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THE COMPUTER EVALUATION OF THE NATURAL FREQUENCIES OF VIBRATING CIRCULAR PLATES WITH FREE, FIXED AND SIMPLY SUPPORTED EDGES

BY

THOMAS MICHAEL JULIANO

A THESIS

PRESENTED IN PARTIAL FULFILLMENT OF

THE REQUIREMENTS FOR THE DEGREE

OF

MASTER OF SCIENCE IN MECHANICAL ENGINEERING

AT

NEWARK COLLEGE OF ENGINEERING

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Newark, New Jersey
1970

APPROVAL OF THESIS

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FOR

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ABSTRACT

The natural frequencies of the transverse vibration of a thin, isotropic, circular plate with free, clamped, and simply supported edge conditions were studied extensively. The frequency equation for each edge condition was derived from the classical partial differential equation of plate vibration. These equations, which are in terms of Bessel functions, were then solved numerically to find the natural frequencies. Since the accuracy of the Bessel function values is very important in evaluating these frequencies, a comprehensive digital computer program was devised to calculate these values to eleven digit accuracy. In this Bessel function program four different methods were required to insure a rapid convergence. They are: (a) Infinite Series; (b) Asymptotic Series; (c) Recursion Formula; (d) Approximate Numerical Method.

The nodal patterns are known from the form of the solution to the fourth order partial differential equation of the vibrating plate. The order of the Bessel functions in the frequency equation corresponds to the number of equally spaced nodal diametral lines. The eigenvalues of this equation determine the number of concentric nodal circles

which are present in the various nodal patterns. For each edge condition, twenty-six frequencies were computed for each of the first twenty-six orders of the frequency equation. The accuracy of these computations has been carried out to ten significant figures. Methods to be used in computing the radii of the nodal circles corresponding to these frequencies were also discussed. However, these values were not obtained.

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INTRODUCTION

The analytical solution of the vibration of a solid elastic plate was not developed until the early part of the nineteenth century. Experimental work had been carried on during previous years by the German acoustician E.F.F. Chladni. He produced figures of the nodal pattern shapes by sprinkling sand on vibrating plates. The Emperor Napoleon of France provided a prize of 3000 francs to be awarded by the Institute of France for satisfactory completion of the mathematical theory of the vibrations of plates. This prize was awarded in 1815 to Mlle. Sophie Germain. She presented the correct fourth order differential equation, but her choice of boundary conditions proved to be incorrect.[1] Most of the difficulty in obtaining the correct boundary conditions arose when the free edge was considered. Poisson gave three equations which were to be satisfied at all points of a free edge. Kirchhoff later proved that in general it would be impossible to satisfy all three of Poisson's equations. However, he also noted that

[1] J.W.S. Rayleigh, The Theory of Sound (New York, 1945), Vol. I, p. XVI.

one of Poisson's equations is true identically for the symmetrical vibrations of a circular plate. Thus, Poisson's theory was correct even though he used three boundary conditions instead of two. In 1850, Kirchhoff resumed his work and completed the theory of the vibration of circular plates.[2]

Kirchhoff calculated the first few solutions of the equation for the natural frequencies of a free plate. This equation was found by substituting the solution of the fourth order differential equation of motion into the two boundary conditions and equating the results. After the first few values, the roots of the frequency equation were extremely difficult to find. The smaller roots were obtained by a trial and error method using interpolated values of Bessel functions taken from available tables. This type of solution is very time consuming, and the hand calculations were of limited accuracy. For the larger roots, asymptotic or semiconvergent descending series were used for the Bessel functions, and the frequency equation was written in the form of a series. This method,first used

[2] Ibid., pp. 369-370.

by Kirchhoff himself, was later used by Lord Rayleigh and Airey. Rayleigh gives an extensive study of vibrating plates with free and clamped edges. He also presents some of Kirchhoff's original work and gives his own results to Kirchhoff's equations.[3] Airey used a different semiconvergent series and presented the first ten roots for each of the first four orders of the frequency equation for clamped and free plates.[4]

With advances in mathematics, new semiconvergent series were developed to produce larger values with greater accuracy. H. Carrington, in 1925, utilized more extensive Bessel function tables to find all the roots less than sixteen with five digit accuracy. He then proceeded to develop the frequency equation in terms of an asymptotic or semiconvergent series, but he did not give any further results.[5]

Some of the more recent works have dealt with elastic end restraints. Since the two extreme values of this type

[3] Ibid., pp. 352-372.

[4] John R. Airey, "The Vibration of Circular Plates and Their Relation to Bessel Functions," Proceedings of the Physical Society of London, Vol. 23 (1911), pp. 225-232.

[5] H. Carrington, "The Frequencies of Vibration of Flat, Circular Plates, Fixed at the Circumference," Philosophical Magazine, Series 6, Vol. 50 (1925), pp. 1261-1264.

of analysis are the clamped and simply supported edges, some results could be used for the natural frequencies of the simply supported case. C. Lakshmi Kantham gives the first root for each of the first four orders.[6] N. Gajendar solved the problem of a vibrating plate with initial displacement and velocity. He gives very little numerical results.[7] R.Y. Bodine has calculated the natural frequencies of a plate which is simply supported along a circle of arbitrary radius. He gives results for supports varying from the center of the plate to the edge of the plate. His results include the first four orders with arguments up to eighteen.[8]

Thus far the range of orders and arguments for which the frequency equation could be solved has been limited by the difficulty in evaluating the Bessel functions.

6 C.L. Kantham, "Bending and Vibration of Elastically Restrained Circular Plates," Journal of the Franklin Institute, Vol. 265 (1958), pp. 483-491.

7 N. Gajendar, "Free Vibrations of a Circular Plate," Journal of the Royal Aeronautical Society, Vol. 69 (May, 1965), pp. 345-347.

8 R.Y. Bodine, "Vibrations of a Circular Plate Supported by a Concentric Ring of Arbitrary Radius," The Journal of the Acoustical Society of America, Vol. 41 (June, 1967), p. 1551.

For the first few orders, the infinite series could be used for small arguments, and the asymptotic or semiconvergent series could be used for the larger arguments. The values which could be obtained by these methods have been tabulated in various Bessel functions handbooks. To find Bessel functions of higher orders, the recursion formulas may be used if the argument is greater than the order. If the argument is less than the order, the number of significant digits decreases rapidly with each recursion. For this reason most Bessel function tables do not contain values for small arguments of large orders. The development of the high speed digital computer allowed a entirely new approach to the problem. A numerical technique was given by F.W.J. Olver, who utilized a reverse recursion process to find the Bessel functions of arguments which are less than the order.[9]

Chapter II contains a complete development of the equations for the natural frequencies of a vibrating circular plate with its edge either clamped, simply supported

[9]Milton Abramowitz and Irene Stegun, Handbook of Mathematical Functions, (Washington, 1968), pp. 355-433.

or free. It also contains the derivation of the equations to find the radii of the nodal circles, and a method of obtaining these radii from ratios of the natural frequencies. Chapter III contains a study of the various methods of computing Bessel functions of the first kind, and Modified Bessel functions of the first kind. An analysis was made of the range of orders and arguments for which each method is accurate. This information was then used in devising a computer program to evaluate these Bessel functions for orders from zero to thirty-three, and for any argument whose Bessel function does not exceed the capacity of the computer being employed. An analysis of the entire computer program used to find the natural frequencies is given in Chapter IV. The accuracy of this program is carried out to ten significant figures.

FREQUENCY AND NODAL PATTERN EQUATIONS

General Displacement Relationships

The differential equation of motion of a thin, flat, circular plate with no external load, which is homogeneous and isotropic, and which experiences small displacements in the vertical direction is[1]

$$\frac{EIg}{(1-v^2)\gamma H}\nabla^4 W = -\frac{\partial^2 W}{\partial t^2} \qquad (1)$$

where

$$\nabla^4 = \nabla^2\nabla^2 = \left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial^2}{\partial r \partial \theta} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right]^2$$

and

I	=	moment of inertia of a unit width of cross section
H	=	plate thickness
E	=	Young's Modulus of Elasticity
v	=	Poison's ratio, 0.3 is used
γ	=	specific weight of the material
g	=	acceleration due to gravity
t	=	time

[1]E. Volterra and E.C. Zachmanoglou, Dynamics of Vibrations (Columbus, 1965), p. 379.

W = displacement of the middle surface of the plate in the vertical direction

r and θ = polar coordinates whose origin is at the center of the plate

Any consistent set of units may be used.

Equation (1) can be reduced to three total differential equations by using the method of separation of variables. The displacement can be written as,

$$W(r,\theta,t) = R(r)\Theta(\theta)T(t) \tag{2}$$

If the proper substitutions are made into the evolving equations, the final result is the following set of total differential equations.

$$\frac{d^2T(t)}{dt^2} + \hat{\omega}^2 T(t) = 0 \tag{3}$$

$$\frac{d^2\Theta(\theta)}{d\theta^2} + k^2\Theta(\theta) = 0 \tag{4}$$

$$\frac{d^2R(r)}{dr^2} + \frac{1}{r}\frac{dR(r)}{dr} + \left[\pm\lambda^2 - \frac{k^2}{r^2}\right]R(r) = 0 \tag{5}$$

Where

$\hat{\omega}$ = natural frequency

k = a constant

$$\lambda^2 = \frac{\hat{\omega}}{\beta} \tag{6}$$

$$\beta^2 = \frac{Dg}{\gamma H} \tag{7}$$

$$D = \frac{EI}{(1-\nu^2)} \tag{8}$$

Which is called the flexural rigidity of the plate.

The general solutions of Eq. (3) and Eq. (4) are respectively,

$$T(t) = A\cos(\hat{\omega}t) + B\sin(\hat{\omega}t) \tag{9}$$

$$\Theta(\theta) = C\cos(k\theta) + D\sin(k\theta) \tag{10}$$

where A, B, C, and D are constants depending on the initial values and the boundary conditions of the particular plate. In order for the displacements to be continuous, $\Theta(\theta)$ must be periodic with a period of 2π. Therefore, k must be a integer.

$$k = n = 1,2,3,... \tag{11}$$

If a change of variables is made in Eq. (5) with $\rho = \lambda r$ in conjunction with the positive sign, and $\rho = i\lambda r$ in conjunction with the negative sign, the resulting equation is

$$\frac{d^2R(\rho)}{d\rho^2} + \frac{1}{\rho}\frac{dR(\rho)}{d\rho} + \left[1-\frac{n^2}{\rho^2}\right]R(\rho) = 0 \tag{12}$$

Equation (12) is of the form of Bessel's Differential Equation, and its solution is of the standard form associated with equations of this type. The general solution is, after replacing ρ by r.

$$R(r) = A_n J_n(\lambda r) + B_n J_n(i\lambda r) + E_n Y_n(\lambda r) + F_n Y_n(i\lambda r) \qquad (13)$$

where

$J_n(x)$ = Bessel function of the first kind

$Y_n(x)$ = Bessel function of the second kind

n = the order of the Bessel function

A, B, E, F = constants

Bessel functions of the second kind become infinite for zero arguments. Thus, if a plate does not have a hole at its center, the constants E_n and F_n must be equal to zero since the displacement at the center of the plate must be a finite value.

The time independent solution of Eq. (1), that is, displacement as a function of position alone is

$$W(r,\theta) = \left[A_n J_n(\lambda r) + B_n J_n(i\lambda r)\right]\left[C_n \cos(n\theta) + D_n \sin(n\theta)\right] \qquad (n = 0,1,2,\ldots) \qquad (14)$$

The constants A_n, B_n, C_n, and D_n must be evaluated from the boundary conditions of the plate.

Frequency Equations

Clamped Edge. The boundary conditions for a circular plate of radius a, which is fixed at the circumference are:

(a) The deflection at the edge is zero.

$$W(a,\theta) = 0 \qquad (15)$$

(b) The slope at the edge is zero.[2]

$$\frac{\partial W(a,\theta)}{\partial r} = 0 \tag{16}$$

The substitution of Eq. (14) into Eq. (15) yields,

$$\left[A_n J_n(\lambda a)+B_n J_n(i\lambda a)\right]\left[C_n \cos(n\theta)+D_n \sin(n\theta)\right] = 0 \tag{17}$$

The second term cannot be equal to zero for all values of θ unless C_n and D_n are identically equal to zero. This would result in a trivial solution with no motion. Thus, the second term can be eliminated, and the resulting equation is

$$A_n J_n(\lambda a)+B_n J_n(i\lambda a) = 0 \tag{18}$$

The substitution of Eq. (14) into Eq. (16) yields by simiiliar reasoning,

$$A_n J_n'(\lambda a)+iB_n J_n'(i\lambda a) = 0 \tag{19}$$

where the primes denote differentiation with respect to r.

The Bessel functions with complex arguments can be replaced by Modified Bessel functions according to the following relationships.[3]

[2] S. Timoshenko and S. Woinowsky-Krieger, Theory of Plates and Shells (New York, 1959), pp. 283-284.

[3] George Arfken, Mathematical Methods for Physicists (New York, 1966), p. 397.

$$J_n(ix) = i^n I_n(x) \tag{20}$$

$$iJ_n'(ix) = i^n I_n'(x) \tag{21}$$

The above relationships are substituted into Eq. (18) and Eq. (19) to give,

$$A_n J_n(\lambda a) + i^n B_n I_n(\lambda a) = 0 \tag{22}$$

$$A_n J_n'(\lambda a) + i^n B_n I_n'(\lambda a) = 0 \tag{23}$$

Both of these equations can be solved for the ratio (B_n/A_n), and set equal to each other,

$$\frac{B_n}{A_n} = \frac{-i^{-n} J_n(\lambda a)}{I_n(\lambda a)} = \frac{-i^{-n} J_n'(\lambda a)}{I_n'(\lambda a)} \tag{24}$$

or after cross multiplication,

$$J_n(\lambda a) I_n'(\lambda a) - I_n(\lambda a) J_n'(\lambda a) = 0 \tag{25}$$

The derivatives can be eliminated by the following relationships[4]

$$J_n'(x) = \frac{n}{x} J_n(x) - J_{n+1}(x) \tag{26}$$

$$I_n'(x) = \frac{n}{x} I_n(x) + I_{n+1}(x) \tag{27}$$

The resulting equation is the frequency equation for a clamped plate

$$J_n(\lambda a) I_{n+1}(\lambda a) + I_n(\lambda a) J_{n+1}(\lambda a) = 0 \tag{28}$$

[4] Ibid., pp. 374, 397.

