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IMPROVING THE RELIABILITY OF ELECTRONIC SYSTEMS
BY MINIMIZATION OF THE VARIANCES AND STABILIZATION
OF THE MEAN VALUES OF THE SYSTEM PERFORMANCE CRITERIA

BY

EMIL CARL NEU

A DISSERTATION
PRESENTED IN PARTIAL FULFILLMENT OF
THE REQUIREMENTS FOR THE DEGREE
OF
DOCTOR OF ENGINEERING SCIENCE IN ELECTRICAL ENGINEERING
AT
NEWARK COLLEGE OF ENGINEERING

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Newark, New Jersey
1966

ABSTRACT

The parameters of an electronic system are not deterministic variables; but rather because of variations in manufacturing processes, they are randomly distributed variables. In addition, aging mechanisms will cause these parameters to drift with time. As a result of both of these factors, the system performance criteria deviate from their initial design center values. This research studied the effect of component parameter variations upon these electronic system performance criteria and then presented a thesis to minimize the effect of these variations.

The distributions of the performance criteria were obtained as functions of the parameter distributions, by first approximating the performance criteria by the linear terms of a Taylor's series. This technique made it possible to obtain relatively simple expressions for the mean values and variances of the performance criteria. Then by assuming that the performance criteria were normally distributed, the probabilities of the performance criteria being within the required limits were determined.

This analysis showed that the probabilities of the system performance criteria being out of tolerance could be minimized, if the mean values of the performance criteria were held at their design center levels and if the variances of these criteria were kept as small as possible. It was seen that the mean values of the performance criteria could be set at their design center levels by proper choice of the parameter mean values. The variances of the criteria were kept as small as possible by first minimizing these variances with respect to the system parameter mean values with the constraint that the initial performance criteria means assume their design center levels. Increases in the variances of the performance criteria with time, as well as drift of the mean values of these criteria, were prevented by satisfying certain relationships among the parameter drift rates.

The results of this research include the extension and refinement of techniques for determining the distributions of system performance criteria as functions of system parameter distributions. In addition, methods were developed for the selection of the mean values of the parameters and the parameter drift rates, so as to minimize the variances of the system performance

criteria, while at the same time preventing drift of the mean values of these criteria. Expressions for computing the coefficient of linear correlation between two variables whose values were selected in a non-random manner were also obtained.

APPROVAL OF DISSERTATION

IMPROVING THE RELIABILITY OF ELECTRONIC SYSTEMS
BY MINIMIZATION OF THE VARIANCES AND STABILIZATION
OF THE MEAN VALUES OF THE SYSTEM PERFORMANCE CRITERIA

BY

EMIL CARL NEU

FOR

DEPARTMENT OF ELECTRICAL ENGINEERING

NEWARK COLLEGE OF ENGINEERING

BY

FACULTY COMMITTEE

APPROVED: _____ CHAIRMAN

NEWARK, NEW JERSEY

JUNE, 1966

PREFACE

This doctoral dissertation is a result of research performed under the direction of Dr. R. P. Misra in the Department of Electrical Engineering at Newark College of Engineering. The dissertation considers methods of analyzing the distributions of the performance criteria of electronic systems, and then develops techniques for increasing the probabilities that these criteria will fall within the required range of values. Both the initial values and the time varying aspects of this problem are studied.

This research was prompted by a suggestion by Dr. Misra that the reliability of electronic systems could be improved by circuit modifications, such as the introduction of feedback. I would like to express my appreciation to Dr. Misra for not only suggesting this problem, but also for giving so generously of his time as my thesis advisor.

The assistance of the other members of my doctoral committee; Dr. L. B. Andersen, Dr. F. A. Russell and Dr. M. H. Zambuto is also gratefully acknowledged. I should also like to thank Dr. Russell for guiding my doctoral studies prior to the initiation of this research. —

EMIL C. NEU

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CHAPTER 1: INTRODUCTION

1-1. The Problem

The parameters of an electronic system are not deterministic variables; but rather because of variations in manufacturing processes, these parameters are randomly distributed variables. Hence the system performance criteria, which are functions of the system parameters, are functions of randomly distributed variables. Thus a system, whose performance criteria must be within certain limits to insure proper operation of the system, generally has a non-zero probability of being out of tolerance. The problem then is to analyze the effect of the above variations of system parameters on the system performance criteria and to determine methods of minimizing this effect.

The above situation is compounded by aging mechanisms that cause the parameters of a system to drift with time. The problem is further complicated by the fact that the parameters which describe the aging mechanisms are in themselves random variables. The net effect is that the system performance criteria are functions of random variables whose distributions change with time. This means that the probability of

the performance criteria being out of tolerance varies with time. Thus in attempting to analyze and reduce the effects of parameter variations on the system performance, the aging of the components must also be considered.

1-2. The Method of Solution

The first step in the solution of the above problem is to find methods of determining the effects of parameter variations, due to both manufacturing processes and aging mechanisms, on system performance criteria. This requires methods for determining the distribution of a function of a number of random variables. Techniques for handling this type of problem have been developed (Ref. 16, 20, 27). Hence the real task at hand is to refine and extend these techniques to the analysis of electronic systems.

Once the above analysis has been completed, the problem of minimizing the effect of parameter variations on system performance can be considered. First, the minimizing of this effect due to manufacturing processes should be handled. In other words, the initial probability of the system performance being out of tolerance must be minimized. This can be done

by noting that in the design of electronic systems there are usually an infinite number of combinations of parameter values which will give the required values of the system performance criteria. Hence that combination of parameter values which minimizes the effect of the initial parameter variations on the initial system performance criteria should be selected.

After this has been accomplished, attention must be given to minimizing the effect of parameter drift. Generally speaking, this means obtaining combinations of parameter drift rates which prevent the probability of the system performance criteria's being out of tolerance from increasing with time. In some cases, this goal may be difficult to achieve; while in others it may, in fact, be possible to reduce this probability of being out of tolerance as time increases. In any event that combination of drift rates which makes the probability of being out of tolerance as a function of time as small as possible should be chosen.

CHAPTER 2: COMPONENT PARAMETER DISTRIBUTIONS

2-1. Introduction

The purpose of this chapter is to discuss parameter distributions of devices, as manufactured, as well as the parameter distributions of the devices as they are supplied to the customer. The mean values and the variances of these latter distributions will be studied. In addition, models for the parameters as functions of time will be considered.

2-2. Parameter Distributions of Manufacturing Processes

In this research the parameter values of devices, as manufactured, will be assumed to be continuous random variables within a specified range. This is a reasonable assumption, for if a sufficiently large number of units is manufactured, it will be possible to obtain almost any value of parameter desired (to say, three significant figures) within the specified range. The distribution of the output of most manufacturing processes can be approximated by the normal distribution. If the normal distribution is used, it must be truncated at the origin since parameters have values of but one sign (See Fig. 2-1). If a distribution other than the normal distribution is used

to represent these parameters, this distribution must also have such a range that parameters of but one sign are represented.

When a particular device is characterized by more than one parameter, these parameters will often be correlated. For example, the h-parameters of a transistor are correlated. If the parameters of a device are correlated, it will be assumed that this correlation is linear.

2-3. Parameter Distributions as Supplied by the Manufacturer

One of the basic assumptions made throughout most of this research is that the standard deviation of each parameter is considerably smaller than the mean value of that parameter. For high quality components this assumption will often be met by parameters of the components taken directly from the manufacturing process. Most manufacturers will, however, apply some type of selection process to their product. Thus, even if the above assumption is not met by the manufactured product, it will be assumed that it is met by the product after the selection process.

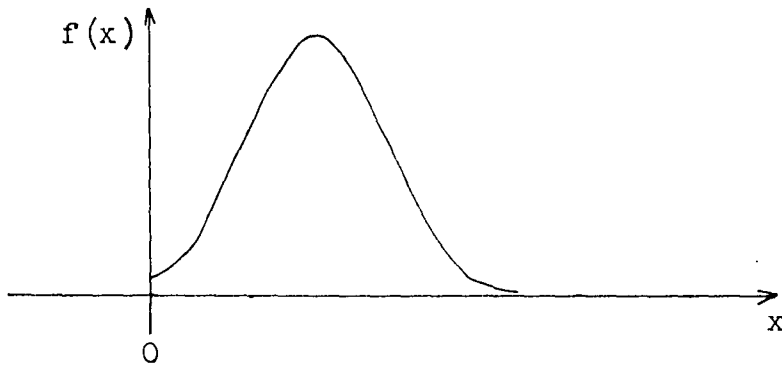


Fig. 2-1. The normal density function, $f(x)$, for a variable x , with single-sided truncation at the origin.

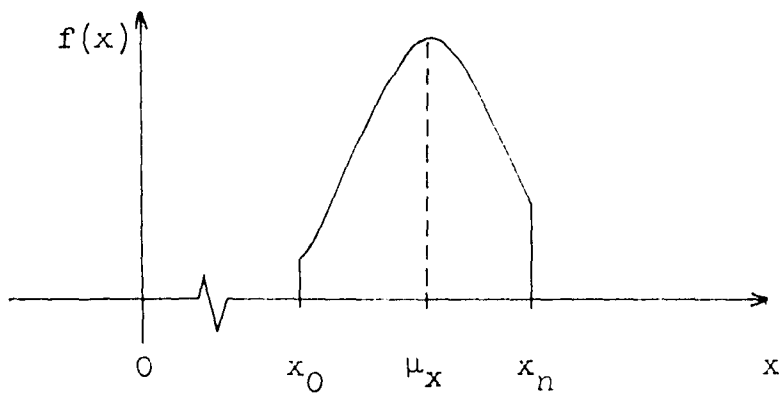


Fig. 2-2. The normal density function with asymmetrical truncation.

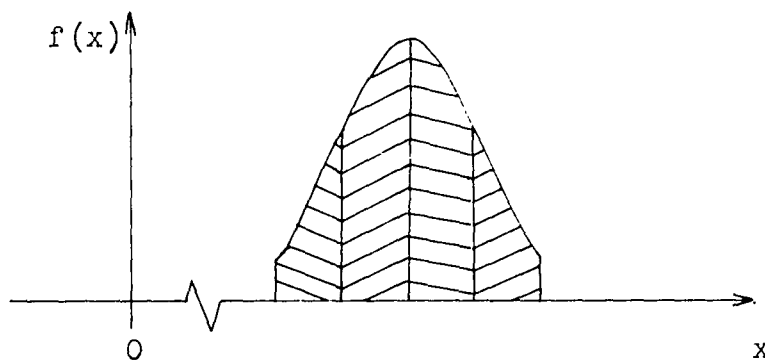


Fig. 2-3. Sections of a normal density function.

When a selection process is applied to a normal distribution, one of a number of types of distributions can result. One possibility is the single-sided truncated normal distribution. In fact, as was previously noted, the normal distribution even before any selection process is of this type. Double-sided truncation may also be applied to the normal distribution (See Fig. 2-2). In this case the truncation is usually symmetrical, but asymmetrical truncation is sometimes used. It should be noted that the distribution remaining¹ after the double-sided truncation of a normal distribution will not be considered in this investigation. The reason for this restriction is that such a distribution does not meet the assumption that the standard deviation of a parameter is considerably smaller than its mean value.

In the above selection processes the purpose of the truncation is to eliminate units whose parameter values differ by too great an amount from the required nominal values of the parameter. The manufacturer will in some instances, however, desire to obtain components with a number of different nominal parameter values from a single normal distribution. In this case the distribution of each parameter value

¹This refers to the common situation where two relatively small tails remain after double-sided truncation.

will be a section of a normal distribution (See Fig. 2-3). These sections can be viewed as being cases of extreme asymmetrical truncation. It will be assumed that these distributions meet the requirement that the standard deviation of a parameter be much smaller than its mean value. The assumption of a normal distribution cannot, however, be made in this case.

2-4. Moments of Parameter Distributions as Supplied by the Manufacturer

(a) The normal distribution. The density function of a normally distributed variable x is given by

$$f(x) = \frac{1}{\sigma_x \sqrt{2\pi}} e^{-\frac{(x-\mu_x)^2}{2\sigma_x^2}} \quad (-\infty < x < \infty) \quad (2-1)$$

where μ_x is the mean value of x ,

σ_x is the standard deviation of x .

As discussed in Sec. 2-2 the normal distribution is always truncated before it is used to represent parameter distributions. If the points of truncation are sufficiently far from the mean value of the untruncated distribution - three standard deviations is often considered sufficiently far - then the mean

value and the standard deviation of the untruncated distribution are good approximations to the mean value and standard deviation respectively, of the truncated distribution.

(b) The asymmetrically truncated normal distribution. For a continuous random variable x with a density function $f(x)$, the mean value of x in the range $d_1 \leq x \leq d_2$ is given by

$$\mu_{xt} = \frac{\int_{d_1}^{d_2} x f(x) dx}{\int_{d_1}^{d_2} f(x) dx} \quad (2-2)$$

If this definition is applied to the density function of Fig. 2-2, $d_1 = x_0$, $d_2 = x_n$ and $f(x)$ is given by Eq. (2-1).

Now $f(x)$ may be transformed to the unit normal density function, $f(k)$, by the following equation:

$$k = \frac{x - \mu_x}{\sigma_x} \quad (2-3)$$

Application of this transformation to the mean value of the distribution of Fig. 2-2 shows that the mean of the truncated distribution is given by

$$\mu_{xt} = \frac{\sigma_x \int_{R_0}^{R_n} R f(R) dR}{\int_{R_0}^{R_n} f(R) dR} + \mu_x \quad (2-4)$$

where

$$R_0 = \frac{x_0 - \mu_x}{\sigma_x} \quad (2-5)$$

$$R_n = \frac{x_n - \mu_x}{\sigma_x} \quad (2-6)$$

Also note that the symbols μ_x and σ_x will be used for the untruncated distribution while μ_{xt} and σ_{xt} will be used for the truncated distribution.

The denominator of Eq. (2-4) can be evaluated from any unit normal distribution table (Ref. 2). On the other hand, while there is a table in existence

(Ref. 32) from which the numerator of Eq. (2-4) can be evaluated, this table is not generally available. It is possible, however, to obtain the following identity (Ref. 14):

$$\int_{k_0}^{k_n} k f(k) dk = f(k_0) - f(k_n) \quad (2-7)$$

Use of this identity in Eq. (2-4) gives

$$\mu_{xt} = \frac{\sigma_x [f(k_0) - f(k_n)]}{\int_{k_0}^{k_n} f(k) dk} + \mu_x \quad (2-8)$$

With this equation it is possible to evaluate μ_{xt} by use of only the unit normal distribution table.

For a continuous random variable x with a density function $f(x)$, the variance of x in the range $d_1 \leq x \leq d_2$ is given by

$$\sigma_x^2 = \frac{\int_{d_1}^{d_2} (x - \mu_{xt})^2 f(x) dx}{\int_{d_1}^{d_2} f(x) dx} \quad (2-9)$$

Expanding $(x - \mu_{xt})^2$ and simplifying yields

$$\sigma_x^2 = \frac{\int_{d_1}^{d_2} x^2 f(x) dx}{\int_{d_1}^{d_2} f(x) dx} - \mu_{xt}^2 \quad (2-10)$$

If this definition is applied to the distribution of Fig. 2-2, the following expression for the variance of the truncated distribution is obtained (Ref. 14):

$$\sigma_{xt}^2 = \sigma_x^2 \left[\left(1 + \frac{k_0 f(k_0) - k_n f(k_n)}{\int_{k_0}^{k_n} f(k) dk} \right) - \left(\frac{f(k_0) - f(k_n)}{\int_{k_0}^{k_n} f(k) dk} \right)^2 \right] \quad (2-11)$$

It should be pointed out that this is the variance about the mean value of the truncated distribution. Also, notice this expression allows σ_{xt}^2 to be evaluated in terms of quantities in the unit normal distribution table.

The means and variances of sections of a normal distribution can also be determined by use of

Eqs. (2-8) and (2-11) respectively. This is true because these sections are merely extreme cases of asymmetrical truncation of a normal distribution.

(c) The symmetrically truncated normal distribution. In the case of symmetrical truncation, it is noticed from Fig. 2-2 that

$$(x_o - \mu_x) = -(x_n - \mu_x)$$

Hence Eqs. (2-5) and (2-6) reveal that in this case

$$k_o = -k_n$$

Consequently, Eq. (2-8) shows that $\mu_{xt} = \mu_x$ for the symmetrical case. Furthermore, the variance as given by Eq. (2-11) becomes (Ref. 14)

$$\sigma_{xt}^2 = \sigma_x^2 \left(1 - \frac{k_n f(k_n)}{\int_0^{k_n} f(k) dk} \right) \quad (2-12)$$

(d) Other distributions. The truncated normal distribution and sections of a normal distribution will be the only distributions treated specifically in this investigation. The mean values and variances of other distributions of continuous random variables may, however, be obtained by use of Eqs. (2-2) and (2-9) respectively.

2.5 Models for Parameters that Drift

In general, parameters which drift will be non-linear functions of time. Thus a parameter x_j which drifts may be represented by the following function of time:

$$x_j(t) = \mu_{x_j}(0) (a_{j0} + a_{j1}t + a_{j2}t^2 + \dots + a_{jq}t^q + \dots - a_{jm}t^m + \dots) \quad (2-13)$$

where $\mu_{x_j}(0)$ is the mean of the initial value of x_j ,
 a_{jq} is the drift rate² associated with the q^{th}
 power of t ,
 t is time.

Also note that the a_{jq} 's are continuous random variables while t is not. In general, there will be correlation among the a_{jq} 's.

An important special case of the above model is obtained when the drift is assumed to be a linear function of time (Ref. 42). In this case Eq. (2-13) becomes

$$x_j(t) = \mu_{x_j}(0) (a_{j0} + a_{j1}t) \quad (2-14)$$

² a_{j0} is not a drift rate, but rather just a random variable whose mean value is unity and whose variance is equal to the initial variance of x_j divided by $\mu_{x_j}^2(0)$. Also note that a_{jq} may be regarded as a generalized drift rate in that it is a change per unit time to the q^{th} power.

A further simplification of this linear model results when it is assumed that the initial value and the drift rate are uncorrelated. It is this linear model with no correlation which will be used in most of the work in this research.

There are a number of reasons why this simplified model will be emphasized. First is the fact that the primary object of this research is to develop new techniques for designing more reliable electronic systems. Consequently, an approximate model, which yields results that are sufficiently simple so as to be useful for design purposes, is preferred to a more exact model, which yields results that are so cumbersome that they are virtually worthless in design. Another reason for the use of the simplified model is the lack of component drift data. At the present state of the art, there is relatively little data which gives drift as a function of stress, and most of what is available pertains to resistors. Furthermore, data relating to the effect of correlation on drift is practically non-existent. This does not mean, however, that correlation can always be neglected in drift models. To the contrary, it can be quite important particularly when feedback is present.

It is also important to note that the above linear model provides a reasonable approximation for many

practical devices. As an illustration of this point, consider the MIL-R-10509D metal film resistor. Curves of the drift of this type resistor for various stresses (percentage of rated power in this case) are available for an ambient temperature of 125°C (Refs. 33, 41 and 46). By approximating these curves with straight lines, the following drift rate vs. stress data is obtained:

Table of the Mean Values of the Drift Rates

Percentage of rated power	Drift rate as percentage of initial resistance per hour
25	2×10^{-6}
50	5×10^{-6}
100	10×10^{-6}

Now the actual power taken by any individual resistor, if it is supplied by a constant current source, is directly proportional to its resistance. Since the drift rate is influenced by the power, the initial resistance and the drift rate are then correlated. For example, suppose that a particular resistor is 5% above the mean resistance of distribution from which it was drawn. Then this resistor

will consume 5% more power than the average resistor. By use of linear interpolation in the above table, it is found that if the average resistor drifts at the rate of $2 \times 10^{-6}\%$ per hour - then the resistor which is 5% high in resistance drifts at the rate of $2.15 \times 10^{-6}\%$ per hour.

From this example it should be noted that if none of the resistors in the distribution differs from the mean value by too great a degree (say 5% or less), then the correlation between the initial resistance and the drift rate is small and may thus be neglected. Now it has already been assumed that the standard deviation of parameters will be much smaller than the respective mean values of those parameters. Therefore, in this research the correlation between the initial resistance and the drift rate may be neglected for resistors of the above type.

CHAPTER 3: DISTRIBUTIONS OF
FUNCTIONS OF RANDOM VARIABLES

3-1. Introduction

When a device is characterized by more than one parameter, these parameters may be correlated. In this chapter functions of correlated variables will be studied in order to determine their distributions. Functions of uncorrelated variables will be treated by specialization of the correlated case.

3-2. Correlated Random Variables

It is possible to divide linearly correlated variables into three classes. In the first class, the variables are correlated for physical reasons. For example, the h-parameters of a transistor maybe correlated¹ because they are used to characterize the same device. The second class pertains to those variables which are correlated because they are functions of the same set of variables. It should be noted, however, that the distinction between these two types of correlation is really artificial. In the so-called physical case, the correlated variables are also functions of the same set of variables. The functions in this case are generally so complicated,

¹Tommerdahl and Nelson (Ref.42) have taken data on 200 type 2N526 transistors. From this data they have found that the coefficient of linear correlation between h_{oe} and h_{fe} for this type transistor is 0.512.

however, that it is easier to measure the correlation experimentally than to compute it. The third class pertains to parameters which are correlated because they were not selected at random. An example of this is a situation where a low value of one parameter is purposely matched with a high value of another parameter.

The degree of correlation between variables may be specified in various ways. One measure of the amount of linear correlation between two variables x_1 and x_2 is their covariance. The covariance between x_1 and x_2 in the range $d_1 \leq x_1 \leq d_2$, $d_3 \leq x_2 \leq d_4$ is defined as

$$\sigma_{x_1, x_2} = \frac{\int_{d_1}^{d_2} \int_{d_3}^{d_4} (x_1 - \mu_{x_1})(x_2 - \mu_{x_2}) f(x_1, x_2) dx_1 dx_2}{\int_{d_1}^{d_2} \int_{d_3}^{d_4} f(x_1, x_2) dx_1 dx_2} \quad (3-1)$$

where $f(x_1, x_2)$ is the joint density function of x_1 and x_2 . This equation can be written in the following form by using the symbol E to denote the process of taking the mean:

$$\sigma_{x_1 x_2} = E[(x_1 - \mu_{x_1})(x_2 - \mu_{x_2})] \quad (3-2)$$

The covariance can also be written as (Ref. 45)

$$\sigma_{x_1 x_2} = E(x_1 x_2) - \mu_{x_1} \mu_{x_2} \quad (3-3)$$

The coefficient of linear correlation between x_1 and x_2 can be defined in terms of this covariance (Ref. 45) as

$$\rho_{x_1 x_2} = \frac{\sigma_{x_1 x_2}}{\sigma_{x_1} \sigma_{x_2}} \quad (3-4)$$

In describing the degree of linear correlation between two random variables, it is the coefficient of linear correlation which is usually specified (Refs. 23 and 39).

