A STUDY OF THE PROPERTIES OF THE BICONICAL ANTENNA

BY

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A THESIS

PRESENTED IN PARTIAL FULFILLMENT OF

THE REQUIREMENTS FOR THE DEGREE

OF

MASTER OF SCIENCE IN ELECTRICAL ENGINEERING

AT

NEWARK COLLEGE OF ENGINEERING

Newark, New Jersey
1968

ABSTRACT

Expressions are found for the fields produced by TM and TEM modes along a finitely conducting biconical antenna. Expressions are also found for the TM and TEM fields along a biconical antenna having a lossy dielectric coating. It is shown that the antenna losses produce an attenuation of the fields along the antenna. It is found that because there is a dispersion of the fields along the biconical antenna, the attenuation produced by the losses is, in general, not characterized by simple exponential.

A solution is obtained for the fields of the TEM wave along a biconical antenna coated with a dielectric cone having a permittivity significantly different from that of the surrounding media. It is shown that the principal effect of dielectric coating is to reduce the wave length along the antenna.

The energy transfer along the biconical antenna is investigated and a physical interpretation is given for the behavior of the fields along the antenna, based on the results obtained.

APPROVAL OF THESIS

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FOR

DEPARTMENT OF ELECTRICAL ENGINEERING

NEWARK COLLEGE OF ENGINEERING

BY

FACULTY COMMITTEE

APPROVED: ______________________

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NEWARK, NEW JERSEY

MAY, 1968

ACKNOWLEDGMENTS

I wish to express my sincere gratitude to the following persons who have aided me in the preparation of this thesis:

To Dr. Garland Ching, whose counsel has greatly added to the quality of the overall thesis.

To the U. S. Navy Underwater Sound Laboratory, for their cooperation in allowing time to complete this work.

Kurt F. Hafner, Jr.

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INTRODUCTION

The biconical antenna was first studied by S. A. Schelkunoff as a basis for extension to analysis of antennas of arbitrary size and shape.[1 2] This antenna was chosen by Schelkunoff as being particularly suited for analysis, because its boundaries in the spherical coordinate system lie along the coordinate surfaces, reducing the mathematical complexity, which is the limiting factor in much antenna analysis. Using perturbation methods, Schelkunoff was able to deform the biconical antenna to obtain approximate solutions for the fields of a number of other antennas which would have been difficult to study directly. The biconical antenna is also important as a practical broadband antenna, providing an antenna impedance of close to 50 ohms resistance over a wide range of frequencies. This makes the biconical antenna particularly useful as a transmitting antenna, since it does not require complex tuning networks to match the impedance of the antenna to that of the transmission line with which it must be used.

The importance of the biconical antenna as a practical antenna and as a tool in the study of other types of antennas makes a good understanding of its behavior useful to the engineer interested in antenna design. It is the purpose of this paper to give a physical insight into the behavior of the fields produced by the biconical

[1]S. A. Schelkunoff, "Theory of Antennas of Arbitrary Size and Shape," I.R.E. Proceedings (Sept. 9, 1941), pp.493-521.

[2]C. T. Tai, "On the Theory of Biconical Antennas," Journal of Applied Physics (Dec. 12, 1948), pp. 1155-1160.

antenna. A better understanding of these fields is gained by considering the effects produced by losses along the biconical antenna due to finitely conducting surfaces and dielectric loss. Consideration of antenna losses depends on an investigation of energy transfer along the antenna, providing a good basis for physical interpretation of the field equations. An investigation of the effect of dielectric coating on the biconical is also found to be of interest. These considerations, in addition to an investigation of the influence of the input and output boundaries, serve to give an intuitive understanding of the behavior of the fields along the antenna, as well as to extend the existing information concerning these fields.

In studying the biconical antenna, it is useful to begin with the infinite case or biconical transmission line. This provides a simple case, which serves as a basis for comparison when the biconical antenna of finite length is considered. The energy transfer along the antenna will then be studied, leading to an examination of the antenna losses. The effect of a dielectric coating in the fields about the antenna will be presented as a final consideration.

SCHELKUNOFF'S APPROACH TO THE BICONICAL ANTENNA PROBLEM

The general approach employed by Schelkunoff applies to whether the antenna is considered to be of finite or infinite length. To examine this method of solution, consider the configuration of the symmetric biconical antenna of half cone angle δ which is shown in Fig. 1. The biconical antenna consists of two Fig. 1 half cones terminated in spherical caps which are centered at the verticies. Only

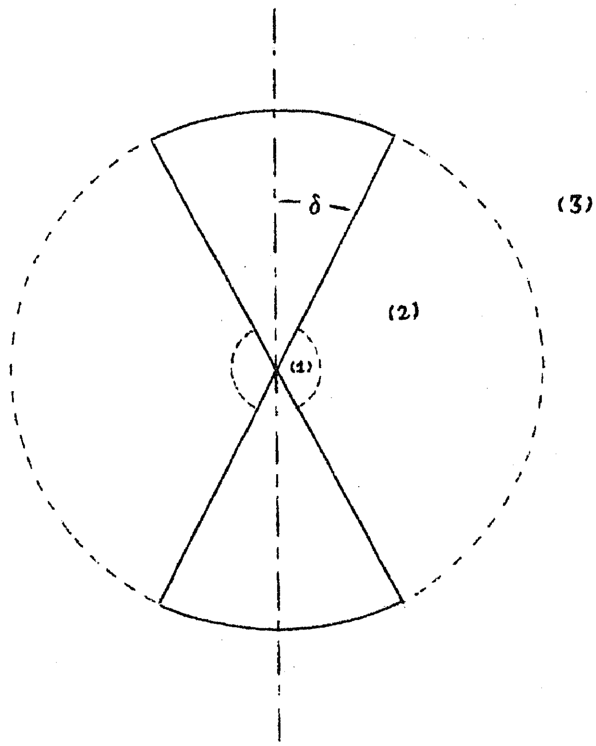

Fig. 1. The Biconical Antenna

the symmetric antenna with coaxial half cones will be considered here. This does not restrict the conceptual presentation, but serves to simplify the mathematics. The results are easily extended to include the nonsymmetric case.

Schelkunoff has divided the antenna into three sections for the purpose of study. The first section, region 1, is bounded by a sphere centered at the origin large enough to include all sources. The second, region 2, is the antenna region bounded by the source region and the sphere defined by the terminating caps. The free space region outside the antenna constitutes region 3. Because the boundaries of the antenna lie along coordinate surfaces, the mathematical expression of the boundary conditions is greatly simplified, which is a principal advantage in choosing this configuration.

The fields surrounding the antenna may be found by obtaining a general solution to the Helmholtz equations in the spherical coordinate system. Since the half cones are coaxial, the antenna is ϕ symmetric which allows the six Helmholtz equations, given below in spherical coordinates, to be divided into two groups of three:

$$\begin{aligned}
&\text{a.} \quad \frac{\partial}{\partial r}(r\, E_\theta) - \frac{\partial}{\partial \theta} E_r = -\, j\, \omega\, \mu\, H_\phi \\
&\text{b.} \quad \frac{\partial}{\partial \theta}(\sin\theta\, H_\phi) = j\, \omega\, \epsilon\, r\, \sin\theta\, E_r \\
&\text{c.} \quad \frac{\partial}{\partial r}(r\, H_\phi) = -\, j\, \omega\, \epsilon\, r\, E_\theta \\
&\text{d.} \quad \frac{\partial}{\partial r}(r\, H_\theta) - \frac{\partial}{\partial \theta} H_r = -\, j\, \omega\, \epsilon\, E_\phi \\
&\text{e.} \quad \frac{\partial}{\partial \theta}(\sin\theta\, E_\phi) = -\, j\, \omega\, \mu\, r\, \sin\theta\, H_r \\
&\text{f.} \quad \frac{\partial}{\partial r}(r\, E_\phi) = -\, j\, \omega\, \mu\, r\, H_\theta
\end{aligned} \tag{1}$$

The first group, consisting of equations (a), (b), and (c), involves only E_r, E_θ and H_ϕ. The second group, equations (d), (e), and (f), involves only H_r, H_θ and E_ϕ. These two groups of equations may be solved independently and the solutions added together to give a complete solution. The first group of variables represents a wave which has no magnetic field component in the radial direction. This defines a transverse magnetic, TM, wave in spherical coordinates. The latter wave, which results from currents circulating in the ϕ direction about the antenna, will not be considered here. The TM wave depends on radial currents along the antenna surface and will be of primary interest. A wave which is transverse electric and magnetic, TEM, may also be supported by the biconical antenna. The expressions for the fields produced by this wave may be found separately, using a quasi-static approach.[3] This solution may be added to the solution of the Helmholtz equations for the TM waves found by Schelkunoff,[4] to give the general expression of the fields along the biconical antenna subject to the boundary conditions. The resulting solutions are given below, where the time variation, $e^{j\omega t}$, has been suppressed, as is customary.

$$
\begin{aligned}
r^2 E_r &= \frac{j}{2\pi\omega\epsilon} \sum_n a_n M_n(\cos\theta) \left[\overline{H}_n^{(2)}(\beta r) + g_n \overline{H}_n^{(1)}(\beta r) \right] \\
r H_\phi &= A \frac{e^{-j\beta r} + K e^{j\beta r}}{\sin\theta} + \\
&\quad \frac{1}{2\pi} \sum_n \frac{a_n}{n(n+1)} \frac{d}{d\theta} M_n(\cos\theta) \left[\overline{H}_n^{(2)}(\beta r) + g_n \overline{H}_n^{(1)}(\beta r) \right] \\
r E_\theta &= A Z_0 \frac{e^{-j\beta r} - K e^{j\beta r}}{\sin\theta} + \\
&\quad \frac{j\eta}{2\pi} \sum_n \frac{a_n}{n(n+1)} \frac{d}{d\theta} M_n(\cos\theta) \left[\overline{H}_n^{(2)'}(\beta r) + g_n \overline{H}_n^{(1)'}(\beta r) \right],
\end{aligned}
\qquad (2)
$$

where $M_n(\cos\theta)$ is an odd Legendre function defined by $M_n(\cos\theta) = \frac{1}{2}\left[P_n(\cos\theta) - P_n(\cos\theta)\right]$. η is the intrinsic impedance of free space. $\overline{H}_n^{(1)}(\beta r)$ and $\overline{H}_n^{(2)}(\beta r)$ are normalized Hankel functions of the first

[3] J. Weeks, " Electromagnetic Theory for Engineering Applications (New York: Wiley, 1964).

[4] S. A. Schelkunoff, Advanced Antenna Theory (New York: Wiley, 1952), p. 42.

and second kind respectively of order n defined by

$$\bar{H}_n^{(1)}(\beta r) = \left[\frac{1}{2}\pi \beta r\right]^{1/2} H_{n+1/2}^{(1)}$$

$$\bar{H}_n^{(2)}(\beta r) = \left[\frac{1}{2}\pi \beta r\right]^{1/2} H_{n+1/2}^{(2)} .$$

Z_0 is the characteristic impedance of the biconical transmission line to the TEM wave given by

$$Z_0 = \frac{\ln(\cot \delta/2)}{\pi}\sqrt{\frac{\mu}{\epsilon}} .$$

The first term in the expression for H_ϕ and the first term in the expression for E_θ represent the TEM wave. The summation in each of the expressions represents the superposition of the fields of the various possible TM modes. The specific modes summed depend on the boundary conditions which determine the possible values of n. The constants A and a_n represent the amplitude of the TEM and TM modes respectively. It will be seen that K represents the reflection coefficient of the TEM wave and g_n the reflection coefficient of the n^{th} order TM wave.

