}$$

Since $\hat{\sigma}_{\lambda}{}^{\mu\nu}$ is antisymmetric in the indices μ, ν, the brackets can be dropped from the first term on the right. The application of (2.35) to (A-2) yields

$$\hat{\sigma}_{\lambda}{}^{\mu\nu}\delta\left(\hat{S}_{\mu\nu}{}^{\lambda}\right) = \hat{\sigma}_{\lambda}{}^{\mu\nu}B^{\lambda}{}_{\alpha}\delta_{1}\left(A^{\alpha}{}_{\nu|\mu}\right) - \hat{\sigma}_{\lambda}{}^{\mu\nu}B^{\lambda}{}_{\alpha}D\left(A^{\alpha}{}_{\nu|\mu}\right)\delta t + \hat{\sigma}_{\lambda}{}^{\omega\nu}\hat{S}_{\nu\omega}{}^{\mu}B^{\lambda}{}_{\alpha}\delta A^{\alpha}{}_{\mu} \quad \text{(A-3)}$$

where the sign has been changed on the last term by using the antisymmetry property of $\hat{\sigma}_{\lambda}{}^{\omega\nu}$. The dummy indices have also been rearranged.

Since the order of variation and differentiation can be interchanged, (A-3) can be written

$$\hat{\sigma}_{\lambda}{}^{\mu\nu}\delta\left(\hat{S}_{\mu\nu}{}^{\lambda}\right) = \left(\hat{\sigma}_{\lambda}{}^{\mu\nu}B^{\lambda}{}_{\alpha}\delta_{1}A^{\alpha}{}_{\nu}\right)_{|\mu} - \left(\hat{\sigma}_{\lambda}{}^{\mu\nu}B^{\lambda}{}_{\alpha}\right)_{|\mu}\delta_{1}A^{\alpha}{}_{\nu} - \left(\hat{\sigma}_{\lambda}{}^{\mu\nu}B^{\lambda}{}_{\alpha}DA^{\alpha}{}_{\nu}\right)_{|\mu}\delta t + \left(\hat{\sigma}_{\lambda}{}^{\mu\nu}B^{\lambda}{}_{\alpha}\right)_{|\mu}DA^{\alpha}{}_{\mu}\delta t + \hat{\sigma}_{\lambda}{}^{\omega\nu}\hat{S}_{\nu\omega}{}^{\mu}B^{\lambda}{}_{\alpha}\delta A^{\alpha}{}_{\mu} \quad \text{(A-4)}$$

This can be simplified, by again applying (2.35), to

$$\hat{\sigma}_{\lambda}{}^{\mu\nu}\delta\left(\hat{S}_{\mu\nu}{}^{\lambda}\right) = -\left(x^{\beta}{}_{\lambda}\hat{\sigma}_{\nu}{}^{\mu\lambda}B^{\nu}{}_{\alpha}\delta_{1}A^{\alpha}{}_{\mu}\right)_{|\beta} + \left(\hat{\sigma}_{\lambda}{}^{\mu\nu}B^{\lambda}{}_{\alpha}\right)_{|\nu}\delta A^{\alpha}{}_{\mu}$$

$$+ \left(\hat{\sigma}^{\nu\mu\lambda}\hat{\pi}_{\nu\mu}x^{\beta}{}_{\lambda}\right)_{|\beta}\delta t + \hat{\sigma}_{\lambda}{}^{\omega\nu}S_{\nu\omega}{}^{\mu}B^{\lambda}{}_{\alpha}\delta A^{\alpha}{}_{\mu} \tag{A-5}$$

where signs were changed by using the antisymmetry property of $\hat{\sigma}^{\nu\mu\lambda}$, and $\hat{\pi}_{\nu\mu}$ was introduced by the definition (2.30). Equation (A-5) is the desired form.

REFERENCES

1. Berdichevskii, V. L. and Sedov, L. I., *Dynamic Theory of Continuously Distributed Dislocations. Its Relation to Plasticity Theory*, PMM, Vol. 31, No. 6, 1967, 981-1000.

2. Eringen, A. C., *Nonlinear Theory of Continuous Media*, McGraw-Hill, New York, 1962.

3. Green, A. E. and Naghdi, P. M., *Some Remarks on the Linear Theory of Shells*, Quart. Jour. Mech. and Appl. Math., Vol. XVIII, Pt. 3, 1965, 257-276.

4. Green, A. E. and Zerna, W., *Theoretical Elasticity*, Oxford, The Clarendon Press, 1954.

5. Habip, L. M., *Theory of Elastic Shells in the Reference State*, Ing. Arch., Vol. 34, No. 4, 1965, 228-237.

6. Habip, L. M., and Ebcioglu, J. K., *On the Equations of Motion of Shells in the Reference State*, Ing. Arch., Vol. 34, No. 1, 1965, 28-32.

7. McConnell, A. J., *Applications of Tensor Analysis*, Dover Publications, Inc., New York, 1957.

8. Mura, T., *Method of Continuously Distributed Dislocations*, Mathematical Theory of Dislocations, Edited by T. Mura, ASME, United Engineering Center, New York, 1969, 25-48.

9. Naghdi, P. M., *A New Derivation of the General Equations of Elastic Shells*, Intern. Jour. of Eng. Sci., Vol. 1, 1963, 509-522.

10. Naghdi, P. M., *Foundations of Elastic Shell Theory*, Progress in Solid Mechanics, Edited by I. N. Sneddon and R. Hill, Vol. 4, 1963, 1-90.

11. Naghdi, P. M., *Further Results in the Derivation of the General Equations of Elastic Shells*, Intern. Jour. of Eng. Sci., Vol. 2, 1964, 269-273.

12. Naghdi, P. M., *On a Variational Theorem in Elasticity and its Application to Shell Theory*, Journal of Appl. Mech., Vol. 31, No. 4, 1964, 647-653.

13. Schroeder, J. and Sherbourne, A. N., *A General Theorem for Thin Shells in Classical Plasticity*, Journal of Math. and Phys., Vol. 47, No. 1, Mar. 1968, 95-108.

14. Sedov, L. I., *Introduction in the Mechanics of Continuous Media*, Addison-Wesley, Mass., 1965.

15. Sedov, L. I., *Models of Continuous Media with Internal Degrees of Freedom*, PMM, Vol. 32, No. 5, 1968, 771-785.

16. Teutonico, L. J., *Dislocations in Anisotropic Media*, Mathematical Theory of Dislocations, Edited by T. Mura, ASME, United Engineering Center, New York, 1969, 49-69.

17. Washizu, K., *Variational Methods in Elasticity and Plasticity*, Pergamon Press, Oxford, 1968.

VITA

Author: Francis X. Prendergast

Education:

New York University M.S.E.M. June, 1963

This degree was obtained as part of the three-year Bell Telephone Laboratories "Communication Development Training Program" which was completed in June, 1964.

Newark College of Engineering

B.S. in M.E. June, 1961

Professional Experience:

January, 1951 to September, 1969

Bell Telephone Laboratories,
Whippany, New Jersey

1961 - 1969 - Member of Technology Group of the Environmental Engineering Department

1959 - 1961 - Member of Electro-Mechanical Design Department

1954 - 1959 - Instructor in Education Center of the Personnel Department

1951 - 1954 - Designer

Dissertation:

The work leading to this dissertation was done by the author while on an educational leave of absence from Bell Telephone Laboratories, Whippany, N. J. during the 1969-1970 school year, and on a part-time basis during the 1969 summer term.