3-3. Functions of Correlated Random Variables

In general, system performance criteria will be functions of correlated variables. These criteria will be both linear functions and non-linear functions. The non-linear functions may, however, be approximated by linear functions. This is done by expanding these functions in a Taylor's series about the point at which all the variables assume their mean values and

then neglecting the non-linear terms of the series. This technique provides a reasonable approximation to non-linear functions, if the standard deviations of the respective variables are considerably smaller than their mean values (Ref. 37). It should be noted that this method also applies to the special cases where all or some of the variables are uncorrelated.

3-4. Moments of Distributions of Functions of Correlated Random Variables

Consider the following function:

$$y = g(x_1, x_2, \dots, x_j, \dots, x_n) \quad (3-5)$$

In general, it will be assumed that the x's in this equation are correlated.

Expressions for the mean and the variance of y are readily available (Ref. 20). The mean value of y is given by

$$\mu_y = g(\mu_{x_1}, \mu_{x_2}, \dots, \mu_{x_j}, \dots, \mu_{x_n}) \quad (3-6)$$

The expression for the variance of y is

$$\sigma_y^2 = \sum_{j=1}^n \left(\frac{\partial y}{\partial x_j} \right)^2 \sigma_{x_j}^2 + 2 \sum_{j=1}^{n-1} \sum_{l=j+1}^n \left(\frac{\partial y}{\partial x_j} \frac{\partial y}{\partial x_l} \right) \rho_{x_j x_l} \sigma_{x_j} \sigma_{x_l} \quad (3-7)$$

$\mu_{x_1}, \mu_{x_2}, \dots, \mu_{x_n}$

Eqs. (3-6) and (3-7) are exact expressions for μ_y and σ_y^2 respectively, if y is a linear function. If y is a non-linear function, then these equations are reasonable approximations to μ_y and σ_y^2 as long as the standard deviations of the respective variables are much smaller than their mean values.

These expressions also apply when all or some of the x 's are uncorrelated. This case is handled by simply setting the coefficient of linear correlation between them equal to zero.

3-5. Functions of Normally Correlated Random Variables

Two random variables, x_1 and x_2 , are said to be normally correlated if their joint density function is given by (Ref. 44)

$$f(x_1, x_2) = \frac{1}{2\pi\sigma_{x_1}\sigma_{x_2}\sqrt{1-\rho_{x_1, x_2}^2}} e^{-Q/2} \quad \begin{array}{l} (-\infty < x_1 < \infty) \\ (-\infty < x_2 < \infty) \end{array} \quad (3-8)$$

where

$$Q = \frac{1}{(1-\rho_{x_1, x_2}^2)} \left[\frac{(x_1 - \mu_{x_1})^2}{\sigma_{x_1}^2} - \frac{2\rho_{x_1, x_2}(x_1 - \mu_{x_1})(x_2 - \mu_{x_2})}{\sigma_{x_1}\sigma_{x_2}} + \frac{(x_2 - \mu_{x_2})^2}{\sigma_{x_2}^2} \right]$$

It can be shown that the marginal distributions of x_1 and x_2 are both normal with moments μ_{x_1} , σ_{x_1} , and μ_{x_2} , σ_{x_2} respectively. In addition, $\rho_{x_1x_2}$ is found to be the coefficient of linear correlation which was defined by Eq. (3-4) (Ref. 30).

The converse of the above statements is also true (Ref. 28). In other words, if x_1 and x_2 are each normally distributed with moments μ_{x_1} , σ_{x_1} , and μ_{x_2} , σ_{x_2} and if the coefficient of linear correlation between x_1 and x_2 is $\rho_{x_1x_2}$, then the joint density function of x_1 and x_2 is given by Eq. (3-8).

All of the above ideas may also be extended to a multivariate normal distribution of n dimensions (Refs. 28 and 30).

Now consider the function y given by Eq. (3-5) for the special case where $n = 2$. Let x_1 and x_2 both be normally distributed with moments μ_{x_1} , σ_{x_1} and μ_{x_2} , σ_{x_2} and a coefficient of linear correlation $\rho_{x_1x_2}$ between them. Assume that y is a linear function of x_1 and x_2 or that it may be approximated by a linear function of x_1 and x_2 . In this case, y is then normally distributed with μ_y and σ_y^2 given by

Eqs. (3-6) and (3-7) respectively. This idea can also be extended to the case where y is a function of n variables (Ref. 28).

It should be emphasized that all the concepts expressed in this section apply to the special cases where all or some of the variables are uncorrelated.

3-6. The Coefficient of Linear Correlation Between Two Functions of the Same Correlated Random Variables

Consider the following functions:

$$y_1 = g_1(x_1, x_2, \dots, x_j, \dots, x_n) \quad (3-9)$$

$$y_2 = g_2(x_1, x_2, \dots, x_j, \dots, x_n) \quad (3-10)$$

It will be assumed that in general there is correlation between the x 's. Now since y_1 and y_2 are functions of the same variables, there will be correlation between y_1 and y_2 - even in the special case, where all the x 's are uncorrelated. The problem is to determine the coefficient of linear correlation between y_1 and y_2 in terms of the functional relationships g_1 and g_2 , the standard deviations of the x 's and the coefficients of linear correlation among the x 's.

Expanding y_1 and y_2 in a Taylor's series about the point at which all the variables assume their mean values and neglecting the non-linear terms yields

$$y_1 = \left[g_1(x_1, x_2, \dots, x_n) \right]_{\mu_{x_1}, \mu_{x_2}, \dots, \mu_{x_n}} + \sum_{j=1}^n \left(\frac{\partial g_1}{\partial x_j} \right) \bigg|_{\mu_{x_1}, \mu_{x_2}, \dots, \mu_{x_n}} (x_j - \mu_{x_j}) \quad (3-11)$$

and

$$y_2 = \left[g_2(x_1, x_2, \dots, x_n) \right]_{\mu_{x_1}, \mu_{x_2}, \dots, \mu_{x_n}} + \sum_{j=1}^n \left(\frac{\partial g_2}{\partial x_j} \right) \bigg|_{\mu_{x_1}, \mu_{x_2}, \dots, \mu_{x_n}} (x_j - \mu_{x_j}) \quad (3-12)$$

As discussed in Sec. 3-3, these expansions will be exact representations of linear functions and approximate representations of non-linear functions.

The covariance between y_1 and y_2 can be obtained by application of Eq. (3-2) to Eqs. (3-11) and (3-12).

Thus

$$\sigma_{y_1 y_2} = E \left\{ \left[\sum_{j=1}^n \left(\frac{\partial g_1}{\partial x_j} \right) \Big|_{\mu_{x_1}, \mu_{x_2}, \dots, \mu_{x_n}} (x_j - \mu_{x_j}) \right] \left[\sum_{j=1}^n \left(\frac{\partial g_2}{\partial x_j} \right) \Big|_{\mu_{x_1}, \mu_{x_2}, \dots, \mu_{x_n}} (x_j - \mu_{x_j}) \right] \right\} \quad (3-13)$$

Since variance and covariance respectively are defined as

$$\sigma_{x_j} = E (x_j - \mu_{x_j})^2 \quad (3-14)$$

$$\sigma_{x_j x_l} = E [(x_j - \mu_{x_j})(x_l - \mu_{x_l})] \quad (3-15)$$

the expression for $\sigma_{y_1 y_2}$ can then be rewritten as

$$\sigma_{y_1 y_2} = \sum_{j=1}^n \left(\frac{\partial g_1}{\partial x_j} \frac{\partial g_2}{\partial x_j} \right) \Big|_{\mu_{x_1}, \mu_{x_2}, \dots, \mu_{x_n}} \sigma_{x_j}^2 + \sum_{j=1}^{j=n-1} \sum_{l=j+1}^n \left(\frac{\partial g_1}{\partial x_j} \frac{\partial g_2}{\partial x_l} + \frac{\partial g_1}{\partial x_l} \frac{\partial g_2}{\partial x_j} \right) \Big|_{\mu_{x_1}, \mu_{x_2}, \dots, \mu_{x_n}} \sigma_{x_j x_l} \quad (3-16)$$

Using the definition of the coefficient of linear correlation, it is seen that

$$\sigma_{y_1 y_2} = \sum_{j=1}^n \left(\frac{\partial g_1}{\partial x_j} \frac{\partial g_2}{\partial x_j} \right) \Big|_{\mu_{x_1}, \mu_{x_2}, \dots, \mu_{x_n}} \sigma_{x_j}^2 + \sum_{j=1}^{j=n-1} \sum_{l=j+1}^n \left(\frac{\partial g_1}{\partial x_j} \frac{\partial g_2}{\partial x_l} + \frac{\partial g_1}{\partial x_l} \frac{\partial g_2}{\partial x_j} \right) \Big|_{\mu_{x_1}, \mu_{x_2}, \dots, \mu_{x_n}} \rho_{x_j x_l} \sigma_{x_j} \sigma_{x_l} \quad (3-17)$$

Finally, the coefficient of linear correlation between y_1 and y_2 is

$$r_{y_1, y_2} = \frac{\sigma_{y_1 y_2}}{\sigma_{y_1} \sigma_{y_2}} \quad (3-18)$$

where σ_{y_1} and σ_{y_2} are obtained by application of Eq. (3-7) to Eqs. (3-9) and (3-10).

This is the desired result in that the coefficient of linear correlation between y_1 and y_2 can now be computed from information on the distributions of the x 's. It might also be noted that Wilks (Ref. 48) presents an expression similar to Eq. (3-16) for m linear functions of n variables. Wilks does not consider, however, the case where the functions may be nonlinear. Since Eq. (3-16) is an approximation for the nonlinear case, it really is not any more general than Wilks' result but it is in a more convenient form (see its application in Sec. 4-5).

CHAPTER 4: INITIAL DISTRIBUTIONS OF THE PERFORMANCE CRITERIA OF ELECTRONIC SYSTEMS

4-1. Introduction

In this chapter the techniques of Chapter 3 will be used to determine the initial distributions of the performance criteria of a number of electronic systems. It will then be shown how these distributions can be used to determine the initial probability of the system operating satisfactorily. Throughout this discussion, it will be assumed that the performance criteria are either linear functions of the system parameters or that they are non-linear functions of the system parameters, which may be approximated by linear functions. In this latter case the standard deviations of the respective parameters will be considered to be much smaller than their mean values.

4-2. Resistors in Series

A simple type of electronic system which has a variety of applications is the series system. As an example of a series system, a network in which n resistors are placed in series will be considered. The total series resistance of this arrangement is

$$R = \sum_{j=1}^n R_j \quad (4-1)$$

where R_j is the initial value of the resistance of the j^{th} resistor.

Application of Eq. (3-6) to this expression yields the following equation for the mean value¹ of R :

$$\mu_R = \sum_{j=1}^n \mu_{R_j} \quad (4-2)$$

The variance of R can be obtained by use of Eq. (3-7).

This gives

$$\sigma_R^2 = \sum_{j=1}^n \sigma_{R_j}^2 + 2 \sum_{j=1}^{n-1} \sum_{l=j+1}^n \rho_{x_j x_l} \sigma_{x_j} \sigma_{x_l} \quad (4-3)$$

¹In discussions where only initial values are being considered, the (0) to denote initial value in $\mu(0)$ and $\sigma(0)$ will be omitted.

Since each resistor is a separate device², there will be no correlation among the resistances because of any physical relationship between them. Consequently, if each resistor is selected at random all the resistances comprising the series resistance will be uncorrelated. In this case, Eq. (4-3) becomes

$$\sigma_R^2 = \sum_{j=1}^n \sigma_{R_j}^2 \quad (4-4)$$

4-3. Voltage Divider

Another simple but widely used electronic system is the resistive voltage divider shown in Fig. 4-1.

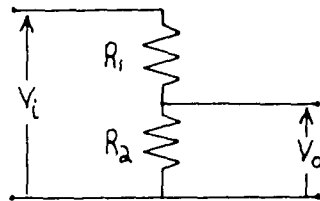


Fig. 4-1. Resistive voltage divider.

If the effect of loading can be neglected, the divider ratio of this circuit is

$$K_v = \frac{V_o}{V_i} = \frac{R_2}{R_1 + R_2} \quad (4-5)$$

²This requires that the resistors be thermally insulated from one another, a condition which can be approximated if the amount of power they dissipate is small and the ventilation is good.

The mean value of K_V can be obtained by use of Eq. (3-6) which gives

$$\mu_{K_V} = \frac{\mu_{R_2}}{\mu_{R_1} + \mu_{R_2}} \quad (4-6)$$

Application of Eq. (3-7) yields the following expression for the variance of K_V :

$$\sigma_{K_V}^2 = \left(\frac{\partial K_V}{\partial \mu_{R_1}} \right)_{\mu_{R_1}, \mu_{R_2}}^2 \sigma_{R_1}^2 + \left(\frac{\partial K_V}{\partial \mu_{R_2}} \right)_{\mu_{R_1}, \mu_{R_2}}^2 \sigma_{R_2}^2 + 2 \left(\frac{\partial K_V}{\partial \mu_{R_1}} \right)_{\mu_{R_1}, \mu_{R_2}} \left(\frac{\partial K_V}{\partial \mu_{R_2}} \right)_{\mu_{R_1}, \mu_{R_2}} \rho_{R_1 R_2} \sigma_{R_1} \sigma_{R_2}$$

If the indicated operations are carried out, it is seen that

$$\sigma_{K_V}^2 = \frac{\mu_{R_2}^2}{(\mu_{R_1} + \mu_{R_2})^4} \sigma_{R_1}^2 + \frac{\mu_{R_1}^2}{(\mu_{R_1} + \mu_{R_2})^4} \sigma_{R_2}^2 - 2 \frac{\mu_{R_1} \mu_{R_2}}{(\mu_{R_1} + \mu_{R_2})^4} \rho_{R_1 R_2} \sigma_{R_1} \sigma_{R_2} \quad (4-7)$$

If, as was discussed in Sec. 4-2, the values of resistance are uncorrelated then this expression reduces to

$$\sigma_{K_V}^2 = \frac{\mu_{R_2}^2}{(\mu_{R_1} + \mu_{R_2})^4} \sigma_{R_1}^2 + \frac{\mu_{R_1}^2}{(\mu_{R_1} + \mu_{R_2})^4} \sigma_{R_2}^2 \quad (4-8)$$

4-4. Single Stage Transistor Amplifier

In order to illustrate how the techniques of Chapter 3 may be applied to a transistor circuit, consider the transistor amplifier stage whose mid-frequency equivalent circuit is shown in Fig. 4-2 (Ref. 8).

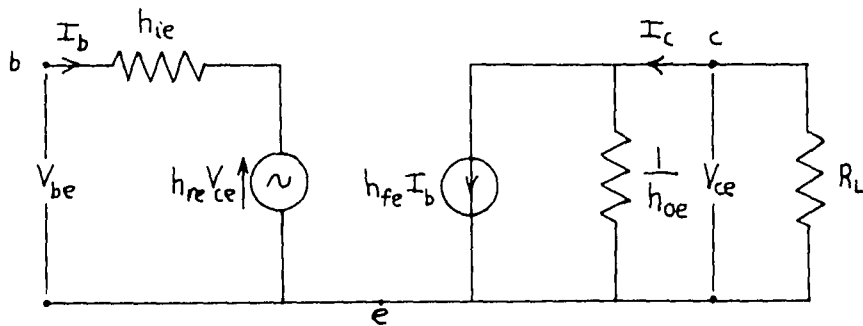


Fig. 4-2. Mid-frequency equivalent circuit of a transistor amplifier.

The symbols used on this circuit are defined as follows:

h_{ie} , h_{re} , h_{fe} and h_{oe} are the common emitter h parameters of the transistor.

R_L is the load resistance.

I_b is the ac base current and also the input signal current.

I_c is the ac collector current and also the output signal current.

V_{be} is the ac base voltage and is also the input signal voltage.

V_{ce} is the ac collector voltage and is also the output signal voltage.

The current gain of this amplifier is given by

$$K_i = \frac{I_c}{I_b} = \frac{h_{fe}}{1 + h_{oe}R_L} \quad (4-9)$$

The mean value of the current gain is, by application of Eq. (3-6),

$$\mu_{K_i} = \frac{\mu_{h_{fe}}}{1 + \mu_{h_{oe}} \mu_{R_L}} \quad (4-10)$$

Use of Eq. (3-7) shows that the variance of the current gain is

$$\begin{aligned} \sigma_{K_i}^2 &= \left(\frac{\partial K_i}{\partial h_{fe}} \right)^2 \sigma_{h_{fe}}^2 + \left(\frac{\partial K_i}{\partial h_{oe}} \right)^2 \sigma_{h_{oe}}^2 + \left(\frac{\partial K_i}{\partial R_L} \right)^2 \sigma_{R_L}^2 \\ &\quad + 2 \left(\frac{\partial K_i}{\partial h_{fe}} \right) \left(\frac{\partial K_i}{\partial h_{oe}} \right) \rho_{h_{fe}h_{oe}} \sigma_{h_{fe}} \sigma_{h_{oe}} \end{aligned}$$

where each partial derivative is evaluated at the mean values of the amplifier parameters. This expression in turn yields

$$\sigma_{K_i}^2 = \frac{1}{(1 + \mu_{hoe} \mu_{R_L})^2} \sigma_{h_{fe}}^2 + \frac{\mu_{h_{fe}}^2 \mu_{R_L}^2}{(1 + \mu_{hoe} \mu_{R_L})^4} \sigma_{hoe}^2 + \frac{\mu_{h_{fe}}^2 \mu_{hoe}^2}{(1 + \mu_{hoe} \mu_{R_L})^4} \sigma_{R_L}^2 \quad (4-11)$$

$$- 2 \frac{\mu_{h_{fe}} \mu_{R_L}}{(1 + \mu_{hoe} \mu_{R_L})^3} \rho_{h_{fe} hoe} \sigma_{h_{fe}} \sigma_{hoe}$$

In this analysis it has been assumed that the h parameters of the transistor are correlated with a coefficient of linear correlation $\rho_{h_{fe} hoe}$. The value of resistance R_L and the h parameters, however, have been assumed to be uncorrelated.

4-5. Correlation Between the z Parameters in Terms of Correlation Between the h Parameters

At the present state of the art there is very little data available on the correlation between transistor parameters. Furthermore, much of the data that is available is in terms of h parameters. On the other hand, it is desirable to be able to exercise some flexibility with respect to the choice of parameters used to analyze a circuit. For example, con-

sider the circuit of Fig. 4-2. If this circuit contained an unbypassed emitter resistor, it could be analyzed most easily by use of z parameters. This is true because the addition of the emitter resistor can be taken into account by simply adding the value of this resistance to each of the z parameters (Ref. 17).

In view of this situation it would be desirable to be able to determine the correlation among any set of parameters in terms of the correlation among the h parameters. Fortunately, this can be done by use of Eq. (3-18).

As an example of this application of Eq. (3-18) consider the circuit of Fig. 4-2. The current gain of this amplifier in terms of z parameters is (Ref. 19)

$$K_i = \frac{-z_{21e}}{z_{22e} + R_L} \quad (4-12)$$

From this expression it can be seen, if the variance of this current gain is to be obtained, that the coefficient of linear correlation between z_{21e} and z_{22e} is required. These z parameters can be expressed in terms of h parameters as follows (Ref. 18):

$$z_{21e} = - \frac{h_{21e}}{h_{22e}} \quad (4-13)$$

$$z_{22e} = \frac{1}{h_{22e}} \quad (4-14)$$

Now the use of Eq. (3-17) to obtain the covariance between z_{21e} and z_{22e} shows that

$$\begin{aligned} \sigma_{z_{21e}z_{22e}} = & \left(\frac{\partial z_{21e}}{\partial h_{21e}} \frac{\partial z_{22e}}{\partial h_{21e}} \right) \sigma_{h_{21e}}^2 + \left(\frac{\partial z_{21e}}{\partial h_{22e}} \frac{\partial z_{22e}}{\partial h_{22e}} \right) \sigma_{h_{22e}}^2 + \left(\frac{\partial z_{21e}}{\partial h_{21e}} \frac{\partial z_{22e}}{\partial h_{22e}} \right. \\ & \left. + \frac{\partial z_{21e}}{\partial h_{22e}} \frac{\partial z_{22e}}{\partial h_{21e}} \right) \rho_{h_{21e}h_{22e}} \sigma_{h_{21e}} \sigma_{h_{22e}} \end{aligned}$$

where all the partial derivatives are evaluated at the mean values of h_{21e} and h_{22e} . This expression in turn yields

$$\sigma_{z_{21e}z_{22e}} = - \frac{\mu_{h_{21e}}}{\mu_{h_{22e}}^4} \sigma_{h_{22e}}^2 + \frac{1}{\mu_{h_{22e}}^3} \rho_{h_{21e}h_{22e}} \sigma_{h_{21e}} \sigma_{h_{22e}} \quad (4-15)$$

The variances of z_{21e} and z_{22e} can be obtained by application of Eq. (3-7) to Eqs. (4-13) and (4-14) respectively. Thus these variances are given by

$$\sigma_{z_{21e}}^2 = \frac{1}{\mu_{h_{22e}}^2} \sigma_{h_{21e}}^2 + \frac{\mu_{h_{21e}}^2}{\mu_{h_{22e}}^4} \sigma_{h_{22e}}^2 - 2\rho_{h_{21e}h_{22e}} \frac{\mu_{h_{21e}} \sigma_{h_{21e}} \sigma_{h_{22e}}}{\mu_{h_{22e}}^3} \quad (4-16)$$

and

$$\sigma_{z_{22e}}^2 = \frac{1}{\mu_{h_{22e}}^4} \sigma_{h_{22e}}^2 \quad (4-17)$$

Substituting Eqs. (4-15), (4-16), and (4-17) into and simplifying gives

$$\rho_{z_{21e}z_{22e}} = \frac{-\mu_{h_{21e}} \sigma_{h_{22e}}^2 + \rho_{h_{21e}h_{22e}} \mu_{h_{22e}} \sigma_{h_{21e}}}{\left(\mu_{h_{22e}}^2 \sigma_{h_{21e}}^2 + \mu_{h_{21e}}^2 \sigma_{h_{22e}}^2 - 2\rho_{h_{21e}h_{22e}} \mu_{h_{21e}} \mu_{h_{22e}} \sigma_{h_{21e}} \sigma_{h_{22e}} \right)^{1/2}} \quad (4-18)$$

This expression is of the desired form; it gives the coefficient of linear correlation between z_{21e} and z_{22e} in terms of information describing h_{21e} and h_{22e}

4-6 Single Loop Feedback Amplifier System

The techniques of Chapter 3 can also be applied to a single feedback system.