The problem of finding the fields along the biconical antenna is now reduced to a boundary value problem. Specific solutions will first be obtained by applying the boundary conditions imposed by the infinite case or biconical transmission line, since the problem is somewhat simplified by doing so.

THE INFINITE BICONICAL TRANSMISSION LINE

In the case of the infinite biconical antenna or biconical transmission line, shown in Fig. 2, there is no free space region. The boundary condition requiring the continuity of the fields between the free space region, region 3, in Fig. 1, and the antenna region, region 2, is replaced by the requirement that there be no reflected waves along the transmission line.

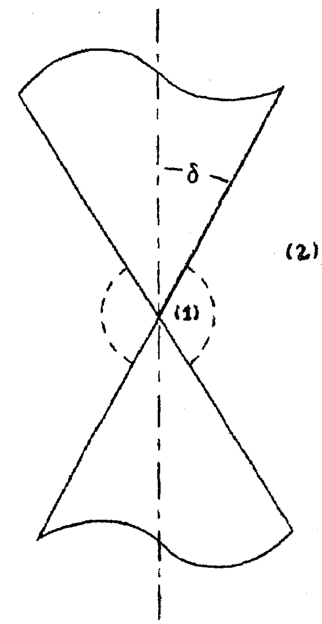

Fig. 2. The Infinite Biconical Transmission Line

It is seen in equation (1) that the TEM wave is represented by the sum of positive and negative exponentials of r, which are the familiar expressions for waves traveling inward and outward respectively, in the radial direction. In the infinite case there is no discontinuity in the radial direction, which means there will be no reflected wave. The coefficient of the term representing the reflected wave, K, must then be zero. Consider the limiting forms of the normalized Hankel functions which represent the TM wave. As r becomes large,

$$\lim_{r \to \infty} \bar{H}_n^{(1)}(\beta r) = -j\, e^{j(\beta r - n\pi/2)}$$
$$\lim_{r \to \infty} \bar{H}_n^{(2)}(\beta r) = j\, e^{-j(\beta r - n\pi/2)} \quad (3)$$

It is seen that as r approaches infinity the first order Hankel reduces to the exponential form of a wave traveling in the negative r direction, and the second order Hankel function a wave traveling in positive r direction. The constant g_n in the expression for the TM wave, then represents the reflection coefficient of n^{th} order wave and must be zero if there is to be no reflected wave in that mode.[5] It is shown in Appendix B that the terms in the expressions for both the TEM and TM waves, which have been called incident and reflected waves here, actually do represent energy flow away from and toward the origin respectively for any value of r. With the reflection coefficients equal to zero the expressions for the fields along the biconical transmission line become

$$r^2 E_r = \frac{j}{2\pi\omega\epsilon} \sum_n a_n M_n(\cos\theta)\, \bar{H}_n^{(2)}(\beta r)$$

$$r H_\phi = \frac{A\, e^{-j\beta r}}{\sin\theta} + \frac{1}{2\pi} \sum_n \frac{a_n}{n(n+1)} \frac{d}{d\theta} M_n(\cos\theta)\, \bar{H}_n^{(2)}(\beta r) \quad (4)$$

$$r E_\theta = \frac{A Z_0\, e^{-j\beta r}}{\sin\theta} + \frac{j\eta}{2\pi} \sum_n \frac{a_n}{n(n+1)} \frac{d}{d\theta} M_n(\cos\theta)\, \bar{H}_n^{(2)\prime}(\beta r).$$

[5] Note that this is not the same as setting $p_n = 0$ in the expression $\bar{J}_n(\beta r) + p_n\ \bar{N}_n(\beta r)$ as done by Schelkunoff. Setting $p_n = 0$ is based on the assumption that the input region becomes infinitely small and would result in standing rather than traveling waves in this case.

The requirement that the tangential component of the electric field reduces to zero along the surfaces of the perfectly conducting half cones, determines the possible TM modes. In order that this field vanishes at the surfaces, n must be such that $M_n(\cos \delta) = 0$ or $P_n(\cos \delta) - P_n(\cos \delta) = 0$. Since $M_n(\cos \theta)$ is an odd function, this will also satisfy the requirement that $M_n[\cos(\pi - \delta)] = 0$. There are an infinite number of values of n which satisfy this condition. Therefore there are an infinite number of possible TM modes. Note that in the case in which n is an intéger $P_n(\cos \theta) = P_n(-\cos \theta)$ for all values of θ which gives the trivial solution that the fields are everywhere zero. Thus, only nonintegral values of n are possible.

The amplitudes of the various modes depend on the excitation which may be expressed as a boundary condition imposed by the source region. Let the source region be of radius ρ and have a specified ϕ component of the magnetic field at its boundary defined by $H_\phi(\beta \rho) = I(\theta)$. Continuity of the tangential component of the magnetic field at the boundary of the source region then requires that

$$I(\theta) = \frac{A\, e^{-j\beta\rho}}{\sin \theta} + \frac{1}{2\pi} \sum_n \frac{a_n}{n(n+1)} \frac{d}{d\theta} M_n(\cos \theta)\, \bar{H}_n^{(2)}(\beta r),$$

The amplitude coefficients are then found by applying the orthogonal properties of the Legendre functions and their derivatives shown in Appendix A to equation (5), giving

$$a_n = \frac{-\pi\, n\,(n+1)\,(2\,n+1) \displaystyle\int_{\delta}^{\pi-\delta} \left[I(\theta) \sin \theta - A\, e^{-j\beta\rho} \right] \frac{d}{d\theta} M_n(\cos \theta)\, d\theta}{\bar{H}_n^{(2)}(\beta r) \sin \delta \dfrac{\partial}{\partial \theta} M_n(\cos \delta) \dfrac{\partial}{\partial n} M_n(\cos \delta)},$$

where $\frac{\partial}{\partial\theta} M_n(\cos\theta)$ and $\frac{\partial}{\partial n} M_n(\cos\theta)$ indicate the partial derivatives of $M_n(\cos\theta)$ with respect to θ and n respectively evaluated at $\theta = \delta$.

Integrating the first term in the numerator gives

$$\int_{\delta}^{\pi-\delta} A\, e^{-j\beta\rho} \frac{d}{d\theta} M_n(\cos\theta)\, d\theta = \Big[A\, e^{-j\beta\rho} M_n(\cos\theta) \Big]_{\delta}^{\pi-\delta} = 0 \quad (6)$$

The right hand side of equation (6) must be zero since n was defined such that $M_n(\cos\theta)$ is zero at the boundaries. The expression for a_n then becomes

$$a_n = \frac{-\pi\, n\,(n+1)\,(2\,n+1) \int_{\delta}^{\pi-\delta} I(\theta)\, \sin\theta \frac{d}{d\theta} M_n(\cos\theta)\, d\theta}{\bar{H}_n^{(2)}(\beta r)\, \sin\delta \frac{\partial}{\partial\theta} M_n(\cos\delta) \frac{\partial}{\partial n} M_n(\cos\delta)} \quad (7)$$

The magnitude of the TEM wave, A, is found by direct integration of equation (5) with respect to θ which gives

$$\int_{\delta}^{\pi-\delta} I(\theta)\, d\theta = \int_{\delta}^{\pi-\delta} \frac{A\, e^{-j\beta\rho}}{\sin\theta}\, d\theta + \quad (8)$$

$$\frac{1}{2\pi} \int_{\delta}^{\pi-\delta} \sum_n \frac{a_n}{n\,(n+1)} \bar{H}_n^{(2)}(\beta r) \frac{d}{d\theta} M_n(\cos\theta)\, d\theta ,$$

The second term on the right hand side must be equal to zero from the definition of n. A is then determined from equation (9) as

$$A = \frac{e^{-j\beta\rho}}{2\ln(\cot\delta/2)} \int_{\delta}^{\pi-\delta} I(\theta)\, d\theta \quad (9)$$

Thus, the amplitude of each of the modes has been determined as a function of the magnetic field at the boundary of the source region. E_θ or E_r rather than H_ϕ, might have been specified along this boundary, and the amplitudes of the various modes determined in a similar manner.

It is significant that in this case the modes which exist, and their respective amplitudes, depend solely on the input, or source region. If, for example, the input is such that $I(\theta) = \frac{1}{\sin\theta}$, only the TEM wave will be excited and the other TM waves will not be present. The m^{th} order TM mode may be excited by itself if $I(\theta) = \frac{d}{d\theta} M_n(\cos\theta)$. Consider the effect of allowing the radius of the source region to approach zero as might be the case with a very narrow feed. As $\rho \to 0$, $\bar{H}_n^{(2)}(\beta r) \to \infty$ for all n and thus it is seen from equation (7) that the amplitude of each of the TM waves approaches zero. Thus with an infinitely small or point input region only the TEM wave will be present. It is of interest to compare these properties with those of the biconical antenna of finite length.

THE LOSSLESS BICONICAL ANTENNA

In finding the fields of the biconical antenna, it is necessary to consider the free space region which was not present in the case of the biconical transmission line. One additional boundary which separates the free space region from the antenna region is now involved. The same modes will exist in this region as previously found for the biconical transmission line. The reflection coefficients are no longer zero, however; but remain to be determined from the boundary conditions.

The general expression for the TM wave in the free space region determined by Schelkunoff may be written as

$$r^2 E_r = \frac{j}{2\pi\omega\epsilon} \sum_k b_k P_k(\cos\theta) \bar{H}_k^{(2)}(\beta r)$$

$$r H_\phi = \frac{1}{2\pi} \sum_k \frac{b_k}{k(k+1)} \frac{d}{d\theta} P_k(\cos\theta) \bar{H}_k^{(2)}(\beta r) \qquad (10)$$

$$r E_\theta = \frac{j\eta}{2\pi} \sum_k \frac{b_k}{k(k+1)} \frac{d}{d\theta} P_k(\cos\theta) \bar{H}_k^{(2)\prime}(\beta r) \quad ,$$

where $P_k(\cos\theta)$ is a Legendre polynomial of order k and b_k is a constant which represents the amplitude of the k^{th} order TM mode. The values of K, which specify the possible TM modes in this region, are determined from the restriction that the fields remain everywhere finite. When θ is equal to zero or π, the argument of the Legendre function is 1 or -1 respectively. Only the Legendre polynomial of integer order remains finite for these arguments. Therefore, only TM waves of integer order will exist in the free space region. This differs from the antenna region in which only fractional order modes may exist. The symmetry of the problem further restricts the possible values to only the odd integers. The Hankel function of the first kind has not been included since the free space region is infinite and thus no reflected wave, which this function represents, can exist. A TEM wave cannot be supported in the free space region since there are no conducting surfaces on which such a wave must terminate.