This equation will have an infinite number of solutions, each of the form λ_{nm}, if the radius is chosen to be unity. The subscript n is the order of the equation, which corresponds to the number of evenly spaced diametral node lines on the plate. The subscript m is the numerical rank of the root, which corresponds to the number of concentric nodal circles. For example, the second root of the second order equation will yield a nodal pattern of two nodal diametral lines and two nodal circles. In this case one of the nodal circles occurs at the fixed edge. The natural frequencies of a specific plate with known physical properties can be found by substituting the solution of Eq. (28) into Eq. (6), Eq. (7) and Eq. (8), that is, Eq. (6) may be written as

$$\hat{\omega} = \lambda^2 \sqrt{\frac{Dg}{\gamma H}} \tag{6}$$

Simply Supported Edge. The boundary conditions for a circular plate of radius a with a simply supported edge are:[5]

(a) The deflection at the edge is zero.

$$W(a,\theta) = 0 \tag{29}$$

[5]Timoshenko, p. 284.

(b) The bending moment at the edge is zero.

$$M_r(a,\theta) = 0 \tag{30}$$

where

$$M_r = -D\left[\frac{\partial^2 W}{\partial r^2} + \nu\left(\frac{1}{r}\frac{\partial W}{\partial r} + \frac{1}{r^2}\frac{\partial^2 W}{\partial \theta^2}\right)\right] \tag{31}$$

The first boundary condition has already been evaluated in the previous section. The result was

$$\frac{B_n}{A_n} = -i^{-n}\frac{J_n(\lambda a)}{I_n(\lambda a)} \tag{32}$$

The substitution of Eq. (14) into the second boundary condition for a simply supported plate yields after considerable factoring,

$$\frac{1}{\nu}\left[\lambda^2 J_n''(\lambda a) - \lambda^2\frac{B_n}{A_n}J_n''(i\lambda a)\right] + \frac{1}{a}\left[\lambda J_n'(\lambda a) + \frac{iB_n}{A_n}J_n'(i\lambda a)\right]$$
$$-\frac{n^2}{a^2}\left[J_n(\lambda a) + \frac{B_n}{A_n}J_n(i\lambda a)\right] = 0 \tag{33}$$

By using the following recursive relationships[6]

$$\frac{2n}{x}J_n(x) = J_{n-1}(x) + J_{n+1}(x) \tag{34}$$

$$J_n'(x) = \frac{1}{2}J_{n-1}(x) - J_{n+1}(x) \tag{35}$$

[6] Arfken, pp. 373-374.

and the Bessel function derivative formula Eq. (26), the following relationship can be derived for the second order derivative.

$$J_n''(x) = \left(\frac{n^2}{x^2}-1\right)J_n(x)-\frac{1}{x}J_n'(x) \tag{36}$$

This equation holds for both real and complex arguments. The substitution of Eq. (36) into Eq. (33) gives after collecting like terms,

$$\frac{B_n}{A_n} = \frac{J_n'(\lambda a)\lambda a(1-\nu)+J_n(\lambda a)\left[\lambda^2a^2-(1-\nu)n^2\right]}{-J_n'(i\lambda a)i\lambda a(1-\nu)+J_n(i\lambda a)\left[\lambda^2a^2+(1-\nu)n^2\right]} \tag{37}$$

The Bessel functions with complex arguments may be replaced by Modified Bessel functions using Eq. (20) and Eq. (21).

$$\frac{B_n}{A_n} = \frac{J_n'(\lambda a)\lambda a(1-\nu)+J_n(\lambda a)\left[\lambda^2a^2-(1-\nu)n^2\right]}{-i^nI_n'(\lambda a)\lambda a\ (1-\nu)+i^nI_n(\lambda a)\left[\lambda^2a^2+(1-\nu)n^2\right]} \tag{38}$$

After the first order derivatives are eliminated by Eq. (26) and Eq. (27), the resulting form is then set equal to Eq. (32). The frequency equation is then obtained by clearing fractions and factoring like terms. The final result is

$$2\lambda a\left[I_n(\lambda a)J_n(\lambda a)\right]-(1-\nu)\left[I_n(\lambda a)J_{n+1}(\lambda a)+J_n(\lambda a)I_{n+1}(\lambda a)\right] = 0 \tag{39}$$

As before the solution consists of an infinite number of eigenvalues, λ_{nm}, from which the natural frequencies are

able to be found.

Free Edge. The boundary conditions for a circular plate of radius a with a free edge are:[7]

(a) The bending moment at the edge is zero.

$$M_r(a,\theta) = 0 \tag{40}$$

(b) The effective shear at the edge is zero.

$$V(a,\theta) = Q_r - \frac{1}{r}\frac{\partial}{\partial\theta}(M_{rt}) = 0 \tag{41}$$

where

$$Q_r = -D\frac{\partial}{\partial r}(\nabla^2 W) \tag{42}$$

and

$$M_{rt} = (1-\nu)D\left[\frac{1}{r}\frac{\partial^2 W}{\partial r\partial\theta} - \frac{1}{r^2}\frac{\partial W}{\partial\theta}\right] \tag{43}$$

The first boundary condition has already been evaluated in the previous section. The resulting ratio was given by Eq. (38). The substitution of Eq. (14) into the second boundary condition for a free edge yields,

$$\frac{B_n}{A_n} = \frac{\begin{array}{r}-\lambda^3 J_n'''(\lambda a) - \frac{1}{a}\lambda^2 J_n''(\lambda a) + J_n'(\lambda a)\lambda\frac{1}{a^2}\left[1+n^2+n^2(1-\nu)\right] \\ -J_n(\lambda a)\frac{1}{a^3}\left[2n^2+n^2(1-\nu)\right]\end{array}}{\begin{array}{r}-i\lambda^3 J_n'''(i\lambda a) - \frac{1}{a}J_n''(i\lambda a) - J_n'(i\lambda a)i\lambda\frac{1}{a^2}\left[1+n^2+n^2(1-\nu)\right] \\ +J_n(i\lambda a)\frac{1}{a^3}\left[2n^2+n^2(1-\nu)\right]\end{array}} \tag{44}$$

[7] Timoshenko, p. 284.

Using the previously stated relationships for the second order and first order derivatives, and the recursion formula for Bessel functions, an equation for the third order derivative may be derived. It is

$$J_n'''(x) = \left[\frac{n^2}{x^2} - 1 + \frac{2}{x^2}\right] J_n'(x) - \left[\frac{3n^2}{x^2} - \frac{1}{x}\right] J_n(x) \qquad (45)$$

After the elimination of the second and third order derivatives, Eq. (44) can be written as,

$$\frac{B_n}{A_n} = \frac{J_n'(\lambda a)\lambda a\left[\lambda^2 a^2 + (1-\nu)n^2\right] - J_n(\lambda a)(1-\nu)n^2}{J_n'(i\lambda a)i\lambda a\left[\lambda^2 a^2 - (1-\nu)n^2\right] + J_n(i\lambda a)(1-\nu)n^2} \qquad (46)$$

The Bessel functions with complex arguments can be replaced by Modified Bessel functions using Eq. (20) and Eq. (21). The resulting form is then set equal to Eq. (38), and the first order derivatives are eliminated by Eq.(26) and Eq. (27). The final form is the frequency equation for a circular plate with a free edge.

$$\begin{aligned}
&\left[J_n(\lambda a)\left[\lambda^2 a^2 - n(n-1)(1-\nu)\right] - J_{n+1}(\lambda a)\lambda a(1-\nu)\right] \qquad (47)\\
&\cdot\left[nI_n(\lambda a)\left[\lambda^2 a^2 - n(n-1)(1-\nu)\right] + I_{n+1}(\lambda a)\lambda a\left[\lambda^2 a^2 - (1-\nu)n^2\right]\right]\\
&-\left[nJ_n(\lambda a)\left[\lambda^2 a^2 + n(n-1)(1-\nu)\right] - J_{n+1}(\lambda a)\lambda a\left[\lambda^2 a^2 + (1-\nu)n^2\right]\right]\\
&\cdot\left[I_n(\lambda a)\left[\lambda^2 a^2 + n(n-1)(1-\nu)\right] - I_{n+1}(\lambda a)\lambda a(1-\nu)\right] = 0
\end{aligned}$$

The solution of Eq. (47), as in the previous cases, consists of an infinite number of eigenvalues, λ_{nm}, from which the natural frequencies may be found. For the particular

cases of the zero and first order vibrations of a free plate, the first root of Eq. (47) is equal to zero. This peculiarity occurs because the only terms remaining in Eq. (47) when n is equal to zero or one are those which have λ as a factor. Thus, λ_{00} and λ_{10} equal to zero are roots.

Thus the three natural frequency equations are:

(a) Clamped Edge — Eq.(28)

(b) Simply Supported Edge — Eq.(39)

(c) Free Edge — Eq.(47)

The solutions of these equations were found by a numerical iteration technique used on a digital computer. This program is discussed in the last chapter of this text.

Equations of Nodal Diametral Lines

The equation for the displacement in the vertical direction of a point on the middle surface of the plate as a function of position alone was given in the first section of this chapter. A similiar form is given here as

$$W_{nm}(r,\theta) = \left[A_n J_n(\lambda_{nm} r) + B_n J_n(i\lambda_{nm} r)\right] \cdot \left[C_n \cos(n\theta) + D_n \sin(n\theta)\right] \qquad (48)$$

where λ_{nm} and W_{nm} replace λ and W of Eq. (14). It can be shown that the number of nodal diameters corresponds to

the value of n, which is the order of the frequency equation. For a specific value of λ_{nm} and r, Eq. (48) can be written as

$$W_n(\theta) = R_n \left[C_n \cos(n\theta) + D_n \sin(n\theta)\right] \tag{49}$$

where R_n is a constant equal to the value of the first bracketed quantity of Eq. (48). Equation (49) can also be written in the following form.

$$W_n(\theta) = K_n \cos(n\theta - \alpha) \tag{50}$$

where

$$K_n = R_n \sqrt{C_n^2 + D_n^2} \tag{51}$$

and

$$\alpha = \arctan\left[D_n / C_n\right] \tag{52}$$

The displacement W will be equal to zero when the angle in Eq. (50) is equal to an odd integer multiple of $\pi/2$, that is

$$(n\theta - \alpha) = k(\pi/2) \qquad \text{for} \quad k = 1,3,5,\ldots \tag{53}$$

or

$$\theta = \frac{k\pi}{2n} + \frac{\alpha}{n} \tag{54}$$

where $0 \leq \theta \leq 2\pi$ and k and n are integers. Then as an illustration n is chosen to be unity, and furthermore, the angle α is chosen to be zero. A radial node line would then occur at

$$\theta = \pi/2,\ 3\pi/2 \tag{55}$$

Similarly, if n were equal to two, nodal lines will occur at

$$\theta = \pi/4,\ 3\pi/4,\ 5\pi/4,\ 7\pi/4 \tag{56}$$

Thus, it is seen that the number of nodal diametral lines corresponds to the value of n, the order of the equation. It is noted that Eq. (54) does not apply when n is equal to zero, since Eq. (48) is then no longer a function of θ, and there are no nodal diameter lines present.

Equations of Nodal Circles

If the value of θ is held constant in Eq. (48) it can be written as

$$W_{nm}(r) = \left[A_n J_n(\lambda_{nm} r) + B_n J_n(i\lambda_{nm} r)\right] \cdot K_1 \tag{57}$$

To find the radii of the nodal circles, this equation is set equal to zero. The constant K_1 can then be divided out since θ may be chosen such that it does not give a nodal diameter line. The resulting equation is

$$A_n J_n(\lambda_{nm} r) + B_n J_n(i\lambda_{nm} r) = 0 \tag{58}$$

or

$$\frac{B_n}{A_n} = \frac{-J_n(\lambda_{nm} r)}{J_n(i\lambda_{nm} r)} \tag{59}$$

The Bessel function of a complex argument can be expressed as a Modified Bessel function using Eq. (20). Then Eq. (59) can be written as

$$\frac{B_n}{A_n} = -i^{-n} \frac{J_n(\lambda_{nm} r)}{I_n(\lambda_{nm} r)} \tag{60}$$

This particular form is desirable because the ratios B_n/A_n have been found in the derivation of the frequency equations for each edge condition. Both boundary conditions for each of the three plate edge conditions were solved for this ratio. Equation (60) can be equated to one of these ratios for each one of the edges.

Clamped Edge. The equating of Eq. (60) to Eq. (24) yields

$$\frac{J_n(\lambda_{nm} r)}{I_n(\lambda_{nm} r)} = \frac{J_n(\lambda_{nm} a)}{I_n(\lambda_{nm} a)} \tag{61}$$

or

$$J_n(\lambda_{nm} r) I_n(\lambda_{nm} a) - I_n(\lambda_{nm} r) J_n(\lambda_{nm} a) = 0 \tag{62}$$

After substituting the values of λ_{nm}, which have been found for a clamped plate, into Eq. (62), the values of r for which the equation equals zero may be found. These values are the radii of concentric nodal circles. For each λ_{nm} there will be m values of r which satisfy the equation.