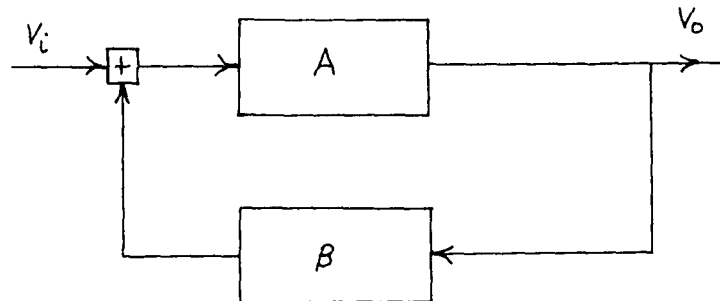


Fig. 4-3. Single loop feedback amplifier system.

In this diagram A is the voltage gain of the amplifier without feedback and β is the fraction of the output voltage which is feedback. The voltage gain of this system can be shown to be

$$K_V = \frac{A}{1 - A\beta} \quad (4-19)$$

Application of Eq. (3-6) to this expression shows that the mean value of the voltage gain is

$$\mu_{K_V} = \frac{\mu_A}{1 - \mu_A \mu_\beta} \quad (4-20)$$

By using Eq. (3-7) the variance of the voltage gain may be written as

$$\sigma_{K_V}^2 = \left(\frac{\partial K_V}{\partial A} \right)_{\mu_A, \mu_B}^2 \sigma_A^2 + \left(\frac{\partial K_V}{\partial \beta} \right)_{\mu_A, \mu_B}^2 \sigma_B^2 + 2 \left(\frac{\partial K_V}{\partial A} \right)_{\mu_A, \mu_B} \left(\frac{\partial K_V}{\partial \beta} \right)_{\mu_A, \mu_B} \rho_{AB} \sigma_A \sigma_B$$

which then yields

$$\sigma_{K_V}^2 = \frac{\sigma_A^2 + \mu_A^4 \sigma_B^2 + 2\rho_{AB} \mu_A^2 \sigma_A \sigma_B}{(1 - \mu_A \mu_B)^4} \quad (4-21)$$

- This expression can also be used for the situation where A and β are uncorrelated, if ρ_{AB} is set equal to zero.

4-7. Cascade System

Another commonly used system to which the methods of Chapter 3 can be applied is the cascade system. The gain of a cascade system is equal to the product of the gains of the individual stages. Thus if n

stages are placed in cascade the overall gain is given by

$$K = \prod_{j=1}^n K_j \quad (4-22)$$

where K_j is the gain of the j^{th} stage.

By use of Eq. (3-6), the mean value of the overall gain is found to be

$$\mu_K = \prod_{j=1}^n \mu_{K_j} \quad (4-23)$$

The variance of the overall gain is obtained by application of Eq. (3-7) which results in

$$\sigma_K^2 = \sum_{l=1}^n \prod_{\substack{j=1 \\ j \neq l}}^n \mu_{K_j}^2 \sigma_{K_l}^2 + 2 \sum_{j=1}^{n-1} \sum_{m=j+1}^n \rho_{j,m} \prod_{\substack{l=1 \\ l \neq j,m}}^n \mu_{K_l}^2 \mu_{K_j} \mu_{K_m} \sigma_{K_j} \sigma_{K_m} \quad (4-24)$$

When there is no correlation between the gain of any of the stages, $\rho_{j,m}$ becomes zero for all combinations of j and m .

4-8. The Probability of the System Performance Being Within Tolerance

The principal reason for obtaining the means and the variances of the system performance criteria is so that the probability of these criteria being within tolerance can be determined. Since the above expressions represent the initial values of the means and variances, they can be used to determine the initial probabilities of performance criteria being within tolerance.

As an example, assume that the mean and the variance of the current gain of an amplifier are given by μ_{K_1} and $\sigma_{K_1}^2$, respectively. Furthermore, assume that the distributions of the amplifier parameters are

truncated sufficiently far from their mean values, so that the current gain is normally distributed.

Now suppose that the current gain must be between the limits of K_{i1} and K_{i2} if the system is to be considered operating. The probability of the system operating is then given by

$$\Pr(K_{i1} \leq K_i \leq K_{i2}) = \int_{K_{i1}}^{K_{i2}} \frac{1}{\sigma_{K_i} \sqrt{2\pi}} e^{-\frac{(K_i - \mu_{K_i})^2}{2\sigma_{K_i}^2}} dK_i \quad (4-25)$$

This integral can be put in a form so that it can be obtained from a unit normal distribution table by the following transformations:

$$A = \frac{K_i - \mu_{K_i}}{\sigma_{K_i}} \quad (4-26)$$

$$A_1 = \frac{K_{i1} - \mu_{K_i}}{\sigma_{K_i}} \quad (4-27)$$

$$A_2 = \frac{K_{i2} - \mu_{K_i}}{\sigma_{K_i}} \quad (4-28)$$

This results in

$$\Pr(K_{i1} \leq \mu_{K_i} \leq K_{i2}) = \int_{K_{i1}}^{K_{i2}} \frac{1}{\sqrt{2\pi}} e^{-R^2/2} dR \quad (4-29)$$

which can be evaluated by use of any unit normal distribution table.

It should be pointed out that there exist several commonly used design criteria which make the above probability unity. One such criteria is the worst case method, which insures that the system performance be within specifications, even when all the system parameters assume the worst possible values permitted by their tolerances. This method is advantageous when only a few units of a system are being produced and no defective systems can be tolerated. Furthermore, this method is more generally applicable because it is not limited by the restrictions previously stated for the statistical method. For mass produced systems, however, the worst case method is often not economically feasible as it usually results in too conservative a design.

CHAPTER 5: MINIMIZATION OF THE VARIANCES OF THE
DISTRIBUTIONS OF FUNCTIONS OF RANDOM VARIABLES

5-1. Introduction

Investigation of the example in Sec. 4-8 reveals that an increase in the variance of a system performance criterion generally¹ means an increase in the probability of the criterion being out of tolerance. Hence, it will be advantageous to minimize the variances of the system performance criteria. From Eq. (3-7) it will be seen that the variances of these criteria are often functions of the mean values of the parameters of the systems. Furthermore, there are generally an unlimited number of combinations of parameter mean values, which will make the mean of a particular performance criterion have the value required by the specifications of the system. Therefore, that combination of parameter means which yields the minimum variance of the performance criterion, while still giving the mean of the performance criterion its desired value, should be used.

¹This statement is true for most practical cases. However, it would not be true, for example, if the mean value of the criterion was outside of the specified limits of this criterion.

This chapter will investigate the variance minimization of three elementary functions; while Chapter 6 will apply this technique to a number of electronic systems.

5-2. The Standard Deviation to Mean Ratio of System Parameters²

In determining the combination of parameter means required in the above analysis, it should be noted that Eq. (3-7) shows that the variance of the performance criterion depends upon the standard deviations of the parameters and coefficients of linear correlation between them, as well as the parameter means. It will be assumed that the coefficients of linear correlation are independent of the parameter means. On the other hand, this assumption cannot be made with respect to the variances of the parameters. Clearly, the larger the mean value of a parameter, the larger its variance. In this work it will be assumed that the standard deviation of a parameter is directly proportional to its mean value.

²This is also the standard deviation of a_{j0} in Eq. (2-14), but the symbol c_j will be used for this quantity in the present context.

The physical meaning of the above assumption is that the standard deviation to mean ratio for a particular type of device is independent of the mean value(s) of the device parameter(s). For example, consider the following density functions of two resistors:

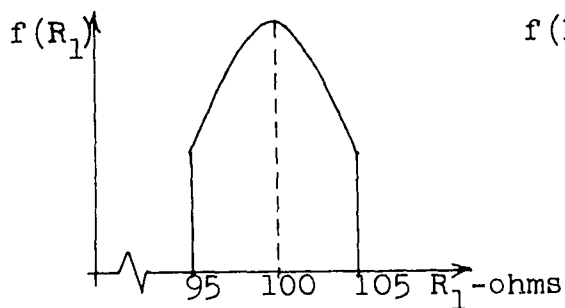


Fig. 5-1. Density function of a 100-ohm 5% resistor with a high variance.

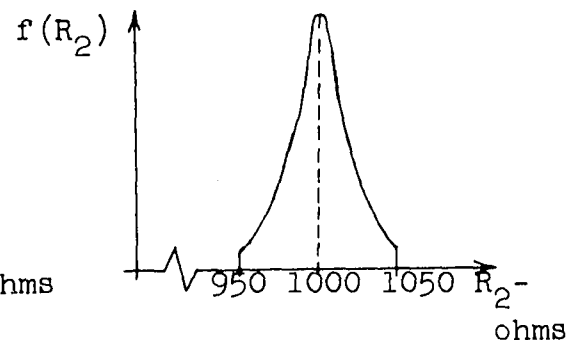


Fig. 5-2. Density function of a 1000-ohm 5% resistor with a low variance.

Obviously, the above assumption would not be valid for this situation though both resistors have the same tolerance, because the standard deviation to mean ratio of the manufacturing processes are different. Thus it is seen that it is a combination of the manufactured standard deviation to mean ratio and the tolerance which determines the standard deviation to mean ratio for a parameter.

Though theoretically this hypothesis may appear very limiting indeed from a purely mathematical point of view, there are various practical engineering situations for which the above assumption is satisfied. The most obvious situation is the case where the standard deviation to parameter mean ratio of the manufacturing process is independent of the parameter mean and the distributions corresponding to each nominal value are truncated to the same degree. An example of this is provided by the distributions of 5% carbon composition resistors such as the IRC type GBT. For this type of resistor the manufacturer will usually remove from the manufactured distribution those resistors which are within 5% of the required nominal value (which is approximately the mean of the manufactured distribution) and label them as being 5% tolerance (Ref. 33).

Resistors of 1% tolerance may be obtained by selecting those resistors which are within 1% of the required nominal value. If this nominal value is sufficiently close to the mean value of the manufactured distribution, then the situation will be identical to that of the 5% resistors discussed above and the standard deviation to mean ratio will be constant. On the other hand, 1% resistors are sometimes obtained by

selecting those resistors which are within 1% of the respective nominal values from appropriate positions on the manufactured distributions for various nominal values. Since each 1% value will then contain units from a number of the manufactured distributions, these 1% values may well have a uniform distribution. This procedure will then yield a constant standard deviation to mean ratio. Unfortunately, however, all the units of each nominal value are not always mixed. Thus, a customer may receive units drawn from only one distribution, in which case the standard deviation to mean ratio will not be a constant, since the distribution of each nominal value will then be a different section of a normal distribution.

As discussed in Sec. 2-3 a manufacturer will in some instances obtain a number of different nominal parameter values from a single normal distribution. In this case the distribution of each parameter will be a section of a normal distribution. Since each of these sections usually includes an equal range of values of the parameter, the standard deviation to mean ratio will vary with the nominal value of the parameter.

In general, manufacturers will not supply the customer with any specific parameter distribution information beyond the nominal value and the tolerance. The reason that this vital information is usually withheld is that it forms part of the cost-price structure of a device and thus would be valuable to the manufacturer's competitors. Thus, if a customer requires this information under present market conditions, he will have to obtain it by making measurements himself. This has been done in some cases and a set of results of such measurements is given in the tables below.

The data presented in these tables has its limitations. For example, the resistor data was taken from one production run and is not necessarily representative of what a customer might receive from a vendor. The transistor data leaves much to be desired in that not enough types of transistors were tested to establish any connection between the standard deviation to mean ratio and transistor types. Furthermore there is no evidence that the data was taken on a representative sample of transistors. Nevertheless, the resistor data does give strong evidence that the standard deviation to mean ratio is constant for the type of

resistor tested; while the transistor data gives some hope that this relationship can be established for various types of transistors.

Table of Standard Deviation to Mean Ratios for IRC Type GBT-1/2 Carbon Composition Resistors (Ref. 42)

Nominal value in Ohms	330	390	470	4.7K	68K	27K	47K
Sample Size	767	341	353	369	363	350	354
σ/μ	0.0242	0.0224	0.0238	0.0283	0.0242	0.0234	0.0206

Table of Standard Deviation to Mean Ratios for h_{fe} of a Number of Types of Transistors (Ref. 43)

Transistor Type	2N220 PNP German. Alloy Junction	2N240 PNP German. Surface Barrier	2N343 NPN Silicon	2N128 PNP German. Surface Barrier	2N299 PNP German. Surface Barrier	2N384 PNP German. Drift
Mean h_{fe}	77.7	39.2	51.8	34.1	16.8	77.6
Sample Size	39	109	36	30	35	39
σ/μ	0.157	0.222	0.224	0.236	0.271	0.385

5-3 Minimization of a Function of n Variables

In order to minimize the variances of system performance criteria, a technique will be required to minimize a function of a number of variables with the constraint that another function of these variables have a prescribed value. Such a technique has been developed (Ref. 1).

This technique will be explained with reference to the following function of n variables:

$$y = g(x_1, x_2, \dots, x_n) \quad (5-1)$$

The necessary conditions for y to have a minimum at the point whose coordinates are $(x_{10}, x_{20}, \dots, x_{n0})$ are given by the following equations (Ref. 15):

$$\begin{aligned} g_{x_1}(x_{10}, x_{20}, \dots, x_{n0}) &= 0 \\ g_{x_2}(x_{10}, x_{20}, \dots, x_{n0}) &= 0 \\ \text{-----} & \\ g_{x_j}(x_{10}, x_{20}, \dots, x_{n0}) &= 0 \\ \text{-----} & \\ g_{x_n}(x_{10}, x_{20}, \dots, x_{n0}) &= 0 \end{aligned} \quad (5-2)$$

The notation used in these equations is

$$g_{x_j}(x_{10}, x_{20}, \dots, x_{n0}) \equiv \left. \frac{\partial g(x_1, x_2, \dots, x_n)}{\partial x_j} \right|_{x_{10}, x_{20}, \dots, x_{n0}} \quad (5-3)$$

The sufficient conditions for the point $(x_{10}, x_{20}, \dots, x_{n0})$ to be a minimum of y have been derived (Ref. 1). These conditions may be explained with reference to a matrix $\|G\|$ which is defined as follows:

$$\|G\| = \begin{vmatrix} g_{x_1 x_1} & g_{x_1 x_2} & \dots & g_{x_1 x_j} & \dots & g_{x_1 x_n} \\ g_{x_2 x_1} & g_{x_2 x_2} & \dots & g_{x_2 x_j} & \dots & g_{x_2 x_n} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ g_{x_l x_1} & g_{x_l x_2} & \dots & g_{x_l x_j} & \dots & g_{x_l x_n} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ g_{x_n x_1} & g_{x_n x_2} & \dots & g_{x_n x_j} & \dots & g_{x_n x_n} \end{vmatrix} \quad (5-4)$$

The notation in this case is

$$g_{x_l x_j} = \frac{\partial^2 g(x_1, x_2, \dots, x_n)}{\partial x_l \partial x_j} \bigg|_{x_{10}, x_{20}, \dots, x_{n0}} \quad (5-5)$$

Now let $\|G\|_j$ be the determinant obtained from $\|G\|$ when the last j rows and the last j columns are deleted. The necessary conditions for y to have a minimum at the point $(x_{10}, x_{20}, \dots, x_{n0})$ are that $|G|_0, |G|_1, |G|_2, \dots, |G|_{n-1}$ all be positive.

This result can easily be specialized to yield the sufficient conditions for a minimum of functions of one and two variables. For the case where $n = 1$, the result is

$$g_{x_1 x_1} > 0 \quad (5-6)$$

For $n = 2$ the conditions are:

$$g_{x_1 x_1} > 0 \quad (5-7)$$

$$g_{x_1 x_1} g_{x_2 x_2} - g_{x_1 x_2}^2 > 0 \quad (5-8)$$

In obtaining Eq. (5-8), the fact that $g_{x_1 x_2} = g_{x_2 x_1}$ was used.

The above results can be modified to include the constraint that another function of (x_1, x_2, \dots, x_n) must have a particular value, L . Suppose that this constraint is stated as follows:

$$\phi(x_1, x_2, \dots, x_n) = L \quad (5-9)$$

This expression may now be solved for x_n and the result used to eliminate x_n from Eq. (5-1). The above procedure for obtaining the necessary and sufficient conditions for a minimum of y can now be applied again. In this case there are only $n-1$ variables.

In some situations it is impossible to solve Eq. (5-9) for any of the x 's. In this event a more general procedure for the minimization of a function with a constraint must be used. One such procedure is the method of Lagrange multipliers (Ref. 1). The above difficulty did not arise in this research, however, and thus this latter method will not be discussed.

5-4. Minimization of the Variance of a Linear Function of Uncorrelated Random Variables

The technique of the preceding section will now be used to minimize the variance of a linear function of n uncorrelated random variables. The following function will be studied:

$$Y = \sum_{j=1}^n b_j x_j \quad (5-10)$$

In this expression the b_j 's are constants and the x_j 's are random variables. This equation can also be written as

$$Y = \sum_{j=1}^{n-1} b_j x_j + b_n x_n \quad (5-11)$$

By application of Eq. (3-6) the mean value of this function is seen to be

$$\mu_y = \sum_{j=1}^{n-1} b_j \mu_{x_j} + b_n \mu_{x_n} \quad (5-12)$$

and by Eq. (3-7) the variance of y is

$$\sigma_y^2 = \sum_{j=1}^{n-1} b_j^2 \sigma_{x_j}^2 + b_n^2 \sigma_n^2 \quad (5-13)$$

Assuming that the standard deviation to mean ratio of the j^{th} term is c_j , this expression can be rewritten as

$$\sigma_y^2 = \sum_{j=1}^{n-1} b_j^2 c_j^2 \mu_{x_j}^2 + b_n^2 c_n^2 \mu_{x_n}^2 \quad (5-14)$$

Now suppose that this variance is to be minimized with respect to the mean values of the x's with the constraint that the mean value of y be given by Eq. (5-12). This constraint means that Eq. (5-12) can be used to eliminate one of the mean values of the x's from Eq. (5-14). Solving Eq. (5-12) for μ_{x_n} and substituting the result into Eq. (5-14) gives

$$\sigma_y^2 = \sum_{j=1}^{n-1} b_j^2 c_j^2 \mu_{x_j}^2 + c_n^2 \left(\mu_y - \sum_{j=1}^{n-1} b_j \mu_{x_j} \right)^2 = 0 \quad (5-15)$$

The variance is now a function of the mean values $\mu_{x_1}, \mu_{x_2}, \dots, \mu_{x_{n-1}}$. Setting the derivatives of σ_y^2 with respect to each of these means equal to zero yields $n-1$ equations. These equations can be solved simultaneously for $\mu_{x_1}, \mu_{x_2}, \dots, \mu_{x_{n-1}}$. The expressions for these means can then be substituted into Eq. (5-12) which may then be solved for μ_n . The details of these manipulations are given in Appendix A. The result is

$$\mu_{x_j} = \frac{\prod_{\substack{m=1 \\ m \neq j}}^n c_m^2}{b_j \sum_{q=1}^n \prod_{\substack{m=1 \\ m \neq q}}^n c_m^2} \mu_y \quad (5-16)$$

It is also shown in Appendix A that the value of σ_y^2 defined by Eq. (5-16) meets the sufficient conditions for a minimum of σ_y^2 . From Appendix A the minimum σ_y^2 is

$$\sigma_y^2 = \frac{\prod_{l=1}^n c_l^2}{\sum_{q=1}^n \prod_{\substack{m=1 \\ m \neq q}}^n c_m^2} \mu_y^2 \quad (5-17)$$

If all the c's are equal, this expression becomes

$$\sigma_y^2 = \frac{c^{2n}}{n c^{2(n-1)}} \mu_y^2 = \frac{c^2}{n} \mu_y^2 \quad (5-18)$$

The standard deviation to mean ratio is then

$$\frac{\sigma_y}{\mu_y} = \frac{c}{\sqrt{n}} \quad (5-19)$$

Thus, if y has n components and the parameter means of the components are chosen so that σ_y^2 is a minimum, the standard deviation to mean ratio of y is smaller than the same ratio for one of the components of y by a factor of \sqrt{n} .

5-5. Minimization of the Variance of a Linear Function of Correlated Random Variables

In this section the analysis of Sec. 5-4 will be extended to the case where there is correlation between the variables. Theoretically, the problem can be solved for the general case of a linear function of n correlated variables. As a practical matter, however, this is not feasible since the equation corresponding to Eq. (A-3) for the uncorrelated case will have coefficients which are complicated functions of the coefficients of linear correlation among the variables. Consequently, it is better to handle each case as it arises.

As an illustration, the following function of two variables will be considered:

$$y = b_1 x_1 + b_2 x_2 \quad (5-20)$$

Application of Eq. (3-6) reveals that the mean value of y is

$$\mu_y = b_1 \mu_{x_1} + b_2 \mu_{x_2} \quad (5-21)$$

By Eq. (3-7) the variance of y is

$$\sigma_y^2 = b_1^2 \sigma_{x_1}^2 + b_2^2 \sigma_{x_2}^2 + 2\rho_{x_1 x_2} b_1 b_2 \sigma_{x_1} \sigma_{x_2} \quad (5-22)$$

If c_1 and c_2 are used to denote the standard deviation to mean ratio of x_1 and x_2 respectively, then Eq. (5-22) becomes

$$\sigma_y^2 = b_1^2 c_1^2 \mu_{x_1}^2 + b_2^2 c_2^2 \mu_{x_2}^2 + 2 r_{x_1 x_2} b_1 b_2 c_1 c_2 \mu_{x_1} \mu_{x_2} \quad (5-23)$$

Substituting Eq. (5-21) into Eq. (5-23) gives

$$\begin{aligned} \sigma_y^2 = & b_1^2 (c_1^2 + c_2^2 - 2 r_{x_1 x_2} c_1 c_2) \mu_{x_1}^2 \\ & - 2 b_1 \mu_y (c_2^2 - r_{x_1 x_2} c_1 c_2) \mu_{x_1} + c_2^2 \mu_y^2 \end{aligned} \quad (5-24)$$

Differentiating with respect to μ_{x_1} yields

$$\frac{\partial \sigma_y^2}{\partial \mu_{x_1}} = 2 b_1^2 (c_1^2 + c_2^2 - 2 r_{x_1 x_2} c_1 c_2) \mu_{x_1} - 2 b_1 \mu_y (c_2^2 - r_{x_1 x_2} c_1 c_2) \quad (5-25)$$

If this derivative is set equal to zero and the resulting equation is solved for μ_{x_1} , the following expression is obtained:

$$\mu_{x_1} = \frac{(c_2^2 - r_{x_1 x_2} c_1 c_2)}{b_1 (c_1^2 + c_2^2 - 2 r_{x_1 x_2} c_1 c_2)} \mu_y \quad (5-26)$$

Substituting this expression for μ_{x_1} into Eq. (5-21) yields the following expression for μ_{x_2} :

$$\mu_{x_2} = \frac{(c_1^2 - \rho_{x_1 x_2} c_1 c_2)}{b_2 (c_1^2 + c_2^2 - 2\rho_{x_1 x_2} c_1 c_2)} \mu_y \quad (5-27)$$

Differentiating Eq. (5-25) with respect to μ_{x_1} gives

$$\frac{\partial^2 \sigma_y^2}{\partial \mu_{x_1}^2} = 2b_1^2 (c_1^2 + c_2^2 - 2\rho_{x_1 x_2} c_1 c_2) \quad (5-28)$$

If μ_{x_1} as given by Eq. (5-26) defines a minimum of σ_y^2 then the right hand side of Eq. (5-28) must be positive. Since $-1 \leq \rho_{x_1 x_2} \leq 1$ (Ref. 24)

$\partial^2 \sigma_y^2 / \partial \mu_{x_1}^2$ will assume its most negative value with respect to $\rho_{x_1 x_2}$ when $\rho_{x_1 x_2} = 1$. In this case,

Eq. (5-28) becomes

$$\frac{\partial^2 \sigma_y^2}{\partial \mu_{x_1}^2} = 2b_1^2 (c_1^2 + c_2^2 - 2c_1 c_2) = 2b_1^2 (c_1 - c_2)^2 \quad (5-29)$$

Hence $\partial^2 \sigma_y^2 / \partial \mu_{x_1}^2$ is positive for all possible values of $\rho_{x_1 x_2}$ and thus Eq. (5-26) defines a minimum of σ_y^2 .