To determine the amplitudes of the fields in each region, consider first the excitation provided by the source region. Again, let the source region specify H_ϕ along its boundary such that $H_\phi(\beta\rho) = I(\theta)$. Equating the tangential component of the magnetic field at the boundary between the source region and the antenna region gives the equation

$$I(\theta) = A\,\frac{e^{-j\beta\rho} + K\,e^{j\beta\rho}}{\sin\theta} + \frac{1}{2\pi}\sum_n \frac{a_n}{n\,(n+1)}\,\frac{d}{d\theta}M_n(\cos\theta)\left[\bar{H}_n^{(2)}(\beta\rho) + g_n\,\bar{H}_n^{(1)}(\beta\rho)\right]. \tag{11}$$

Applying the orthogonality properties of the Legendre functions as in the previous section gives

$$a_n = \frac{-\pi\,n\,(n+1)\,(2\,n+1)\displaystyle\int_\delta^{\pi-\delta} I(\theta)\,\sin\theta\,\frac{d}{d\theta}M_n(\cos\theta)\,d\theta}{\sin\delta\,\dfrac{\partial}{\partial\theta}M_n(\cos\delta)\,\dfrac{\partial}{\partial n}M_n(\cos\delta)\left[\bar{H}_n^{(2)}(\beta\rho) + g_n\,\bar{H}_n^{(1)}(\beta\rho)\right]} \tag{12}$$

$$A = \frac{\displaystyle\int_\delta^{\pi-\delta} I(\theta)\,d\theta}{2\,\ln(\cot\delta/2)\left[e^{-j\beta\rho} + K\,e^{j\beta\rho}\right]}\,. \tag{13}$$

In this case A and a_n are not completely defined by the input since the reflection coefficients K and g_n, which appear in the expressions for these terms given above, remain undetermined. A second set of equations relating the amplitude coefficients of the fields to the reflection coefficients, is obtained from the requirement that the fields be continuous between the antenna region and the free space region.

Let the radius of the spherical caps which determine the boundary between region 2 and region 3 be R. At this boundary the fields given by the free space equations must be the same as those given by the equations which apply in the antenna region. Equating the expressions for H_ϕ and E_θ at r = R gives [6]

$$A \frac{e^{-j\beta R} + K e^{j\beta R}}{\sin\theta} + \frac{1}{2\pi} \sum_n \frac{a_n}{n(n+1)} \frac{d}{d\theta} M_n(\cos\theta) \left[\bar{H}_n^{(2)}(\beta R) + g_n \bar{H}_n^{(1)}(\beta R) \right] = \frac{1}{2\pi} \sum_k \frac{b_k}{k(k+1)} P_k(\cos\theta)\, \bar{H}_k^{(2)}(\beta R) \tag{14}$$

$$A Z_0 \frac{e^{-j\beta R} - K e^{j\beta R}}{\sin\theta} + \frac{1}{2\pi} \sum_n \frac{a_n}{n(n+1)} \frac{d}{d\theta} M_n(\cos\theta) \left[\bar{H}_n^{(2)'}(\beta R) + g_n \bar{H}_n^{(1)'}(\beta R) \right] = \frac{1}{2\pi} \sum_k \frac{b_k}{k(k+1)} P_k(\cos\theta)\, \bar{H}_k^{(2)'}(\beta R) \tag{15}$$

Applying the orthogonal properties of the Legendre functions given in Appendix A to equation (14), the coefficient of the n^{th} order TM wave in the antenna region may be related to the amplitudes of the free space waves by

[6] Equating H_ϕ across this boundary imposes the same conditions as equating D_r.

$$a_n = \sum_k \frac{\bar{H}_k^{(2)}(\beta R)}{\bar{H}_n^{(2)}(\beta R) + g_n \bar{H}_n^{(1)}(\beta R)} \frac{2n+1}{k(k+1) - n(n+1)} \frac{P_k(\cos\delta)}{\frac{\partial}{\partial n} M_n(\cos\delta)} b_k . \tag{16}$$

Letting

$$U_{nk} = \frac{2n+1}{k(k+1) - n(n+1)} \frac{P_k(\cos\delta)}{\frac{\partial}{\partial n} M_k(\cos\delta)} , \tag{17}$$

a_n may be expressed as

$$a_n = \sum_k \frac{\bar{H}_k^{(2)}(\beta R)}{\bar{H}_n^{(2)}(\beta R) + g_n \bar{H}_n^{(1)}(\beta R)} U_{nk} b_k . \tag{18}$$

The orthogonality properties of the Legendre functions may be applied to equation (15) to express b_k in terms of values of a_n by

$$b_k = \frac{2k+1}{\bar{H}_k^{(2)'}(\beta R)} \eta \pi A Z_0 \left[e^{-j\beta R} - K e^{j\beta R} \right] P_k(\cos\delta) + \sum_n \eta \frac{\bar{H}_n^{(2)'}(\beta R) + g_n \bar{H}_n^{(1)'}(\beta R)}{\bar{H}_k^{(2)'}(\beta R)} V_{kn} a_n , \tag{19}$$

where

$$V_{kn} = \frac{k(k+1)}{n(n+1)} \frac{2k+1}{k(k+1) - n(n+1)} \sin\delta \, P_k(\cos\delta) \frac{d}{d\theta} M_n(\cos\delta) . \tag{20}$$

Equations (12), (13), (18), and (19) represent three sets of equations with an equal number of unknowns, which related the amplitude and reflection coefficient of the various modes to the boundary conditions imposed by the source region. These three sets of equations may be solved simultaneously to give the amplitudes of each of the waves and the reflection coefficients of the waves in the antenna region.

The equations are now coupled in such a way that a single mode can no longer exist in the antenna region nor in the free space region. A single mode existing in the antenna region would require a series of terms in the free space region in order to maintain continuity of E_θ between the two regions, but this series would not necessarily produce a continuous H_ϕ. Thus, an additional series of terms is required in the antenna region to satisfy all the boundary conditions and in general all TM modes will exist. This argument applies to the TEM wave as well. A source region producing an $H_\phi(\beta\rho) = \frac{1}{\sin\theta}$ will no longer produce only a TEM wave, as in the case of the biconical transmission line. If only specified modes are excited by the source region, the amplitudes of the field component defined, at the input, H in the case considered here, must be zero at the input boundary. Either $\bar{H}_n^{(2)}(\beta\rho) + g_n \bar{H}_n^{(1)}(\beta\rho)$ or its derivative must then equal zero, if the fields in the n^{th} order mode are not zero everywhere. This requires that $|g_n| = 1$. The angle of the reflection coefficient, g_n, will depend on the antenna parameters, specifically the radius of the source region, the propagation constant in the antenna region and the half cone angle.

It is shown in Appendix A that the total energy which is transferred along the biconical antenna may be given by the sum of the energies supplied by each of the individual modes. It is also shown that the energy transferred along the antenna in a given mode depends on the difference of the squares of the absolute value of the amplitude of the incident and reflected waves. When the magnitude of the reflection coefficient is unity, no energy is transferred in that mode, which is the case in each of the modes except those excited by the source region. Thus, the energy which is scattered back along the antenna in the other modes is totally reflected at the boundary of the source region, setting up standing waves in these modes. Only the modes which were initially excited by the source region contribute to the energy transferred along the antenna. It was seen for the infinite biconical transmission line, in the special case in which the radius of the input region approached zero, only the TEM wave could exist. In the case of the finite biconical antenna then, only the TEM wave will transfer energy within the antenna region.

More than one mode exists along the antenna for a particular excitation due to the discontinuity at the antenna cap. The waves generated by the source region propagate along the antenna and are scattered at the point where the conical conducting section meets the spherical cap. The summation of the various TM modes in each region is the mathematical expression of the result of this scattering. Since a single mode excited by the source region will give rise to additional modes along the antenna, some of the scattered energy is therefore reflected back along the antenna.

The biconical antenna may be considered to be a biconical transmission line, terminated in an impedance presented by free space. The scattered waves, which are reflected back along the antenna, producing the standing waves, have the effect of a number of tuning stubs in parallel with the load impedance. Since the medium is linear, each of the modes excited by the source region may be considered separately, as consisting of a transmitting mode, in which energy is transferred, and an infinite number of modes which cause the antenna to act as a tuning stub. The complete expression for the fields along the biconical antenna may then be considered to be the sum of each of the transmitting waves and their associated scattered waves. It should be noted that energy is also reflected in the source excited mode. Thus, fields in this mode may be separated into a standing and a traveling wave. The traveling wave determines the energy transferred along the antenna. The standing wave may be considered as contributing to the tuning effect, along with the other waves produced by scattering.

A further insight is gained by considering the effects of energy absorption along the antenna, which results from the finite conductivity of the material from which a practical antenna must be constructed.

THE LOSSY BICONICAL ANTENNA

The properties of the lossy biconical antenna differ from those of the loss free antenna in such a way that it will be advantageous to use a different approach to find the field along this antenna. In considering the loss free antenna it was found that the mathematical expressions for the fields were orthogonal, which provides a convenient means of satisfying mathematically the continuity of the fields between the antenna region and the free space region. The possible modes in the antenna region, in the loss free case, are determined by the restriction that the tangential component of the electric field approach zero at the conditioning surfaces. These conditions do not apply when the antenna is lossy. It, therefore, becomes difficult to find an expression for the fields along the lossy antenna using the technique of matching the boundary conditions employed in the previous section.

An approximate solution for the expressions for fields of the practical biconical antenna may be found, however, if the conductivity of the surfaces is considered to be high, as it is for real conductors. In this case, the expressions for the fields which exist along the loss free antenna serve as a good first approximation to the solution being sought. The loss produced at the surface by the assumed fields may be found and a second approximation generated, by accounting for this loss in the form of attenuation along the antenna. If a better approximation is desired the process may be repeated, using the new expression in place of the loss free fields, as the first approximation to the desired solution. Most real conductors are of sufficiently high conductivity that a single iteration provides a satisfactory accuracy.

In applying this approach it must be remembered that the principle of superposition no longer applies when there are losses present. Therefore, it is not possible to find expressions for the fields in each of the modes separately and then add the results to give a complete solution for the fields along the antenna. However, the expression for the fields may be found in the case where only a single mode exists or is dominant along the antenna. If more than one mode exists, all modes must be considered together, for the particular combination of modes present. Such cases will not be considered here. However, consideration of certain special cases will provide a basis for anticipating the effect of antenna losses in the more general case.

The fact that the expressions for the fields are no longer orthogonal implies that the waves are coupled together and that a single mode can not exist along the antenna. If the conductivity is high, however, the modes are loosely coupled and it is possible for one mode to dominate sufficiently that it may be treated as a single mode in obtaining an approximate solution.

It was seen that, in the case of the biconical antenna of finite length, the discontinuity at the boundary between the antenna region and the free space region,caused a scattering in such a way that when one mode was excited standing waves would be set up in other modes along the antenna. If, however, the angle of the antenna cone is small,the configuration of the fields in antenna region approaches that of the fields in free space. Thus, the discontinuity at the boundary is less severe. As a result,loss energy is reflected back, at the boundary, in the form of other modes. A transmitting TM mode excited by the source region,will then be dominant and the fields due to the other modes existing along the antenna will be sufficiently small that they may be neglected. This case is of particular interest if the results obtained are to be extended to the linear by perturbation or some other method.