Simply Supported Edge. The equation for the nodal radii of a plate with simply supported edge is the same as that for a clamped edge. This is the case because both share the same boundary condition, that is, the displacement at

the edge is equal to zero. The values of λ_{nm} used in this equation are those which satisfy the frequency equation of a simply supported plate. Equation (62) may be written again here noting that the λ_{nm} are not the same as those for a clamped plate.

$$J_n(\lambda_{nm}r)I_n(\lambda_{nm}a)-I_n(\lambda_{nm}r)J_n(\lambda_{nm}a) = 0 \qquad (63)$$

Free Edge. To find the radii of the nodal circles of a plate with a free edge Eq. (60) is equated to Eq. (38), which was derived from the condition that the moment at the edge was zero.

$$-i^{-n}\frac{J_n(\lambda r)}{I_n(\lambda r)} = \frac{J_n'(\lambda a)\lambda a(1-v)+J_n(\lambda a)\left[\lambda^2a^2-(1-v)n^2\right]}{-i^nI_n'(\lambda a)\lambda a(1-v)+i^nI_n(\lambda a)\left[\lambda^2a^2+(1-v)n^2\right]} \qquad (64)$$

The first order derivatives can be eliminated by Eq. (26) and Eq. (27).

$$\frac{-J_n(\lambda r)}{I_n(\lambda r)} = \frac{J_n(\lambda a)\left[\lambda^2a^2-n(1-v)(n-1)\right]-J_{n+1}(\lambda a)\lambda a(1-v)}{I_n(\lambda a)\left[\lambda^2a^2+n(1-v)(n-1)\right]-I_{n-1}(\lambda a)\lambda a(1-v)} \qquad (65)$$

For simplicity of calculation the subscripts nm are omitted from the two preceding equations. After clearing the fractions and simplifying the final equation for the nodal radii of a free plate is, with the subscripts replaced,

$$\begin{aligned} &J_n(\lambda_{nm}r)\left[I_n(\lambda_{nm}a)\left[\lambda_{nm}^2a^2+n(1-v)(n-1)\right]-I_{n+1}(\lambda_{nm}a)\lambda_{nm}a(1-v)\right] \\ &+I_n(\lambda_{nm}r)\left[J_n(\lambda_{nm}a)\left[\lambda_{nm}^2a^2-n(1-v)(n-1)\right]\right. \\ &\left.-J_{n+1}(\lambda_{nm}a)\lambda_{nm}a(1-v)\right] = 0 \end{aligned} \qquad (66)$$

The values of λ_{nm} which satisfy the frequency equation for a free plate are substituted into Eq. (66), and the radii of the nodal circles can then be found.

Ratio Method for Nodal Circles

A method utilizing the ratios of the natural frequency eigenvalues may be employed to find the radii of the nodal circles of vibrating plates. It can be shown that for each λ_{nm} there are m values of r which satisfy the nodal circle equations. For the clamped and the simply supported edges the minimun value which m can have is unity, because the edge must be a node. However, beginning with the second order, the free plate is capable of assuming a mode shape without any nodal circles. Thus, m is equal to zero.

The maximum value which r can have is a, the radius of the plate. For the cases of the clamped and simply supported plates, the edge will always be a node, or r equal to a is a solution to the nodal circle equations. The value of the first root of the n^{th} order equation is $\lambda_{n1}a$. When m is equal to two, $\lambda_{n2}a$ is a solution. However, a root will also occur when r is such that the product $\lambda_{n2}r$ is equal to $\lambda_{n1}a$. Since λ_{n2} is greater than λ_{n1}, and r is less than a, this equality is known to be

attainable. Thus, for λ_{n2} there is one nodal circle present in addition to the edge. It follows that for each λ_{nm} a root will occur at $\lambda_{nm}a$, for the edge, and whenever the product $\lambda_{nm}r$ is equal to one of the previous products $\lambda_{nm-1}a$, $\lambda_{nm-2}a$,...$\lambda_{n1}a$. Therefore, there are m values of r which satisfy the nodal circle equations for each λ_{nm}. This relationship between the roots may be used to produce a scheme to find the radii of the nodal circles. For a given value of n, the following equalities are determined for the m values of $\lambda_{nm}r$.

m=1	m=2	m=3	m=4
$a\lambda_{n1}$ =	$r_{22}\lambda_{n2}$ =	$r_{33}\lambda_{n3}$ =	$r_{44}\lambda_{n4}$
	$a\lambda_{n2}$ =	$r_{32}\lambda_{n3}$ =	$r_{43}\lambda_{n4}$
		$a\lambda_{n3}$ =	$r_{42}\lambda_{n4}$
			$a\lambda_{n4}$

(67)

Since the values of λ_{nm} have been determined from the frequency equations, the radii of the nodal circles may be found from the ratios given in the group of equations called Eq. (67). For the first case,

$$r_{22} = \frac{\lambda_{n1}\, a}{\lambda_{n2}} \tag{68}$$

and in the general case,

$$r_{m,k} = \left[\frac{\lambda_{n,m-1}}{\lambda_{n,m}}\right] r_{m-1,k-1} \qquad \text{for } m \geq 1, \text{ and } k \geq 1 \qquad (69)$$

where $r_{m1} = a$, for all m, for the clamped and simply supported edge. Thus, once the natural frequency eigenvalues are known, the radii of the nodal circles may be found from them directly.

The case of the free edge is more difficult to handle than the two preceding edge conditions. Since the edge is not constrained, its displacement is not equal to zero. Therefore r equal to a is not a solution to the equation for the nodal radii of a free plate, that is, Eq. (66). In order to use the above scheme for a free plate, r_{m1} must be evaluated for each value of λ_{nm}. Once these values are known, the same procedure may be followed to find the nodal radii. The values of r_{m1} are obtained from Eq. (66).

Due to the unconstrained boundary, the first mode does not exist for the zero order vibration or for the first order vibration. It was noted in the previous section that λ_{00} and λ_{10} are equal to zero. This phenomena can also be explained by observing the physical characteristics of a vibrating free plate, and the behavior of Eq. (66). The first mode of the zero order vibration would have the center point

as a node since the edge is free. However, this is clearly impossible for the zero order case, since both $J_0(0)$ and $I_0(0)$ are equal to unity. Equation (66) cannot equal zero unless λ_{00} equals zero. Thus, when there are no nodal diameters, at least one nodal circle must be present during vibration. The first mode of the first order vibration would consist of one nodal diameter and no nodal circles. This situation is physically impossible, because the ends of the plate are unconstrained. It is analogous to the vibration of a free rod in an unsymmetrical mode with a node at the middle, and no other nodes present. The physical reasoning can be confirmed in both cases by the actual calculation of the first root of Eq. (66) with n equal to zero and one. The first root for both orders is different from zero and a, thus indicating the presence of a nodal circle. The radius of the first nodal circle is given as 0.6802a, for the zero order vibration, and 0.781a, for the first order vibration.[8]

[8]John Prescott, Applied Elasticity (New York, 1961), pp. 588,596.

BESSEL FUNCTIONS

The evaluation of Bessel functions of the first kind and Modified Bessel functions of the first kind was accomplished by using four separate methods. Each method was found to have a specific range of arguments for which accuracy could be maintained at eleven significant figures. Overlapping values were found to insure that this degree of accuracy was retained when transferring from one method to the next. The four methods used were:

(a) Infinite Series

(b) Asymptotic Series

(c) Recursion Formula

(d) Approximate Numerical Method

Infinite Series

The infinite series defining the Bessel function of the first kind is[1]

$$J_n(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!\,(n+s)!} \left[\frac{x}{2}\right]^{n+2s} \qquad (1)$$

and the series for the Modified Bessel function of the first kind is[2]

[1] Arfken, p. 372.

[2] Ibid., p. 397.

$$I_n(x) = \sum_{s=0}^{\infty} \frac{1}{s!(n+s)!}\left[\frac{x}{2}\right]^{n+2s} \qquad (2)$$

Both of these series are useful if the value of the argument x is nearly equal to the order n. If the argument is much greater than or less than the order, then the rates of convergence of both series are very slow, and a enormous number of terms are required to give acceptable accuracy.

Asymptotic Series

In order to circumvent the difficulty posed when the argument is large relative to the order, the following Asymptotic expansions were used.[3]

$$J_n(x) = \left[\frac{2}{\pi x}\right]^{1/2}\left[P_n(x)\left[\cos x - \left[n+\frac{1}{2}\right]\frac{\pi}{2}\right] + Q_n(x)\left[\sin x - \left[n+\frac{1}{2}\right]\frac{\pi}{2}\right]\right] \qquad (3)$$

where

$$P_n(x) = 1 - \frac{(4n^2-1)(4n^2-3^2)}{2!(8x)^2} + \frac{(4n^2-1)(4n^2-3^2)(4n^2-5^2)(4n^2-7^2)}{4!(8x)^4} - \ldots \qquad (4)$$

and

$$Q_n(x) = -\frac{(4n^2-1)}{1!(8x)} + \frac{(4n^2-1)(4n^2-3^2)(4n^2-5^2)}{3!(8x)^3} - \ldots \qquad (5)$$

[3] Ibid., p. 405.

and

$$I_n(x) = C_n(x)e^x\left[\frac{1}{2\pi x}\right]^{1/2} \tag{6}$$

where

$$C_n(x) = 1 - \frac{(4n^2-1)}{1!(8x)} + \frac{(4n^2-1)(4n^2-3^2)}{2!(8x)^2} - \ldots \tag{7}$$

If the number of terms is taken to be greater than a certain value specified for each order by the relationship,

$$k \geq \frac{1}{4}(2n-5) \tag{8}$$

then the error becomes smaller than the first term which is omitted.[4] The minimum error will be obtained at the smallest term. Thus, each term must be compared to the previous one. Once the terms begin to increase in value accuracy will be lost.

The overlap area between the infinite series and the asymptotic expansion becomes smaller as the order increases. In fact, after the twentieth order vacant areas occur, that is, arguments for which Bessel functions cannot be accurately found by either method.

[4] Eugene Jahnke and Fritz Emde, Tables of Functions with Formulae and Curves 4th ed. (New York, 1945), p. 137-138.

Recursion Formula

Since the two previous methods give excellent results for the lower orders, a technique was devised to find the higher order values from the zero and first order values. The infinite series is used for arguments less than thirteen, and the asymptotic series for arguments which are greater than or equal to thirteen. The Bessel functions of higher orders can then be found by using the appropriate recursion formulas.[5] These formulas are obtained from the standard Bessel function recursion formulas by replacing n by n-1 .

$$J_n(x) = \frac{2(n-1)}{x} J_{n-1}(x) - J_{n-2}(x) \qquad (9)$$

and

$$I_n(x) = I_{n-2}(x) - \frac{2(n-1)}{x} I_{n-1}(x) \qquad (10)$$

By starting with the zero and first orders any higher order value can be found by using these recursion formulas to advance one order for each cycle. To find the Bessel function of the tenth order, for example, nine cycles would have to be made, that is one cycle to find the value for each of the orders from the second to the tenth inclusively.

[5] Arfken, pp. 373, 397.

Approximate Numerical Method

The recursion formulas work well if the order is less than the argument, however, when the argument is less than the order a rapid accumulation of round off errors destroys the accuracy. This problem has plagued the users of Bessel function tables for many years. Most tables offer a large amount of values for arguments which are greater than the order. As the order increases a greater number of arguments were omitted. Until very recently this problem was insurmountable, and these values were never used.

An analysis of the situation shows the reason for this round off error is the fact that $J_n(x)$ and $I_n(x)$ are decreasing functions of n if n is greater than x . The Bessel function of each succeeding order is smaller than its predecessor, and eventually its contribution is lost. However, a recursion process may be carried out in the direction of decreasing n and still maintain accuracy. A scheme for a numerical approach to evaluating Bessel functions by this reverse recursion technique is given by F.W.J. Olver.[6] The procedure is as follows. To evaluate

[6]Abramowitz, pp. 385-386.

the Bessel function of the first kind of order p, for some argument x, such that p is greater than x, the value of a test order q must first be chosen. Then $J_q(x)$ is set equal to zero, and $J_{q-1}(x)$ is set equal to unity. These two values are then used to initiate the reverse recursion process. The formula used is obtained by replacing n by n+2 in Eq. (9) to give,

$$J_n(x) = \frac{2(n+1)}{x} J_{n+1}(x) - J_{n+2}(x) \qquad (11)$$

Each value of $J_n(x)$ obtained from Eq. (11) is stored or retained until $J_p(x)$ is reached. The number of digits in this trial value for $J_p(x)$ is the number of accurate digits in the final result, that is the actual value of $J_p(x)$. Thus, the test order q must be chosen large enough to yield the desired number of accurate digits. If the number of digits in the trial value $J_p(x)$ is equal to the number of accurate digits needed in the actual value of $J_p(x)$, then the value of q is sufficiently large. If this is not the case, then a larger value of q is used, and the process is repeated until the required number of digits is achieved. The recursion process is continued until $J_0(x)$ is found. To find the actual value of $J_p(x)$, the trial value for $J_p(x)$ found above is multiplied by a normalization factor K.

This factor K is found by substituting the trial values into the following relationship.

$$\frac{1}{K} = J_0(x) + 2J_2(x) + 2J_4(x) + 2J_6(x) + \ldots \qquad (12)$$

The test order q, by actual evaluation, was found to be a function of both the order and the argument. The value of q necessary to maintain eleven digits in the trial value of $J_p(x)$ was found to be large if x approached p, and not much greater than p itself when p was greater than x. The minimum value of q was approximately p+15 for all orders.

The Modified Bessel function of the first kind is treated in a similar manner. The procedure is the same except the relationship for the normalization factor K is

$$\frac{e^x}{K} = I_0(x) + 2I_1(x) + 2I_2(x) + 2I_3(x) \quad \ldots \qquad (13)$$

where e is the base of the natural logarithms.

Computer programs were devised to evaluate the Bessel functions by the recursion formula technique, and by the approximate numerical method. The range of arguments for which these methods give accurate results had to be evaluated for each order, and an overlap area had to be found between the two methods for each order. The approximate

numerical technique using the reverse recursion method was found to be applicable to all arguments. However, if the argument was greater than the order, the test order q was very large, and the calculation by this method is time consuming. In this area the direct recursion formulas are more efficient than the reverse technique. A combination of the two methods gives the best accuracy in the least amount of time.

The minimum arguments which have to be reached before the direct recursion formulas can be used were found for each order, and a plot was made of the straight line envelope of these points. This was performed for both types of Bessel functions, and the results are given in Fig. (1) on page 36. The calculation of the Bessel functions for each order begins with the approximate method. When the value of the argument is greater than the transfer point for that particular order, the calculations are shifted to the direct recursion method. The empirical equations of the straight line envelopes of these transfer points were found to be:

$$\text{Transfer Point} = \frac{4}{7}(\text{Order} + 35) \qquad (14)$$

for the Bessel function of the first kind, and

$$\text{Transfer Point} = 3.88\ (\text{Order} + 1.94) \qquad (15)$$

for the Modified Bessel function of the first kind. Both of these empirical relationships are applicable for orders which are less than or equal to thirty-three, since that was the maximum order for which the transfer points were evaluated. Bessel functions of arguments which are to the left of the transfer point line in Fig. (1) are evaluated by the approximate numerical method. If the argument is to the right of the line, the recursion formulas are used. The line designated $J_n(x)$ is the transfer point line for the Bessel function of the first kind, and the line designated $I_n(x)$ is that for the Modified Bessel function of the first kind.