5-6. Minimization of the Variance of the Product of Random Variables

The technique of variance minimization will now be applied to a function which is the product of random variables. This will be done by considering the following function of n correlated random variables:

$$Y = \prod_{j=1}^n x_j \quad (5-30)$$

Application of Eq. (3-6) to this function shows that the mean value of y is

$$\mu_y = \prod_{j=1}^n \mu_{x_j} \quad (5-31)$$

Use of Eq. (3-7) shows that the variance of y is given by

$$\sigma_y^2 = \sum_{j=1}^n \prod_{\substack{l=1 \\ l \neq j}}^n \mu_{x_l}^2 \sigma_{x_j}^2 + 2 \sum_{j=1}^{n-1} \sum_{l=j+1}^n \rho_{x_j x_l} \prod_{\substack{m=1 \\ m \neq j}}^n \mu_{x_m} \prod_{\substack{q=1 \\ q \neq l}}^n \mu_{x_q} \sigma_{x_j} \sigma_{x_l} \quad (5-32)$$

Assuming that the standard deviation to mean ratio for the j^{th} variable is a constant c_j , this equation may be rewritten as

$$\sigma_y^2 = \sum_{j=1}^n \prod_{l=1}^n \mu_{x_l}^2 c_j^2 + 2 \sum_{j=1}^{n-1} \sum_{l=j+1}^n \rho_{x_j x_l} \prod_{m=1}^n \mu_{x_m}^2 c_j c_l \quad (5-33)$$

Factoring this equation and using Eq. (5-31) leads to

$$\sigma_y^2 = \mu_y^2 \left[\sum_{j=1}^n c_j^2 + 2 \sum_{j=1}^{n-1} \sum_{l=j+1}^n \rho_{x_j x_l} c_j c_l \right] \quad (5-34)$$

This result shows that once the mean value of y is specified, the variance of y is independent of the mean values of the x 's. Hence it is not possible to minimize the variance of the product of correlated random variables by adjusting the mean values of the various factors. It should also be noted that since the case of the product of uncorrelated random variables can be treated by letting $\rho_{j\ell} = 0$ in Eq. (5-34), the above result applies equally well to the uncorrelated case.

5-7. Minimization of the Variance of the Quotient of Random Variables

In this section variance minimization will be applied to a function which is the quotient of a number of random variables. The following function of correlated random variables will be used:

$$y = \frac{\prod_{j=1}^n x_j}{\prod_{l=n+1}^w x_l} \quad (5-35)$$

Eq. (3-6) shows that the mean value of y is

$$\mu_y = \frac{\prod_{j=1}^n \mu_{x_j}}{\prod_{l=n+1}^w \mu_{x_l}} \quad (5-36)$$

Use of Eq. (3-7) shows that the variance of y may be written as:

$$\begin{aligned}
\sigma_y^2 = & \frac{1}{\prod_{l=n+1}^w \mu_{x_l}^2} \left[\sum_{j=1}^n \prod_{\substack{q=1 \\ q \neq j}}^n \mu_{x_q}^2 \sigma_{x_j}^2 \right] + \prod_{j=1}^n \mu_{x_j}^2 \left[\sum_{l=n+1}^w \prod_{q=n+1}^w \frac{1}{\mu_{x_q}^2 \mu_{x_l}^2} \sigma_{x_l}^2 \right] \\
& + 2 \frac{1}{\prod_{l=n+1}^w \mu_{x_l}^2} \left[\sum_{j=1}^{n-1} \sum_{q=j+1}^n \rho_{x_j x_q} \prod_{\substack{r=1 \\ r \neq j}}^n \mu_{x_r} \prod_{\substack{s=1 \\ s \neq q}}^n \mu_{x_s} \sigma_{x_j} \sigma_{x_q} \right] \\
& + 2 \left[\prod_{j=1}^n \mu_{x_j}^2 \sum_{l=n+1}^{w-1} \sum_{q=l+1}^w \rho_{x_l x_q} \prod_{r=n+1}^n \frac{1}{\mu_{x_r} \mu_{x_l}} \prod_{s=n+1}^n \frac{1}{\mu_{x_s} \mu_{x_q}} \sigma_{x_l} \sigma_{x_q} \right] \\
& - 2 \left[\sum_{j=1}^n \sum_{l=n+1}^w \rho_{x_j x_l} \frac{\prod_{\substack{r=1 \\ r \neq j}}^n \mu_{x_r} \prod_{r=1}^n \mu_{x_r}}{\prod_{q=n+1}^w \mu_{x_q} \prod_{q=n+1}^w \mu_{x_q} \mu_{x_l}} \sigma_{x_j} \sigma_{x_l} \right] \quad (5-37)
\end{aligned}$$

The respective terms of this equation represent the following components of σ_y^2 : the components due to the variables in the numerator of y , the components due to the variables in the denominator of y , the components due to correlation between the variables in

the numerator of y , the components due to correlation between the variables in the denominator of y and the components due to correlation between the variables in the numerator of y and the variables in the denominator of y .

This equation can be simplified by representing the standard deviation to mean ratio of the respective variables by the symbol c and then factoring. This gives

$$\sigma_y^2 = \frac{\prod_{g=1}^m \mu_{x_g}^2}{\prod_{l=m+1}^n \mu_{x_l}^2} \left[\sum_{j=1}^n c_j^2 + \sum_{l=n+1}^w c_l^2 + 2 \sum_{j=1}^{n-1} \sum_{g=j+1}^n \rho_{x_j x_g} c_j c_g \right. \\ \left. + 2 \sum_{l=n+1}^{w-1} \sum_{g=l+1}^w \rho_{x_l x_g} c_l c_g - 2 \sum_{j=1}^n \sum_{l=n+1}^w \rho_{x_j x_l} c_j c_l \right] \quad (5-38)$$

Now simplifying the term in brackets and using Eq.

(5-36) yields the following expression of σ_y^2 :

$$\sigma_y^2 = \mu_y^2 \left[\sum_{j=1}^w c_j^2 + 2 \sum_{j=1}^{n-1} \sum_{q=j+1}^n \rho_{x_j x_q} c_j c_q + 2 \sum_{l=n+1}^{w-1} \sum_{g=l+1}^w \rho_{x_l x_g} c_l c_g - 2 \sum_{j=1}^n \sum_{l=n+1}^w \rho_{x_j x_l} c_j c_l \right] \quad (5-39)$$

This result shows that if the mean value of y is specified, then the variance of y is independent of the mean values of the x 's. Hence, under these circumstances it is not possible to minimize the variance of the quotient of random variables with respect to the mean values of the variables. It should also be noted that this result applies to the situation where the variables are uncorrelated, since this case can be handled by simply letting $\rho_{x_j x_l} = 0$.

5-8. Minimization of the Variance of Other Functions of Random Variables

There are an unlimited number of functions to which the above minimization of variance procedure may be applied. In general, the procedure for a function of n variables is:

- (1). Use Eq. (3-7) to obtain the variance of the function.

(2). Assume that the standard deviations of the respective n variables are directly proportional to their mean values.

(3). Apply the constraint that the mean value of the function is specified by using Eq. (3-6) to eliminate the mean value of one of the variables in the expression for the variance of the function.

(4). Differentiate the resulting variance expression with respect to the mean value of each of the remaining $n-1$ variables.

(5). Form $n-1$ equations by setting each of these derivatives equal to zero.

(6). Solve these equations for the mean values of the $n-1$ variables.

(7). Use the technique of Sec. 5-3 to see if these mean values specify a minimum of the variance of the function.

(8). Obtain the mean value of the variable that was eliminated in Step 3 by substituting the above mean values into Eq. (3-6).

This procedure will not be applied to any more specific functions in this chapter. However, it will be used to minimize the variance of the performance criteria of a number of electronic systems in Chapter 6.

CHAPTER 6: MINIMIZATION OF THE VARIANCES
OF THE INITIAL DISTRIBUTIONS OF THE
PERFORMANCE CRITERIA OF ELECTRONIC SYSTEMS

6-1. Introduction

In Chapter 5 a technique for minimizing the variance of a function of random variables was developed. This method was then applied to three elementary functions. In this chapter the variances of the performance criteria of a number of electronic systems will be minimized. For some of these systems it will be possible to make use of the results obtained for the above-mentioned elementary functions; while in other cases the variance of the performance criteria under consideration will be minimized directly.

6-2. Resistors in Series

As an example of the application of variance minimization to an electronic system, the variance of the series resistance in Sec. 4-2 will be minimized. If it is assumed that the resistance values of the individual resistors are uncorrelated and that the standard deviation to mean value ratio of each of these resistances is c , then Eq. (4-4) for the total series resistance becomes

$$\sigma_R^2 = c^2 \sum_{j=1}^n \mu_{R_j}^2 \quad ((-1))$$

Now let it be required that this variance be minimized with respect to the mean values of the individual resistances, while still maintaining the mean of the total series resistance at its required value. This can be accomplished by applying the results of the minimization of the variance of a linear function of n variables in Sec. 5-4 to this situation. In this case all the b 's are unity and all the c 's are identical. Thus, from Eq. (5-16), the mean values of the resistances required to make the variance of R a minimum are given by

$$\mu_{R_j} = \frac{c^{2(n-1)}}{nc^{2(n-1)}} \mu_R = \frac{\mu_R}{n} \quad (6-2)$$

From Eq. (5-18) the minimum variance of R is

$$\sigma_R^2 = \frac{c^2}{n} \mu_R^2 \quad (6-3)$$

These results show that the minimum variance of R (when all the c 's are the same) is achieved when the mean values of the series resistors are all equal. In addition, it is seen that this procedure reduces the variance of R by a factor of n from what it would have been if R had consisted of just one resistor.

The above results do not necessarily mean that a number of resistors in series should be used in preference to a single resistor. Other factors to be considered are the cost of a number of relatively high tolerance resistors vs. one low tolerance resistor, the increased probability of a catastrophic failure¹ in the series circuit, and the higher assembly cost for the series circuit. In addition, the availability of components with low parameter tolerances must be considered. Therefore, the decision as whether to use a number of resistors in series or to use a single quality resistor must be made on the merits of each case.

¹This assumes that failure is due to an open circuit. If the mode of failure is by a short circuit, the series configuration is actually more reliable than a single resistor.

6-3. Voltage Divider

In this section the minimization of variance technique will be applied to the voltage divider studied in Sec. 4-3. The first step in this procedure will be to assume that

$$\sigma_{R_1} = c_1 \mu_{R_1} \quad (6-4)$$

$$\sigma_{R_2} = c_2 \mu_{R_2} \quad (6-5)$$

Substitution of these equations in Eq. (4-7) leads to

$$\sigma_{K_V}^2 = \frac{\mu_{R_1}^2 \mu_{R_2}^2}{(\mu_{R_1} + \mu_{R_2})^4} (c_1^2 + c_2^2 - 2\rho_{R_1 R_2} c_1 c_2) \quad (6-6)$$

Now suppose that the mean value of the divider ratio has been specified. This constraint can be applied to the problem by substituting Eq. (4-6) into Eq. (6-6). Thus the expression for the variance of the divider ratio becomes

$$\sigma_{K_V}^2 = \mu_{K_V}^2 (1 - \mu_{K_V})^2 (c_1^2 + c_2^2 - 2\rho_{R_1 R_2} c_1 c_2) \quad (6-7)$$

From this expression it is seen that the variance of the divider ratio is independent of the variances of the resistances once the mean value of the divider ratio and the c 's have been specified. Thus it is not

possible to minimize the variance of the divider ratio with respect to the mean values of the resistances under these circumstances². It should also be noted that a positive coefficient of linear correlation between R_1 and R_2 will reduce the variance of the divider ratio. Ordinarily, however, the resistors are selected at random and thus their values will not be correlated. A method for introducing correlation between uncorrelated parameters will be discussed in Chapter 7.

6-4 Single Stage Transistor Amplifier

The minimization of variance technique may also be applied to active circuits. This will be illustrated by minimizing the variance of the current gain of the single stage transistor amplifier discussed in Sec. 4-4. This minimization will be carried out under the assumption that the standard deviations of the various parameters may be expressed as follows:

$$\sigma_{h_{fe}} = c_f \mu_{h_{fe}} \quad (6-8)$$

$$\sigma_{h_{oe}} = c_o \mu_{h_{oe}} \quad (6-9)$$

$$\sigma_{R_L} = c_L \mu_{R_L} \quad (6-10)$$

²It should be noted that this variance can be minimized by using another method; namely, by considering this as a feedback system and minimizing the return difference (Ref. 4).

If these relationships are used in Eq. (4-11), the expression for the variance of the current gain becomes

$$\sigma_{K_i}^2 = \frac{\mu_{h_{fe}}^2}{(1 + \mu_{h_{oe}} \mu_{R_L})^2} \left[c_f^2 + \frac{\mu_R^2}{(1 + \mu_R)^2} (c_o^2 + c_L^2) - 2 \frac{\mu_R}{(1 + \mu_R)} \rho_{h_{fe} h_{oe}} c_f c_o \right] \quad (6-11)$$

where³

$$\mu_R \equiv \mu_{h_{oe}} \mu_{R_L} \quad (6-12)$$

Now assume that the mean value of the current gain is specified. This constraint can be applied by substitution of Eq. (4-10) for the mean value of the current gain into Eq. (6-11). This gives

$$\sigma_{K_i}^2 = \mu_{K_i}^2 \left[c_f^2 + (c_o^2 + c_L^2) \frac{\mu_R^2}{(1 + \mu_R)^2} - 2 \rho_{h_{fe} h_{oe}} c_f c_o \frac{\mu_R}{(1 + \mu_R)} \right] \quad (6-13)$$

³This definition is useful since $\mu_{h_{oe}}$ and μ_{R_L} only appear as the product $\mu_{h_{oe}} \mu_{R_L}$ in Eq. (6-11).

where $\mu_{h_{fe}}$ has been eliminated. This expression will now be minimized with respect to μ_R . Differentiating with respect to μ_R yields

$$\frac{\partial \sigma_{K_i}^2}{\partial \mu_R} = 2\mu_{K_i}^2 \left[(c_o^2 + c_L^2) \frac{\mu_R}{(1 + \mu_R)^3} - \rho_{h_{fe} h_{oe}} c_f c_o \frac{1}{(1 + \mu_R)^2} \right] \quad (6-14)$$

Setting this derivative equal to zero and solving for μ_R results in

$$\mu_R = \frac{c_f c_o \rho_{h_{fe} h_{oe}}}{c_o^2 + c_L^2 - c_f c_o \rho_{h_{fe} h_{oe}}} \quad (6-15)$$

and

$$\mu_R \rightarrow \infty \quad (6-16)$$

The latter value for μ_R will be discarded as being impractical, since an infinite μ_R would require an infinite $\mu_{h_{fe}}$ in order to achieve the required current gain of the amplifier. Furthermore, if μ_R were simply made much greater than unity, this would still require an impractically high $\mu_{h_{fe}}$.

The expression for μ_R given by Eq. (6-15) will now be checked to see if it specifies a minimum of μ_R .

Differentiating Eq. (6-14) with respect to μ_R gives

$$\frac{\partial^2 \sigma_{K_i}^2}{\partial \mu_R^2} = \frac{2\mu_{K_i}^2}{(1+\mu_R)^3} \left[(C_0^2 + C_L^2) \frac{1-2\mu_R}{1+\mu_R} + 2C_F C_0 R_{h_{fe} h_{oe}} \right] \quad (6-17)$$

Substituting μ_R as given by Eq. (6-15) into this expression yields

$$\frac{\partial^2 \sigma_{K_i}^2}{\partial \mu_R^2} = 2\mu_{K_i}^2 \frac{(C_0^2 + C_L^2 - 2R_{h_{fe} h_{oe}})^4}{(C_0^2 + C_L^2)^3} \quad (6-18)$$

Hence the second derivative of $\sigma_{K_1}^2$ with respect to μ_R is positive at the point defined by Eq. (6-15) and, therefore, this point is a minimum of $\sigma_{K_1}^2$. Use of Eq. (6-15) in Eq. (6-11) yields the following expression for the minimum value of $\sigma_{K_1}^2$:

$$\sigma_{K_i}^2 = \mu_{K_i}^2 \left[C_F^2 - \frac{(C_F C_0 R_{h_{fe} h_{oe}})^2}{C_0^2 + C_L^2} \right] \quad (6-19)$$

Now if Eq. (6-12) is substituted in Eq. (6-15), the relationship between μ_{R_L} and $\mu_{h_{oe}}$ which must be satisfied in order that $\sigma_{K_1}^2$ be a minimum is thereby obtained:

$$\mu_{R_L} = \frac{c_F c_o \rho_{h_{fe} h_{oe}}}{c_o^2 + c_L^2 - c_F c_o \rho_{h_{fe} h_{oe}}} \frac{1}{\mu_{h_{oe}}} \quad (6-20)$$

This expression can be shown to define a practical relationship between μ_{R_L} and $\mu_{h_{oe}}$. For example, suppose that $c_F = c_o = 3c_L$. In addition, a typical value of $\rho_{h_{fe} h_{oe}}$ is 0.5 (Ref. 42). Using this information in Eq. (6-20) gives

$$\mu_{R_L} = \frac{(3c_L)(3c_L)(0.5)}{(3c_L)^2 + c_L^2 - (3c_L)(3c_L)(0.5)} \frac{1}{\mu_{h_{oe}}} = \frac{0.818}{\mu_{h_{oe}}} \quad (6-21)$$

This is a relationship which it is possible, as a practical matter⁴, to satisfy.

6-5. Single Loop Feedback Amplifier System

As an illustration of the application of the minimization of variance method to a feedback system, the variance of the voltage gain of the system of Sec. 4-6 will be minimized. In performing this minimization, it will be assumed that there is not any correlation between the voltage gain of the amplifier without feedback and the fraction of the output voltage which is feedback. It will also be assumed that

⁴It should be noted that this condition gives a mean current gain of only 0.55 $\mu_{h_{fe}}$. Also indications are that better results could be obtained by designing by the sensitivity method, and then using the above technique to improve this design. This will result however in a more complex circuit.

$$\sigma_A = c_A \mu_A \quad (6-22)$$

and

$$\sigma_B = c_B \mu_B \quad (6-23)$$

Under the above conditions, Eq. (4-21) for the variance of voltage gain of the feedback amplifier becomes:

$$\sigma_{K_V}^2 = \frac{\mu_A^2}{(1 - \mu_A \mu_B)^4} (c_A^2 + c_B^2 \mu_A^2 \mu_B^2) \quad (6-24)$$

Now suppose that the mean value of the overall voltage gain is specified. This constraint can be applied by substituting Eq. (4-20) into Eq. (6-24) to eliminate either μ_A or μ_B . If μ_B is eliminated, the result is:

$$\sigma_{K_V}^2 = (c_A^2 + c_B^2) \frac{\mu_{K_V}^4}{\mu_A^2} - 2c_B^2 \frac{\mu_{K_V}^3}{\mu_A} + c_B^2 \mu_{K_V}^2 \quad (6-25)$$

Differentiating with respect to μ_A gives

$$\frac{\partial \sigma_{K_V}^2}{\partial \mu_A} = -2(c_A^2 + c_B^2) \frac{\mu_{K_V}^4}{\mu_A^3} + 2c_B^2 \frac{\mu_{K_V}^3}{\mu_A^2} \quad (6-26)$$

Setting this derivative equal to zero and solving for μ_A yields

$$\mu_A = \frac{c_A^2 + c_B^2}{c_B^2} \mu_{K_V} \quad (6-27)$$

In order to see if this expression for μ_A defines a minimum of $\sigma_{K_V}^2$, the second derivative of $\sigma_{K_V}^2$ with respect to μ_A will be evaluated at the point defined by Eq. (6-27). Differentiation of Eq. (6-26) gives

$$\frac{\partial^2 \sigma_{K_V}^2}{\partial \mu_A^2} = 6(c_A^2 + c_B^2) \frac{\mu_{K_V}^4}{\mu_A^4} - 4c_B^2 \frac{\mu_{K_V}^3}{\mu_A^3} \quad (6-28)$$

Using Eq. (6-27) in this expression shows that

$$\frac{\partial^2 \sigma_{K_V}^2}{\partial \mu_A^2} = \frac{2c_B^8}{(c_A^2 + c_B^2)} > 0 \quad (6-29)$$

Since this second derivative is positive, Eq. (6-27) does define a minimum of $\sigma_{K_V}^2$.

The value of μ_B required if $\sigma_{K_V}^2$ is to be a minimum can be obtained by substituting Eq. (6-27) into Eq. (4-20) and solving for μ_B . The result is

$$\mu_B = \frac{-c_A^2}{c_A^2 + c_B^2} \frac{1}{\mu_{K_V}} \quad (6-30)$$

Now if β is determined by only the parameters of passive elements, $c_\beta^2 \ll c_a^2$ and thus

$$\mu_{K_V} \approx -\frac{1}{\mu_\beta} \quad (6-31)$$

This is the familiar expression for gain of a feedback amplifier, which results when $\mu_A \mu_\beta$ is made considerably larger than unity in order to eliminate the dependence of this gain upon μ_A .

The variance of the overall voltage gain when the criterion for a minimum is met can be obtained by substitution of Eq. (6-27) in Eq. (6-25). The result is

$$\sigma_{K_V}^2 = \frac{c_A^2 c_\beta^2}{c_A^2 + c_\beta^2} \mu_{K_V}^2 \quad (6-32)$$

Conversely, if there is no feedback

$$\mu_{K_V} = \mu_A$$

and thus Eq. (6-25) becomes

$$\sigma_{K_V}^2 = c_A^2 \mu_{K_V}^2 \quad (6-33)$$

Comparison of Eqs. (6-32) and (6-33) shows that feedback with variance minimization always reduces the variance of the overall gain below its value without feedback. In fact, if $c_B^2 \ll c_A^2$, $\sigma_{K_V}^2$ as given by

Eq. (6-32) becomes

$$\sigma_{K_V}^2 \approx c_B^2 \mu_{K_V}^2 \quad (6-34)$$

which is much smaller than the variance without feedback in practical circuits.