It was seen that if the biconical antenna is considered to be of infinite length,that the various modes exist independently. However, in this case no reflected wave will be present. Although this case is not of practical importance, it does represent a theoretical configuration, which produces a single mode for the purposes of study.

Consider now the losses at the conducting surface. For highly conducting materials the power dissipated per unit area may be expressed approximately in terms of a surface current density and a surface resistance. A true surface current can exist only on a perfect conductor. However, when the skin depth is small, as it is when the conductivity is high, the current in the conducting material may be represented by an equivalent surface current. The surface resistance is then defined such that the product of the square of the equivalent surface current density and the surface resistance gives the same power loss as the actual current in the conducting media. This resistance is a function of frequency,since the skin depth and resultant loss depend on frequency. The average power loss over the surface of the antenna,due to the currents in the conducting material, is then given by

$$P_L = \frac{1}{2}\int R_s \left|I_s\right|^2 ds \quad , \tag{21}$$

where P_L is the average loss power, R_s is the surface resistance and I_s is the surface current density. The average power refers to the time average over one cycle. The instantaneous power loss as a function of time, is not of particular interest since only the crest value of the fields is being considered.

The surface resistance is given by [7]

$$R_s = \sqrt{\frac{\omega \mu}{2 \sigma}} \quad , \tag{22}$$

where μ and σ are the permeability and conductivity respectively of the conducting material. The surface current density at a perfectly conducting surface is given by $\mathbf{I}_s = \mathbf{n} \times \mathbf{H}$. Where $\mathbf{n}$ is a unit vector normal to the surface and $\mathbf{H}$ is the magnetic field intensity at the surface. This expression is a good approximation to the equivalent surface current density, which is used in equation (21). Since the TM waves or the special case of the TEM wave have only an H_ϕ component, the expression for the surface current density becomes

$$I_s = H_\phi(\delta) \quad . \tag{23}$$

Equations (21) and (23) would give the total average power loss along the surface of the antenna if the magnetic field were known at every point. An approximate expression, however, may be found in an incremental length radially along the antenna from these expressions by considering the fields to be of the same form as in the loss free

[7] Edward C. Jordan, Electromagnetic Waves and Radiating Systems (New York: Wiley, 1952).

case.

By requiring that the law of conservation of energy be satisfied along the antenna, an attenuation factor may be obtained from the approximate expression for the average power loss per unit length, giving a variation of the average power transferred. The amplitude coefficients of the loss free fields may then be modified to include this attenuation. This gives the approximation being sought for the expressions for the fields along the lossy antenna.

The average power loss per unit length in the radial direction is given by

$$L_s = \frac{1}{2}\int_0^{2\pi} R_s \left|H_\phi\right|^2 r \sin\delta \, d\phi \quad . \tag{24}$$

Since the fields are ϕ symmetric, the integral becomes

$$L_s = \pi R_s \left|H_\phi\right|^2 r \sin\delta \quad . \tag{25}$$

Consider the case in which the TEM wave serves as the first approximation to the solution being sought for the fields along the lossy antenna.

<u>The TEM Wave</u>

The expression for the incident TEM wave may be written as

$$\begin{aligned} r H_\phi &= A(r)\frac{e^{-j\beta r}}{\sin\theta} \\ r E_\theta &= A(r)\frac{Z_0 e^{-j\beta r}}{\sin\theta} \quad . \end{aligned} \tag{26}$$

The amplitude is now a function of r which is to be determined from the loss. The average power loss over an incremental radial length given in equation (25) becomes

$$L_s = \frac{\pi \ R_s \ |A(r)|^2}{r \sin \delta} \quad . \tag{27}$$

The average power passing outward in the radial direction across a spherical surface, due to the incident TEM wave, is found in appendix B, and may be written as

$$W = 2 \ \pi \ |A(r)|^2 Z_0 \ \ln(\cot \delta/2) \quad ,$$

where the amplitude has now been written as a function of the radius of the sphere chosen. In the loss free case, the average power passing outward through any closed surface, which includes the source region, must be independent of the surface chosen, as required by the conservation of energy. Thus, when there is no loss, the average power passing through a spherical surface centered at the origin is independent of the radius of the sphere, as given in Appendix B. If the source region is not included, the average power passing through the surface must be zero.

When the antenna is lossy, the average power transmitted must be a function of the radial position. Although, in this case, power also passes through the spherical surface within the conducting cones, when the conductivity is high the fields attenuate rapidly inside the conductor and the contribution to the total power transferred is negligible. Thus, the expression given in appendix B for the average power passing through the spherical surface for the loss free case, in which the fields are zero everywhere within the conducting material, may be used.

When there is loss along the antenna surfaces, conservation of energy requires that the average power loss in some incremental length equals the change in average power passing outward over the spherical boundary. Thus,

$$L_s = \frac{-\partial W}{\partial r} \quad . \tag{28}$$

Comparing the average power transferred with its derivative at some radial distance r gives

$$\frac{\partial W}{\partial r} = \frac{-\,W\,R_s}{2\,r\,Z_o\,\ln(\cot\delta/2)\,\sin\delta} \quad . \tag{29}$$

Letting

$$B = \frac{R_s}{Z_0\,\ln(\cot\delta/2)\,\sin\delta} \quad , \tag{30}$$

equation (29) becomes

$$\frac{\partial W}{\partial r} = -\,\frac{2\,B\,W}{r} \quad . \tag{31}$$

Equation (31) may be solved for W to give

$$W = D\,e^{-2B\,\ln r} \quad , \tag{32}$$

where D is an arbitrary constant which depends on the boundary conditions. The amplitude is then determined as a function of r as

$$A(r) = A\,e^{-B\,\ln r} \quad . \tag{33}$$

Thus, the average power loss per unit length is

$$L_s = \frac{\pi\,R_s\,|A|^2\,e^{-2B\,\ln r}}{r\,\sin\delta} \quad . \tag{34}$$

The expressions for the attenuated TEM fields then become

$$r\,H_\phi = A\,\frac{e^{-j\beta r}\,e^{-B\ln r}}{\sin\theta}$$

$$r\,E_\theta = A\,Z_o\,\frac{e^{-j\beta r}\,e^{-B\ln r}}{\sin\theta}\ . \tag{35}$$

Because of the finite conductivity of the surface, the radial currents must also give rise to a radial component of the electric field. This produces a θ component of the poynting vector. Conservation of energy requires that the average power passing through any closed surface within the antenna region be zero. This condition is satisfied if the radial component of the electric field is given by

$$r\,E_r = A\,R_s\,\frac{e^{-j\beta r}\,e^{-B\ln r}}{\sin\theta}\ . \tag{36}$$

Since the surface current density is in the radial direction, E_r at the surface of the cones is related to the surface current density by

$$E_r = I_s\,R_s\ , \tag{37}$$

The surface current density of the attenuated wave may be found from equation (37) and the magnetic field intensity as found in equation (35). The radial component of the electric field at the surface, calculated from the surface current density then becomes

$$E_{rs} = A\,\frac{e^{-j\beta r}\,e^{-B\ln r}}{\sin\delta}\ , \tag{38}$$

This is the same result as would be obtained by letting $\theta=\delta$ in equation (36).

Consider now the θ component of the poynting vector, which is by the radial component of the electric field. If the surface integral of the poynting vector is taken over the surface of a truncated cone, the conical sides of which are coincident with the antenna surface, only the θ component contributes to the integral along the conical surface. Within the region of the conducting cones the fields attenuate rapidly and have only a small component normal to the truncating surface. Thus, the contribution of the fields in this region to the total power crossing the surface may be neglected. The average power, P_s, passing over this surface is then given in terms of the fields at the conducting surfaces by

$$P_s = \frac{1}{2} \operatorname{Re} \int_0^{2\pi} \int_{r_1}^{r_2} E_r^*(\delta)\, H_\phi(\delta) \sin\delta \, dr \, d\phi \quad , \tag{39}$$

where the asterisk is used to indicate the complex conjugate. Equation (39) may be differentiated with respect to r, to give the average power crossing this surface per unit radial length, giving

$$\frac{\partial P_s}{\partial r} = \frac{1}{2} \operatorname{Re} \int_0^{2\pi} E_r^* H_\phi \, r \sin\delta \, d\phi \quad . \tag{40}$$

Substituting the expressions for the attenuated electric and magnetic fields found in equation (35), and performing the integration, equation (40) gives

$$\frac{\partial P_s}{\partial r} = \frac{1}{2} \operatorname{Re} \int_0^{2\pi} E_r^* H_\phi \, r \sin\delta \, d\phi = \frac{\pi R_s |A|^2 e^{-2B \ln r}}{2 r \sin\delta} \quad . \tag{41}$$

Equation (41) represents the average power supplied to one of the conducting half cones per unit length by the fields along the antenna. Because the antenna is symmetric, this is half of the total energy supplied by the fields per unit radial length. This is the same expression for the average power loss per unit length along the antenna as determined previously from the surface current density, as is required by the conservation of energy. Both are approximate expressions, but the approximations made were not the same in each case. Thus, the approximations made give results which are self consistent.

A reflected component of the TEM wave may also exist. The expression for the reflected TEM wave may be easily found in a similar manner. The reflected wave, however, will attenuate with decreasing radius.

$$r\,E_r = A\,K\,R_s \frac{K\,e^{j\beta r}\,e^{B \ln r}}{\sin\theta}$$

$$r\,H_\phi = A \frac{K\,e^{j\beta r}\,e^{B \ln r}}{\sin\theta} \qquad (42)$$

$$r\,E_\theta = A\,Z_o \frac{-K\,e^{j\beta r}\,e^{B \ln r}}{\sin\theta}$$

It is seen that the wave which has been called the TEM wave along the lossy antenna is no longer actually transverse electric and magnetic, but now contains a radial component of the electric field. This radial component must actually result from other TM modes which exist along the antenna. Thus, the waves are now coupled together as noted previously. A biconical transmission line, excited in such a way that in the loss free case only the TEM wave would exist, must have additional TM modes when the antenna is lossy even though there is no scattering from a terminating end. This coupling is due to the fact that the wave in the antenna region no longer propagates only in the radial direction. Instead it is defracted at the conducting boundary. A small portion of the energy propagates into the conducting material nearly normal to the surface. If surface resistance is small, which is the case when the conductivity is high, the radial component of the electric field will be small and the resulting wave is, as was originally assumed, very nearly a TEM wave.

Consider next the fields which are approximated by the n^{th} order TM wave.