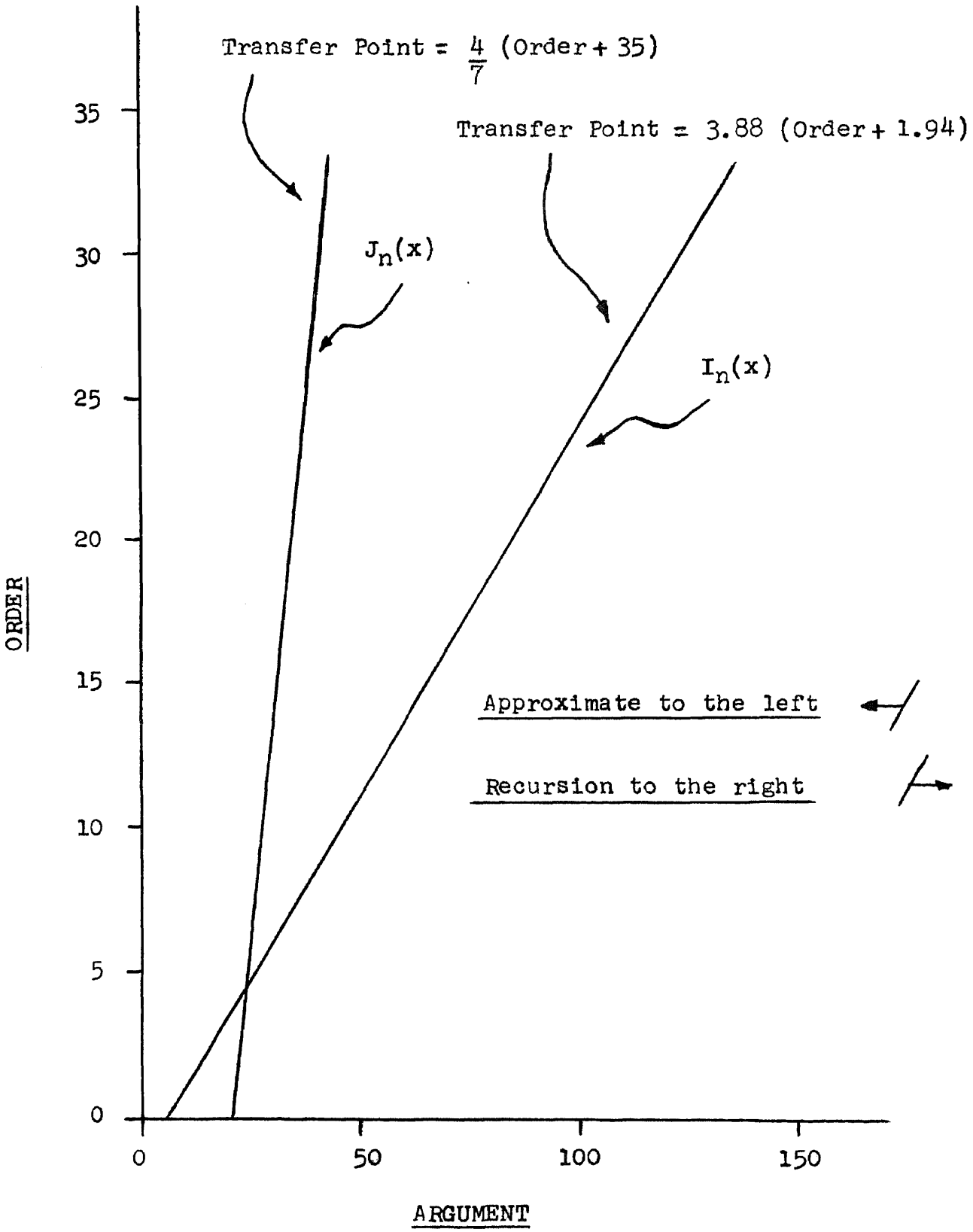


Fig. (1) Bessel function subprogram transfer points.

COMPUTER PROGRAM ANALYSIS

Bessel Function Subprogram

The Bessel function subprogram is composed of four main divisions, that is, one for each of the four methods discussed in the previous chapter. Each one of these is composed of two subdivisions to find $J_n(x)$ and $I_n(x)$. Thus, there are eight separate programs used to compute the Bessel function values. The selection of the proper technique is based on both the order and the argument. A flow chart of the Bessel function subprogram is given in Fig. (2) on page 39, and a copy of the actual program is given in the Appendix. This program may be used to calculate the value of the Bessel function, and the Modified Bessel function of the first kind of any argument to eleven digits accuracy for the first thirty-three orders. These four methods may be used for higher orders, but the values of the Modified Bessel function may become exceedingly large, and care must be taken to avoid exceeding the capacity of the computer.

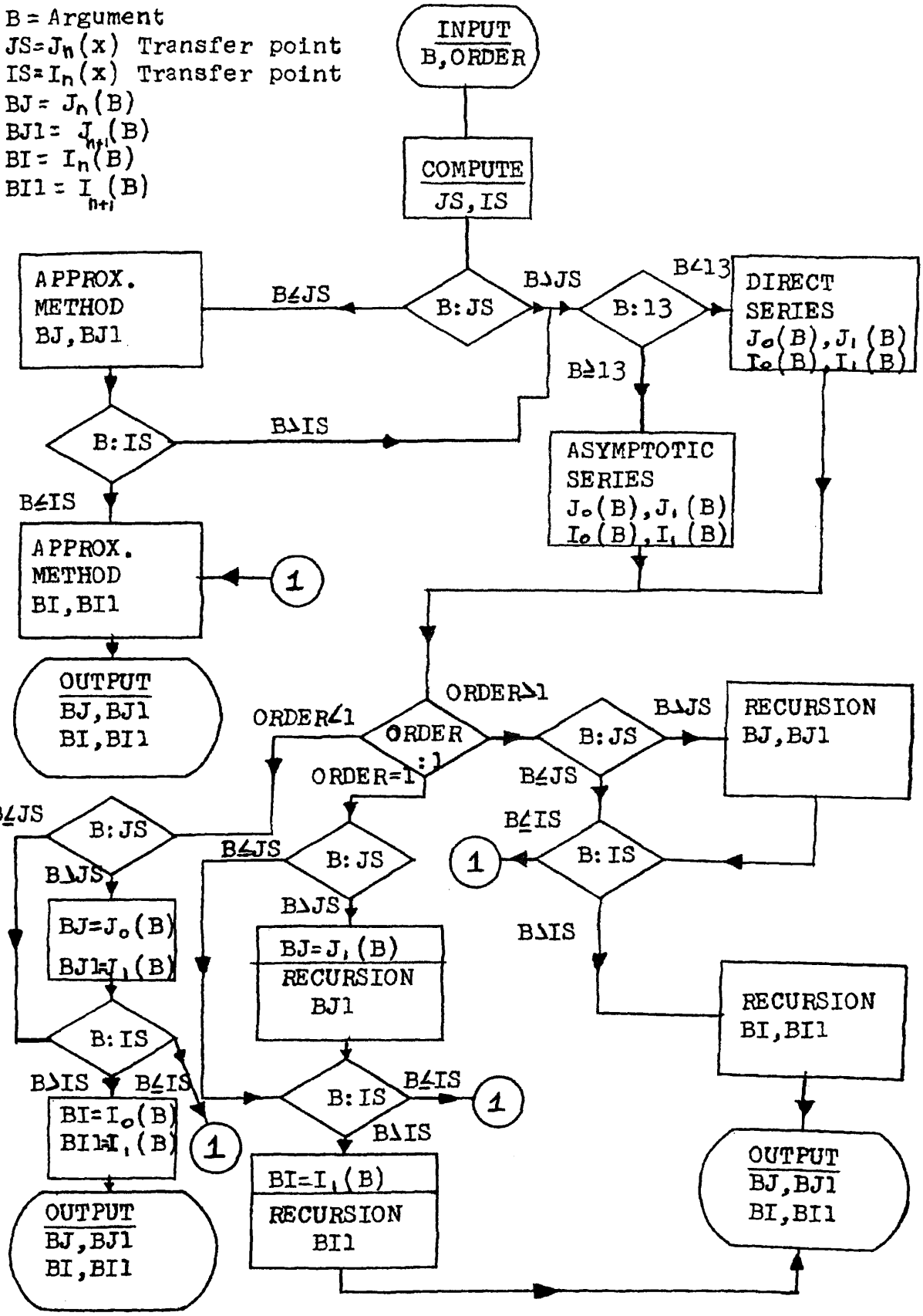
The error of each method of finding the Bessel functions was restricted to be less than 0.00000000001. A detailed

comparison of values yielded overlap regions among the four different methods. All the values in these overlap regions were checked to agree to eleven digits, and they were also checked with Bessel function tables wherever possible. This agreement of values obtained by totally different methods insures the uninteruption of accuracy when transferring from one method to the other. It also provides an opportunity to check the reliability of the expressions used to evaluate the error of each of the four methods.

Main Program

The main program consists of an iteration procedure which evaluates the roots of an equation by searching for a sign change of the function, and narrowing the interval of arguments between sign changes until the root is reached. In this scheme an interval of some specified length is chosen, and the value of the function is noted at the starting point and the end point. For this case, the function is the value of the right side of the frequency equation. Only those arguments which give the function a zero value are solutions to the frequency equation. The sign of the function at each of the two points is compared. If the function exhibits a sign change, a root is present in that

Fig. (2) Bessel function subprogram flow chart.



interval. The function is then evaluated at the midpoint of the interval, and its sign is compared to each of the end points. The half interval which contains the sign change is taken as the new interval, and the process is repeated. However, if the sign at the starting point is identical to that of the end point then a new interval of the same length is chosen with the end point of the previous interval taken as the starting point of the new interval. The above process of comparing signs is repeated, and the entire procedure is continued until a root is established. Since the halving process only approximates the roots, some criteria must be introduced to decide which values are considered to be roots. In this program an argument is considered as a solution if either the value of the function corresponding to it is in absolute value less than 0.0000000001, or the length of the interval has decreased to less than this amount. This situation indicates the occurrence of a sign change within the interval between two arguments which differ by less than 0.0000000001. The end point of this interval is chosen as the root, and it is also used as the starting point of the next interval, so that this sign change is not considered twice.

An educated guess must be made of the frequency of occurrence of the roots, so that the interval length is chosen to avoid the inclusion of more than one root. In this program the interval length was chosen to be one unit to eliminate the possibility of double roots within a single interval. This value was used because Bessel functions of the first kind are approximately periodic with a period nearly equal to 2π. The frequency equations exhibit a similar behavior, since the Modified Bessel function of the first kind is divergent. Thus, the periodicity is entirely dependent upon the Bessel function of the first kind.

A flow chart of the main program is given in Fig. (3) on page 42. The suffixes L and R refer to the left and right of the interval, and FUNCT(X) refers to the function evaluation subprogram, which is analyzed in the next section.

Function Evaluation Subprogram

In order to evaluate the function used in the main program, the values of the Bessel functions must first be obtained from the Bessel function subprogram. These values are then substituted into the frequency equation, which yields the value of the required function. The subprogram

Fig. (3) Main program flow chart.