It is also interesting to note that feedback (even without variance minimization) will always reduce the variance of the overall gain below its value without feedback, if the following condition is met:

$$\frac{c_B^2 - c_A^2}{c_B^2 + c_A^2} < \frac{\mu_{K_V}}{\mu_A} < 1 \quad (6-35)$$

This condition is derived in Appendix B and shows that only negative feedback ($|\mu_{K_V}/\mu_A| < 1$ defines negative feedback) within certain limits will reduce the variance of the overall gain below its value without feedback.

This treatment touches on only one phase of this topic. For example, the design obtained here should be compared with that obtained by the sensitivity method. In addition, a design which allows for the possibility of the system being unstable should be considered.

6-6. Cascade Voltage Amplifier

In this section the variance minimization technique will be applied to the voltage gain of an amplifier consisting of n stages in cascade. The overall voltage gain of such an arrangement is (Ref. 10)

$$K_V = \prod_{j=1}^n K_{V_j} \quad (6-36)$$

where K_{V_j} is the voltage gain of the j^{th} stage.

Now the problem is to determine that combination of mean values of voltage gains of the respective stages, which will yield the minimum variance of the overall voltage gain, while giving the required mean value of this gain. This is to be done under the assumption that the standard deviation to mean ratio of the voltage gain of the j^{th} stage is c_j . With this assumption, the analysis of Sec. 5-6 for the minimization of the variance of the product of n random variables applies. This analysis reveals that if the mean value of the product is specified, it is not possible to minimize the variance of the product with respect to the mean values of the various factors. Thus in the present problem it is not possible to minimize the variance of the overall voltage gain

with respect to the mean values of the voltage gains of the respective stages. It should also be noted that this result holds for the case where the voltage gains of the stages are uncorrelated, as well as the situation where they are correlated.

It should be emphasized that the above result applies only under the limitations previously stated; namely, that the propagation of variance formula can be applied to the gain of the amplifier, and that the standard deviation to mean ratio of the parameters is constant. Thus, if another technique under another set of assumptions is applied, it may be possible to improve this design. For example, a better design for this amplifier can be obtained by use of the sensitivity method. Furthermore, it may be possible to improve upon this latter design by applying the above statistical techniques to it.

CHAPTER 7: CORRELATION DUE TO SELECTION PROCESSES

7-1. Introduction

In Sec. 6-3, it was seen that it was not possible to minimize the variance of the divider ratio of the voltage divider with respect to the mean values of the resistances. This was true because once the mean value of the divider ratio was specified, the variance of the divider ratio was independent of the mean values of the resistances. It was noted, however, (see Eq. (6-6)) that the divider ratio variance could be reduced if the coefficient of linear correlation between the two resistances is positive. Now ordinarily when the voltage divider units are assembled, the respective resistors are selected at random. In this case, there would be no correlation between the resistances.

It is the purpose of this chapter to discuss methods of selecting components during the assembly of systems, so that the system parameters will be correlated in such a manner, that the variance of the system performance criteria will be reduced. Expressions for the coefficients of linear correlation between these parameters will also be derived.

7-2. Selection Processes

In general, the selection processes mentioned above involve dividing the distributions of the parameters to be correlated into a number of cells and then matching the respective cells of the parameters during the assembly of the systems. For example, consider the parameter distributions in Figs. 7-1 and 7-2.

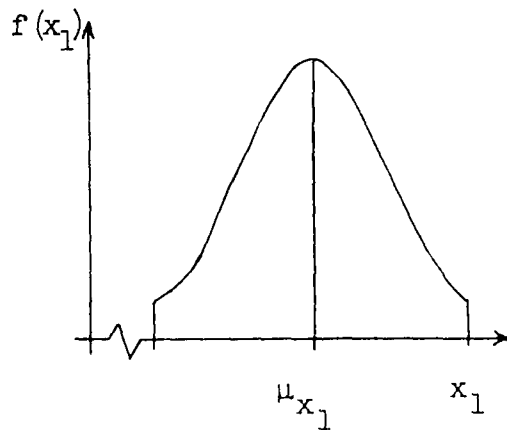


Fig. 7-1

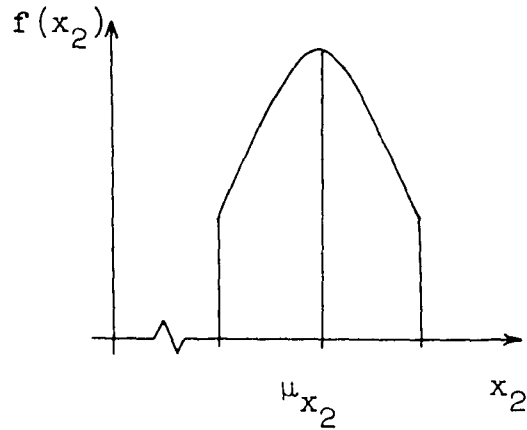


Fig. 7-2

Suppose that each of these distributions is divided into two cells with the mean values of the respective parameters as boundaries between the cells. Then, as the system units are assembled, values of $x_1 < \mu_{x_1}$ are matched with values of $x_2 < \mu_{x_2}$. In a similar manner, values of $x_1 > \mu_{x_1}$ are matched with values of $x_2 > \mu_{x_2}$.

This procedure results in a positive coefficient of linear correlation between x_1 and x_2 . If instead the smaller values of x_1 are matched with the larger values of x_2 and vice versa, then the coefficient of linear correlation between x_1 and x_2 will be negative. It should also be noted that the magnitude of this coefficient can be increased by increasing the number of cells. As mentioned in Sec. 5-6, the coefficient of linear correlation ranges from a minimum of -1 to a maximum of +1. This statement is true, in general, and does not depend on the process under consideration.

7-3. Choosing the Cell Boundaries

In choosing the boundaries of the cells, the basic principle, which should be kept in mind, is that the respective cells to be matched must contain an equal number of components. In terms of the density functions, this means that the cells to be matched must contain equal fractions of the total areas under their respective density functions. For example, if cell 1 of $f(x_1)$ and cell 1 of $f(x_2)$ in Figs. 7-3 and 7-4 are to be matched, this would require that

$$\frac{\int_{x_{10}}^{x_{11}} f(x_1) dx_1}{\int_{x_{10}}^{x_{14}} f(x_1) dx_1} = \frac{\int_{x_{20}}^{x_{21}} f(x_2) dx_2}{\int_{x_{20}}^{x_{24}} f(x_2) dx_2} \quad (7-1)$$

(This condition is not met for the boundaries shown.).

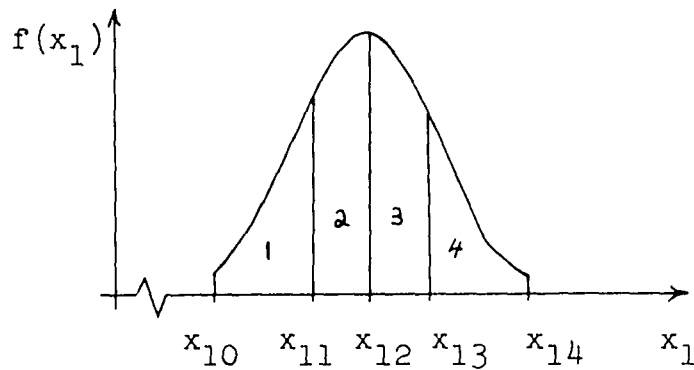


Fig. 7-3. Distribution with four cells, each of which has an equal number of components.

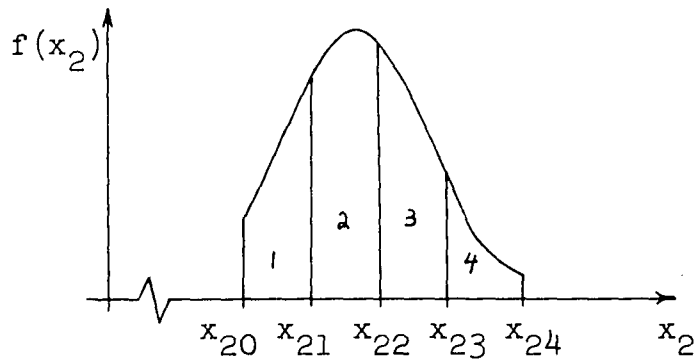


Fig. 7-4. Distribution with four cells, each of which covers an equal range of values of x_2 .

Within the framework of the above basic principle, there are two general techniques for selecting cell boundaries. The first involves fixing the boundaries, so that each cell of the parameter distribution contains an equal number of components. This is equivalent to saying that each cell of the distribution must have an equal portion of the area under the density function associated with it. This situation is illustrated in Fig. 7-3.

Now suppose that it is desired to divide a normal distribution which is truncated at $x_1 = x_{10}$ and at $x_1 = x_{1n}$ into n cells, each of which contains an equal number of components. The area per cell then is the total area of the truncated distribution divided by n . Thus S_{1j} , the area of the j^{th} cell of $f(x_1)$ can be written as

$$S_{1j} = \frac{\int_{x_{10}}^{x_{1n}} f(x_1) dx_1}{n} \quad (7-2)$$

S_{1j} may be written as

$$S_{1j} = \int_{x_{1j}}^{x_{1j+1}} f(x_1) dx_1 \quad (7-3)$$

In Eqs. (7-2) and (7-3), $f(x_1)$ can be transformed to the unit normal distribution, $f(k_1)$, by the following relationships:

$$k_1 = \frac{x_1 - \mu_{x_1}}{\sigma_{x_1}} \quad (7-4)$$

and

$$k_{1j} = \frac{x_{1j} - \mu_{x_1}}{\sigma_{x_1}} \quad (7-5)$$

If after this transformation is made, Eqs. (7-2) and (7-3) are combined, the result is

$$S_j = \frac{1}{n} \int_{k_{10}}^{k_{1j+1}} f(k_1) dk_1 = \int_{k_{1j}}^{k_{1j+1}} f(k_1) dk_1 \quad (7-6)$$

A unit normal distribution table can now be used to select the cell boundaries k_{1j} ($j = 1, 2, \dots, n-1$), so that Eq. (7-6) is satisfied.

A second method of choosing cell boundaries is to fix the boundaries, so that each cell covers an equal range of values of the variable. This technique was used to draw the boundaries in Fig. 7-4. In terms of the notation of this figure, the following relationships would be required to hold, if a distribution is to be divided into n equal range cells:

$$\begin{aligned}
 (X_{21} - X_{20}) = (X_{22} - X_{21}) = \dots = (X_{2j} - X_{2,j-1}) \dots \dots \\
 \dots = (X_{2n} - X_{2,n-1})
 \end{aligned}
 \tag{7-7}$$

It should be noted that the equal area method may be applied to any pair of truncated normal distributions. On the other hand, the equal range method usually¹ may not be applied to normal distributions which are not truncated in the same manner, since the cells to be matched would contain unequal numbers of components. For example, the distributions of Figs. 7-3 and 7-4 could not be matched by the equal cell method for this reason.

7-4. Expressions for the Coefficient of Linear Correlation

In this section expressions for computing the coefficient of linear correlation between two variables selected by the methods discussed above will be derived. The equal range method will be considered first, since

¹An exception to this rule occurs if $(\mu_{x_1} - x_{10}) = (x_{2n} - \mu_{x_2})$ and $(x_{1n} - \mu_{x_1}) = (\mu_{x_2} - x_{20})$ and if it is desired to match cells, so that a negative coefficient of linear correlation is obtained.

the results of this case can be specialized to yield the expressions for the equal area case.

In general, the coefficient of linear correlation between two random variables, x_1 and x_2 , can be computed by use of Eq. (3-4). Since the standard deviations of x_1 and x_2 can be obtained by using Eq. (2-11), the real problem here is to derive an expression for the covariance between x_1 and x_2 .

An expression for this covariance can be obtained by application of Eq. (3-3). Using the notation of Figs. 7-3 and 7-4, Eq. (3-3) becomes

$$\sigma_{x_1, x_2} = \frac{\int_{x_{10}}^{x_{1n}} \int_{x_{20}}^{x_{2n}} x_1 x_2 f(x_1, x_2) dx_1 dx_2}{\int_{x_{10}}^{x_{1n}} \int_{x_{20}}^{x_{2n}} f(x_1, x_2) dx_1 dx_2} - \mu_{x_1} \mu_{x_2} \quad (7-8)$$

Because of the selection process, the contribution to the first term on the right-hand side of this expression by each pair of cells must be evaluated separately.

The entire term is then the summation of the contributions of all pairs of cells. This results in

$$\sigma_{X_1 X_2} = \frac{\sum_{j=0}^{n-1} \int_{x_{1j}}^{x_{1,j+1}} f(x_1) dx_1 \frac{\int_{x_{1j}}^{x_{1,j+1}} x_1 f(x_1) dx_1 \int_{x_{2j}}^{x_{2,j+1}} x_2 f(x_2) dx_2}{\int_{x_{1j}}^{x_{1,j+1}} f(x_1) dx_1 \int_{x_{2j}}^{x_{2,j+1}} f(x_2) dx_2}}{\int_{x_{10}}^{x_{1n}} f(x_1) dx_1} - \mu_{X_{1t}} \mu_{X_{2t}} \quad (7-9)$$

In obtaining the above expression, it was noted that since x_1 and x_2 are independent before the selection process,

$$f(x_1, x_2) = f(x_1) f(x_2)$$

It should also be noted that the contribution to the first term on the right-hand side by each pair of cells is weighted according to the number of components in each cell.

The evaluation of the integrals in Eq. (7-9) can be facilitated by introducing the following transformations:

$$R_1 = \frac{x_1 - \mu_{x_1}}{\sigma_{x_1}} \quad (7-10)$$

$$R_2 = \frac{x_2 - \mu_{x_2}}{\sigma_{x_2}} \quad (7-11)$$

$$R_{1j} = \frac{x_{1j} - \mu_{x_1}}{\sigma_{x_1}} \quad (7-12)$$

$$R_{2j} = \frac{x_{2j} - \mu_{x_2}}{\sigma_{x_2}} \quad (7-13)$$

where the quantities μ_{x_1} , μ_{x_2} , σ_{x_1} , and σ_{x_2} pertain to the untruncated distributions. With these transformations and the definition of S_j in Eq. (7-3), it can be seen that Eq. (7-9) becomes

$$\sigma_{x_1 x_2} = \frac{\sum_{j=0}^{n-1} S_{1j} \left[\frac{\sigma_{x_1} \int_{R_{1j}}^{R_{1j+1}} R_1 f(R_1) dR_1}{S_{1j}} + \mu_{x_1} \right] \left[\frac{\sigma_{x_2} \int_{R_{2j}}^{R_{2j+1}} R_2 f(R_2) dR_2}{S_{2j}} + \mu_{x_2} \right]}{\int_{R_{10}}^{R_{1n}} f(R_1) dR_1} - \mu_{x_1 t} \mu_{x_2 t} \quad (7-14)$$

If the identity given by Eq. (2-7) is used in this expression, the result is

$$\sigma_{x_1, x_2} = \frac{\sum_{j=0}^{n-1} S_{1j} \left\{ \frac{\sigma_{x_1} [f(R_{1j}) - f(R_{1, j+1})]}{S_{1j}} + \mu_{x_1} \right\} \left\{ \frac{\sigma_{x_2} [f(R_{2j}) - f(R_{2, j+1})]}{S_{2j}} + \mu_{x_2} \right\}}{\int_{R_{10}}^{R_{1n}} f(R_1) dR_1 - \mu_{x_1} \mu_{x_2}} \quad (7-15)$$

This equation can now be evaluated with the aid of a unit normal distribution table.

With this expression for the covariance between x_1 and x_2 , Eq. (3-4) can be used in conjunction with Eq. (2-11) to compute the coefficient of linear correlation between x_1 and x_2 for the equal range selection process.

The covariance between x_1 and x_2 for the equal area case can also be obtained from Eq. (7-15). In this situation, however, S_{1j} and S_{2j} are the same for each cell of their respective distributions. Hence Eq. (7-15) may be rewritten as

$$\sigma_{x_1 x_2} = \frac{\frac{1}{S_2} \sum_{j=0}^{n-1} \left\{ \sigma_{x_1} [f(k_{1j}) - f(k_{1,j+1})] + S_1 \mu_{x_1} \right\} \left\{ \sigma_{x_2} [f(k_{2j}) - f(k_{2,j+1})] + S_2 \mu_{x_2} \right\}}{\int_{k_{10}}^{k_{1n}} f(k_1) dk_1 - \mu_{x_1} \mu_{x_2}} \quad (7-16)$$

The computation of the coefficient of linear correlation between two variables correlated by a selection process is illustrated in Appendix C. Both the equal range process and the equal area process are considered. The results of these computations are summarized in the table of Appendix C. Inspection of this table reveals that for the distributions considered, both processes yield the same value of the coefficient of linear correlation in the two-cell case. For three, four, and five cells, the equal area case gives a larger $\rho_{x_1 x_2}$. For six or more cells, both processes yield essentially the same results. It should also be noted that the rate of increase of $\rho_{x_1 x_2}$ with respect to the number of cells decreases,

as the number of cells increases. Thus the use of more than three or four cells would not in most situations be economically feasible; that is, the small increases in $\rho_{x_1 x_2}$ would not justify the cost of adding more cells.

7-5. Application to a Voltage Divider

It was found in Sec. 6-3 that while the variance of the divider ratio of the voltage divider could not be minimized, it could be reduced by making the coefficient of linear correlation between R_1 and R_2 positive. In order to illustrate this point, assume that $c_1 = c_2 = c$ in Eq. (6-7). Then the variance of the divider ratio becomes

$$\sigma_{K_V}^2 = 2c^2 \mu_{K_V}^2 (1 - \mu_{K_V})^2 (1 - \rho_{R_1 R_2}) \quad (7-17)$$

Without correlation, this variance is

$$\sigma_{K_V}^2 = 2c^2 \mu_{K_V}^2 (1 - \mu_{K_V})^2 \quad (7-18)$$

The percentage decrease in the variance because of correlation is then

$$\Delta \sigma_{K_V}^2 = \frac{2c^2 \mu_{K_V}^2 (1 - \mu_{K_V})^2 - 2c^2 \mu_{K_V}^2 (1 - \mu_{K_V})^2 (1 - \rho_{R_1 R_2})}{2c^2 \mu_{K_V}^2 (1 - \mu_{K_V})^2} \times 100 = 100 \rho_{R_1 R_2}$$

Now assume that both R_1 and R_2 are normally distributed with symmetrical truncation at $R = \mu_R \pm 3\sigma_R$ for each resistance. If a three-cell equal area selection process is used, $\rho_{R_1 R_2}$ from the table in Appendix C is equal to 0.800. This means that there is an 80.0% reduction in the variance of the divider ratio because of the selection process.

In conclusion, it is seen that this method provides an assembly process which is intermediate between exact matching of the parameter values of the components and a completely random method of selection. The choice of the selection method is dependent upon such factors as the tolerances available in the components to be used, the relative costs of the selection processes, and the tolerance required for the performance criteria.

CHAPTER 8: DRIFT ANALYSIS OF ELECTRONIC SYSTEMS

8-1. Introduction

In Sec. 2-5, the drift of the parameters of electronic systems was discussed. Now if the parameters of a system drift, then system performance criteria which depend on these parameters will also drift (Refs. 40, 46, 47). The purpose of this chapter is to analyze this drift of the system performance criteria. In order to perform this analysis, the first step will be to derive expressions for the mean values and the variances of the models of Sec. 2-5. Next, the mean value and variance of a function of these parameters will be obtained. These results will then be used to study the effect of parameter drift on a number of electronic systems.

It should also be noted that the drift rates of the parameters of the electronic systems to be considered in this chapter and in Chapter 9 will be assumed to be extremely small fractions of the initial mean values of these parameters (Refs. 4 and 12). Some typical values for the drift rates of a resistor are given in the table of Sec. 2-5.

8-2. Moments of System Parameters as Functions of Time

In this section, expressions for the mean values and the variances of the parameter models of Sec. 2-5 will be derived. Since these parameters are varying with time, their means and variances will also vary with time.

First of all, the model given by Eq. (2-13) will be studied. This will be done with the assumption that $x_j(t)$ may be approximated by the first $m + 1$ terms of the series in Eq. (2-13). Thus, $x_j(t)$ can be written as

$$x_j(t) = \mu_{x_j(0)} \sum_{q=0}^m a_{jq} t^q \quad (8-1)$$

Then by Eq. (3-6) the mean value of $x_j(t)$ is

$$\mu_{x_j(t)} = \mu_{x_j(0)} \sum_{q=0}^m \mu_{a_{jq}} t^q \quad (8-2)$$

where it should be noted that t is not a random variable. Application of Eq. (3-7) yields the following expression for the variance of $x_j(t)$:

$$\sigma_{x_j^2}^2(t) = \mu_{x_j^2(0)} \left[\sum_{q=0}^m t^{2q} \sigma_{a_{jq}}^2 + 2 \sum_{q=0}^{m-1} \sum_{l=q+1}^m t^{2q+l} \rho_{a_{jq} a_{jl}} \sigma_{a_{jq}} \sigma_{a_{jl}} \right] \quad (8-3)$$

The linear model of Eq. (2-14) with no correlation between the initial value and drift rate of $x_j(t)$ can be handled by letting $m = 1$ and the coefficients of linear correlation equal zero in Eqs. (8-2) and (8-3). This gives

$$\mu_{x_j}(t) = \mu_{x_j}(0) (1 + \mu_{a_{j1}} t) \quad (8-4)$$

and

$$\sigma_{x_j}^2(t) = \mu_{x_j}^2(0) (\sigma_{a_{j0}}^2 + \sigma_{a_{j1}}^2 t^2) \quad (8-5)$$

In deriving Eq. (8-4), it was recalled that the mean value of a_{j0} is unity.