The TM Wave

The same procedure may be used to find an approximate solution for the fields of a TM mode along the lossy biconical antenna as was used in the case of the TEM mode. The fields along the loss free antenna may be used in a similar manner to calculate the loss per unit length along the conducting surface for a particular value of r, which is given below:

$$L_s = -\frac{\partial W}{\partial r} = \frac{R_s}{4\pi r} \frac{|a_n|^2}{n^2 (n+1)^2} \left|\bar{H}_n^{(2)}(\beta r)\right|^2 \left[\frac{d}{d\theta} M_n(\cos\delta)\right]^2 \sin\delta \quad (43)$$

The amplitude of the wave a_n, will be a function of the radius to be determined by the losses.

The average power transmitted along the antenna by the n^{th} order TM mode is found in appendix B for a particular amplitude as

$$W = \frac{1}{2n+1} \frac{\eta}{2\pi} \frac{|a_n|^2}{n^2 (n+1)^2} \left[-\frac{\partial}{\partial\theta} M_n(\cos\delta) \frac{\partial}{\partial n} M_n(\cos\delta) \sin\delta\right] . \quad (44)$$

The transmitted power is then related to its derivative by

$$\frac{\partial W}{\partial r} = \frac{W(2n+1) R_s \left|\bar{H}_n^{(2)}(\beta r)\right|^2 \frac{d}{d\theta} M_n(\cos\delta)^2}{2\eta \left[-\frac{\partial}{\partial\theta} M_n(\cos\delta) \frac{\partial}{\partial n} M_n(\cos\delta) \sin\delta\right] r} . \quad (45)$$

Solving for W gives

$$W = C \exp\left[-B_n \int_\rho^r \frac{\bar{H}_n^{(2)}(\beta r)}{r} dr\right] , \quad (46)$$

where

$$B_n = \frac{(2n+1) R_s \left[\frac{d}{d\theta} M_n(\cos\delta)\right]^2}{2\eta \left[-\frac{\partial}{\partial\theta} M_n(\cos\delta) \frac{\partial}{\partial n} M_n(\cos\delta)\right]} ,$$

and C is an arbitrary constant. The amplitudes of the E_θ and H_ϕ fields are written as functions for r to correspond to the transmitted energy. They become

$$r\,H_\phi = \frac{1}{2\pi}\;\frac{a_n}{n\,(n+1)}\;\bar{H}_n^{(2)}(\beta r)\,\exp\left[-\frac{B_n}{2}\int_\rho^r \frac{\left|\bar{H}_n^{(2)}(\beta r)\right|^2}{r}\,dr\right]\frac{d}{d\theta}M_n(\cos\theta)$$

$$r\,E_\theta = \frac{j\,\eta}{2\pi}\;\frac{a_n}{n\,(n+1)}\;\bar{H}_n^{(2)'}(\beta r)\,\exp\left[-\frac{B_n}{2}\int_\rho^r \frac{\bar{H}_n^{(2)}(\beta r)^{\,2}}{r}\,dr\right]\frac{d}{d\theta}M_n(\cos\theta) \tag{47}$$

The radial component of the electric field will be similarly attenuated. In order that energy be conserved within the antenna region, between the cones the integral of the poynting vector over any closed surface within this region must be zero. To satisfy this condition another term may be added to the expression for E_r giving

$$r^2E_r = \frac{j\,a_n}{2\,\pi\,\omega}\;\bar{H}_n^{(2)}(\beta r)\,\exp\left[-\frac{B_n}{2}\int_\rho^r \frac{\left|\bar{H}_n^{(2)}(\beta r)\right|^2}{r}\,dr\right]M_n(\cos\theta)\;+ \tag{48}$$

$$\frac{a_n\,r\,R_s}{2\,\pi\,n\,(n+1)}\;\bar{H}_n^{(2)}(\beta r)\,\exp\left[-\frac{B_n}{2}\int_\rho^r \frac{\left|\bar{H}_n^{(2)}(\beta r)\right|^2}{r}\,dr\right]\frac{d}{d\theta}M_n(\cos\theta)\,.$$

It is noted that at the surface of the half cones the first term on the right hand side of equation (48) becomes zero. The second term satisfies the condition, at the boundary, that the tangential component of the electric field is equal to the product of the surface current density and the surface resistance. This additional term also creates θ component of the poynting vector at the surface such that the average power supplied to the conducting material per unit radial length, is equal to the average power dissipation per unit length, calculated from the product of the surface resistance and the square of the surface current density. This may be verified directly as was done for the TEM wave.

An expression may also be found for the reflected waves in the same manner. The expressions for the reflected wave are the same as those given in equation (47) and (48) for the incident wave if $\overline{H}_n^{(1)}(\beta r)$ is replaced by $\overline{H}_n^{(2)}(\beta r)$, and the wave attenuates with decreasing radius, making the exponential positive.

It is seen from the expressions obtained that these waves do not attenuate according to a simple exponential as in the more familiar cases of the rectangular wave guide or the coaxial transmission line. However, in the case of the biconical transmission line, unlike the other wave guides, there is a dispersion of the wave in the direction of propagation. Near the verticies of the half cones the energy transmitted is concentrated in much smaller volume than it is at large radial distances. The resulting surface currents are high and the losses are greater in proportion to the total energy than at greater distances where the surface current is smaller.

It is seen from the additional term in the expression for the radial component of the electric field that a single TM mode cannot exist alone in the antenna region. The losses, therefore, cause a coupling between the fields of the various modes as noted previously. When the conductivity of the antenna is high, as has been assumed in this analysis, the amplitude of the fields produced in the other modes will be small compared with that of fields in the primary mode, which was one of the original assumptions.

It is of interest to compare these effects with those produced by losses introduced in another manner, specifically dielectric loss. The biconical antenna having a lossy dielectric coating will be investigated next.

THE BICONICAL ANTENNA WITH A LOSSY COATING

The symmetric biconical antenna of half cone angle δ, having a dielectric coating of half cone angle α, is shown in Fig. 3.

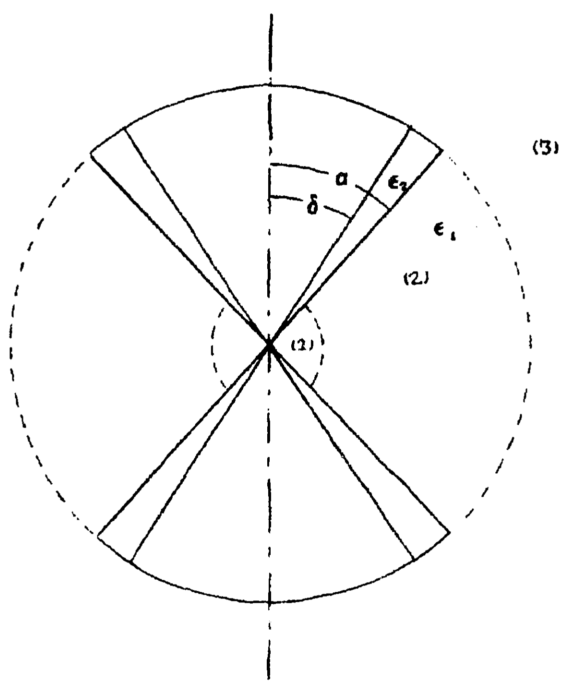

Fig. 3. The Dielectric Coated Biconical Antenna

If the permittivity of the dielectric coating is close to that of the sorrounding space ($\epsilon_1 \approx \epsilon_2$ in Fig. 3)and the loss tangent is small, the fields of the loss free antenna with no coating may serve as a first approximation to the actual fields. This approximation applies to the fields inside the dielectric as well. It will be assumed in this analysis that the only loss results from the dielectric, and that the antenna is perfectly conducting. Such an approximation would apply when the dielectric loss is small, but still must be greater than the conductor loss.

The power loss may then be calculated from the conductivity of the dielectric material by assuming the fields are of the same form as in the loss free case. The power dissipated per unit volume in a lossy media is given by

$$P_v = \frac{1}{2} \sigma |E|^2 \quad , \tag{45}$$

where Pv is the power loss per unit volume and σ is the conductivity of the media. The average power loss per unit length is found by taking the surface integral of the crest value of the assumed wave with respect to θ and ϕ . Consider first the wave which is approximated by the TEM wave in the loss free case.

<u>The TEM Wave</u>

For the TEM wave the average power loss per unit length at some particular radial distance is given by

$$L_s = \beta \ \pi \ |A|^2 \ Z_0 \ \sigma \ \ln \ \frac{\cot \alpha/2}{\cot \delta/2} = \frac{-\partial W}{\partial r} \quad . \tag{46}$$

The average power transferred outward along the biconical antenna as found in appendix B is then related to its derivative by

$$\frac{\partial W}{\partial r} = - W D \quad , \tag{47}$$

where

$$D = \frac{Z_0 \ln \frac{\cot \alpha/2}{\cot \delta/2} \sigma}{\ln(\cot \delta/2)} \quad . \tag{48}$$

Equation (48) may be solved for W to give the average power transferred outward across a spherical boundary as a function of the radius. The amplitude coefficient in the expression for the fields may then be written as a function of the radius to correspond to calculated power transfer. The expression for the fields then becomes

$$r H_\phi = A \frac{e^{-j\beta r} e^{-Dr/2}}{\sin \theta}$$

$$r E_\theta = A Z_0 \frac{e^{-j\beta r} e^{-Dr/2}}{\sin \theta} \quad . \tag{49}$$

The expression for the reflected wave is found in the same manner; however, the fields attenuate with decreasing radius:

$$r H_\phi = A \frac{K e^{j\beta r} e^{Dr/2}}{\sin \theta}$$

$$r E_\theta = A Z_0 \frac{- K e^{j\beta r} e^{Dr/2}}{\sin \theta} \tag{50}$$

It is seen that in this case the TEM wave varies as a simple exponential, which was not the case when the loss was due to the surface conductivity. It should be remembered, however, that this is only an approximate expression. Consider next the case in which the fields along the antenna may be approximated by the n^{th} order TM mode.