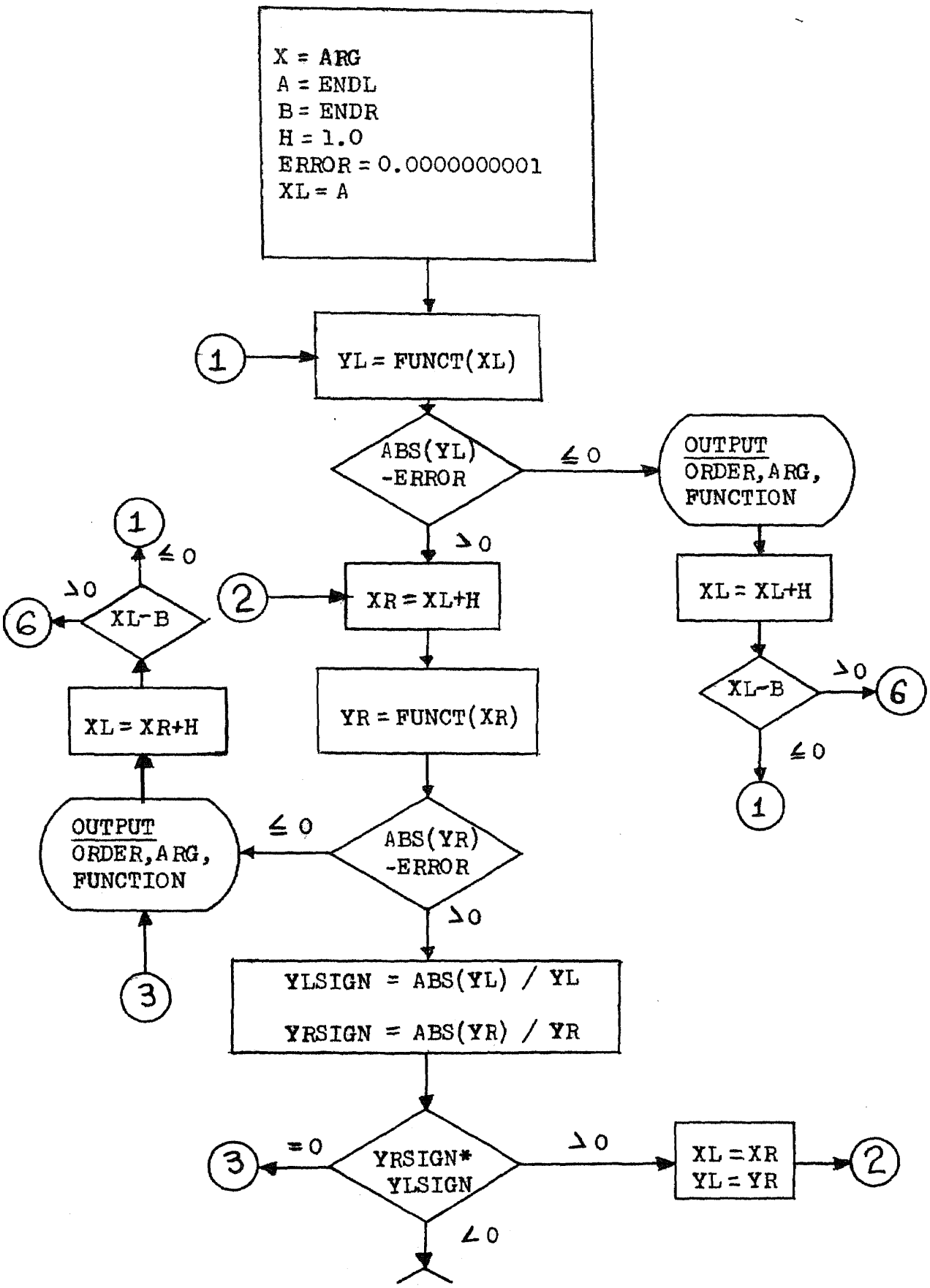
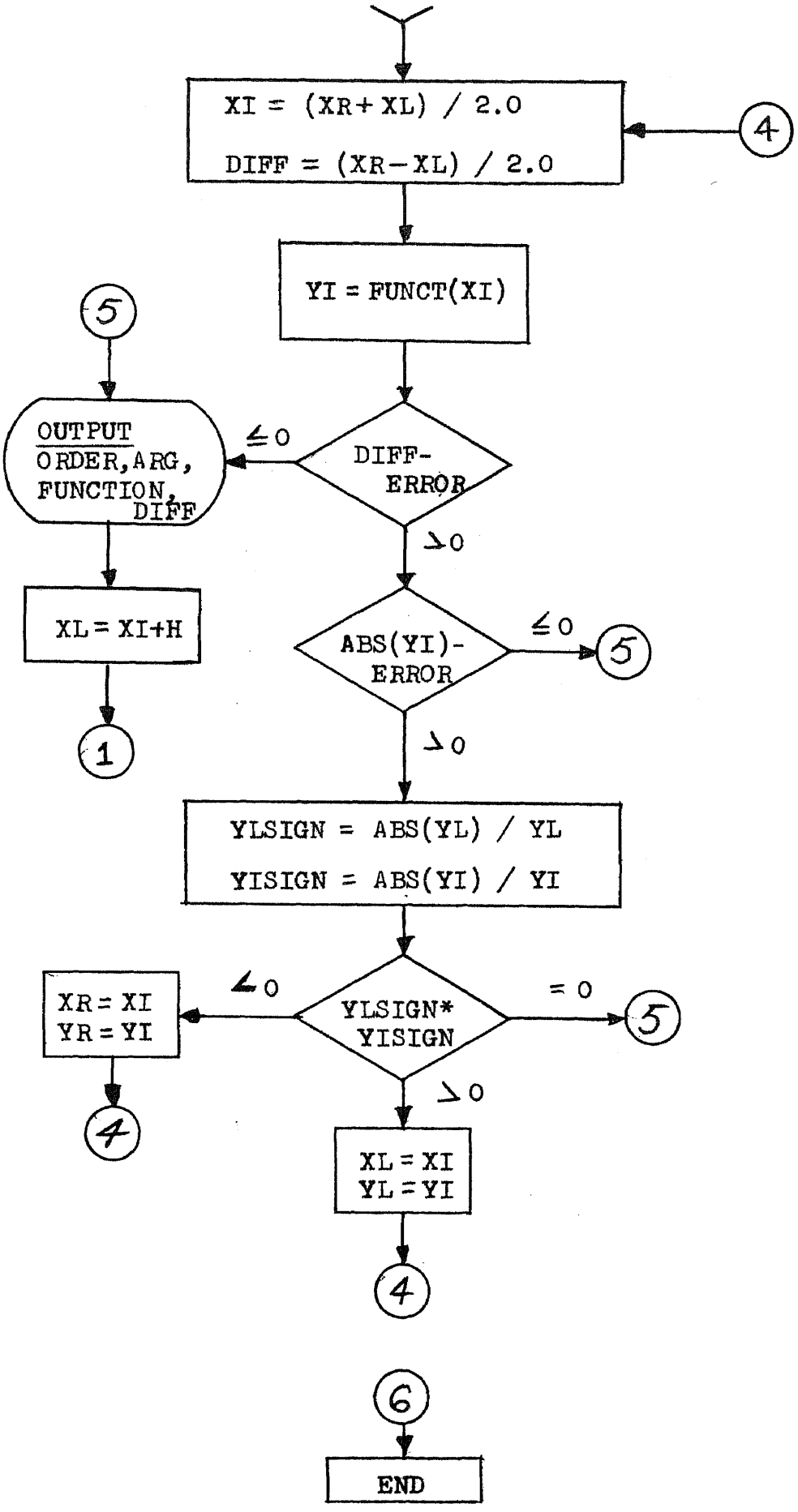


Fig. (3) continued.



which contains the frequency equation is called SUBROUTINE EPSLON. The solution of the frequency equation for each boundary condition was accomplished by using the same main program and Bessel function subprogram for each, but in the SUBROUTINE EPSLON only the specific frequency equation for each edge condition was used.

A single function subprogram called FUNCT(X) was used to call the other two subprograms- BESSEL and EPSLON. It allows for the proper selection of the dummy variables which are used in the subroutines, and it facilitates an easy method of transferring control to a series of subprograms. A single "Function" statement will result in the evaluation of all the necessary Bessel functions, and the calculation of the value of FUNCT(X) to be returned to the main program. It eliminates the necessity of having a series of "Call" statements through out the main program with a different set of variables in each one.

Accuracy of Results

The accuracy of the entire program was maintained at ten significant digits. This was done by imposing the previously stated criteria for defining the value of an argument as a root. Since the only errors which could occur in

the computation are round off errors, it can be shown that this control value represents the actual error of the results. All calculations done prior to the evaluation of the function are accurate to eleven significant figures, since the only calculations performed were the evaluation of the Bessel functions, and these are specified to have this accuracy.

Round off error may be produced in the evaluation of the function in the subprogram titled EPSLON. However, this computation involves only one equation, and the number of operations is not so great as to introduce a round off error large enough to have any effect on the tenth digit of a sixteen digit number. It must also be noted that this calculation is performed only once for each argument, and the error is not accumulated during the iterations, since the Bessel functions are independently calculated for each argument. Some round off error is produced in the main program. However, the only calculation performed is the division of the interval in half, and this computation will also have no effect on the tenth digit of a sixteen digit number. Thus, the only factor which has any detectable effect upon the accuracy of the results is the setting of the control

value to determine when an argument is a root. Since this value was chosen to be 0.0000000001, the results are accurate to ten significant figures.

CONCLUSIONS AND RESULTS

The eigenvalues of the natural frequency equation are tabulated for three edge conditions: clamped; simply supported; free. The first twenty-six roots of the first twenty-six orders were found with Poisson's ratio equal to 0.300. The roots of the lower order vibrations are in good agreement with existing results to the degree of accuracy used at the time. Tables of these frequency values given by other authors are also listed here.

In the evaluation of the natural frequencies ten digit accuracy was maintained through out the program. The Bessel function subprogram may be used to find the solutions of other problems which require the evaluation of Bessel functions up to the thirty-third order. The values of higher order Bessel functions may be found if the transfer points for these orders are obtained.

FREE EDGE

m \ n	0	1	2	3
0			2.320	3.530
1	3.000	4.530	5.900	7.300
2	6.30	7.600	9.200	10.40
3	9.500	10.95	12.37	13.801

Kirchhoff's values converted from the form in Volterra p. 399.

CLAMPED EDGE

m \ n	0	1	2	3
1	3.1955	4.611	5.906	7.144
2	6.3064	7.799	9.197	10.536
3	9.4395	10.958	12.402	13.795
4	12.5771	14.109	15.579	17.005
5	15.7164	17.256	18.745	20.192
6	18.8565	20.401	21.901	23.366
7	21.9971	23.545	25.055	26.532
8	25.1379	26.689	28.205	29.693
9	28.2790	29.832	31.354	32.849
10	31.4200	32.975	34.502	36.003

Airey's values

CLAMPED EDGE

m \ n	0	1	2	3
1	3.1961	4.6110	5.9056	7.1433
2	6.3064	7.7993	9.1967	10.537
3	9.4395	10.958	12.402	13.795
4	12.577	14.108	15.579	
5	15.716			

From Carrington's paper in 1925.

SIMPLY SUPPORTED

m \ n	0	1	2	3
1	2.2	3.7	5.1	6.4
2	5.4	6.8	8.4	9.8
3	8.8	10.1	11.6	13.0
4	11.9	13.3	14.8	16.2

Values taken from chart in Bodine's paper in 1967.

n	0	1	2	3
Clamped	3.20	4.61	5.90	6.30
Simply Supported	2.22	3.73	5.06	5.45

Fundamental modes only, from Kantham's paper in 1958.

CLAMPED EDGE

m \ n	0	1	2	3	4
1	3.196220617	4.610899880	5.905678236	7.143531024	8.346605939
2	6.306437048	7.799273801	9.196882600	10.53666987	11.83671846
3	9.439499138	10.95806719	12.40222097	13.79506360	15.14987010
4	12.57713064	14.10862781	15.57949149	17.00529018	18.39595702
5	15.71643853	17.25572701	18.74395810	20.19231303	21.60844831
6	18.85654552	20.40104490	21.90148516	23.36627975	24.80149223
7	21.99709516	23.54532554	25.05482216	26.53214306	27.98220170
8	25.13791541	26.68894922	28.20543287	29.69262100	31.15457239
9	28.27891311	29.83213054	31.35416937	32.84933383	34.32103153
10	31.42003345	32.97499985	34.50156168	36.00330909	37.48314260
11	34.56124206	36.11764083	37.64795700	39.15523056	40.64195984
12	37.70251633	39.26010971	40.79359228	42.30557131	43.79822079
13	40.84384075	42.40244566	43.93863487	45.45466916	46.95245728
14	43.98520433	45.54467679	47.08320631	48.60277191	50.10506273
15	47.12659909	48.68682381	50.22739700	51.75006548	53.25633453
16	50.26801907	51.82890232	53.37127560	54.89669212	56.40650173
17	53.40945973	54.97092428	56.51489519	58.04276256	59.55574365
18	56.55091758	58.11289902	59.65829744	61.18836426	62.70420271
19	59.69238985	61.25483392	62.80151556	64.33356723	65.85199347
20	62.83387435	64.39673490	65.94457629	67.47842821	68.99920914
21	65.97536930	67.53860678	69.08750144	70.62299362	72.14592634
22	69.11687326	70.68045348	72.23030897	73.76730187	75.29220863
23	72.25838504	73.82227826	75.37301381	76.91138496	78.43810921
24	75.39990364	76.96408384	78.51562845	80.05526980	81.58367290
25	78.54142824	80.10587251	81.65816343	83.19897918	84.72893780
26	81.68295815	83.24764621	84.80062773	86.34253250	87.87393643

CLAMPED EDGE

m \ n	5	6	7	8	9
1	9.525701356	10.68702586	11.83453021	12.97090865	14.09809354
2	13.10736372	14.35515634	15.58455188	16.79874060	18.00009791
3	16.47507757	17.77643378	19.05805844	20.32302126	21.57368079
4	19.75827660	21.09712081	22.41612475	23.71808431	25.00520347
5	22.99787246	24.36470172	25.71210576	27.04258559	28.35815492
6	26.21165995	27.60028082	28.97011748	30.32339603	31.66194085
7	29.40878990	30.81490500	32.20296412	33.57494944	34.93250995
8	32.59449808	34.01498922	35.41817203	36.80581630	38.17941341
9	35.77201399	37.20454027	38.62049165	40.02145421	41.40877956
10	38.94344482	40.38620082	41.81308485	43.22552423	44.62474724
11	40.64195984	42.11022715	44.99814499	46.42058364	47.83022501
12	45.27337561	46.73260106	48.17724501	49.60847780	51.02732320
13	48.43362799	49.89958418	51.35154430	52.79057443	54.21761365
14	51.59153406	53.06345010	54.52191726	55.96791031	57.40229320
15	54.74751212	56.22474315	57.68903650	59.14128653	60.58229044
16	57.90188624	59.38388723	60.85342821	62.31133247	63.75833764
17	61.05491147	62.54121752	64.01551021	65.47854975	66.93102045
18	64.20679131	65.69700263	67.17561864	68.64334335	70.10081319
19	67.35769020	68.85146040	70.33402709	71.80604395	73.26810431
20	70.50774242	72.00476943	73.49096055	74.96692433	76.43321509
21	73.65705879	75.15707753	76.64660584	78.12621155	79.59641370
22	76.80573153	78.30850813	79.80111935	81.28409624	82.75792575
23	79.95383804	81.45916508	82.95463305	84.44073963	85.91794252
24	83.10144377	84.60913645	86.10725908	87.59627902	89.07662724
25	86.24860443	87.75849738	89.25909338	90.75083211	92.23412009
26	89.39536776	90.90731239	92.41021847	93.90450038	95.39054216