8-3. Moments of a Function of Parameters that Vary with Time

In this section, the mean value and the variance of the following function of parameters that drift will be obtained:

$$Y(t) = g[x_1(t), x_2(t), \dots, x_j(t), \dots, x_n(t)] \quad (8-6)$$

Substitution of Eq. (8-1) in this expression yields

$$Y(t) = g \left[\mu_{x_1}(0) \sum_{q=0}^m a_{1q} t^q, \mu_{x_2}(0) \sum_{q=0}^m a_{2q} t^q, \dots, \mu_{x_j}(0) \sum_{q=0}^m a_{jq} t^q, \right. \\ \left. \dots, \mu_{x_n}(0) \sum_{q=0}^m a_{nq} t^q \right] \quad (8-7)$$

The mean value of $y(t)$ can now be obtained by use of Eq. (3-6). This yields

$$\mu_y(t) = g \left[\mu_{x_1}(0) \sum_{q=0}^m \mu_{a_{1q}} t^q, \mu_{x_2}(0) \sum_{q=0}^m \mu_{a_{2q}} t^q, \dots, \mu_{x_j}(0) \sum_{q=0}^m \mu_{a_{jq}} t^q, \right. \\ \left. \dots, \mu_{x_n}(0) \sum_{q=0}^m \mu_{a_{nq}} t^q \right] \quad (8-8)$$

This result can be applied to the linear drift model by letting $m = 1$. This gives

$$\mu(t) = g \left[\mu_{x_1}(0) (1 + \mu_{a_{11}} t), \mu_{x_2}(0) (1 + \mu_{a_{21}} t), \dots, \mu_{x_j}(0) (1 + \mu_{a_{j1}} t), \right. \\ \left. \dots, \mu_{x_n}(0) (1 + \mu_{a_{n1}} t) \right] \quad (8-9)$$

Application of Eq. (3-7) to Eq. (8-7) shows that the variance of $y(t)$ is given by

$$\sigma_y^2(t) = \sum_{j=1}^n \sum_{q=0}^m \left(\frac{\partial y(t)}{\partial a_{jq}} \right)^2 \sigma_{a_{jq}}^2 + 2 \sum_{j=1}^n \sum_{q=0}^{m-1} \sum_{s=q+1}^m \left(\frac{\partial y(t)}{\partial a_{jq}} \right) \left(\frac{\partial y(t)}{\partial a_{js}} \right) \rho_{a_{jq} a_{js}} \sigma_{a_{jq}} \sigma_{a_{js}}$$

$$+ 2 \sum_{j=1}^{n-1} \sum_{q=0}^{m-1} \sum_{l=j+1}^n \sum_{s=0}^m \left(\frac{\partial y(t)}{\partial a_{jq}} \right) \left(\frac{\partial y(t)}{\partial a_{ls}} \right) \rho_{a_{jq} a_{ls}} \sigma_{a_{jq}} \sigma_{a_{ls}} \quad (8-10)$$

where all the derivatives are evaluated at the mean values of the parameters for the drift model of $y(t)$. In obtaining Eq. (8-10) it should be noted that the second and third terms respectively, of this equation represent the correlation between the a_{jq} coefficients of a component and the correlation between the a_{jq} coefficients of the various components. The complexity of the expression emphasizes the need for a simplified drift model in order to obtain an engineering solution to the drift problem.

The linear model with no correlation among any of the parameters can be handled by letting $m = 1$ and all the coefficients of linear correlation equal zero

in Eq. (8-10). This results in

$$\sigma_Y^2(t) = \sum_{j=1}^n \left(\frac{\partial \gamma(t)}{\partial a_{j0}} \right)^2 \sigma_{a_{j0}}^2 + \left(\frac{\partial \gamma(t)}{\partial a_{j1}} \right)^2 \sigma_{a_{j1}}^2 \quad (8-11)$$

8-4. Resistors in Series

In this section, the above techniques for determining the mean values and variances of functions of parameters that drift will be applied to the series circuit of Sec. 4-2. If the linear model of Eq. (2-14) is used for each resistance, then from Eq. (4-1) the resistance of the series circuit as a function of time is

$$R(t) = \sum_{j=1}^n \mu_{R_j}(0) (a_{j0} + a_{j1}t) \quad (8-12)$$

Use of Eq. (3-6) shows that the mean value of $R(t)$ is

$$\mu_R(t) = \sum_{j=1}^n \mu_{R_j}(0) (1 + \mu_{a_{j1}}t) \quad (8-13)$$

If, in accordance with the discussion of Sec. 2-5, it is assumed that there is no correlation between any of the parameters of the model of Eq. (8-12), then Eq. (3-7) shows that the variance of $R(t)$ is given by

$$\sigma_R^2(t) = \sum_{j=1}^n \mu_{R_j}^2(0) (\sigma_{a_{j0}}^2 + \sigma_{a_{j1}}^2 t^2) \quad (8-14)$$

These results show that the direction of the drift of the mean value of the series resistance will depend on the signs of the drift rates of the individual resistors. On the other hand, however, the variance of $R(t)$ will always increase with time.

8-5. Voltage Divider

The techniques of Sec. 8-3 will now be applied to the voltage divider discussed in Sec. 4-3. If the linear model of Eq. (2-14) is again used for the resistors, then the divider ratio, as given by Eq. (4-5), becomes

$$K_V(t) = \frac{\mu_{R_2}(0)(a_{20} + a_{21}t)}{\mu_{R_1}(0)(a_{10} + a_{11}t) + \mu_{R_2}(0)(a_{20} + a_{21}t)} \quad (8-15)$$

By application of Eq. (3-6) the mean value of this divider ratio is

$$\mu_{K_V}(t) = \frac{\mu_{R_2}(0)(1 + \mu_{a_{21}}t)}{\mu_{R_1}(0)(1 + \mu_{a_{11}}t) + \mu_{R_2}(0)(1 + \mu_{a_{21}}t)} \quad (8-16)$$

With the assumption that there is no correlation between any of the parameters in Eq. (8-15), Eq. (3-7) shows that the variance of the divider ratio is

$$\sigma_{K_V}^2(t) = \left(\frac{\partial K_V(t)}{\partial a_{10}} \right)^2 \sigma_{a_{10}}^2 + \left(\frac{\partial K_V(t)}{\partial a_{11}} \right)^2 \sigma_{a_{11}}^2 + \left(\frac{\partial K_V(t)}{\partial a_{20}} \right)^2 \sigma_{a_{20}}^2 + \left(\frac{\partial K_V(t)}{\partial a_{21}} \right)^2 \sigma_{a_{21}}^2$$

where the partial derivatives are evaluated at the mean values of the parameters in Eq. (8-15). If these operations are carried out, the result is

$$\sigma_{K_V}^2(t) = \mu_{R_1}^2(0) \mu_{R_2}^2(0) \frac{(1 + \mu_{a_{21}} t)^2 (\sigma_{a_{10}}^2 + \sigma_{a_{11}}^2 t^2) + (1 + \mu_{a_{11}} t)^2 (\sigma_{a_{20}}^2 + \sigma_{a_{21}}^2 t^2)}{[\mu_{R_1}(0)(1 + \mu_{a_{11}} t) + \mu_{R_2}(0)(1 + \mu_{a_{21}} t)]^4} \quad (8-17)$$

Inspection of the expressions for the mean value and the variance of the divider ratio shows both of these quantities may drift in either direction, depending upon the magnitudes and the signs of the various parameters in these equations. A more detailed study of these expressions will be made in Chapter 9.

8-6. Single Loop Feedback Amplifier System

As an example of the application of drift analysis to a feedback system, the drift of the single loop feedback amplifier system of Sec. 4-6 will now be analyzed. The primary object of this analysis is to

obtain expressions which will be useful in the design methods to be developed in Chapter 9. Therefore, consistent with the remarks of Sec. 2-5, the linear model of Eq. (2-14) will be used to represent the drift of both the gain of the amplifier without feedback and the fraction of the output which is feedback. In addition, it will be assumed that there is no correlation among any of the parameters in the drift model of the system.

If the model of Eq. (2-14) is now applied to Eq. (4-19), the drift model of the system becomes

$$K_V(t) = \frac{\mu_A(0)(a_{A0} + a_{A1}t)}{1 - \mu_A(0)\mu_B(0)(a_{A0} + a_{A1}t)(a_{B0} + a_{B1}t)} \quad (8-18)$$

The mean value of this gain obtained by use of Eq. (3-6) is

$$\mu_{K_V}(t) = \frac{\mu_A(0)(1 + \mu_{a_{A1}}t)}{1 - \mu_A(0)\mu_B(0)(1 + \mu_{a_{A1}}t)(1 + \mu_{a_{B1}}t)} \quad (8-19)$$

Eq. (3-7) shows that the variance of K_V can be expressed as

$$\sigma_{K_V}^2(t) = \left(\frac{\partial K_V(t)}{\partial a_{A0}}\right)^2 \sigma_{a_{A0}}^2 + \left(\frac{\partial K_V(t)}{\partial a_{A1}}\right)^2 \sigma_{a_{A1}}^2 + \left(\frac{\partial K_V(t)}{\partial a_{B0}}\right)^2 \sigma_{a_{B0}}^2 + \left(\frac{\partial K_V(t)}{\partial a_{B1}}\right)^2 \sigma_{a_{B1}}^2$$

where each of the derivatives is evaluated at the mean values of the system drift model parameters. From this expression the variance of the overall voltage gain of the system becomes

$$\sigma_{K_V}^2(t) = \mu_A^2(0) \frac{(\sigma_{a_{A0}}^2 + \sigma_{a_{A1}}^2 t^2) + \mu_A^2(0) \mu_B^2(0) (1 + \mu_{a_{A1}} t)^4 (\sigma_{a_{B0}}^2 + \sigma_{a_{B1}}^2 t^2)}{\left[1 - \mu_A(0) \mu_B(0) (1 + \mu_{a_{A1}} t) (1 + \mu_{a_{B1}} t) \right]^4} \quad (8-20)$$

— The above results show that both the mean value and the variance of the overall voltage gain may either increase or decrease with time, depending on the values of the parameters of the system drift model. These expressions for the mean value and the variance will be analyzed in more detail in Chapter 9 in connection with design to limit the drift of the system performance criteria.

CHAPTER 9: PREVENTING DRIFT OF THE
PERFORMANCE CRITERIA OF ELECTRONIC SYSTEMS

9-1. Introduction

In Sec. 5-1, it was noted that an electronic system is generally designed so that the mean values of the system performance criteria have the values called for in the system specifications. In addition, it was seen that the probabilities of the performance criteria being out of tolerance should be made as small as possible by minimizing the variance of these criteria.

Now in Chapter 8, it was found that, due to the drift of the system parameters, the mean values and the variances of the system performance criteria vary with time. Thus the system should be designed so that these time variations do not cause the probabilities of the system performance criteria being out of tolerance to increase with time - in fact, a decrease with time would be desirable. This can be accomplished if¹ the mean values of the criteria are held constant

¹If the drift of a performance criterion in a particular direction cannot be prevented, it is often compensated for by biasing the initial value of this criterion in the direction opposite to the drift (Ref. 50).

with time while the variances of the criteria are prevented from increasing with time.

The approach in this chapter will be to devise techniques for achieving the above goals by working with a number of elementary functions whose parameters vary with time. These techniques will then be applied to a number of electronic systems. In all cases, the linear model of Eq. (2-14) will be considered to be valid. In addition, it will be assumed that there is no correlation among any of the initial values of the system parameters and the parameter drift rates (Ref. 4c).

9-2. Linear Functions

In this section, the following linear function of time-varying parameters will be considered:

$$y(t) = \sum_{j=1}^n b_j x_j(t) \quad (9-1)$$

where the b_j 's are constants. If the $x_j(t)$'s are given by Eq. (2-14), then $y(t)$ can be written as

$$y(t) = \sum_{j=1}^n b_j \mu_{x_j}(0) (a_{j0} + a_{j1}t) \quad (9-2)$$

Application of Eq. (3-6) to this expression yields the following equation for the mean value of $y(t)$:

$$\mu_y(t) = \sum_{j=1}^n b_j \mu_{x_j}(0) (1 + \mu_{a_{j1}} t) \quad (9-3)$$

The variance of $y(t)$ can be obtained by use of Eq. (3-7).

This gives

$$\sigma_y^2(t) = \sum_{j=1}^n b_j^2 \mu_{x_j}^2(0) (\sigma_{a_{j0}}^2 + \sigma_{a_{j1}}^2 t^2) \quad (9-4)$$

Now if the mean value of $y(t)$ is to remain constant at its initial value, then from Eq. (9-3) this requires that

$$\sum_{j=1}^n b_j \mu_{x_j}(0) (1 + \mu_{a_{j1}} t) = \sum_{j=1}^n b_j \mu_{x_j}(0) \quad (9-5)$$

or

$$\sum_{j=1}^n b_j \mu_{x_j}(0) \mu_{a_{j1}} = 0 \quad (9-6)$$

Thus the mean value of $y(t)$ can be held constant at its initial value, if the drift rates of the parameters are such that Eq. (9-6) is satisfied. Inspection of Eq. (9-4), however, reveals that no combination of

parameter variances and drift rates will prevent the variance of $y(t)$ from increasing with time. Hence it is important that variance minimization be used to make the initial variance as small as possible.

9-3. Product Functions

The following function, consisting of the product of n randomly distributed variables which vary with time, will be studied next:

$$y(t) = \prod_{j=1}^n x_j(t) \quad (9-7)$$

Since it will be assumed that $x_j(t)$ is given by Eq. (2-14), this expression can be written as

$$y(t) = \prod_{j=1}^n \mu_{x_j(0)} (a_{j0} + a_{j1} t) \quad (9-8)$$

Now by Eq. (3-6) the mean value of $y(t)$ is

$$\mu_y(t) = \prod_{j=1}^n \mu_{x_j(0)} (1 + \mu_{a_{j1}} t) \quad (9-9)$$

Application of Eq. (3-7) to Eq. (9-8) shows that

$$\sigma_y^2(t) = \sum_{j=1}^n \left[\left(\frac{\partial y(t)}{\partial a_{j0}} \right)^2 \sigma_{a_{j0}}^2 + \left(\frac{\partial y(t)}{\partial a_{j1}} \right)^2 \sigma_{a_{j1}}^2 \right]$$

$\mu_{a_{j0}}, \mu_{a_{j1}} \qquad \mu_{a_{j0}}, \mu_{a_{j1}}$

from which the variance of $y(t)$ is seen to be

$$\sigma_y^2(t) = \sum_{j=1}^n \mu_{x_j}^2(0) (\sigma_{a_{j0}}^2 + \sigma_{a_{j1}}^2 t) \prod_{\substack{\lambda=1 \\ \lambda \neq j}}^n \mu_{x_\lambda}^2(0) (1 + \mu_{a_{\lambda 1}} t)^2 \quad (9-10)$$

In order to determine under what conditions the mean value of $y(t)$ will remain constant with time, $\mu_y(t)$ will be approximated as follows:

$$\mu_y(t) \approx \prod_{j=1}^n \mu_{x_j}(0) + \prod_{j=1}^n \mu_{x_j}(0) \sum_{\lambda=1}^n \mu_{a_{\lambda 1}} t \quad (9-11)$$

In writing this expression, it was assumed that those terms involving the product of more than one $\mu_{a_{j1}} t$ type factor could be neglected. This generally can be done if the parameter drifts only a small percentage of its initial value during the period of interest. Now on the basis of Eq. (9-11), $\mu_y(t)$ will remain constant at its initial value if

$$\prod_{j=1}^n \mu_{x_j}(0) \sum_{\lambda=1}^n \mu_{a_{\lambda 1}} = 0$$

or

$$\sum_{\lambda=1}^n \mu_{a_{\lambda 1}} = 0 \quad (9-12)$$

Thus if the summation of the mean values of all the parameter drift rates is zero, the mean value of $y(t)$ will not drift.

Inspection of Eq. (9-10) shows that there are two binomial factors which affect the drift of the variance of $y(t)$. The first of these necessarily increases in magnitude with time. The second factor will decrease with time if $\mu_{a_{j1}}$ is negative. Whether or not the variance of $y(t)$ increases or decreases with time, if $\mu_{a_{j1}}$ is negative for at least some components, depends on the relative magnitudes of all the parameters in Eq. (9-10). In any event, it is desirable, from the standpoint of keeping the variance of $y(t)$ small, to make as many as possible of the mean values of the component drift rates negative.

9-4. Quotient Functions

A quotient function of the following form will now be considered:

$$\gamma(t) = \frac{\prod_{j=1}^n x_j(t)}{\prod_{\lambda=n+1}^w x_\lambda(t)} \quad (9-13)$$

If $x_j(t)$ and $x_\lambda(t)$ are represented by the model of Eq. (2-14), then this function can be written as

$$\gamma(t) = \frac{\prod_{j=1}^n \mu_{x_j(0)}(a_{j0} + a_{j1}t)}{\prod_{\lambda=n+1}^w \mu_{x_\lambda(0)}(a_{\lambda0} + a_{\lambda1}t)} \quad (9-14)$$

From Eq. (3-6) the mean value of $y(t)$ is

$$\mu_y(t) = \frac{\prod_{j=1}^n \mu_{x_j(0)}(1 + \mu_{a_{j1}}t)}{\prod_{\lambda=n+1}^w \mu_{x_\lambda(0)}(1 + \mu_{a_{\lambda1}}t)} \quad (9-15)$$

The variance of $y(t)$ can be expressed as

$$\sigma_y^2(t) = \left(\frac{\partial \gamma(t)}{\partial a_{j0}} \right)^2 \sigma_{a_{j0}}^2 + \left(\frac{\partial \gamma(t)}{\partial a_{j1}} \right)^2 \sigma_{a_{j1}}^2 + \left(\frac{\partial \gamma(t)}{\partial a_{l0}} \right)^2 \sigma_{a_{l0}}^2 + \left(\frac{\partial \gamma(t)}{\partial a_{l1}} \right)^2 \sigma_{a_{l1}}^2$$

where the derivatives are evaluated at the mean values of the parameters in Eq. (9-14). From this expression the variance of $y(t)$ is

$$\sigma_y^2(t) = \frac{\sum_{j=1}^n \mu_{x_j}^2(0) \left[\prod_{\substack{q=1 \\ q \neq j}}^n \mu_{x_q}^2(0) (1 + \mu_{a_{q1}} t)^2 \sigma_{a_{j0}}^2 + \prod_{\substack{q=1 \\ q \neq j}}^n \mu_{x_q}^2(0) (1 + \mu_{a_{q1}} t)^2 \sigma_{a_{j1}}^2 \right]}{\prod_{l=n+1}^w \mu_{x_l}^2(0) (1 + \mu_{a_{l1}} t)^2}$$

$$+ \prod_{j=1}^n \mu_{x_j}^2(0) (1 + \mu_{a_{j1}} t)^2 \left[\sum_{l=n+1}^w \frac{\sigma_{a_{l0}}^2}{(1 + \mu_{a_{l1}} t)^2 \prod_{r=n+1}^w \mu_{x_r}^2(0) (1 + \mu_{a_{r1}} t)^2} \right]$$

$$+ \sum_{l=n+1}^w \frac{\sigma_{a_{l1}}^2 t^2}{(1 + \mu_{a_{l1}} t)^2 \prod_{r=n+1}^w \mu_{x_r}^2(0) (1 + \mu_{a_{r1}} t)^2}$$

or

$$\sigma_y^2(t) = \frac{\sum_{j=1}^n \mu_{x_j}^2(0) (\sigma_{a_{j0}}^2 + \sigma_{a_{j1}}^2 t^2) \prod_{\substack{q=1 \\ q \neq j}}^n \mu_{x_q}^2(0) (1 + \mu_{a_{q1}} t)^2}{\prod_{l=n+1}^w \mu_{x_l}^2(0) (1 + \mu_{a_{l1}} t)^2} + \frac{\prod_{j=1}^n \mu_{x_j}^2(0) (1 + \mu_{a_{j1}} t)^2}{\prod_{r=n+1}^n \mu_{x_r}^2(0) (1 + \mu_{a_{r1}} t)^2} \sum_{l=n+1}^w \frac{\sigma_{a_{l0}}^2 + \sigma_{a_{l1}}^2 t^2}{(1 + \mu_{a_{l1}} t)^2} \quad (9-16)$$

Now if the mean value of $y(t)$ is to remain constant at its initial value, it is seen from Eq. (9-15) that

$$\frac{\prod_{j=1}^n \mu_{x_j}(0) (1 + \mu_{a_{j1}} t)}{\prod_{l=n+1}^w \mu_{x_l}(0) (1 + \mu_{a_{l1}} t)} = \frac{\prod_{j=1}^n \mu_{x_j}(0)}{\prod_{l=n+1}^n \mu_{x_l}(0)} \quad (9-17)$$

By inspection, Eq. (9-17) will be satisfied if

$$\frac{\prod_{j=1}^n \mu_{x_j}(0) \mu_{a_{j1}}}{\prod_{l=n+1}^w \mu_{x_l}(0) \mu_{a_{l1}}} = \frac{\prod_{j=1}^n \mu_{x_j}(0)}{\prod_{l=n+1}^n \mu_{x_l}(0)} \quad (9-18)$$

or

$$\prod_{j=1}^n \mu_{a_{j1}} = \prod_{l=n+1}^n \mu_{a_{l1}} \quad (9-19)$$

If the relationship among the mean values of the drift rates given by Eq. (9-19) can be satisfied, then the mean value of $y(t)$ will not drift.

Inspection of Eq. (9-16) shows that if the variance of $y(t)$ is to be kept as small as possible at all times, then the mean values of the drift rates of the parameters in the numerator of Eq. (9-13) should be negative and the mean values of the drift rates of the parameters in the denominator of this equation should be positive. While it may not be feasible to satisfy these conditions for all the parameters, they should be met for as many parameters as possible. It also should be noted that satisfying these conditions does not insure that the variance of $y(t)$ will not increase with time but at least if these conditions are met, the tendency for $\sigma_y^2(t)$ to increase with time will be kept at a minimum.

9-5. Resistors in Series

The above techniques will now be applied to the circuit of Sec. 4-2 consisting of n resistors in series. The expressions for the mean and the variance of the series resistance as a function of time of this circuit are given by Eqs. (8-13) and (8-14), respectively.

Since the series resistance is a linear function of the individual resistances, the results of the linear function analysis of Sec. 9-2 may be applied here. Comparison of Eqs. (8-13), (9-3), and (9-6) reveals that if the mean value of the series resistance is to remain constant, then the drift rates of the individual resistances must meet the following condition:

$$\sum_{j=1}^n \mu_{R_j(0)} \mu_{a_{j1}} = 0 \quad (9-20)$$

On the other hand, as was noted in Sec. 9-2, it is not possible to prevent the variance of a linear function from increasing with time by proper choice of the parameter drift rates. Hence it is doubly important that the initial variance of the series resistance be as small as possible. Thus the

procedure of Sec. 6-2 should be used to minimize this variance.

9-6 Controlling the Drift Rates of Resistances

From the condition stated by Eq. (9-20), it is seen that it is important to be able to control the drift rates of resistances. In considering methods of adjusting the drift rates of resistances, it should be noted that at constant ambient temperature the rate of drift of a resistance depends on the type resistor and the percentage of its rated power taken by the resistor (Ref. 33, 41 and 46).

Now for the series circuit of Sec. 9-5, the same current flows through each resistance. Thus once the mean values of the resistances are determined (by say, the variance minimization conditions), the amount of power each resistor takes with respect to the other resistors is fixed. Hence the drift rates must be adjusted by selection of the types of resistors used and the percentages of rated power taken by the resistors. This latter factor can be controlled by choosing resistors with the proper wattage rating. Thus if two resistors of the same type are connected in series and one resistance is twice the other then, as has been determined by other methods, the larger resistance should have twice the power rating of the smaller one.