The TM Wave

In the case of the TM wave, the total electric field is the sum of E_r and E_θ components, each of which will produce a current density in the conducting medium contributing to the loss. The average power loss per unit length becomes

$$L_s = \int_0^{2\pi} \int_\delta^\alpha \frac{\left|E_r^2 + E_\theta^2\right|^2}{\sigma} r^2 \sin\theta \, d\theta \, d\phi \tag{51}$$

The value of the integral in equation (51) from $\pi - \alpha$ to $\pi - \delta$ is the same as that from δ to α, due to symmetry. This equation has been multiplied by two to represent the sum of the losses in each of the dielectric half cones. Using the loss free fields as a first approximation for the incident TM wave, the above integral becomes

$$L_s = \int_0^{2\pi} \int_\alpha^\delta \frac{a_n \left|\overline{H}_n^{(2)}(\beta r)\right|^2}{4 r^4 \pi^2 \omega^2 2} M_n(\cos\theta)^2 \sin\theta \, d\theta \, d\phi + \tag{52}$$

$$\frac{1}{n^2 (n+1)^2} \int_0^{2\pi} \int_\alpha^\delta \frac{a_n \left|\overline{H}_n^{(2)\prime}(\beta r)\right|^2}{4 r^2 \pi^2} \frac{d}{d\theta} M_n(\cos\theta) \, r^2 \sin\theta \, d\theta \, d\phi .$$

The loss per unit length is then related to its derivative by

$$L_s = \frac{-\partial W}{\partial r} = 2\ W\ d_n \left[\left| \bar{H}_n^{(2)'}(\beta r) \right|^2 + \frac{\left| \bar{H}_n^{(2)}(\beta r) \right|^2 n\ (n+1)}{r^2\ \omega^2\ \epsilon^2} \right], \quad (53)$$

where

$$d_n = \left[\frac{\partial}{\partial\theta} M_n \cos a) \frac{\partial}{\partial n} M_n(\cos a) - M_n(\cos a) \frac{\partial^2}{\partial\theta\partial n} M_n(\cos a) - \right.$$

$$\left. \frac{\partial}{\partial\theta} M_n(\cos \delta) \frac{\partial}{\partial n} M_n(\cos \delta) + M_n(\cos \delta) \frac{\partial^2}{\partial\theta\partial n} M_n(\cos \delta) \right] \times \quad (54)$$

$$n\ (n+1)\ (\sin a - \sin \delta) \div \left[- \frac{\partial}{\partial\theta} M_n(\cos \delta) \frac{\partial}{\partial n} M_n(\cos \delta) \sin \delta \right] \eta .$$

solving for W gives

$$W = C \exp \left[- 2\ d_n \int_\rho^r \left| \bar{H}_n^{(2)'}(\beta r) \right|^2 + \frac{\left| \bar{H}_n^{(2)}(\beta r) \right|^2 n\ (n+1)}{r^2\ \omega^2\ \epsilon^2}\ dr \right], \quad (55)$$

or

$$W = \exp - 2\ d_n\ f(r) ,$$

where

$$f(r) = \int_\rho^r \left| \bar{H}_n^{(2)'}(\beta r) \right|^2 + \frac{\left| \bar{H}_n^{(2)}(\beta r) \right|^2 n\ (n+1)}{r^2\ \omega^2\ \epsilon^2}\ dr . \quad (56)$$

When the attenuation found in equation (55) is accounted for, the expression for the TM wave becomes

$$r^2\ E_r = \frac{j\ a_n}{2\ \pi\ \omega\ \epsilon} M_n(\cos\theta)\ \bar{H}_n^{(2)}(\beta r) \exp\left[- d_n\ f(r) \right]$$

$$r\ E_\theta = \frac{j\ \eta\ a_n}{2\ \pi\ n\ (n+1)} \frac{d}{d\theta} M_n(\cos\theta)\ \bar{H}_n^{(2)'}(\beta r) \exp\left[- d_n\ f(r) \right] \quad (57)$$

$$r\ H_\phi = \frac{a_n}{2\ \pi\ n\ (n+1)} \frac{d}{d\theta} M_n(\cos\theta)\ \bar{H}_n^{(2)}(\beta r) \exp\left[- d_n\ f(r) \right] .$$

A similar expression could be obtained for the reflected wave, which would be a positive exponential indicating an attenuation with decreasing radius. In the expression for the reflected wave, the Hankel functions of the second kind in Equation (57) were replaced by Hankel function of the first kind, which represent a wave traveling in the negative r direction.

The expression for the attenuation of the TM wave with radial distance is seen to be quite complex, due to the two components of the electric field, which produce losses in the dielectric. The expression is simplified somewhat if the antenna is considered to be filled with the lossy dielectric, in which case $\alpha = \frac{\pi}{2}$. Still further complexity would be introduced should a second approximation be sought.

The assumption that the fields remain the same as those in the loss free case, altered only by amplitude variation, requires that energy will be transferred only in the radial direction. This cannot be the case, since losses in the dielectric must be supplied from the energy of the fields. In the case of the biconical antenna having a finite conductivity, the energy supplied by the fields to the antenna losses was accounted for by the addition of another term to the expression for the radial component of the electric fields. This term created a θ component of the poynting vector such that the conservation of energy was satisfied within the antenna region. Such a term might also be added to the expressions for the fields outside the dielectric. Within the dielectric the amplitude of this added term would be a function of θ , becoming zero at the conducting surface.

All of the fields, in fact, must attenuate in the θ direction. Since the losses are small, the additional E_r term and the θ attenuation of the fields are third order effects, and the approximate solution obtained here is a good approximation to the desired solution. It is noted that when the conductivity is high, as in the case of the finitely conducting antenna surfaces, the energy could be considered to propagate normal to the surface, and when the conductivity was low it could be considered to propagate parallel to the surface.

It has been assumed here that the only effect caused by the dielectric was the introduction of loss along the antenna, resulting in attenuation of the waves, since permittivity of the dielectric coating was close to that of the dielectric comprising the remainder of the antenna region. To investigate the effect of the differing permittivities of the coating and the remaining region, consider a biconical antenna coated with a perfect dielectric having a permittivity which is significantly different than that of the surrounding region.

THE DIELECTRIC COATED ANTENNA

The problem of the biconical antenna having two or more dielectrics bounded by spherical surfaces has been studied extensively and may be found elsewhere. When the dielectric boundary is in the radial direction the problem becomes more difficult. The orthogonality properties of the Legendre functions, which greatly simplify the application of the boundary conditions along the spherical surface, do not exist for the Bessel functions, which determine the continuity along the radial boundaries. An approximate solution for the TEM wave may be obtained easily, however, using the quasi-static approach. This method will be presented here and a solution obtained for the special case of the TEM wave, since an effect similar to that produced on the TEM wave would be expected on each of the other TM modes.

Consider the biconical antenna shown in Fig. 3, in which there are no losses. To find the expression for the fields of the TEM wave, using a quasi-static approach, the capacitance and inductance per unit length in the radial direction are found. The current and voltage along the antenna are then found as a function of radial position in terms of these parameters, as in transmission line analysis.

Applying this procedure, the electric flux density in the region between the two charged half cones is given by

$$D_\theta = \frac{K}{\sin\theta} \quad , \tag{58}$$

where K is a constant determined by the charge on the half cones from $q_s = 2\pi K$ and q_s is the surface charge density.

If the dielectric coating has a permittivity ϵ_2, and the remaining volume a permittivity ϵ_1, the potential difference between the two cones is determined by

$$V = \frac{1}{\epsilon_2} \int_{\delta}^{\alpha} \frac{K}{\sin\theta}\, d\theta + \frac{1}{\epsilon_1} \int_{\alpha}^{\pi-\alpha} \frac{K}{\sin\theta}\, d\theta + \frac{1}{\epsilon_1} \int_{\pi-\alpha}^{\pi-\delta} \frac{K}{\sin\theta}\, d\theta , \qquad (59)$$

which gives

$$V = \frac{2\,K}{\epsilon_2} \ln(\cot \alpha/2) + \frac{2\,K}{\epsilon_1} \ln \frac{\cot \delta/2}{\cot \alpha/2} \quad . \qquad (60)$$

The capacitance per unit length in the radial direction in then given by

$$C = \frac{q_s}{V} = \frac{\pi\, \epsilon_1 \epsilon_2}{\epsilon_2 \ln(\cot \alpha/2) + \epsilon_1 \ln \dfrac{\cot \delta/2}{\cot \alpha/2}} \quad . \qquad (61)$$

The inductance per unit radial length is found in a similar manner as

$$L = \frac{\mu}{\pi} \ln(\cot \delta/2) \quad . \qquad (62)$$

The new expressions for the characteristic impedance and phase velocity then become, respectively

$$Z_0 = \sqrt{\frac{L}{C}} = \frac{1}{\pi} \sqrt{\frac{\mu \ln(\cot \delta/2)}{\epsilon_2 \ln(\cot \alpha/2) + \epsilon_1 \ln \dfrac{\cot \delta/2}{\cot \delta/2}}} \qquad (63)$$

$$v = \sqrt{\frac{1}{L\,C}} = \sqrt{\frac{\epsilon_2 \ln(\cot \alpha/2) + \epsilon_1 \ln \dfrac{\cot \delta/2}{\cot \alpha/2}}{\epsilon_2 \ln(\cot \delta/2)}} \sqrt{\frac{1}{\mu\, \epsilon_1}} \quad , \qquad (64)$$

The resulting expression for the TEM wave is then given by

$$r\, H_\phi = A \frac{e^{-j\beta r} - K\, e^{j\beta r}}{\sin\theta} \qquad r\, D_\theta = A\, Z_0 \frac{e^{-j\beta r} + K\, e^{j\beta r}}{\sin\theta} \quad , \qquad (65)$$

where Z_0 is as defined in equation (63) and $\beta = \frac{\omega}{v}$.

It is seen that the phase velocity along the antenna is now lower than that of the uncoated antenna, due to a factor resulting from the increased capacitance. As the angle of the dielectric coating becomes smaller α approaches δ , and the characteristic impedance and phase velocity approach that of the uncoated case.

It is seen that the principle effect of the dielectric is to reduce the phase velocity and, therefore, the wave length along the antenna. A similar effect would then be repeated in the case of the TM waves. Since the characteristic impedance was also changed by the coating, it would be expected that the ratio of the electric and magnetic fields would be altered in the TM case.

CONCLUSIONS

Approximate expressions have been found for the fields produced by a single TM mode existing along a finitely conducting biconical antenna. Expressions have also been found for the TM fields along a biconical antenna coated with a lossy dielective having a dielective constant very close to that of the surrounding medium. It is seen that when the antenna losses are small, the loss free fields serve a first approximation to the actual fields. Modifying these expressions, by an attenuation factor, a good approximation is obtained to the fields along the biconical antenna along which there are losses. It is seen that, except in the case of the TEM mode along the biconical antenna coated lossy dielectric, these fields do not attenuate according to a simple exponential along the antenna. This is due to the natural dispersion of the wave. It was seen that the antenna losses create a coupling between the modes, such that a single mode may not exist along the antenna. When the losses are small, however, the coupling between the modes is not strong and a single mode may be dominant.

An expression was also found, using a quasi-static approach, for the TEM wave along the loss free dielectric coated antenna, in which the permittivity of the coating differed significantly from that of the surrounding medium. It is seen that the principal effect of the coating is to shorten the wave length along the antenna.

The biconical antenna may be considered as a transmission line terminated in an impedance, which is presented by free space in parallel with a number of tuning stubs created by the reflection of energy back along the antenna in other modes.

RECOMMENDATIONS

The expressions obtained for the fields along the biconical antenna, having either surface or dielectric losses, may be extended to other types of antennas using perturbation methods. The linear antenna would be of particular interest in such a further investigation because of its wide use and the difficulty in analyzing this antenna directly.

It would be of interest to obtain solutions for the TM waves along the dielectric coated antenna, in which the permittivity of the coating is significantly different from that of the surrounding media. It is expected that the effect of such a coating on the TM wave is similar to the effect on the TEM wave, in which case the wave length along the antenna was reduced. In this the expressions for the fields in the various modes could be added together to give a complete solution for the fields along the antenna. It is found to be extremely difficult to obtain a solution for the fields of the various TM modes in this case by writing the general expression for the fields in each region and then matching the boundary conditions across the dielectric boundary, since the Bessel functions, which describe the variation of the TM modes in the radial direction, are not orthogonal. Some other approaches might simplify the mathematics involved, and give a complete solution to this problem, which is worthy of further investigation.