CLAMPED EDGE

m \ n	10	11	12	13	14
1	15.21752515	16.33031005	17.43731958	18.53925394	19.63668548
2	19.19044779	20.37122616	21.54358686	22.70847325	23.86666801
3	22.81189466	24.03915635	25.25668738	26.46550169	27.66645157
4	26.27925538	27.54169121	28.79371613	30.03634370	31.27043579
5	29.66046293	30.95088001	32.23055928	33.50048170	34.76148989
6	32.98726880	34.30065690	35.60319205	36.89580819	38.17931472
7	36.27703464	37.60970619	38.93154128	40.24342134	41.54611633
8	39.54023383	40.88937011	42.22776976	43.55626075	44.87557148
9	42.78363050	44.14701578	45.49981714	46.84281051	48.17668289
10	46.01181982	47.38767394	48.75312995	50.10891435	51.45567409
11	49.22804829	50.61491559	51.99159058	53.35875346	54.71701314
12	52.43468281	53.83135544	55.21805273	56.59541184	57.96400591
13	55.63349411	57.03895733	58.43466733	59.82122150	61.19915953
14	58.82583590	60.23922806	61.64309019	63.03798297	64.42441500
15	62.01276237	63.43334495	64.84461897	66.24711129	67.64130164
16	65.19510762	66.62224244	68.04028653	69.44973564	70.85104269
17	68.37354089	69.80667239	71.23092612	72.64676917	74.05462962
18	71.54860577	72.98724714	74.41721802	75.83895907	77.25287536
19	74.72074860	76.16447074	77.59972349	79.02692314	80.44645341
20	77.89033953	79.33876215	80.77891005	82.21117698	83.63592688
21	81.05768818	82.51047260	83.95517041	85.39215456	86.82177055
22	84.22305567	85.67989889	87.12883707	88.57022388	90.00438773
23	87.38666378	88.84729359	90.30019349	91.74569926	93.18412337
24	90.54870218	92.01287310	93.46948291	94.91885080	96.36127440
25	93.70933404	95.17682388	96.63691529	98.08991202	99.53609784
26	96.86870053	98.33930755	99.80267287	101.2590859	102.7088174

CLAMPED EDGE

m \ n	15	16	17	18	19
1	20.73008895	21.81986310	22.90634657	23.98982973	25.07056371
2	25.01882878	26.16551430	27.30720392	28.44431239	29.57720140
3	28.86026146	30.04755307	31.22886460	32.40466553	33.57536824
4	32.49673238	33.71587428	34.92842070	36.13486298	37.33563555
5	36.01431379	37.25959062	38.49788048	39.72967870	40.95542586
6	39.45441847	40.72174089	41.98183176	43.23518018	44.48222354
7	42.84030343	44.12658189	45.40544850	46.67748962	47.94302447
8	46.18634671	47.48916032	48.78452574	50.07290444	51.35471295
9	49.50204590	50.81944688	52.12937784	53.43228299	54.72856495
10	52.79398822	54.12437738	55.44731170	56.76321738	58.07248219
11	56.06691722	57.40896026	58.74359069	60.07121656	61.39221043
12	59.32435263	60.67692148	62.02213971	63.36039751	64.69205226
13	62.56897090	63.93110118	65.28595726	66.63391194	67.97530775
14	65.80284928	67.17370870	68.53738077	69.89422153	71.24455908
15	69.02762822	70.40649260	71.77826382	73.14328196	74.50186120
16	72.24462277	73.63085738	75.01009813	76.38266985	77.74887341
17	75.45490098	76.84794595	78.23409965	79.61367249	80.98695262
18	78.65934029	80.05869891	81.45127084	82.83735280	84.21722085
19	81.85866894	83.26389827	84.66244642	86.05459717	87.44061509
20	85.05349689	86.46420008	87.86832772	89.26615138	90.65792473
21	88.24433924	89.66015918	91.06950874	92.47264794	93.86982011
22	91.43163422	92.85224833	94.26649627	95.67462717	97.07687456
23	94.61575722	96.04087303	97.45972558	98.87255370	100.2795817
24	97.79703179	99.22638324	100.6495728	102.0668295	103.4783689
25	100.9757384	102.4090827	103.8363646	105.2578041	106.6736084
26	104.1521214	105.5892364	107.0203866	108.4457834	109.8656261

CLAMPED EDGE

m \ n	20	21	22	23	24
1	26.14876738	27.22463280	28.29832958	29.37000831	30.43980343
2	30.70618850	31.83155425	32.95354794	34.07239220	35.18828683
3	34.74133721	35.90289640	37.06033527	38.21391363	39.36386574
4	38.53112467	39.72167556	40.90759819	42.08917210	43.26665040
5	42.17551579	43.39030224	44.60010431	45.80521098	47.00588495
6	45.72335474	46.95892831	48.18926539	49.41465796	50.63537238
7	49.20247625	50.45619546	51.70450075	52.94768291	54.18600811
8	52.63032874	53.90009513	55.16432543	56.42330650	57.67730169
9	56.01858998	57.30269239	58.58117828	59.85432875	61.12240266
10	59.37546012	60.67247534	61.96382561	63.24978513	64.53060710
11	62.70691353	64.01563931	65.31867650	66.61629170	67.90873170
12	66.01743230	67.33684012	68.65055506	69.95883573	71.26192209
13	69.31046024	70.63966090	71.96317960	73.28126677	74.59415530
14	72.58869649	73.92691437	75.25947319	76.58661515	77.90856598
15	75.85429248	77.20084584	78.54177244	79.87730636	81.20766618
16	79.10898807	80.46327360	81.81197212	83.15530970	84.49349785
17	82.35420809	83.71568876	85.07162800	86.42224400	87.76774144
18	85.59113232	86.95932755	88.32203139	89.67945459	91.03179497
19	88.82074728	90.19522496	91.56426483	92.92807032	94.28683269
20	92.04388513	93.42425507	94.79924341	96.16904654	97.53384940
21	95.26125336	96.64716181	98.02774685	99.40319809	100.7736943
22	98.47345774	99.86458289	101.2504442	102.6312248	104.0070976
23	101.6810204	103.0770685	104.4679132	105.8537316	107.2346907
24	104.8843939	106.2850959	107.6806558	109.0712443	110.4570233
25	108.0839728	109.4890819	110.8891100	112.2842225	113.6745759
26	111.2801027	112.6893914	114.0936606	115.4930701	116.8877712

CLAMPED EDGE

m \ n	25
1	31.50783550
2	36.30141186
3	40.51040364
4	44.44026311
5	48.20236580
6	51.85165243
7	55.41972068
8	58.92655351
9	62.38563899
10	65.80652586
11	69.19622548
12	72.56003728
13	75.90206219
14	79.22553641
15	82.53305633
16	85.82673476
17	89.10831230
18	92.37923849
19	95.64073203
20	98.89382634
21	102.1394044
22	105.3782260
23	108.6109490
24	111.8381461
25	115.0603188
26	118.2779081

SIMPLY SUPPORTED EDGE

m \ n	0	1	2	3	4
1	2.221519535	3.728024286	5.060958083	6.321179804	7.539336856
2	5.451605702	6.962811055	8.373591729	9.723629861	11.03188014
3	8.611391038	10.13771896	11.58869380	12.98749078	14.34751183
4	11.76087250	13.29666282	14.77168204	16.20138094	17.59566277
5	14.90687908	16.44889214	17.93992189	19.39103177	20.80982233
6	18.05129414	19.59767616	21.10013002	22.56697493	24.00422096
7	21.19484757	22.74445660	24.25547164	25.74378251	27.18604446
8	24.33788191	25.88996883	27.40763849	28.89608970	30.35934396
9	27.48057930	29.03462741	30.55761825	32.05381264	33.52658769
10	30.62304556	32.17868394	33.70602685	35.20863015	36.68936944
11	33.76534638	35.32229993	36.85326849	38.36126464	39.84876610
12	36.90752470	38.46558388	39.99961955	41.51221674	43.00553235
13	40.04960979	41.60861132	43.14527542	44.66184458	46.16021322
14	43.19162227	44.75143652	46.29037820	47.81041115	49.31321242
15	46.33357711	47.89409943	49.43503371	50.95811405	52.46483543
16	49.47548541	51.03663011	52.57932236	54.10510459	55.61531780
17	52.61735560	54.17905166	55.72330638	57.25150065	58.76484410
18	55.75919420	57.32138208	58.86703466	60.39739543	61.91356102
19	58.90100631	60.46363560	62.01054617	63.54286360	65.06158664
20	62.04279598	63.60582367	65.15387232	66.68796568	68.20901711
21	65.18456650	66.74795553	68.29703875	69.83275124	71.35593150
22	68.32632052	69.89003881	71.44006657	72.97726129	74.50239546
23	71.46806025	73.03207977	74.58297332	76.12153005	77.64846392
24	74.60978750	76.17408367	77.72577370	79.26558629	80.79418323
25	77.75150379	79.31605492	80.86848014	82.40945436	83.93959277
26	80.89321041	82.45799726	84.01110318	85.55315505	87.08472618

SIMPLY SUPPORTED EDGE

m \ n	5	6	7	8	9
1	8.729438345	9.899220082	11.05346553	12.19536662	13.32717366
2	12.30927231	13.56273564	14.79697976	16.01537263	17.22041810
3	15.67731529	16.98275134	18.26802317	19.53626785	20.78989834
4	18.96131755	20.30323612	21.62507678	22.92965878	24.21920802
5	22.20177754	23.57100544	24.92067602	26.25329502	27.57088296
6	25.41637525	26.80691920	28.17860714	29.53366185	30.87390737
7	28.61423340	30.02192442	31.41152349	32.78500611	34.14401765
8	31.80058705	33.22239304	34.62687657	36.01579964	37.39064855
9	34.97867334	36.41231325	37.82937766	39.23144480	40.61986069
10	38.15060897	39.59432023	41.02216741	42.43555703	43.83575160
11	41.31783963	42.77022303	44.20742746	45.63071384	47.04120378
12	44.48138802	45.94133874	47.38672429	48.81870864	50.23831060
13	47.64199881	49.10859524	50.56121377	52.00091475	53.42863264
14	50.80022755	52.27271291	53.73176855	55.17836404	56.61335911
15	53.95649741	55.43423801	56.89906046	58.35185446	59.79341335
16	57.11113652	58.59359627	60.06361552	61.52201338	62.96952396
17	60.26440335	61.75112459	63.22585171	64.68934115	66.14227404
18	63.41650431	64.90709315	66.38610552	67.85424200	69.31213609
19	66.56760625	68.06172124	69.54465100	71.01704612	72.47949729
20	69.71788456	71.21518870	72.70171362	74.17802608	75.64467801
21	72.86733485	74.36764447	75.85748072	77.33740888	78.80794578
22	76.01616783	77.51921298	79.01210917	80.49538514	81.96952579
23	79.16442327	80.66999900	82.16573144	83.65211616	85.12960899
24	82.31216773	83.82009138	85.31846012	86.80773938	88.28835843
25	85.45945794	86.96956596	88.47039155	89.96237265	91.44591417
26	88.60634251	90.11848790	91.62160867	93.11611758	94.60239722

SIMPLY SUPPORTED EDGE

m \ n	10	11	12	13	14
1	14.45054067	15.56672374	16.67670230	17.78125678	18.88102052
2	18.41403560	19.59773363	20.77272231	21.93998942	23.10035317
3	22.03081654	23.26055211	24.48035648	25.69126842	26.89416109
4	25.49551745	26.76005592	28.01404418	29.25850951	30.49432584
5	28.87509666	30.16731436	31.44869679	32.72023213	33.98276971
6	32.20086204	33.51580538	34.81982738	36.11386536	37.39873216
7	35.48994537	36.82397160	38.14711354	39.46025365	40.76416332
8	38.75269053	40.10301619	41.44257210	42.77218590	44.09258615
9	41.99578420	43.36022137	44.71405218	46.05805142	47.39290548
10	45.22377310	46.60056373	47.96694163	49.32363173	50.67127990
11	48.43987233	49.82757849	51.20508350	52.57306576	53.93213283
12	51.64642780	53.04385596	54.43130425	55.80940795	57.17873867
13	54.84519589	56.25134301	57.64773563	59.03496919	60.41358184
14	58.03752028	59.45153436	60.85601954	62.25153461	63.63858660
15	61.22444806	62.64559858	64.05744348	65.46050780	66.85526971
16	64.40680820	65.83446367	67.25303271	68.66300931	70.06484492
17	67.58526622	69.01887672	70.44361475	71.85994573	73.26829634
18	70.76036293	72.19944643	73.62986548	75.05205917	76.46643122
19	73.93254265	75.37667413	76.81234280	78.23996345	79.65991853
20	77.10217397	78.55097665	79.99151155	81.42417105	82.84931784
21	80.26956539	81.72270360	83.16776241	84.60511343	86.03510106
22	83.43497714	84.89215052	86.34142621	87.78315665	89.21766917
23	86.59863035	88.05956897	89.51278510	90.95861338	92.39736527
24	89.76071419	91.22517456	92.68208128	94.13175244	95.57448473
25	92.92139145	94.38915316	95.84952386	97.30280628	98.74928331
26	96.08080303	97.55166587	99.01529437	100.4719770	101.9219827

SIMPLY SUPPORTED EDGE

m \ n	15	16	17	18	19
1	19.97651548	21.06817761	22.15637527	23.24142291	24.32359140
2	24.25450006	25.40301250	26.54638945	27.68506211	28.81940604
3	28.08977651	29.27875138	30.46163672	31.63891305	32.81102308
4	31.72224376	32.94291344	34.15690235	35.36470916	36.56677481
5	35.23704572	36.48370307	37.72330707	38.95635784	40.18330028
6	38.67513802	39.94370772	41.20499422	42.45948963	43.70763415
7	42.05952136	43.34692878	44.62692063	45.89997566	47.16652424
8	45.40441810	46.70825635	48.00461519	49.29395704	50.57669946
9	48.71922575	50.03755958	51.34849923	53.65218927	53.94933282
10	52.01046446	53.34170566	54.66547343	55.98219395	57.29225509
11	55.28283132	56.62565509	57.96105210	59.28943009	60.61116149
12	58.53981299	59.89309947	61.23902473	62.57797845	63.91031772
13	61.78406183	63.14685372	64.50236366	65.85096383	67.19299630
14	65.01763723	66.38910838	67.75338669	69.11082756	70.46175852
15	68.24216616	69.62159767	79.99393240	72.35950971	73.71864318
16	71.45895336	72.84571511	74.22548086	75.59857474	76.96529697
17	74.66905887	76.06259498	77.44923893	78.82930037	80.20306681
18	77.87335386	79.27317113	80.66620181	82.05274188	83.43306674
19	81.07256159	82.47822017	83.87719846	85.26977952	86.65622729
20	84.26728804	85.67839377	87.08292551	88.48115411	89.87333261
21	87.45804512	88.87424329	90.28397315	91.68749405	93.08504872
22	90.64526844	92.06623858	93.48084505	94.88933633	96.29194537
23	93.82933127	95.25478278	96.67397389	98.08714278	99.49451318
24	97.01055538	98.44022390	99.86373360	101.2813130	102.6931771
25	100.1892198	101.6228864	103.0504492	104.4721948	105.8883073
26	103.3655677	104.8029668	106.2344046	107.6600920	109.0802277