9-7. Voltage Divider

As a second illustration of the techniques for preventing the drift of the performance criterion of an electronic system, the divider ratio of a resistive voltage divider will be considered. This circuit was first discussed in Sec. 4-3 and the drift of the divider ratio was analyzed in Sec. 8-5.

If the mean value of the divider ratio is to remain constant, then from Eq. (8-16) the following relationship must hold:

$$\frac{\mu_{R_2}(0)(1 + \mu_{a_{21}}t)}{\mu_{R_1}(0)(1 + \mu_{a_{11}}t) + \mu_{R_2}(0)(1 + \mu_{a_{21}}t)} = \frac{\mu_{R_2}(0)}{\mu_{R_1}(0) + \mu_{R_2}(0)} \quad (9-21)$$

This is equivalent to writing

$$\frac{\mu_{R_2}(0)\mu_{a_{21}}}{\mu_{R_1}(0)\mu_{a_{11}} + \mu_{R_2}(0)\mu_{a_{21}}} = \frac{\mu_{R_2}(0)}{\mu_{R_1}(0) + \mu_{R_2}(0)}$$

which upon simplification yields

$$\mu_{a_{11}} = \mu_{a_{21}} \quad (9-22)$$

Therefore, drift of the mean value of the divider ratio can be prevented, if both of the resistors have the same drift rate.

Now, if the resistors are of the same type and power rating, Eq. (9-22) will be satisfied when both resistors take the same amount of power. This occurs only if the two resistances are equal, which means that the divider ratio must be 0.5. The condition stated by Eq. (9-22) can be met, however, even if the divider ratio is not 0.5, by using different types of resistors for R_1 and R_2 and/or by using a resistor with a larger power rating for the larger resistance. For example, if μ_{K_V} is to be 0.1 then from Eq. (4-6) μ_{R_1} / μ_{R_2} must be 9. Hence R_1 takes 9 times the amount of power that R_2 does. Thus in this case, Eq. (9-22) can be approximately satisfied if R_1 is a two-watt resistor and R_2 is 1/4-watt resistor of the same type; provided, of course, that the input voltage is such that neither resistor takes more than its rated amount of power.

The conditions under which the variance of the divider ratio will not increase with time will now be found. In performing this analysis, it will be assumed that the drift rates of both resistors are equal; i.e., Eq. (9-22) is satisfied. In addition, it will be assumed that

$$\sigma_{a_{11}} = \sigma_{a_{21}} \quad (9-23)$$

and that

$$\sigma_{a_{10}} = \sigma_{a_{20}} \quad (9-24)$$

The assumption of Eq. (9-24) really amounts to saying that the standard deviation to mean ratios for R_1 and R_2 are equal (see Footnote No. 1, Chapter 2).

Using Eqs. (9-22), (9-23), and (9-24) in Eq. (8-17) results in

$$\sigma_{K_V}^2(t) = \frac{2\mu_{R_1}^2(0)\mu_{R_2}^2(0)(\sigma_{a_{10}}^2 + \sigma_{a_{11}}^2 t^2)}{(\mu_{R_1}(0) + \mu_{R_2}(0))^4 (1 + \mu_{a_{11}} t)^2}$$

or by Eq. (4-6)

$$\sigma_{K_V}^2(t) = \mu_{K_V}^2 (1 - \mu_{K_V})^2 \frac{(\sigma_{a_{10}}^2 + \sigma_{a_{11}}^2 t^2)}{(1 + \mu_{a_{11}} t)^2} \quad (9-25)$$

Now since $\mu_{K_V}(t)$ is being held constant, this variance will not rise above its initial value if

$$\frac{\sigma_{a_{10}}^2}{1} \geq \frac{\sigma_{a_{10}}^2 + \sigma_{a_{11}}^2 t^2}{(1 + \mu_{a_{11}} t)^2}$$

or

$$\sigma_{a_{10}}^2 \geq \frac{\sigma_{a_{11}}^2 t^2}{2\mu_{a_{11}} t + \mu_{a_{11}}^2 t^2}$$

This last expression may also be written as

$$\sigma_{a_{10}}^2 \geq \frac{\sigma_{a_{11}}^2}{\mu_{a_{11}} \left(\frac{2}{t} + \mu_{a_{11}} \right)} \quad (9-26)$$

Since the drift rates of the parameters are assumed to be extremely small fractions of the respective initial values of the parameters, there is no danger of the denominator of Eq. (9-26) going to zero if $\mu_{a_{11}}$ is negative. From the standpoint of keeping $\sigma_{K_V}^2(t)$ small, however, it can be seen from Eq. (9-25) that it is desirable to make $\mu_{a_{11}}$ positive. Regardless of the sign of $\mu_{a_{11}}$, Eq. (9-26) will be most difficult to satisfy when t assumes its largest value of interest. Thus if the inequality of Eq. (9-26) holds at this value of t , the variance of the divider ratio will never rise above its initial value during the useful life of the system.

9-8. Single Loop Feedback Amplifier System

In this section, methods of preventing the drift of the overall voltage gain of a feedback amplifier will be considered. This same system was discussed in Sec. 4-6 and expressions for the mean value and the variance of its overall voltage gain as functions of

time were developed in Sec. 8-6.

If the mean value of the overall voltage gain of the amplifier is to remain constant at its initial value, then from Eq. (8-18) it can be seen that

$$\frac{\mu_A(0)(1 + \mu_{a_{A1}}t)}{1 - \mu_A(0)\mu_B(0)(1 + \mu_{a_{A1}}t)(1 + \mu_{a_{B1}}t)} = \frac{\mu_A(0)}{1 - \mu_A(0)\mu_B(0)} \quad (9-27)$$

Since the term $\mu_A(0)\mu_B(0)\mu_{a_{A1}}\mu_{a_{B1}}t^2$ may be neglected if the percentages of drift of A and B are both very small, Eq. (9-27) can be written as

$$\frac{\mu_A(0) + \mu_A(0)\mu_{a_{A1}}t}{1 - \mu_A(0)\mu_B(0) - \mu_A(0)\mu_B(0)(\mu_{a_{A1}} + \mu_{a_{B1}})t} = \frac{\mu_A(0)}{1 - \mu_A(0)\mu_B(0)}$$

which means that

$$\frac{\mu_{a_{A1}}}{-\mu_B(0)(\mu_{a_{A1}} + \mu_{a_{B1}})} = \frac{\mu_A(0)}{1 - \mu_A(0)\mu_B(0)}$$

This expression can now be solved to obtain the following condition for zero drift of the mean value of the overall voltage gain of the feedback amplifier:

$$\frac{\mu_{a_{A1}}}{\mu_{a_{B1}}} = -\frac{\mu_A(0)\mu_B(0)}{1 - \mu_A(0)\mu_B(0)} \quad (9-28)$$

In attempting to satisfy this relationship, it should be noted that both $\mu_A(0)$ and $\mu_\beta(0)$ are generally determined by the specification of the mean value of the overall voltage gain and the condition for the minimization of $\sigma_{K_V}^2$. Furthermore, $\mu_{a_{A1}}$ is not always a quantity which is easily controlled. On the other hand, however, β is in many cases determined by the parameters of a resistive network. Thus Eq. (9-28) can usually be satisfied by using the methods of Sec. 9-6 to select the proper value of $\mu_{a_{\beta 1}}$.

It now remains to investigate the possibility of preventing the variance of the overall voltage gain from increasing with time. In order to do this, Eq. (8-19) will first be substituted into Eq. (8-20). This yields

$$\sigma_{K_V}^2(t) = \mu_{K_V}^4(t) \left[\frac{(\sigma_{a_{A0}}^2 + \sigma_{a_{A1}}^2 t^2)}{\mu_A^2(0)(1 + \mu_{a_{A1}} t)^4} + \mu_\beta^2(0) (\sigma_{a_{\beta 0}}^2 + \sigma_{a_{\beta 1}}^2 t^2) \right] \quad (9-29)$$

Now if it is assumed that Eq. (9-28) has been satisfied, then $\mu_{K_V}(t)$ does not vary with time. Thus but for the binomial involving $\mu_{a_{A1}}$, $\sigma_{K_V}^2(t)$ necessarily increases with time. If $\mu_{a_{A1}}$ is made positive, then

$\sigma_{K_V}^2(t)$ will either decrease with time or at least its tendency to increase with time will be reduced. Therefore, in designing the subsystem A, it is desirable to make $\mu_{a_{A1}}$ positive.

The importance of minimizing the initial values of the variances of the A, β , and K_V should also be emphasized. If this is done, then even if the variance of K_V increases with time, it will start increasing from a relatively small value. Hence at any instant of time $\sigma_{K_V}^2(t)$ will be smaller than if the above initial variances had not been minimized.

The design criteria derived in this section are actually conditions on the parameters of the systems A and B, rather than on the parameters of specific components such as resistors. Thus, rather than merely purchasing components with the desired parameters, it may be necessary to design the systems A and B to meet the above conditions or to work with the manufacturer in meeting these conditions.

CHAPTER 10: CONCLUSIONS

10-1 Introduction

This dissertation is concerned with the application of the techniques of mathematical statistics in the design phase of the reliability problem. A diligent search has not brought to attention any work which has applied this field of mathematics to reliability engineering. Instead, previous design efforts have focused on the use of special configurations, such as feedback, to insure that with the parameter tolerances given, the initial performance criteria of each system produced will fall within specifications. Such techniques, however, have ignored the reliability problem in that they have not considered the system performance as a function of time.

10-2 Analysis of the Distributions of the Performance

Criteria of Electronic Systems

In the course of the investigation to formulate a thesis for the design of more reliable electronic systems, several conclusions were reached indicating specific mathematical techniques and the practical value of the approach. In particular, it was found that:

- (a) The propagation of variance formula and the corresponding expression for the mean value can be used to determine the initial mean values and variances of the

performance criteria of a wide class of electronic systems. Thus, if the distributions of the system performance may be approximated by the normal distribution, the initial probabilities that these performance criteria are within tolerance can be determined. Although this technique is limited to systems whose performance criteria may be approximated by the linear terms of a Taylor's series, many electronic systems such as quality amplifiers and wave filters fall within this category.

- (b) The expression for the coefficient of linear correlation between functions of the same correlated variables can be used to determine the correlation among one set of two part parameters, when information is given describing another set. In addition, the variances and means of the second set of parameters can be obtained by the propagation of variance formula and the corresponding expression for the mean. Therefore, even when the two port parameters are considered as random variables the circuit analyst can select the set which is best suited to his problem. Though the above technique may be applied to most passive circuits with low tolerance parameters, its real usefulness lies in its application to transistors since manufacturers generally supply information on

only one set of transistor parameters. The limitations of this method are the same as those stated for the propagation of variance formula. Such transistor data as is available, indicates that the parameter distributions of many transistors meet these requirements.

- (c) The propagation of variance formula and the corresponding expression for the mean value can also be applied to the analysis of electronic systems whose parameters vary slowly with time. Furthermore, these parameters may be linear functions of time or they may be represented by a power series in time. Therefore, the mean values and variances of the performance criteria for a variety of types of electronic systems can be obtained as functions of time, provided of course, that these performance criteria meet the previously stated conditions for the application of the propagation of variance formula. This then permits the determination of the reliability characteristics of these systems. This information in turn can be used to design more reliable systems.

10-3 Improving the Reliability of Electronic Systems

This research resulted in the development of the following design techniques:

- (a) The variances of the system performance criteria were minimized with respect to the mean values of the system parameters, while constraining the criteria means to equal their prescribed values. This technique allows the designer to make sizeable reductions in the initial variances of the system performance criteria without changing the system configuration or the parameter tolerances. This technique was developed under the assumption that the parameter standard deviations are proportional to their respective mean values. While the parameters of a number of quality components meet this assumption the method is not restricted to these components, since the basic technique may be extended to parameters with other standard deviation-to-mean relationships.
- (b) Otherwise uncorrelated parameters were correlated by dividing the distributions of these parameters into cells and then matching the components from certain cells. This technique permits a manufacturer to reduce the variances of the system performance criteria by use of an assembly process which is intermediate between the random selection of components and an exact matching procedure. The final decision on which process to use is an economic problem since the cost of the assembly process must be weighed against the component cost.

(c) Both the drift of the mean values of the performance criteria of electronic systems and the tendency of the variances of these criteria to increase with time can be reduced by satisfying certain relationships among the drift parameters of the system. As an illustration, when the linear drift model was used to represent the parameters of certain systems, it was possible to hold the mean values of the performance criteria of these systems at their initial values while preventing their variances from increasing with time. Therefore, this method can be used to design more reliable electronic systems without necessarily using more expensive components or elaborate system configurations. Furthermore, since most electronic circuits contain components whose drift rates can be controlled (such as resistors), this method can be applied to a wide variety of systems.

In conclusion, it is seen that several techniques were developed to solve the following practical problems: minimization of the initial variance of the performance criteria of electronic systems; correlation of uncorrelated parameters; and prevention of the drift of the performance criteria of electronic systems. Though these techniques, as developed herein, are limited to systems whose performance criteria may be analyzed by the propagation of variance formula and whose parameter standard

deviations are proportional to their mean values, they cover many practical situations. Furthermore, they can be extended (see Chapter 11: Recommendations) to include a variety of additional systems.

10-4. Summary

This dissertation has shown that it is possible to design for reliability. Up to this time systems either have been designed to insure that their performance criteria would be within tolerance by use of expensive worst case methods or the probabilities that these performance criteria would be within tolerance was left to chance. The techniques developed here make it possible to determine the effect of parameter variations on the system performance criteria as a function of time and then to take these variations into account in design. Thus, a powerful new tool has been made available which permits the designer to meet the evergrowing demands for greater reliability in mass produced systems.

CHAPTER 11: RECOMMENDATIONS

11-1. Introduction

In this chapter recommendations for future work will be considered. These recommendations will be divided into two categories: 1. suggestions for the extension of the methods of analysis and design formulated in this research; 2. proposals for obtaining more data pertaining to the distributions of system parameters.

11-2. Extension of this Research

One of the most limiting assumptions made in this research is that the standard deviation of each parameter is directly proportional to its respective mean value. The variance minimization procedure discussed in Chapters 5 and 6 was performed on the basis of this assumption. Now as pointed out in Sec. 5-2, this assumption is true for the parameters of components such as certain types of resistors. However, it will undoubtedly be found to be untrue for certain other components. For example, it will not be true for parameters whose distributions are sections of a normal distribution. Hence it would be useful to devise a method for minimizing the variances of the performance criteria of a system whose parameter standard deviations are not proportional to their mean values. Furthermore, there are instances when certain systems

will become subsystems of a larger system. Thus it would be helpful to determine the relationship between the standard deviations and mean values of the performance criteria of various systems. For example, in the case of the transistor amplifier studied in this dissertation, Eqs. (6-13) and (6-19) reveal that $\sigma_{K_i}^2$ has been minimized.

Another assumption made in this research is that the standard deviations of the system parameters are considerably smaller than their respective mean values. While this assumption is true for a great many high quality components, it certainly would be useful if the methods of analysis and design formulated in this research were to be extended to include systems with relatively high tolerance components. Then these design procedures could be applied to situations where economic or technical difficulties preclude the use of low tolerance components. Some possible techniques which might prove helpful in this investigation are the inclusion of the first non-linear term in the Taylor's series approximation for non-linear functions (Ref. 38), moment-generating techniques (Ref. 29) and the Monte Carlo method (Ref. 31).

It also should be noted that there are situations where the Taylor's series for the system performance criteria may not converge, even when the standard deviations of the parameters are considerably smaller than

their mean values. An example of this is the single loop feedback amplifier when the system is unstable, in which case the denominator of the gain equation is zero. Another example is the parallel resonant circuit. While the analysis and design techniques of this dissertation should prove useful with respect to the resonant frequency (for example, the variance of this frequency might be minimized with respect to the parameter means of this circuit) the magnitude of the impedance of this circuit becomes very large at the resonant frequency and thus its Taylor's series may not converge. The investigation of situations such as these should prove quite worth while.

At the present stage of development of the techniques formulated in this research, algebra has posed a serious problem. In particular, it was noted that while in principle the variance of a system of any complexity can be minimized; in practice it was found that for a system of more than three or four parameters the amount of algebra become prohibitive. There are a number of possible solutions to this problem. One solution would be to reduce the number of parameters with respect to which the variance is to be minimized. Another solution is to divide the system into a number of subsystems, as was done in the case of the single loop feedback amplifier. The

use of a digital computer to perform the minimization of a function of a large number of variables should be explored.

A comparison of the sensitivity method of designing electronic systems with the techniques developed in this research should prove to be worth while. The sensitivity method is useful when the initial performance criteria of one or only a few units of a system are involved, for example, the performance criteria of a missile system. By contrast, while the methods formulated in this dissertation consider time variations, they are restricted to systems which are mass produced. In this latter situation, even though the probability of the system operating properly should be high, a few failures can generally be tolerated. In a missile system, however, even one failure can result in disaster. In view of the above remarks, it would be desirable to apply the techniques developed in this research to a system designed by use of the sensitivity method.

11-3 The Need for More Component Data

This work shows the need for making available to the designer component parameter data which are generally withheld or not requested. Up to now it has been customary that this information not be available since

designers did not consider statistical reliability characteristics.

To design for reliability the following data is of importance:

1. The initial distributions of the system parameters.
2. The distributions of the coefficients describing the drift of the system parameters.
3. The correlation among the system parameters and the coefficients describing their drift.

It should be emphasized that this information must include data for all types of components used in electronic systems. Most of the data which is now available pertains to resistors, a fact which restricted much of the work of this dissertation to resistive circuits.

As a result of these observations it appears useful to present the following proposals: First the designer must be impressed with the importance of the above data so that he will request it. Secondly, specifications for complete parameter information should be written into every contract for components. Finally, improved techniques for obtaining drift data must be developed. This latter task must necessarily be a cooperative effort between the manufacturer and the system designer.

APPENDIX A: MINIMIZATION OF THE VARIANCE OF A
LINEAR FUNCTION OF n UNCORRELATED RANDOM VARIABLES

The purpose of this appendix is to discuss the details of the variance minimization in Sec. 5-4. Both the necessary and the sufficient conditions for a minimum will be derived.

For convenience Eq. (5-15) may be rewritten as:

$$\sigma_y^2 = \sum_{j=1}^{n-1} b_j^2 (c_j^2 + c_n^2) \mu_{x_j}^2 - 2c_n^2 \mu_y \sum_{j=1}^{n-1} b_j \mu_{x_j} \quad (A-1)$$

$$+ 2c_n^2 \sum_{j=1}^{n-2} \sum_{l=j+1}^{n-1} b_j b_l \mu_{x_j} \mu_{x_l} + c_n^2 \mu_y^2$$

If this expression is differentiated with respect to the variables $\mu_{x_1}, \mu_{x_2}, \dots, \mu_{x_{n-1}}$ derivatives of the following type result:

$$\frac{\partial \sigma_y^2}{\partial \mu_{x_j}} = 2b_j^2 (c_j^2 + c_n^2) \mu_{x_j} - 2c_n^2 b_j \mu_y + 2c_n^2 b_j \sum_{\substack{l=1 \\ l \neq j}}^{n-1} b_l \mu_{x_l} \quad (A-2)$$

If each of these derivatives is set equal to zero, $n-1$ equations are obtained. The j^{th} equation is

$$c_n^2 \mu_y = b_1 c_n^2 \mu_{x_1} + b_2 c_n^2 \mu_{x_2} + \dots + b_j (c_j^2 + c_n^2) \mu_{x_j} + \dots + b_{n-1} c_n^2 \mu_{x_{n-1}} \quad (\text{A-3})$$

Solving these equations simultaneously for μ_{x_j} yields

$$\mu_{x_j} = \frac{\Delta_j}{\Delta} \quad (\text{A-4})$$

where

$$\Delta_j = \begin{vmatrix} b_1(c_1^2 + c_n^2) & b_2 c_n^2 & \dots & c_n^2 \mu_y & \dots & b_{n-1} c_n^2 \\ b_1 c_n^2 & b_2(c_2^2 + c_n^2) & \dots & c_n^2 \mu_y & \dots & b_{n-1} c_n^2 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ b_1 c_n^2 & b_2 c_n^2 & \dots & c_n^2 \mu_y & \dots & b_{n-1} c_n^2 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ b_1 c_n^2 & b_2 c_n^2 & \dots & c_n^2 \mu_y & \dots & b_{n-1}(c_{n-1}^2 + c_n^2) \end{vmatrix} \quad (\text{A-5})$$

and

$$\Delta = \begin{vmatrix} b_1(c_1^2 + c_n^2) & b_2 c_n^2 & \dots & b_j c_n^2 & \dots & b_{n-1} c_n^2 \\ b_1 c_n^2 & b_2(c_2^2 + c_n^2) & \dots & b_j c_n^2 & \dots & b_{n-1} c_n^2 \\ \hline b_1 c_n^2 & b_2 c_n^2 & \dots & b_j(c_j^2 + c_n^2) & \dots & b_{n-1} c_n^2 \\ \hline b_1 c_n^2 & b_2 c_n^2 & \dots & b_j c_n^2 & \dots & b_{n-1}(c_{n-1}^2 + c_n^2) \end{vmatrix} \quad (A-6)$$

Δ_j can be simplified by factoring out the quantities $b_1, b_2, \dots, b_{j-1}, b_{j+1}, \dots, b_{n-1}$ and $c_n^2 \mu_y$ and then subtracting the j^{th} row from each of the other rows.