APPENDIX A

ORTHOGONALITY PROPERTIES OF LEGENDRE POLYNOMIALS

To demonstrate the orthogonality properties of the Legendre functions used in this paper, consider two odd Legendre functions, M_n and M_k, which vanish at the boundaries λ and $\pi - \lambda$. The defining equations for these functions are

$$\frac{d}{d\theta}\left[\sin\theta \frac{d}{d\theta} M_n(\cos\theta)\right] = \left[-n(n+1)\sin\theta\right] M_n(\cos\theta) \qquad \text{(A-1)}$$

and

$$\frac{d}{d\theta}\left[\sin\theta \frac{d}{d\theta} M_k(\cos\theta)\right] = \left[-k(k+1)\sin\theta\right] M_k(\cos\theta) \quad .$$

Multifplying the first equation by M_k and the second by M_n, and subtracting, gives

$$\left[n(n+1) - k(k+1)\right] \sin\theta\, M_n(\cos\theta)\, M_k(\cos\theta) = \qquad \text{(A-2)}$$
$$\frac{d}{d\theta}\left[\sin\theta\left(M_n(\cos\theta)\frac{d}{d\theta}M_k(\cos\theta) - M_k(\cos\theta)\frac{d}{d\theta}M_n(\cos\theta)\right)\right] \quad .$$

If both sides are integrated with respect to θ between the limits of λ and $\pi - \lambda$ the equation becomes

$$\left[n(n+1) - k(k+1)\right]\int_{\lambda}^{\pi-\lambda} \sin\theta\, M_n(\cos\theta)\, M_k(\cos\theta)\, d\theta =$$
$$\left[\sin\theta\;\; M_n(\cos\theta)\frac{d}{d\theta}M_k(\cos\theta) - M_k(\cos\theta)\frac{d}{d\theta}M_n(\cos\theta)\right]_{\lambda}^{\pi-\lambda} \quad . \qquad \text{(A-3)}$$

The right hand side of this equation is zero, since it was originally specified that $M_n(\cos\theta)$ and $M_k(\cos\theta)$ vanished at the boundaries.

Since n and K are not zero

$$\int_{\lambda}^{\pi-\lambda} M_n(\cos\theta)\, M_k(\cos\theta)\, \sin\theta\, d\theta = 0 \,, \qquad \text{(A-4)}$$

where $k \neq n$.

The Legendre functions are, therefore, orthogonal with respect to $\sin\theta$.

To investigate the orthogonality properties of the derivatives of the even Legendre functions just described, integrating the product of the derivatives by parts, between λ and $\pi - \lambda$ gives

$$\int_{\lambda}^{\pi-\lambda} \frac{d}{d\theta} M_n(\cos\theta) \frac{d}{d\theta} M_k(\cos\theta)\, d\theta =$$

$$\left[\sin\theta\, M_n(\cos\theta) \frac{d}{d\theta} M_k(\cos\theta)\right]_{\lambda}^{\pi-\lambda} \qquad \text{(A-5)}$$

$$- \int_{\lambda}^{\pi-\lambda} M_n(\cos\theta) \frac{d}{d\theta}\left[\sin\theta \frac{d}{d\theta} M_k(\cos\theta)\right] d\theta \,.$$

The first term on the right hand side is zero, since $M_n(\cos\theta) = 0$ at the limits. Thus

$$\int_{\lambda}^{\pi-\lambda} \frac{d}{d\theta} M_n(\cos\theta) \frac{d}{d\theta} M_k(\cos\theta)\, d\theta = \qquad \text{(A-6)}$$

$$\int_{\lambda}^{\pi-\lambda} M_n(\cos\theta) \frac{d}{d\theta}\left[\sin\theta \frac{d}{d\theta} M_k(\cos\theta)\right] d\theta \,.$$

From the defining equation, however,

$$\frac{d}{d\theta} \sin\theta \frac{d}{d\theta} M_k(\cos\theta) = -k(k+1)\sin\theta\, M_k(\cos\theta) \tag{A-7}$$

Making this substitution, the equation becomes

$$\int_{\lambda}^{\pi-\lambda} \frac{d}{d\theta} M_n(\cos\theta) \frac{d}{d\theta} M_k(\cos\theta)\, d\theta = k(k+1) \int_{\lambda}^{\pi-\lambda} M_n(\cos\theta)\, M_k(\cos\theta) \sin\theta\, d\theta \, . \tag{A-8}$$

This last term has already been shown to be equal to zero when $n \neq k$, giving the final result

$$\int_{\lambda}^{\pi-\lambda} \frac{d}{d\theta} M_n(\cos\theta) \frac{d}{d\theta} M_k(\cos\theta)\, d\theta = 0 \, . \tag{A-9}$$

Consider next

$$\int_{\lambda}^{\pi-\lambda} \sin\theta \left[M_n(\cos\theta)\right]^2 d\theta \, .$$

By writing the defining equation as a partial derivative and then taking the derivative of both sides with respect to n, the following two equations are obtained

$$\frac{\partial}{\partial\theta}\left[\sin\theta \frac{\partial}{\partial\theta} M_n(\cos\theta)\right] = -n(n+1)\sin\theta\, M_n(\cos\theta) \tag{A-10}$$

$$\frac{\partial}{\partial\theta}\left[\sin\theta \frac{\partial^2}{\partial n \partial\theta} M_n(\cos\theta)\right] = -(2n+1)\sin\theta\, M_n(\cos\theta) - n(n+1)\sin\theta \frac{\partial}{\partial\theta} M_n(\cos\theta) \, . \tag{A-11}$$

Multiplying the first of these equations by $\frac{\partial}{\partial \theta} M_n(\cos\theta)$ and the second by $M_n(\cos\theta)$, and subtracting, gives

$$\frac{\partial}{\partial \theta}\left[\sin\theta \frac{\partial}{\partial \theta} M_n(\cos\theta)\right]\frac{\partial}{\partial \theta} M_n(\cos\theta) - \frac{\partial}{\partial \theta}\left[\sin\theta\, M_n(\cos\theta)\, \frac{\partial^2}{\partial \theta\, \partial n} M_n(\cos\theta)\right] = (2\,n+1)\sin\theta\left[M_n(\cos\theta)\right]^2 , \tag{A-12}$$

or rewriting

$$\frac{\partial}{\partial \theta}\left[\sin\theta\left(\frac{\partial}{\partial \theta} M_n(\cos\theta)\frac{\partial}{\partial n} M_n(\cos\theta) - M_n(\cos\theta)\frac{\partial^2}{\partial\theta\partial n} M_n(\cos\theta)\right)\right] = (2\,n+1)\sin\theta\left[M_n(\cos\theta)\right]^2 . \tag{A-13}$$

Integrating both sides with respect to θ ,gives

$$\int_{\lambda}^{\pi-\lambda} \sin\theta\left[M_n(\cos\theta)\right]^2 d\theta = \frac{\sin\theta}{2\,n+1}\left[\frac{\partial}{\partial \theta} M_n(\cos\theta)\frac{\partial}{\partial n} M_n(\cos\theta) - M_n(\cos\theta)\frac{\partial^2}{\partial\theta\,\partial n} M_n(\cos\theta)\right]_{\lambda}^{\pi-\lambda} . \tag{A-14}$$

The second term on the right hand side is zero at the limits, thus

$$\int_{\lambda}^{\pi-\lambda} \sin\theta\left[M_n(\cos\theta)\right]^2 d\theta = \frac{\sin\theta}{2\,n+1}\left[\frac{\partial}{\partial \theta} M_n(\cos\theta)\frac{\partial}{\partial n} M_n(\cos\theta)\right]_{\lambda}^{\pi-\lambda} \tag{A-15}$$

$$\int_{\lambda}^{\pi-\lambda} \sin\theta\left[M_n(\cos\theta)\right]^2 d\theta = -\frac{2\sin\lambda}{2\,n+1}\frac{\partial}{\partial \theta} M_n(\cos\lambda)\frac{\partial}{\partial n} M_n(\cos\lambda) . \tag{A-16}$$

The value of

$$\int_{\lambda}^{\pi-\lambda} \sin\theta \left[M_n(\cos\theta)\right]^2 d\theta$$

is found easily by integration by parts.

$$\int_{\lambda}^{\pi-\lambda} \left[\frac{d}{d\theta} M_n(\cos\theta)\right]^2 \sin\theta \, d\theta = \int_{\lambda}^{\pi-\lambda} \frac{d}{d\theta} M_n(\cos\theta) \frac{d}{d\theta} M_n(\cos\theta) \sin\theta \, d\theta =$$

$$\left[M_n(\cos\theta) \frac{d}{d\theta} M_n(\cos\theta) \sin\theta\right]_{\lambda}^{\pi-\lambda} - \int_{\lambda}^{\pi-\lambda} M_n(\cos\theta) \frac{d}{d\theta}\left[\sin\theta \frac{d}{d\theta} M_n(\cos\theta)\right] d\theta. \tag{A-17}$$

The first term on the right hand side is zero at the limits of integration. Substituting into the second term from the defining equation gives

$$\int_{\lambda}^{\pi-\lambda} \left[\frac{d}{d\theta} M_n(\cos\theta)\right]^2 \sin\theta \, d\theta = n(n+1) \int_{\lambda}^{\pi-\lambda} \left[M_n(\cos\theta)\right]^2 \sin\theta \, d\theta. \tag{A-18}$$

The integral of the square of the Legendre function has already been found, thus the final result becomes

$$\int_{\lambda}^{\pi-\lambda} \left[\frac{d}{d\theta} M_n(\cos\theta)\right]^2 \sin\theta \, d\theta =$$

$$-\frac{2n(n+1)}{2n+1} \frac{\partial}{\partial\theta} M_n(\cos\lambda) \frac{\partial}{\partial n} M_n(\cos\lambda) \sin\lambda \tag{A-19}$$

APPENDIX B

ENERGY TRANSFER ALONG THE BICONICAL ANTENNA

The investigation of the energy flow provides a good basis for physical interpretation of the expressions for the fields along the antenna. The total power passing into a closed surface is given by the integral of the poynting vector over that surface. By choosing an appropriate surface, considerable information is obtained concerning the energy flow.

The average power passing through a given surface is given by the real part of the average value of the poynting vector. If the average power passing through some surface is W, then

$$W = \frac{1}{2}\,\mathrm{Re}\int \hat{E}^{*} \times \hat{H} \cdot ds \qquad \text{(B-1)}$$

where $\hat{E}$ and $\hat{H}$ represent the complex amplitudes of the expression for the electric and magnetic fields respectively.