SIMPLY SUPPORTED EDGE

m \ n	20	21	22	23	24
1	25.40311591	26.48020217	27.55503124	28.62776350	29.69854172
2	29.94975057	31.07638638	32.19957147	33.31953603	34.43648644
3	33.97827731	35.14106931	36.29967416	37.45435735	38.60535815
4	37.76349134	38.95520916	40.14224286	41.32487618	42.50336596
5	41.40453222	42.62041102	43.83125908	45.03736843	46.23900451
6	44.94982338	46.18641431	47.41773041	48.64406582	49.86568893
7	48.42695488	49.68161968	50.93083891	52.17490487	53.41408515
8	51.85322095	53.12386589	54.38894865	55.64875707	56.90355555
9	55.24019670	56.52511158	57.80439692	59.07832176	60.34714987
10	58.59601107	59.89378634	61.18587898	62.47256360	63.75409380
11	61.92658748	63.23602157	64.53975258	65.83804731	67.13115277
12	65.23637064	66.55644958	67.87080383	69.17972203	70.48343421
13	68.52877630	69.85859507	71.18272230	72.50140833	73.81488602
14	71.80648223	73.14527898	74.47840896	75.80611421	77.12862032
15	75.07162328	76.41871966	77.76018317	79.09624764	80.42713145
16	78.32592630	79.68072208	81.02992605	82.37376402	83.71244725
17	81.57088058	82.93276667	84.28918244	85.64027106	86.98623684
18	84.80743319	86.17608109	87.53923491	88.89710503	90.24988900
19	88.03678833	89.41169337	90.78115869	92.14538739	93.50457041
20	91.25969785	92.64047183	94.01586300	95.38606739	96.75126962
21	94.47686471	95.86315570	97.24412265	98.61995481	99.99083067
22	97.68889093	99.08037874	100.4666026	101.8477452	103.2239792
23	100.8962955	102.2926878	103.6838771	105.0700400	106.4513432
24	104.0995283	105.5005576	106.8964452	108.2873619	109.6734691
25	107.2989817	108.7044021	110.1047426	111.5001681	112.8908347
26	110.4949996	111.9045848	113.3091518	114.7088600	116.1038605

SIMPLY SUPPORTED EDGE

m \ n	25
1	30.76749360
2	35.55060851
3	39.75289307
4	43.67794563
5	47.43640948
6	51.08284538
7	54.64862547
8	58.15358753
9	61.61112089
10	65.03070434
11	68.41929818
12	71.78216360
13	75.12337236
14	78.44613795
15	81.75303890
16	85.04617375
17	88.32727158
18	91.59777253
19	94.85888760
20	98.11164379
21	101.3569188
22	104.5954678
23	107.8279450
24	111.0549197
25	114.2768908
26	116.1038605

FREE EDGE

m \ n	0	1	2	3	4
0	———	———	2.009524802	3.115921966	4.176852520
1	3.000522846	4.524881227	5.892050377	7.189832951	8.444916203
2	6.200257918	7.733795398	9.166760558	10.53907278	11.86939309
3	9.367509371	10.90675641	12.37183066	13.78540518	15.16047485
4	12.52271181	14.06669269	15.55136854	16.99158158	18.39685326
5	15.67270058	17.22033862	18.71836650	20.17733558	21.60452366
6	18.81998447	20.37045988	21.87820913	23.35114935	24.79502783
7	21.96568789	23.51840649	25.03356472	26.51733578	27.97437572
8	25.11038835	26.66491663	28.18591490	29.67833791	31.14603573
9	28.25441311	29.81042818	31.33615103	32.83564939	34.31216077
10	31.39796063	32.95521739	34.48484253	35.99023722	37.47416235
11	34.54115885	36.09946718	37.63237060	39.14275618	40.63300651
12	37.68409369	39.24330351	40.77900024	42.29366585	43.78937762
13	40.82682504	42.38681594	43.92492149	45.44329864	46.94377461
14	43.96939590	45.53006990	47.07027423	48.59190100	50.96570061
15	47.11183798	48.67311431	50.21511637	51.73965963	53.24804796
16	50.25417511	51.81598647	53.35967074	54.88671858	56.39842866
17	53.39642563	54.95871531	56.50385825	58.03319081	59.54788583
18	56.53866038	58.10132366	59.64777613	61.17916616	62.69655820
19	59.68072097	61.24382970	62.79146437	64.32471697	65.84455797
20	62.82278617	64.38624817	65.93495542	67.46990218	68.99197687
21	65.96480676	67.52859107	69.07827587	70.61477028	72.13889065
22	69.10678878	70.67086834	72.22144776	73.75936155	75.28536241
23	72.24873721	73.81308821	75.36448947	76.90370969	78.43144512
24	75.39065623	76.95525763	78.50741647	80.04784316	81.57718361
25	78.53254934	80.09738243	81.65024186	83.19178613	84.72261608

FREE EDGE

m \ n	5	6	7	8	9
0	5.217448488	6.246056300	7.266624576	8.281376190	9.291701338
1	9.670223343	10.87326564	12.05887907	13.23039421	14.39021869
2	13.16835519	14.44273982	15.69719788	16.93508779	18.15892963
3	16.50533613	17.82567945	19.12561414	20.40822422	21.67589504
4	19.77373135	21.12695084	22.46006879	23.77583842	25.07644273
5	23.00520392	24.38334295	25.74201401	27.08365568	28.41024151
6	26.21417384	27.61194510	28.99100809	30.35352225	31.70126532
7	29.40829928	30.82197941	32.21774420	33.59751137	34.96288277
8	32.59207057	34.01892545	35.42864679	36.82294470	38.20326554
9	35.76831183	37.20626841	38.62784121	40.03456223	41.42774127
10	38.93889542	40.38634125	41.81811202	43.23558654	44.63995484
11	42.10511502	43.56076944	45.00141358	46.42829395	47.84249544
12	45.26789479	46.73072275	48.17916149	49.61434271	51.03725881
13	48.42791338	49.89706580	51.35240812	52.79497201	54.22566817
14	51.58568086	53.06045219	54.52195320	55.97112789	57.40881464
15	54.74158826	56.22138563	57.68841594	59.14354601	60.58754992
16	57.89594059	59.38026087	60.85228365	62.31280755	63.76254927
17	61.04897931	62.53739173	64.01394533	65.47937808	66.93435537
18	64.20089815	65.69303104	67.17371540	68.64363520	70.10340966
19	67.35185442	68.84738475	70.33185091	71.80588846	73.27007508
20	70.50197716	72.00062242	73.48856407	74.96639421	76.43465292
21	73.65137327	75.15288491	76.64403171	78.12536656	79.59739546
22	76.80013206	78.30429025	79.79840237	81.28298583	82.75851564
23	79.94832876	81.45493811	82.95180176	84.43940497	85.91819445
24	83.09602716	84.60491322	86.10433704	87.59475460	89.07658678
25	86.24328173	87.75428808	89.25610008	90.74914697	92.23382596

FREE EDGE

m \ n	10	11	12	13	14
0	10.29853357	11.30253297	12.30418503	13.30385801	14.30183838
1	15.54015828	16.68160727	17.81566794	18.94322889	20.06501844
2	19.37067187	20.57185760	21.76373332	22.94732276	24.12347863
3	22.93051676	24.17361719	25.40645207	26.63006820	27.84534891
4	26.36364728	27.63899040	28.90342389	30.15822940	31.40419304
5	29.72339532	31.02447218	32.31461670	33.59480607	34.86588231
6	33.03572156	34.35814526	35.66960764	36.97103212	38.26322130
7	36.31521263	37.65565797	38.98521652	40.30475571	41.61503530
8	39.57084646	40.92675078	42.27190884	43.60713209	44.93313697
9	42.80850848	44.17784705	45.53661852	46.88558283	48.22541431
10	46.03225245	47.41338701	48.78415930	50.14528003	51.49738334
11	49.24496920	50.63655457	52.01799660	53.38996025	54.75304183
12	52.44878570	53.84970103	55.24069894	56.62240208	57.99537143
13	55.64530534	57.05460547	58.45421627	59.84472139	61.22664896
14	58.83576203	60.25264173	61.66005905	63.05856181	64.44864763
15	62.02112232	63.44488931	64.85941759	66.26522200	67.66277190
16	65.20215493	66.63220936	68.05324395	69.46574323	70.87015042
17	68.37947953	69.81529789	71.24230957	72.66097120	74.07170176
18	71.55360154	72.99472406	74.42724675	75.85160054	77.26818198
19	74.72493761	76.17095812	77.60857906	79.03820761	80.46021956
20	77.89383440	79.34439226	80.78674402	82.22127499	83.64834163
21	81.06058286	82.51535629	83.96211031	85.40121001	86.83299392
22	84.22542906	85.68412967	87.13499089	88.57835905	90.01455602
23	87.38858257	88.85095089	90.30565327	91.75301867	93.19335342
24	90.55022307	92.01602473	93.47432755	94.92544430	96.36966691
25	93.71050550	95.17952809	96.64121276	98.09585728	99.54374005

FREE EDGE

m \ n	15	16	17	18	19
0	15.29835370	16.29358788	17.28769184	18.28079099	19.27299081
1	21.18164196	22.29360884	23.40135232	24.50524435	25.60560694
2	25.29291995	26.45625940	27.61402404	28.76667100	29.91459973
3	29.05304761	30.25381290	31.44820778	32.63672463	33.81979690
4	32.64206634	33.87250202	35.09607114	36.31327674	37.52456452
5	36.12857703	37.38353060	38.63130726	39.87240715	41.10727604
6	39.54687790	40.82262131	42.09100073	43.35250578	44.60757510
7	42.91672509	44.21041916	45.49664725	46.77588412	48.04855711
8	46.25055907	47.55996530	48.86186378	50.15671205	51.44492378
9	49.55671454	50.88002283	52.19582488	53.50455992	54.80662666
10	52.84103788	54.17675583	55.50500044	56.82619230	58.14071465
11	56.10777857	57.45465645	58.79411675	60.12656161	61.45235869
12	59.36011456	60.71709241	62.06672508	63.40939671	64.74545962
13	62.60047868	63.96664780	65.32555615	66.67757052	68.02302830
14	65.83077018	67.20534433	68.57275070	69.93333944	71.28743353
15	69.05249665	70.43479013	71.81001474	73.17850481	74.54056952
16	72.26687218	73.65628272	75.03872728	76.41452513	77.78397229
17	75.47488683	76.87088217	78.26001683	79.64259586	81.01890273
18	78.67735700	80.07946407	81.47481704	82.86370752	84.24640703
19	81.87496250	83.28275877	84.68390788	86.07868874	87.46736158
20	85.06827450	86.48138078	87.88794655	89.28823872	90.68250680
21	88.25777667	89.67585129	91.08749121	92.49295202	93.89247308
22	91.44388158	92.86661547	94.28301923	95.69333778	97.09780088
23	94.62694345	96.05405604	97.47494154	98.88983481	100.2989565
24	97.80726828	99.23850404	100.6636140	102.0828235	103.4963446
25	100.9851219	102.4202473	103.8493464	105.2726354	106.6903182

FREE EDGE

m \ n	20	21	22	23	24
0	26.70272098	27.79683314	28.88816137	29.97689931	31.06321989
1	31.05816160	32.19766759	33.33339441	34.46558953	35.59447530
2	34.99780849	36.17110126	37.33998118	38.50472332	39.66557602
3	38.73033159	39.93093349	41.12669005	42.31789012	43.50479564
4	42.33631318	43.55987781	44.77829451	45.99185771	47.20083538
5	45.85660350	47.09994778	48.33793167	49.57084991	50.79897175
6	49.31505257	50.57572106	51.83088185	53.08082665	54.32582282
7	52.72688744	54.00290601	55.27333104	56.53843600	57.79848428
8	56.10238835	57.39217698	58.67629693	59.95502807	61.22862841
9	59.44891781	60.75112300	62.04762500	63.33869797	64.62459185
10	62.77184515	64.08533107	65.39310239	66.69542344	67.99253914
11	66.07523785	67.39903026	68.71711314	70.02974255	71.33715630
12	69.36224065	70.69549531	72.02305894	73.34517923	74.66208673
13	72.63533165	73.97731062	75.31362760	76.64452197	77.97021698
14	75.89649545	77.24654887	78.59097762	79.93001289	81.26387073
15	79.14734375	80.50489553	81.85686647	83.20347976	84.54494435
16	82.38920134	83.75373788	85.11274246	86.46643050	87.81500404
17	85.62316890	86.99422991	88.35981174	89.72012232	91.07535696
18	88.85016964	90.22734068	91.59908832	92.96561323	94.32710422
19	92.07098445	93.45389081	94.83143177	96.20380104	97.57118113
20	95.28627894	96.67458053	98.05755763	99.43545337	100.8083881
21	98.49662440	99.89001146	101.2781535	102.6612308	104.0394147
22	101.7025142	103.1007037	104.4937096	105.8817062	107.2648587
23	104.9043771	106.3071096	107.7047202	109.0973775	110.4852413
24	108.1025872	109.5096243	110.9116014	112.3086815	113.7010191
25	111.2974732	112.7085952	114.1147190	115.5160021	116.9125941

FREE EDGE

m \ n	25
0	32.14727825
1	36.72025232
2	40.82276438
3	44.68764496
4	48.40547229
5	52.02254392
6	55.56611611
7	59.05371865
8	62.49733638
9	65.90554049
10	69.28467696
11	72.63957570
12	75.97399643
13	79.29092125
14	82.59275338
15	85.88145619
16	89.15865285
17	92.42569939
18	95.68373917
19	98.93374425
20	102.1765472
21	105.4128660
22	108.6433233
23	111.8684628
24	115.0887608
25	118.3046372