Δ_j thus becomes

$$\Delta_j = c_n^2 \mu_y \prod_{\substack{m=1 \\ m \neq j}}^{n-1} b_m \begin{vmatrix} c_1^2 & 0 & \dots & 0 & \dots & 0 \\ 0 & c_2^2 & \dots & 0 & \dots & 0 \\ \hline c_n^2 & c_n^2 & \dots & 1 & \dots & c_n^2 \\ \hline 0 & 0 & \dots & 0 & \dots & c_{n-1}^2 \end{vmatrix} \quad (A-7)$$

Expansion of this determinant by the method of minors shows that

$$\Delta_j = c_n^2 \mu_y \prod_{\substack{m=1 \\ m \neq j}}^{n-1} b_m c_m^2 \quad (A-8)$$

Now consider the determinant Δ given by Eq. (A-6). In order to simplify this determinant, the quantities b_1, b_2, \dots, b_{n-1} may be factored out. In addition, the first column may be considered as being made up of the following binomials:

$$(b_1 c_1^2 + b_1 c_n^2), (b_1 c_n^2 + 0), \dots, (b_1 c_n^2 + 0)$$

Thus Δ may be written as the sum of two determinants (Ref. 49):

$$\Delta = \prod_{\lambda=1}^{n-1} b_{\lambda} \left[\begin{array}{cccc} c_n^2 & c_n^2 & \dots & c_n^2 \\ c_n^2 & (c_2^2 + c_n^2) & \dots & c_n^2 \\ \dots & \dots & \dots & \dots \\ c_n^2 & c_n^2 & \dots & (c_j^2 + c_n^2) \\ \dots & \dots & \dots & \dots \\ c_n^2 & c_n^2 & \dots & c_n^2 \\ \dots & \dots & \dots & \dots \\ c_n^2 & c_n^2 & \dots & c_n^2 \\ \dots & \dots & \dots & \dots \\ c_n^2 & c_n^2 & \dots & (c_{n-1}^2 + c_n^2) \end{array} \right] +$$

$$\begin{vmatrix}
 c_1^2 & c_n^2 & \dots & c_n^2 & \dots & c_n^2 \\
 0 & (c_2^2 + c_n^2) & \dots & c_n^2 & \dots & c_n^2 \\
 \dots & \dots & \dots & \dots & \dots & \dots \\
 0 & c_n^2 & \dots & (c_j^2 + c_n^2) & \dots & c_n^2 \\
 \dots & \dots & \dots & \dots & \dots & \dots \\
 0 & c_n^2 & \dots & c_n^2 & \dots & (c_{n-1}^2 + c_n^2)
 \end{vmatrix} \quad (A-9)$$

The first determinant in this equation can be evaluated by subtracting the first column from each of the other columns and then expanding by the method of minors.

This result is

$$\Delta = \prod_{l=1}^{n-1} b_l \left[\prod_{m=2}^n c_m^2 + \Delta_1 \right] \quad (A-10)$$

where Δ_1 is the second determinant in Eq. (A-9).

Δ_1 can be evaluated by continuing the process used in obtaining Eq. (A-9). For example, the next step is to consider the second column of Δ_1 as consisting of binomial elements. Thus Δ_1 may be written as the sum of two determinants, the first of which is equal to

$$\prod_{\substack{m=1 \\ m \neq 2}}^n c_m^2$$

This procedure results in the following expression

for Δ :

$$\Delta = \prod_{\ell=1}^{n-1} b_{\ell} \left[\prod_{m=2}^n c_m^2 + \prod_{\substack{m=1 \\ m \neq 2}}^n c_m^2 + \prod_{\substack{m=1 \\ m \neq 3}}^n c_m^2 + \dots + \prod_{m=1}^{n-1} c_m^2 \right] \quad (\text{A-11})$$

or

$$\Delta = \prod_{\ell=1}^{n-1} b_{\ell} \sum_{q=1}^n \prod_{\substack{m=1 \\ m \neq q}}^n c_m^2 \quad (\text{A-12})$$

Finally, substitution of Eqs. (A-3) and (A-12) into Eq. (A-4) gives

$$\mu_{x_j} = \frac{\prod_{\substack{m=1 \\ m \neq j}}^n c_m^2}{\sum_{q=1}^n b_j \prod_{\substack{m=1 \\ m \neq q}}^n c_m^2} \mu_y \quad (\text{A-13})$$

In spite of the fact that Eq. (A-3) involves only variables μ_{x_1} thru $\mu_{x_{n-1}}$, Eq. (A-13) applies to μ_{x_1} thru μ_{x_n} . The truth of this statement can be seen if it is noted that in Eq. (5-10) x_n has the same status as all the other variables. The choice of μ_{x_n} as the variable to be eliminated in Eq. (5-15) was an arbitrary one.

Now it must be shown that the point defined by Eq. (A-13) satisfies the sufficient conditions for a minimum of σ_y^2 . In order to accomplish this, the determinant of the matrix $\|G\|$ of Eq. (5-4) must be obtained. Differentiation of Eq. (A-2) with respect to μ_{x_j} and μ_{x_r} respectively yields

$$\frac{\partial^2 \sigma_y^2}{\partial \mu_{x_j}^2} = 2b_j^2 (c_j^2 + c_n^2) \quad (\text{A-14})$$

and

$$\frac{\partial^2 \sigma_y^2}{\partial \mu_{x_j} \partial \mu_{x_r}} = 2c_n^2 b_j b_r \quad (\text{A-15})$$

Comparing Eqs. (A-14) and (A-15) with Eq. (A-6) shows that for this problem

$$|G| = (a)^{n-1} \prod_{\lambda=1}^{n-1} b_{\lambda} \Delta \quad (\text{A-16})$$

From Eqs. (A-12) and (A-16) it is seen that $|G|$ is positive for all $n \geq 2$. This corresponds to saying that $|G|_0, |G|_1, |G|_2, \dots, |G|_{n-2}$ are all positive. Hence the sufficient conditions for a minimum, as stated in Sec. 5-3, are met.

The minimum value of σ_y^2 can be obtained by substituting Eq. (A-13) into Eq. (5-14):

$$\sigma_y^2 = \sum_{j=1}^n \left\{ b_j^2 c_j^2 \frac{\left[\prod_{\substack{m=1 \\ m \neq j}}^n c_m^2 \right]^2}{b_j \sum_{q=1}^n \left[\prod_{\substack{m=1 \\ m \neq q}}^n c_m^2 \right]} \mu_y \right\} \quad (\text{A-17})$$

This may be rewritten as

$$\sigma_y^2 = \frac{\left[\prod_{\lambda=1}^n c_{\lambda}^2 \right] \sum_{j=1}^n \left[\prod_{\substack{m=1 \\ m \neq j}}^n c_m^2 \right]}{\left[\sum_{q=1}^n \left[\prod_{\substack{m=1 \\ m \neq q}}^n c_m^2 \right] \right]^2} \mu_y^2 \quad (\text{A-18})$$

or

$$\sigma_y^2 = \frac{\prod_{l=1}^n c_l^2}{\sum_{q=1}^n \prod_{\substack{m=1 \\ m \neq q}}^n c_m^2} \mu_y^2 \quad (\text{A-19})$$

APPENDIX B: THE CONDITION FOR FEEDBACK TO REDUCE
THE VARIANCE OF THE GAIN OF AN AMPLIFIER

It has been shown in Sec. 6-5 that feedback with minimization of the variance of the overall voltage gain always reduces the variance of this gain below its value without feedback. The conditions under which feedback without minimization of the variance of the overall voltage gain will reduce the variance of this gain will be investigated in this appendix.

In performing this analysis, Eq. (6-25) will be used for the variance of the overall voltage gain with feedback, since the constraint that the mean value of this gain is specified has been incorporated in this equation. The variance of the voltage gain without feedback is given by Eq. (6-33). Now referring to Eqs. (6-25) and (6-33), the following inequality must hold if feedback is to reduce the variance of the voltage gain

$$(c_A^2 + c_B^2) \frac{\mu_{KV}^4}{\mu_A^2} - 2c_B^2 \frac{\mu_{KV}^3}{\mu_A} + c_B^2 \mu_{KV}^2 < c_A^2 \mu_{KV}^2 \quad (B-1)$$

Upon simplification, this equation becomes

$$\left(C_A^2 + C_B^2\right) \left(\frac{\mu_{K_V}}{\mu_A}\right)^2 - 2C_B^2 \left(\frac{\mu_{K_V}}{\mu_A}\right) + (C_B^2 - C_A^2) < 0 \quad (\text{B-2})$$

The boundary conditions where the variance of the voltage gain with feedback just equals the variance of the voltage gain without feedback can be obtained by setting the left-hand side of Eq. (B-2) equal to zero and then using the quadratic formula to solve for (μ_{K_V}/μ_A) . This procedure shows that if

$$\left(\frac{\mu_{K_V}}{\mu_A}\right) = 1, \quad \frac{C_B^2 - C_A^2}{C_B^2 + C_A^2} \quad (\text{B-3})$$

then feedback has no effect on the variance of the voltage gain.

The range of values of (μ_{K_V}/μ_A) for which feedback reduces the variance of the voltage gain can be obtained by evaluating the derivative of the left-hand side of the Eq. (B-2) with respect to (μ_{K_V}/μ_A) at the points specified by Eq. (B-3). The derivative of Eq. (B-2) is

$$\frac{\partial M}{\partial (\mu_{K_V}/\mu_A)} = 2(C_A^2 + C_B^2) \left(\frac{\mu_{K_V}}{\mu_A}\right) - 2C_B^2 \quad (\text{B-4})$$

where M is defined as the left-hand side of Eq. (B-2).

When $(\mu_{K_V}/\mu_A) = 1$, this derivative becomes

$$\frac{\partial M}{\partial(\mu_{K_V}/\mu_A)} = 2c_A^2 \quad (\text{B-5})$$

Thus the rate of change of M with respect to (μ_{K_V}/μ_A) when $(\mu_{K_V}/\mu_A) = 1$ is positive. Hence if the inequality of Eq. (B-2) is to be satisfied

$$(\mu_{K_V}/\mu_A) < 1 \quad (\text{B-6})$$

When $(\mu_{K_V}/\mu_A) = (c_B^2 - c_A^2)/(c_B^2 + c_A^2)$ the derivative of M with respect to (μ_{K_V}/μ_A) is

$$\frac{\partial M}{\partial(\mu_{K_V}/\mu_A)} = -2c_A^2 \quad (\text{B-7})$$

This shows that at this point, the rate of change of M with respect to (μ_{K_V}/μ_A) is negative. Thus if the condition of Eq. (B-2) is to be met

$$(\mu_{K_V}/\mu_A) > (c_B^2 - c_A^2)/(c_B^2 + c_A^2) \quad (\text{B-8})$$

The requirements stated by Eqs. (B-6) and (B-8) can be combined to specify the following range of values of

(μ_{K_V}/μ_A) :

$$\frac{c_B^2 - c_A^2}{c_B^2 + c_A^2} < \left(\frac{\mu_{K_V}}{\mu_A} \right) < 1 \quad (\text{B-9})$$

Since $\left| \mu_{K_V} / \mu_A \right| < 1$ defines negative feedback, Eq. (B-9) states that feedback will reduce the variance of the voltage gain only if the feedback is negative and within the range specified by this equation. In many practical situations $c_B^2 \ll c_A^2$ in which case the above condition becomes

$$-1 < \left(\frac{\mu_{K_V}}{\mu_A} \right) < 1 \quad (\text{B-10})$$

APPENDIX C: COMPUTATION OF THE
COEFFICIENT OF LINEAR CORRELATION

The coefficient of linear correlation between two random variables, correlated by a selection process, will now be computed for a specific numerical example. The computation will include both the equal range and the equal area processes. It will be assumed that the distributions of both variables are symmetrically truncated normal distributions with the truncation taking place at $x = \mu_x \pm 3\sigma_x$ in each case.

The equal range process will be considered first. Since both distributions are symmetrically truncated to the same degree the following observations can be made:

$$\mu_{x_{1t}} = \mu_{x_1} \quad (C-1)$$

$$\mu_{x_{2t}} = \mu_{x_2} \quad (C-2)$$

$$k_{1j} = k_{2j} \equiv k_j \quad (C-3)$$

$$k_0 = -k_n \quad (C-4)$$

$$S_{1j} = S_{2j} \equiv S_j \quad (C-5)$$

Eq. (7-15) may thus be rewritten as

$$\sigma_{x_1 x_2} = \frac{\sum_{j=0}^{n-1} S_j \left\{ \frac{\sigma_{x_1} [f(k_j) - f(k_{j+1})]}{S_j} + \mu_{x_1} \right\} \left\{ \frac{\sigma_{x_2} [f(k_j) - f(k_{j+1})]}{S_j} + \mu_{x_2} \right\}}{2 \int_0^{k_n} f(k) dk} - \mu_{x_1} \mu_{x_2} \quad (C-6)$$

Expansion of the numerator of the first term on the right-hand side yields

$$\sigma_{x_1 x_2} = \frac{\sum_{j=0}^{n-1} S_j \left\{ \frac{\sigma_{x_1} \sigma_{x_2} [f(k_j) - f(k_{j+1})]^2}{S_j^2} + (\sigma_{x_1} \mu_{x_2} + \sigma_{x_2} \mu_{x_1}) \frac{[f(k_j) - f(k_{j+1})]}{S_j} + \mu_{x_1} \mu_{x_2} \right\}}{2 \int_0^{k_n} f(k) dk} - \mu_{x_1} \mu_{x_2} \quad (C-7)$$

In view of Eq. (C-4), the middle term of this expansion becomes zero when the summation is carried from $j = 0$ thru $j = n-1$. Since the last term of the expansion contains no j 's, the summation of the S_j 's and the integral in the denominator cancel for this term, leaving $\mu_{x_1} \mu_{x_2}$. This in turn cancels the $\mu_{x_1} \mu_{x_2}$ outside the summation. Thus for the equal range case when the distributions are truncated to the same degree the covariance may be written as

$$\sigma_{x_1 x_2} = \sigma_{x_1 x_2} \frac{\sum_{j=0}^{n-1} \frac{[f(k_j) - f(k_{j+1})]^2}{S_j}}{2 \int_0^{k_n} f(k) dk} \quad (C-8)$$

Now by use of Eq. (2-12), the variances of the truncated distributions are found to be

$$\sigma_{x_{it}}^2 = \sigma_{x_i}^2 \left[1 - \frac{k_n f(k_n)}{\int_0^{k_n} f(k) dk} \right] \quad (C-9)$$

and

$$\sigma_{x_2}^2 = \sigma_{x_2}^2 \left[1 - \frac{k_n f(k_n)}{\int_0^{k_n} f(k) dk} \right] \quad (C-10)$$

The coefficient of linear correlation between x_1 and x_2 can now be obtained by substituting Eqs. (C-8), (C-9), and (C-10) into Eq. (3-4). The result is

$$\rho_{x_1 x_2} = \frac{\sum_{j=0}^{n-1} \frac{[f(k_j) - f(k_{j+1})]^2}{S_j}}{2 \left[\int_0^{k_n} f(k) dk - k_n f(k_n) \right]} \quad (C-11)$$

For $n = 3$, this equation becomes

$$\rho_{x_1 x_2} = \frac{\frac{[f(k_0) - f(k_1)]^2}{S_0} + \frac{[f(k_1) - f(k_2)]^2}{S_1} + \frac{[f(k_2) - f(k_3)]^2}{S_2}}{2 \left[\int_0^{k_3} f(k) dk - k_3 f(k_3) \right]}$$

The values of k_0 and k_3 can be obtained as follows by use of Eq. (7-5):

$$k_0 = k_{10} = k_{20} = \frac{x_{10} - \mu_{x_1}}{\sigma_{x_1}} = \frac{\mu_{x_1} - 3\sigma_{x_1} - \mu_{x_1}}{\sigma_{x_1}} = -3$$

$$k_3 = k_{13} = k_{23} = \frac{x_{13} - \mu_{x_1}}{\sigma_{x_1}} = \frac{\mu_{x_1} + 3\sigma_{x_1} - \mu_{x_1}}{\sigma_{x_1}} = 3$$

Now the width of each cell is

$$\Delta k = \frac{k_3 - k_0}{h} = \frac{3 - (-3)}{3} = 2$$

Hence

$$k_1 = k_{11} = k_{21} = k_0 + \Delta k = -3 + 2 = -1$$

$$k_2 = k_{12} = k_{22} = k_0 + 2\Delta k = -3 + 4 = 1$$

The covariance thus becomes

$$r_{x_1 x_2} = \frac{\frac{[f(-3) - f(-1)]^2}{S_0} + \frac{[f(-1) - f(1)]^2}{S_1} + \frac{[f(1) - f(3)]^2}{S_2}}{2 \left[\int_0^3 f(k) dk - 3f(3) \right]}$$

Since $f(k) = f(-k)$ this equation can be written as

$$R_{X_1 X_2} = \frac{\frac{[f(1) - f(-3)]^2}{S_2}}{\int_0^3 f(k) dk - 3f(3)}$$

Use of a unit normal distribution table (Ref. 2) yields

$$R_{X_1 X_2} = \frac{\frac{[0.0044 - 0.2400]^2}{0.1574}}{0.4987 - 3(0.0044)} = 0.739$$

Similar computations can be performed for various values of n . The results for n ranging from 0 thru 10 are given in the table at the end of this appendix.

Eq. (C-8) may also be used to compute the coefficient of linear correlation for the equal area process. In this case S_j is, by definition of the process, equal to the same value for each cell. This area and the cell boundaries can be obtained as follows from Eq. (7-6):

$$S_j = \frac{1}{3} \int_{-3}^3 f(k) dk = \int_{k_j}^{k_{j+1}} f(k) dk \quad (C-12)$$

Use of a unit normal distribution table then shows that

$$S_j = \frac{0.99730}{3} = 0.3324$$

Now choosing the boundaries between the two regions so that

Eq. (C-12) is satisfied results in $k_1 = -0.430$, $k_2 = 0.430$, and $k_3 = 3$. Substitution of

$k_1 = -0.430$, $k_2 = 0.430$, and $k_3 = 3$ into Eq. (C-11) gives

$$P_{X_1 X_2} = \frac{\frac{[F(-3) - F(-0.430)]^2}{0.3324} + \frac{[F(0.430) - F(0.430)]^2}{0.3324} + \frac{[F(0.430) - F(3)]^2}{0.3324}}{2 \left[\int_0^3 f(k) dk - 3f(3) \right]}$$

Since $f(k) = f(-k)$, this equation becomes

$$P_{X_1 X_2} = \frac{\frac{[F(-3) - F(-0.430)]^2}{0.3324}}{2 \left[\int_0^3 f(k) dk - 3f(3) \right]}$$

Use of the unit normal distribution table gives

$$P_{X_1 X_2} = \frac{\frac{[0.0044 - 0.3637]^2}{0.3324}}{0.4987 - 3(0.0044)} = 0.800$$

Use of a unit normal distribution table then shows that

$$S_j = \frac{0.99730}{3} = 0.3324$$

Now choosing the boundaries between the cells so that Eq. (C-12) is satisfied results in $k_0 = -3$, $k_1 = -0.430$, $k_2 = 0.430$, and $k_3 = 3$. Substituting in Eq. (C-11) gives

$$P_{X_1, X_2} = \frac{\frac{[f(-3) - f(-0.430)]^2}{0.3324} + \frac{[f(-0.430) - f(0.430)]^2}{0.3324} + \frac{[f(0.430) - f(3)]^2}{0.3324}}{2 \left[\int_0^3 f(k) dk - 3f(3) \right]}$$

Since $f(k) = f(-k)$, this equation becomes

$$P_{X_1, X_2} = \frac{\frac{[f(-3) - f(-0.430)]^2}{0.3324}}{3 \int_0^3 f(k) dk - 3f(k)}$$

Use of the unit normal distribution table yields

$$P_{X_1, X_2} = \frac{\frac{[0.0044 - 0.3637]^2}{0.3324}}{0.4987 - 3(0.0044)} = 0.800$$

These computations have also been performed for various other values of n . The results are given in the following table:

Table of the Coefficients of Linear Correlation Between Two Random Variables Correlated by a Selection Process

Number of cells	Equal ranges $k_0 = -3; k_3 = 3$	Equal areas $k_0 = -3; k_3 = 3$
1	0	0
2	0.643	0.643
3	0.739	0.800
4	0.837	0.868
5	0.889	0.903
6	0.921	0.924
7	0.941	0.942
8	0.954	0.951
9	0.964	0.960
10	0.969	0.964

APPENDIX D: LIST OF SYMBOLS

A list of the symbols used in this dissertation is given below. Since it is impractical to list every symbol, in some instances only a representative type of symbol is given. For example, μ_x indicates the method of specifying the mean value of a quantity.

Unless otherwise noted, it will be assumed that the MKS system of units is to be used. One exception to this rule is that drift rates are generally given on a per hour basis.

A	gain of single loop feedback system when feedback is removed
a_{j0}	random variable whose mean value is unity and whose variance is initial variance of j^{th} parameter
a_{j1}	drift rate of j^{th} parameter
a_{jq}	drift rate of j^{th} parameter associated with q^{th} power of t
b_j	constant used as coefficient of j^{th} parameter in linear function
c_j	standard deviation to mean ratio of j^{th} parameter
$E(y)$	expected value of y
$f(x)$	probability density function of x

$g(x)$	function of x
h_{fe}	short circuit forward-current transfer ratio of common emitter configuration
h_{ie}	short circuit input impedance of common emitter configuration
h_{oe}	open circuit output admittance of common emitter configuration
h_{re}	open circuit reverse-voltage transfer ratio of common emitter configuration
I_b	ac component of base current
I_c	ac component of collector current
K_i	current gain
K_v	voltage gain
K_{v_j}	voltage gain of the j^{th} stage of a cascade amplifier
k	normally distributed random variable whose mean value is zero and whose standard deviation is unity
n	number of variables, also number of stages in cascade system
R	total series resistance
R_j	the j^{th} resistance in series circuit
R_L	load resistance
R_1, R_2	voltage divider resistances
S_{ij}	area under $f(x_1)$ corresponding to j^{th} cell
t	time
V_{be}	ac component of base voltage with respect to emitter
V_{ce}	ac component of collector voltage with respect to base

$x_j, x_j(0)$	initial ¹ value of the j^{th} random variable
$x_j(t)$	value of j^{th} random variable at time t
$x_{j\ell}$	ℓ^{th} value of j^{th} random variable
γ_j	j^{th} function of random variables
z_{ie}	forward-transfer impedance of common emitter configuration with input open-circuited
β	fraction of output of single loop feedback system which is feedback
$\mu_{x_j}, \mu_{x_j}(0)$	initial mean value of x_j
$\mu_{x_j}(t)$	mean value of x_j at time t
$\rho_{x_j x_\ell}$	coefficient of linear correlation between x_j and x_ℓ
$\sigma_{x_j}, \sigma_{x_j}(0)$	initial value of standard deviation of x_j
$\sigma_{x_j}^2, \sigma_{x_j}^2(0)$	initial value of variance of x_j
$\sigma_{x_j x_\ell}$	covariance between x_j and x_ℓ

¹The "(0)" is included only when it is necessary to differentiate between $x_j(0)$ and $x_j(t)$.

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VITA

Emil C. Neu was born in _____, _____ on _____ . He was graduated with honor from Stevens Institute of Technology in 1955 and received the Master of Science degree in Electrical Engineering from Stevens in 1957. He has been in the doctoral program in electrical engineering at Newark College of Engineering since 1961.

From 1955 through 1957 he was a graduate assistant in the department of electrical engineering at Stevens. During the summers of 1955 and 1956, he was employed as an electrical engineer at ITT Federal Laboratories. In 1957 Mr. Neu was appointed an instructor in electrical engineering at Stevens. He held this post until 1962 at which time he was promoted to his present position of assistant professor of electrical engineering at the same institution. From 1957 through 1965 he has also been on the teaching staff of Stevens summer school.

Mr. Neu is a member of the Institute of Electrical and Electronics Engineers, the Society of the Sigma Xi and the American Society of Engineering Education. In 1962 he was awarded the United Engineers and Constructors preceptorship at Stevens.