Consider the average power passing over the surface of a sphere centered at the vertex of a biconical antenna. Equation (B-1) in this case becomes

$$W = \frac{1}{2}\,\mathrm{Re}\int_{0}^{2\pi}\int_{\delta}^{\pi-\delta} E_{\theta}^{*}\, H_{\phi}\, r^{2} \sin\theta \; d\theta \; d\phi \qquad \text{(B-2)}$$

The term $E_{\theta}^{*} H_{\phi}$ represents the product of two infinite summations. The result is an infinite sum of products, which include cross products between terms of unlike modes as well as like modes. This entire summation must then be integrated over the surface in order to obtain the average power leaving the enclosed volume. The integration of the summation with respect to θ involves a weighting function of $\sin\theta$. It is shown in Appendix A that the odd Legendre functions are orthogonal with respect to $\sin\theta$. Thus, the integral of the products between terms of unlike modes over this surface will be zero. The integral of the product of the TEM term and the terms of the TM waves is also zero, since the odd Legendre function vanishes at the boundary. The only terms which remain are then the products of terms involving like modes. The integral of the sum of these terms is equal to the sum of the integrals. This, then, is the same result that would be obtained if the power contributed by each wave were considered separately, and then added to give the total power. It is important to note that this is not an application of the principle of superposition, which does not apply to power, but is a result of the orthogonal properties of the functions involved. Once it has been established, however, that the powers of the individual modes may be added together, it is convenient to do so.

Consider first the TEM wave. Equation (B-2) becomes

$$W = \frac{1}{2}\,\mathrm{Re}\int_0^{2\pi}\int_\delta^{\pi-\delta} \frac{A^2 Z_0}{r^2}\;\frac{e^{j\beta r} - K^* e^{-j\beta r}}{\sin\theta}\;\frac{e^{-j\beta r} + K e^{j\beta r}}{\sin\theta}\; r^2 \sin\theta \, d\theta \, d\phi . \tag{B-3}$$

Multiplying the terms within the integral gives

$$W = \frac{1}{2} \operatorname{Re} \int_0^{2\pi} \int_\delta^{\pi-\delta} \frac{|A|^2 Z_0}{\sin\theta} \left[1 - |K|^2 + K e^{j2\beta r} - K^* e^{-j2\beta r}\right] d\theta \, d\phi . \quad \text{(B-4)}$$

Letting $K = C + jB$, the expression reduces to

$$W = \frac{1}{2} \operatorname{Re} \int_0^{2\pi} \int_\delta^{\pi-\delta} \frac{|A|^2 Z_0}{\sin\theta} \left[1 - |K|^2 + j\, 2\, C \sin(2\beta r) + j\, 2\, B \cos(2\beta r)\right] d\theta \, d\phi . \quad \text{(B-5)}$$

Since Z_0 is real, taking only the real part of the integral gives

$$W = \frac{1}{2} \int_0^{2\pi} \int_\delta^{\pi-\delta} \frac{|A|^2 Z_0}{\sin\theta} \left[1 - |K|^2\right] d\theta \, d\phi . \quad \text{(B-6)}$$

Performing the integrations the average power due to the TEM wave is given by

$$W = 2\pi\, |A|^2\, Z_0 \ln(\cot \delta/2) \left[1 - |K|^2\right].$$

$\ln(\cot\delta)$ is positive when $0 < \delta < \pi$ Since Z_0 is positive, all terms in this expression, with the possible exception of the last one, are positive. This last term is of particular interest. The 1 is a result of the product of the first term in the expressions for H_ϕ and E_θ. The first term in these expressions has been said to represent a wave traveling outward in the radial direction. It is seen that these terms contribute a positive average power, or power passing outward from the closed surface. Thus, the first term does in fact represent an outward propagation of energy.

The $-|K|^2$ term results from taking the product of the second term in the expressions E_θ and H_ϕ. This gives a negative average power or power passing into the closed surface. This second term was said to represent a reflected wave or one propagating inward, and it is seen that this is the case. If the antenna is radiating energy, the $|K|<1$. If energy is being absorbed in this mode by the antenna, then $|K|>1$ when $|K| = 1$, a standing wave exists along the antenna and no energy is transferred.

Consider next the average power passing over the spherical surface due to the n^{th} order TM mode. The product $E_\theta^* H_\phi$ in this case becomes

$$E_\theta^* H_\phi = \frac{-j\eta}{4\pi^2} \frac{|a_n|^2}{n^2 (n+1)^2} S_n'^*(\beta r) S_n(\beta r) \left[\frac{d}{d\theta} M_n(\cos\theta)\right]^2, \qquad \text{(B-8)}$$

where

$$S_n(\beta r) = \overline{H}_n^{(2)}(\beta r) + g_n \overline{H}_n^{(1)}(\beta r).$$

The product $S_n'(\beta r) S_n(\beta r)$ is then given by

$$S_n'^*(\beta r) S_n(\beta r) = \overline{H}_n^{(2)'*}(\beta r) \overline{H}_n^{(2)}(\beta r) + |g_n|^2 \overline{H}_n^{(1)'*}(\beta r) \overline{H}_n(\beta r) + g_n^* \overline{H}_n^{(1)'*}(\beta r) \overline{H}_n^{(2)}(\beta r) + g_n \overline{H}_n^{(2)'*}(\beta r) \overline{H}_n^{(1)}(\beta r) \qquad \text{(B-9)}$$

Expressing the Hankel functions in terms of Bessel functions, the first term becomes

$$\overline{H}_n^{(2)'*}(\beta r) \overline{H}_n^{(2)}(\beta r) = \left[\overline{J}_n'(\beta r) + j \overline{J}_{-n}(\beta r)\right]\left[\overline{J}_n(\beta r) - j \overline{J}_n'(\beta r)\right] \qquad \text{(B-10)}$$

or

$$\overline{H}_n^{(2)'*}(\beta r)\,\overline{H}_n^{(2)}(\beta r) = \overline{J}'_n(\beta r)\,\overline{J}_n(\beta r) + \overline{J}'_{-n}(\beta r)\,\overline{J}_{-n}(\beta r) + j\left[\overline{J}'_{-n}(\beta r)\,\overline{J}_n(\beta r) - \overline{J}'_n(\beta r)\,\overline{J}_{-n}(\beta r)\right], \tag{B-11}$$

but

$$\overline{J}'_{-n}(\beta r)\,\overline{J}_n(\beta r) - \overline{J}'_n(\beta r)\,\overline{J}_{-n}(\beta r) = 1\,. \tag{B-12}$$

Thus

$$\overline{H}_n^{(2)'*}(\beta r)\,\overline{H}_n^{(2)}(\beta r) = \overline{J}'_n(\beta r)\,\overline{J}_n(\beta r) + \overline{J}'_{-n}(\beta r)\,\overline{J}_{-n}(\beta r) + j\,. \tag{B-13}$$

The first two terms are both real, since r and β are real, and Bessel functions are real for real arguments. Similarly it may be shown that the second term in equation (B-9) is given by

$$\overline{H}_n^{(1)'*}(\beta r)\,\overline{H}_n^{(1)}(\beta r) = \overline{J}'_n(\beta r)\,\overline{J}_n(\beta r) + \overline{J}'_{-n}(\beta r)\,\overline{J}_{-n}(\beta r) - j\,, \tag{B-14}$$

letting $g_n = E - jF$, the cross products become upon combining

$$g_n^*\,\overline{H}_n^{(1)'*}(\beta r)\,\overline{H}_n^{(2)}(\beta r) + g_n\,\overline{H}_n^{(2)'*}(\beta r)\,\overline{H}_n^{(1)}(\beta r) = E\left[\overline{J}'_n(\beta r)\,\overline{J}_n(\beta r) - \overline{J}'_{-n}(\beta r)\,\overline{J}_{-n}(\beta r)\right] - F\left[\overline{J}'_{-n}(\beta r)\,\overline{J}_n(\beta r) + \overline{J}'_n(\beta r)\,\overline{J}_{-n}(\beta r)\right]. \tag{B-15}$$

All of the terms on the right hand side are real since they are Bessel functions of real arguments. Since the product in equation (B-8) is multiplied by -j, taking the real part of the poynting vector retains only imaginary part of $S'^{*}_n(\beta r)\,S_n(\beta r)$.

Combining the results obtained above the real part of the poynting vector becomes

$$\text{Re } p_r = \frac{\eta}{4\pi^2} \frac{|a_n|^2}{n^2 (n+1)^2} \left[\frac{d}{d\theta} M_n(\cos\theta)^2\right]\left[1 - |g_n|^2\right] \quad \text{(B-16)}$$

The average power transferred along the biconical antenna in wave is then given by (B-17)

$$W = \frac{\eta}{8\pi^2} \int_0^{2\pi}\int_{\delta}^{\pi-\delta} \frac{|a_n|^2}{r^2 n^2 (n+1)^2} \left[1 - |g_n|^2\right]\left[\frac{d}{d\theta} M_n(\cos\theta)\right]^2 r^2 \sin\theta \, d\theta \, d\phi.$$

Making use of the results obtained in Appendix B, the required result is found (B-18)

$$W = \frac{\eta}{2\pi} \frac{|a_n|^2}{n^2 (n+1)^2} \left[\frac{- M_n(\cos\delta)}{\theta} \frac{M_n(\cos\delta)}{n} \sin\delta\right]\left[\frac{1 - |g_n|^2}{2n+1}\right].$$

The term in brackets involving functions of δ must be positive although it includes a negative sign. This term is the result of integrating $\left[\frac{d}{d\theta} M_n(\cos\theta)\right]^2 \sin\theta$. M_n is a real number since the argument is real. The derivative is, therefore, also real which when squared must be positive number. Since $\sin\theta$ is positive over the entire range of integration, the integrand is always positive. Therefore, the integral will be positive.

All terms in equation (B-18), with the possible exception of the last, are positive. The 1 in the last term is a result of the product of the Hankel functions of the second kind in the expressions for the E and H fields, which were said to represent an outward traveling wave, and it is seen that these terms do contribute a positive term to the total outward energy flow. The product of the first order Hankel function terms in the H and E fields contribute the $-g_n{}^2$ term in the expression for the transferred power. The negative sign represents power transferred inward along a radial path. The second order Hankel function does represent a reflected coefficient. If g_n is greater than unity, the antenna is absorbing rather than transmitting energy. This does not imply, however, that the fields in the region of a biconical antenna, which is acting as a receiving antenna, would be those given here for the transmitting antenna with $|g_n| > 1$. Although the patterns and impedance of a receiving antenna are the same as that of a transmitting antenna, the fields and resulting current distributions are quite different and are not discussed here. A standing wave results when the absolute value of the reflection coefficient is unity.

Consider now the remaining component of the poynting vector for the TM wave

$$p_r = E_\theta^* H_\phi \qquad \text{(B-19)}$$

Note that the poynting vector of TEM wave has no such component since $E_r = 0$. In the case of the TM wave

$$E_r^* H_\phi = \frac{1}{2\pi^2 \omega \epsilon} \frac{|a_n|^2}{n(n+1)} \left[\bar{H}_n^{(2)}(\beta r) + g_n \bar{H}_n^{(1)}(\beta r)^2\right]\left[\frac{d}{d\theta} M_n(\cos\theta)\right]^2 \qquad \text{(B-20)}$$

The right hand side of this equation is purely an imaginary number. Since the average of the poynting vector has no real part, the real part of its integral is zero, and no average power is transferred in the θ direction. This is as expected since in the case of the lossless antenna energy traveling within the region of the antenna in the θ direction will be totally reflected from the perfectly conducting boundary.

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