APPENDIX

```
      PROGRAM PLATE
C     TO FIND THE ROOTS OF THE FREQUENCY EQUATION
      IMPLICIT REAL*8 (A-H, O-Z), INTEGER*4 (I-N)
      COMMON XORD
  500 FORMAT (' ROOTS OF THE FREQUENCY EQUATION'/)
      PRINT 500
      XORD = 0.0
      ENDORD = 25.0
  502 FORMAT (6HORD =,3X,F5.1,3X,5HTO    ,F5.1/)
      PRINT 502, XORD,ENDORD
  100 FORMAT (3X,5HORD =,F5.1,3X,6HROOT =,F15.11,3X,5HEP1 =,D19.12)
  200 FORMAT (3X,5HORD =,F5.1,3X,6HROOT =,F15.11,3X,5HEP1 =,D19.12,
     1     3X,6HDIFF =,D19.12)
      ERROR = 0.10D-09
  504 FORMAT (8H ERROR =,3X,D23.16//)
      PRINT 504
      A = 1.0
      GO TO 3
    2 CONTINUE
      A = 1.0
    3 CONTINUE
      B = 150.0
      H = 1.0
      XL = A
    4 YL = FUNCT(XL)
      IF (ABS(YL)-ERROR)  10,10,20
   10 PRINT 100,XORD,XL,YL
      XL= XL+ H
      IF (XL-B)  4,4,70
   20 XR = XL+ H
      IF (XR-B)  22,22,70
   22 YR = FUNCT(XR)
      IF (ABS(YR)-ERROR)  30.30.24
   24 CONTINUE
      YLSIGN = ABS(YL)/YL
      YRSIGN = ABS(YR)/YR
      IF (YRSIGN*YLSIGN)  40,30,60
   30 PRINT 100,XORD,XR,YR
      XL = XR + H
      IF (XL-B)  4,4,70
   40 XI = (XR + XL)/2.0
      DIFF=(XR-XL)/2.0
      YI = FUNCT(XI)
  300 FORMAT (7D18.11)
      PRINT 300,XL,XI,XR,DIFF,YL,YI,YR
```

```
45 IF (DIFF-ERROR)  46,46,48
46 CONTINUE
47 PRINT 200,XORD,XI,YI,DIFF
   XL= XI + H
   GO TO 4
48 CONTINUE
   IF (ABS(YI)-ERROR)  47,47,50
50 CONTINUE
   YLSIGN = ABS(YL)/YL
   YISIGN = ABS(YI)/YI
   IF (YLSIGN*YISIGN)  52,47,54
52 XR = XI
   YR = YI
   GO TO 40
54 XL = XI
   YL = YI
   GO TO 40
60 XL = XR
   YL = YR
   GO TO 20
70 CONTINUE
   XORD = XORD + 1.0
   IF (XORD-ENDORD)  2,2,80
80 CONTINUE
   STOP
   END

   DOUBLE PRECISION FUNCTION FUNCT(B)
   IMPLICIT REAL*8 (A-H, O-Z), INTEGER*4 (I-N)
   COMMON XORD
   V = 0.300
   CALL BESSEL (B,XORD,BJ,BI,BJ1,BI1)
   CALL EPSLON (B,XORD,BJ,BI,BJ1,BI1,V,EP1)
   FUNCT = EP1
   RETURN
   END

   SUBROUTINE BESSEL (B,XORD,BJ,BI,BJ1,BI1)
   IMPLICIT REAL*8 (A-H, O-Z), INTEGER*4 (I-N)
   DIMENSION BJR(250),BIR(250),TERM(150)
   NTERM = 150
   SHIFTJ = 4.0/7.0*(XORD+35.0)
   SHIFTI = 3.88*(XORD+1.94)
   IF (B-SHIFTJ)  145,145,1
```

```
    1 CONTINUE
      IF (B-13.0)  2,40,40
C     DIRECT SERIES FOR J(X) AND I(X)
C     FOR X LESS THAN 13.0
    2 ORD= 0.0
    3 CONTINUE
      TEST1= 0.1D-12
      Z = B/2.0
      Z2 = Z*Z
      SIGN = (-1.0)
      TEMPI = 1.0
      IF (ORD-1.0)  8,4,4
    4 NORD = ORD
      DO 5 I = 1,NORD
      XI = I
    5 TEMPI = TEMPI*Z/XI
    8 CONTINUE
      TEMPJ = TEMPI
      SUMI = TEMPI
      SUMJ = TEMPJ
   10 DO 28 NS = 1,NTERM
      S1 = NS
      S2 = ORD + S1
      Z2S12 = Z2/(S1*S2)
      TEMPI = TEMPI*Z2S12
      TEMPJ = TEMPI
      SUMIT = SUMI + TEMPI
      SUMJT = SUMJ + SIGN*TEMPJ
      IF (TEST1-ABS(TEMPI))  26,26,30
   26 SUMI = SUMIT
      SUMJ = SUMJT
      SIGN = (-SIGN)
   28 CONTINUE
   30 BIT = SUMIT
      BJT= SUMJT
   35 CONTINUE
      IF (ORD)  36,36,37
   36 BJR(1) = BJT
      BIR(1) = BIT
      ORD = 1.0
      GO TO 3
   37 BJR(2)= BJT
      BIR(2)= BIT
      GO TO 60
```

```
   40 CONTINUE
C     ASYMPTOTIC SERIES FOR J(X) AND I(X)
C     FOR X GREATER THAN OR EQUAL TO 13.0
      PI = 3.141592653589793
      TEST2 = 0.1D-12
      CN = 4.0
      ORD = 0.0
   41 CONTINUE
      TERML = 1.0
      TERMP = 0.10D 50
      IF (ORD)  420,420,42
  420 DO 430 N = 1,NTERM
      XN = N
      TERM(N) = TERML*(-1.0)*((2.0*XN-1.0)**2)/(8.0*B*XN)
      TERML = TERM(N)
      IF (ABS(TERMP)-ABS(TERML))  421,421,425
  421 TERM(N) = 0.0
      GO TO 325
  425 TERMP = TERML
      IF (ABS(TERML)-TEST2)  325,325,430
  430 CONTINUE
      GO TO 325
   42 DO 320 N = 1,NTERM
      XN = N
      TERM(N) = TERML*(4.0*ORD**2-(2.0*XN-1.0)**2)/(8.0*B*XN)
      TERML = TERM(N)
      IF (ABS(TERMP)-ABS(TERML))  305,305,310
  305 ZN = N
      IF (CN-ZN)  309,309,310
  309 TERM(N) = 0.0
      GO TO 325
  310 TERMP = TERML
      IF (ABS(TERML)-TEST2)  325,325,320
  320 CONTINUE
  325 CONTINUE
      M = XN
      IF ((-1.0)**M)  43,43,44
   43 LIMP = XN-1.0
      LIMQ = XN
      GO TO 45
   44 LIMP = XN
      LIMQ = XN-1.0
   45 P = 1.0
      DO 46 N2 = 2,LIMP,2
```

```
      N = N2/2
      P = P + ((-1.0)**N)*TERM(N2)
   46 CONTINUE
      Q = 0.0
      DO 47 N21 = 1,LIMQ,2
      N = (N21 + 1)/2
      Q = Q + ((-1.0)**N)*TERM(N21)
   47 CONTINUE
      C = 1.0
      K = XN
      DO 48 N = 1,K
      C = C + ((-1.0)**N)*TERM(N)
   48 CONTINUE
      DEL = B-(ORD + 0.5)*0.5*PI
      BJT = (2.0/(PI*B))**0.5*(P*COS(DEL)+ Q*SIN(DEL))
      BIT = C*EXP(B)/((2.0*PI*B)**0.5)
      IF (ORD)  50,50,55
   50 BJR(1) = BJT
      BIR(1) = BIT
      ORD = 1.0
      GO TO 41
   55 BJR(2) = BJT
      BIR(2) = BIT
   60 IF (XORD-1.0)  75,80,61
   61 XORD1 = XORD + 1.0
      XORD2 = XORD1 + 1.0
      NXORD2 = XORD2
      IF (B-SHIFTJ)  100,100,62
   62 CONTINUE
C     RECURSION FORMULA
      DO 70 N =3,NXORD2
      XN = N
      BJR(N) = (2.0*XN-4.0)/B*BJR(N-1)-BJR(N-2)
      IF (XN-XORD1)  70,63,65
   63 BJ = BJR(N)
      GO TO 70
   65 BJ1 = BJR(N)
   70 CONTINUE
      GO TO 100
   75 IF (B-SHIFTJ)  77,77,76
   76 BJ = BJR(1)
      BJ1 = BJR(2)
   77 IF (B-SHIFTI)  170,170,78
   78 BI = BIR(1)
```

```
      BI1 = BIR(2)
      GO TO 230
   80 IF (B-SHIFTJ)  812,812,811
  811 BJ = BJR(2)
      BJ1 = 2.0*BJR(2)/B - BJR(1)
  812 IF (B-SHIFTI)  170,170,82
   82 CONTINUE
      BI = BIR(2)
      BI1 = BIR(1)-2.0*BIR(2)/B
      GO TO 230
  100 CONTINUE
      IF (B-SHIFTI)  170,170,820
  820 CONTINUE
  821 DO 95 N = 3,NXORD2
      XN = N
      BIR(N) = BIR(N-2)-(2.0*XN-4.0)/B*BIR(N-1)
      IF (XN-XORD1)  95,83,85
   83 BI= BIR(N)
      GO TO 95
   85 BI1 = BIR(N)
   95 CONTINUE
      GO TO 230
C     APPROXIMATE NUMERICAL METHOD
  145 CONTINUE
      ORD = XORD
      TORDJ = ORD + 15.0
  146 CONTINUE
      JORD1 = ORD + 1.0
      JTORD = TORDJ
      TORD1J = TORDJ + 1.0
      JTORD1 = TORD1J
      BJR(JTORD1) = 0.0
      BJR(JTORD) = 1.0
      TORD2J = TORDJ + 2.0
      JTORD2 = TORD2J
      KJ = JTORD-1
      DO 155 N = 1,KJ
      XN = N
      NN1 = JTORD-N
      NN2 = JTORD1-N
      NN3 = JTORD2-N
      BJR(NN1) = 2.0*(TORDJ-XN)*BJR(NN2)/B - BJR(NN3)
      IF (NN1-JORD1)  155,147,155
  147 IF (ABS(BJR(JORD1+ 1))-0.10D 12)  148,148,155
```

```
148 TORDJ = TORDJ + 5.0
    IF (TORDJ-250.0)  146,146,150
150 CONTINUE
152 FORMAT (3X,5HORD =,F5.1,3X,3HB =,F8.3,3X,7HTORDJ =,F8.2)
    PRINT 152,ORD,B,TORDJ
    GO TO 230
155 CONTINUE
    BJNORM = BJR(1)
    DO 160 N = 1,JTORD1
    N21 = 2*N + 1
    IF (JTORD1-N21)  161,157,157
157 BJNORM = BJNORM + 2.0*BJR(N21)
160 CONTINUE
161 CONTINUE
    BJFAC = 1.0/BJNORM
    BJ =  BJFAC*BJR(JORD1)
    BJ1 = BJFAC*BJR(JORD1 + 1)
    IF (B-SHIFTI)  170,170,1
170 CONTINUE
    ORD = XORD
    TORDI = ORD + 15.0
171 CONTINUE
    IORD1 = ORD +1.0
    ITORD = TORDI
    TORD1I = TORDI+ 1.0
    ITORD1 = TORD1I
    BIR(ITORD1) = 0.0
    BIR(ITORD) = 1.0
    TORD2I = TORDI + 2.0
    ITORD2 = TORD2I
    KI = ITORD-1
    DO 180 N = 1,KI
    XN = N
    NN1 = ITORD-N
    NN2 = ITORD1-N
    NN3 = ITORD2-N
    BIR(NN1) = 2.0*(TORDI-XN)*BIR(NN2)/B + BIR(NN3)
    IF (NN1-IORD1)  180,172,180
172 IF (ABS(BIR(IORD1+ 1))-0.10D 12)  173,173,180
173 TORDI = TORDI + 5.0
    IF (TORDI-250.0)  171,171,175
175 CONTINUE
176 FORMAT (3X,5HORD =,F5.1,3X,3HB =,F8.3,3X,7HTORDI =,F8.2)
    PRINT 176,ORD,B,TORDI
```

```
      GO TO 230
  180 CONTINUE
      BINORM = BIR(1)
      DO 185 N = 2,ITORD1
      BINORM = BINORM + 2.0*BIR(N)
  185 CONTINUE
      BIFAC = EXP(B)/BINORM
      BI = BIFAC*BIR(IORD1)
      BI1 = BIFAC*BIR(IORD1+ 1)
  230 CONTINUE
      RETURN
      END

C     BOUNDARY CONDITION ********** CLAMPED EDGE *****************
      SUBROUTINE EPSLON (B,ORD,BJ,BI,BJ1,BI1,V,EP1)
      IMPLICIT REAL*8 (A-H, O-Z), INTEGER*4 (I-N)
      EP1 = BJ + BJ1*(BI/BI1)
      RETURN
      END

C     BOUNDARY CONDITION ********** SIMPLY SUPPORTED EDGE *******
      SUBROUTINE EPSLON (B,ORD,BJ,BI,BJ1,BI1,V,EP1)
      IMPLICIT REAL*8 (A-H, O-Z), INTEGER*4 (I-N)
      EP1= 2.0*B*B*BJ-B*(1.0-V)*(BJ1+ BJ*(BI1/BI))
      RETURN
      END

C     BOUNDARY CONDITION ********** FREE EDGE ********************
      SUBROUTINE EPSLON (B,ORD,BJ,BI,BJ1,BI1,V,EP1)
      IMPLICIT REAL*8 (A-H, O-Z), INTEGER*4 (I-N)
      AAF = BJ*(B**2-ORD*(ORD-1.0)*(1.0-V))-BJ1*B*(1.0-V)
      BB1= ORD*BI*(B**2-ORD*(ORD-1.0)*(1.0-V))
      BB2= BI1*B*(B**2-(1.0-V)*(ORD**2))
      BBF = BB1 + BB2
      CC1= ORD*BJ*(B**2 + ORD*(ORD-1.0)*(1.0-V)
      CC2 = BJ1*B*(B**2 + (1.0-V)*(ORD**2))
      CCF = CC1-CC2
      DDF = BI*(B**2+ORD*(1.0-ORD)*(1.0-V))-BI1*B*(1.0-V)
      EP1= AAF-CCF*(DDF/BBF)
      RETURN
      END
```

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